RAZVIJANJE, PROJICIRANJE IN RAZPENJANJE UKRIVLJENIH PLOSKEV NA RAVNINO

Sandi Berk
Inštitut za geodezijo in fotogrametrijo FGG, Ljubljana
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Izvleček

Obravnavana je problematika prikazovanja ukrivljenih ploskev na ravnini. S postopkom Delaunayjeve triangulacije lahko iz niza zajetih točk tvorimo žično ogrodje ploskve. Ukrivljeno ploskev torej aproksimiramo z ravnimi trikotnimi ploskvicami. Predstavljena je metoda razpenjanja žičnega ogrodja ploskve na ravnino. Razpenjanje izvedemo tako, da vsota kvadratov linijskih deformacij vzdolž celotnega ogrodja doseže najmanjšo vrednost. Ideja zanj izhaja iz Airyjevega merila za projekcijo najmanjših deformacij.
Ključne besede: *Airyjevo merilo, Delaunayjeva triangulacija, izravnava, linijska deformacija, razpenjanje, ukrivljena ploskev, žično ogrodje*

1 UVOD

Prvi kartografski prikazi zemeljskega površja so bili narejeni še v dobri veri, da je le-to ravno. Začetek druge stopnje v razvoju kartografije sega v obdobje antike, v čas Pitagore (6. stol. pr. n. š.), ko je dozorelo spoznanje, da je zemeljsko površje ukrivljena ploskev. Pojavili so se dokazi, da ima Zemlja obliko krogle, in posledica teh spoznanj je bil pojav prvih kartografskih projekcij (Jovanović, 1983). Seveda pa se s problemi preslikavanja ploskev na ravnino ne srečujemo le v kartografiji. Takšne potrebe se pojavljajo tudi v konservatorstvu (npr. pri restavriranju stropnih fresk in ornamentov), v medicini in drugje. Izkaže se, da je preslikavanje takšnih, ne vnaprej opredeljivih ploskev dokaj zahtevna naloga. Namen prispevka je prikazati problematiko preslikavanja ploskev na ravnino in predstaviti eno izmed aplikativnih rešitev, uporabljenih v nekartografske namene.

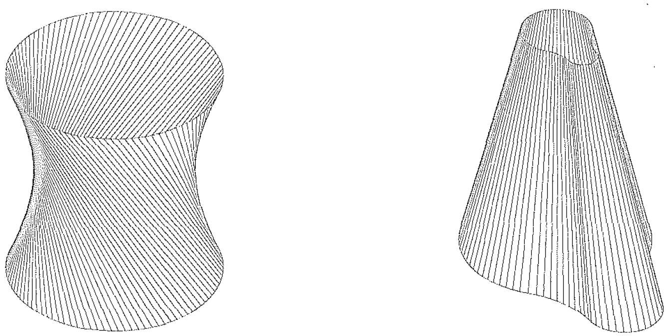
2 PRIKAZOVANJE UKRIVLJENIH PLOSKEV NA RAVNINI

Spomnimo se nekaterih osnovnih lastnosti gladkih ukrivljenih ploskev (npr. Jovanović, 1983). Preseke z ravninami, ki vsebujejo normalo na ploskev v dani točki, imenujemo normalni preseki. Normalni preseki imajo lahko v dani točki različne krivinske polmere. Normalna preseka z največjim in najmanjšim krivinskim polmerom sta vedno medsebojno pravokotna in ju imenujemo glavna normalna preseka. Ustrezna krivinska polmera označimo z R_1 in R_2 in ju imenujemo glavna krivinska polmera ploskve v dani točki. S pomočjo teh dveh polmerov definiramo polno ali Gaußovo ukrivljenost ploskve v dani točki kot

$$K = (R_1 \cdot R_2)^{-1}.$$

Posebna skupina ploskev so tiste, ki jih lahko tvorimo s premikanjem premice v prostoru. Imenujemo jih premonosne ploskve; skozi vsako točko takšne ploskve lahko položimo vsaj eno premico, ki v celoti leži na ploskvi. Vendar pa to še ni zadosten pogoj, da lahko ploskev razvijemo na ravnino; premonosna ploskev je na primer tudi enodelni hiperboloid (Slika 1, levo). Ploskev, ki jo lahko brez deformacij razvijemo na ravnino, imenujemo odvojna ploskev. Za vsako točko na njej je Gaußova ukrivljenost (K) enaka 0.

$$K = 0$$



Slika 1: Enodelni hiperboloid (levo) in stožčasta ploskev (desno); obe ploskvi sta premonosni, stožčasta je tudi odvojna

Odvojne ploskve tvorimo s premikanjem premice vzdolž neke krivulje (vodilje). Če jo premikamo vzporedno v dano smer, dobimo valjasto ploskev, če pa ima premica nepremično točko, dobimo stožčasto ploskev (Slika 1, desno). Le pri takšnih ploskvah lahko torej govorimo o razvijanju oziroma razgrnitvi na ravnino. Za vse ostale ploskve prehod na ravnino ni možen brez popačenja vsebine (Jovanović, 1983). Srečamo se torej z osnovnim problemom matematične kartografije: kako ukrivljeno ploskev oziroma elemente na njej preslikati na ravnino tako, da bodo deformacije čim manjše? Preslikava ploskve na ravnino brez deformacij bi pomenila ohranitev dolžin vseh linijskih elementov, s tem pa tudi ohranitev kotov in površin.

