ARS MATHEMATICA CONTEMPORANEA

Also available at http://amc-journal.eu
ISSN 1855-3966 (printed edn.), ISSN 1855-3974 (electronic edn.)
ARS MATHEMATICA CONTEMPORANEA 11 (2016) 35–47

Z_3-connectivity of $K_{1,3}$-free graphs without induced cycle of length at least 5

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Received 26 September 2014, accepted 2 June 2015, published online 18 August 2015

Abstract

Jaeger *et al.* conjectured that every 5-edge-connected graph is Z_3-connected. In this paper, we prove that every 4-edge-connected $K_{1,3}$-free graph without any induced cycle of length at least 5 is Z_3-connected, which partially generalizes the earlier results of Lai [Graphs and Combin. 16 (2000) 165–176] and Fukunaga [Graphs and Combin. 27 (2011) 647–659].

Keywords: Z_3-connectivity, $K_{1,3}$-free, nowhere-zero 3-flow.

Math. Subj. Class.: 05C40

1 Introduction

Graphs in this paper are finite, loopless, and may have multiple edges. Terminology and notations not defined here are from [1].

For a graph G and $v \in V(G)$, denote by $N_G(v)$ (or shortly $N(v)$) the set of neighbors of v in G. Let $d_G(v) = |N_G(v)|$ and $N[v] = N(v) \cup \{v\}$. For $A \subset V(G)$, let $N(A) = \cup_{v \in A} N(v) \setminus A$. A graph G is *trivial* if $|V(G)| = 1$, and *non-trivial* otherwise. An n-*cycle* is a cycle of length n. A path P_n is a path on n vertices. The complete graph on n vertices is denoted by K_n, and K_n^- is obtained from K_n by deleting an edge. For two vertex-disjoint subgraphs H_1 and H_2 of G, denote by $e_G(H_1, H_2)$ (or simply $e(H_1, H_2)$) the number of edges with one end vertex in H_1 and the other one in H_2. If $V(H_1) = \{a\}$, we use $e_G(a, H_2)$(or simply $e(a, H_2)$) instead of $e_G(H_1, H_2)$. For simplicity, if V_1, V_2 are two disjoint subsets of $V(G)$, we use $e_G(V_1, V_2)$ for $e_G(G[V_1], G[V_2])$. Similarly, we define $e(V_1, V_2)$ and $e(a, V_2)$. For graphs $H_1, \ldots, H_s$, a graph G is $\{H_1, \ldots, H_s\}$-*free* if for each $i \in \{1, 2, \ldots, s\}$, G has no induced subgraph H_i.

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Let G be a graph and let D be an orientation of G. If an edge $e = uv \in E(G)$ is directed from a vertex u to a vertex v, then u is a tail of e, v is a head of e. For a vertex $v \in V(G)$, let $E^+(v)$={$e \in E(D)$: v is a tail of e }, and $E^-(v)$={$e \in E(D)$: v is a head of e }. Let A be an abelian group with identity 0 and $A^* = A - \{0\}$. Define $F(G, A) = \{f : E(G) \to A\}$ and $F^*(G, A) = \{f : E(G) \to A^*\}$. For each $f \in F(G, A)$, the boundary of f is a function $\partial f : V(G) \to A$ given by,

$$\partial f(v) = \sum_{e \in E^+(v)} f(e) - \sum_{e \in E^-(v)} f(e),$$

where "$\sum$" refers to the addition in A.

A function $b : V(G) \to A$ is called an *A-valued zero-sum function* on G if $\sum_{v \in V(G)} b(v) = 0$. The set of all A-valued zero-sum functions on G is denoted by $Z(G, A)$. A graph G is *A-connected* if G has an orientation D such that for any $b \in Z(G, A)$, there is a function $f \in F(G, A^*)$ such that $\partial f(v) = b$. In particular, if $\partial f(v) = 0$ for each vertex $v \in V(G)$, then f is called a nowhere-zero A-flow of G. More specifically, a nowhere-zero k-flow is a nowhere-zero Z_k-flow, where Z_k is the cyclic group of order k. Tutte [16] proved that G admits a nowhere-zero A-flow with $|A| = k$ if and only if G admits a nowhere-zero k-flow.

Integer flow problems were introduced by Tutte in [16]. Group connectivity was introduced by Jaeger *et al.* in [7] as a generalization of nowhere-zero flows. The following longstanding conjecture is due to Jaeger *et al.* and is still open.

Conjecture 1.1. *(Jaeger* et al. *[7]) Every 5-edge-connected graph is Z_3-connected.*

Conjecture 1.1 was extensively studied over thirty years. For the literature, some results can be seen in [3, 4, 10, 13, 17, 18] and so on. Recently, Thomassen [15] proved that every 8-edge-connected graph is Z_3-connected, which improved by Lovász, Thomassen, Wu and Zhang [12] as follows.

Theorem 1.2. *Every 6-edge-connected graph is Z_3-connected.*

However, Conjectures 1.1 is still open. A graph is *chordal* if every cycle of length at least 4 has a chord. A graph G is *bridged* if every cycle C of length at least 4 has two vertices x, y such that $d_G(x, y) < d_C(x, y)$. A graph is *HHD-free* if any k-cycle for $k \geq 5$ in the graph has at least two chords. Lai [9] characterized Z_3-connectivity of 3-edge-connected chordal graphs. Li *et al.* [11] and Fukunaga [6] generalized this result to bridged graphs and 4-edge-connected HHD-free graphs.

Theorem 1.3. *(Fukunaga[6]) Every 4-edge-connected HHD-free graph is Z_3-connected.*

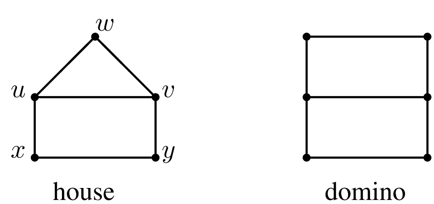

Figure 1: 2 forbidden graphs

On the other hand, it is easy to see that a graph G is HHD-free if and only if G contains no induced subgraph isomorphic to house, domino and k-cycle where $k \geq 5$. Note that a domino contains a $K_{1,3}$ as a subgraph. One naturally ask whether both house and domino may be replaced by a $K_{1,3}$. On the other hand, Xu [14] proved that Conjecture 1.1 is true if and only if every 5-edge-connected $K_{1,3}$-free graph is Z_3-connected. Thus, we consider Z_3-connectivity of $K_{1,3}$-free graphs without induced cycle of length at least 5 and prove the following theorem in this paper.

Theorem 1.4. *Let G be a 4-edge-connected, $K_{1,3}$-free simple graph. If G does not contain any induced cycle of length at least 5, then G is Z_3-connected.*

Theorem 1.4 cannot be implied by Theorem 1.2 in the sense that there are infinite graphs which is Z_3-connected by Theorem 1.4 but not by Theorem 1.2 as follows. Let H_1 be a copy of K_5 and H_2 be a copy of K_m where $m \geq 5$. Pick a vertex u of H_1 and a vertex v of H_2. Define G_m to be the graph obtained from H_1 and H_2 by identifying u and v. It is easy to see that for each $m \geq 5$, G_m is a 4-edge-connected $K_{1,3}$-free graph without any induced cycle of length at least 5. Thus, G_m is Z_3-connected by Theorem 1.4. Clearly, G_m has an edge cut of size 4 which implies Theorem 1.2 does not show that G_m is Z_3-connected.

