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3 HS YM and CS Theories in Flat Spacetime

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Abstract. It is shown that in a flat background one can define higher spin (HS) gauge theories with an infinite number of fields. In particular here HS YM-like in any dimension and HS CS-like theories in any odd dimension are introduced and analyzed. They are invariant under HS gauge transformations which include ordinary U(1) gauge transformations and diffeomorphisms. It is also shown how to recover local Lorentz invariance. The action, equations of motion and conserved currents in the HS YM-like theories are explicitly exhibited.

Povzetek. Avtor v prispevku pokaže, da lahko definira na ravnem ozadju umeritvene teorije višjih spinov z neskončnim številom polj. Kot poseben primer uvede in analizira teorije Yang-MIllsovega tipa z višjim spinom v poljubni dimenziji in teorije Cherna-Simmonsa z višjim spinom v poljubni lihi dimenziji. Te teorije so invariantne na umeritvene transfomacije za višje spine, ki vključujejo običajne transformacije U(1) in difeomorfizme. Pokaže še, kako znova vpeljati lokalno Lorentzovo invarianco. V teorijah Yang-Millsovega tipa z višjim spinom zapiše akcijo ter enačbe gibanja in ohranitvene tokove.

Keywords: Higher spin theories, Yang-Mills like theories, Chern-Simmons theories, flat spacetime

3.1 Introduction. . .

There are compelling motivations for research to study spin (HS) theories, that is theories with an infinite number of fields with increasing spin. In a theory that unifies all the forces of nature such a feature seems to be inevitable. First (super)string theories have this characteristic. It is well known that the infinite number of fields with increasing spins is related to their good UV behavior. Also the AdS/CFT correspondence indicates that if we wish to resolve the singularities of the theory on the boundary we have to turn to the dual theory, which is a (super)string theory. Other arguments suggest that, when gravity is involved, infinite many local fields of increasing spins are needed in order to avoid possible conflicts with causality [1].

Starting from on these general motivations, in this contribution I will focus on a specific problem, for which for a long time there have been no answers, or only negative ones, in the literature: can one formulate a sensible local massless

HS theory in a flat space-time? The standard lore in the literature may be summarized by two objections: first, there are the so-called no-go theorems, which prevent the existence of such theories under rather general conditions; second, the construction of massless HS theories has been so far only successful in AdS spaces. However here I will exhibit examples of HS theories defined in flat spacetime in any dimension, which are massless, gauge invariant and, at least classically, consistent.

In [3] and, later on, in [4,7] a method has been proposed to produce HS effective actions by integrating out matter fields coupled to external potentials and quantized according to the worldline quantization. The method consists in computing current correlators, see [5,6], and explicitly determine the effective action. Barring anomalies, we are guaranteed that the result is HS gauge invariant. Unfortunately the method is very cumbersome and the resulting effective action is not guaranteed to be local.

In this paper I would like to show that there exists a shortcut. Exploiting the analogy of the HS gauge transformations with the gauge transformations in ordinary non-Abelian gauge theories, one can construct analogous local HS invariants and covariant objects, and in particular actions. In this way one can define (perturbatively) local HS Yang-Mills theories in any dimension and HS Chern-Simons theories in any odd dimension. I will focus in particular on the former. They are characterized by a coupling constant, like the ordinary YM theories. I will show how to define the action, their equations of motion and their conserved quantities. The HS gauge transformations contains in particular the ordinary $U(1)$ gauge transformations and the diffeomorphisms. They do not include the local Lorentz transformations. Since the HS YM-like theories are formulated in a frame-like formalism, local Lorentz transformations are relevant in order to permit their gravitational interpretation. Below I will show how to local Lorentz invariance is hidden in the formalism and how to recover it.

3.2 Higher spin effective action

This section is devoted to a concise presentation of the effective action method. The effective action here is defined via the worldline quantization method. This method consists, roughly speaking, in considering the coordinates on which the field depends, as the position of a quantum particle, while the latter is quantized according to the Weyl-Wigner quantization.

Let us consider a free fermion theory

$$S_0 = \int d^d x\, \overline{\psi}(i\gamma\cdot\partial - m)\psi, \tag{3.1}$$

coupled to external sources. According to the Weyl quantization method for a particle worldline, the full action is expressed as an expectation value of operators

$$S = \langle\overline{\psi}| - \gamma^a(\widehat{P}_a - \widehat{H}_a) - m|\psi\rangle \tag{3.2}$$

We recall that a quantum operator $\widehat{O}$ can be represented with a symbol $O(x,u)$ through the Weyl map

$$\widehat{O} = \int d^dx\, d^dy \frac{d^dk}{(2\pi)^d} \frac{d^du}{(2\pi)^d}\, O(x,u)\, e^{ik\cdot(x-\widehat{X})-iy\cdot(u-\widehat{P})} \tag{3.3}$$

where $\widehat{X}$ is the position operator. The symbol of the product of two operators is the $*$ product (or Moyal product) of the corresponding symbols.

In (3.2) $\widehat{P}_a$ is the momentum operator whose symbol is the classical momentum u_a[1] . $\widehat{H}_a$ is an operator whose symbol is $h_a(x,u)$, where

$$h_a(x,u) = \sum_{n=0}^{\infty} \frac{1}{n!}\, h_a^{\mu_1\dots\mu_n}(x)\, u_{\mu_1}\dots u_{\mu_n} \tag{3.4}$$

$s = n+1$ is the spin and the tensors are assumed to be symmetric in $\mu_1, \dots, \mu_n$. Any field like $h_a(x,u)$, which depends also from the momentum u, will be referred to as *master field*.

One should notice that there are two kind of labels a and μ_i. They will be interpreted later as flat and curved indices, respectively, but in a flat background they play the same role. Their true nature will illustrated later on.