Postopek, ki nas privede do želenega rezultata, imenujemo projiciranje oziroma preslikavanje ukrivljene ploskve na ravnino. Oba pojma običajno dojemamo kot sinonima, čeprav projiciranje v ožjem pomenu izraža geometrijski postopek, kjer odnose med točkami na dani ploskvi in ustreznimi točkami na neki pomožni odvojni ploskvi (npr. plašč valja ali stožca) ali pa neposredno na ravnini vzpostavimo s pomočjo centralnega ali vzporednega snopa premic. Pojem preslikava pa izraža matematično zvezo med točkami na ploskvi in ustreznimi točkami (sliko) na ravnini. Zelo malo kartografskih projekcij je v resnici mogoče obravnavati kot geometrijsko projiciranje; poimenovanja posameznih projekcij (npr. stožčasta, valjna, horizontna), ki nakazujejo nanj, imajo bolj didaktični značaj (Jovanović, 1983). Včasih se s tem v zvezi pojavlja delitev na geometrijske in matematične kartografske projekcije.

3 DEFORMACIJE PRI PRESLIKAVAH UKRIVLJENIH PLOSKEV

Velikost deformacije v preslikani točki (na ravnini) izražamo s tako imenovanim linijskim merilom. To je razmerje med neskončno majhnim linijskim elementom (daljico) na projekcijski ravnini in ustreznim linijskim elementom na ploskvi

$$c_A = \lim_{B \to A} \frac{A'B'}{AB}, \qquad 3.1$$

kjer sta AB dolžina loka na ploskvi, $A'B'$ pa dolžina slike le-tega na ravnini (Maling, 1973, Jovanović, 1983). Linijskega merila ne smemo zamenjevati z glavnim merilom oziroma deklariranim merilom karte. To je pomanjšava vsebine karte, ki jo lahko izvedemo pred ali po preslikavi na ravnino in nima vpliva na deformacije vsebine. Strogo vzeto se linijsko merilo nanaša na točno določeno smer v dani točki, odvisno je namreč od smeri, iz katere točko B v enačbi 3.1 približujemo točki A, torej velja $c_A = c_A(\alpha)$.

želimo seveda, da se razmerja med linijskimi elementi na ploskvi in ustreznimi slikami le-teh na ravnini čimbolj ohranjajo. Odstopanje linijskega merila od enote

$$e_A = 1 - c_A \qquad 3.2$$

imenujemo linijska deformacija. Ta je lahko negativna ali pozitivna, odvisno od tega, ali gre za skrček ali raztezek linijskega elementa.

4 IZBIRA OPTIMALNIH PROJEKCIJ UKRIVLJENIH PLOSKEV

Kako torej izberemo najustreznejšo projekcijo? Za vsako ploskev lahko skonstruiramo poljubno število različnih konformnih (kotno pravilnih) in tudi ekvivalentnih (površinsko pravilnih) projekcij. Nikoli pa za ploskev, ki ni odvojna, projekcija na ravnino ne more biti hkrati konformna in ekvivalentna. Izbiro najustreznejše projekcije določajo oblika in velikost območja preslikave, oblika ploskve, ki jo oziroma s katere preslikujemo in predvsem naš namen. Za potrebe različnih kartometričnih nalog (merjenje kotov oz. smeri, merjenje površin, merjenje dolžin) izbiramo projekcije glede na vrsto oziroma značaj deformacij (Maling, 1989). Kadar poleg ploskve, ki jo preslikujemo na ravnino, opredelimo oziroma ustrezno omejimo tudi območje preslikave, lahko govorimo o optimalni projekciji. To je projekcija, za katero so – v celoti gledano – deformacije minimalne. Gre za tako imenovano Airyjevo merilo, ki terja, da je vsota kvadratov linijskih deformacij vzdolž celotnega preslikanega območja minimalna (Maling, 1973). Zapišemo ga lahko v obliki

$$\iint_S \left(\int_{\alpha=0}^{2\cdot\pi} e^2(\alpha) \cdot d\alpha \right) \cdot dS = \min,$$

kjer je $e(\alpha)$ linijska deformacija v dani točki in dani smeri, S pa območje preslikave. Projekcijo, ki zadovoljuje zgoraj navedeni pogoj, imenujemo tudi projekcija najmanjših deformacij. Poleg projekcije (absolutno) najmanjših deformacij lahko govorimo tudi o konformni projekciji najmanjših deformacij in ekvivalentni projekciji najmanjših deformacij. Pogoj je isti tudi za ti dve projekciji, seveda pa morata hkrati izpolnjevati še ustrezni pogoj konformnosti oziroma ekvivalentnosti. Izpeljava takšnih optimalnih projekcij je tudi za relativno enostavne ploskve in enostavno opredeljena območja običajno prezahtevna in analitično neizvedljiva naloga. Konformna projekcija najmanjših deformacij za preslikavo območja na elipsoidu, omejenega z zaključenim poligonom, je bila na primer predstavljena šele pred kratkim (Nestorov, 1997) in je v praksi izvedljiva le z numeričnimi metodami.

5 NEKARTOGRAFSKE PROJEKCIJE UKRIVLJENIH PLOSKEV

Postopki preslikavanja zemeljskega površja na ravnino so predmet matematične kartografije. Temeljijo na tem, da je naša ploskev relativno enostavna: krogla (sfera) ali rotacijski elipsoid (sferoid). Kljub temu so izpeljave kartografskih projekcij – kot smo videli – velikokrat zelo zahtevne. Če se ne omejujemo več na kroglo oziroma elipsoid, ampak obravnavamo ploskve na splošno, se torej srečamo z velikimi problemi. Vsaka malo bolj kompleksna ploskev postane z vidika preslikavanja na ravnino v praksi analitično neobvladljiva, kar pomeni, da trud, ki bi bil potreben za izpeljavo ustreznih enačb projekcij, po vsej verjetnosti ne bi bil povrnjen. Če želimo nalogo reševati analitično, moramo našo ploskev aproksimirati s takšno, za katero so ustrezne enačbe projekcij že izpeljane. Pri tem pa seveda lahko pride do precejšnjih odstopanj. Ena izmed rešitev problema, pri kateri se odpovemo analitičnemu pristopu reševanja naloge, bo opisana v nadaljevanju.

