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FORMULAS FOR VARIOUS DOMINATION NUMBERS OF PRODUCTS OF PATHS AND CYCLES

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Formulas for various domination numbers of products of paths and cycles

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Abstract. The existence of a constant time algorithm for solving different domination problems on the subclass of polygraphs, rotagraphs and fasciagraphs, is shown by means of path algebras. As these graphs include products (the Cartesian, strong, direct, lexicographic) of paths and cycles, we implement the algorithm to get formulas in the case of the domination numbers, the Roman domination numbers and the independent domination numbers of products of paths and cycles where the size of one factor is fixed, i.e. independently of the size of the second factor. We also show that the values of the investigated graph invariants on the fasciagraphs and the rotagraphs with the same monograph can only differ for a constant value.

AMS subject classifications: 05C25, 05C69, 05C85, 68R10

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1. Introduction

Over years, extensive work has been done concerning the domination number and its variations, also from the algorithmic point of view [20, 21]. It is well known that the problem of determining the domination number and different variations of the domination number of arbitrary graphs is NP–complete [21]. It is therefore worthwhile to consider algorithms for some classes of graphs, including all four standard products of paths and cycles. Known results on the domination, Roman domination, total domination and independent domination of the Cartesian product of paths and cycles are shown in Table 1 and known results on the strong and the direct product of paths and cycles can be found in Table 2. In cases where more general results are known, only those are listed. For the lexicographic product of graphs, exact formulas for the domination, the total domination and the independent domination numbers can be easily obtained and can be found, for instance, in [33]. Recently also the Roman domination number of the lexicographic product of graphs was investigated in [31]. The domination number of the Cartesian and the strong graph bundles were studied in [42]. Graph bundles represent a well known generalization of graph products [36].

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A general $O(\log n)$ algorithm based on path algebras, which can be used to compute various graph invariants on the fasciagraphs and rotagraphs, has been proposed in [26]. The algorithm of [26] can in most cases, including the computation of distance based invariants [25], the domination numbers [34, 40], the Roman domination numbers [35] and others [24, 41] be turned into a constant time algorithm, i.e. the improved algorithm can find closed formulas for arbitrary number of monographs in a fasciagraph or a rotagraph. The existence of an algorithm that provides closed formulas for the domination numbers on grid graphs has been observed or claimed also in [16, 32]. Other approaches for investigating graph invariants on polygraphs can be found in [7].

Here we generalize the algorithm of [40] to compute so–called $*$–domination numbers of fasciagraphs and rotagraphs. This means that for (almost) any kind of domination type (whether we look for domination number, Roman domination number, total domination number, independent domination number, k-domination number, etc.), we are able to find exact formulas on fasciagraph and rotagraphs, for any number of monographs. As fasciagraphs and rotagraphs include different products of paths and cycles, the results of the implementation of this generalized algorithm, which can be found in Tables 3 - 10, complement previously known results of Tables 1 and 2.

In the rest of this paper we first recall the background for the main algorithm from [26] and [40]. In Section 3, the algorithm, which generalizes and improves the space complexity of the algorithm from [26, 40] is precisely presented. Summary of results is given in Section 4. In particular, the properties of the new algorithm allow us to improve best known results in several cases. A short conclusion then ends the paper.

Table 1: Known results for the Cartesian product

Graph invariant	**The Cartesian product**
Domination	$\gamma(P_n \Box P_k) = \left\lfloor \frac{(n+2)(k+2)}{5} \right\rfloor - 4$ for $n,k \geq 16$, [17], other results: [1, 10, 13, 16, 19, 38], $\gamma(C_n \Box P_k)$, for $k \leq 11$ and $n \in \mathbb{N}$; [34], $\gamma(P_n \Box C_k)$, $\gamma(C_n \Box C_k)$ for $k \leq 6$ and $n \in \mathbb{N}$, [34].
Roman domination	$\gamma_R(P_n \Box P_k)$, $\gamma_R(C_n \Box P_k)$, for $k \leq 8$ and $n \in \mathbb{N}$, [35], $\gamma_R(P_n \Box C_k)$, $\gamma_R(C_n \Box C_k)$ for $k \leq 6$ and $n \in \mathbb{N}$, [35].
Total domination	$\gamma_t(P_n \Box P_k)$ for $k \leq 4$ and $n \in \mathbb{N}$ [18], $\gamma_t(P_n \Box P_k)$ for $k = 5, 6$ and $n \in \mathbb{N}$ [29], bounds for $\gamma_t(P_n \Box P_k)$ for $k, n > 16$ [18].
Independent domination	$i(P_n \Box P_k)$ for $k \leq 14$ and $n \in \mathbb{N}$ [14]

Table 2: Known results for the strong and the direct product

Graph invariant	The strong product	The direct product
Domination	$\gamma(G \boxtimes H) \leq \gamma(G)\gamma(H)$, $\gamma(T \boxtimes H) = \gamma(T)\gamma(H)$, where T is a tree, [33].	$\gamma(P_n \times P_k)$ for $k \leq 6$, [27, 28], for $k \leq 9$ [11], further results for $k \leq 33$ and $n \leq 40$ [11].
Roman domination	$\gamma_R(G \boxtimes H) \leq 2\gamma(G)\gamma(H)$, [39].	–
Total domination	$\gamma_t(G \boxtimes H) \leq \gamma_t(G)\gamma(H)$, [33].	$\gamma_t(T \times H) = \gamma_t(T)\gamma_t(H)$, where T is a tree, [37]. $\gamma_t(C_n \times C_k)$ for $n, k \in \mathbb{N}$ [15]
Independent domination	$i(P_n \boxtimes P_m)$, $i(C_n \boxtimes C_m)$, $i(P_n \boxtimes C_m)$, $n, m \in \mathbb{N}$, [30].	–

2. Preliminaries

We consider finite undirected graphs and directed graphs (digraphs). An edge between vertices u and v in an undirected graph will be denoted uv while in a digraph, an arc between vertices u and v will be denoted (u, v). P_n stands for a path on n vertices and C_n for a cycle on n vertices. For $u \in V(G)$, $N(u) = \{v \in V(G) \mid uv \in E(G)\}$ is an *open neighborhood* of u and $N[u] = N(u) \cup \{u\}$ a *closed neighborhood*.

Here we show that different variations of the graph domination problems can be solved in constant time on fasciagraphs and rotagraphs. Because there are so many graph invariants, related to the domination number, many authors tried to unite them or present them in a way. Some of the classifications can be found in [5, 6, 8, 20, 21]. We will refer to them as defined in the sequel:

Definition 1. *Let $G = (V(G), E(G))$ be a graph, $a_i \geq 0$ for $i = 0, \ldots, l$ and $f^* : V(G) \longrightarrow \{a_0, a_1, \ldots, a_l\}$ a function. Let $(V_0, V_1, V_2, \ldots, V_l)$ be ordered partition of $V(G)$ induced by f^*, where $V_i = \{v \in V(G) \mid f^*(v) = a_i\}$ for $i = 0, 1, \ldots, l$. Note that there exists a 1–1 correspondence between the functions $f^* : V(G) \longrightarrow \{a_0, a_1, \ldots, a_l\}$ and ordered partitions $(V_0, V_1, \ldots, V_l)$ of $V(G)$. Thus, we write $f^* = (V_0, V_1, \ldots, V_l)$. The weight of f^* is defined as:*

$$w(f^*) = \sum_{v \in V(G)} f^*(v).$$

A function $f^* = (V_0, V_1, \ldots, V_l)$ is called a $*$–dominating function if a "particular statement $*$" holds for f^*. The $*$–domination number equals

$$\gamma^*(G) = \inf\{w(f^*) \mid f^* \text{ is a } *\text{–dominating function of } G\}. \tag{1}$$

We say that a function f is γ^*–function if it is a $*$–dominating function of weight $\gamma^*(G)$.

