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ROMAN DOMINATION NUMBER OF THE CARTESIAN PRODUCTS OF PATHS AND CYCLES

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Abstract

Roman domination is a historically inspired variety of general domination such that every vertex is labeled with labels from $\{0, 1, 2\}$. Roman domination number is the smallest of the sums of labels fulfilling condition that every vertex, labeled 0, has a neighbor, labeled 2. Using algebraic approach we give $O(C)$ time algorithm for computing Roman domination number of special classes of polygraphs (rota– and fasciagraphs). By implementing the algorithm we give formulas for Roman domination number of the Cartesian products of paths and cycles $P_n \Box P_k$, $P_n \Box C_k$ for $k \leq 8$ and $n \in \mathbb{N}$ and for $C_n \Box P_k$ and $C_n \Box C_k$ for $k \leq 5$, $n \in \mathbb{N}$. We also give a list of Roman graphs among investigated families.

1 Introduction

Domination and its variations have been intensively studied and its algorithmic aspects have been widely investigated [15, 16]. It is well known that the problem of determining domination number of arbitrary graphs is NP–complete [15]. It is therefore interesting to consider algorithms for some classes of graphs, including grid graphs. Exact domination numbers of the Cartesian products of paths $P_n \Box P_k$ with fixed k was established in [1, 5, 7, 13, 14], of the Cartesian product of cycles in [9, 19, 30] and of the Cartesian products of cycles and paths in [25]. A general O($\log n$) algorithm based on path algebra approach, which can be used to compute domination number of $P_n \Box P_k$ for a fixed k, has been proposed in [20]. The algorithm of [20] can in most cases, including the computation of distance based invariants [18] and the domination numbers [31], be turned into a constant time algorithm, i.e. the algorithm can find closed formulas for arbitrary n. The existence of an algorithm that provides closed formulas for domination numbers on grid graphs has been observed or claimed also in [11, 23].

An interesting variety of the graph domination that is popular because of its historical motivation [26, 29] is called Roman domination. The history of the problem goes back to the 4th century, when Emperor Constantine tried to secure the Roman Empire by placing armies in the cities in a way that the area would be secured with minimum possible number of armies, some of which could also be sent to defend neighboring cities without leaving the "home" city unsecured. While the problem is still of interest in military operations research [2] it also has obvious applications anywhere when a time critical service is supposed to be provided with certain backup. (For example, firemen brigade should never send all cars to answer the first emergency call.) Roman domination is a variety of the general domination such that different types of guards are used. Every vertex of a graph must be labeled with numbers from $\{0, 1, 2\}$ so that every vertex labeled 0 has a neighbor labeled 2. Roman domination number of a graph is the smallest of the sums of labels, such that they fulfill the above condition. Formal definition was proposed in [6] and is recalled in Section 2.

As the problem of determining Roman domination number of a graph is NP–complete [8], it is interesting to determine Roman domination number of some classes of graphs [6, 12, 17, 21, 22, 27, 28]. Also Vizing's–like conjecture for Roman domination [33] and some properties of γ_R–functions [4, 10, 24] were studied. One of the open problems posed in the first article on this variety of domination [6] was to determine exact Roman domination number for arbitrary grid graph. Roman domination numbers for $P_n \Box P_k$ for $k \in \{1, 2, 3, 4\}$ and $n \in \mathbb{N}$ have been computed in [6, 8]. An algorithm for computing Roman domination number of grid graphs for a fixed k in linear time was also presented in [8]. Here we use path algebra approach to design an $O(\log n)$ algorithm for Roman domination numbers of grid graphs and show how it can be turned into a constant time algorithm that provides closed formulas for Roman domination numbers of grid graphs. More precisely, the algorithm's time complexity is independent of n and has superpolynomial time complexity in terms of k. We use the algorithm to find formulas for Roman domination number of $P_n \Box P_k$ and $P_n \Box C_k$ for $k \leq 8$ and $n \in \mathbb{N}$ and for $C_n \Box P_k$ and $C_n \Box C_k$ for $k \leq 5$ and $n \in \mathbb{N}$.

In the rest of this paper we first summarize the background for the main algorithm. Section 2 is dedicated to the concept of polygraphs, which has been widely used in chemical graph theory and elsewhere. In Section 3 we summarize a general algorithm for solving different problems on polygraphs, which was proposed in [20]. The algorithm for computing Roman domination number for faciagraphs and rotagraphs is presented in Section 4. Section 5 then summarizes some results obtained by implementing the algorithm. Roman graphs (i.e. graphs, satisfying $\gamma_R(G) = 2\gamma(G)$) among graphs we investigate are listed in Section 6. Finally, constructions for γ_R– functions of graphs are presented.

2 Preliminaries

We consider finite undirected and directed graphs. A graph will always mean an undirected graph, a digraph will stand for a directed graph. An edge in an undirected graph will be denoted uv while in directed graph, an arc between vertices u and v will be denoted (u, v). P_n will stand for a path on n vertices and C_n for a cycle on n vertices.

For a graph $G = (V, E)$, a set D is a *dominating set* if every vertex in $V \setminus D$ is adjacent to a vertex in D. The *domination number* $\gamma(G)$ is the minimum cardinality of a dominating set of G. A dominating set of cardinality $\gamma(G)$ is called a *minimum dominating set*, or shortly a γ–set.

Roman domination has been formally defined in [6] as follows: For a graph $G = (V, E)$, let $f : V \longrightarrow \{0, 1, 2\}$ and let (V_0, V_1, V_2) be the ordered partition of V induced by f, where $V_i = \{v \in V(G) \mid f(v) = i\}$. Let $|V_i| = n_i$ for $i = 0, 1, 2$. Note that there exists a 1–1 correspondence between the functions $f : V \longrightarrow \{0, 1, 2\}$ and ordered partitions (V_0, V_1, V_2) of V. Thus, we will write $f = (V_0, V_1, V_2)$. A function $f = (V_0, V_1, V_2)$ is a *Roman dominating function* (RDF) if $V_2 \succ V_0$, in other words, if the set V_2 dominates the set V_0. The weight of f is defined as:

$$w(f) = \sum_{v \in V} f(v) = n_1 + 2n_2.$$

The Roman domination number, $\gamma_R(G)$, equals the minimum weight of an RDF of G. We will also say that a function $f = (V_0, V_1, V_2)$ is a γ_R–function, if it is an RDF and $w(f) = \gamma_R(G)$. Obviously, $\gamma(G) \leq \gamma_R(G) \leq 2\gamma(G)$. Only graphs that satisfy $\gamma_R(G) = \gamma(G)$ are edgeless graphs and a graph G is called a *Roman graph* if $\gamma_R(G) = 2\gamma(G)$. Finding classes of Roman graphs was one of the open problems posed in [6]. For instance, among paths, graphs P_{3k} and P_{3k+2} are Roman graphs since $\gamma_R(P_{3k+1}) = 2k + 1 < 2\gamma(P_{3k+1}) = 2k + 2$. The difference between γ_R and 2γ can be arbitrary large, for example on the family of subdivided stars. Subdivided star $\tilde{K}_{1,n}$ is obtained from the star $S_{n+1} = K_{1,n}$ by subdivision of each edge. We have $\gamma_R(\tilde{K}_{1,n}) = 2 + n < 2\gamma(\tilde{K}_{1,n}) = 2n$. Construction of minimum dominating set and γ_R–function for $\tilde{K}_{1,5}$ can be seen on Figure 1 where full circs represent vertices in the domination set on the left side and vertices of weight 1 in γ_R–function on the right side. Vertex of weight 2 in the γ_R–function is presented with double circ.

The *Cartesian product* of graphs G and H, denoted $G \square H$, is a graph with vertex set $V(G) \times V(H)$ and two vertices (g, h) and (g', h') are connected if $g = g'$ and $hh' \in E(H)$ or $gg' \in E(G)$ and $h = h'$. Examples of the Cartesian product graphs include the grid graphs, which are products of paths $P_n \square P_k$, and tori, which are products of cycles $C_n \square C_k$.

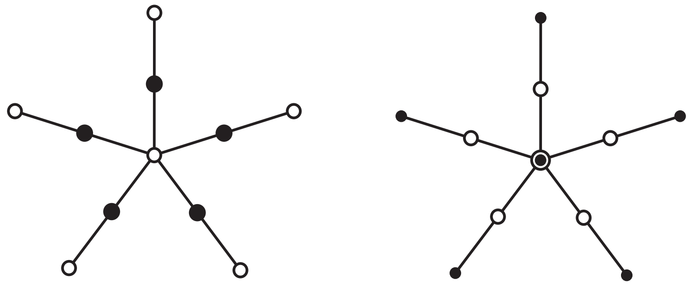

Figure 1: Minimum dominating set and γ_R–function of $\tilde{K}_{1,5}$.

Let $G_1, \ldots, G_n$ be arbitrary mutually disjoint graphs and $X_1, \ldots, X_n$ a sequence of sets of edges such that an edge of X_i joins a vertex of $V(G_i)$ with a vertex of $V(G_{i+1})$. For convenience we also set $G_0 = G_n$, $G_{n+1} = G_1$ and $X_0 = X_n$. This in particular means that edges in X_n join vertices of G_n with vertices of G_1. A *polygraph* $\Omega_n = \Omega_n(G_1, \ldots G_n; X_1, \ldots X_n)$ over monographs $G_1, \ldots, G_n$ is defined in the following way:

$$V(\Omega_n) = V(G_1) \cup \ldots \cup V(G_n),$$

$$E(\Omega_n) = E(G_1) \cup X_1 \cup \ldots \cup E(G_n) \cup X_n.$$

For a polygraph Ω_n and for $i = 1, \ldots, n$ we also define

$$D_i = \{u \in V(G_i) \mid \exists v \in G_{i+1} : uv \in X_i\},$$

$$R_i = \{u \in V(G_{i+1}) \mid \exists v \in G_i : uv \in X_i\}.$$

In general, $R_i \cap D_{i+1}$ does not have to be empty. If all graphs G_i are isomorphic to a fixed graph G and all sets X_i are equal, then we call such a graph *rotagraph* and denote it $\omega_n(G; X)$. A rotagraph without edges between the first and the last copy of G (formally, $X_n = \emptyset$) is *fasciagraph*, $\psi_n(G; X)$. In rotagraph as well as in fasciagraph, all sets D_i and R_i are equal. We will denote those two sets with D and R, respectively. Observe that Cartesian products of paths $P_n \Box P_k$ are examples of fasciagraphs and that Cartesian products of cycles $C_n \Box C_k$ are examples of rotagraphs. Graphs $P_n \Box C_k$ can be treated as fasciagraphs or as rotagraphs.

3 Path algebras and the algorithm

Let us now summarize a general framework for solving different problems on the class of fasciagraphs and rotagraphs, which was proposed in [20] and also used in [31]:

A semiring $\mathcal{P} = (P, \oplus, \circ, e^{\oplus}, e^{\circ})$ is a set P on which two binary operations, $\oplus$ and $\circ$ are defined such that:

1. $(P, \oplus)$ is a commutative monoid with $e^{\oplus}$ as a unit;

2. $(P, \circ)$ is a monoid with e° as a unit;

3. $\circ$ is left– and right–distributive over $\oplus$;

4. $\forall x \in P, x \circ e^\oplus = e^\oplus = e^\oplus \circ x$.

An idempotent semiring is called a *path algebra*. It is easy to see that a semiring is a path algebra if and only if $e^\circ \oplus e^\circ = e^\circ$ holds for e°, the unit of the monoid $(P, \circ)$. An important example of a path algebra for our work is $\mathcal{P}_1 = (\mathbb{N}_0 \cup \{\infty\}, \min, +, \infty, 0)$. Here $\mathbb{N}_0$ denotes the set of nonnegative integers and $\mathbb{N}$ the set of positive integers. For more examples of path algebras we refer to [3].