6 TVORBA ŽIČNIH OGRODIJ UKRIVLJENIH PLOSKEV

Za mersko (metrično) obravnavo ploskve je treba najprej določiti dovolj primerno razporejenih točk na njej. Gostota točk je odvisna od ukrivljenosti ploskve na danem območju in od zahtevane natančnosti. Običajno izberemo dobro določljive točke detajla na površini ploskve. Koordinate teh točk (x, y, z) lahko določimo na primer s postopki dvoslikovne fotogrametrije. Pri tem uporabimo poljuben (lokalni) pravokotni koordinatni sistem. Ploskev lahko sedaj opišemo analitično (kot aproksimacijsko ploskev določenega tipa), ali pa jo predstavimo v obliki neenakomerne trikotniške mreže. Zadnja je definirana kot mreža stikajočih se ravninskih trikotnikov, ki se glede na razporeditev točk (odvisno od geometrije ploskve) razlikujejo v velikosti, obliki in naklonu.

Trikotniško mrežo tvorimo s pomočjo niza raztresenih točk, običajno z Delaunayjevo triangulacijo. Gre za najboljšo trikotniško aproksimacijo dane ploskve; dobljeni trikotniki se kar najbolj približajo enakostraničnim (Clarke, 1995). Osnovni algoritem Delaunayjeve triangulacije v trirazsežnem prostoru je zelo enostaven. Iz niza zajetih točk na ploskvi izberemo poljubno trojico. Tvorimo najmanjšo kroglo, ki še vsebuje vse tri izbrane točke. Nato preverimo, ali je še kaka druga točka iz niza znotraj takšne krogle. Če takšnih točk ne najdemo, potem izbrana trojica predstavlja Delaunayjev trikotnik. S takšnimi preverjanji lahko poiščemo vse Delaunayjeve trikotnike. Z uporabo algoritma Delaunayjeve triangulacije je torej tvorba trikotniške mreže enolična in avtomatična. Če predhodno ne določimo roba triangulacije (obodnega poligona), je rezultat triangulacije konveksna lupina. Z ustreznim merilom za izločanje robnih trikotnikov z zelo ostrimi koti v ogliščih lahko tudi iskanje smiselnega roba območja prepustimo računalniku. Dobljeno trikotniško mrežo imenujemo tudi žično ogrodje ploskve.

7 RAZPENJANJE ŽIČNIH OGRODIJ UKRIVLJENIH PLOSKEV

Po zgoraj opisanem postopku ploskev aproksimiramo z ravnimi trikotnimi ploskvicami; robovi le-teh (stranice trikotnikov) tvorijo palično konstrukcijo. Osnovna ideja je v tem, da dobljeno konstrukcijo preoblikujemo v ravninsko tako, da jo pri tem čim manj deformiramo. Zato bomo v nadaljevanju namesto o projekciji govorili o razpenjanju žičnega ogrodja ploskev na ravnino. Nakazano rešitev v praksi zelo enostavno prevedemo na izravnavo ustrezne trilateracijske mreže. Prave (prostorske) dolžine stranic trikotnikov dobijo vlogo dolžinskih opazovanj. Rezultat izravnave so popravki dolžin, in sicer takšni, da žično ogrodje postane ravninsko in da je vsota kvadratov popravkov dolžin stranic, pomnoženih z ustreznimi utežmi, minimalna

$$\sum_{i=1}^{N} p_i \cdot v_i^2 = \min, \tag{7.1}$$

kjer so N število vseh stranic mreže, v_i popravek dolžine i-te stranice in p_i utež i-te stranice. Pomembna je torej pravilna izbira uteži. Linijska merila stranic po razpenjanju lahko izrazimo kot

$$c_i = \frac{d_i + v_i}{d_i}, \tag{7.2}$$

kjer je d_i prava dolžina stranice. Vstavimo enačbo 7.2 v enačbo 3.2 in izrazimo linijsko deformacijo i-te stranice

$$e_i = 1 - c_i = -\frac{v_i}{d_i}. \tag{7.3}$$

Za optimalno razpenjanje smiselno uporabimo Airyjevo načelo, da je vsota kvadratov linijskih deformacij vzdolž celotne konstrukcije minimalna

$$\sum_{i=1}^{N} e_i^2 = \min. \tag{7.4}$$

Enačba 7.4 je le drugačen zapis enačbe 7.1 in upoštevaje enačbo 7.3 dobimo

$$e_i^2 = p_i \cdot v_i^2 \Rightarrow p_i = \frac{1}{d_i^2}.$$

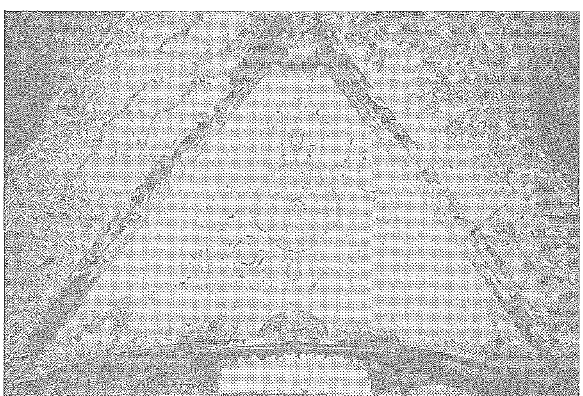
Utež dolžine i-te stranice mora torej biti obratnosorazmerna kvadratu dolžine. Takšen izbor uteži nam zagotavlja, da je vsota kvadratov linijskih deformacij vzdolž celotne konstrukcije pri razpenjanju minimalna. Rezultat postopka je optimalno razpeto žično ogrodje ploskve, kar je zelo dobra aproksimacija optimalne projekcije (projekcija najmanjših deformacij) dane ploskve na ravnino; z zgoščevanjem točk na ploskvi bi se namreč vse bolj bližali rezultatu, kakršnega bi dobili s takšno optimalno projekcijo.