Now one makes the above formalism explicit in (3.2), where we also insert two completenesses $\int d^dx|x\rangle\langle x|$, and make the identification $\psi(x) = \langle x|\psi\rangle$. Expressing S in terms of symbols one finds

$$\begin{aligned} S &= S_0 + \int \frac{d^du}{(2\pi)^d}\, d^dx\, d^dz\, e^{iu\cdot z}\, \overline{\psi}\Big(x+\frac{z}{2}\Big)\, \gamma\cdot h(x,u)\, \psi\Big(x-\frac{z}{2}\Big) \\ &= S_0 + \sum_{s=1}^{\infty} \int d^dx\, J^{(s)}_{\mu_1\dots\mu_s}(x)\, h^{\mu_1\dots\mu_s}_{(s)}(x) \end{aligned} \tag{3.5}$$

The tensor field $h_a^{\mu_1\dots\mu_n}$ is linearly coupled to the HS current

$$J^a_{\mu_1\dots\mu_n}(x) = \frac{i^n}{n!} \frac{\partial}{\partial z^{(\mu_1}} \cdots \frac{\partial}{\partial z^{\mu_n)}} \overline{\psi}\left(x+\frac{z}{2}\right) \gamma_a \psi\left(x-\frac{z}{2}\right)\Big|_{z=0}. \tag{3.6}$$

For instance, for $s=1$ and $s=2$ one obtains

$$J^{(1)}_a = \overline{\psi}\gamma_a\psi \tag{3.7}$$

$$J^{(2)}_{a\mu_1} = \frac{i}{2}\left(\partial_{(\mu_1}\overline{\psi}\gamma_{a)}\psi - \overline{\psi}\gamma_a\partial_{\mu_1)}\psi\right) \tag{3.8}$$

The HS currents are on-shell conserved in the free theory (3.1)

$$\partial_a J^{a\mu_1\cdots\mu_{s-1}}_{(s)} = 0 \tag{3.9}$$

[1] Throughout the paper the position in the phase space are denoted by couples of letters (x,u), (y,v), (z,t), (w,r), the first letter refers to the space-time coordinate and the second the the momentum of the worldline particle. The letters k, p, q will be reserved to the momenta of the (Fourier-transformed) physical amplitudes.

3.2.1 HS gauge symmetries

The action (3.2) is trivially invariant under the operation

$$S = \langle\overline{\psi}|\widehat{O}\widehat{O}^{-1}\widehat{G}\widehat{O}\widehat{O}^{-1}|\psi\rangle \tag{3.10}$$

where $\widehat{G} = -\gamma\cdot(\widehat{P} - \widehat{H}) - m$. So it is invariant under

$$\widehat{G} \longrightarrow \widehat{O}^{-1}\widehat{G}\widehat{O}, \qquad |\psi\rangle \longrightarrow \widehat{O}^{-1}|\psi\rangle \tag{3.11}$$

Writing $\widehat{O} = e^{-i\widehat{E}}$ we easily find the infinitesimal version.

$$\delta|\psi\rangle = i\widehat{E}|\psi\rangle, \qquad \delta\langle\overline{\psi}| = -i\langle\overline{\psi}|\widehat{E}, \tag{3.12}$$

and

$$\delta\widehat{G} = i[\widehat{E}\,,\widehat{G}] = i[\gamma\cdot(\widehat{P} - \widehat{H})\,,\widehat{E}] = \gamma\cdot\delta\widehat{H} \tag{3.13}$$

Let the symbol of $\widehat{E}$ be $\varepsilon(x,u)$, then the symbol of $[i\gamma\cdot\widehat{P},\widehat{E}]$ is

$$\int d^d y\langle x - \frac{y}{2}|[i\gamma\cdot\widehat{P},\widehat{E}]|x + \frac{y}{2}\rangle\, e^{iy\cdot u} = -i\gamma\cdot\partial_x\varepsilon(x,u) \tag{3.14}$$

Similarly

$$\text{Symb}\big([\widehat{H}_a,\widehat{E}]\big) = [h_a(x,u) \overset{*}{,} \varepsilon(x,u)] \tag{3.15}$$

where $[a \overset{*}{,} b] \equiv a * b - b * a$ is the $*$-commutator. Therefore, in terms of symbols,

$$\delta_\varepsilon h_a(x,u) = \partial_a^x\varepsilon(x,u) - i[h_a(x,u) \overset{*}{,} \varepsilon(x,u)] \equiv \mathcal{D}_a^{*x}\varepsilon(x,u) \tag{3.16}$$

where the covariant derivative defined by

$$\mathcal{D}_a^{*x} = \partial_a^x - i[h_a(x,u) \overset{*}{,} \quad] \tag{3.17}$$

has been introduced.

The variation in eq.(3.16) will be referred to hereafter as HS gauge transformation, and the corresponding symmetry *HS gauge symmetry*. For the transformations of ψ, see [4] .

It is easy to see that the conservation law in the classical interacting theory

$$\mathcal{D}_x^{*a} J_a(x,u) = 0 \qquad\qquad (\text{on} - \text{shell}) \tag{3.18}$$

follows from the above.

Using the $*$-Jacobi identity (which holds also for the Moyal product, because the latter is associative) one can easily get

$$\begin{aligned}(\delta_{\varepsilon_2}\delta_{\varepsilon_1} - \delta_{\varepsilon_1}\delta_{\varepsilon_2})\, h^\mu(x,u) &= i\,(\partial_a^x[\varepsilon_1 \overset{*}{,} \varepsilon_2](x,u) - i[h_a(x,u) \overset{*}{,} [\varepsilon_1 \overset{*}{,} \varepsilon_2](x,u)]]) \\ &= i\,\mathcal{D}_a^{*x}[\varepsilon_1 \overset{*}{,} \varepsilon_2](x,u)\end{aligned} \tag{3.19}$$

i.e. the HS ε-transform is of Lie algebra type.