Let $\mathcal{P} = (P, \oplus, \circ, e^\oplus, e^\circ)$ be a path algebra and let $\mathcal{M}_n(\mathcal{P})$ be the set of all $n \times n$ matrices over P. Let $A, B \in \mathcal{M}_n(\mathcal{P})$ and define operations $\oplus$ and $\circ$ in the usual way:

$$\begin{aligned}(A \oplus B)_{ij} &= A_{ij} \oplus B_{ij},\\ (A \circ B)_{ij} &= \bigoplus_{k=1}^{n} A_{ik} \circ B_{kj}.\end{aligned}$$

$\mathcal{M}_n(\mathcal{P})$ equipped with above operations is a path algebra with the zero and the unit matrix as units of semiring. In our example $\mathcal{P}_1 = (\mathbb{N}_0 \cup \{\infty\}, \min, +, \infty, 0)$, all elements of the zero matrix are ∞, the unit of the monoid $(P, \min)$, and the unit matrix is a diagonal matrix with diagonal elements equal to $e^\circ = 0$ and all other elements equal to $e^\oplus = \infty$.

Let $\mathcal{P}$ be a path algebra and let G be a labeled digraph, that is a digraph together with a labeling function ℓ which assigns to every arc of G an element of P. Let $V(G) = \{v_1, v_2, \ldots, v_n\}$. The labeling ℓ of G can be extended to paths in the following way: For a path $Q = (v_{i_0}, v_{i_1})(v_{i_1}, v_{i_2}) \ldots (v_{i_{k-1}}, v_{i_k})$ of G let

$$\ell(Q) = \ell\left(v_{i_0}, v_{i_1}\right) \circ \ell\left(v_{i_1}, v_{i_2}\right) \circ \ldots \circ \ell\left(v_{i_{k-1}}, v_{i_k}\right)$$

Let S_{ij}^k be the set of all paths of order k from v_i to v_j in G and let $A(G)$ be the matrix defined by:

$$A(G)_{ij} = \begin{cases} \ell\left(v_i, v_j\right); & \text{if } (v_i, v_j) \text{ is an arc of } G\\ e^\oplus; & \text{otherwise}\end{cases}$$

It is well-known (see for example [3]) that

$$\left(A(G)^k\right)_{ij} = \bigoplus_{Q \in S_{ij}^k} \ell(Q).$$

Let $\omega_n(G; X)$ be a rotagraph and $\psi_n(G; X)$ a fasciagraph. Set $U = D_i \cup R_i = D \sqcup R$ and let $N = 2^{|U|}$. Define a labeled digraph $\mathcal{G} = \mathcal{G}(G; X)$ as follows: The vertex set of $\mathcal{G}$ is formed by the subsets of U which will be denoted by V_i. An arc joins a subset V_i with a subset V_j if V_i is not in a "conflict" with V_j. Here a conflict of V_i with V_j means that using V_i and V_j as a part of a solution in consecutive copies of G would violate the problem assumption. For instance, if we look for a domination number of a graph, such a conflict would be a nonempty intersection between sets V_i and V_j, or if we look for an independence number of a graph, such a conflict would be an edge between sets V_i and V_j. Let finally $\ell : E(\mathcal{G}) \longrightarrow P$ be a labeling of $\mathcal{G}$ where $\mathcal{P}$ is a path algebra on the set P. The general scheme for the algorithm as proposed in [20] is:

Algorithm 1 [20]

1. Select the appropriate path algebra $\mathcal{P} = (P, \oplus, \circ, e^{\oplus}, e^{\circ})$.
2. Determine an appropriate labeling ℓ of a graph $\mathcal{G}(G; X)$.
3. In $\mathcal{M}(\mathcal{P})$ calculate $A(\mathcal{G})^n$.
4. Among admissible coefficients of $A(\mathcal{G})^n$ select one which optimizes the corresponding goal function.

It is well known that, in general, Step 3 of the algorithm can be implemented to run in $O(\log n)$ time. However, computing the powers of $A(\mathcal{G})^n = A_n$ in $O(C)$ time is possible using special structure of the matrices in some cases, including the distance based invariants [18], the domination numbers [31], and others [32]. Here we prove that $A(\mathcal{G})^n = A_n$ can be computed in $O(C)$ time for Roman domination number (see Section 4).

4 Roman domination number of fasciagraphs and rotagraphs

Let $\omega_n(G; X)$ be a rotagraph and $\psi_n(G; X)$ a fasciagraph as defined above. Set $U = D_i \cup R_i = D \sqcup R$. (Keep in mind that $D_i \subseteq G_i$ and $R_i \subseteq G_{i+1}$, but since $R_i = R$ and $D_i = D$ for all i, we can write $U = D_i \cup R_i = D \sqcup R$). A labeled digraph $\mathcal{G} = \mathcal{G}(G; X)$ is a graph with vertex set:

$$V(\mathcal{G}) = \{(V_i, W_i) \mid V_i, W_i \subseteq U, V_i \cap W_i = \emptyset\}$$

For convenience we sometimes refer to a vertex of $\mathcal{G}$ shortly by $v_i = (V_i, W_i)$. In particular, $v_0 = (V_0, W_0)$ stands for $(\emptyset, \emptyset)$.

Let $v_i, v_j \in V(\mathcal{G})$ and consider for a moment $\psi_3(G; X)$. Let $V_i \cup W_i \subseteq D_1 \cup R_1$ and $V_j \cup W_j \subseteq D_2 \cup R_2$. Let $\gamma_{R_{i,j}}(G; X)$ be the weight of a γ_R-function of a graph $G_2 \setminus (((V_i \cup W_i) \cap R_1) \cup (D_2 \cap (V_j \cup W_j)))$, such that $V_i \cup V_j \subseteq V'$ and $W_i \cup W_j \subseteq W'$, where (V'_0, V', W') is an RDF of a graph G_2. For consistency, we introduce an arc between

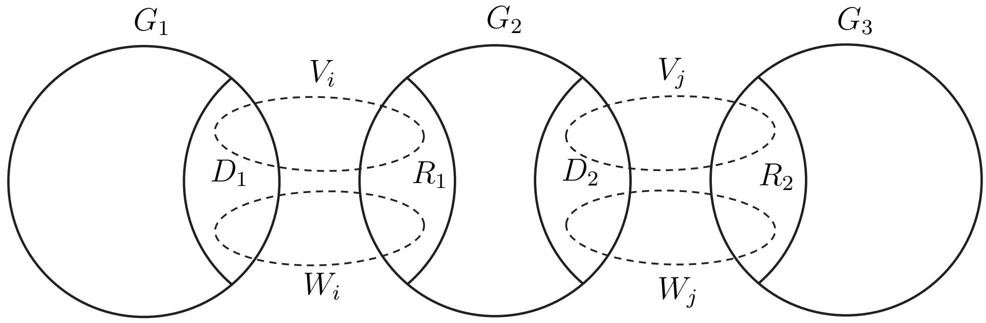

Figure 2: $\psi_3(G;X)$ with the above notation

vertices v_i and v_j only if $R \cap V_i \cap W_j \cap D = \emptyset$ and $R \cap W_i \cap V_j \cap D = \emptyset$. Set

$$\begin{aligned}\ell(v_i, v_j) = {} & |R \cap V_i| + 2\,|R \cap W_i| + |V_j \cap D| + 2\,|W_j \cap D| - \\ & - |R \cap V_i \cap V_j \cap D| - 2\,|R \cap W_i \cap W_j \cap D| + \gamma_{R_{i,j}}(G;X). \end{aligned} \tag{1}$$

Then we have an algorithm which computes Roman domination number of rotagraphs and fasciagraphs:

Algorithm 2

1. For a path algebra select $\mathcal{P} = (\mathbb{N}_0 \cup \{\infty\}, \min, +, \infty, 0)$.
2. Label $\mathcal{G} = \mathcal{G}(G;X)$ as defined above.
3. In $\mathcal{M}(\mathcal{P})$ calculate $A\left(\mathcal{G}\right)^n$.
4. Let $\gamma_R\left(\psi_n(G;X)\right) = \left(A\left(\mathcal{G}\right)^n\right)_{00}$ and $\gamma_R\left(\omega_n(G;X)\right) = \min_i \left(A\left(\mathcal{G}\right)^n\right)_{ii}$.

Theorem 4.1 *The Algorithm 2 correctly computes Roman domination number of rotagraphs and fasciagraphs:*

$$\gamma_R\left(\psi_n(G;X)\right) = \left(A\left(\mathcal{G}\right)^n\right)_{00} \tag{2}$$

$$\gamma_R\left(\omega_n(G;X)\right) = \min_i \left(A\left(\mathcal{G}\right)^n\right)_{ii} \tag{3}$$

in $O(\log n)$ time.

Proof. Let G_1 and G_2 be arbitrary graphs, X_1 a set of edges between vertices of G_1 and G_2 and let $\Omega_2(G_1, G_2; X_1, \emptyset)$ be a polygraph. Let also $\mathcal{P} = (\mathbb{N}_0 \cup \{\infty\}, \min, +, \infty, 0)$ be a path algebra and let $\mathcal{G}'$ be a labeled digraph for Ω_2 defined as above. Then, by the definition of labeling, we have

$$\begin{aligned}\gamma_R\left(\Omega_2\left(G_1, G_2; X_1, \emptyset\right)\right) &= \left[A(G_1) + A(G_2)\right]_{00} \\ &= \left[\min_{v_k \in V(\mathcal{G})} \{\ell(0, v_k) + \ell(v_k, 0)\}\right]_{00}.\end{aligned}$$

Let $G_1 = G$, $X_1 = X$ and $G_2 = \psi_{n-1}(G; X)$. Then (2) follows by induction.

For (3), similarly, consider $\Omega_2(G_1, G_2; X_1, X_2)$ and let $G_1 = G$, $X_1 = X_2 = X$ and $G_2 = \psi_{n-1}(G; X)$.

Time complexity of the algorithm was already discussed for general case in Section 3. □

As mentioned before we prove that calculating powers of matrices $A(\mathcal{G})^n = A_n$ (and therefore implementing the algorithm) is possible in $O(C)$ time based on the following lemma:

Lemma 4.2 *Let $k = |V(\mathcal{G}(G; X))|$ and $K = |V(G)|$. Then there is an index $q \leq (4K + 2)^{k^2}$ such that $D_q = D_p + C$ for some index $p < q$ and some constant matrix C. Let $P = q - p$. Then for every $r \geq p$ and every $s \geq 0$ we have*

$$A_{r+sP} = A_r + sC\,.$$

Proof. First observe that for any $l \geq 1$, the difference between any pair of entries of A_l, both different from ∞, is bounded by $4K$: Assume $(A_l)_{ij} \neq \infty$. Then

$$\begin{aligned}(A_l)_{ij} &= \gamma_R\left((V(G_1) \setminus (V_i \cup W_i)) \cup V(G_2) \cup \ldots \cup V(G_{l-1}) \cup (V(G_l) \setminus (V_j \cup W_j))\right) \\ &\leq \gamma_R(\psi_l(G; X)).\end{aligned}$$

Since $V_i \cap W_i = \emptyset$ and $V_j \cap W_j = \emptyset$ it follows that $|R \cap V_i| + 2\,|R \cap W_i| + |V_j \cap D| + 2\,|W_j \cap D| \leq 4|V(G)|$. According to (1) we have

$$\begin{aligned}\ell(v_i, v_j) &\leq 4|V(G)| + \gamma_{R_{i,j}} = 4|V(G)| + (A_l)_{ij} \\ (A_l)_{ij} &\geq \ell(v_i, v_j) - 4|V(G)| \geq \gamma_R\left(\psi_l(G; X)\right) - 4|V(G)|.\end{aligned}$$

Therefore

$$\gamma_R\left(\psi_l(G; X)\right) - 4|V(G)| \leq (A_l)_{ij} \leq \gamma_R(\psi_l(G; X)).$$

For $l \geq 1$, let $K_l = \min\{(A_l)_{ij}\}$ and let $A'_l = A_l - (K_l)J$, where J is the matrix with all entries equal to 0 (recall that we are still in the path algebra $\mathcal{P} = (\mathbb{N}_0 \cup \{\infty\}, \min, +, \infty, 0)$). Since the difference between any two elements of A_l, different from ∞, cannot be greater than $4K$, the entries of A'_l can have only values $0, 1, \ldots, 4K, \infty$. Hence there are indices $p < q \leq (4K + 2)^{k^2}$ such that $A'_p = A'_q$. This proves the first part of the proposition.