Theorem 1.3 cannot imply Theorem 1.4 in the sense that there are infinite graphs which is Z_3-connected by Theorem 1.4 but not by Theorem 1.3 as follows. Let H_i be a copy of K_{n_i} where $1 \leq i \leq 4$ and $n_i \geq 5$ for $i \in \{1, 2, 3, 4\}$. Pick two distinct vertices u_i and v_i of H_i. Denote by Γ_n the graph obtained from H_1, H_2, H_3, H_4 by identifying v_i with u_{i+1} for $i = 1, 2, 3$, and v_4 with u_1. It is easy to verify that Γ_n contains a house and so Theorem 1.3 cannot guarantee that Γ_n is Z_3-connected but Theorem 1.4 does.

The paper is organized as follows: In Section 2, the former related results are presented, and some lemmas are established. In Section 3, the main theorem is proved.

2 Lemmas

For a subset $X \subseteq E(G)$, the contraction G/X denotes the graph obtained from G by identifying the two ends of each edge in X and then deleting all the resulting loops. Note that even if G is simple, G/X may have multiple edges. For convenience, we write G/e for $G/\{e\}$, where $e \in E(G)$. If H is a subgraph of G, then we write G/H for $G/E(H)$.

For $k \geq 2$, a *wheel* W_k is the graph obtained from a k-cycle by adding a new vertex, called the *center* of the wheel, which is adjacent to every vertex of the k-cycle. A wheel W_k is *odd* (*even*) if k is odd (or even). For technical reasons, we refer the wheel W_1 to a 3-cycle.

In order to prove Theorem 1.4, we need some lemmas. Some results [2, 5, 7, 8, 9, 10] on group connectivity are summarized as follows.

Lemma 2.1. Let A be an abelian group and G a simple graph. Then each of the following holds:
(1) K_1 is Z_3-connected.
(2) If $e \in E(G)$ and if G is A-connected, then G/e is A-connected.
(3) If H is a subgraph of G and if both H and G/H are A-connected, then G is A-connected.
(4) For $n \geq 5$, K_n^- and K_n are Z_3-connected;
(5) An n-cycle is A-connected if and only if $|A| \geq n + 1$;

(6) For every positive integer k, W_{2k} is Z_3-connected and W_{2k+1} is not Z_3-connected.
(7) Let H be a Z_3-connected subgraph of G. If $e(v, V(H)) \geq 2$ for $v \in V(G-H)$, then the subgraph induced by $V(H) \cup \{v\}$ is Z_3-connected.
(8) Let H_1, H_2 be subgraphs of G such that H_1 and H_2 are A-connected, If $V(H_1) \cap V(H_2) \neq \emptyset$, then $H_1 \cup H_2$ is A-connected.

For a graph G with $u, v, w \in V(G)$ such that $uv, uw \in E(G)$, let $G_{[uv,uw]}$ denote the graph obtained from G by deleting two edges uv and uw, and then adding a new edge vw, that is, $G_{[uv,uw]} = G \cup \{vw\} - \{uv, uw\}$.

Lemma 2.2. (Chen *et al.* and Lai, [2, 9]) Let A be an abelian group, let G be a graph and u, v, w be three vertices of G such that $d(u) \geq 4$ and $v, w \in N(u)$. If $G_{[uv,uw]}$ is A-connected, then so is G.

A graph G satisfies the *Ore-condition* if $d_G(u) + d_G(v) \geq n$ for every pair of nonadjacent vertices u and v of G.

Theorem 2.3. (Luo *et al.*[13]) *Let G be a simple graph on n vertices, where $n \geq 3$. If G satisfies the Ore-condition, then G is not Z_3-connected if and only if G is one of $\{G_1, G_2, \ldots, G_{12}\}$ shown in Figure 2.*

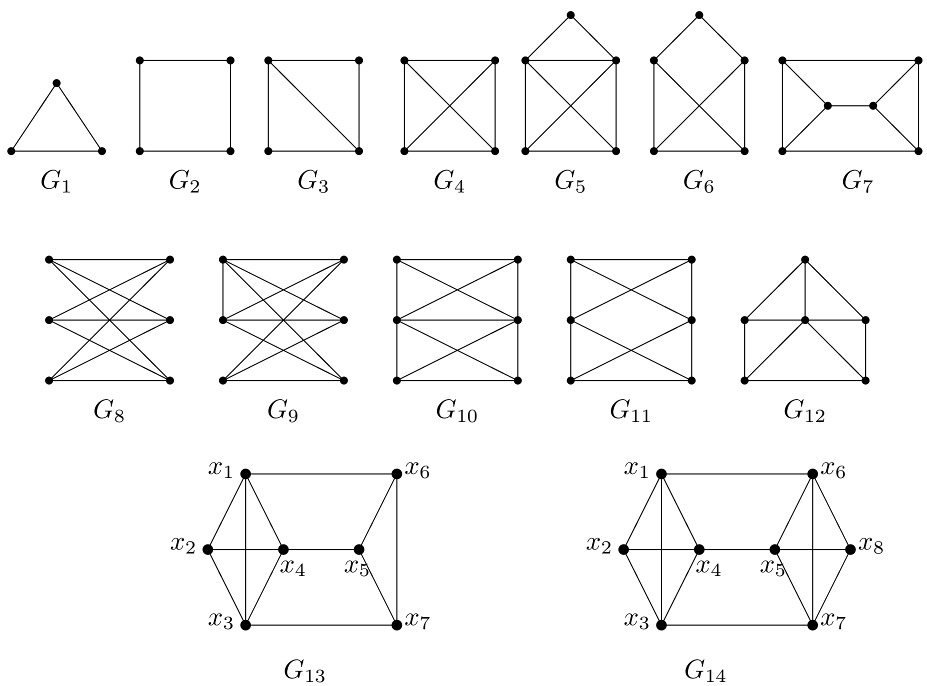

Figure 2: 14 specified graphs

Lemma 2.4. Suppose that H is one graph of $\{G_7, G_{13}, G_{14}\}$. Denote by G the graph obtained from H by adding an edge $e = xy$ which is neither of H nor parallel to any existing edge of H. Then G is Z_3-connected.

Proof. We use the same notation of G_{13}, G_{14} shown in Figure 2. Let $H = G_7$, then G satisfies the Ore-condition. By Theorem 2.3, G is Z_3-connected.

Let $H = G_{13}$. If $x_2 \in \{x, y\}$, then G satisfies the Ore-condition. By Theorem 2.3, G is Z_3-connected. Thus, assume that $x_2 \notin \{x, y\}$. By symmetry, let $e = x_1x_5$. Contracting 2-cycle in $G_{[x_1x_2,x_1x_3]}$ and contracting all 2-cycles generated in the process, we get an even wheel W_4 with the center at x_5, which is Z_3-connected by Lemma 2.1 (6) and so G is Z_3-connected by Lemma 2.2.

