The Cayley isomorphism property for the group $C_2^5 \times C_p$

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Abstract

A finite group G is called a DCI-group if two Cayley digraphs over G are isomorphic if and only if their connection sets are conjugate by a group automorphism. We prove that the group $C_2^5 \times C_p$, where p is a prime, is a DCI-group if and only if $p \neq 2$. Together with the previously obtained results, this implies that a group G of order $32p$, where p is a prime, is a DCI-group if and only if $p \neq 2$ and $G \cong C_2^5 \times C_p$.

Keywords: Isomorphisms, DCI-*groups, Schur rings.*

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1 Introduction

Let G be a finite group and $S \subseteq G$. The *Cayley digraph* $\mathrm{Cay}(G, S)$ over G with connection set S is defined to be the digraph with vertex set G and arc set $\{(g, sg) : g \in G, s \in S\}$. Two Cayley digraphs over G are called *Cayley isomorphic* if there exists an isomorphism between them which is also an automorphism of G. Clearly, two Cayley isomorphic Cayley digraphs are isomorphic. The converse statement is not true in general (see [3, 10]). A subset $S \subseteq G$ is called a CI-*subset* if for each $T \subseteq G$ the Cayley digraphs $\mathrm{Cay}(G, S)$ and $\mathrm{Cay}(G, T)$ are isomorphic if and only if they are Cayley isomorphic. A finite group G is called a DCI-*group* (CI-*group*, respectively) if each subset of G (each inverse-closed subset of G, respectively) is a CI-subset.

The investigation of DCI-groups was initiated by Ádám [1] who conjectured, in our terms, that every cyclic group is a DCI-group. This conjecture was disproved by Elspas and Turner in [10]. The problem of determining of finite DCI- and CI-groups was suggested by Babai and Frankl in [5]. For more information on DCI- and CI-groups we refer the readers to the survey paper [21].

In this paper we are interested in abelian DCI-groups. The cyclic group of order n is denoted by C_n. Elspas and Turner [10] and independently Djoković [8] proved that every cyclic group of prime order is a DCI-group. The fact that C_{pq} is a DCI-group for distinct primes p and q was proved by Alspach and Parsons in [3] and independently by Klin and Pöschel in [17]. The complete classification of cyclic DCI-groups was obtained by Muzychuk in [23, 24]. He proved that a cyclic group of order n is a DCI-group if and only if $n = k$ or $n = 2k$, where k is square-free.

Denote the class of all finite abelian groups where every Sylow subgroup is elementary abelian by $\mathcal{E}$. From [18, Theorem 1.1] it follows that every DCI-group is the coprime product (i.e. the direct product of groups of coprime orders) of groups from the following list:

$$C_p^k, \ C_4, \ Q_8, \ A_4, \ H \rtimes \langle z \rangle,$$

where p is a prime, H is a group of odd order from $\mathcal{E}$, $|z| \in \{2, 4\}$, and $h^z = h^{-1}$ for every $h \in H$. One can check that the class of DCI-groups is closed under taking subgroups. So one of the crucial steps towards the classification of all DCI-groups is to determine which groups from $\mathcal{E}$ are DCI.

The following non-cyclic groups from $\mathcal{E}$ are DCI-groups (p and q are assumed to be distinct primes): C_p^2 [2, 14]; C_p^3 [2, 9]; C_2^4, C_2^5 [7]; C_p^4, where p is odd [15] (a proof for C_p^4 with no condition on p was given in [22]); C_p^5, where p is odd [13]; $C_p^2 \times C_q$ [18]; $C_p^3 \times C_q$ [27]; $C_p^4 \times C_q$ [20]. The smallest example of a non-DCI-group from $\mathcal{E}$ was found by Nowitz [28]. He proved that C_2^6 is non-DCI. This implies that C_2^n is non-DCI for every $n \geq 6$. Also C_3^n is non-DCI for every $n \geq 8$ [33] and C_p^n is non-DCI for every prime p and $n \geq 2p + 3$ [32].

In this paper we find a new infinite family of DCI-groups from $\mathcal{E}$ which are close to the smallest non-DCI-group from $\mathcal{E}$. The main result of the paper can be formulated as follows.

Theorem 1.1. *Let p be a prime. Then the group $C_2^5 \times C_p$ is a DCI-group if and only if $p \neq 2$.*

Theorem 1.1 extends the results obtained in [18, 20, 27] which imply that the group $C_p^k \times C_q$ is a DCI-group whenever p and q are distinct primes and $k \leq 4$. Note that the "only if" part of Theorem 1.1, in fact, was proved by Nowitz in [28]. The next corollary immediately follows from [18, Theorem 1.1] and Theorem 1.1.