3.2.2 The HS effective action

The general formula for the effective action is

$$W[h] = W[0] + \sum_{n=1}^{\infty} \frac{1}{n!} \int \prod_{i=1}^{n} d^d x_i \frac{d^d u_i}{(2\pi)^d} \mathcal{W}^{(n)}_{a_1,\dots,a_n}(x_1,u_1,\dots,x_n,u_n,\epsilon) \times h^{a_1}(x_1,u_1)\dots h^{a_n}(x_n,u_n) \tag{3.20}$$

where $\mathcal{W}^{(n)}_{a_1,\dots,a_n}(x_1,u_1,\dots,x_n,u_n,\epsilon)$ are the n-point functions of the currents $J_{a_1}(x_1,u_1),\dots,J_{a_n}(x_n,u_n)$. $W[0]$ is the constant 0-point contribution, which will be disregarded in the sequel. There are various ways to compute these amplitudes. The most popular is by means of Feynman diagrams. For instance, the 3-point function can be calculated via the Feynman diagram integral

$$\begin{aligned}
&\langle J_{a_1}(x_1,u_1) J_{a_2}(x_2,u_2) J_{a_3}(x_3,u_3)\rangle \\
&= -i \int \frac{d^d q_1}{(2\pi)^d} \frac{d^d q_2}{(2\pi)^d} e^{i(q_1+q_2)\cdot x_1} e^{-iq_1\cdot x_2} e^{-iq_2\cdot x_3} \\
&\times \delta\left(u_1 - \frac{2p-q_1-q_2}{2}\right) \delta\left(u_2 - \frac{2p-q_1}{2}\right) \delta\left(u_3 - \frac{2p-2q_1-q_2}{2}\right) \\
&\times \int \frac{d^d p}{(2\pi)^d} \mathrm{tr}\left(\gamma_{a_1} \frac{1}{\not p + m} \gamma_{a_2} \frac{1}{\not p - \not q_1 + m} \gamma_{a_3} \frac{1}{\not p - \not q_1 - \not q_2 + m}\right),
\end{aligned} \tag{3.21}$$

to which one must add the cross term. q_1, q_2 are the momenta of two external outgoing legs. The third one has incoming momentum $q_1 + q_2$.

These amplitudes have cyclic symmetry. The invariance of the effective action under (3.16) is expressed by

$$0 = \delta_\varepsilon W[h] = \sum_{n=1}^{\infty} \frac{1}{(n-1)!} \int \prod_{i=1}^{n} d^d x_i \frac{d^d u_i}{(2\pi)^d} \tag{3.22}$$
$$\times \mathcal{W}^{(n)}_{a_1,\dots,a_n}(x_1,u_1,\dots,x_n,u_n) \mathcal{D}^{*\mu_1}_x \varepsilon(x_1,u_1) h^{a_2}(x_2,u_2)\dots h^{a_n}(x_n,u_n)$$

The generalized equations of motion are obtained by varying $W[h]$ with respect to the master field $h_a(x,u)$. Let us write them in the compact form

$$\mathcal{F}_a(x,u) = 0 \tag{3.23}$$

where

$$\mathcal{F}_a(x,u) \equiv \sum_{n=0}^{\infty} \frac{1}{n!} \int \prod_{i=1}^{n} d^d x_i \frac{d^d u_i}{(2\pi)^d} \mathcal{W}^{(n+1)}_{aa_1\dots a_n}(x,u,x_1,u_1,\dots,x_n,u_n,\epsilon) \times h^{a_1}(x_1,u_1)\dots h^{a_n}(x_n,u_n)$$

The EoM's (3.23) are covariant under HS gauge transformation

$$\delta_\varepsilon \mathcal{F}_a(x,u) = i[\varepsilon(x,u) \overset{*}{,} \mathcal{F}_a(x,u)] \tag{3.24}$$

3.3 Yang-Mills-like theories

3.3.1 The gauge transformation in the fermion model

Let us return to the gauge transformation (3.16)

$$\delta_\varepsilon h_a(x,u) = \partial_a^x \varepsilon(x,u) - i[h_a(x,u) \stackrel{*}{,} \varepsilon(x,u)] \equiv \mathcal{D}_a^{x*} \varepsilon(x,u) \tag{3.25}$$

and write it down in components. To avoid a proliferation of numerical indices, let us write the expansion of $h_a(x,u)$ as

$$h_a(x,u) = A_a(x) + \chi_a^\mu(x) u_\mu + \frac{1}{2} b_a^{\mu\nu} u_\mu u_\nu + \frac{1}{6} c_a^{\mu\nu\lambda} u_\mu u_\nu u_\lambda + \dots \tag{3.26}$$

As noted above we use two different types of indices. In the expansion (3.4) the indices $\mu_1, \dots, \mu_n$ are upper (contravariant), as it should be, because in the Weyl quantization procedure the momentum has lower index, since it must satisfy $[x^\mu, p_\nu] = i\,\delta_\nu^\mu$. The index a instead is traditionally reserved for a flat index. Of course when the background metric is flat the indices a and μ_i are on the same footing, but it is useful to keep them distinct. Let us see why.

For the HS gauge parameter we write

$$\varepsilon(x,u) = \epsilon(x) + \xi^\mu u_\mu + \frac{1}{2} \Lambda^{\mu\nu} u_\mu u_\nu + \frac{1}{3!} \Sigma^{\mu\nu\lambda} u_\mu u_\nu u_\lambda + \dots \tag{3.27}$$

The transformation (3.25) to the lowest order reads,

$$\begin{aligned}
\delta A_a &= \partial_a \epsilon + \xi\cdot\partial A_a - \partial_\rho \epsilon\, \chi_a^\rho + \dots \\
\delta \chi_a^\nu &= \partial_a \xi^\nu + \xi\cdot\partial \chi_a^\nu - \partial_\rho \xi_\nu \chi_a^\rho + \partial^\rho A_a \Lambda_\rho{}^\nu - \partial_\lambda \epsilon\, b_a{}^{\lambda\nu} + \dots \\
\delta b_a^{\nu\lambda} &= \partial_a \Lambda^{\nu\lambda} + \xi\cdot\partial b_a{}^{\nu\lambda} - \partial_\rho \xi^\nu b_a{}^{\rho\lambda} - \partial_\rho \xi^\lambda b_a{}^{\rho\nu} + \partial_\rho \chi_a^\nu \Lambda_\rho{}^\lambda + \partial_\rho \chi_a^\lambda \Lambda_\rho{}^\nu \\
&\quad - \chi_a^\rho \partial_\rho \Lambda_{\nu\lambda} + \dots
\end{aligned} \tag{3.28}$$

The next nontrivial order contains terms with three derivatives, and so on.