The equality $A_{r+sP} = A_r + sC$ follows from the fact that for arbitrary matrices D, E and a constant matrix C we have $(D \oplus C) \circ E = D \circ E \oplus C$. This can easily be seen by computing the values of ij-th entries of both sides of the equality:

$$\begin{aligned}((D \oplus C) \circ E)_{ij} &= \min_k\{((D)_{ik} + C) + (E)_{kj}\} = \min_k\{(D)_{ik} + (E)_{kj}\} + C \\ (D \circ E \oplus C)_{ij} &= \min_k\{(D)_{ik} + (E)_{kj}\} + (C)_{ij} = \min_k\{(D)_{ik} + (E)_{kj}\} + C.\end{aligned}$$

□

Hence, if we assume that the size of G is a given constant (and n is a variable), then the algorithm will run in constant time. But it is important to emphasize that the algorithm is useful for practical purposes only if the number of vertices of the monograph G is relatively small, since the time complexity is in general exponential in the number of vertices of the monograph G.

5 Products of paths and cycles

Our aim is to calculate Roman domination number for graphs $P_n \Box P_k$, $P_n \Box C_k$, $C_n \Box P_k$ and $C_n \Box C_k$ for some fixed k. Since these graphs are isomorphic to special classes of fasciagraphs and rotagraphs (i.e. fasciagraphs and rotagraphs where $G = P_k$ or $G = C_k$ and where X is a matching between two copies of G), Lemma 4.2 implies a constant time algorithm for computing their Roman domination numbers, but its straightforward application is not useful in our case since indices q are huge. Because with increasing k, matrices $A(G)^n$ become bigger and bigger, we also omitted straightforward implementation of Algorithm 2. Instead of calculating whole matrices $A(G)^n$, we calculated only those rows which are important for the result and checked the difference of the new row against the previously stored rows until a constant difference was detected. This yields a correct result because of the next lemma, adaptation of Lemma from [32].

Lemma 5.1 *Assume that the j–th row of A_{n+P} and A_n differ for a constant, $a_{ji}^{(n+P)} = a_{ji}^{(n)} + C$ for all i. Then $\min_i a_{ji}^{(n+P)} = \min_i a_{ji}^{(n)} + C$.*

Proof. Let $a_{jk}^{(n+P)} = \min_i a_{ji}^{(n+P)}$ and assume that there exists $l \neq k$ such that $a_{jk}^{(n)} > a_{jl}^{(n)}$. It follows that

$$a_{jl}^{(n+P)} = a_{jl}^{(n)} + C < a_{jk}^{(n)} + C = a_{jk}^{(n+P)},$$

which contradicts the minimality of $a_{jk}^{(n+P)}$. □

By implementation we got formulas presented in the following subsections. For each case also constructions of γ_R–functions are presented. In every figure that follows we only emphasized vertices of V_1 and V_2 of a γ_R–function of a depicted graph in a way that a single full circ represents a vertex of V_1 and a double circ represents a vertex of V_2.

5.1 $\gamma_R(P_n \Box P_k)$ for $k \in \{5, 6, 7, 8\}$

Roman domination number of grid graphs was studied in [6, 8] and the following results have been established:

$$\gamma_R(P_n) = \gamma_R(C_n) = \left\lceil \frac{2n}{3} \right\rceil$$
$$\gamma_R(P_n \Box P_2) = n + 1$$
$$\gamma_R(P_n \Box P_3) = \begin{cases} \left\lfloor \frac{6n}{4} \right\rfloor + 2; & \text{if } n \in \{4k+3 \mid k \in \mathbb{N} \cup \{0\}\} \\ \left\lfloor \frac{6n}{4} \right\rfloor + 1; & \text{otherwise} \end{cases}$$
$$\gamma_R(P_n \Box P_4) = \begin{cases} 2n+1; & \text{if } n \in \{1,2,3,5,6\} \\ 2n; & \text{otherwise} \end{cases}$$

No formulas were given for $k > 4$. However, the author also proposed an algorithm for computing $\gamma_R(P_n \Box P_k)$ for fixed k in $O(n)$ time. By implementing Algorithm 2 as already discussed above, we obtained formulas given below. We also looked for the constructions for every n. Roman dominating sets of weight γ_R are depicted for every case on Figures 3 to 6.

$$\gamma_R(P_n \Box P_5) = \begin{cases} 8; & \text{if } n = 3 \\ \left\lfloor \frac{12n}{5} \right\rfloor + 2; & \text{otherwise} \end{cases}$$
$$\gamma_R(P_n \Box P_6) = \begin{cases} \left\lfloor \frac{14n}{5} \right\rfloor + 2; & \text{if } n < 5 \text{ or } n \in \{5k, 5k+3, 5k+4 \mid k \in \mathbb{N}\} \\ \left\lfloor \frac{14n}{5} \right\rfloor + 3; & \text{otherwise} \end{cases}$$
$$\gamma_R(P_n \Box P_7) = \begin{cases} \left\lfloor \frac{16n}{5} \right\rfloor + 2; & \text{if } n \in \{1,2,4,7,5k \mid k \in \mathbb{N}\} \\ \left\lfloor \frac{16n}{5} \right\rfloor + 3; & \text{otherwise} \end{cases}$$
$$\gamma_R(P_n \Box P_8) = \begin{cases} 9; & \text{if } n = 2 \\ 16; & \text{if } n = 4 \\ \left\lfloor \frac{18n}{5} \right\rfloor + 4; & \text{if } n \in \{5k+3 \mid k \in \mathbb{N}\} \\ \left\lfloor \frac{18n}{5} \right\rfloor + 3; & \text{otherwise} \end{cases}$$

Roman domination numbers for small grids are presented in Table 1.

5.2 $\gamma_R(P_k \Box C_n)$ for $k \in \{3,4,5,6,7,8\}$

In the literature we found no formulas for Roman domination numbers in these cases. Our formulas are given below. Constructions for each case are depicted on Figures 7 to 12.

Table 1: Roman domination number of some $P_n \Box P_k$.

$k \setminus n$	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
1	1	2	2	3	4	4	5	6	6	7	8	8	9	10	10
2		3	4	5	6	7	8	9	10	11	12	13	14	15	16
3			6	7	8	10	12	13	14	16	18	19	20	22	24
4				8	11	13	14	16	18	20	22	24	26	28	30
5					14	16	18	21	23	26	28	30	33	35	38
6						19	22	24	27	30	33	36	38	41	44
7							24	28	31	34	38	41	44	47	50
8								32	35	39	42	46	50	53	57

$$\gamma_R(P_n \Box C_3) = \left\lfloor \frac{6n}{4} \right\rfloor + 1$$

$$\gamma_R(P_n \Box C_4) = \begin{cases} 3; & \text{if } n = 1 \\ 2n; & \text{else} \end{cases}$$

$$\gamma_R(P_n \Box C_5) = 2n + 2$$

$$\gamma_R(P_n \Box C_6) = \left\lfloor \frac{8n}{3} \right\rfloor + 2$$

$$\gamma_R(P_n \Box C_7) = \begin{cases} 3n+2; & \text{if } n \in \{1,2,4\} \\ 3n+3; & \text{otherwise} \end{cases}$$

$$\gamma_R(P_n \Box C_8) = \begin{cases} 8; & \text{if } n = 2 \\ \left\lfloor \frac{7n}{2} \right\rfloor + 2; & \text{if } n \in \{3,4,8\} \\ \left\lfloor \frac{7n}{2} \right\rfloor + 3; & \text{otherwise} \end{cases}$$

5.3 $\gamma_R(C_n \Box P_k)$ for $k \in \{2,3,4,5\}$ and $n \geq 3$

We implemented this case as a rotagraph. From (3) we know that calculations in this case take much more time than calculations for fasciagraphs. Therefore we covered only cases for $k \in \{2,3,4,5\}$. As in former cases, formulas for Roman domination number are presented below and constructions can be found on Figures 13 to 17.

$$\gamma_R\left(C_n \Box P_2\right) = \begin{cases} n; & \text{if } n \in \{4k \mid k \in \mathbb{N}\} \\ n+1; & \text{otherwise} \end{cases}$$

$$\gamma_R\left(C_n \Box P_3\right) = \begin{cases} 5; & n = 3 \\ \left\lceil \frac{6n}{4} \right\rceil; & \text{if } n \in \{4k, 4k+1 \mid k \in \mathbb{N}\} \\ \left\lceil \frac{6n}{4} \right\rceil + 1; & \text{otherwise} \end{cases}$$

$$\gamma_R\left(C_n \Box P_4\right) = \begin{cases} 7; & \text{if } n = 3 \\ 2n; & \text{otherwise} \end{cases}$$

$$\gamma_R\left(C_n \Box P_5\right) = \begin{cases} \left\lceil \frac{12n}{5} \right\rceil + 1; & \text{if } n \in \{5k+2 \mid k \in \mathbb{N}\} \\ \left\lceil \frac{12n}{5} \right\rceil; & \text{otherwise} \end{cases}$$

5.4 $\gamma_R(C_n \Box C_k)$ for $k \in \{3,4,5\}$

In [12], authors showed that $\gamma_R\left(C_{5n} \Box C_{5m}\right) = 10mn$, which is consistent with our calculations. We found no other formulas in the literature for these cases. Constructions for each case can be found on Figures 18 to 20.

$$\gamma_R\left(C_n \Box C_3\right) = \left\lceil \frac{3n}{2} \right\rceil$$

$$\gamma_R\left(C_n \Box C_4\right) = 2n$$

$$\gamma_R\left(C_n \Box C_5\right) = \begin{cases} 2n; & \text{if } n \in \{5k \mid k \in \mathbb{N}\} \\ 2n+2; & \text{otherwise} \end{cases}$$

6 Roman graphs

Combining results obtained here and known results on the domination number [1, 5, 13, 19, 25] we also looked for Roman graphs (graphs, satisfying $\gamma_R(G) = 2\gamma(G)$). In cases where domination numbers of graphs have not been calculated, we also refer to a simple observation that a graph cannot be Roman if its Roman domination number is odd and to Proposition 16 in [6] which implies that a graph G is Roman if and only if it has a γ_R–function $f = (V_0, V_1, V_2)$ with $V_1 = \emptyset$. Except in cases $P_n \Box C_5$, $P_n \Box C_8$, $C_n \Box P_5$ and $C_n \Box C_5$, the following are characterizations of Roman graphs among investigated.