Corollary 1.2. *Let p be a prime. Then a group G of order $32p$ is a DCI-group if and only if $p \neq 2$ and $G \cong C_2^5 \times C_p$.*

To prove Theorem 1.1, we use the S-ring approach. An S-*ring* over a group G is a subring of the group ring $\mathbb{Z}G$ which is a free $\mathbb{Z}$-module spanned by a special partition of G. If every S-ring from a certain family of S-rings over G is a CI-S-ring then G is a DCI-group (see Section 4). The definition of an S-ring goes back to Schur [31] and Wielandt [34]. The usage of S-rings in the investigation of DCI-groups was proposed by

Klin and Pöschel [17]. Most recent results on DCI-groups were obtained using S-rings (see [15, 18, 19, 20, 27]).

The text of the paper is organized in the following way. In Section 2 we provide definitions and basic facts concerned with S-rings. Section 3 contains a necessary information on isomorphisms of S-rings. In Section 4 we discuss CI-S-rings and their relation with DCI-groups. We also prove in this section a sufficient condition of CI-property for S-rings (Lemma 4.4). Section 5 is devoted to the generalized wreath and star products of S-rings. Here we deduce from previously obtained results two sufficient conditions for the generalized wreath product of S-rings to be a CI-S-ring (Lemma 5.5 and Lemma 5.8). Section 6 and 7 are concerned with p-S-rings and S-rings over a group of order pk, where p is a prime and $\mathrm{GCD}(p,k) = 1$, (so-called non-powerful order) respectively. In Section 8 we provide properties of S-rings over the groups C_2^n, $n \leq 5$, and prove that all S-rings over these groups are CI. The material of this section is based on computational results obtained with the help of the GAP package COCO2P [16]. Finally, in Section 9 we prove Theorem 1.1.

Notation. Let G be a finite group and $X \subseteq G$. The element $\sum_{x \in X} x$ of the group ring $\mathbb{Z}G$ is denoted by $\underline{X}$.

The set $\{x^{-1} : x \in X\}$ is denoted by X^{-1}.

The subgroup of G generated by X is denoted by $\langle X \rangle$; we also set $\mathrm{rad}(X) = \{g \in G : gX = Xg = X\}$.

Given a set $X \subseteq G$ the set $\{(g, xg) : x \in X, g \in G\}$ of arcs of the Cayley digraph $\mathrm{Cay}(G, X)$ is denoted by $A(X)$.

The group of all permutations of G is denoted by $\mathrm{Sym}(G)$.

The subgroup of $\mathrm{Sym}(G)$ consisting of all right translations of G is denoted by G_{right}.

The set $\{K \leq \mathrm{Sym}(G) : K \geq G_{\mathrm{right}}\}$ is denoted by $\mathrm{Sup}(G_{\mathrm{right}})$.

For a set $\Delta \subseteq \mathrm{Sym}(G)$ and a section $S = U/L$ of G we set $\Delta^S = \{f^S : f \in \Delta, S^f = S\}$, where $S^f = S$ means that f permutes the L-cosets in U and f^S denotes the bijection of S induced by f.

If $K \leq \mathrm{Sym}(\Omega)$ and $\alpha \in \Omega$ then the stabilizer of α in K and the set of all orbits of K on Ω are denoted by K_α and $\mathrm{Orb}(K, \Omega)$ respectively.

If $H \leq G$ then the normalizer of H in G is denoted by $N_G(H)$.

The cyclic group of order n is denoted by C_n.

The class of all finite abelian groups where every Sylow subgroup is elementary abelian is denoted by $\mathcal{E}$.

2 S-rings

In this section we give a background of S-rings. In general, we follow [20], where the most part of the material is contained. For more information on S-rings we refer the readers to [6, 25].

Let G be a finite group and $\mathbb{Z}G$ the integer group ring. Denote the identity element of G by e. A subring $\mathcal{A} \subseteq \mathbb{Z}G$ is called an S-*ring (a Schur ring)* over G if there exists a partition $\mathcal{S}(\mathcal{A})$ of G such that:

(1) $\{e\} \in \mathcal{S}(\mathcal{A})$,

(2) if $X \in \mathcal{S}(\mathcal{A})$ then $X^{-1} \in \mathcal{S}(\mathcal{A})$,

(3) $\mathcal{A} = \mathrm{Span}_{\mathbb{Z}}\{\underline{X} : X \in \mathcal{S}(\mathcal{A})\}$.

The elements of $\mathcal{S}(\mathcal{A})$ are called the *basic sets* of $\mathcal{A}$ and the number $\mathrm{rk}(\mathcal{A}) = |\mathcal{S}(\mathcal{A})|$ is called the *rank* of $\mathcal{A}$. If $X, Y \in \mathcal{S}(\mathcal{A})$ then $XY \in \mathcal{S}(\mathcal{A})$ whenever $|X| = 1$ or $|Y| = 1$.