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This general definition unites many known domination types, let us mention some:

1. Let $l = 1$, $a_i = i$ for $i = 0, 1$ and let statement $*$ be that:
   (a) every vertex of V_0 has a neighbor in V_1. Then Definition 1 is a definition of the *domination number*, $\gamma(G)$.
   (b) every vertex of $V(G)$ has a neighbor in V_1. Then Definition 1 is a definition of the *total domination number*, $\gamma_t(G)$ [23].
   (c) every vertex of V_0 has a neighbor in V_1 and the set V_1 is an independent set (i.e. no edge joins two vertices of V_1). Then Definition 1 is a definition of the *independent domination number*, $i(G)$ [2].
2. Let $l = 2$, $a_i = i$ for $i = 0, 1, 2$ and let statement $*$ be that every vertex of V_0 has a neighbor in V_2. Then Definition 1 is a definition of the *Roman domination number*, $\gamma_R(G)$ [12].
3. Let $l = k + 1$, $a_i = i$ for $i = 0, 1, \ldots, l$ and let statement $*$ be that every vertex of V_0 is defended (i.e. has a neighbor in V_i for some $i \geq 1$) and for any sequence $v_1, \ldots, v_k$ of (not necessarily distinct) vertices, there exists a sequence of functions $f = f_0, f_1, \ldots, f_k$ such that for $i = 1, \ldots, k$, (i) either $f_{i-1}(v_i) > 0$, in which case $f_i = f_{i-1}$, or $f_{i-1}(v_i) = 0$, in which case f_i is obtained from f_{i-1} by one movement to v_i, and (ii) f_i has no undefended vertex. For more detailed definition see [22]. Then Definition 1 is definition of the *k–Roman domination number*, $\gamma_R^k(G)$.

In the sequel we will focus our attention to finding $*$–domination numbers of four standard products of graphs. For graphs G and H, the vertex set of these products is always the same, $V(G) \times V(H)$. Two vertices are adjacent in the:

1. *Cartesian product*, $G \Box H$, if and only if $g = g'$ and $hh' \in E(H)$ or $gg' \in E(G)$ and $h = h'$;
2. *strong (normal) product*, $G \boxtimes H$, if and only if $g = g'$ and $hh' \in E(H)$ or $gg' \in E(G)$ and $h = h'$ or $gg' \in E(G)$ and $hh' \in E(H)$;
3. *direct (also known as tensor, cardinal, categorical, Kronecker) product*, $G \times H$, if and only if $gg' \in E(G)$ and $hh' \in E(H)$;
4. *lexicographic product*, $G[H]$, if and only if $gg' \in E(G)$ or $g = g'$ and $hh' \in E(H)$.

Let $G_1, \ldots, G_n$ be arbitrary mutually disjoint graphs and $X_1, \ldots, X_n$ a sequence of sets of edges such that an edge of X_i joins a vertex of $V(G_i)$ with a vertex of $V(G_{i+1})$ ($X_i \subseteq V(G_i) \times V(G_{i+1})$ for $i = 1, \ldots, n$). For convenience we set $G_{n+1} = G_1$. A *polygraph* $\Omega_n = \Omega_n(G_1, \ldots G_n; X_1, \ldots X_n)$ over monographs $G_1, \ldots, G_n$ has the vertex set $V(\Omega_n) = V(G_1) \cup \ldots \cup V(G_n)$, and the edge set $E(\Omega_n) = E(G_1) \cup X_1 \cup \ldots \cup E(G_n) \cup X_n$. For a polygraph Ω_n and for $i = 1, \ldots, n$ we also define

$$D_i = \{u \in V(G_i) \mid \exists v \in G_{i+1} : uv \in X_i\},$$

$$R_i = \{u \in V(G_{i+1}) \mid \exists v \in G_i : uv \in X_i\}.$$

In general, $R_i \cap D_{i+1}$ does not have to be empty. If all graphs G_i are isomorphic to a fixed graph G (i.e. there exists an isomorphism $\varphi_i : V(G_i) \longrightarrow V(G)$ for $i = 1, \ldots, n+1$, and $\varphi_{n+1} = \varphi_1$) and all sets X_i are equal to a fixed set $X \subseteq V(G) \times V(G)$ ($(u, v) \in X \iff \left(\varphi_i^{-1}(u), \varphi_{i+1}^{-1}(v)\right) \in X_i$ for all i), we call such a graph *rotagraph*, $\omega_n(G; X)$. A rotagraph without edges between the first and the last copy of G is *fasciagraph*, $\psi_n(G; X)$. More precisely, in fasciagraph, $X_n = \emptyset$ and $X_1 = X, \ldots, X_{n-1} = X$. In rotagraph as well as in fasciagraph, all sets D_i and R_i are equal to fixed sets D and R, respectively ($D_i = \varphi_i^{-1}(D)$ and $R_i = \varphi_{i+1}^{-1}(R)$). Of course, in a case of fasciagraphs, $D_n = \emptyset$ and $R_n = \emptyset$. Let for a moment $G \cdot H$ denote any of the graph products, $\square, \boxtimes, \times$ or the lexicographic. Observe that products of paths $P_n \cdot P_k$ are examples of fasciagraphs and that products of cycles $C_n \cdot C_k$ are examples of rotagraphs. Products of paths and cycles can, except in the case of non-commutative products (the lexicographic product), be treated either as fasciagraphs or as rotagraphs. Polygraphs, which were first studied in [3], also include a generalization of a graph product, that is graph bundles, introduced in [36].

A *semiring* $\mathcal{P} = (P, \oplus, \circ, e^{\oplus}, e^{\circ})$ is a set P on which two binary operations, $\oplus$ and $\circ$ are defined such that $(P, \oplus)$ is a commutative monoid with $e^{\oplus}$ as a unit, $(P, \circ)$ is a monoid with e° as a unit, $\circ$ is left– and right–distributive over $\oplus$ and for every $x \in P$, $x \circ e^{\oplus} = e^{\oplus} = e^{\oplus} \circ x$. An idempotent semiring is called a *path algebra*. It is easy to see that a semiring is a path algebra if and only if $e^{\circ} \oplus e^{\circ} = e^{\circ}$ holds for e°, the unit of the monoid $(P, \circ)$. An important example of a path algebra for our work is $\mathcal{P}_1 = (\mathbb{N}_0 \cup \{\infty\}, \min, +, \infty, 0)$. Here $\mathbb{N}_0$ denotes the set of nonnegative integers and $\mathbb{N}$ the set of positive integers.

