

Splošna siva prenosno matrična metoda in uporaba v izračunu naravnih frekvenc sistemov

The Universal Grey Transfer Matrix Method and Its Application in Calculating the Natural Frequencies of Systems

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Za analizo ter vodenje načrtovanja dinamičnih sistemov potrebujemo učinkovite metode izračuna naravnih frekvenc sistemov. V prispevku predlagamo splošno sivo prenosno matrično metodo. Pri tej metodi sta za izračun naravnih frekvenc dinamičnega sistema uporabljeni splošna siva teorija ter metoda z združevanjem splošne sive matematike in prenosne matrike. Na osnovi predlagane metode smo znotraj programa Matlab razvili posebno orodje. Opisani so tudi trije primeri, ki potrjujejo izbiro predlagane metode.

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(Ključne besede: sistemi dinamični, frekvence naravne, matrične metode prenosne, matrične metode sive)

In order to analyze dynamic systems and to guide dynamic systems' design, effective methods for the calculation of the natural frequencies of systems are needed. The universal grey transfer matrix method is proposed in this paper. In this method, the universal grey theory and method are used to calculate the natural frequencies of dynamic systems by combining universal grey mathematics with a transfer matrix. A specific Matlab toolbox based on the proposed method is developed. Three examples are given to verify the proposed method.

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0 INTRODUCTION

Natural frequency is considered in many engineering fields, such as structural dynamic systems design, when we either want to avoid resonance or to utilize resonance. Interval analysis (or so-called interval mathematics) was originally put forward to estimate the natural frequency calculating errors. Since the first literature on interval mathematics was published in 1966 [1], interval mathematics has developed very quickly. But when interval data are calculated, different operation sequences for calculating independent variables result in different extending intervals. Though this issue has been studied by some scholars, the optimum calculating sequence is still required ([2], [3], [11] and [12]). Zhang et al studied the extending problems of trigonometric function monotony and

interval analysis methods on mechanism error analysis. However, the errors decided by means of the interval analysis method can sometimes be larger than the errors obtained with the extremum method. Chen obtained some useful results using the perturbation approach for the upper and lower parameters in the structural interval and the perturbation approach for an interval matrix based on the perturbation approach [5]. However, the interval operation criterion used by Chen is unable to analyze uncertainty issues. The grey systematic theory and its application have been rapidly developed in the control field, since it was introduced in 1982 ([6], [7] and [9]). The growth of this theory is determined by its mathematical depiction. The grey set was creatively put forward by Wang et al., and based on which, the mathematical operating laws were introduced in [6]. But like interval mathematics,

some of the algebraic nature cannot be extended when the grey data are analyzed. This limits its application. Therefore, Wang et al put forward the concept of the universal grey set, the foundation of the universal grey algebra, and the foundation of the universal grey mathematics analysis [7]. These universal theories are more powerful than the fuzzy set theory and the ordinary grey theory when dealing with uncertainty issues. At the same time, the application based on the above universal theories can be extended to overcome the disadvantages of interval analysis or the ordinary grey operation. Therefore, these extended means (methods) to deal with engineering problems using the universal grey theory have several practical applications. Based on the basic concepts in the universal grey data [7], in this paper the transferring between the universal grey data and interval grey data, and the interval estimating using the universal grey data are studied. The Matlab scripts that combine universal theories and a matrix-transfer approach are developed. The natural frequencies of a dynamic system are calculated by the developed Matlab scripts. Examples are given to verify the proposed methods. The conclusions are drawn in the final section.

1 FUNDAMENTALS OF THE UNIVERSAL GREY MATHEMATICS

1.1 Concept of universal grey data [7]

Definition 1: Assume an universe of discourse $U=R$ (the set of real numbers), then the universal grey set in R is called the universal grey data set, denoted as $g(R)$. The elements in $g(R)$ are called universal grey data, denoted as:

$$g = (x, [\underline{\mu}, \bar{\mu}]), x \in R, \underline{\mu}, \bar{\mu} \in R \quad (1),$$

where x is an observed value, $[\underline{\mu}, \bar{\mu}]$ is the grey information part of x . $g(0)=(0, [0, 0])$ and $g(1)=(1, [1, 1])$ are the zero element and the unit element in $g(R)$, respectively. $g'(0)$, referred to as the sub-zero element, denotes the universal grey data with a zero observing part and a non-zero grey information part.