Let $\mathcal{A}$ be an S-ring over a group G. A set $X \subseteq G$ is called an $\mathcal{A}$-*set* if $\underline{X} \in \mathcal{A}$. A subgroup $H \leq G$ is called an $\mathcal{A}$-*subgroup* if H is an $\mathcal{A}$-set. From the definition it follows that the intersection of $\mathcal{A}$-subgroups is also an $\mathcal{A}$-subgroup. One can check that for each $\mathcal{A}$-set X the groups $\langle X \rangle$ and $\mathrm{rad}(X)$ are $\mathcal{A}$-subgroups. By the *thin radical* of $\mathcal{A}$ we mean the set defined as

$$\mathbf{O}_\theta(\mathcal{A}) = \{x \in G : \{x\} \in \mathcal{S}(\mathcal{A})\}.$$

It is easy to see that $\mathbf{O}_\theta(\mathcal{A})$ is an $\mathcal{A}$-subgroup.

Lemma 2.1 ([11], Lemma 2.1])**.** *Let $\mathcal{A}$ be an S-ring over a group G, H an $\mathcal{A}$-subgroup of G, and $X \in \mathcal{S}(\mathcal{A})$. Then the number $|X \cap Hx|$ does not depend on $x \in X$.*

Let $L \trianglelefteq U \leq G$. A section U/L is called an $\mathcal{A}$-*section* if U and L are $\mathcal{A}$-subgroups. If $S = U/L$ is an $\mathcal{A}$-section then the module

$$\mathcal{A}_S = \mathrm{Span}_\mathbb{Z} \left\{ \underline{X}^\pi : X \in \mathcal{S}(\mathcal{A}), \ X \subseteq U \right\},$$

where $\pi \colon U \to U/L$ is the canonical epimorphism, is an S-ring over S.

3   Isomorphisms and schurity

Let $\mathcal{A}$ and $\mathcal{A}'$ be S-rings over groups G and G' respectively. A bijection $f \colon G \to G'$ is called an *isomorphism* from $\mathcal{A}$ to $\mathcal{A}'$ if

$$\{A(X)^f : X \in \mathcal{S}(\mathcal{A})\} = \{A(X') : X' \in \mathcal{S}(\mathcal{A}')\},$$

where $A(X)^f = \{(g^f, \ h^f) : (g, \ h) \in A(X)\}$. If there exists an isomorphism from $\mathcal{A}$ to $\mathcal{A}'$ then we say that $\mathcal{A}$ and $\mathcal{A}'$ are *isomorphic* and write $\mathcal{A} \cong \mathcal{A}'$.

The group of all isomorphisms from $\mathcal{A}$ onto itself contains a normal subgroup

$$\{f \in \mathrm{Sym}(G) : A(X)^f = A(X) \text{ for every } X \in \mathcal{S}(\mathcal{A})\}$$

called the *automorphism group* of $\mathcal{A}$ and denoted by $\mathrm{Aut}(\mathcal{A})$. The definition implies that $G_{\mathrm{right}} \leq \mathrm{Aut}(\mathcal{A})$. The S-ring $\mathcal{A}$ is called *normal* if G_{right} is normal in $\mathrm{Aut}(\mathcal{A})$. One can verify that if S is an $\mathcal{A}$-section then $\mathrm{Aut}(\mathcal{A})^S \leq \mathrm{Aut}(\mathcal{A}_S)$. Denote the group $\mathrm{Aut}(\mathcal{A}) \cap \mathrm{Aut}(G)$ by $\mathrm{Aut}_G(\mathcal{A})$. It easy to check that if S is an $\mathcal{A}$-section then $\mathrm{Aut}_G(\mathcal{A})^S \leq \mathrm{Aut}_S(\mathcal{A}_S)$. One can verify that

$$\mathrm{Aut}_G(\mathcal{A}) = (N_{\mathrm{Aut}(\mathcal{A})}(G_{\mathrm{right}}))_e.$$

Let $K \in \mathrm{Sup}(G_{\mathrm{right}})$. Schur proved in [31] that the $\mathbb{Z}$-submodule

$$V(K, G) = \mathrm{Span}_\mathbb{Z}\{\underline{X} : X \in \mathrm{Orb}(K_e, \ G)\},$$

is an S-ring over G. An S-ring $\mathcal{A}$ over G is called *schurian* if $\mathcal{A} = V(K, G)$ for some $K \in \mathrm{Sup}(G_{\mathrm{right}})$. One can verify that given $K_1, K_2 \in \mathrm{Sup}(G_{\mathrm{right}})$,

$$\text{if } K_1 \leq K_2 \text{ then } V(K_1, G) \geq V(K_2, G). \tag{3.1}$$

If $\mathcal{A} = V(K, G)$ for some $K \in \mathrm{Sup}(G_{\mathrm{right}})$ and S is an $\mathcal{A}$-section then $\mathcal{A}_S = V(K^S, S)$. So if $\mathcal{A}$ is schurian then $\mathcal{A}_S$ is also schurian for every $\mathcal{A}$-section S. It can be checked that

$$V(\mathrm{Aut}(\mathcal{A}), G) \geq \mathcal{A} \tag{3.2}$$

and the equality is attained if and only if $\mathcal{A}$ is schurian.