Let $H = G_{14}$. If $e = x_2x_8$, then G satisfies the Ore-condition. Since $|V(H)| = 8$, by Lemma 2.3, G is Z_3-connected. Thus, assume that $e \neq x_2x_8$. By symmetry, assume that $e = x_1x_5$ or $e = x_2x_6$. In the former case, contracting 2-cycle in $G_{[x_1x_2,x_1x_3]}$ and contracting all 2-cycles generated in the process, we obtain an even wheel W_4 induced by $\{x_1, x_4, x_5, x_6, x_7\}$ with the center at x_5. Contracting this W_4 into one vertex and contracting 2-cycle generated in the process, finally we get a K_1 which is Z_3-connected. By Lemmas 2.1 (7) and 2.2, G is Z_3-connected. In the latter case, contracting 2-cycle in $G_{[x_1x_2,x_1x_3]}$ and contracting all 2-cycles generated in the process, we obtain an even wheel W_4 induced by $\{x_4, x_5, x_6, x_7, x_8\}$ with the center at x_5, which is Z_3-connected by Lemma 2.1. Note that x_1 has two neighbors in this even wheel. By Lemma 2.1(7), $G_{[x_1x_2,x_1x_3]}$ is Z_3-connected. By Lemma 2.2, G is Z_3-connected. □

3 Proof of Theorem 1.4

Throughout this section, we assume that $\kappa'(G) \geq 4$, $K_{1,3}$-free simple graph and G does not contain any induced cycle of length at least 5. We argue our proof by contradiction, assume that G is a counterexample to Theorem 1.4 with $|V(G)|$ minimized.

Lemma 3.1. Suppose that H is a maximal Z_3-connected subgraph of G and H_i is a component of $G - V(H)$. Let $x_1 \in V(H)$ such that $x_1y_1, \ldots, x_1y_k$, where $y_1, \ldots, y_k \in V(H_i)$ and $2 \leq k \leq 3$. Then each of $y_1, \ldots, y_k$ is not a cut vertex of H_i.

Proof. We only prove the case that $k = 3$. The proof for that $k = 2$ is similar. Without loss of generality, we will prove that y_3 is a cut vertex of H_i. Suppose otherwise that y_3 is not a cut vertex of H_i. Since the maximality of H, $e(y_i, H) = 1$ by Lemma 2.1 (7). Since G is $K_{1,3}$-free, $y_1y_2, y_1y_3, y_2y_3 \in E(G)$. Since $\kappa'(G) \geq 4$, let $x_4 \in V(H)$ and $y_4 \in V(H_i)$ such that $x_4y_4 \in E(G)$, and y_4 is not in the component of $H_i - y_3$ containing y_1 and y_2.

Consider the neighbors of y_1 and y_2. Let $N(y_1) \setminus \{x_1, y_2, y_3\} = \{u_1, u_2, \ldots, u_a\}$ and $N(y_2) \setminus \{x_1, y_1, y_3\} = \{v_1, v_2, \ldots, v_b\}$. Since G is $K_{1,3}$-free, both subgraphs induced by $\{u_1, \ldots, u_a\}$ and by $\{v_1, \ldots, v_b\}$ are complete graphs. We assume, without loss of generality, that $a \geq b$. Since G is 4-edge-connected, $a \geq 1$ and $b \geq 1$. Note that y_3 is a cut vertex of H_i and G is $K_{1,3}$-free. The following claim is straightforward.

Claim. All neighbors of y_3 are y_1, y_2 in the component of $H_i - y_3$ containing $\{y_1, y_2\}$.

Case 1. $\{u_1, \ldots, u_a\} \cap \{v_1, v_2, \ldots, v_b\} \neq \emptyset$.

If $a \geq 4$, then the subgraph induced by $\{y_1, u_1, u_2, \ldots, u_a\}$ is a complete graph K_{a+1}, which is Z_3-connected by Lemma 2.1 (4). By Lemma 2.1 (7), G contains a Z_3-connected subgraph induced by $V(H) \cup \{y_1, y_2, y_3, u_1, u_2, \ldots, u_a\}$, contrary to the maximality of H. Thus, $a \leq 3$.

Assume that $a = 3$. If $|\{u_1, u_2, \ldots, u_a\} \cap \{v_1, v_2, \ldots, v_b\}| \geq 2$, then the subgraph induced by $\{y_1, y_2, u_1, \ldots, u_a\}$ is K_5 or K_5^-, which is Z_3-connected by Lemma 2.1 (4). By Lemma 2.1 (7), G contains a Z_3-connected subgraph induced by $V(H) \cup \{y_1, y_2, y_3, u_1, u_2, \ldots, u_a\}$ which is larger than H, contrary to the choice of H. Thus, $|\{u_1, u_2, \ldots, u_a\} \cap \{v_1, v_2, \ldots, v_b\}| = 1$ and let $u_1 = v_1$. Assume that $3 \geq b \geq 2$. Since $\kappa'(G) \geq 4$, there is a path from $\{u_2, u_3\}$ to v_2 avoiding each vertex of $\{y_1, y_2, u_1\}$. Since G contains no induced cycle of length at least 5, $u_i v_2 \in E(G)$ where $i \in \{2, 3\}$. In this case, G contains an even wheel W_4 induced by $\{y_1, y_2, u_1, u_i, v_2\}$ with the center at u_1, which is Z_3-connected by Lemma 2.1 (6). By Lemma 2.1 (7), G contains a Z_3-connected subgraph induced by $V(H) \cup \{y_1, y_2, y_3, u_1, u_2, \ldots, u_a\}$, contrary to the maximality of H. Thus, $b = 1$. In this case, since $\kappa'(G) \geq 4$, let $u_2 p_1, u_3 q_1 \in E(G)$ where $p_1 \notin \{u_1, u_3, y_1\}$ and $q_1 \notin \{u_1, u_2, y_1\}$. Since $\kappa'(G) \geq 4$ and G contains no cycle of length at least 5, $p_1 q_1, p_1 u_3, q_1 u_2 \in E(G)$. We replace p_1 with u_2 and replace q_1 with u_3. By argument above, we obtain p_2, q_2 such that $p_2 q_2, p_2 p_1, q_2 q_1, p_2 q_1, q_2 p_1 \in E(G)$. Repeating such a way, we can obtain two infinite sequences of $p_1, p_2, \ldots$ and $q - 1, q_2 \ldots$ such that $p_i p_{i+1}, q_i q_{i+1}, p_i q_i, p_i q_{i+1}, q_i, q_{i+1} \in E(G)$ for $i = 1, 2, \ldots$. This contradicts that G is finite.

We are left to consider that $a \leq 2$. In this case, since G is 4-edge-connected, $a = b = 2$ and $\{u_1, u_2\} = \{v_1, v_2\}$. As the proof above, we also obtain a contradiction.

Case 2. $\{u_1, \ldots, u_a\} \cap \{v_1, v_2, \ldots, v_b\} = \emptyset$.

We claim that $a + b \geq 4$. Suppose otherwise that $a + b \leq 3$. It follows that either $a = 2, b = 1$ or $a = b = 1$. We only prove the case when $a = 2$ and $b = 1$. The proof is similar for the case that $a = b = 1$. Since $a = 2$ and $b = 1$, $y_1 u_1, y_1 u_2, y_2 v_1 \in E(G)$. By the Claim, y_3 is not adjacent to one of u_1, u_2 and v_1. Thus, $\{y_1 u_1, y_1 u_2, y_2 v_1\}$ is an edge cut of size 3, contrary to that $\kappa'(G) \geq 4$.

Assume that $a \geq 4$. If $b \geq 4$, then G contains a path from $\{u_1, \ldots, u_a\}$ to $\{v_1, \ldots, v_b\}$. Note that $\kappa'(G) \geq 4$ and G has no cycle of length at least 5. If $2 \leq b \leq 3$, then each vertex of $\{v_1, v_2 \ldots, v_b\}$ has a neighbor in $\{u_1, u_2, \ldots, u_a\}$. If $b = 1$, then v_1 has three neighbors in $\{u_1, \ldots, u_a\}$. By Lemma 2.1 (4), G contains a Z_3-connected subgraph K_{a+1}. By Lemma 2.1 (7), G contains a Z_3-connected subgraph induced by $V(H) \cup \{y_1, y_2, y_3, u_1, \ldots, u_a, v_1, \ldots, v_b\}$, contrary to the maximality of H.