Let $\mathcal{P} = (P, \oplus, \circ, e^{\oplus}, e^{\circ})$ be a path algebra and let $\mathcal{M}_n(\mathcal{P})$ be the set of all $n \times n$ matrices over P. Let $A, B \in \mathcal{M}_n(\mathcal{P})$ and define operations $\oplus$ and $\circ$ in the usual way:

$$\begin{aligned} (A \oplus B)_{ij} &= A_{ij} \oplus B_{ij}, \\ (A \circ B)_{ij} &= \bigoplus_{k=1}^{n} A_{ik} \circ B_{kj}. \end{aligned}$$

$\mathcal{M}_n(\mathcal{P})$ equipped with above operations is a path algebra with the zero and the unit matrix as units of semiring. In our example $\mathcal{P}_1 = (\mathbb{N}_0 \cup \{\infty\}, \min, +, \infty, 0)$, all elements of the zero matrix are ∞, the unit of the monoid $(P, \min)$, and the unit matrix is a diagonal matrix with diagonal elements equal to $e^{\circ} = 0$ and all other elements equal to $e^{\oplus} = \infty$. Sometimes, we will also need ordinary matrix summation in $\mathbb{R}$ (i.e. $(A + B)_{ij} = A_{ij} + B_{ij}$). We denote it with ordinary $+$.

Let $\mathcal{P}$ be a path algebra and let G be a labeled digraph, that is a digraph together with a labeling function ℓ which assigns to every arc of G an element of P. Let $V(G) = \{v_1, v_2, \ldots, v_n\}$. The labeling ℓ of G can be extended to walks in the following way: For a walk $Q = (v_{i_0}, v_{i_1})(v_{i_1}, v_{i_2}) \ldots (v_{i_{k-1}}, v_{i_k})$ of G let

$$\ell(Q) = \ell\left(v_{i_0}, v_{i_1}\right) \circ \ell\left(v_{i_1}, v_{i_2}\right) \circ \ldots \circ \ell\left(v_{i_{k-1}}, v_{i_k}\right).$$

Let S_{ij}^k be the set of all walks of order k from v_i to v_j in G and let $A(G)$ be the matrix defined by:

$$(A(G))_{ij} = \begin{cases} \ell(v_i, v_j); & \text{if } (v_i, v_j) \text{ is an arc of } G \\ e^{\oplus}; & \text{otherwise} \end{cases} \tag{2}$$

It is well-known [9] that

$$\left(A(G)^k\right)_{ij} = \bigoplus_{Q \in S_{ij}^k} \ell(Q).$$

3. Algorithm for determining ∗–domination numbers of fasciagraphs and rotagraphs

Let us now present a constant time algorithm for determining ∗–domination numbers of fasciagraphs and rotagraphs. As mentioned before, the algorithm which computes different graph invariants on fasciagraphs and rotagraphs in $O(\log n)$ time was proposed in [26] and then improved to run in $O(C)$ time for the domination number [40] and also for some other graph invariants in [25, 35, 41]. We now generalize this algorithm in the following way:

Let $\omega_n(G;X)$ be a rotagraph and $\psi_n(G;X)$ a fasciagraph as defined above. Set $U = D \sqcup R$. (Keep in mind that $D_i \subseteq G_i$ and $R_i \subseteq G_{i+1}$, but since $R_i = R$ and $D_i = D$ for $i = 1, \ldots, n$ in case of rotagraphs and for $i = 1, \ldots, n-1$ in case of fasciagraphs, we can write $U = D_i \cup R_i = D \sqcup R$). A labeled digraph $\mathcal{G} = \mathcal{G}(G;X)$ is a graph with a vertex set:

$$V(\mathcal{G}) = \left\{ v_i = \left(V_0^i, V_1^i, \ldots, V_l^i\right) \mid V_j^i \subseteq U \text{ and } V_j^i \cap V_h^i = \emptyset \text{ for } 0 \leq j, h \leq l, j \neq h \right\}.$$

In particular, $v_0 = \left(V_0^0, V_1^0, \ldots, V_l^0\right)$ stands for $(\emptyset, \emptyset, \ldots, \emptyset)$.

Let $v_i, v_j \in V(\mathcal{G})$ and consider for a moment $\psi_3(G;X)$. Let $V_0^i \cup V_1^i \cup \ldots \cup V_l^i \subseteq D_1 \cup R_1$ and $V_0^j \cup V_1^j \cup \ldots \cup V_l^j \subseteq D_2 \cup R_2$. (Note that we use the notation $D_1 \cup R_1$ and $D_2 \cup R_2$ instead of $D \sqcup R$ here only to be clear about the general idea.) Let $\gamma_{i,j}^*(G;X)$ be the weight of a γ^*– function of a graph $G_2 \setminus \left(\left(R_1 \cap \left(V_0^i \cup V_1^i \cup \ldots \cup V_l^i\right)\right) \cup \left(\left(V_0^j \cup V_1^j \cup \ldots \cup V_l^j\right) \cap D_2\right)\right)$, such that $V_h^i \cup V_h^j \subseteq V_h$ for $h = 0, \ldots, l$ where $(V_0, V_1, \ldots V_l)$ is a ∗–dominating function of a graph G_2. For consistency, we introduce an arc between vertices v_i and v_j only if $V_{h_1}^i \cap V_{h_2}^j = \emptyset$ for all $0 \leq h_1, h_2 \leq l$, $h_1 \neq h_2$. Set

$$\ell(v_i, v_j) = \sum_{h=0}^{l} a_h \left|V_h^i \cap R\right| + \gamma_{i,j}^*(G;X) + \sum_{h=0}^{l} a_h \left|V_h^j \cap D\right| - \sum_{h=0}^{l} a_h \left|V_h^i \cap D \cap R \cap V_h^j\right|. \tag{3}$$

Remark 1. *If $D \cap R = \emptyset$, then equation (3) is reduced to*

$$\ell(v_i, v_j) = \sum_{h=0}^{l} a_h \left|V_h^i \cap R\right| + \gamma_{i,j}^*(G;X) + \sum_{h=0}^{l} a_h \left|V_h^j \cap D\right|.$$

Now, considering graph $\mathcal{G}(G;X)$ and labeling (3), form a matrix $A(\mathcal{G})$ as defined in (2). Then, according to [26], we have an algorithm which computes $*$–domination number of rotagraphs and fasciagraphs in $O(\log n)$ time:

Algorithm 1

1. For a path algebra select $\mathcal{P} = (\mathbb{N}_0 \cup \{\infty\}, \min, +, \infty, 0)$.
2. Label $\mathcal{G} = \mathcal{G}(G;X)$ as defined above.
3. In $\mathcal{M}(\mathcal{P})$ calculate $A(\mathcal{G})^n$.
4. Let $\gamma^*(\psi_n(G;X)) = (A(\mathcal{G})^n)_{00}$ and $\gamma^*(\omega_n(G;X)) = \min_i (A(\mathcal{G})^n)_{ii}$.