Definition 2: $\forall g_1 = (x_1, [\underline{\mu}_1, \bar{\mu}_1]), g_2 = (x_2, [\underline{\mu}_2, \bar{\mu}_2])$, define $g_1 = g_2$ if and only if $x_1 = x_2, \underline{\mu}_1 = \underline{\mu}_2, \bar{\mu}_1 = \bar{\mu}_2$.

Definition 3: For any $g = (x, [\underline{\mu}, \bar{\mu}]) \in g(R)$, define the negative element of g in $g(R)$ to be $-g = (-x, [\underline{\mu}, \bar{\mu}])$. If $\mu \neq 0$, define the inverse element of g in $g(R)$ to be $g^{-1} = (x^{-1}, [\underline{\mu}^{-1}, \bar{\mu}^{-1}])$.

Based on the above definitions, addition and multiplication can be defined in $g(R)$. More details can be found in [7] and [9]. Using the negative element and addition, the subtraction operation can also be defined. Similarly, the division operation can be defined using inverse elements and multiplication.

It can also be determined that the universal grey addition is a closed loop satisfying the associative law and the commutative law in mathematics. The universal grey multiplication is a closed loop that copes with the associative law and the commutative law. The universal grey multiplication operation satisfies the distribution law to the universal grey addition operation [7]. It should be noted that when there is universal grey data like are $(0, [-0.3/0, -0.2/0])$ in the universal grey data operation, the 0 in the denominator should be eliminated. When programmed in the Matlab environment, the 0 in the denominator should be replaced with a minimum data such as 10 to 18. The calculated results are not then influenced too much.

1.2 Transformation between universal grey data and interval data

In an application, $\underline{\mu}$ and $\bar{\mu}$ in the universal grey data $(x, [\underline{\mu}, \bar{\mu}])$ can be considered as the lowest or uppermost degrees of the belief to x . For example, if $\underline{\mu} = 0.6, \bar{\mu} = 0.8$, then the degree of the belief to x is in the range of $[0.6x, 0.8x]$. Therefore, a universal grey data can be remarked using an interval data as $(x, [\underline{\mu}, \bar{\mu}]) = [x\underline{\mu}, x\bar{\mu}]$, $\underline{\mu}$ and $\bar{\mu} \in [-1, 1]$. An interval grey data $[a, b]$ can also be remarked using a universal grey data $(x, [\underline{\mu}, \bar{\mu}])$ [6]. We speak concretely:

(1) when $a > 0$, there is $[a, b] = (b, [a/b, 1])$.

Let the interval grey number be $[1, 2] = (2, [0.5, 1])$.

(2) when $ab < 0$ and $\max\{|a|, |b|\} = b$, there is $[a, b] = (b, [a/b, 1])$.

Let the interval grey number be $[-1, 2]$. Here, $a = -1, b = 2, ab = -2, \max\{|-1|, |2|\} = 2$. So, $[-1, 2] = (2, [-0.5, 1])$.

(3) when $ab < 0$ and $\max\{|a|, |b|\} = |a|$, there is $[a, b] = (a, [b/a, 1])$.

Let the interval grey number be $[-2, 1]$. Here, $a = -2, b = 1, ab = -2, \max\{|-2|, |1|\} = 2$. So, $[-2, 1] = (-2, [-0.5, 1])$.

(4) when $b < 0$, there is $[a, b] = (a, [b/a, 1])$.

Let the interval grey number be $[-2, -1]$. Here, $a = -2, b = -1 < 0$. So, $[-2, -1] = (-2, [0.5, 1])$.