It is natural to compare the previous HS gauge variations with the ordinary gauge, diff, ... transformations. To this end let us denote by $\tilde{A}_a$ the standard U(1) gauge field and by $\tilde{e}_a^\mu = \delta_a^\mu - \tilde{\chi}_a^\mu$ the standard inverse vielbein, and let us restrict the previous general transformation to gauge and diff transformations alone. We have

$$\begin{aligned}
\delta \tilde{A}_a &\equiv \delta\left(\tilde{e}_a^\mu \tilde{A}_\mu\right) \equiv \delta\left((\delta_a^\mu - \tilde{\chi}_a^\mu)\tilde{A}_\mu\right) \\
&= \left(-\xi\cdot\partial\tilde{\chi}_a^\mu + \partial_\lambda \xi^\mu \tilde{\chi}_a^\lambda\right) \tilde{A}_\mu + (\delta_a^\mu - \tilde{\chi}_a^\mu)\left(\partial_\mu \epsilon + \xi\cdot\tilde{A}_\mu\right) \\
&\approx \partial_a \epsilon + \xi\cdot\tilde{A}_a - \tilde{\chi}_a^\mu \partial_\mu \epsilon
\end{aligned} \tag{3.29}$$

and

$$\delta \tilde{e}_a^\mu \equiv \delta(\delta_a^\mu - \tilde{\chi}_a^\mu) = \xi\cdot\partial\tilde{e}_a^\mu - \partial_\lambda \xi^\mu \tilde{e}_a^\lambda = -\xi\cdot\tilde{\chi}_a^\mu - \partial_a \xi^\mu + \partial_\lambda \xi^\mu \tilde{\chi}_a^\lambda \tag{3.30}$$

so that

$$\delta \tilde{\chi}_a^\mu = \xi\cdot\partial\tilde{\chi}_a^\mu + \partial_a \xi^\mu - \partial_\lambda \xi^\mu \tilde{\chi}_a^\lambda \tag{3.31}$$

where we have retained only the terms at most linear in the fields. From the above we see that the natural identifictions are

$$A_a = \tilde{A}_a, \qquad \chi_a^\mu = \tilde{\chi}_a^\mu \tag{3.32}$$

The transformations (3.28) are consistent with the ordinary gauge and diffeomorphism transformations. Therefore the master field h_a can describe in particular the geometry of the gauge theories and the geometry of gravity. The above does not explain the nature of the index $_a$. It is natural to interpret it as a flat index, but this calls for local Lorentz symmetry. This issue will be resumed later on.

3.3.2 Analogy with gauge transformations in gauge theories

It should be remarked that in eq.(3.25) and (3.28) the derivative ∂_a means $\partial_a = \delta_a^\mu \partial_\mu$, not $\partial_a = e_a^\mu \partial_\mu = (e_a^\mu - \chi_a^\mu + \ldots)\,\partial_\mu$. In fact the linear correction $-\chi_a^\mu \partial_\mu$ is contained in the term $-i[h_a(x,u) \stackrel{*}{,} \varepsilon(x,u)]$, see for instance the second term in the RHS of the first equation (3.28). The obvious remark is that the transformation (3.25) looks similar to the ordinary gauge transformation of a non-Abelian gauge field

$$\delta_\lambda A_a = \partial_a \lambda + [A_a, \lambda] \tag{3.33}$$

where $A_a = A_a^\alpha T^\alpha, \lambda = \lambda^\alpha T^\alpha$, T^α being the Lie algebra generators.

In gauge theories it is useful to represent the gauge potential as a connection one form $\mathbf{A} = A_a dx^a$, so that (3.33) becomes

$$\delta_\lambda \mathbf{A} = \mathbf{d}\lambda + [\mathbf{A}, \lambda] \tag{3.34}$$

We can do the same for (3.25)

$$\delta_\varepsilon \mathbf{h}(x,u) = \mathbf{d}\varepsilon(x,u) - i[\mathbf{h}(x,u) \stackrel{*}{,} \varepsilon(x,u)] \equiv \mathbf{D}\varepsilon(x,u) \tag{3.35}$$

where $\mathbf{d} = \partial_a\, dx^a, \mathbf{h} = h_a dx^a$ and x^a are coordinates in the tangent spacetime, and it is understood that

$$[\mathbf{h}(x,u) \stackrel{*}{,} \varepsilon(x,u)] = [h_a(x,u) \stackrel{*}{,} \varepsilon(x,u)]dx^a$$

We will apply this formalism to the construction of HS CS or YM-like actions.

3.3.3 HS Yang-Mills action

In analogy with the ordinary Yang-Mills theory one can introduce the curvature 2-form

$$\mathbf{G} = \mathbf{dh} - \frac{i}{2}[\mathbf{h} \stackrel{*}{,} \mathbf{h}], \tag{3.36}$$

whose components are

$$G_{ab} = \partial_a h_b - \partial_b h_a - i[h_a \stackrel{*}{,} h_b] \tag{3.37}$$

Their transformation rule is

$$\delta_\varepsilon G_{ab} = -i[G_{ab} \overset{*}{,} \varepsilon] \tag{3.38}$$

Next we will consider functionals which are integrated polynomials of $\mathbf{G}$ or of its components G_{ab}. In order to exploit the transformation property (3.16) in the construction we need the 'trace property', analogous to the trace of polynomials in ordinary non-Abelian gauge theories. The only object with trace properties we can define in the HS context is

$$\begin{aligned}\langle\!\langle f * g \rangle\!\rangle &\equiv \int d^d x \int \frac{d^d u}{(2\pi)^d} f(x,u) * g(x,u) \\ &= \int d^d x \int \frac{d^d u}{(2\pi)^d} f(x,u) g(x,u) = \langle\!\langle g * f \rangle\!\rangle \end{aligned} \tag{3.39}$$

From this, plus associativity, it follows that

$$\langle\!\langle f_1 * f_2 * \ldots * f_n \rangle\!\rangle = (-1)^{\epsilon_1(\epsilon_2+\ldots+\epsilon_n)} \langle\!\langle f_2 * \ldots * f_n * f_1 \rangle\!\rangle \tag{3.40}$$

where ϵ_i is the Grassmann degree of f_i. In particular

$$\langle\!\langle [f_1 \overset{*}{,} f_2 * \ldots * f_n \} \rangle\!\rangle = 0 \tag{3.41}$$

where $[\ \overset{*}{,}\ \}$ is the $*$-commutator or anti-commutator, as appropriate.