1. Roman graphs among $P_n \Box P_k$:

$k = 1$: $n \in \{3l+2, 3l+3 \mid l \in \mathbb{N}_0\}$

$k = 2$: n odd

$k = 3$: $n \in \{4l + 1, 4l + 2, 4l + 3 \mid l \in \mathbb{N}\}$

$k = 4$: $n \in \mathbb{N} \setminus \{1, 2, 3, 5, 6, 9\}$

$k = 5$: $n \in \{1, 2, 3, 7, 5l, 5l + 1 \mid l \in \mathbb{N}\}$

$k = 6$: $n \in \{1, 3, 5, 7, 8, 12, 15, 22\}$

$k = 7$: $n \in \{2, 3, 4, 5, 6, 7, 8, 10, 11, 13, 16\}$

$k = 8$: $n \in \{1, 4, 6, 7, 8\}$

2. Roman graphs among $P_n \square C_k$:

$k = 3$: $n \in \{4l + 1, 4l + 2 \mid l \in \mathbb{N}_0\}$

$k = 4$: $n \geq 2$

$k = 5$: $n \in \{1, 2, 3\}$

$k = 6$: $n \in \{1, 3, 4, 6, 6l + 1, 6l + 3, 6l + 4, 6l + 6 \mid l \in \mathbb{N}\}$

$k = 7$: $n \in \{2, 4, 2l + 1 \mid l \in \mathbb{N}_0\}$

$k = 8$: $n \in \{1, 2, 3, 4, 5, 6\}$

3. Roman graphs among $C_n \square P_k$:

$k = 2$: $n \in \{4l, 4l + 1, 4l + 3 \mid l \in \mathbb{N}\}$

$k = 3$: $n \geq 4$

$k = 4$: $n \in \mathbb{N} \setminus \{3, 5, 9\}$

$k = 5$: $n \in \{3, 4, 7, 8, 10l, 10l + 4, 10l + 7, 10l + 8 \mid l \in \mathbb{N}\}$

4. Roman graphs among $C_n \square C_k$:

$k = 3$: $n \in \{4l, 4l + 1 \mid l \in \mathbb{N}\}$

$k = 4$: $n \in \mathbb{N}$

$k = 5$: $n \in \{3, 4, 5l, 5l + 1, 5l + 2, 5l + 4 \mid l \in \mathbb{N}\}$

Remark 6.1 *It was proven in [25] that* $n + \lceil \frac{n}{5} \rceil \leq \gamma(P_5 \square C_n) \leq n + \lceil \frac{n}{4} \rceil$. *In fact, we show here that since* $C_{10k} \square P_5$, $C_{10k+4} \square P_5$, $C_{10k+7} \square P_5$ *and* $C_{10k+8} \square P_5$ *are Roman graphs (see Figure 17), their dominatin number equals* $\frac{\gamma_R}{2}$.

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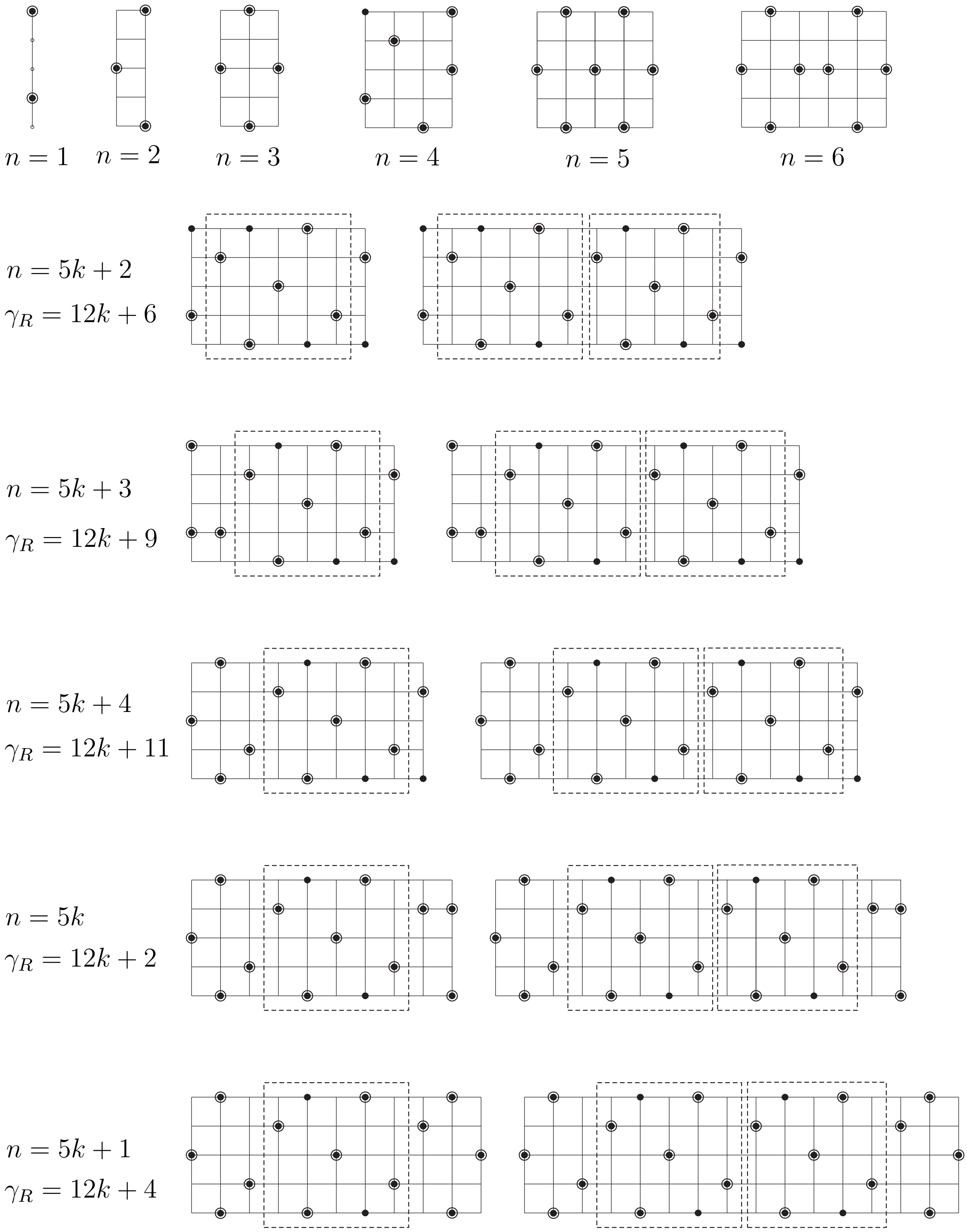
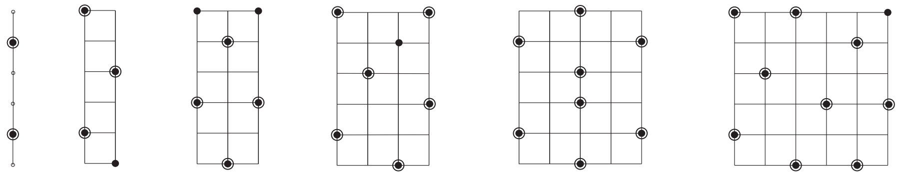
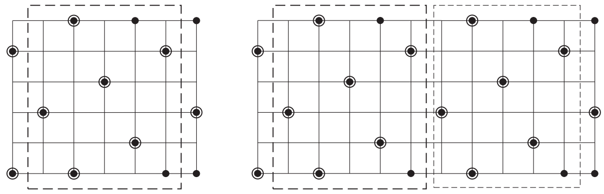