Theorem 1. *The Algorithm 1 correctly computes $*$–domination number of rotagraphs and fasciagraphs:*

$$\gamma^*(\psi_n(G;X)) = (A(\mathcal{G})^n)_{00} \tag{4}$$

$$\gamma^*(\omega_n(G;X)) = \min_i (A(\mathcal{G})^n)_{ii} \tag{5}$$

in $O(\log n)$ time.

Proof. Let G_1 and G_2 be arbitrary graphs, X_1 a set of edges between vertices of G_1 and G_2 and let $\Omega_2(G_1, G_2; X_1, \emptyset)$ be a polygraph. Let also $\mathcal{P} = (\mathbb{N}_0 \cup \{\infty\}, \min, +, \infty, 0)$ be a path algebra and let $\mathcal{G}'$ be a labeled digraph for Ω_2 defined similarly as above. Then, by the definition of labeling, we have

$$\begin{aligned}\gamma^*(\Omega_2(G_1, G_2; X_1, \emptyset)) &= [A(G_1) + A(G_2)]_{00} \\ &= \min_{v_k \in V(\mathcal{G})} \{\ell(v_0, v_k) + \ell(v_k, v_0)\}.\end{aligned}$$

Let $G_1 = G$, $X_1 = X$ and $G_2 = \psi_{n-1}(G;X)$. Then (4) follows by induction.

For (5), similarly, consider $\Omega_2(G_1, G_2; X_1, X_2)$ and let $G_1 = G$, $X_1 = X_2 = X$ and $G_2 = \psi_{n-1}(G;X)$.

It is well known that, in general, Step 3 of the algorithm can be implemented to run in $O(\log n)$ time and other steps can be done in a constant time. Therefore Algorithm 1 can run in $O(\log n)$ time. □

This algorithm can be improved: computing the powers of $A(\mathcal{G})^n = A^n$ in $O(C)$ time is possible using special structure of the matrices, so called "cyclicity lemma", which was proposed in [4] and used in a similar way in [34, 35, 40, 41].

Lemma 1. *Let $N = |V(\mathcal{G}(G;X))|$, $K = |V(G)|$ and $a = \max\{a_0, \ldots, a_l\}$. Then there is an index $q \leq (2aK+2)^{N^2}$ such that $A^q = A^p + C$ for some index $p < q$ and some constant matrix $C = [c]_{ij}$. Let $P = q - p$. Then for every $r \geq p$ and every $s \geq 0$ we have*

$$A^{r+sP} = A^r + sC\,.$$

Proof. First observe that for any $k \geq 1$, the difference between any pair of entries of A^k, both different from ∞, is bounded by $2aK$:

Assume $(A^k)_{ij} \neq \infty$. Then

$$\begin{aligned}\left(A^k\right)_{ij} &= \gamma^*\left(\left(V(G_1) \setminus \left(\bigcup_{h=0}^{l} V_h^i\right)\right) \cup V(G_2) \cup \ldots \cup V(G_{k-1}) \cup \left(V(G_k) \setminus \left(\bigcup_{h=0}^{l} V_h^j\right)\right)\right) \\ &\leq \gamma^*(\psi_k(G;X)).\end{aligned}$$

Since $V_{h_1}^i \cap V_{h_2}^i = \emptyset$ and $V_{h_1}^j \cap V_{h_2}^j = \emptyset$ for $h_1, h_2 = 0, \ldots, l$ and because of the choice of a it follows that

$$\sum_{h=0}^{l} a_h \left|V_h^i \cap R\right| + \sum_{h=0}^{l} a_h \left|V_h^j \cap D\right| \leq 2aK.$$

According to (3) we have

$$\begin{aligned}\ell(v_i, v_j) &\leq 2aK + \gamma_{i,j}^*(G;X) - \sum_{h=0}^{l} a_h \left|V_h^i \cap D \cap R \cap V_h^j\right| \\ &\leq 2aK + \gamma_{i,j}^*(G;X) = 2aK + \left(A^k\right)_{ij} \\ \left(A^k\right)_{ij} &\geq \ell(v_i, v_j) - 2aK \geq \gamma^*\left(\psi_k(G;X)\right) - 2aK.\end{aligned}$$

Therefore

$$\gamma^*\left(\psi_k(G;X)\right) - 2aK \leq (A^k)_{ij} \leq \gamma^*(\psi_k(G;X)).$$

For $k \geq 1$, let $M_k = \min\{(A^k)_{ij}\}$ and let $\left(A^k\right)' = A^k - (M_k)J$, where J is the matrix with all entries equal to 0 (recall that we are still in the path algebra $\mathcal{P} = (\mathbb{N}_0 \cup \{\infty\}, \min, +, \infty, 0))$. Since the difference between any two elements of A^k, different from ∞, cannot be greater than $2aK$, the entries of $\left(A^k\right)'$ can have only values $0, 1, \ldots, 2aK, \infty$. Hence there are indices $p < q \leq (2aK+2)^{N^2}$ such that $(A^p)' = (A^q)'$. This proves the first part of the lemma.

The equality $A^{r+sP} = A^r + sC$ follows from the fact that for arbitrary matrices D, E and a constant matrix C we have $(D+C) \circ E = D \circ E + C$, where $+$ is the ordinary matrix addition, i.e. $(A+B)_{ij} = A_{ij} + B_{ij}$ for all i, j:

$$\begin{aligned}((D+C) \circ E)_{ij} &= \min_k\{((D)_{ik} + C) + (E)_{kj}\} = \min_k\{(D)_{ik} + (E)_{kj}\} + C \\ (D \circ E + C)_{ij} &= \min_k\{(D)_{ik} + (E)_{kj}\} + (C)_{ij} = \min_k\{(D)_{ik} + (E)_{kj}\} + C.\end{aligned}$$

Therefore, let $A^q = A^p + C$ for some index $p < q$ and some constant matrix $C = [c]_{ij}$. Then

$$\begin{aligned}A^{q+1} &= (A^p + C) \circ A \\ &= (A^p \circ A) + C \\ &= A^{p+1} + C.\end{aligned}$$

Let $P = q - p$ and $r \geq p$. Then also

$$\begin{aligned} A^{r+P} &= A^{r+q-p} = A^q \circ A^{r-p} \\ &= (A^p + C) \circ A^{r-p} = A^{p+r-p} + C \\ &= A^r + C, \end{aligned}$$

and by induction on s we have

$$\begin{aligned} A^{r+sP} &= A^{r+P+(s-1)P} = (A^r + C) \circ A^{(s-1)P} = A^{r+(s-1)P} + C \\ &= A^r + (s-1)C + C = A^r + sC \end{aligned}$$

for every $s \geq 0$. □

Remark 2. *Assume that one looks for:*

1. *a domination or total domination or independent domination number of a fasciagraph or a rotagraph. Then* $q < (2K+2)^{N^2}$. *For instance, if fasciagraph is a grid graph* $P_n \Box P_k$*, then* $q < (2k+2)^{2^{2k}}$. *In particular, if* $k = 4$, $q \leq 10^{256}$.
2. *a Roman domination number of a fasciagraph or a rotagraph. Then* $q < (4K+2)^{N^2}$. *For instance, if fasciagraph is a grid graph* $P_n \Box P_k$*, then* $q < (4k+2)^{2^{3k}}$. *In particular, if* $k = 2$, $q \leq 10^{64}$.