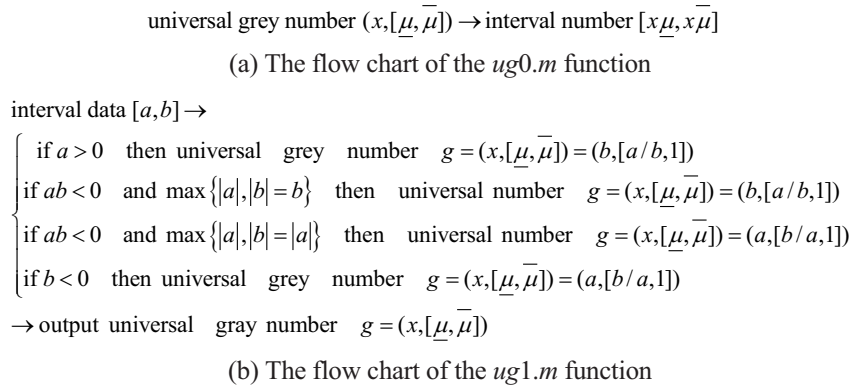


Fig. 1. The flow charts of the function programs

Based on the above procedure, we developed two Matlab functions named *ug0.m* and *ug1.m*. The flowcharts of both functions are shown in Figs. 1 (a) and (b). The universal grey data can be obtained using both of the functions.

For example, $a=[3,8]$ and $b=ug1(a)$, then the result $b=(8,[0.3750,1.0000])$ will be obtained. Then $a1=ug0(b)=[3.0000,8.0000]$ will be the final result.

1.3 Section analysis function of universal grey functions ([6], [7] and [9])

Universal grey data can extend the real data, and universal grey functions can extend ordinary real-data functions. Assume, $x=f(x)$ is an ordinary real-data function, and x and y are real data. x can be extended by universal grey, denoted as $g(x)$. $g(x) = (x, [\underline{\mu}, \bar{\mu}])$. Then the universal grey extension of $f(x)$, $f(gx)$, can be calculated using:

$$f(g_x) = (f(x), [f(\underline{\mu x})/f(x), f(\bar{\mu x})/f(x)]) \quad (2).$$

After extending, the real foundational elementary functions are called universal grey foundational elementary functions, which keep the properties of foundational elementary functions. However, interval mathematics loses these properties. Universal grey data can also realize an interval analysis.

Example: testify $f(x)=x^2f(x-7)-6-1/(x(x-4)-30)$ has no roots in the interval of $[8, 10]$, and estimate its maximum and minimum values in $[7]$, $[8]$ and $[10]$.

The solution (with the extending interval method) is:

$$F[8,10] = ([8,10])([8,10]) - 7 - 6 - \frac{1}{([8,10])([8,10] - 4) - 30} = [1.5, 23.9667], 0 \notin F[8,10]$$

This testifies that $f(x)$ has no real roots in $[8, 10]$ and its maximum value is no more than 23.9667, its minimum value is no less than 1.5.

But a rational function is different because of its operating sequence, it could have a different interval extend function. For example, make the related formula write into the formula $f(x)=x^2-7x-6-1/(x(x-4)-30)$ in this case. $F[8, 10]$ is again calculated according to the changed formula. Its solution is different: if its form is changed, its resolution is different, and it is even unable to be calculated.

The universal grey number could eliminate this shortcoming. For the above example, as $[8, 10] = (10, [0.8, 1])$, there is $F[8, 10] = [1.5, 23.9667]$. If $f(x)$ is re-written as $f(x)=(x^2-11x^3-8x^2+234x+179)/(x^2-4x-30)$, then there still is $F[8, 10] = [1.5, 23.9667]$. Therefore, the results will not be affected by the changes of formula forms using the universal grey number.