An S-ring $\mathcal{A}$ over a group G is defined to be *cyclotomic* if there exists $K \leq \mathrm{Aut}(G)$ such that $\mathcal{S}(\mathcal{A}) = \mathrm{Orb}(K, G)$. In this case we write $\mathcal{A} = \mathrm{Cyc}(K, G)$. Obviously, $\mathcal{A} = V(G_{\mathrm{right}}K, G)$. So every cyclotomic S-ring is schurian. If $\mathcal{A} = \mathrm{Cyc}(K, G)$ for some $K \leq \mathrm{Aut}(G)$ and S is an $\mathcal{A}$-section then $\mathcal{A}_S = \mathrm{Cyc}(K^S, S)$. Therefore if $\mathcal{A}$ is cyclotomic then $\mathcal{A}_S$ is also cyclotomic for every $\mathcal{A}$-section S.

Two permutation groups K_1 and K_2 on a set Ω are called 2-*equivalent* if $\mathrm{Orb}(K_1, \Omega^2) = \mathrm{Orb}(K_2, \Omega^2)$ (here we assume that K_1 and K_2 act on Ω^2 componentwise). In this case we write $K_1 \approx_2 K_2$. The relation $\approx_2$ is an equivalence relation on the set of all subgroups of $\mathrm{Sym}(\Omega)$. Every equivalence class has a unique maximal element with respect to inclusion. Given $K \leq \mathrm{Sym}(\Omega)$, this element is called the 2-*closure* of K and denoted by $K^{(2)}$. If $\mathcal{A} = V(K, G)$ for some $K \in \mathrm{Sup}(G_{\mathrm{right}})$ then $K^{(2)} = \mathrm{Aut}(\mathcal{A})$. An S-ring $\mathcal{A}$ over G is called 2-*minimal* if

$$\{K \in \mathrm{Sup}(G_{\mathrm{right}}) : K \approx_2 \mathrm{Aut}(\mathcal{A})\} = \{\mathrm{Aut}(\mathcal{A})\}.$$

Two groups $K_1, K_2 \leq \mathrm{Aut}(G)$ are said to be *Cayley equivalent* if $\mathrm{Orb}(K_1, G) = \mathrm{Orb}(K_2, G)$. In this case we write $K_1 \approx_{\mathrm{Cay}} K_2$. If $\mathcal{A} = \mathrm{Cyc}(K, G)$ for some $K \leq \mathrm{Aut}(G)$ then $\mathrm{Aut}_G(\mathcal{A})$ is the largest group which is Cayley equivalent to K. A cyclotomic S-ring $\mathcal{A}$ over G is called *Cayley minimal* if

$$\{K \leq \mathrm{Aut}(G) : K \approx_{\mathrm{Cay}} \mathrm{Aut}_G(\mathcal{A})\} = \{\mathrm{Aut}_G(\mathcal{A})\}.$$

It is easy to see that $\mathbb{Z}G$ is 2-minimal and Cayley minimal.

4   CI-S-rings

Let $\mathcal{A}$ be an S-ring over a group G. Put

$$\mathrm{Iso}(\mathcal{A}) = \{f \in \mathrm{Sym}(G) : f \text{ is an isomorphism from } \mathcal{A} \text{ onto an } S\text{-ring over } G\}.$$

One can see that $\mathrm{Aut}(\mathcal{A})\,\mathrm{Aut}(G) \subseteq \mathrm{Iso}(\mathcal{A})$. However, the converse statement does not hold in general. The S-ring $\mathcal{A}$ is defined to be a CI-S-*ring* if $\mathrm{Aut}(\mathcal{A})\,\mathrm{Aut}(G) = \mathrm{Iso}(\mathcal{A})$. It is easy to check that $\mathbb{Z}G$ and the S-ring of rank 2 over G are CI-S-rings.

Put

$$\mathrm{Sup}_2(G_{\mathrm{right}}) = \{K \in \mathrm{Sup}(G_{\mathrm{right}}) : K^{(2)} = K\}.$$

The group $M \leq \mathrm{Sym}(G)$ is said to be *G-regular* if M is regular and isomorphic to G. Following [15], we say that a group $K \in \mathrm{Sup}(G_{\mathrm{right}})$ is *G-transjugate* if every G-regular subgroup of K is K-conjugate to G_{right}. Babai proved in [4] the statement which can be formulated in our terms as follows: a set $S \subseteq G$ is a CI-subset if and only if the group $\mathrm{Aut}(\mathrm{Cay}(G, S))$ is G-transjugate. The next lemma provides a similar criterion for a schurian S-ring to be CI.