Figure 3: $P_n \Box P_5$

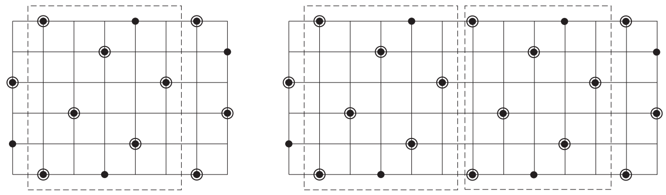


$n = 1$ $n = 2$ $n = 3$ $n = 4$ $n = 5$ $n = 6$

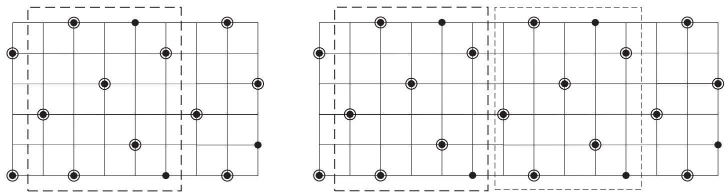
$n = 5k + 2$
$\gamma_R = 14k + 8$



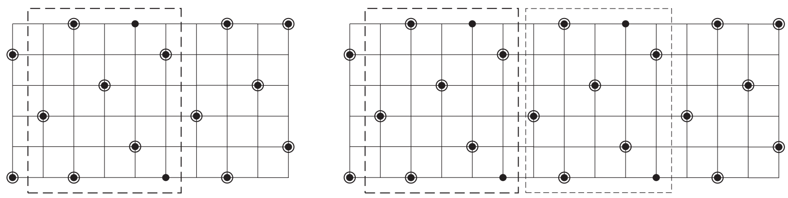
$n = 5k + 3$
$\gamma_R = 14k + 10$



$n = 5k + 4$
$\gamma_R = 14k + 13$



$n = 5k$
$\gamma_R = 14k + 2$



$n = 5k + 1$
$\gamma_R = 14k + 5$

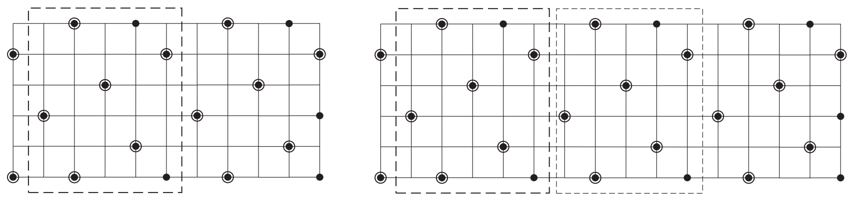


Figure 4: $P_n \Box P_6$