Assume that $a = 3$. If $b = 3$, denote by F the subgraph induced by $\{u_1, u_2, u_3, v_1, v_2, v_3, y_1, y_2\}$. Since $\kappa'(G) \geq 4$ and G contains no cycle of length at least 5, each vertex of $\{u_1, u_2, u_3\}$ is adjacent to one of $\{v_1, v_2, v_3\}$ and each vertex of $\{v_1, v_2, v_3\}$ is adjacent to each vertex of $\{u_1, u_2, u_3\}$. Since $\kappa'(G) \geq 4$, $e(\{u_1, u_2, u_3\}, \{v_1, v_2, v_3\}) \geq 3$ and each vertex of F is of degree 4 and this subgraph satisfies the Ore-condition. By Theorem 2.3, F is Z_3-connected. By Lemma 2.1 (7), G contains a Z_3-connected subgraph induced by $V(H) \cup V(F)$, contrary to the maximality of H.

Let $b = 2$. Since $\kappa'(G) \geq 4$ and G contains no cycle of length at least 5, each vertex of $\{u_1, u_2, u_3\}$ is adjacent to one of $\{v_1, v_2\}$ and each vertex of $\{v_1, v_2\}$ is adjacent to two vertices of $\{u_1, u_2, u_3\}$. It follows that one, say u_3, of $\{u_1, u_2, u_3\}$ has two neighbors in $\{v_1, v_2\}$. It implies that the subgraph induced by $\{u_1, u_2, u_3, v_1, v_2\}$ is an even wheel W_4 with the center at u_3, which is Z_3-connected by Lemma 2.1 (6). By Lemma 2.1 (7), G contains a Z_3-connected subgraph induced by $V(H) \cup \{y_1, y_2, y_3, u_1, u_2, u_3, v_1, v_2\}$, contrary to the maximality of H.

Let $b = 1$. Since $\kappa'(G) \geq 4$ and G contains no cycle of length at least 5, v_1 is adjacent to each vertex of $\{u_1, u_2, u_3\}$. The subgraph induced by $\{u_1, u_2, u_3, v_1, y_1\}$ is K_5^-, which is Z_3-connected by Lemma 2.1 (4). By Lemma 2.1 (7), G contains a Z_3-connected subgraph induced by $V(H) \cup \{y_1, y_2, y_3, u_1, u_2, u_3, v_1\}$, contrary to the maximality of H.

Next, assume that $a = 2$. Let $b = 2$. Since $\kappa'(G) \geq 4$ and G contains no cycle of length at least 5, each vertex of $\{u_1, u_2\}$ is adjacent to two of $\{v_1, v_2\}$ and each vertex of $\{v_1, v_2\}$ is adjacent to two vertices of $\{u_1, u_2\}$. Denote by F the subgraph induced by $\{y_1, y_2, u_1, u_2, v_1, v_2\}$. It follows that F satisfies the Ore-condition and each of 4 vertices of F is of degree 4. By Theorem 2.3, F is Z_3-connected. By Lemma 2.1 (7), G contains a Z_3-connected subgraph induced by $V(H) \cup V(F)$, contrary to the maximality of H. □

Lemma 3.2. G does not contain a nontrivial Z_3-connected subgraph H.

Proof. Suppose that our lemma fails and H is a maximal Z_3-connected subgraph of G. Suppose that $H_1, H_2, \ldots, H_k$ are components of $G - V(H)$, where $k \geq 1$. Let $G' = G/H$ and v' be the vertex into which H is contracted.

Observe H_i, where $i \in \{1, 2, \ldots, k\}$. Let $E(H, H_i) = \{x_1y_1, x_2y_2, \ldots, x_ty_t\}$, where $x_i \in V(H)$ and $y_j \in V(H_i)$ for $i, j \in \{1, 2, \ldots, t\}$. Since G is 4-edge-connected, $t \geq 4$. By the maximality and by Lemma 2.1 (7), $y_1, \ldots, y_t$ are distinct t vertices of H_i. Let $e_i = x_iy_i$ for $i \in \{1, 2, \ldots, t\}$.

Claim 1. $E(H, H_i)$ does not contain 4 edges having a common end-vertex.

Proof of Claim 1. Suppose otherwise that without loss of generality, that e_1, e_2, e_3, e_4 have a common vertex x_1, that is, $x_1 = x_2 = \ldots = x_4$. Then the subgraph induced by $\{x_1, y_1, \ldots, y_4\}$ is a complete graph K_5 since G is $K_{1,3}$-free. By Lemma 2.1 (4), K_5 is Z_3-connected. By Lemma 2.1 (8), G contains a Z_3-connected subgraph induced by $V(H) \cup \{x_1, y_1, \ldots, y_4\}$, contrary to the choice of H. Thus, $E(H, H_i)$ contains at most three edges having a common vertex. This proves Claim 1.

Claim 2. $E(H, H_i)$ does not contain 4 independent edges.

Proof of Claim 2. Suppose otherwise that $E(H, H_i)$ contains 4 independent edges. We assume,without loss of generality, that e_1, e_2, e_3, e_4 are independent edges. Since G has no induced cycle of length at least 5, as the argument above, $y_iy_j \in E(G)$ for $1 \leq i < j \leq 4$. This means that the subgraph the subgraph induced by $\{y_1, y_2, y_3, y_4\}$ is a K_4. In the graph G', the subgraph induced by $\{v', y_1, y_2, y_3, y_4\}$ is a K_5 which is Z_3-connected by Lemma 2.1 (4). By Lemma 2.1 (3), the subgraph induced by $V(H) \cup \{y_1, y_2, y_3, y_4\}$ is Z_3-connected, contrary the maximality of H. This proves Claim 2.

Claim 3. $E(H, H_i)$ does not contain 2 edges having a common end-vertex.

Proof of Claim 3. By Claim 2, we assume that $t = 4$ and e_1, e_2, e_3, e_4 have at least a pair of two edges sharing a vertex in H. Suppose otherwise that we assume, without loss of generality, that e_1, e_2 have a common vertex x_1, that is, $x_1 = x_2$. Since $t = 4$, we need to consider e_3 and e_4 do not share a common end-vertex or e_3 and e_4 share a common end-vertex.

In the former case, the subgraph induced by $\{x_1, y_1, y_2\}$ is a K_3 since G is $K_{1,3}$-free. Since G has no induced cycle of length at least 5, $y_3y_4 \in E(G)$, $y_3y_i, y_4y_j \in E(G)$ where $i, j \in \{1, 2\}$. By Lemma 3.1, the subgraph induced by $\{y_1, y_2, y_3, y_4\}$ is a K_4 since G has no induced cycle of length at least 5. In the graph G', the subgraph induced by $\{v', y_1, y_2, y_3, y_4\}$ is a K_5 which is Z_3-connected by Lemma 2.1 (4). By Lemma 2.1 (3), the subgraph induced by $V(H) \cup \{y_1, y_2, y_3, y_4\}$ is Z_3-connected, contrary the choice of H.