This property holds also when the f_i are valued in a Lie algebra, provided the symbol $\langle\!\langle\ \rangle\!\rangle$ includes also the trace over the Lie algebra generators.

Let us return to G_{ab}. From the propery (3.41) it follows that

$$\delta_\varepsilon \langle\!\langle G^{ab} * G_{ab} \rangle\!\rangle = -i \langle\!\langle G^{ab} * G_{ab} * \varepsilon - \varepsilon * G^{ab} * G_{ab} \rangle\!\rangle = 0 \tag{3.42}$$

Therefore

$$\mathcal{YM}(\mathbf{h}) = -\frac{1}{4g^2} \langle\!\langle G^{ab} * G_{ab} \rangle\!\rangle \tag{3.43}$$

is invariant under the HS gauge transformation and it is a well defined functional in any dimension.

This construction can be easily generalized to the non-Abelian case, that is when the master field $\mathbf{h}_a$ is valued in a Lie algebra with generators T^α: $\mathbf{h}_a = h_a^\alpha T^\alpha$. See [8] .

3.3.4 HS CS action

Using the above properties it is not hard to prove, [7] that

$$\mathcal{CS}(\mathbf{h}) = n \int_0^1 dt \langle\!\langle \mathbf{h} * \mathbf{G}_t * \ldots * \mathbf{G}_t \rangle\!\rangle \tag{3.44}$$

where

$$\mathbf{G}_t = \mathbf{d}\mathbf{h}_t - \frac{i}{2}[\mathbf{h}_t \overset{*}{,} \mathbf{h}_t], \qquad \mathbf{h}_t = t\mathbf{h}, \tag{3.45}$$

is HS gauge invariant in a space of odd dimension $d = 2n - 1$. It defines the HS CS action in any odd-dimensional spacetime.

3.3.5 Covariant YM-type eom's

From(3.43) we get the following eom:

$$\partial_b G^{ab} - i[h_b \overset{*}{,} G^{ab}] \equiv \mathcal{D}_b^* G^{ab} = 0 \tag{3.46}$$

which is covariant under the HS gauge transformation

$$\delta_\varepsilon \left(\mathcal{D}_b^* G^{ab}\right) = -i[\mathcal{D}_b^* G^{ab}, \varepsilon] \tag{3.47}$$

In components this equation splits into an infinite set according to the powers of u. Let us expand G_{ab} in the notation of sec.3.3.1. We have

$$G_{ab} = F_{ab} + X_{ab}^{\mu} u_\mu + \frac{1}{2} B_{ab}^{\mu\nu} u_\mu u_\nu + \frac{1}{6} C_{ab}^{\mu\nu\lambda} u_\mu u_\nu u_\lambda + \dots \tag{3.48}$$

and express them in terms of the component fields of $h_a(x, u)$.

For instance, the first eom ($\mathcal{O}(u^0)$) is

$$\begin{aligned}
0 = &\, \Box A_b - \partial_b \partial\cdot A + \frac{1}{2}\left(\partial_\sigma \partial\cdot A \chi_b^\sigma + \partial_\sigma A^a \partial_a \chi_b^\sigma - \partial_\sigma \partial^a A_b \chi_a^\sigma - \partial_\sigma A_b \partial\cdot\chi^\sigma\right) \\
&+ \frac{1}{2}\partial_\sigma A^a \left(\partial_a \chi_b^\sigma - \partial_b \chi_a^\sigma + \frac{1}{2}\left(\partial_\lambda A_a b_b^{\lambda\sigma} - \partial_\lambda A_b b_a^{\lambda\sigma} + \partial_\lambda \chi_a^\sigma \chi_b^\lambda - \partial_\lambda \chi_b^\sigma \chi_a^\lambda\right)\right) \\
&- \frac{1}{2}\chi_a^\sigma \Big(\partial_\sigma \partial^a A_b - \partial_\sigma \partial_b A^a \\
&+ \frac{1}{2}\left(\partial_\sigma \partial_\lambda A^a \chi_b^\lambda + \partial_\lambda A^a \partial_\sigma \chi_b^\lambda - \partial_\sigma \partial_\lambda A_b \chi^{a\lambda} - \partial_\lambda A_b \partial_\sigma \chi^{a\lambda}\right) \Big) \\
&+ \dots\dots
\end{aligned} \tag{3.49}$$

The second ($\mathcal{O}(u^1)$)

$$\begin{aligned}
\Box \chi_a^\mu - \partial_a \partial^b \chi_b^\mu = &\, \frac{1}{2}\Big(\partial^b (\partial_\sigma A_a\, b_b^{\sigma\mu} - \partial_\sigma A_b\, b_a^{\sigma\mu} + \partial_\sigma \chi_a^\mu \chi_b^\sigma - \partial_\sigma \chi_b^\mu \chi_a^\sigma) \\
&+ \partial_\tau A^b \partial_a b_b^{\mu\tau} - \partial_\tau A^b \partial_b b_a^{\mu\tau} + \partial_\tau \chi^{b\mu} \partial_a \chi_b^\tau - \partial_\tau \chi^{b\mu} \partial_b \chi_a^\tau \\
&- \partial_\tau \partial_a A_b\, b^{b\tau\mu} + \partial_\tau \partial_b A_a\, b^{b\tau\mu} - \partial_\tau \partial_a \chi_b^\mu \chi^{b\tau} + \partial_\tau \partial_b \chi_a^\mu \chi^{b\tau} \Big) + \dots
\end{aligned} \tag{3.50}$$

Ellipses denote terms with a larger number of spacetime derivatives.