Tako zastavljeno trilateracijsko mrežo izravnamo kot prosto ravninsko mrežo. Postopek izravnave je iterativen. Kot začetne približne koordinate vzamemo kar koordinate zajetih točk (oglišč trikotnikov) brez z-koordinate. Prvi približek je torej pravokotna projekcija žičnega ogrodja na ravnino xy. Ogrodje prej v lokalnem koordinatnem sistemu obrnemo tako, da je regresijska ravnina vozlišč ogrodja (oglišč trikotnikov) čimbolj horizontalna. Opozoriti je treba, da pri izravnavi trilateracijskih mrež pričakujemo majhne popravke opazovanj in zato smemo sisteme enačb le-teh linearizirati. V našem primeru lahko to storimo le, če je ploskev dovolj blizu odvojni. V nasprotnem primeru je zadostitev pogoja minimalnih deformacij nekoliko bolj zahtevna naloga.

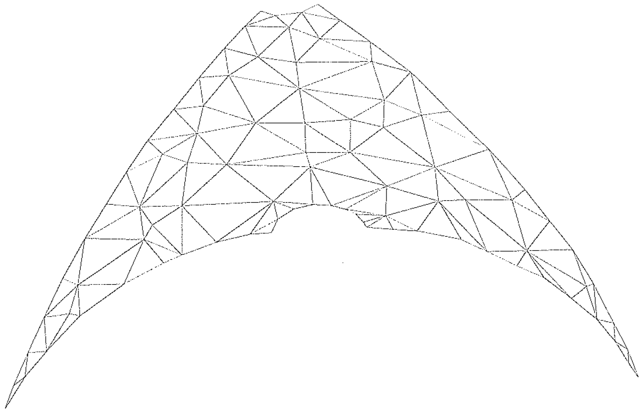
8 PRAKTIČNI PRIMER RAZPENJANJA NA RAVNINO

Opisani postopek razpenjanja na ravnino je bil razvit in uporabljen za ravninske prikaze fresk v cerkvi sv. Marije Alietske v Izoli. Med drugimi izdelki sta naročnika, Občina Izola in Medobčinski zavod za varstvo naravne in kulturne dediščine Piran, za potrebe restavratorskih del zahteval osem takšnih na ravnino razpetih fotografij fresk. Izvajalec projekta je bil Inštitut za geodezijo in fotogrametrijo FGG (Oven, Berk, 1998). Mersko dokumentiranje objektov kulturne dediščine danes temelji na trirazsežnih modelih objektov, ki jih tvorijo žična ogrodja posameznih stranskih ploskev (Kosmatin Fras et al., 1998). Zato so vse faze zajema in predstavitve tovrstnih ploskev že utečena zadeva. Prikazani so rezultati zajema in obdelave ene izmed fresk, ki je na polkrožnem oboku na stropu cerkve.

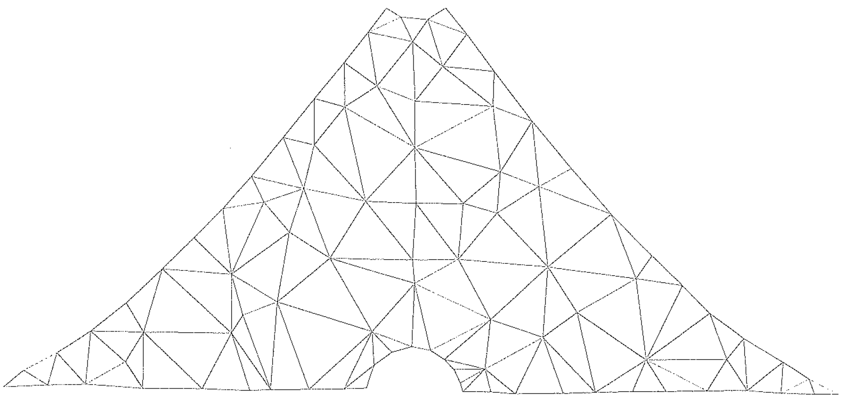
Freska je bila zajeta z 92 detajlnimi točkami. Z Delaunayjevo triangulacijo smo dobili žično ogrodje freske (Slika 3), ki ga tvori 126 trikotnikov in 217 stranic trikotnikov. Razpenjanje je torej v našem primeru pomenilo razrešitev predoločenega sistema 217 enačb s 184 (= 2x92) neznankami. Postopek je terjal 5 ponovitev izravnave. Razdalja med skrajnim levim in skrajnim desnim vogalom freske se je z razpetjem na ravnino povečala s 4,792 m na 6,113 m, torej za 1,321 m (tj. 27,6 %). Največja linijska deformacija stranice ogrodja je pri tem znašala 0,0094 oziroma 9,4 ‰. Po izvedenem razpenjanju žičnega ogrodja smo dobili ravninsko ogrodje (Slika 4), ki je služilo kot osnova za ravninski prikaz dane vsebine. Šlo je za skanirano fotografijo freske oziroma ornamenta (Slika 2). Vklop je bil izveden s prevzorčenjem, in sicer po odsekih glede na izbrane detajlne točke (vozlišča žičnega ogrodja). Uporabljena je bila bilinearna transformacija. Končni rezultat je bil na ravnino razpeta fotografija freske (Slika 5), natisnjena v merilu 1:10.