In the latter case, we assume, without loss of generality, that e_3 and e_4 share a common end-vertex x_3. Since G is $K_{1,3}$-free, the subgraph induced by $\{x_1, y_1, y_2\}$ is a complete graph and so is the subgraph induced by $\{x_3, y_3, y_4\}$. Since G has no induced cycle of length at least 5, as the argument above, $y_iy_j \in E(G)$ for some $i \in \{1, 2\}$ and some $j \in \{3, 4\}$. We assume, without loss of generality, that $i = 2, j = 3$. By Lemma 3.1, each vertex of $\{y_1, y_2, y_3, y_4\}$ is not a cut vertex. Since G has no induced cycle of length at least 5 and G is 4-edge-connected, y_2 is adjacent to y_4, and y_3 is adjacent to y_1. In the graph G', the subgraph induced by $\{v', y_1, y_2, y_3, y_4\}$ is a K_5^- which is Z_3-connected by Lemma 2.1 (4). By Lemma 2.1 (3), the subgraph induced by $V(H) \cup \{y_1, y_2, y_3, y_4\}$ is Z_3-connected, contrary the maximality of H. This proves Claim 3.

By Claims 1, 2, and 3, we assume, without loss of generality, that e_1, e_2, e_3 have a common vertex x_1, that is, $x_1 = x_2 = x_3$. Thus, $t = 4$ and $x_4 \neq x_1$. It follows that the subgraph induced by $\{x_1, y_1, y_2, y_3\}$ is a complete graph K_4. Consider the cycle $x_1Px_4y_4Qy_j$, where $V(P) \subset V(H)$, $V(Q) \subset V(H_i)$ and $j \in \{1, 2, 3\}$. Since G contains no any induced cycle of length at least 5, $V(P) = V(Q) = \emptyset$ and $x_1x_4, y_4y_j \in E(G)$. We assume, without loss of generality, that $j = 3$, that is, $y_3y_4 \in E(G)$. By Lemma 3.1, each of $\{y_1, y_2, y_3\}$ is not cut vertex. Since G contains no any induced cycle of length at least 5 and $\kappa'(G) \geq 4$, $y_1y_4, y_2y_4 \in E(G)$. This, in the graph G', the subgraph induced by $\{v', y_1, y_2, y_3, y_5\}$ is a K_5, which is Z_3-connected by Lemma 2.1 (4). By Lemma 2.1 (3), the subgraph induced by $V(H) \cup \{y_1, y_2, y_3, y_4\}$ is Z_3-connected, contrary the maximality of H. □

Proof of Theorem 1.4

Since domino contains an induced $K_{1,3}$ and G contains no induced $K_{1,3}$, G contains no induced domino. By Theorem 1.3 and the choice of G, G contains an induced house. We use the same notations depicted in Figure 2. By symmetry, assume that $d(u) \leq d(v)$.

Claim 1. $|N(u) \cap N(v) \setminus \{w\}| \leq 1$.

Proof of Claim 1. Suppose otherwise that $|N(u) \cap N(v) \setminus \{w\}| \geq 2$. Let $u_1, v_1 \in N(u) \cap N(v) \setminus \{w\}$. Denote by F the subgraph induced by $\{u_1, v_1, w\}$. Since G is $K_{1,3}$-free, F contains at least one edge. If F contains two edges, then the subgraph induced by $\{u_1, v_1, w, u, v\}$ contains an even wheel W_4, which is Z_3-connected by Lemma 2.1 (6), contrary to Lemma 3.2. Thus, F contains only one edge e. By symmetry, assume that $e = wu_1$ or $e = u_1v_1$. In each case, since G is $K_{1,3}$-free, $xv_1, yv_1 \in E(G)$. This means that the subgraph induced by $\{v_1, u, v, x, y\}$ is an even wheel W_4 with the center at v_1, which is Z_3-connected by Lemma 2.1 (6), contrary to Lemma 3.2. This proves Claim 1.

Claim 2. $|N(u) \cap N(v) \setminus \{w\}| \neq 0$.

Proof of Claim 2. Suppose otherwise that $|N(u) \cap N(v) \setminus \{w\}| = 0$. Since $\kappa'(G) \geq 4$, $\delta(G) \geq 4$. First, we claim that $\max\{d(u), d(v)\} \leq 5$. Suppose otherwise that $d(u) \geq 6$. Let $u_1, u_2, u_3 \in N(u) \setminus \{w, v, x\}$. Since G is $K_{1,3}$-free, either $G[\{u, x, u_1, u_2, u_3\}]$ or $G[\{u, w, u_1, u_2, u_3\}]$ is a complete subgraph K_5 which is Z_3-connected by Lemma 2.1, contrary to Lemma 3.2. Thus, $4 \leq d(u), d(v) \leq 5$.

Assume first that $d(u) = d(v) = 4$. Let $N(u) \setminus \{w, v, x\} = \{u_1\}$ and $N(v) \setminus \{w, u, y\} = \{v_1\}$. Since G is $K_{1,3}$-free and $u_1v, v_1u \notin E(G))$, $u_1x, v_1y \in E(G)$.

Since G contains no induced cycle of length at least 5 and $\kappa'(G) \geq 4$, $u_1v_1 \in E(G)$. If $u_1y \in E(G)$ or $xv_1 \in E(G)$, then $G[\{u, v, u_1, v_1, x, y\}]$ contains a subgraph isomorphic to $G_7 + e$ which is Z_3-connected by Lemma 2.4, contrary to Lemma 3.2. Thus, assume that $u_1y, xv_1 \notin E(G)$. Since G contains no induced cycle of length at least 5, $wv_1, wu_1 \notin E(G)$. Since $\kappa'(G) \geq 4$, there exists a shortest (u_1, w)-path P such that $N_P(u_1) \notin \{u, x, v_1\}$. Since $wu_1 \notin E(G)$, $u_2 \in V(P)$ such that $u_1u_2, u_2w \in E(G)$ since G contains no induced cycle of length at least 5. Consider the cycle wu_2u_1xyvw. Since G contains no induced cycle of length at least 5, $u_2y, u_2x \in E(G)$. Since $|N(u) \cap N(v) \setminus \{w\}| = 0$, $u_2v \notin E(G)$. This implies that G contains a $K_{1,3}$ induced by $\{u_2, u_1, w, y\}$, a contradiction.

Next, assume that $d(u) = 4$ and $d(v) = 5$. Let $N(u) \setminus \{w, v, x\} = \{u_1\}$ and $N(v) \setminus \{w, v, y\} = \{v_1, v_2\}$. Since G is $K_{1,3}$-free and $|N(u) \cap N(v) \setminus \{w\}| = 0$, $u_1x, v_1y, v_2y, v_1v_2 \in E(G)$. If $wv_1, wv_2 \in E(G)$, then G contains a K_5^- induced by $\{w, v, v_1, v_2, y\}$ which is Z_3-connected by Lemma 2.1 (4), contrary to Lemma 3.2. Thus, assume that $wv_1 \notin E(G)$. Since G contains no induced cycle of length at least 5 and $\kappa'(G) \geq 4$, $u_1v_1 \in E(G)$. If $u_1y \in E(G)$ or $u_1v_2 \in E(G)$, then $G[\{u, v, x, y, u_1, v_1, v_2\}]$ contains a subgraph isomorphic to $G_{13} + e$ which is Z_3-connected by Lemma 2.4, contrary to Lemma 3.2. Thus, assume that $u_1y, u_1v_2 \notin E(G)$. As the proof above, there is u_2 such that such that $u_1u_2, u_2w \in E(G)$ and $u_2y, u_2x \in E(G)$. It follows that G contains a $K_{1,3}$ induced by $\{u_2, u_1, w, y\}$, a contradiction.