1.4 A non-linear equation homology universal grey algorithm based on homology perturbation

The Newtonian iteration formula $x_{n+1}=x_n - f(x_n)/f'(x_n)$ is a famous method for solving the non-linear equation $f(x)=0$. Based on homology perturbation, reference [8] constructs homology functions and shows that the famous Newtonian iteration formula $x_{n+1}=x_n-f(x_n)/f'(x_n)$ can only research an approximate solution of the homology perturbation. A new iteration formula that has faster convergence was proposed by Zen et al [8]:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} - \frac{f''(x_n)}{2f'(x_n)} \left(\frac{f(x_n)}{f'(x_n)} \right)^2 \quad (3).$$

Eq. (3) is the universal grey extension employed in this paper. A given precision is used as

a condition to stop the iteration through $\|d(x_{n+1}, x_n)\| < \varepsilon$ (see [7]). d is the universal grey distance.

2 THE APPLICATION OF THE UNIVERSAL GREY MATRIX TRANSFERRING METHOD IN CALCULATING THE NATURAL FREQUENCIES OF SYSTEMS

2.1 The universal grey matrix transferring method

When any element of a matrix is a universal grey number, the matrix is called a universal grey matrix. When the matrix transferring method is actualized, its elements are either real or complex numbers. If any element is a universal grey number, the matrix transferring method is called the universal grey matrix transferring method.

The procedure for calculating natural frequencies using the universal grey matrix transferring theory can be given as below.

- (1) Construct the sub-transfer matrix C_p , find the symbol formula of C (C is a transferring matrix) using the means of a symbol by deducing [10].
- (2) Obtain Z_n (the subsequent state vector) and Z_0 (the original state vector) according to the boundary conditions.
- (3) Achieve the high-order equation $f(k_p, J_p, m_p, \omega) = 0$ using the Maple symbol reasoning method, and by substituting the mean values of the interval values k_i, J_i , and m_i into the equations $f(k_p, J_p, m_p, \omega) = 0$. Then, find the initial natural frequency ω_i^0 of ω [2].
- (4) Extend the universal grey data. Assume the universal grey data of k_p, J_p, m_i as $\tilde{k}_i, \tilde{J}_i, \tilde{m}_i$ and extend $f(k_p, J_p, m_p, \omega) = 0$ to $\tilde{f}(\tilde{k}_i, \tilde{J}_i, \tilde{m}_i, \tilde{\omega}) = 0$. Here, $\tilde{f}(\cdot)$ denotes the continuation function of $f(\cdot)$. The set $\tilde{\omega}_i^0$, as the original value, came from the universal grey extension based on ω_i^0 . Evaluate the solution $\tilde{\omega}_i$ using the homology universal grey non-linear equation algorithm based on the homology perturbation in Section 1.4.

2.2 Software development for universal calculating

Based on the universal grey mathematical theory and the above procedures, a universal grey operating toolbox was developed in the Matlab environment. The universal grey operating toolbox is based on a universal grey class, referred to as the UgClass. Besides the functions *ug0.m* and *ug1.m*

mentioned before, the functions *ugMatrixPlus.m*, *ugMatrixSubtract.m*, *ugMatrixTimes.m*, and *ugMatrixDivision.m* are developed. The flow chart of the main program is shown in Fig. 2. In this figure, we indicate each Matlab function.

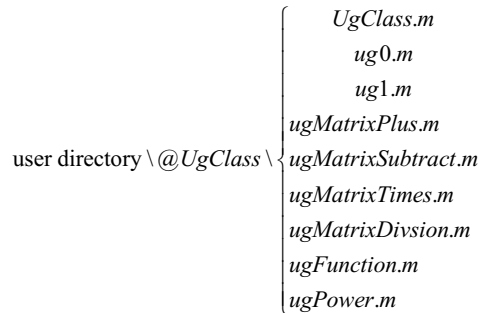


Fig. 2. The flow chart of the main program

2.3 Examples

Example 1 [9]: Fig. 3 is a dynamic model of an equal-diameter discs system. The left end is fixed and the right end is free. The rotating inertia of each disc is J . The sprain stiffness of the axis is k . Find the natural frequencies of the freely sprain vibration.