Lemma 4.1. *Let $K \in \mathrm{Sup}_2(G_{\mathrm{right}})$ and $\mathcal{A} = V(K, G)$. Then $\mathcal{A}$ is a CI-S-ring if and only if K is G-transjugate.*

Proof. The statement of the lemma follows from [15, Theorem 2.6].   □

Let $K_1, K_2 \in \mathrm{Sup}(G_{\mathrm{right}})$ such that $K_1 \leq K_2$. Then K_1 is called a *G-complete subgroup* of K_2 if every G-regular subgroup of K_2 is K_2-conjugate to some G-regular subgroup of K_1 (see [15, Definition 2]). In this case we write $K_1 \preceq_G K_2$. The relation $\preceq_G$ is a partial order on $\mathrm{Sup}(G_{\mathrm{right}})$. The set of the minimal elements of $\mathrm{Sup}_2(G_{\mathrm{right}})$ with respect to $\preceq_G$ is denoted by $\mathrm{Sup}_2^{\min}(G_{\mathrm{right}})$.

Lemma 4.2 ([20, Lemma 3.3]). *Let G be a finite group. If $V(K, G)$ is a CI-S-ring for every $K \in \mathrm{Sup}_2^{\min}(G_{\mathrm{right}})$ then G is a DCI-group.*

Remark 4.3. The condition that $V(K, G)$ is a CI-S-ring for every $K \in \mathrm{Sup}_2^{\min}(G_{\mathrm{right}})$ is equivalent to, say, that every schurian S-ring over G is a CI-S-ring. The latter condition means that every 2-closed overgroup of G_{right} is G-transjugate. However, 2-closed overgroup of G_{right} may not be the automorphism group of a Cayley digraph over G. So the condition that the automorphism group of every Cayley digraph over G is G-transjugate or, equivalently, that G is a DCI-group, seems weaker than the condition of Lemma 4.2. It is a natural question whether there exists a DCI-group for which the condition of Lemma 4.2 does not hold.

We finish the section with the lemma that gives a sufficient condition for an S-ring to be a CI-S-ring. In order to formulate this condition, we need to introduce some further notations. Let $\mathcal{A}$ be a schurian S-ring over an abelian group G and L a normal $\mathcal{A}$-subgroup of G. Then the partition of G into the L-cosets is $\mathrm{Aut}(\mathcal{A})$-invariant. The kernel of the action of $\mathrm{Aut}(\mathcal{A})$ on the latter cosets is denoted by $\mathrm{Aut}(\mathcal{A})_{G/L}$. Since $\mathrm{Aut}(\mathcal{A})_{G/L}$ is a normal subgroup of $\mathrm{Aut}(\mathcal{A})$, we can form the group $K = \mathrm{Aut}(\mathcal{A})_{G/L}G_{\mathrm{right}}$. Clearly, $K \leq \mathrm{Aut}(\mathcal{A})$. From [15, Proposition 2.1] it follows that $K = K^{(2)}$.

Lemma 4.4. *Let $\mathcal{A}$ be a schurian S-ring over an abelian group G, L an $\mathcal{A}$-subgroup of G, and $K = \mathrm{Aut}(\mathcal{A})_{G/L}G_{\mathrm{right}}$. Suppose that both $\mathcal{A}_{G/L}$ and $V(K, G)$ are CI-S-rings and $\mathcal{A}_{G/L}$ is normal. Then $\mathcal{A}$ is a CI-S-ring.*

Proof. Firstly we prove that the group $\mathrm{Aut}(\mathcal{A})^{G/L}$ is G/L-transjugate. Suppose that F is a G/L-regular subgroup of $\mathrm{Aut}(\mathcal{A})^{G/L}$. The S-ring $\mathcal{A}_{G/L}$ is a CI-S-ring by the assumption of the lemma. So Lemma 4.1 implies that the group $\mathrm{Aut}(\mathcal{A}_{G/L})$ is G/L-transjugate. Since $F \leq \mathrm{Aut}(\mathcal{A})^{G/L} \leq \mathrm{Aut}(\mathcal{A}_{G/L})$, we conclude that F and $(G/L)_{\mathrm{right}}$ are $\mathrm{Aut}(\mathcal{A}_{G/L})$-conjugate. However, $\mathcal{A}_{G/L}$ is normal and hence $F = (G/L)_{\mathrm{right}}$. Therefore $\mathrm{Aut}(\mathcal{A})^{G/L}$ is G/L-transjugate.