Finally, assume that $d(u) = d(v) = 5$. Let $N(u) \setminus \{w, v, x\} = \{u_1, u_2\}$ and $N(v) \setminus \{w, u, y\} = \{v_1, v_2\}$. Since G is $K_{1,3}$-free and $|N(u) \cap N(v) \setminus \{w\}| = 0$, $u_1x, u_2x, u_1u_2, v_1y, v_2y, v_1v_2, u_1v_1 \in E(G)$. If $\{u_2y, u_2v_2, u_2v_1\} \cap E(G) \neq \emptyset$, then $G[\{u, v, x, y, u_1, u_2, v_1, v_2\}]$ contains a subgraph isomorphic to $G_{14} + e$ which is Z_3-connected by Lemma 2.4, contrary to Lemma 3.2. Thus, assume that $u_2y, u_2v_2, u_2v_1 \notin E(G)$. Since G contains no induced cycle of length at least 5, $u_2w \notin E(G)$. Since $\kappa'(G) \geq 4$, as the proof above, there exists a vertex $u_3 \in V(P)$ such that $u_2u_3, u_3w \in E(G)$ and $u_3x, u_3y \in E(G)$. In this case, G contains a $K_{1,3}$ induced by $\{u_3, u_2, y, w\}$, a contradiction. This proves Claim 2.

By Claims 1 and 2, assume that $N(u) \cap N(v) \setminus \{w\} = \{z\}$. If $xz, yz \in E(G)$, then $G[\{u, v, x, y, z\}]$ is a Z_3-connected subgraph W_4, contrary to Lemma 3.2. Thus, $xz \notin E(G)$ or $yz \notin E(G)$. Recall that $d(u) \leq d(v)$. We claim that $d(v) \leq 6$. Otherwise, since G is $K_{1,3}$-free, $G[N[v] \setminus \{w, u, z\}]$ contains a complete subgraph K_m, where $m \geq 5$, which is Z_3-connected by Lemma 2.1, contrary to Lemma 3.2. Thus, $4 \leq d(u), d(v) \leq 6$.

Case 1. $xz, yz \notin E(G)$.

Since $G[\{u, w, x, z\}]$ is not an induced $K_{1,3}$, $wz \in E(G)$. We first establish a claim.

Claim 3. If $d(u) = 4$, then $d(x) = 4$; if $d(v) = 4$, then $d(y) = 4$.

Proof of Claim 3. Suppose otherwise that $d(x) \geq 5$. Since $d(u) = 4$, each $s \in N(x) \setminus \{u\}$ is not adjacent to u. Thus, $G[N[x] \setminus \{u\}]$ is a Z_3-connected K_m, where $m \geq 5$, since G is $K_{1,3}$-free, contrary to Lemma 3.2. Since G is 4-edge-connected, $d(x) \geq 4$. Thus, $d(x) = 4$. The proof for the case that $d(y) = 4$ is similar. This proves Claim 3.

Assume that $d(u) = d(v) = 4$. By Claim 3, $d(x) = 4$. Let $N(x) \setminus \{u, y\} = \{x_1, x_2\}$. Since G is $K_{1,3}$-free, $yx_1, yx_2, x_1x_2 \in E(G)$. Since $\kappa'(G) \geq 4$, G contains a path from x_1 to w which does not contains any vertex of $\{x_2, x, y, u, v\}$. Since G contains no induced cycle of length at least 5, this path is an edge, that is, $x_1w \in E(G)$ or $x_1z \in E(G)$. Similarly, we can prove that $x_2z \in E(G)$ or $x_2w \in E(G)$. In each case, $H =$

$G[\{u, v, x, y, x_1, x_2, w, z\}]$ satisfies the Ore-condition. By Lemma 2.3, H is Z_3-connected, contrary to Lemma 3.2.

Assume that $d(u) = 4$ and $d(v) = 5$. Let $N(v) \setminus \{u, w, z, y\} = \{v_1\}$. Since G is $K_{1,3}$-free, $yv_1 \in E(G)$. By the Claim, $d(x) = 4$. Assume that $xv_1 \in E(G)$. Let $xx_1 \in E(G)$. Since G is $K_{1,3}$-free, $x_1y, x_1v_1 \in E(G)$. Let $H = G[\{u, v, x, y, x_1, v_1, w, z\}]$. If $wv_1 \in E(G)$, contract the 2-cycle (v, v_1) in $H_{[wv,wv_1,]}$ and repeatedly contact the 2-cycles generated in the process, eventually, we get a K_1 which is Z_3-connected. By Lemmas 2.1 and 2.2, H is Z_3-connected, contrary to Lemma 3.2. Thus, $wv_1 \notin E(G)$. Since $\kappa'(G) \geq 4$ and G contains no induced cycle of length at least 5, $x_1w \in E(G)$. As the proof above, we can get $H_{[x_1y,x_1v_1]}$ is Z_3-connected. By Lemma 2.2, H is Z_3-connected, contrary to Lemma 3.2.

Thus, $xv_1 \notin E(G)$. Let $xx_1, xx_2 \in E(G)$. Since G is $K_{1,3}$-free, $yx_1, yx_2, x_1x_2 \in E(G)$. Since G contains no induced cycle of length at least 5, $x_1v_1, x_2v_1, wv_1, zv_1 \notin E(G)$. Since G contains no induced cycle of length at least 5 and $\kappa'(G) \geq 4$, $x_1w, x_2z \in E(G)$ or $x_1z, x_2w \in E(G)$. In each case, $L = G[\{u, v, x, y, x_1, x_2, w, z\}]$ satisfies the Ore-condition. By Lemma 2.3, L is Z_3-connected, contrary to Lemma 3.2.

If $d(u) = 4$ and $d(v) = 6$, let $N(v) \setminus \{u, w, z, y\} = \{v_1, v_2\}$. Since G is $K_{1,3}$-free, $v_1y, v_2y, v_1v_2 \in E(G)$. By the Claim, $d(x) = 4$. First assume that $xv_1, xv_2 \in E(G)$. In this case, G contains a Z_3-connected subgraph K_5^- induced by $\{x, y, v, v_1, v_2\}$, contrary to Lemma 3.2. Next, assume that $xv_1 \in E(G)$ and $xv_2 \notin E(G)$. Let $xx_1 \in E(G)$. Since G is $K_{1,3}$-free, $x_1y, x_1v_1 \in E(G)$. Let $H = G[\{w, u, v, x, y, x_1, v_1, v_2\}]$. If $wv_1 \in E(G)$ or $wv_2 \in E(G)$ or $x_1z \in E(G)$, we can prove that $H_{[wv,wv_1]}$ or $H_{[wv,wv_2]}$ or $H_{[x_1y,x_1v_1]}$ is Z_3-connected. By Lemma 2.2, H is Z_3-connected, contrary to Lemma 3.2. If $x_1v_2 \in E(G)$, then G contains a Z_3-connected subgraph K_5^- induced by $\{x_1, y, v, v_1, v_2\}$, a contradiction. Thus, $wv_1, wv_2, x_1z, x_1v_2 \notin E(G)$. Since $\kappa'(G) \geq 4$ and G contains no induced cycle of length at least 5, $wx_1 \in E(G)$. As the argument above, $H_{[x_1y,x_1v_1]}$ is Z_3-connected. By Lemma 2.2, H is Z_3-connected, contrary to Lemma 3.2. Finally, assume that $xv_1, xv_2 \notin E(G)$. Let $xx_1, xx_2 \in E(G)$. Since G is $K_{1,3}$-free, $x_1x_2, yx_1, yx_2 \in E(G)$. Since G contains no induced cycle of length at least 5, $wv_1, wv_2, zv_1, zv_2 \notin E(G)$ and $e(\{x_1, x_2\}, \{v_1, v_2\}) = 0$. Since $\kappa'(G) \geq 4$ and G contains no induced cycle of length at least 5, $wx_1, zx_2 \in E(G)$ or $wx_2, zx_1 \in E(G)$. In each case, $L = G[\{w, u, v, x, y, x_1, x_2, z\}]$ satisfies the Ore-condition, by Lemma 2.3, L is Z_3-connected, contrary to Lemma 3.2.