We see that already in cases of very small monographs, enormously large q *are obtained. That is why the second part of Lemma 1 is useful for practical purposes - once a period is detected, it cannot change. When we implemented the algorithm for various domination problems and smaller monographs, the period was always found much sooner - at latest for* $q = 20$.

So, if we assume that the size of a monograph G is a given constant (and n is a variable), the algorithm will run in constant time. However, straightforward implementation may not be practical due to obvious large space requirements of the algorithm. Fortunately, instead of calculating whole matrices A^n, calculating only those rows which are important for the result and checking the difference of the new row against the previously stored rows until a constant difference is detected yields a correct result because of the following lemma, first presented in [41] and recently generalized in the following way:

Lemma 2. *[35] Assume that the* j*–th row of* A^{n+P} *and* A^n *differ for a constant,* $a_{ji}^{n+P} = a_{ji}^n + C$ *for all* i*. Then* $\min_i a_{ji}^{n+P} = \min_i a_{ji}^n + C$.

This immediately gives an improvement in the case of fasciagraphs: recall that $\gamma^*(\psi_n(G;X)) = (A(\mathcal{G})^n)_{00}$. For computing 00-th element of A^n we only need first rows of matrices A^i for $2 \leq i \leq n-1$:

$$A_{0i}^n = \min_k \left\{ A_{0k}^{n-1} + A_{ki} \right\}.$$

In the case of rotagraphs such an improvement is not crucial because $\gamma^*(\omega_n(G;X)) = \min_i (A(\mathcal{G})^n)_{ii}$. Therefore we prove that once we have a period for a fasciagraph

$\psi_n(G;X)$, the same period is optimal for $\omega_n(G;X)$, in other words, formulas for fasciagraphs and rotagraphs with the same monograph G can only differ for a constant value. In particular, there can be at most P different constants for which fasciagraph and rotagraph differ:

Lemma 3. *Let $A^q = A^p + C$ and $P = q - p$. Then for every $t \in 0, 1, \ldots, P-1$ there is a constant C_t such that for all $n \geq p$ with $t \equiv (n-p) \pmod{P}$ we have*

$$\gamma^*(\psi_n(G;X)) - \gamma^*(\omega_n(G;X)) = C_t.$$

Proof. Let $A^q = A^p + C$ for some $q > p$ and a constant matrix $C = [c]_{ij}$. Such p, q, C exist because of the Lemma 1. We can write $n = p + sP + t$, where $P = q - p$, $s \geq 0$ and $0 \leq t < P$. Then $A^n = A^{p+sP+t} = A^{p+t} + sC$ also by Lemma 1 and we have:

$$\begin{aligned}
\gamma^*(\psi_n(G;X)) - \gamma^*(\omega_n(G;X)) &= (A^n)_{00} - \min_i \{(A^n)_{ii}\} \\
&= \left(A^{p+t} + sC\right)_{00} - \min_i \left\{\left(A^{p+t} + sC\right)_{ii}\right\} \\
&= \left(A^{p+t}\right)_{00} + sC - \min_i \left\{\left(A^{p+t}\right)_{ii}\right\} - sC \\
&= \left(A^{p+t}\right)_{00} - \min_i \left\{\left(A^{p+t}\right)_{ii}\right\} = C_t.
\end{aligned}$$

□

4. Summary of results

Theorem 2. *$*$–domination numbers of fasciagraphs and rotagraphs can be computed in constant time, i.e. independently of the size of a monograph G.*

Proof. Algorithm 1 implies an $O(\log n)$ algorithm for computing $*$–domination numbers of fasciagraphs and rotagraphs. When applying Lemma 1, we get closed expressions for $*$–domination numbers of fasciagraphs and rotagraphs. □

In special cases we have:

Corollary 1. *Domination numbers, Roman domination numbers and independent domination numbers of the Cartesian, strong, direct or lexicographic products of paths and cycles, where the size of one factor is fixed, can be in computed constant time, i.e. independently of the size of the second factor.*

A summary of results of our implementation of the algorithm is given in the next proposition. Additional formulas, found by an earlier version of our algorithm, can be found in [35] for the Roman domination numbers of the Cartesian products of paths and cycles and in [34] for the domination numbers of the Cartesian products of paths and cycles.

Proposition 1. *Closed expressions for:*

1. *$\gamma(P_n \times P_k)$, $\gamma(P_n \times C_k)$, $\gamma(C_n \times P_k)$ and $\gamma(C_n \times C_k)$ for some fixed k are given in Tables 3 and 4;*

2. *$\gamma_R(P_n \times P_k)$, $\gamma_R(P_n \times C_k)$, $\gamma_R(C_n \times P_k)$ and $\gamma_R(C_n \times C_k)$ for some fixed k are given in Tables 5 and 6.*

3. *$i(P_n \Box P_k)$, $i(P_n \Box C_k)$, $i(C_n \Box P_k)$, $i(C_n \Box C_k)$ and $i(P_n \times P_k)$, $i(P_n \times C_k)$, $i(C_n \times P_k)$, $i(C_n \times C_k)$ for some fixed k are given in Tables 7 – 10.*

In the tables, previously known results are shaded gray. Note that the independent domination number is not monotone and therefore it can happen that $i(P_n \Box P_k) > i(P_n \Box C_k)$ in some cases.

5. Conclusion

1. The aim of this paper was to generalize and improve the results of the domination number on polygraphs from [26, 35, 40]. We have shown that almost any variation of the domination number can be solved in constant time on polygraphs. The only restriction, which is applied in the proof of Lemma 1, is actually that the "labels" of the vertices must be nonnegative. If some are negative, then Algorithm 1 still yields an $O(\log n)$ algorithm on polygraphs.

2. When one implements the improved algorithm, calculations for rotagraphs take much more time that the ones for fasciagraphs. Lemma 3 indicates a step forward in a sense that the results for rotagraphs can be deduced from the results for fasciagraphs. Particularly, the values for rotagraph and for fasciagraphs with the same monograph can only differ for a constant value.

3. The improved algorithm allows us to improve many best known results for the domination number and others (see Tables 1, 2). The results of Tables 3 - 10 could, according to the theoretical results, be extended for larger monographs. We implemented the improved algorithm on personal computer and later on a small computer cluster. Therefore, due to time constraints, we did not compute any additional formulas. But with a little help of technology we could produce more results. Moreover, we restricted our attention to the domination, Roman domination and independent domination number. One could go even further and implement the algorithm for other domination types that satisfy the definition of $*$–domination.