Now let us show that $K \preceq_G \mathrm{Aut}(\mathcal{A})$. Let H be a G-regular subgroup of $\mathrm{Aut}(\mathcal{A})$. Then $H^{G/L}$ is abelian transitive subgroup of $\mathrm{Aut}(\mathcal{A})^{G/L}$ and hence $H^{G/L}$ is regular on G/L. Therefore $H^{G/L} \cong (G/L)_{\mathrm{right}} = (G_{\mathrm{right}})^{G/L}$. There exists $\gamma \in \mathrm{Aut}(\mathcal{A})$ such that $(H^{G/L})^{\gamma^{G/L}} = (G/L)_{\mathrm{right}} = (G_{\mathrm{right}})^{G/L}$ because $\mathrm{Aut}(\mathcal{A})^{G/L}$ is G/L-transjugate. This yields that $H^\gamma \leq K$. Thus, $K \preceq_G \mathrm{Aut}(\mathcal{A})$.

Finally, let us prove that $\mathrm{Aut}(\mathcal{A})$ is G-transjugate. Again, let H be a G-regular subgroup of $\mathrm{Aut}(\mathcal{A})$. Since $K \preceq_G \mathrm{Aut}(\mathcal{A})$, there exists $\gamma \in \mathrm{Aut}(\mathcal{A})$ such that $H^\gamma \leq K$. The S-ring $V(K, G)$ is a CI-S-ring by the assumption of the lemma. So K is G-transjugate by Lemma 4.1. Therefore H^γ and G_{right} are K-conjugate and hence H and G_{right} are $\mathrm{Aut}(\mathcal{A})$-conjugate. Thus, $\mathrm{Aut}(\mathcal{A})$ is G-transjugate and $\mathcal{A}$ is a CI-S-ring by Lemma 4.1.   □

It should be mentioned that the proof of Lemma 4.4 is similar to the proof of [20, Lemma 3.6].

5   Generalized wreath and star products

Let $\mathcal{A}$ be an S-ring over a group G and $S = U/L$ an $\mathcal{A}$-section of G. An S-ring $\mathcal{A}$ is called the *S-wreath product* or the *generalized wreath product* of $\mathcal{A}_U$ and $\mathcal{A}_{G/L}$ if $L \trianglelefteq G$ and $L \leq \mathrm{rad}(X)$ for each basic set X outside U. In this case we write $\mathcal{A} = \mathcal{A}_U \wr_S \mathcal{A}_{G/L}$ and omit S when $U = L$. The construction of the generalized wreath product of S-rings was introduced in [12].

The S-wreath product is called *nontrivial* or *proper* if $L \neq \{e\}$ and $U \neq G$. An S-ring $\mathcal{A}$ is said to be *decomposable* if $\mathcal{A}$ is the nontrivial S-wreath product for some $\mathcal{A}$-section S of G; otherwise $\mathcal{A}$ is said to be indecomposable. We say that an $\mathcal{A}$-subgroup $U < G$ has a *gwr-complement* with respect to $\mathcal{A}$ if there exists a nontrivial normal $\mathcal{A}$-subgroup L of G such that $L \leq U$ and $\mathcal{A}$ is the S-wreath product, where $S = U/L$.

Lemma 5.1 ([19, Theorem 1.1]). *Let $G \in \mathcal{E}$, $\mathcal{A}$ an S-ring over G, and $S = U/L$ an $\mathcal{A}$-section of G. Suppose that $\mathcal{A}$ is the nontrivial S-wreath product and the S-rings $\mathcal{A}_U$ and $\mathcal{A}_{G/L}$ are CI-S-rings. Then $\mathcal{A}$ is a CI-S-ring whenever*

$$\mathrm{Aut}_S(\mathcal{A}_S) = \mathrm{Aut}_U(\mathcal{A}_U)^S \, \mathrm{Aut}_{G/L}(\mathcal{A}_{G/L})^S.$$

In particular, $\mathcal{A}$ is a CI-S-ring if

$$\mathrm{Aut}_S(\mathcal{A}_S) = \mathrm{Aut}_U(\mathcal{A}_U)^S \quad or \quad \mathrm{Aut}_S(\mathcal{A}_S) = \mathrm{Aut}_{G/L}(\mathcal{A}_{G/L})^S.$$

Lemma 5.2 ([19, Proposition 4.1]). *In the conditions of Lemma 5.1, suppose that $\mathcal{A}_S = \mathbb{Z}S$. Then $\mathcal{A}$ is a CI-S-ring. In particular, if $U = L$ then $\mathcal{A}$ is a CI-S-ring.*

Lemma 5.3 ([20, Lemma 4.2]). *In the conditions of Lemma 5.1, suppose that at least one of the S-rings $\mathcal{A}_U$ and $\mathcal{A}_{G/L}$ is cyclotomic and $\mathcal{A}_S$ is Cayley minimal. Then $\mathcal{A}$ is a CI-S-ring.*

Lemma 5.4. *Let $\mathcal{A}$ be an S-ring over an abelian group G. Suppose that $\mathcal{A}$ is the nontrivial $S = U/L$-wreath product for some $\mathcal{A}$-section $S = U/L$ and L_1 is an $\mathcal{A}$-subgroup containing L. Then $\mathcal{B} = V(K, G)$, where $K = \mathrm{Aut}(\mathcal{A})_{G/L_1} G_{\mathrm{right}}$, is also the S-wreath product.*

Proof. Since $K \leq \mathrm{Aut}(\mathcal{A})$, from Equations (3.1) and (3.2) it follows that

$$\mathcal{B} = V(K, G) \geq V(\mathrm{Aut}(\mathcal{A}), G) \geq \mathcal{A}.$$

So U and L are also $\mathcal{B}$-subgroups.