Let us see a few elementary examples. Consider the case of a pure U(1) gauge field A alone. The equation of motion is

$$\partial_a F^{ab} = \Box A^b - \partial_b \partial\cdot A = 0 \tag{3.51}$$

In the 'Feynman gauge' $\partial\cdot A = 0$ this reduces to $\Box A^b = 0$.

Let us suppose next that only gravity is present. Eq.(3.50) becomes

$$\partial_a X^{ab\mu} = \Box \chi_b^\mu - \partial_b \partial\cdot\chi^\mu = 0 \tag{3.52}$$

In the 'Feynman gauge' $\partial\cdot\chi^\mu = 0$, (3.52) reduces to $\Box\chi_b^\mu = 0$[2].

[2] In ordinary gravity ($R_{\mu\nu} = 0$) we have to impose the DeDonder gauge in order to obtain the same result.

Finally, keeping only the spin 3 field the eom becomes

$$\partial_a B^{ab\mu\nu} = \Box b_b{}^{\mu\nu} - \partial_b \partial^a b_a^{\mu\nu} = 0 \tag{3.53}$$

Again in the 'Feynman gauge' $\partial^a b_a^{\mu\nu} = 0$ we get $\Box b_b{}^{\mu\nu} = 0$.

In general we can impose for all the fields the Feynman gauge

$$\partial^a h_a(x, u) = 0 \tag{3.54}$$

As is clear from (3.49), for instance, the above eom's are characterized by the fact that at each order, defined by the number of derivatives, there is a finite number of terms. This defines a *perturbatively local* theory.

3.3.6 Conserved currents

The conservation laws of the HS models can be found following the analogy of a current in an ordinary gauge theory or the energy momentum tensor in gravity theories. For instance, if in HS YM we express the invariance of the action under the HS gauge transformation we can write

$$\begin{aligned} 0 &= -\frac{1}{4}\delta_\varepsilon \langle\langle G_{ab} * G^{ab} \rangle\rangle = \langle\langle \delta_\varepsilon h_a * \mathcal{D}_b^* G^{ab} \rangle\rangle \\ &= \langle\langle \mathcal{D}_a^* \varepsilon * \mathcal{D}_b^* G^{ab} \rangle\rangle = -\langle\langle \varepsilon * \mathcal{D}_a^* \mathcal{D}_b^* G^{ab} \rangle\rangle, \end{aligned} \tag{3.55}$$

This implies the off-shell relation or conservation law

$$\mathcal{D}_a^* \mathcal{D}_b^* G^{ab} = 0 \tag{3.56}$$

from which we can identify the conserved master current

$$\mathcal{J}_a = \mathcal{D}_b^* G^{ab} \tag{3.57}$$

These conserved currents vanish on shell and are conserved off-shell. Expanding in u

$$\mathcal{J}_a = \sum_{n=0}^{\infty} \frac{1}{n!} \mathcal{J}_a^{\mu_1 \ldots \mu_n}(x) u_{\mu_1} \ldots u_{\mu_n} \tag{3.58}$$

we find the conserved components.

Remark. The approach to covariance implicit in the HS YM theory (but also in the effective action method) is entirely new. Unlike most HS approaches we do not start from the EH action for gravity, and we do not replace ordinary derivatives with Riemannian covariant derivatives. We obtain nevertheless an action invariant under HS gauge transformations. The gauge transformation (3.16) reproduces both ordinary U(1) gauge transformations and diffeomorphisms, but the action functional is defined in the phase space. It gives nevertheless rise to local (HS gauge covariant) equations of motion that reproduce the ordinary YM eoms, and, although not completely, the metric equations of motion of EH gravity: the linear eom coincide with the ordinary one after gauge fixing. Although the equations (3.50) is very reminiscent of ordinary gravity, this is not yet enough to identify the type of gravity described by it. In fact this problem requires further investigation and will be discussed in a forthcoming paper.

3.4 Local Lorentz symmetry

As pointed out before the HS YM action is fully invariant in particular under diffeomorphisms. This prompted us to interpret the second component of $h_a(s,u)$ in the u expansion, χ_a^μ, as a vielbein fluctuation, and $\delta_a^\mu - \chi_a^\mu$ as a vielbein or local frame. However this implies that $_a$ is a flat index and must transform appropriately under local Lorentz transformations. But, at least at first sight, local Lorentz invariance is absent. Consider simply the case in which only the field A_a is non-vanishing, the form of the Lagrangian is

$$L_A \sim F_{ab}F^{ab}, \qquad F_{ab} = \partial_a A_b - \partial_b A_a \tag{3.59}$$

This is not invariant under a Lorentz transformation, because when $A_a \to A_a + \Lambda_a{}^b A_b$ we generate terms $((\partial_a \Lambda_b{}^c) A_c - (\partial_b \Lambda_a{}^c) A_c) F^{ab}$, that do not vanish. This is a simple example of a general problem in HS YM. It is crucial to clarify it.