4. Another avenue of research was initiated recently in [7] where the properties of invariants, allowing such an algorithm to be applied to polygraphs, were investigated. On the other hand, here restriction to certain type of domination problems led to improvements in speed. It may be interesting to investigate whether and under what additional conditions Lemma 1 and Lemma 3 can be valid for a larger class of graph invariants.

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Table 3: Domination numbers for the direct products $P_n \times P_k$ and $C_n \times P_k$, $k = 3, \ldots, 11$ and $n \geq 3$.

k	$\gamma(P_n \times P_k)$	$\gamma(C_n \times P_k)$
3	n	n
4	n; if $n \equiv 0 \pmod 4$ $n+1$; if $n \equiv 1, 3 \pmod 4$ $n+2$; otherwise	n; if $n \equiv 0 \pmod 4$ $n+1$; if $n \equiv 1, 3 \pmod 4$ $n+2$; otherwise
5	5; if $n = 3$ 6; if $n = 4$ 11; if $n = 7$ $\lceil \frac{4n}{3} \rceil + 1$; if $n \equiv 2 \pmod 3$ $\lceil \frac{4n}{3} \rceil + 2$; otherwise	$\lceil \frac{4n}{3} \rceil + 1$; if $n \equiv 2 \pmod 6$ $\lceil \frac{4n}{3} \rceil$; otherwise
6	5; if $n = 3$ $\lceil \frac{8n}{5} \rceil + 1$; if $n \equiv 3, 4 \pmod 5$ $\lceil \frac{8n}{5} \rceil$; otherwise	$\lceil \frac{8n}{5} \rceil + 1$; if $n \equiv 4, 8 \pmod{10}$ $\lceil \frac{8n}{5} \rceil$; otherwise
7	14; if $n = 7$ $\lceil \frac{9n}{5} \rceil$; if $n \equiv 4 \pmod 5$ $\lceil \frac{9n}{5} \rceil + 1$; if $n \equiv 1, 3 \pmod 5$ $\lceil \frac{9n}{5} \rceil + 2$; otherwise	$\lceil \frac{9n}{5} \rceil + 1$; if $n \equiv 1, 6, 8 \pmod{10}$ $\lceil \frac{9n}{5} \rceil + 2$; if $n \equiv 2 \pmod{10}$ $\lceil \frac{9n}{5} \rceil$; otherwise
8	$2n$; if $n \equiv 0 \pmod 4$ $2n+2$ otherwise	$2n$; if $n \equiv 0 \pmod 4$ $2n+2$ otherwise
9	10; if $n = 4$ 17; if $n = 7$ $\lceil \frac{11n}{5} \rceil + 2$; if $n \equiv 0, 1, 3, 6, 7, 8 \pmod{10}$ $\lceil \frac{11n}{5} \rceil + 3$; otherwise	
10	16; if $n = 6$ 24; if $n = 9$ 26; if $n = 10$ $\lceil \frac{12n}{5} \rceil + 2$; if $n \equiv 3, 4, 8 \pmod{10}$ $\lceil \frac{12n}{5} \rceil + 3$; if $n \equiv 1, 2, 6, 7 \pmod{10}$ $\lceil \frac{12n}{5} \rceil + 4$; otherwise	
11	12; if $n = 4$ $\lceil \frac{13n}{5} \rceil + 3$; if $n = 3, 5, 9, 13, 19$ $\lceil \frac{13n}{5} \rceil + 2$; if $n = 6, 7, 11, 12$ $\lceil \frac{13n}{5} \rceil + 3$; if $n \equiv 1, 4, 7, 8 \pmod{10}$ $\lceil \frac{13n}{5} \rceil + 4$; otherwise	

Table 4: Domination numbers for the direct products $P_n \times C_k$ and $C_n \times C_k$, $k = 3, \ldots, 11$ and $n \geq 3$.

k	$\gamma(P_n \times C_k)$	$\gamma(C_n \times C_k)$
3	$\lceil \frac{2n}{3} \rceil$; if $n \equiv 2 \pmod 3$ $\lceil \frac{2n}{3} \rceil + 1$; otherwise	3; if $n = 3$ 4; if $n = 4$ $\lceil \frac{2n}{3} \rceil$; otherwise
4	n; if $n \equiv 0 \pmod 4$ $n+1$; if $n \equiv 1, 3 \pmod 4$ $n+2$; otherwise	n; if $n \equiv 0 \pmod 4$ $n+1$; if $n \equiv 1, 3 \pmod 4$ $n+2$; otherwise
5	$n+2$	n; if $n \equiv 0 \pmod 5$ $n+1$; if $n \equiv 1 \pmod 5$ $n+2$; otherwise
6	$\lceil \frac{4n}{3} \rceil + 1$; if $n \equiv 2 \pmod 3$ $\lceil \frac{4n}{3} \rceil + 2$; otherwise	8; if $n = 4$ $\lceil \frac{4n}{3} \rceil + 1$; if $n \equiv 2 \pmod 6$ $\lceil \frac{4n}{3} \rceil$; otherwise
7	$\lceil \frac{3n}{2} \rceil + 3$; if $n \equiv 2 \pmod 4$ $\lceil \frac{3n}{2} \rceil + 2$; otherwise	
8	$2n$; if $n \equiv 0 \pmod 4$ $2n+2$ otherwise	$2n$
9	$2n+2$; if $n \equiv 0, 2, 4, 5, 8, 11 \pmod{12}$ $2n+3$ otherwise	
10	$2n+4$	
11	12; if $n = 4$ $\lceil \frac{7n}{3} \rceil + 3$; if $n \equiv 2 \pmod 3$ $\lceil \frac{7n}{3} \rceil + 4$; otherwise	

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Table 5: Roman domination numbers for the direct products $P_n \times P_k$ and $C_n \times P_k$, $k = 3, \ldots, 9$ and $n \geq 2$.

k	$\gamma_R(P_n \times P_k)$		$\gamma_R(C_n \times P_k)$	
3	$\lceil \frac{3n}{2} \rceil + 1$;	if $n \equiv 2 \pmod 4$	$\lceil \frac{3n}{2} \rceil + 1$;	if $n \equiv 2 \pmod 4$
	$\lceil \frac{3n}{2} \rceil$;	otherwise	$\lceil \frac{3n}{2} \rceil$;	otherwise
4	6;	if $n = 2$	$2n$	
	12;	if $n = 5$		
	$2n$;	otherwise		
5	$\lceil \frac{8n}{3} \rceil$;	if $n \equiv 0 \pmod 3$	$\lceil \frac{8n}{3} \rceil$;	if $n \equiv 0, 3, 5 \pmod 6$
	$\lceil \frac{8n}{3} \rceil + 1$;	if $n \equiv 1 \pmod 3$	$\lceil \frac{8n}{3} \rceil + 1$;	if $n \equiv 1, 4 \pmod 6$
	$\lceil \frac{8n}{3} \rceil + 2$;	otherwise	$\lceil \frac{8n}{3} \rceil + 2$;	otherwise
6	12;	if $n = 2$	$3n$	
	24;	if $n = 6$		
	$3n$;	if $n \equiv 0 \pmod 2$		
	$3n + 1$;	otherwise		
7	10;	if $n = 2$		
	11;	if $n = 3$		
	20;	if $n = 5$		
	$\lceil \frac{10n}{3} \rceil$;	if $n \equiv 1 \pmod 3$		
	$\lceil \frac{10n}{3} \rceil + 1$;	if $n \equiv 2 \pmod 3$		
	$\lceil \frac{10n}{3} \rceil + 2$;	otherwise		
8	12	if $n = 2$	$4n$	
	24	if $n = 5$		
	$4n$;	otherwise		
9	12	if $n = 2$		
	14	if $n = 3$		
	$\lceil \frac{13n}{3} \rceil$;	if $n \equiv 4 \pmod 6$		
	$\lceil \frac{13n}{3} \rceil + 1$;	if $n \equiv 1, 2 \pmod 6$		
	$\lceil \frac{13n}{3} \rceil + 2$;	otherwise		