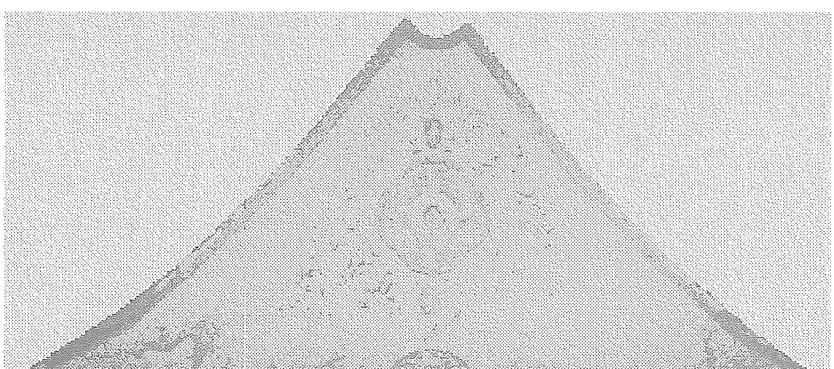
Slika 2: Fotografija freske



Slika 3: Žično ogrodje freske



Slika 4: Na ravnino razpeto žično ogrodje freske



Slika 5: Na ravnino razpeta fotografija freske

Še nekaj podatkov o uporabljenem instrumentariju in programski opremi: Stereopari so bili posneti z mersko kamero Rolleiflex 6006. Stereoizvrednotenje z zajemom karakterističnih točk fresk je bilo izvedeno na analitičnem instrumentu Adam Promap System. Za tvorbo žičnega ogrodja je bil uporabljen (posebej za to izdelani) program Delaunay, za razpenjanje tega ogrodja na ravnino pa program Trim (Berk, Janežič, 1995). Prevzorčenje skaniranih fotografij glede na razpeto ogrodje je bilo izvedeno s programom Adobe Photoshop.

9 ZAKLJUČEK

Prikaz ukrivljenih ploskev na ravnini v splošnem ni možen brez deformacij vsebine. Za enostavne ploskve (plašč krogle in rotacijskega elipsoida) je na voljo široka paleta kartografskih projekcij, s pomočjo katerih imamo deformacije tako ali drugače pod nadzorom. Kadar pa je naša ploskev bolj kompleksna, jo sicer lahko aproksimiramo z lepše prilegajočo se ploskvijo višjega reda, izpeljati ustrezno projekcijo zanjo pa je vse prej kot enostavna naloga. V prispevku je predstavljena možnost, pri kateri se odpovemo analitični rešitvi. Vsako ploskev lahko na podlagi niza raztresenih točk na njej aproksimiramo z ravnimi trikotnimi ploskvicami. Najboljšo možno aproksimacijo dobimo s postopkom Delaunayjeve triangulacije. Stranice dobljenih trikotnikov tvorijo žično ogrodje ploskve, ki ga razpnemo na ravnino tako, da so deformacije minimalne. Takšno razpenjanje je zelo dobra aproksimacija optimalne projekcije ploskve na ravnino. Opisani postopek lahko z zamenjavo vlog posameznih količin prevedemo na izravnavo ustrezne trilateracijske mreže. Za razpenjanje lahko uporabimo obstoječo programsko opremo za izravnavo geodetskih mrež. Na ravnino razpeto žično ogrodje ploskve nam nato služi kot podlaga za ravninski prikaz vsebine, ki se nahaja na njej.

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dr. Bojan Stopar

UNROLLING, PROJECTING, AND STRETCHING OF CURVED SURFACES INTO A PLANE

Sandi Berk
Institute of Geodesy, Cartography and Photogrammetry
FGG, Ljubljana
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Abstract

Problems of representing curved surfaces on the plane are discussed. From a set of captured points, a wire frame of the surface can be created using the Delaunay triangulation. In this way, the curved surface is approximated by small planar facets.
A method of stretching of the wire frame of the surface onto a plane is presented. The stretching is performed in such a manner that the sum of squares of linear distortions throughout the frame, as a whole, reaches a minimum value. The idea comes from the Airy's criterion for the projection of minimal distortions.
Keywords: *adjustment, Airy's criterion, curved surface, Delaunay triangulation, linear distortion, stretching, wire frame*

1 INTRODUCTION

The first cartographic representations of the Earth were made in good faith that it is a flat and disk-shaped form. The beginning of the second step in the development of cartography dates back in the antiquity, the Pythagorean era (6th century BC) when the notion of Earth as a curved surface had begun to ripen. Evidence arose proving the Earth is a sphere. The consequence of this was the appearance of first cartographic projections (Jovanović, 1983). However, problems arising from the projection of surfaces do not occur in cartography only. Such needs also exist in the conservation of cultural heritage (e.g. restoration of frescos and ornaments), medicine and elsewhere. It has become obvious that the mapping of surfaces, which cannot be predefined, represents a serious problem. The purpose of this paper is to illustrate the problems arising in projecting curved surfaces onto a plane. Its aim is also to present one of the applicable solutions used in non-cartographic purposes.

2 REPRESENTING CURVED SURFACES ON A PLANE

Let us recall some of the basic characteristics of smooth curved surfaces (e.g. Jovanović, 1983). The sections with planes including a normal onto the surface at the given point are called normal sections. At the given points, the normal sections may have different radii of curvature. The two normal sections, the one with the highest and the one with the lowest radius of curvature, are always mutually perpendicular. They are called principal normal sections. The corresponding radii of curvature are designated as R_1 and R_2 and are called principal radii of curvature at a given point. By using the two radii we may define the full or the Gaussian curvature of a surface at a given point:

$$K = (R_1 \cdot R_2)^{-1}.$$

A specific group o surfaces consists of those surfaces that can be generated though moving a straight line in space. These are called ruled surfaces; at least one straight line lying wholly in the plane can be placed through each point of such a surface. However, this is not a sufficient condition allowing us to unroll the surface onto a plane. For example, a onesheet hyperboloid is also a ruled surface (Figure 1, left). A surface that can be unrolled onto a plane without any distortions is called a developable surface. For each point lying on such a surface, the Gaussian curvature equals 0.