3.4.1 Inertial frames and connections

Let us start from the definition of trivial frame. A trivial (inverse) frame $e_a^\mu(x)$ is a frame that can be reduced to a Kronecker delta by means of a local Lorentz transformation (LLT), i.e. there exists a (pseudo)orthogonal transformation $O_a{}^b(x)$ such that

$$O_a{}^b(x) e_b{}^\mu(x) = \delta_a^\mu \tag{3.60}$$

As a consequence $e_b{}^\mu(x)$ contains only inertial (non-dynamical) information. A full gravitational (dynamical) frame is the sum of a trivial frame and a nontrivial piece

$$\tilde{E}_a^\mu(x) = e_a{}^\mu(x) - \tilde{\chi}_a^\mu(x) \tag{3.61}$$

By means of a suitable LLT it can be cast in the form

$$E_a^\mu(x) = \delta_a^\mu - \chi_a^\mu(x) \tag{3.62}$$

This is the form we have encountered above in HS theories. But it should not be forgotten that the Kronecker delta is a trivial frame. If we want to recover local Lorentz covariance instead of $\partial_a = \delta_a^\mu \partial_\mu$ we must understand

$$\partial_a = e_a{}^\mu(x)\partial_\mu \tag{3.63}$$

where $e_a{}^\mu(x)$ is a trivial (or purely inertial) vielbein. In particular, under an infinitesimal LLT, it transforms according to

$$\delta_\Lambda e_a{}^\mu(x) = \Lambda_a{}^b(x) e_b{}^\mu(x) \tag{3.64}$$

A trivial connection (or inertial spin connection) is defined by

$$\mathcal{A}^a{}_{b\mu} = \left(O(x)\partial_\mu O^{-1}(x)\right)^a{}_b \tag{3.65}$$

where $O(x)$ is a generic local (pseudo)orthogonal transformation (finite local Lorentz transformation). As a consequence its curvature vanishes

$$\mathcal{R}^a{}_{b\mu\nu} = \partial_\mu \mathcal{A}^a{}_{b\nu} - \partial_\nu \mathcal{A}^a{}_{b\mu} + \mathcal{A}^a{}_{c\mu}\mathcal{A}^c{}_{b\nu} - \mathcal{A}^a{}_{c\nu}\mathcal{A}^c{}_{b\mu} = 0 \tag{3.66}$$

Let us recall that the space of connections is affine. We can obtain any connection from a fixed one by adding to it adjoint-covariant tensors. When the spacetime is topologically trivial we can choose as origin of the affine space the 0 connection. The latter is a particular member in the class of the trivial connections. To see this let us suppose we start with the spin connection (3.65). A Lorentz transformation of a spin connection $\mathcal{A}_\mu = \mathcal{A}_\mu{}^{ab}\Sigma_{ab}$ is

$$\mathcal{A}_\mu(x) \to L(x)D_\mu L^{-1}(x) = L(x)(\partial_\mu + \mathcal{A}_\mu)L^{-1}(x) \tag{3.67}$$

where $L(x)$ is a (finite) LLT. If we choose $L = O^{-1}$ we get

$$\mathcal{A}_\mu(x) \to 0 \tag{3.68}$$

But at this point the LL symmetry is completely fixed. Thus choosing the zero spin connection amounts to fixing the local Lorentz gauge.

The connection $\mathcal{A}_\mu$ contains inertial and no gravitational information. It will be referred to as the *inertial connection*. It is a *non-dynamical* object (its content is pure gauge). The dynamical degrees of freedom will be contained in the adjoint tensor to be added to $\mathcal{A}_\mu$ in order to form a fully dynamical spin connection[3]. $\mathcal{A}_\mu$ is nevertheless a connection and it makes sense to introduce the inertial derivative

$$D_\mu = \partial_\mu - \frac{i}{2}\mathcal{A}_\mu \tag{3.69}$$

which is Lorentz covariant.

It is clear that the results ensuing from the effective action method, as well as the HS YM and HS CS theories, are all formulated in a trivial frame setting, eq.(3.62), with a trivial spin connection. In other words the local Lorentz gauge is completely fixed. However from this formalism it is not difficult to recover explicit local Lorentz covariance.

3.4.2 How to recover local Lorentz symmetry

Let us restart from the definition of $J_a(x,u)$

$$\begin{aligned} J_a(x,u) &= \sum_{n,m=0}^{\infty} \frac{(-i)^n i^m}{2^{n+m} n! m!} \partial_{\mu_1}\ldots\partial_{\mu_m}\bar\psi(x)\gamma_a\partial_{\nu_1}\ldots\partial_{\nu_n}\psi(x) \\ &\qquad \times \frac{\partial^{n+m}}{\partial u_{\mu_1}\ldots\partial u_{\mu_m}\partial u_{\nu_1}\ldots\partial u_{\nu_n}}\delta(u) \\ &= \sum_{s=1}^{\infty}(-1)^{s-1} J^{(s)}_{a\mu_1\ldots\mu_{s-1}}(x)\frac{\partial^{s-1}}{\partial u_{\mu_1}\ldots\partial u_{\mu_{s-1}}}\delta(u) \end{aligned} \tag{3.70}$$

[3] The splitting of vierbein and spin connection into an inertial and a dynamical part is characteristic of teleparallelism, [9]

from which we derive

$$J^{(s)}_{a\mu_1\dots\mu_{s-1}}(x) = \sum_{n=0}^{s-1} \frac{(-1)^n}{s^{s-1}(s-1)!} \partial_{(\mu_1} \dots \partial_{\mu_n} \bar{\psi}(x)\gamma_a \partial_{\mu_{n+1}} \dots \partial_{\mu_{s-1})}\psi(x) \tag{3.71}$$

Assume now the following LLT

$$\delta_\Lambda \psi = -\frac{i}{2}\Lambda\psi, \qquad \Lambda = \Lambda^{ab}\Sigma_{ab}, \qquad \Sigma_{ab} = \frac{i}{4}[\gamma_a, \gamma_b] \tag{3.72}$$
$$\delta_\Lambda \bar{\psi} = \frac{i}{2}\bar{\psi}\Lambda$$

and replace in (3.71) the ordinary derivative on ψ with the inertial covariant derivative

$$\partial_\mu \psi \rightarrow D_\mu \psi = \left(\partial_\mu - \frac{i}{2}\mathcal{A}_\mu\right)\psi \tag{3.73}$$

and on $\bar{\psi}$ with

$$\partial_\mu \bar{\psi} \rightarrow D^\dagger_\mu \bar{\psi} = \partial_\mu \bar{\psi} + \frac{i}{2}\bar{\psi}\mathcal{A}_\mu \tag{3.74}$$