Table 6: Roman domination numbers for the direct products $P_n \times C_k$ and $C_n \times C_k$, $k = 3, \ldots, 9$ and $n \geq 2$.

k	$\gamma_R(P_n \times C_k)$	$\gamma_R(C_n \times C_k)$
3	$\lceil \frac{4n}{3} \rceil$; if $n \equiv 1 \pmod 3$ $\lceil \frac{4n}{3} \rceil + 1$; otherwise	5; if $n = 3$ $\lceil \frac{4n}{3} \rceil$; if $n \equiv 1, 3 \pmod 3$ $\lceil \frac{4n}{3} \rceil + 1$; otherwise
4	6; if $n = 2$ 12; if $n = 5$ $2n$; otherwise	$2n$
5	$2n+2$; if $n = 3, 4$ $2n+3$; if $n = 2, 6, 7$ $2n+4$; otherwise	8; if $n = 3$ $2n$; if $n \equiv 0 \pmod 5$ $2n+2$; if $n \equiv 1, 4 \pmod 5$ $2n+3$; if $n \equiv 2 \pmod 5$ $2n+4$; otherwise
6	$\lceil \frac{8n}{3} \rceil + 1$; if $n \equiv 1 \pmod 3$ $\lceil \frac{8n}{3} \rceil + 2$; otherwise	$\lceil \frac{8n}{3} \rceil + 1$; if $n \equiv 1, 4 \pmod 6$ $\lceil \frac{8n}{3} \rceil + 2$; if $n \equiv 2 \pmod 6$ $\lceil \frac{8n}{3} \rceil$; otherwise
7	$3n+2$; if $n = 3, 4, 5$ $3n+3$; if $n = 6, 7$ $3n+4$; otherwise	
8	12 if $n = 2$ 24; if $n = 5$ $4n$; otherwise	$4n$
9	14 if $n = 3$ $4n+2$; if $n \equiv 1 \pmod 3$ $4n+3$; if $n \equiv 0 \pmod 3$ $4n+4$; otherwise	

Table 7: Independent domination numbers for $P_n \Box P_k$ and $C_n \Box P_k$, $k = 2, \ldots, 8$ and $n \geq 2$.

k	$i(P_n \Box P_k)$	$i(C_n \Box P_k)$
2	$\lceil \frac{n}{2} \rceil$; if $n \equiv 1 \pmod 2$ $\lceil \frac{n}{2} \rceil + 1$; otherwise	$\lceil \frac{n}{2} \rceil$; if $n \equiv 0, 3 \pmod 4$ $\lceil \frac{n}{2} \rceil + 1$; otherwise
3	2 if $n = 2$ $\lceil \frac{3n}{4} \rceil$; if $n \equiv 1 \pmod 2$ $\lceil \frac{3n}{4} \rceil + 1$; otherwise	$\lceil \frac{3n}{4} \rceil$; if $n \equiv 0, 3 \pmod 4$ $\lceil \frac{3n}{4} \rceil + 1$; otherwise
4	$n + 1$ if $n = 2, 3, 5, 6, 9$ n; otherwise	n if $n \equiv 0 \pmod 2$ $n + 1$; otherwise
5	3; if $n = 2$ 4; if $n = 3$ $\lceil \frac{6n}{5} \rceil$; if $n \equiv 1 \pmod 5$ $\lceil \frac{6n}{5} \rceil + 1$; otherwise	$\lceil \frac{6n}{5} \rceil$; if $n \equiv 0 \pmod 2$ $\lceil \frac{6n}{5} \rceil + 1$; otherwise
6	11; if $n = 7$ $\lceil \frac{10n}{7} \rceil$; if $n \equiv 1, 5 \pmod 7$ $\lceil \frac{10n}{7} \rceil + 2$; if $n \equiv 0 \pmod 7$ $\lceil \frac{10n}{7} \rceil + 1$; otherwise	$\lceil \frac{10n}{7} \rceil$; if $n \equiv 0, 4, 5, 8, 10, 12 \pmod{14}$ $\lceil \frac{10n}{7} \rceil + 2$; if $n \equiv 7 \pmod{14}$ $\lceil \frac{10n}{7} \rceil + 1$; otherwise
7	10; if $n = 5$ $\lceil \frac{5n}{3} \rceil + 1$; if $n \equiv 1, 3 \pmod 6$ $\lceil \frac{5n}{3} \rceil$; otherwise	7; if $n = 3$ $\lceil \frac{5n}{3} \rceil + 1$; if $n \equiv 0 \pmod 3$ $\lceil \frac{5n}{3} \rceil$; otherwise
8-14	see [14]	

Table 8: Independent domination numbers for $P_n \Box C_k$ and $C_n \Box C_k$, $k = 3, \ldots, 10$ and $n \geq 2$.

k	$i(P_n \Box C_k)$	$i(C_n \Box C_k)$
3	n	n
4	n	$\begin{cases} 4; & \text{if } n = 3 \\ n; & \text{otherwise} \end{cases}$
5	$n+2$	$\begin{cases} 8; & \text{if } n = 5 \\ n+1; & \text{if } n \equiv 4 \pmod 5 \\ n+2; & \text{otherwise} \end{cases}$
6	$\begin{cases} 6; & \text{if } n = 3 \\ \left\lceil \frac{4n}{3} \right\rceil; & \text{if } n \equiv 1 \pmod 3 \\ \left\lceil \frac{4n}{3} \right\rceil + 1; & \text{otherwise} \end{cases}$	$\begin{cases} 6; & \text{if } n = 3 \\ 12; & \text{if } n = 7 \\ \left\lceil \frac{4n}{3} \right\rceil; & \text{if } n \equiv 0, 4 \pmod 6 \\ \left\lceil \frac{4n}{3} \right\rceil + 1; & \text{otherwise} \end{cases}$
7	$\begin{cases} 4; & \text{if } n = 2 \\ 6; & \text{if } n = 3 \\ 8; & \text{if } n = 4 \\ \left\lceil \frac{3n}{2} \right\rceil + 2; & \text{if } n \equiv 1 \pmod 2 \\ \left\lceil \frac{3n}{2} \right\rceil + 3; & \text{otherwise} \end{cases}$	$\begin{cases} 12; & \text{if } n = 6 \\ \left\lceil \frac{3n}{2} \right\rceil + 1; & \text{if } n \equiv 4, 5, 13 \pmod{14} \\ \left\lceil \frac{3n}{2} \right\rceil + 2; & \text{otherwise} \end{cases}$
8	$\begin{cases} 4; & \text{if } n = 2 \\ 6; & \text{if } n = 3 \\ 8; & \text{if } n = 4 \\ \left\lceil \frac{9n}{5} \right\rceil + 2; & \text{if } n \equiv 0 \pmod 5 \\ \left\lceil \frac{9n}{5} \right\rceil + 1; & \text{otherwise} \end{cases}$	$\begin{cases} 8; & \text{if } n = 3 \\ \left\lceil \frac{9n}{5} \right\rceil; & \text{if } n \equiv 0, 4 \pmod{10} \\ \left\lceil \frac{9n}{5} \right\rceil + 2; & \text{if } n \equiv 1 \pmod{10} \\ \left\lceil \frac{9n}{5} \right\rceil + 1; & \text{otherwise} \end{cases}$
9	$2n+2$	
10	$\begin{cases} 2n+2; & \text{if } n = 2, 4, 5 \\ 2n+3; & \text{if } n = 3, 6, 7, 8, 9 \\ 2n+4; & \text{otherwise} \end{cases}$	