$K = 0$

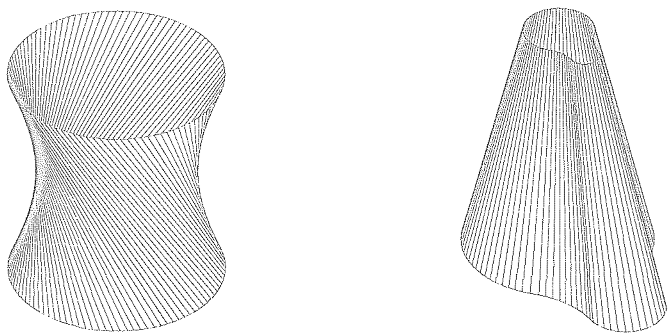

Figure 1: Onesheet hyperboloid (left) and conical surface (right); both surfaces are ruled surfaces, the conical surface is also developable one

Developable surfaces are generated by moving a straight line along a curve (directrix). Provided the straight line is moved parallel to itself in the given direction, a cylindrical surface is obtained. However, provided the straight line has a fixed point or vertex, a conical surface is obtained (Figure 1, right). Only with such surfaces it is possible to speak about unrolling or unfolding onto a plane. All other surfaces are impossible to project onto a plane without deforming the content (Jovanović, 1983). Now, we are facing the basic problem in mathematical cartography: how to project a curved surface and its elements onto a plane by reducing the deformation of the content to a minimum level. A projection of a surface onto a plane without deformation would mean the lengths of all linear elements as well as the angles and areas have been preserved.

The procedure bringing us to the desired result is called projecting or mapping of a curved surface onto o plane. Both terms are usually taken to be synonymous, although in a narrow sense, projecting denotes a geometrical procedure where relations between points in a given surface and their corresponding points in an auxiliary ruled surface (e.g. curved surface of a cylinder or cone) or directly in a plane are set up with a central or parallel bundle of straight lines. The term mapping denotes the mathematical connection between points on a surface and the corresponding points (image) in a plane. There are few cartographic projections that can actually be treated in terms of geometric projecting; names of individual projections (e.g. conical, cylindrical, horizontal) indicating the type serve merely in didactic purposes (Jovanović, 1983). In literature we may encounter the division of cartographic projections into geometrical and mathematical.

3 DEFORMATIONS IN CURVED SURFACE MAPPING

The extent of the deformation in a mapped point (in a plane) is denoted with the linear or particular scale. This is a ration between an infinitesimal linear element (a straight line) in the projection plane and the corresponding linear element in the surface. Therefore,

$$c_A = \lim_{B \to A} \frac{A'B'}{AB}, \qquad 3.1$$

where AB stand for the arc length in a plane and $A'B'$ for the length of the image of the former in the plane (Maling, 1973, Jovanović, 1983). The linear scale should not be mistaken for the principal scale or the stated scale of the map. This is a reduction of the map content which can be performed prior to or after the mapping into a plane. The reduction has no effect on the deformation of the content. Technically speaking, the linear scale is related to a specifically defined direction in a given point, for it depends on the direction from which the point B in the equation 3.1 is neared to the point A. This relation is shown in the following equation: $c_A = c_A(\alpha)$.

The ratios between the linear elements on the surface and their respective images in the plane need to be preserved to the highest extent possible. The deviations of the linear scale from the principal unit

$$e_A = 1 - c_A \qquad 3.2$$

is called linear distortion. Linear distortion may be either positive or negative, depending on whether the linear element is contracted or extended.

4 SELECTING OPTIMAL CURVED SURFACE PROJECTIONS

The question imposing itself is how to select the optimal projection. A random number of different conformal as well as equal-area projections can be constructed for each surface. However, this is not possible for surfaces not being developable for the projection into a plane cannot be conformal and equal-area at the same time. The selection of the optimal projection is dictated by the shape and size of the mapped region, and by the shape of the surface that is being mapped or from which it is being mapped, as well as by our intentions. For the purposes of various cartometric tasks (angle or direction measurement, measurement of area, measurement of length), projections are selected according to the type and the nature of distortions (Maling, 1989). When the mapped region is defined or properly outlined as well, beside the surface being mapped into the plane, we may speak about an optimal projection. On the whole, this is a projection containing minimal distortion. What we have here is the Airy's criterion requiring the sum of squared linear distortions along the entire mapped area to be minimal (Maling, 1973). The mathematical form of the criterion is as follows:

$$\iint_S \left(\int_{\alpha=0}^{2\cdot\pi} e^2(\alpha) \cdot d\alpha \right) \cdot dS = \min,$$

where $e(\alpha)$ denotes linear distortion in a given point and direction, and S denotes the mapped area.

The projection conforming to the abovementioned criterion is called the projection of minimal distortions.

Beside the projection of (absolutely) minimal distortions, we may also speak about a conformal projection of minimal distortions as well as about a equal-area projection of minimal distortions. The condition stands for the two projections as well. However, they also need to comply with the conformity and the equal-area condition. The derivation of such optimal projections is usually too demanding and a task impossible to perform analytically even for relatively simple surfaces and mapped regions with simple definitions. The conformal projection of minimal distortions for the mapping of an area of an ellipsoid bound with a closed polygon was introduced only recently (Nestorov, 1997). In practice, the projection can be performed only through numerical methods.

5 NON-CARTOGRAPHIC PROJECTIONS OF CURVED SURFACES

The procedures of mapping the earth's surface into a plane are a subject of mathematical cartography. The procedures are based upon the fact that we have to deal with a relatively simple surface: sphere or a ellipsoid of revolution (spheroid). Notwithstanding, the derivation of cartographic projections – as we have seen – can be often a rather demanding task. If we do not limit our deliberations to a sphere or ellipsoid but start dealing with surfaces in general, we encounter complex problems. Each even slightly more complex surface becomes in practice analytically insuperable for the point of view of mapping into a plane. This means that the effort needed for the derivation of corresponding projection equations, probably would not pay off. However, if we desire to solve the problem analytically, we need to approximate the surface in question to such a surface for which corresponding projection equations have already been derived. One of the solutions to the problem in which we renounce the analytical approach shall be described in the text that follows.