Eq.(3.71) becomes

$$J'^{(s)}_{a\mu_1\dots\mu_{s-1}}(x) \tag{3.75}$$
$$= \sum_{n=0}^{s-1} \frac{(-1)^n}{s^{s-1}(s-1)!} D^\dagger_{(\mu_1} \dots D^\dagger_{\mu_n} \bar{\psi}(x)\gamma_a D_{\mu_{n+1}} \dots D_{\mu_{s-1})}\psi(x)$$

Now, given

$$\delta_\Lambda \mathcal{A}_\mu = -\partial_\mu \Lambda + \frac{i}{2}[\mathcal{A}_\mu, \Lambda] \tag{3.76}$$

and (3.12), it is easy to prove that

$$\delta_\Lambda (D_\mu \psi) = -\frac{i}{2}\Lambda(D_\mu \psi), \qquad \delta(D^\dagger_\mu \psi) = \frac{i}{2}(D^\dagger_\mu \psi)\Lambda \tag{3.77}$$

The same holds for multiple covariant derivatives

$$\delta_\Lambda (D_{\mu_1} \dots D_{\mu_n} \psi) = \frac{i}{2}\Lambda(D_{\mu_1} \dots D_{\mu_n} \psi), \quad \text{etc.}$$

It follows that

$$\delta_\Lambda J'^{(s)}_{a\mu_1\dots\mu_{s-1}}(x)$$
$$= -\sum_{n=0}^{s-1} \frac{(-1)^n}{s^{s-1}(s-1)!} D^\dagger_{(\mu_1} \dots D^\dagger_{\mu_n} \bar{\psi}(x)[\gamma_a, \Lambda] D_{\mu_{n+1}} \dots D_{\mu_{s-1})}\psi(x)$$
$$= \Lambda_a{}^b(x)\, J'^{(s)}_{b\mu_1\dots\mu_{s-1}}(x) \tag{3.78}$$

Therefore the interaction term

$$S'_{int} = \sum_{s=1}^{\infty} \int d^d x \, J'^{(s)}_{a\mu_1\ldots\mu_{s-1}}(x) h^{a\mu_1\ldots\mu_{s-1}} \tag{3.79}$$

is invariant under (3.12) and (3.76) provided

$$\delta_\Lambda h^{a\mu_1\ldots\mu_n}(x) = \Lambda^a{}_b(x)\, h^{b\mu_1\ldots\mu_n}(x) \tag{3.80}$$

On the other hand, writing

$$S'_0 = \int d^d x \, \bar\psi \left(i\gamma^a \left(\partial_a - \frac{i}{2}\mathcal{A}_a \right) - m \right) \psi \tag{3.81}$$

instead of S_0, also S'_0 turns out to be invariant under LLT. So, provided we define LLT via (3.12) and (3.76), $S' = S'_0 + S'_{int}$ is invariant.

Replacing simple spacetime derivatives ∂_μ with the inertial ones D_μ everywhere is not enough. As pointed out above instead of $\partial_a = \delta_a^\mu \partial_\mu$ we should write $\partial_a = e_a{}^\mu(x)\partial_\mu$, where $e_a{}^\mu(x)$ is a purely inertial frame. Moreover, whenever it appears, we should rewrite $\mathcal{A}_a(x) = e_a{}^\mu(x)\mathcal{A}_\mu(x)$.

With this new recipes all inconsistencies disappear. For instance

$$\delta_\Lambda(D_a J_b) = \Lambda_a{}^c(D_c J_b) + \Lambda_b{}^c(D_a J_c)$$

Therefore $\delta_\Lambda(\eta^{ab} D_a J_b) = 0$. Likewise

$$\delta_\Lambda G_{ab} = \Lambda_a{}^c G_{cb} + \Lambda_b{}^c G_{ac} \tag{3.82}$$

which implies the local Lorentz invariance of $G_{ab}G^{ab}$.

Summary. *The HS effective action approach fixes completely the local Lorentz gauge. This is due the fact that in its formalism (and in the general in the HS YM and CS formalism) the choice* $e_a^\mu = \delta_a^\mu$ *and* $\mathcal{A}_a = 0$ *for the inertial frame and connection, is implicit. However the same formalism offers the possibility to recover the LL invariance by means of a simple recipe:*

1. *replace any spacetime derivative, even in the* $*$ *product, with the inertial covariant derivative,*
2. *interpret any flat index* $_a$ *attached to any object* O_a *as* $e_a^\mu(x)O_\mu$.

Anticipating future developments we add that in the process of quantization $e_a^\mu(x)$ and $\mathcal{A}_a(x)$ will be treated as classical backgrounds.

3.5 Conclusions

The main message of this paper is that it is possible to construct field teory models of Yang-Mills type with infinite many HS fields in flat spacetime in any dimension. It is also possible to construct similar models of Chern-Simons type in any odd dimensional flat spacetime. We have seen that of such models we can define the actions, invariant under HS gauge transformations, which encompass the ordinary gauge transformations and the diffeomorphisms. It was also shown that although the local Lorentz gauge is fixed in this formalism, local Lorentz invariance can be easily implemented. We can derive sensible eom's. A more detailed account and further developments are contained in related papers[7,8] : for instance one can introduce matter master scalar and fermion fields, and realize the analog of Higgs mechanism; one can also introduce ghosts, and carry out the BRST quantization and develop the practical machinery for perturbative calculations via Feynman diagrams.

All these results may be at first surprising, because, as noted in the introduction, there exist no-go (Weiberg-Witten) theorems forbidding the existence of interacting massless HS theories in flat spacetime (for a review see [10]). A full discussion of this problem will be given in [8] . Here let us simply notice that such theorems are based on a set of hypotheses, which are very plausible in ordinary field theories, but can be circumvented in theories like the ones introduced here. For instance two basic requirements are the minimal coupling of the matter fields to gravity and the polynomial structure of the energy-momentum tensor. It turns out that none of these requirements is realized in HS YM-like theories: gravity is non-minimally coupled to HS fields and the energy-momentum tensor is not a polynomial of the fields, but a series.

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