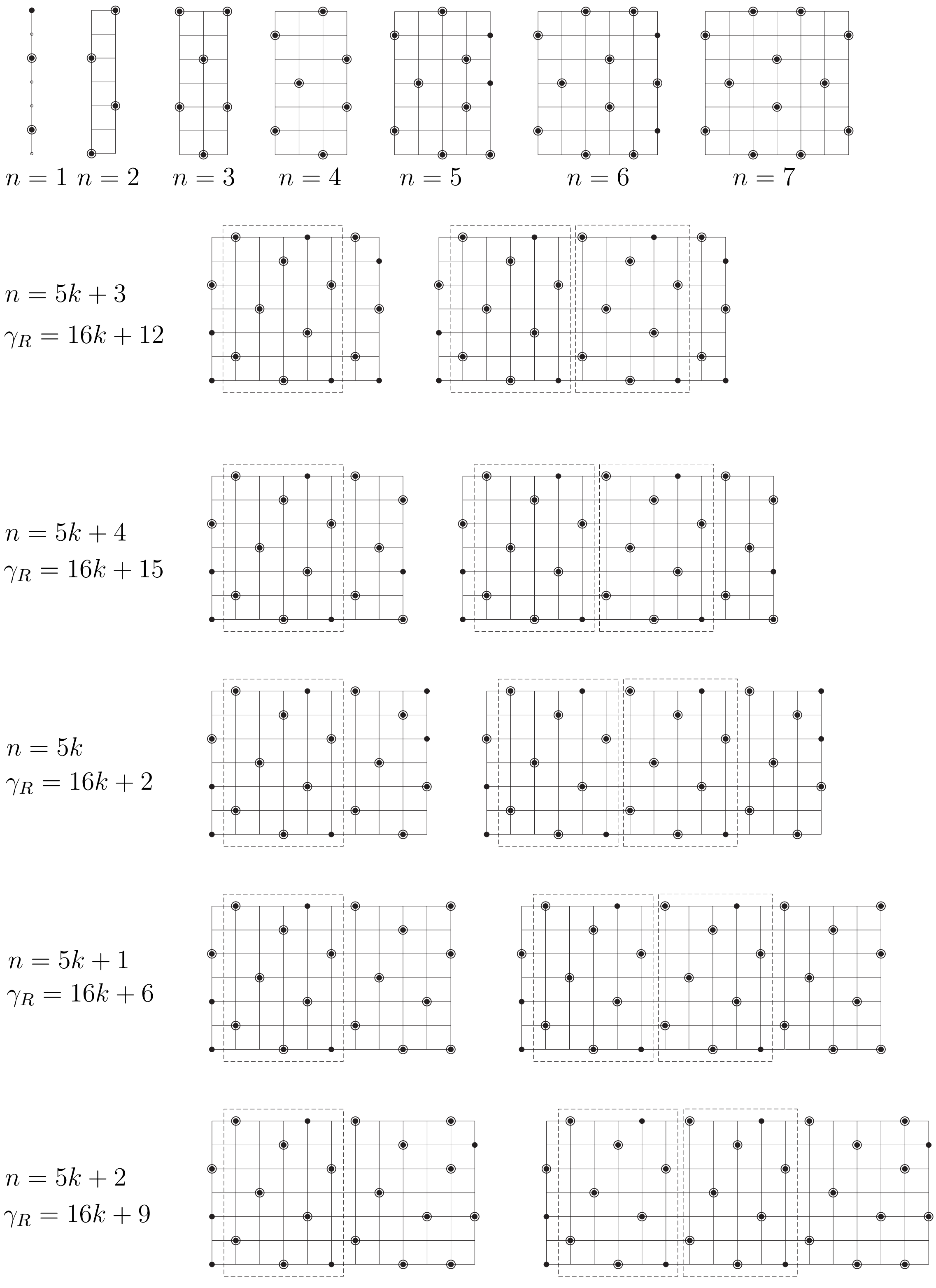


Figure 5: $P_n \Box P_7$

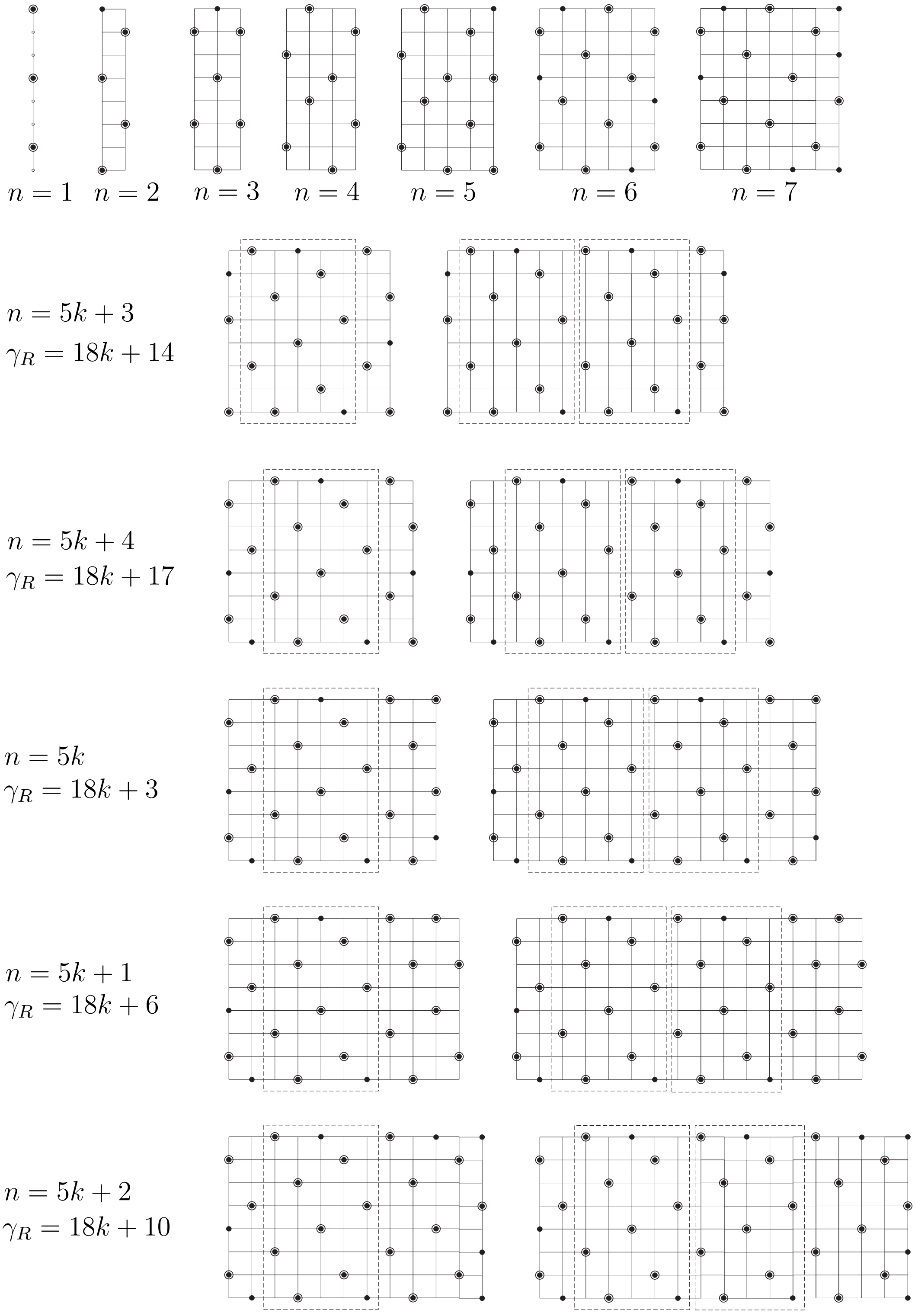


Figure 6: $P_n \Box P_8$

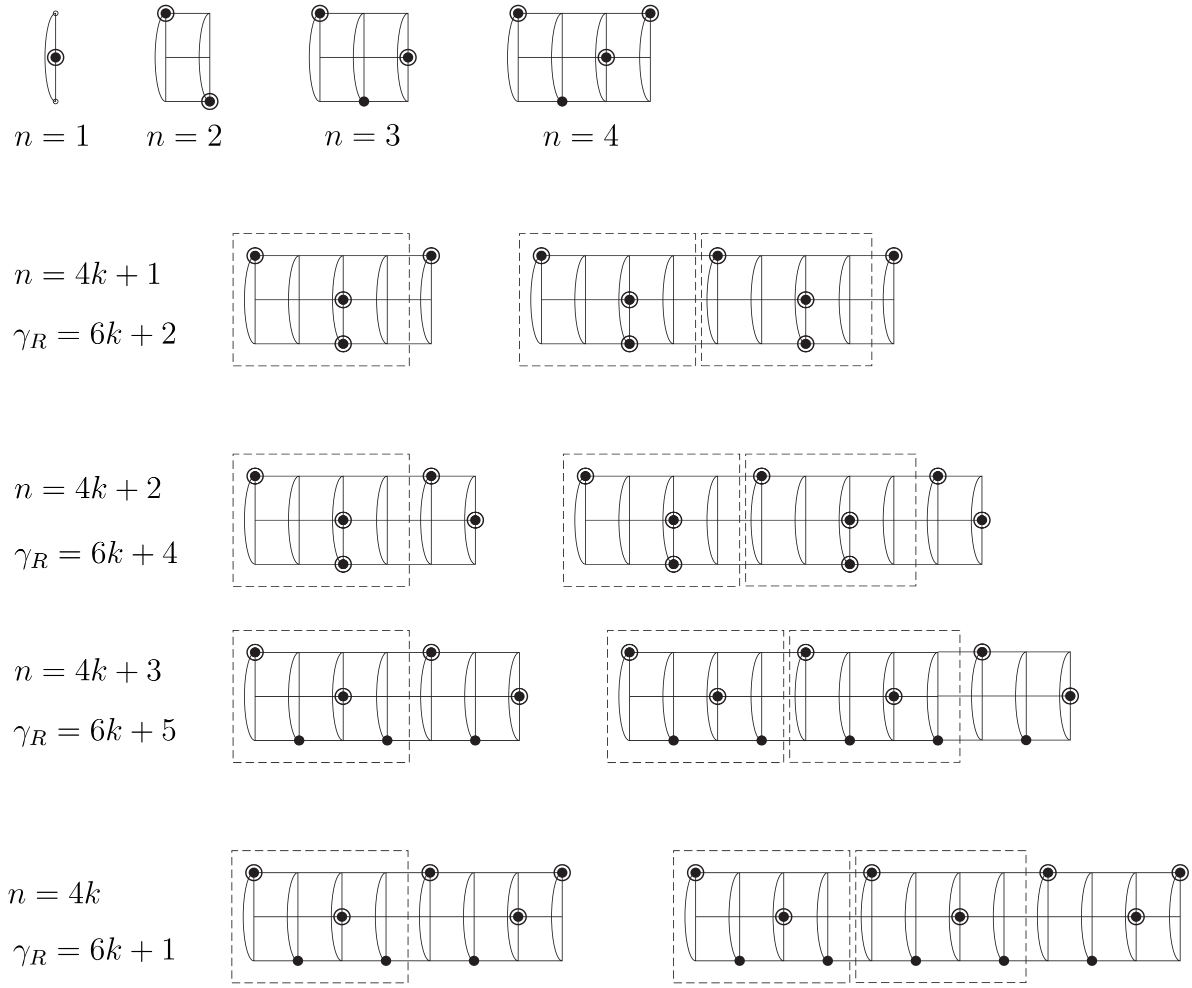


Figure 7: $P_n \square C_3$

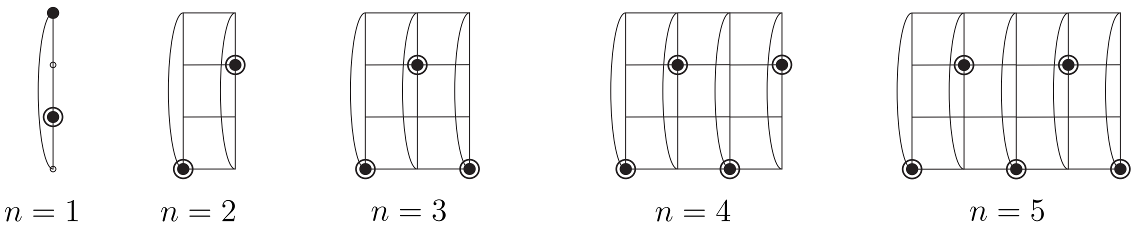


Figure 8: $P_n \square C_4$

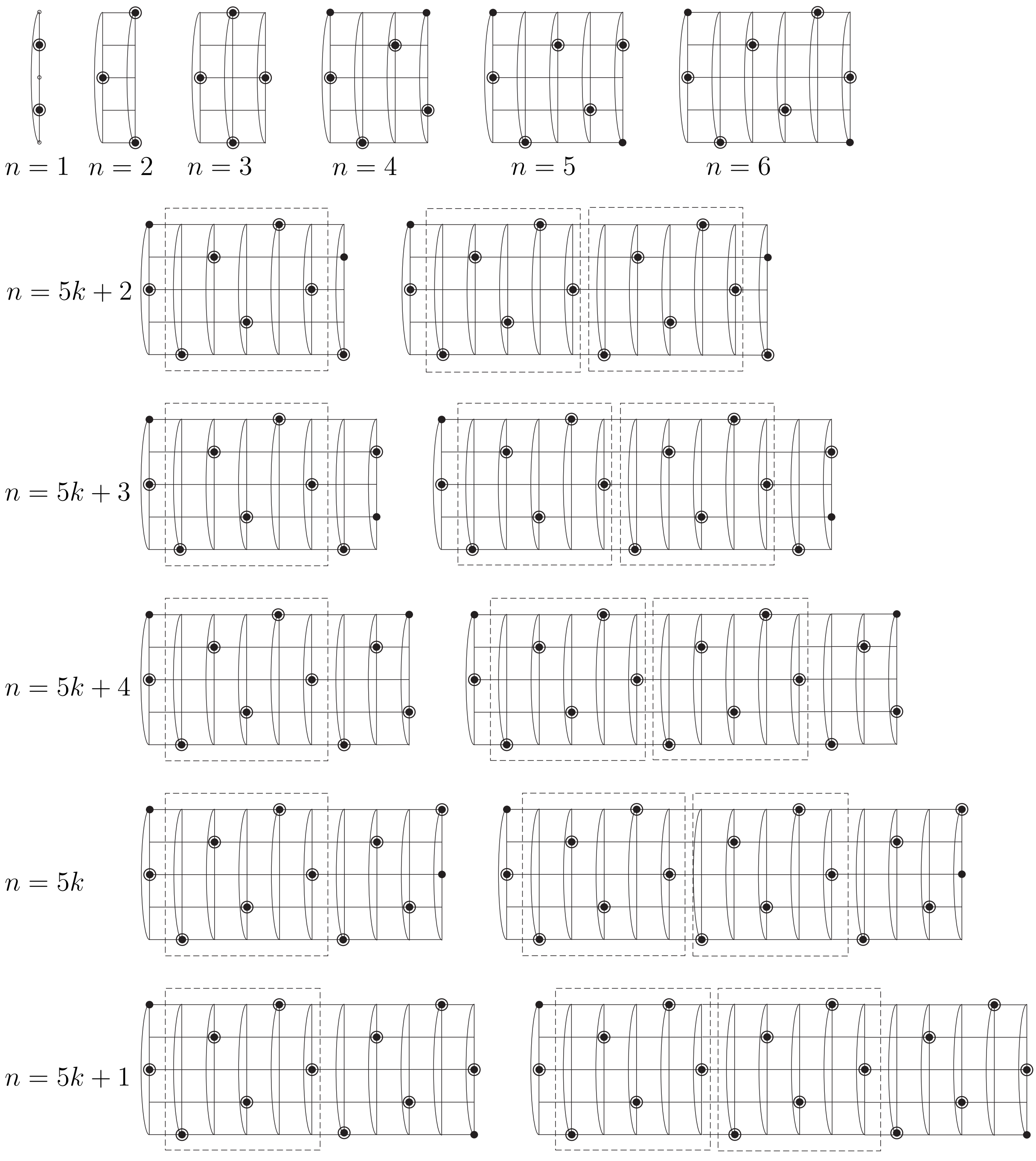


Figure 9: $P_n \square C_5$

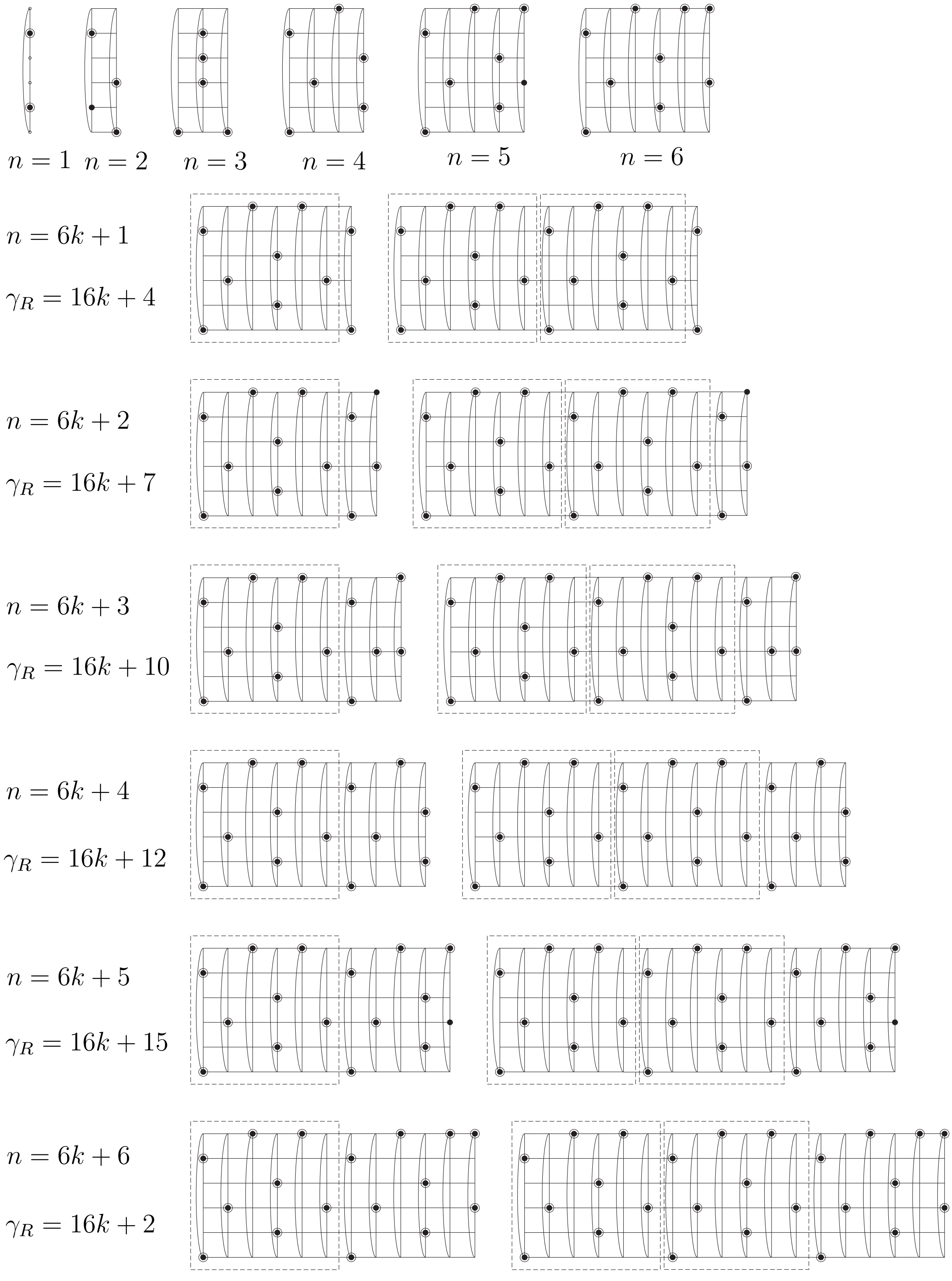


Figure 10: $P_n \Box C_6$

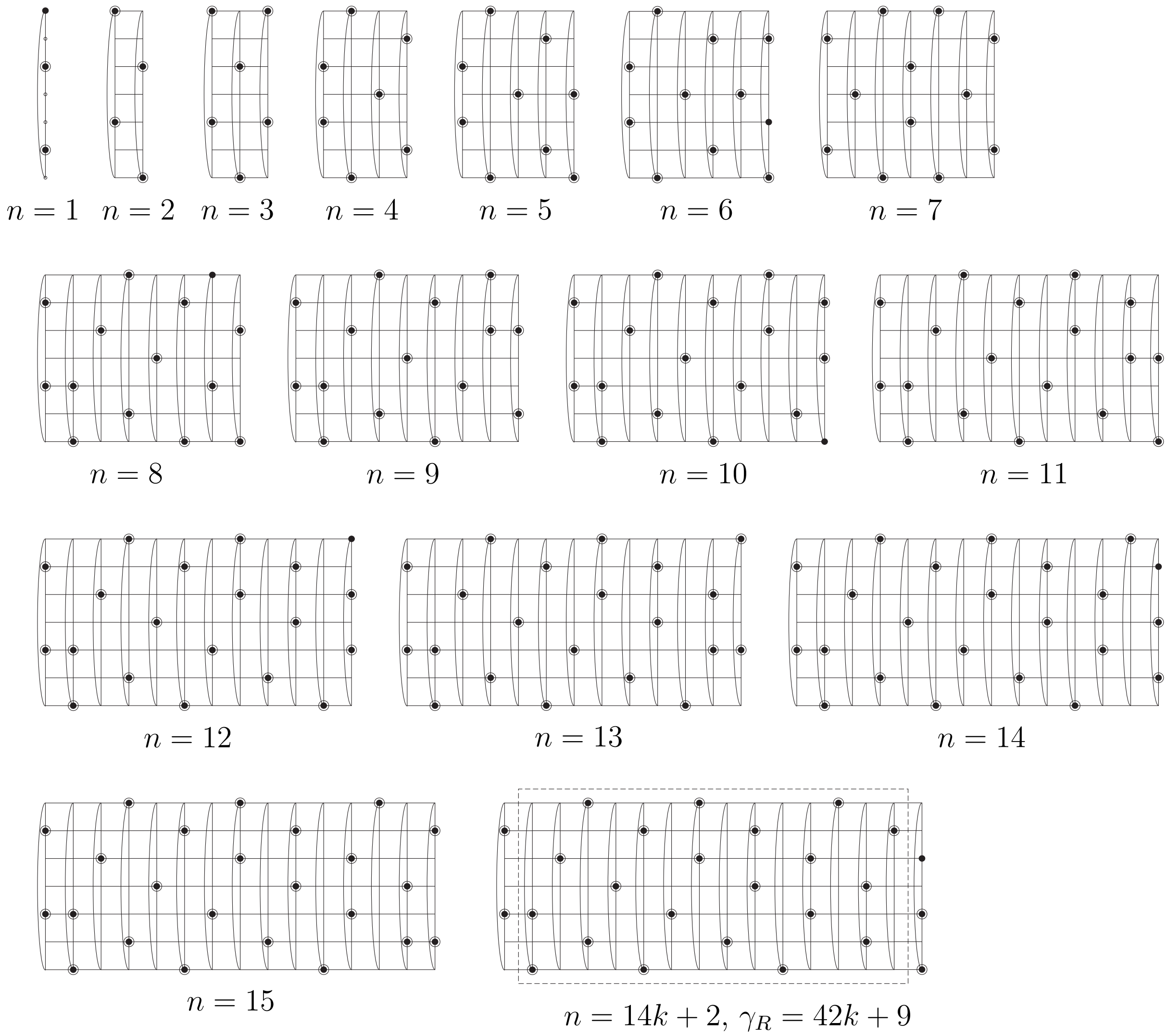