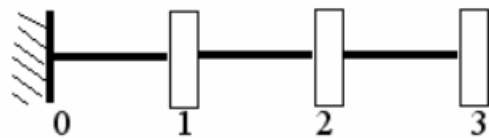


Fig. 3. The equal-diameter disc system

The boundary conditions are:

$$\theta_0 = 0, M_3^R = 0$$

where, θ_0 is the torsion angle of the left-hand end 0, and M_3^R is the torque of the right-hand end 3.

The transferring matrix is:

$$C_i = \begin{bmatrix} 1 & 1/k \\ -\omega_n^2 J & 1 - \omega_n^2 \frac{J}{k} \end{bmatrix}$$

where ω_n is the angular frequency of the discs system [9].

After deducing in the Maple environment we get:

$$M_3^R = \left(1 - \frac{\omega_n^6 J^3}{k^3} + \frac{5\omega_n^4 J^2}{k^2} - \frac{6\omega_n^2 J}{k}\right) M_0 = 0$$

Because of $M_0 \neq 0$ the following is true:

$$f(k, J, \omega_n) = 1 - \frac{\omega_n^6 J^3}{k^3} + \frac{5\omega_n^4 J^2}{k^2} - \frac{6\omega_n^2 J}{k} = 0$$

With the symbolic solution methods, we get:

$$\omega_{n1} = 0.445\sqrt{\frac{k}{J}}, \omega_{n2} = 1.247\sqrt{\frac{k}{J}}, \omega_{n3} = 1.802\sqrt{\frac{k}{J}} \quad (\text{rad/s})$$

Extending them with the universal grey method their values amount to $\tilde{J} = (1, [1, 1])$, $\tilde{k} = (1, [1, 1])$, then $\tilde{\omega}_{n1} = (0.445, [1, 1])$, $\tilde{\omega}_{n2} = (1.247, [1, 1])$, and $\tilde{\omega}_{n3} = (1.802, [1, 1])$. Clearly, the solutions obtained with the symbolic solution methods are the same solutions as obtained with the universal grey method.

Example 2 [4]: As with the frame construction in Fig. 4, the parameters of the construction are shown below.

- (1) The upper limit and lower limit of the stiffness parameters (N/m) are $k_1' = [2000, 2020]$, $k_2' = [1800, 1850]$, $k_3' = [1600, 1630]$, $k_4' = [1400, 1420]$, $k_5' = [1200, 1210]$, and $k_6' = [1000, 1008]$, respectively.

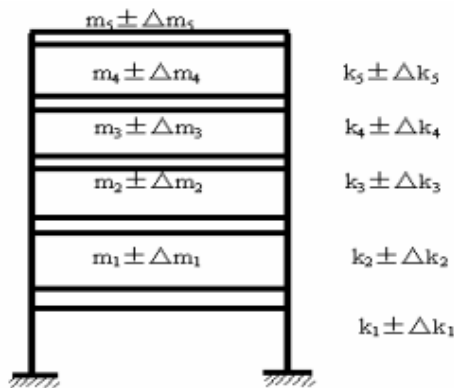


Fig. 4. The frame structure

- (2) The upper limit and lower limit of the mass parameters (unit: kg) are $m_1' = [29, 31]$, $m_2' = [26, 28]$, $m_3' = [26, 28]$, $m_4' = [24, 26]$, and $m_5' = [17, 19]$, respectively.

The sub-transfer matrix and the transfer matrix can be obtained using the transfer matrix method. Considering the boundary conditions, we determine that the power of ω in $f(k_i, J_i, m_i, \omega) = 0$ is 10 using the symbol reasoning method. The coefficient of the high-order equation of ω is a function of known parameters.

We replace the corresponding parameters using the mean values in the equation of $f(k_i, J_i, m_i, \omega) = 0$, $i=1, 2, 3$, and 4. For example, let $k_1' = 2010$, and $m_1' = 30$. We can find the original values to be transferred to the universal grey number. Set $\varepsilon=0.01$, and we solve the above equation. The value of it can, however, be smaller in order to improve the precision. Then, we transfer the universal grey data to the grey data. The results are shown in Table 1. For a comparison study, the interval perturbation solutions based on the Deif method and the perfect solutions are given in Table 1 at the same time. It is clear that the proposed method can get the same or even more precise results than the Deif analysis method is able to provide.