If $d(u) = 5$ and $d(v) = 5$, let $N(u) \setminus \{v, w, z, x\} = \{u_1\}$ and $N(v) \setminus \{u, w, z, y\} = \{v_1\}$. Since G is $K_{1,3}$-free, $u_1x, v_1y \in E(G)$. Since $\kappa'(G) \geq 4$ and G contains no induced cycle of length at least 5, $u_1v_1 \in E(G)$. If $u_1y \in E(G)$ or $v_1x \in E(G)$, then $G[\{u, v, x, y, u_1, v_1\}]$ contains a subgraph isomorphic to $G_7 + e$ which is Z_3-connected by Lemma 2.4, contrary to Lemma 3.2. Thus, $u_1y, v_1x \notin E(G)$. Assume that $u_1z \in E(G)$. Since G is $K_{1,3}$-free, $v_1z \in E(G)$. It follows that G contains a Z_3-connected subgraph W_4 induced by $\{u, v, u_1, v_1, z\}$ with the center at z, contrary to Lemma 3.2. Thus, by symmetry, we assume that $u_1z, v_1z \notin E(G)$ and $wu_1, wv_1 \notin E(G)$. As $\kappa'(G) \geq 4$, there is w_1 such that $u_1w_1, w_1w \in E(G)$. Observe cycle ww_1u_1xyvw. Since G contains no induced cycle of length at least 5, $w_1y \in E(G)$. It follows that G contains a $K_{1,3}$ induced by $\{w_1, u_1, w, y\}$, a contradiction.

If $d(u) = 5$ and $d(v) = 6$, let $N(u) \setminus \{v, w, z, x\} = \{u_1\}$ and $N(v) \setminus \{u, w, z, y\} = \{v_1, v_2\}$. Since G is $K_{1,3}$-free, $u_1x, v_1y, v_2y, v_1v_2 \in E(G)$. Since G contains no induced cycle of length at least 5, $wv_1, wv_2, zv_1, zv_2 \notin E(G)$. Since $\kappa'(G) \geq 4$ and G contains no induced cycle of length at least 5, by symmetry, we assume that $u_1v_1 \in E(G)$. If $\{u_1y, u_1v_2, v_1x, v_2x\} \cap E(G) \neq \emptyset$, then $G[\{u, v, x, y, u_1, v_1, v_2\}]$ contains a subgraph isomorphic to $G_{13} + e$ which is Z_3-connected by Lemma 2.4, contrary to Lemma 3.2. Thus, assume that $u_1y, u_1v_2, v_1x, v_2x \notin E(G)$. Since G has no induced cycle of length at least 5, $u_1z, wu_1 \notin E(G)$. Since $\kappa'(G) \geq 4$ and G contains no induced cycle of length at least 5, there is w_1 such that such that $u_1w_1, w_1w \in E(G)$. Since G is $K_{1,3}$-free, $w_1y, w_1x \in E(G)$. This implies that $G[\{w_1, u_1, w, y\}]$ is an induced $K_{1,3}$, a contradiction.

If $d(u) = 6$ and $d(v) = 6$, let $N(u)\setminus\{v, w, z, x\} = \{u_1, u_2\}$ and $N(v)\setminus\{u, w, z, y\} = \{v_1, v_2\}$. since G is $K_{1,3}$-free, $u_1x, u_2x, u_1u_2, v_1y, v_2y, v_1v_2 \in E(G)$. If either $e(\{u_1, u_2\}, \{v_1, v_2\}) \geq 2$ or $e(\{u_1, u_2\}, \{v_1, v_2\}) = 1$ and $\{u_1y, u_1v_2, u_2y, u_2v_2, u_2v_1\} \cap E(G) \neq \emptyset$, then $G[\{u, v, x, y, u_1, u_2, v_1, v_2\}]$ contains a subgraph isomorphic to $G_{14}+e$ which is Z_3-connected by Lemma 2.4, contrary to Lemma 3.2. Thus, $e(\{u_1, u_2\}, \{v_1, v_2\}) \leq 1$. Moreover, if $e(\{u_1, u_2\}, \{v_1, v_2\} = 1$ and $u_1y, u_1v_2, u_2y, u_2v_2, u_2v_1 \notin E(G)$. In this case, let $u_1v_1 \in E(G)$. Since G contains no induced cycle of length at least 5, $wu_1, wu_2, wv_1, wv_2, u_2z \notin E(G)$. Consider the case that $e(\{u_1, u_2\}, \{v_1, v_2\}) = 0$. By Lemmas 2.4 and 3.2, $e(x, \{v_1, v_2\}) \leq 1$ and $e(y, \{u_1, u_2\}) \leq 1$. Since G contains no induced cycle of length at least 5, $wu_2, u_2z \notin E(G)$. In each case, since $\kappa'(G) \geq 4$ and G contains no induced cycle of length at least 5, there is w_1 such that such that $u_2w_1, w_1w \in E(G)$ and $w_1y, w_1x \in E(G)$. In this case, G contains a $K_{1,3}$ induced by $\{w_1, u_2, w, y\}$, a contradiction.

Case 2. one edge of $\{xz, yz\}$ is not in $E(G)$.

We assume, without loss of generality, that $xz \in E(G)$ and $yz \notin E(G)$. Since G is $K_{1,3}$-free, $wz \in E(G)$. Consider that $d(u) = d(v) = 4$. Since $\delta(G) \geq 4$ and G is $K_{1,3}$-free, $d(y) = 4$. Let $\{y_1, y_2\} \subseteq N(y) \setminus \{x, v\}$. Assume that one edge of y_1z, y_2z is in G, without loss of generality, assume that $y_1z \in E(G)$. Since G is $K_{1,3}$-free, $y_1x, y_2x, y_1y_2 \in E(G)$. Let $H = G[\{u, v, w, x, y, z, y_1, y_2\}]$. Contracting the 2-cycle (y_1, y_2) in $H_{[yy_1, yy_2]}$ and repeatedly contacting the 2-cycles generated in the process, eventually, we get a K_1 which is Z_3-connected. By Lemmas 2.1 and 2.2, H is Z_3-connected, contrary to Lemma 3.2. Thus, $y_1z, y_2z \notin E(G)$. Since $\kappa'(G) \geq 4$ and G contains no induced cycle of length at least 5, $wy_1 \in E(G)$ or $wy_2 \in E(G)$. In each case, Contracting 2-cycle (u, w) and contracting all 2-cycle generated in the process in $H_{[wu, wz]}$, we obtain a K_5^- which is Z_3-connected by Lemma 2.1 (1). By Lemma 2.2, H is Z_3-connected, contrary to Lemma 3.2.