Figure 11: $P_n \Box C_7$

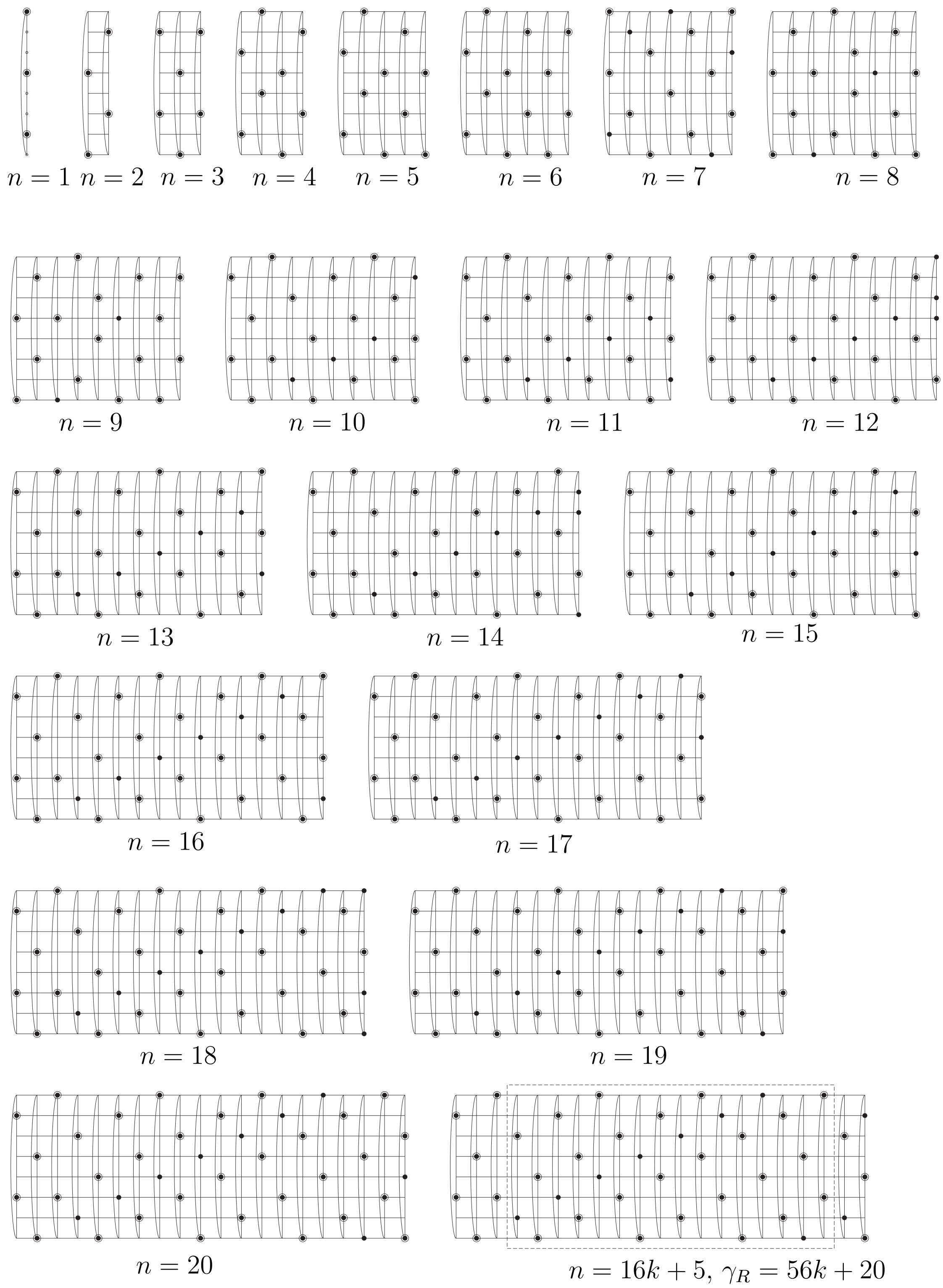


Figure 12: $P_n \Box C_8$

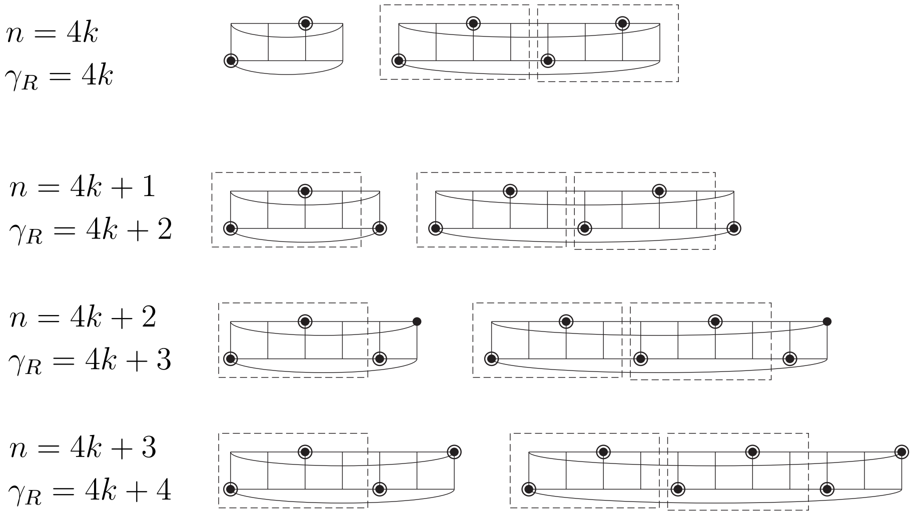


Figure 13: $C_n \square P_2$

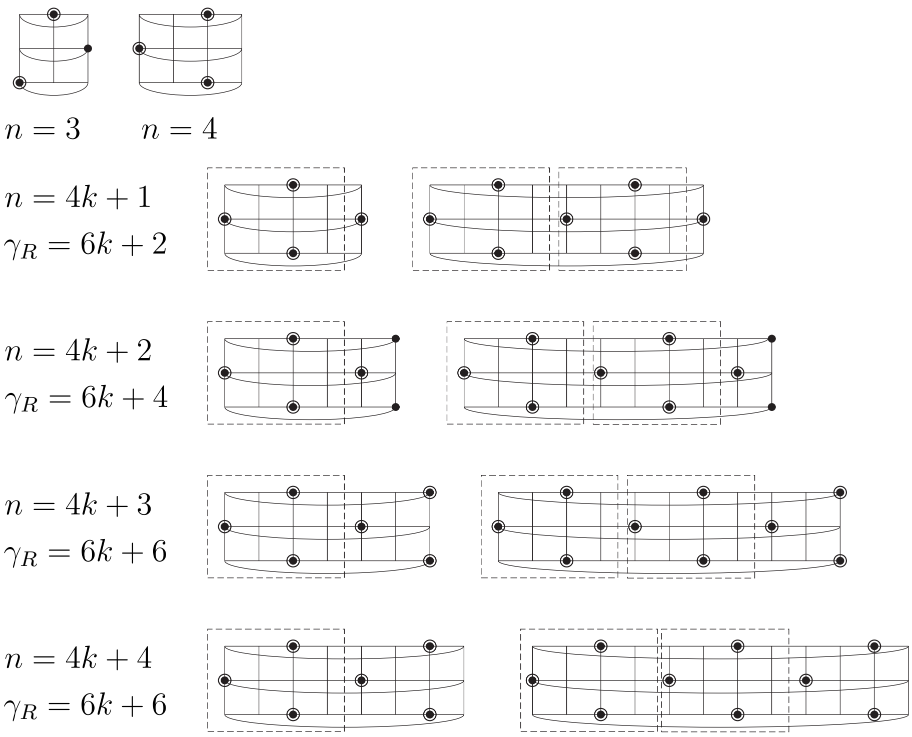


Figure 14: $C_n \square P_3$

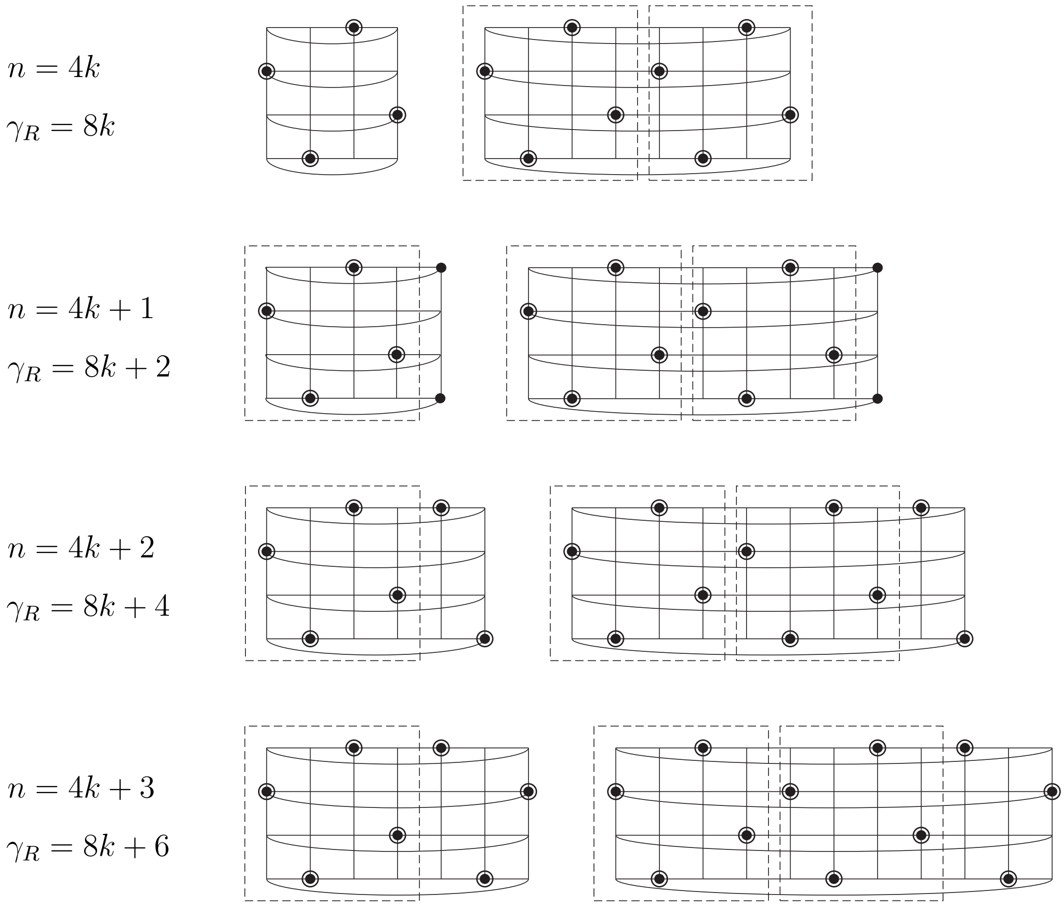


Figure 15: $C_n \Box P_4$

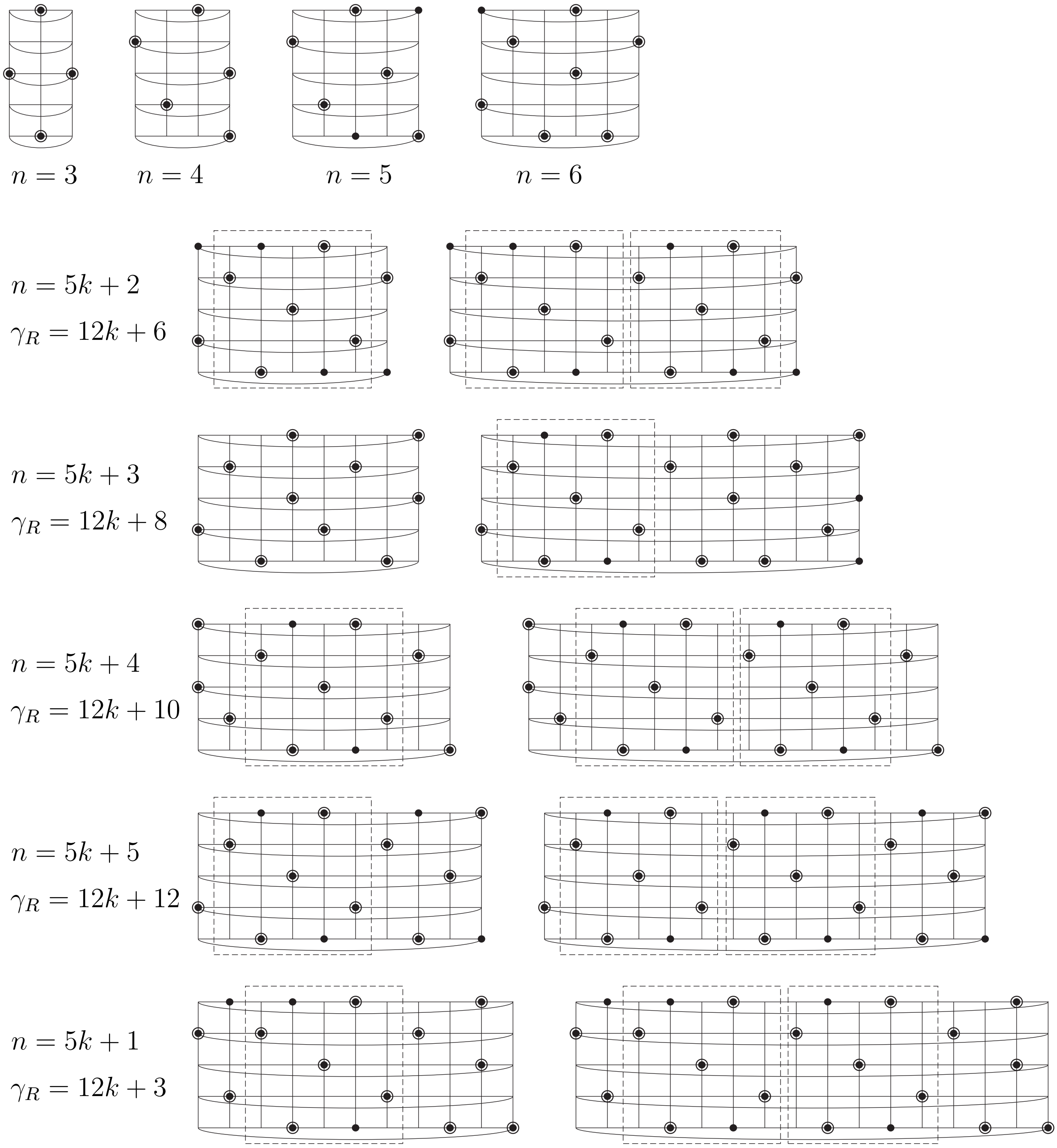


Figure 16: $C_n \Box P_5$

$n = 10k$
$\gamma_R = 24k$

$n = 10k + 4$