Example 3 [4]: A mass-spring system is shown in Fig. 5. The interval expressions of mass and spring stiffness are:

$$[K] = \begin{bmatrix} [3800, 3830] & [-1820, -1800] & 0 & 0 & 0 \\ [-1820, -1800] & [3400, 3430] & [-1610, -1600] & 0 & 0 \\ 0 & [-1610, -1600] & [3000, 3010] & [-1416, -1400] & 0 \\ 0 & 0 & [-1416, -1400] & [2600, 2620] & [-1210, -1200] \\ 0 & 0 & 0 & [-1210, -1200] & [1200, 1210] \end{bmatrix}$$

$$[M] = \begin{bmatrix} [29, 30] & 0 & 0 & 0 & 0 \\ 0 & [26, 28] & 0 & 0 & 0 \\ 0 & 0 & [26, 28] & 0 & 0 \\ 0 & 0 & 0 & [26, 28] & 0 \\ 0 & 0 & 0 & 0 & [17, 19] \end{bmatrix}$$

Table 1. Comparison of natural frequencies

		ω_1	ω_2	ω_3	ω_4	ω_5
Perfect solutions[4]	Upper limit	7.8303	47.820	109.40	174.00	230.08
	Lower limit	4.6166	40.643	98.180	157.84	209.51
Interval perturbation solutions based on Deif [4]	Upper limit	7.7702	47.661	109.17	173.66	229.69
	Lower limit	4.5623	40.643	97.966	157.53	209.15
Proposed method		[4.6166, 7.8303]	[40.6431, 47.821]	[98.181, 109.40]	[157.84, 173.99]	[209.51, 230.08]

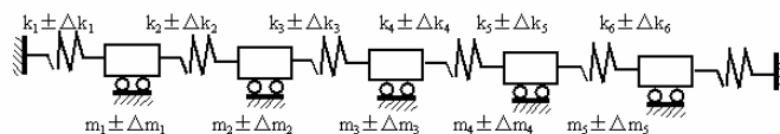


Fig. 5. The spring-mass system

Table 2. Comparison of natural frequencies

		ω_1	ω_2	ω_3	ω_4	ω_5
Perfect solutions [4]	Lower limit	5.05895	42.42973	100.30453	159.81766	211.36365
	Upper limit	7.03611	45.73130	107.32792	172.14525	226.78797
Interval matrix perturbing solution based on Deif [4]	Lower limit	4.82429	41.72490	99.47379	158.93609	210.11230
	Upper limit	7.18635	46.30420	107.93573	172.51133	227.43530
Proposed method		[5.05884, 7.03655]	[42.42854, 45.73256]	[100.30154, 107.33104]	[159.81563, 172.14531]	[211.36258, 226.78912]

In this case $\varepsilon=0.01$. The analyzed results are obtained with the proposed universal grey transfer matrix method. Then, the universal grey data is transferred to the grey data. The solutions are shown in Table 2. The solutions show that the results obtained with the proposed method are the same as that obtained by the perfect method; namely, the lower limit and the upper limit obtained with the perfect one method are almost the same, with the end points of the interval grey data obtained with the proposed methods. The interval perturbation based on the Deif method in [4] completes the work with some errors. Compared with other methods, it is clear that the proposed method is more effective.

3 CONCLUSION

In this paper, universal mathematics is introduced. The high-order equation can be found with the transfer matrix method combined with symbol reasoning. The solution can be obtained with the

universal grey method. A specific Matlab toolbox for the universal grey operation is developed. The calculating process and the examples of the natural frequencies' analysis of the structure are studied using the universal grey transfer matrix method. The paper shows that the proposed method is an effective and reliable method that is easy to program. For this reason it will have a wide range of applications in dynamic system design and vibration analysis. However, if the dynamic system is too complicated, the reasoning will be more complicated and will exceed the memory of the computer. In this case, the universal grey matrix perturbation will be used [6].

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