If $d(u) = 4$ and $d(v) = 5$, let $v_1 \in N(v) \setminus \{w, u, y, z\}$. Since G is $K_{1,3}$-free, $v_1y \in E(G)$. Since $\kappa'(G) \geq 4$, let $yy_1 \in E(G)$. Let H be the subgraph induced by $\{u, v, x, y, w, z, y_1, v_1\}$. Since G is $K_{1,3}$-free, $xy_1 \in E(G)$. Since G contains no induced cycle of length at least 5, $v_1w \notin E(G)$. We claim that $v_1x \notin E(G)$ for otherwise, assume that $v_1x \in E(G)$. Since G is $K_{1,3}$-free, $y_1v_1 \in E(G)$. Contracting 2-cycle (y_1, v_1) and contracting all 2-cycles generated in the process in $H_{[xy_1, xv_1]}$, we get a K_5^- which is Z_3-connected by Lemma 2.1 (4). By Lemma 2.2, H is Z_3-connected, contrary to Lemma 3.2. If $v_1z \in E(G)$, by Lemma 3.2, $v_1u, v_1w \notin E(G)$. In this case, the subgraph induced by $\{z, x, w, v_1\}$ is a $K_{1,3}$, a contradiction. Thus, $v_1z \notin E(G)$. If $wy_1 \in E(G)$, then $H_{[wu, wz]}$ contains a 2-cycle (u, z). Contracting this 2-cycle and contracting all 2-cycles generated in the process, finally we obtain a K_1. By Lemma 2.1 (1) (3) (5), and by Lemma 2.2, H is Z_3-connected, contrary to Lemma 3.2. Thus, $wy_1 \notin E(G)$. Recall that $wx \notin E(G)$. Since $\kappa'(G) \geq 4$, there is a vertex w_1 such that $ww_1, w_1v_1 \in E(G)$. Since $d(u) = 4$

and $d(v) = 5$, $w_1u, w_1v \notin E(G)$. Since G has no induced cycle of length at least 5, $w_1x \in E(G)$. In this case, the subgraph induced by $\{w, w_1, x, v_1\}$ is a $K_{1,3}$, a contradiction.

If $d(u) = 4$ and $d(v) = 6$, let $N(v) \setminus \{w, u, x, z\} = \{v_1, v_2\}$. Since G is $K_{1,3}$-free, $yv_1, yv_2.v_1v_2 \in E(G)$. Assume that $v_1z \in E(G)$. Observe the subgraph $G[\{z, x, w, v_1\}]$. Since G is $K_{1,3}$-free, $xv_1 \in E(G)$ or $wv_1 \in E(G)$. In the former case, G contains a Z_3-connected subgraph W_4 induced by $\{z, u, x, v_1, v\}$ with the center at z, contrary to Lemma 3.2. In the latter case, G contains a Z_3-connected subgraph W_4 induced by $\{w, u, z, v_1, v\}$ with the center at v, contrary to Lemma 3.2. Thus, $v_1z \notin E(G)$. Similarly, $v_2z \notin E(G)$. If $v_1x, v_2x \in E(G)$, then G contains a Z_3-connected subgraph K_5^- induced by $\{y, x, v_1, v, v_2\}$, contrary to Lemma 3.2. Thus,$|\{v_1x, v_2x\} \cap E(G)| \leq 1$. Assume that $v_1x \notin E(G)$. Since G contains no induced cycle of length at least 5, $wv_1, wv_2 \notin E(G)$. Since $\kappa'(G) \geq 4$ and G contains no induced cycle of length at least 5, there exists a vertex w_1 such that $ww_1, w_1v_1 \in E(G)$ and $w_1x \in E(G)$. In this case, G contains a $K_{1,3}$ induced by $\{w_1, w, x, v_1\}$, a contradiction.

If $d(u) = d(v) = 5$, let $N(u) \setminus \{w, v, x, z\} = \{u_1\}$ and $N(v) \setminus \{w, u, y, z\} = \{v_1\}$. Since G is $K_{1,3}$-free, $u_1x, v_1y \in E(G)$. Since G is $K_{1,3}$-free, $zu_1 \in E(G)$. Since G has no induced cycle of length at least 5, $wv_1 \notin E(G)$. We claim that $zv_1 \notin E(G)$. To the contrary, assume that $zv_1 \in E(G)$. Since G is $K_{1,3}$-free, $u_1v_1, xv_1 \in E(G)$. Let $H = G[\{u, v, w, x, y, z, u_1, v_1\}]$. Contracting the 2-cycle (u, x) in $H_{[u_1u, u_1x]}$ and repeatedly contacting the all 2-cycles generated in the process, eventually, we get a K_1 which is Z_3-connected. By Lemmas 2.1 and 2.2, H is Z_3-connected, contrary to Lemma 3.2. Thus, $v_1z \notin E(G)$. In this case, since $\kappa'(G) \geq 4$, there is a path Q from u_1 to v_1 avoiding any vertex in $\{z, w, u, v\}$. Since G has no induced cycle of length at least 5, $|E(Q)| = 1$, that is, $v_1u_1 \in E(G)$. If $u_1y \in E(G)$ or $v_1x \in E(G)$, then $G[\{u, v, x, y, u_1, v_1\}]$ contains a subgraph isomorphic to $G_7 + e$ which is Z_3-connected by Lemma 2.4, contrary to Lemma 3.2. Thus, $u_1y, v_1x \notin E(G)$. Since G has no induced cycle of length at least 5, $wu_1 \notin E(G)$. As $\kappa'(G) \geq 4$, there is a path P from w to v_1. Since $wv_1 \notin E(G)$, there is $w_1 \in V(G)$ such that $w_1w, w_1v_1 \in E(G)$. Since G has no induced cycle of length at least 5, $w_1x, w_1y \in E(G)$. Since G is $K_{1,3}$-free, $xv_1 \in E(G)$. This is a contradiction, as we have proved $xv_1 \notin E(G)$.

If $d(u) = 5$ and $d(v) = 6$, let $N(u) \setminus \{w, v, x, z\} = \{u_1\}$ and $N(v) \setminus \{w, u, y, z\} = \{v_1, v_2\}$. Since G is $K_{1,3}$-free, $u_1x, v_1y, v_2y, v_1v_2, zu_1 \in E(G)$. Since G has no induced cycle of length at least 5, $wv_1, wv_2 \notin E(G)$. We claim that none of $\{zv_1, zv_2\}$ is in $E(G)$. Suppose otherwise that assume that $zv_1 \in E(G)$. Since G is $K_{1,3}$-free, $u_1v_1, xv_1 \in E(G)$. Let $H = G[\{u, v, w, x, y, z, u_1, v_1, v_2\}]$. Then H is isomorphic to $G_{14} + e$, which is Z_3-connected by Lemma 2.4, contrary to Lemma 3.2. Thus, $zv_1, zv_2 \notin E(G)$. As $\kappa'(G) \geq 4$, there is a path P from u_1 to v_1 avoiding any vertex in $\{z, w, u, v, x, y\}$. Since G has no induced cycle of length at least 5, $u_1v_1 \in E(G)$. In this case, the subgraph induced by $\{u, v, x, y, z, u_1, v_1, v_2\}$ is also isomorphic to $G_{14} + e$, which is Z_3-connected by Lemma 2.4, contrary to Lemma 3.2.

If $d(u) = d(v) = 6$, let $N(u) \setminus \{w, v, x, z\} = \{u_1, u_2\}$ and $N(v) \setminus \{w, u, y, z\} = \{v_1, v_2\}$. Since G is $K_{1,3}$-free, $u_1x, u_2x, u_1u_2, v_1y, v_2y, v_1v_2, zu_1, zu_2 \in E(G)$. This means that the subgraph induced by $\{z, u, u_1, u_2, x\}$ is a K_5, which is Z_3-connected by Lemma 2.1, contrary to Lemma 3.2.

Acknowledgements The first author was supported by the Science Foundation of China (11171129) and by Doctoral Fund of Ministry of Education of China (20130144110001).

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