6 FORMATION OF WIRE FRAMES OF CURVED SURFACE

The metric analysis of a surface first requires the determination of a sufficient number of appropriately distributed points on the surface. The density of points depends on the curvature of the surface in the given region and on the required accuracy. Usually, well-definable points of details on the surface are selected. The coordinates of these points (x, y, z) can be determined, for example, with procedures found in stereophotogrammetry. An arbitrary (local) rectangular coordinate system is applied in the procedure. Now, the surface can be defined analytically (as an approximation surface of a specific type) or presented in the form of a triangular irregular network – TIN. The letter is defined as a mesh of adjacent planar triangles differing size, shape and inclination according to the distribution of points (depending on the geometry of the surface).

The triangular network is created with a set of random points, usually with the Delaunay triangulation. The Delaunay triangulation is the most effective triangular approximation of the given surface; the obtained triangles are almost equilateral (Clarke, 1995). The principal algorithm of the Delaunay triangulation in the three-dimensional space is very simple. Three points are selected arbitrarily from a set of acquired points. A minimal sphere is created still including the three selected points. A check is run to verify whether there are any other points from the set fitting into a similar sphere. Provided that no other points are found, the three selected points represent the Delaunay triangle. The creation of a triangular network becomes uniform and automated with the application of the Delaunay triangulation. If the edge of the network (peripheral polygon) is not defined preliminarily, the triangulation results in a convex hull. With an adequate condition for the elimination of edge acute-angled triangles in corners, the searching for a logical edge of the region may be left to a computer. The obtained triangular network is called the surface wire frame.

7 STRETCHING THE WIRE FRAMES OF CURVED SURFACES

After the procedure described above has been carried out, the surface is approximated to small planar facets; the sides of these facets (sides of triangles) form a truss construction. The main idea is to transform the obtained construction into planar form and to cause as little distortion as possible. Therefore, hereinafter the projection of the surface wire frame into a plane shall be referred to as the stretching of the surface wire frame into a plane. In practice, the indicated solution is simple to translate into the adjustment of the corresponding trilateration network. Actual (spatial) lengths of sides of triangles acquire the role of distance measurement. The result of the adjustment are residuals causing the wire frame to acquire planar form and the sum of weighted residuals squared to be minimal. Therefore,

$$\sum_{i=1}^{N} p_i \cdot v_i^2 = \min, \qquad 7.1$$

where N denotes the number of all network sides, v_i denotes the residual of the i-th side and p_i denotes the weight of the i-th side. The choice of adequate weights bears great importance. The linear scales of sides after stretching are expressed as

$$c_i = \frac{d_i + v_i}{d_i}, \qquad 7.2$$

where d_i denotes the actual length of the side. Let us insert the equation 7.2 into the equation 3.2 and the residual of the i-th side is expressed as

$$e_i = 1 - c_i = -\frac{v_i}{d_i}. \qquad 7.3$$

The Airy's criterion is applied for an optimal stretching. The criterion requires the sum of squared linear distortions along the entire construction to be minimal, leading to

$$\sum_{i=1}^{N} e_i^2 = \min. \qquad 7.4$$

The equation 7.4 is merely a different form of the equation 7.1. By taking into consideration the equation 7.3, the following is obtained:

$$e_i^2 = p_i \cdot v_i^2 \Rightarrow p_i = \frac{1}{d_i^2}.$$

The weight of the length of the i-th side must be inversely proportional to the square of the length itself. Such a choice of weights ensures that the sum of squared linear distortions along the entire construction is minimal during stretching. The procedure results in an optimally stretched surface wire frame which is a very favorable approximation of the optimal projection (projection of minimal distortions) of the given surface into a plane; the densification of points in a surface would near the result obtained through such an optimal projection.

A trilateration network set in such a manner is adjusted as a simple planar network. The adjustment procedure is an iterative one. The coordinates of the acquired points (vertices of triangles) without the z-coordinate are taken as initial approximation coordination. The first approximation is a rectangular projection of a wire frame into a xy-plane. The wire frame has to be shifted in the local coordinate system in order to make the regression plane of wire frame vertices (vertices of triangles) as horizontal as possible. It needs to be pointed out that in the adjustment of trilateration networks insignificant adjustment residues are expected to be done. Therefore, the equation systems of these networks may be linearized. In our case, the linearization may be performed only when the surface is approximated enough to a developable surface. Otherwise, meeting the minimal distortion condition poses a rather difficult task.

8 EXAMPLE OF STRETCHING INTO A PLANE

The procedure described above of stretching into a plane has been developed and used for the planar presentation of frescos in the church of St. Mary of Aliete in Izola. Beside other products, the commissioner, the Municipality of Izola, the Intermunicipial Institute for the Preservation of the Environment and Cultural Heritage Piran, ordered the production of eight photographs of frescoes stretched into a plane for the purposes of restoration works. The contractor implementing the project was the Institute of Geodesy, Cartography and Photogrammetry FGG (Oven, Berk, 1998). Nowadays, metric documenting of cultural heritage objects is based upon three-dimensional models of objects forming wire frames of individual lateral sides (Kosmatin Fras et al., 1998). Therefore, all phases of acquisition and presentation of such surfaces are a matter of routine. The results of acquisition and processing one of the frescoes are presented in this paper. The presented fresco is located on a semicircular arch on the church ceiling.

The fresco was acquired with 92 detail points. The Delaunay triangulation produced the wire frame of the fresco (Figure 3), formed by 126 triangles and 217 triangle sides. In this particular case, stretching the surface meant to solve an overdetermined system of 217 equations with 184 (=2 92) unknowns. The procedure required 5 adjustment iterations. With the stretching of the fresco into a plane, the distance between the far left and the far right vertex of the fresco increased from 4.792 m to 6.113 m, i.e. for 1.321 m or 27,6 %. The highest linear distortion of a wire frame side reached the value of 0.0094, i.e. 9.4 ‰. After the stretching of the wire frame had been performed, a planar frame was obtained (Figure 4), serving as a basis for a planar presentation of the given content. The object dealt with was a scanned photograph of the fresco, i.e. ornament (Figure 2). The edge matching was performed through resampling. The object was resampled region by region with regard to the selected detail points (wire frame vertices). Bilinear transformation was applied. The final result was a photograph of a fresco stretched into a plane (Figure 5) and printed on a scale of 1:10.