Table 9: Independent domination numbers for $P_n \times P_k$ and $C_n \times P_k$, $k = 3, \dots, 11$ and $n \geq 2$.

k	$i(P_n \times P_k)$	$i(C_n \times P_k)$
3	n	n
4	$\lceil \frac{4n}{3} \rceil$; if $n \equiv 0 \pmod 3$ $\lceil \frac{4n}{3} \rceil + 1$; if $n \equiv 2 \pmod 3$ $\lceil \frac{4n}{3} \rceil + 2$; otherwise	$\lceil \frac{4n}{3} \rceil$; if $n \equiv 0, 1, 3 \pmod 6$ $\lceil \frac{4n}{3} \rceil + 1$; if $n \equiv 2, 5 \pmod 6$ $\lceil \frac{4n}{3} \rceil + 2$; otherwise
5	5; if $n = 3$ $\lceil \frac{4n}{3} \rceil + 1$; if $n \equiv 2 \pmod 3$ $\lceil \frac{4n}{3} \rceil + 2$; otherwise	$\lceil \frac{4n}{3} \rceil$; if $n \equiv 0, 1, 3 \pmod 6$ $\lceil \frac{4n}{3} \rceil + 1$; if $n \equiv 2, 5 \pmod 6$ $\lceil \frac{4n}{3} \rceil + 2$; otherwise
6	$\lceil \frac{5n}{3} \rceil$; if $n \equiv 2 \pmod 6$ $\lceil \frac{5n}{3} \rceil + 1$; if $n \equiv 3, 4, 5 \pmod 6$ $\lceil \frac{5n}{3} \rceil + 2$; otherwise	$\lceil \frac{5n}{3} \rceil + 1$; if $n \equiv 4 \pmod 6$ $\lceil \frac{5n}{3} \rceil$; otherwise
7	7; if $n = 3$ 12; if $n = 4$ 17; if $n = 7$ $2n + 2$; otherwise	12; if $n = 4$ 11; if $n = 5$ 22; if $n = 10$ $2n$; otherwise
8	$2n + 4$; if $n \equiv 1 \pmod 3$ $2n + 2$; otherwise	
9	$\lceil \frac{7n}{3} \rceil + 1$; if $n \equiv 2 \pmod 6$ $\lceil \frac{7n}{3} \rceil + 3$; if $n \equiv 1 \pmod 6$ $\lceil \frac{7n}{3} \rceil + 2$; otherwise	
10	16; if $n = 4$ $\lceil \frac{8n}{3} \rceil + 2$; if $n \equiv 0, 2 \pmod 3$ $\lceil \frac{8n}{3} \rceil + 3$; otherwise	
11	11; if $n = 3$ $\lceil \frac{8n}{3} \rceil + 2$; if $n \equiv 2 \pmod 3$ $\lceil \frac{8n}{3} \rceil + 4$; if $n \equiv 0 \pmod 3$ $\lceil \frac{8n}{3} \rceil + 5$; otherwise	

Table 10: Independent domination numbers for $P_n \times C_k$ and $C_n \times C_k$, $k = 2, \ldots, 10$ and $n \geq 2$.

k	$i(P_n \times C_k)$	$i(C_n \times C_k)$
3	$\lceil \frac{2n}{3} \rceil$; if $n \equiv 2 \pmod 3$ $\lceil \frac{2n}{3} \rceil + 1$; otherwise	n; if $n = 3, 4, 5$ $\lceil \frac{2n}{3} \rceil$; if $n \equiv 0 \pmod 3$ $\lceil \frac{2n}{3} \rceil + 1$; otherwise
4	$\lceil \frac{4n}{3} \rceil$; if $n \equiv 0 \pmod 3$ $\lceil \frac{4n}{3} \rceil + 1$; if $n \equiv 2 \pmod 3$ $\lceil \frac{4n}{3} \rceil + 2$; otherwise	$\lceil \frac{4n}{3} \rceil$; if $n \equiv 0 \pmod 3$ $\lceil \frac{4n}{3} \rceil + 1$; if $n \equiv 1 \pmod 3$ $\lceil \frac{4n}{3} \rceil + 2$; otherwise
5	4; if $n = 2$ 5; if $n = 3$ 8; if $n = 5$ 9; if $n = 6$ $n + 4$; otherwise	5; if $n = 3$ 8; if $n = 6$ $n + 2$; if $n \equiv 2 \pmod 5$ $n + 4$; if $n \equiv 4 \pmod 5$ $n + 3$; otherwise
6	$\lceil \frac{4n}{3} \rceil + 1$; if $n \equiv 2 \pmod 3$ $\lceil \frac{4n}{3} \rceil + 2$; otherwise	$\lceil \frac{4n}{3} \rceil$; if $n \equiv 0 \pmod 3$ $\lceil \frac{4n}{3} \rceil + 1$; if $n \equiv 1 \pmod 3$ $\lceil \frac{4n}{3} \rceil + 2$; otherwise
7	10; if $n = 4$ $\lceil \frac{5n}{3} \rceil + 1$; if $n \equiv 2 \pmod 3$ $\lceil \frac{5n}{3} \rceil + 2$; otherwise	10; if $n = 4$ 14; if $n = 7$ 18; if $n = 10$ $\lceil \frac{5n}{3} \rceil$; if $n \equiv 0 \pmod 3$ $\lceil \frac{5n}{3} \rceil + 1$; otherwise
8	12; if $n = 4$ $2n + 2$; otherwise	
9	$2n + 2$; if $n \equiv 2 \pmod 3$ $2n + 3$; if $n \equiv 0 \pmod 3$ $2n + 4$; otherwise	
10	$2n + 4$; if $n = 2, 3$ $2n + 6$; if $n = 5, 6$ $2n + 8$; otherwise	

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