Let $\mathcal{C} = \mathbb{Z}U \wr_S \mathbb{Z}(G/L)$. The S-rings $\mathcal{C}_U$ and $\mathcal{C}_{G/L}$ are schurian and $\mathcal{C}_S$ is 2-minimal because $\mathcal{C}_S = \mathbb{Z}S$. So $\mathcal{C}$ is schurian by [26, Corollary 10.3]. This implies that

$$\mathcal{C} = V(\mathrm{Aut}(\mathcal{C}), G). \tag{5.1}$$

Every element from $\mathrm{Aut}(\mathcal{C})_e$ fixes every basic set of $\mathcal{C}$ and hence it fixes every L-coset. Since $L_1 \geq L$, every element from $\mathrm{Aut}(\mathcal{C})_e$ fixes every L_1-coset. We conclude that

$\mathrm{Aut}(\mathcal{C})_e \leq \mathrm{Aut}(\mathcal{A})_{G/L_1}$ and hence $\mathrm{Aut}(\mathcal{C}) \leq K$. Now from Equations (3.1) and (5.1) it follows that

$$\mathcal{C} = V(\mathrm{Aut}(\mathcal{C}), G) \geq V(K, G) = \mathcal{B}. \tag{5.2}$$

The group U is a $\mathcal{B}$- and a $\mathcal{C}$-subgroup. Due to Equation (5.2), every basic set of $\mathcal{B}$ which lies outside U is a union of some basic sets of $\mathcal{C}$ which lie outside U. So $L \leq \mathrm{rad}(X)$ for every $X \in \mathcal{S}(\mathcal{B})$ outside U. Thus, $\mathcal{B}$ is the S-wreath product. $\square$

Lemma 5.5. *In the conditions of Lemma 5.1, suppose that: (1) every S-ring over U is a CI-S-ring; (2) $\mathcal{A}_{G/L}$ is 2-minimal or normal. Then $\mathcal{A}$ is a CI-S-ring.*

Proof. Let $\mathcal{B} = V(K, G)$, where $K = \mathrm{Aut}(\mathcal{A})_{G/L}G_{\mathrm{right}}$. From Lemma 5.4 it follows that $\mathcal{B}$ is the S-wreath product. Since $L_1 = L$, the definition of $\mathcal{B}$ implies that $\mathcal{B}_{G/L} = \mathbb{Z}(G/L)$ and hence $\mathcal{B}_S = \mathbb{Z}S$. Clearly, $\mathcal{B}_{G/L}$ is a CI-S-ring. The S-ring $\mathcal{B}_U$ is a CI-S-ring by the assumption of the lemma. Therefore $\mathcal{B}$ is a CI-S-ring by Lemma 5.2. The S-ring $\mathcal{A}_{G/L}$ is a CI-S-ring by the assumption of the lemma. Thus, $\mathcal{A}$ is a CI-S-ring by [20, Lemma 3.6] whenever $\mathcal{A}_{G/L}$ is 2-minimal and by Lemma 4.4 whenever $\mathcal{A}_{G/L}$ is normal. $\square$

Let V and W be $\mathcal{A}$-subgroups. The S-ring $\mathcal{A}$ is called the *star product* of $\mathcal{A}_V$ and $\mathcal{A}_W$ if the following conditions hold:

(1) $V \cap W \unlhd W$;

(2) each $T \in \mathcal{S}(\mathcal{A})$ with $T \subseteq (W \setminus V)$ is a union of some $V \cap W$-cosets;

(3) for each $T \in \mathcal{S}(\mathcal{A})$ with $T \subseteq G \setminus (V \cup W)$ there exist $R \in \mathcal{S}(\mathcal{A}_V)$ and $S \in \mathcal{S}(\mathcal{A}_W)$ such that $T = RS$.