Additional data on the used equipment and software:
The stereopairs were taken with the Rolleiflex 6006 metric camera. The stereorestitution with the acquisition of characteristic points of the frescoes was performed with the Adam Promap System analytical instrument. The Delaunay software package (produced for this specific purpose) was applied in the creation of the wire frame. The stretching of the frame into a plane was carried out with the Trim software package (Berk, Janežič, 1995). The resampling of scanned photographs with respect to the stretched frame was performed with the Adobe Photoshop software package.

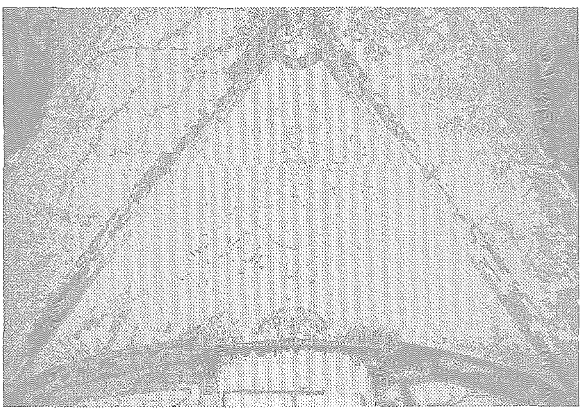

Figure 2: Photograph of the fresco

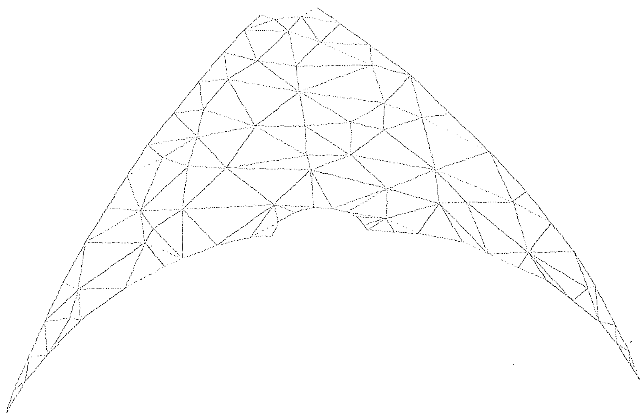

Figure 3: Wire frame of the fresco

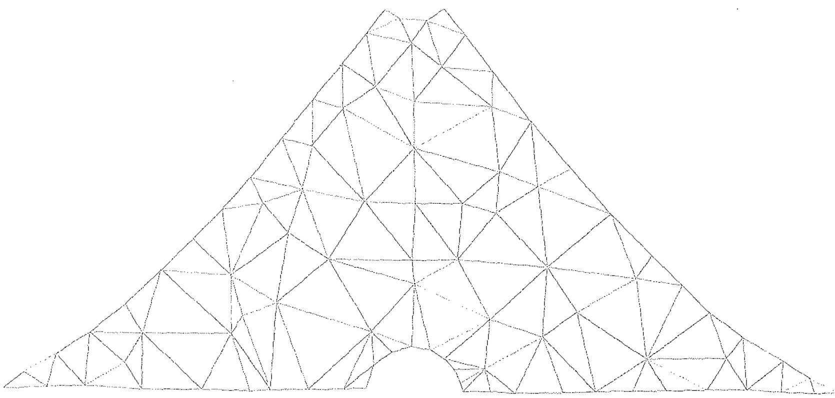

Figure 4: Wire frame of the fresco stretched into a plane

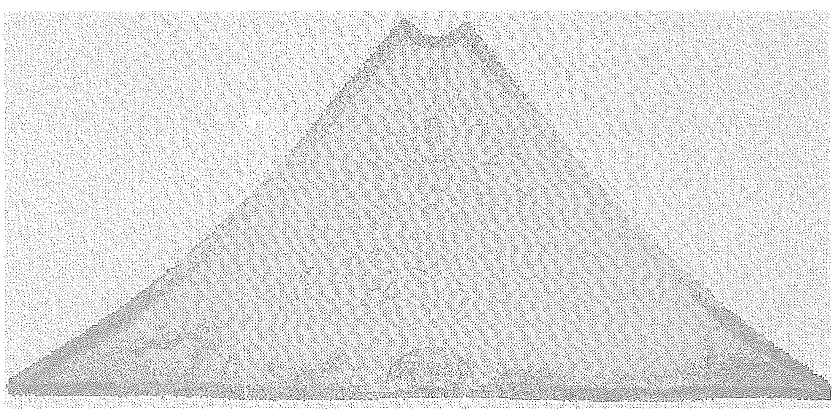

Figure 5: The photograph of the fresco stretched into a plane

9 CONCLUSION

Generally, the presentation of curved surfaces in a plane is not feasible without any distortions of content. A wide array of cartographic projections, enabling us to control the distortions, is available for dealing with simple surfaces (the curved surface of the sphere or the ellipsoid of revolution). However, when the complexity of the surface increases, it can be approximated to a higher-order surface fitting it closely. To derive the corresponding projection for the surface in question is all but an easy task. The paper laid out the possibility in which we renounce the analytical solution. Each surface can be approximated to planar triangular facets on the basis of a set of random points on these surfaces. The optimal approximation is achieved through Delaunay triangulation. The sides of obtained triangles form a surface wire frame which is stretched into a plane keeping the distortions as minimal as possible. Such stretching represents a very favorable approximation of the optimal projection of a surface into a plane. The described procedure can be translated into the adjustment of a corresponding trilateral network by changing the roles of individual values. Existing software packages for the adjustment of geodetic networks can be applied for stretching. The surface wire frame stretched into a plane than serves as a basis for the planar presentation of the surface content.

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