$\gamma_R = 24k + 8$

$n = 10k + 7$
$\gamma_R = 24k + 18$

$n = 10k + 8$
$\gamma_R = 24k + 20$

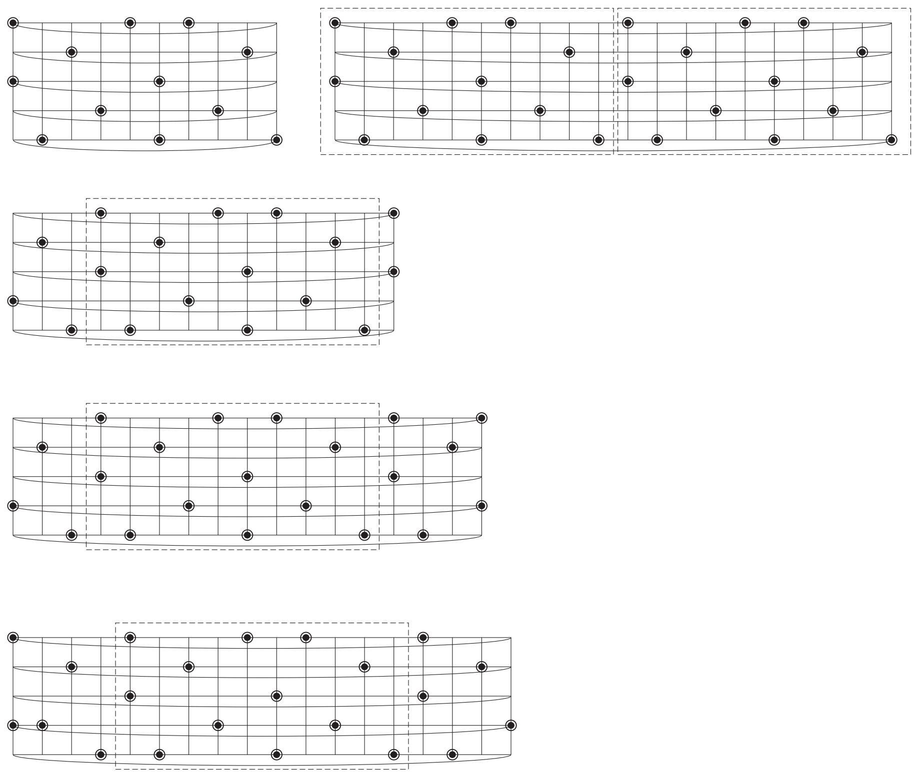


Figure 17: $C_{10k}\Box P_5$, $C_{10k+4}\Box P_5$, $C_{10k+7}\Box P_5$ and $C_{10k+8}\Box P_5$ are Roman graphs.

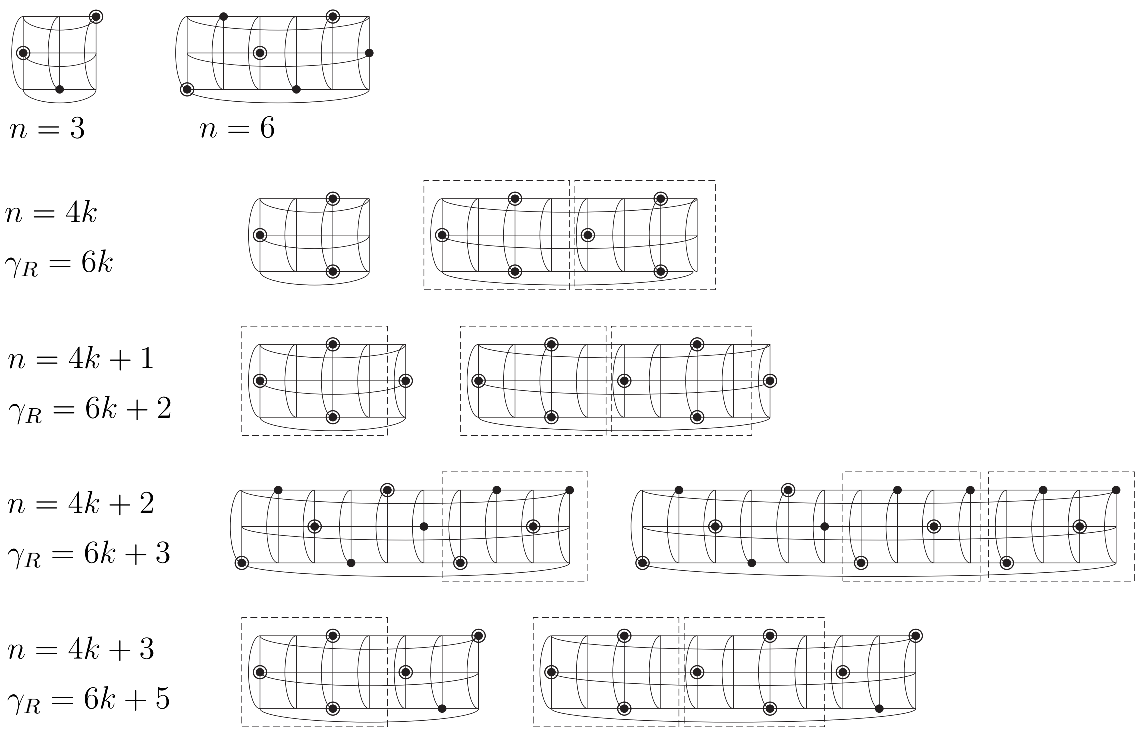


Figure 18: $C_n \Box C_3$

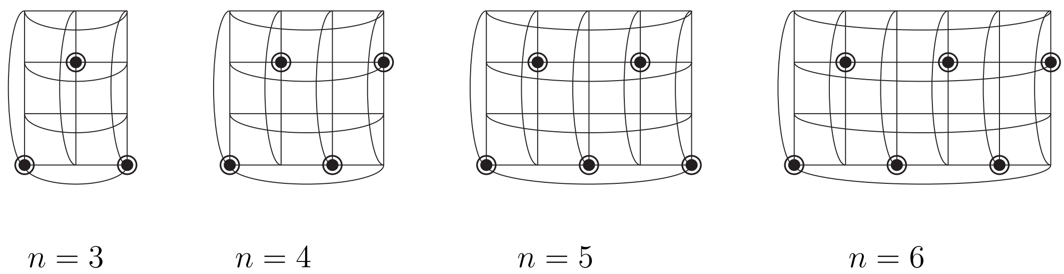


Figure 19: $C_n \Box C_4$

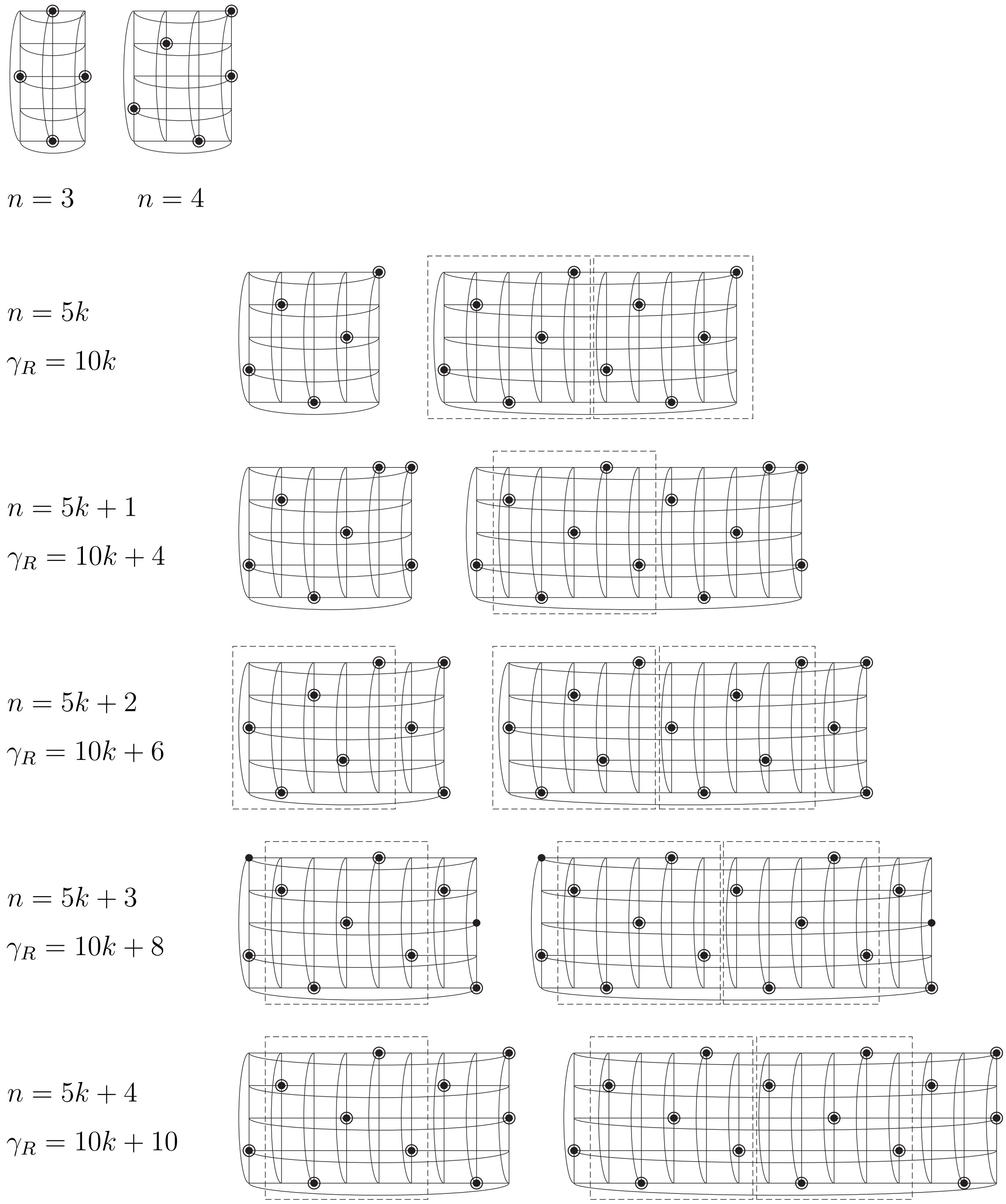


Figure 20: $C_n \Box C_5$