In this case we write $\mathcal{A} = \mathcal{A}_V \star \mathcal{A}_W$. The construction of the star product of S-rings was introduced in [15]. The star product is called *nontrivial* if $V \neq \{e\}$ and $V \neq G$. If $V \cap W = \{e\}$ then the star product is the usual *tensor product* of $\mathcal{A}_V$ and $\mathcal{A}_W$ (see [11, p. 5]). In this case we write $\mathcal{A} = \mathcal{A}_V \otimes \mathcal{A}_W$. One can check that if $\mathcal{A} = \mathcal{A}_V \otimes \mathcal{A}_W$ then $\mathrm{Aut}(\mathcal{A}) = \mathrm{Aut}(\mathcal{A}_V) \times \mathrm{Aut}(\mathcal{A}_W)$. If $V \cap W \neq \{e\}$ then $\mathcal{A}$ is the nontrivial $V/(V \cap W)$-wreath product. Indeed, let $T \in \mathcal{S}(\mathcal{A})$ such that $T \nsubseteq V$. If $T \subseteq W \setminus V$ then $V \cap W \leq \mathrm{rad}(T)$ by Condition (2) of the definition. If $T \subseteq G \setminus (V \cup W)$ then $T = RS$ for some $R \in \mathcal{S}(\mathcal{A}_V)$ and some $S \in \mathcal{S}(\mathcal{A}_W)$ such that $S \subseteq W \setminus V$ by Condition (3) of the definition. Since $V \cap W \leq \mathrm{rad}(S)$, we obtain $V \cap W \leq \mathrm{rad}(T)$.

Lemma 5.6. *Let $G \in \mathcal{E}$ and $\mathcal{A}$ a schurian S-ring over G. Suppose that $\mathcal{A} = \mathcal{A}_V \star \mathcal{A}_W$ for some $\mathcal{A}$-subgroups V and W of G and the S-rings $\mathcal{A}_V$ and $\mathcal{A}_{W/(V \cap W)}$ are CI-S-rings. Then $\mathcal{A}$ is a CI-S-ring.*

Proof. The statement of the lemma follows from [18, Proposition 3.2, Theorem 4.1]. $\square$

Lemma 5.7 ([13, Lemma 2.8]). *Let $\mathcal{A}$ be an S-ring over an abelian group $G = G_1 \times G_2$. Assume that G_1 and G_2 are $\mathcal{A}$-groups. Then $\mathcal{A} = \mathcal{A}_{G_1} \otimes \mathcal{A}_{G_2}$ whenever $\mathcal{A}_{G_1}$ or $\mathcal{A}_{G_2}$ is the group ring.*

Lemma 5.8. *In the conditions of Lemma 5.1, suppose that $|G : U|$ is a prime and there exists $X \in \mathcal{S}(\mathcal{A}_{G/L})$ outside S with $|X| = 1$. Then $\mathcal{A}$ is a CI-S-ring.*

Proof. Let $X = \{x\}$ for some $x \in G/L$. Due to $G \in \mathcal{E}$, we conclude that $|\langle x \rangle|$ is prime. So $|\langle x \rangle \cap S| = 1$ because x lies outside S. Since $|G : U|$ is a prime, $G/L = \langle x \rangle \times S$. Note that $\mathcal{A}_{\langle x \rangle} = \mathbb{Z}\langle x \rangle$. Therefore

$$\mathcal{A}_{G/L} = \mathbb{Z}\langle x \rangle \otimes \mathcal{A}_S$$

by Lemma 5.7.

Let $\varphi \in \mathrm{Aut}_S(\mathcal{A}_S)$. Define $\psi \in \mathrm{Aut}(G/L)$ in the following way:

$$\psi^S = \varphi, \ x^\psi = x.$$

Then $\psi \in \mathrm{Aut}_{G/L}(\mathcal{A}_{G/L})$ because $\mathcal{A}_{G/L} = \mathbb{Z}\langle x \rangle \otimes \mathcal{A}_S$. We obtain that

$$\mathrm{Aut}_{G/L}(\mathcal{A}_{G/L})^S \geq \mathrm{Aut}_S(\mathcal{A}_S),$$

and hence $\mathrm{Aut}_{G/L}(\mathcal{A}_{G/L})^S = \mathrm{Aut}_S(\mathcal{A}_S)$. Thus, $\mathcal{A}$ is a CI-S-ring by Lemma 5.1. □

6 p-S-rings

Let p be a prime. An S-ring $\mathcal{A}$ over a p-group G is called a p-S-*ring* if every basic set of $\mathcal{A}$ has a p-power size. Clearly, if $|G| = p$ then $\mathcal{A} = \mathbb{Z}G$. In the next three lemmas G is a p-group and $\mathcal{A}$ is a p-S-ring over G.

Lemma 6.1. *If $\mathcal{B} \geq \mathcal{A}$ then $\mathcal{B}$ is a p-S-ring.*

Proof. The statement of the lemma follows from [29, Theorem 1.1]. □

Lemma 6.2. *Let $S = U/L$ be an $\mathcal{A}$-section of G. Then $\mathcal{A}_S$ is a p-S-ring.*

Proof. From Lemma 2.1 it follows that for every $X \in \mathcal{S}(\mathcal{A})$ the number $\lambda = |X \cap Lx|$ does not depend on $x \in X$. So λ divides $|X|$ and hence λ is a p-power. Let $\pi \colon G \to G/L$ be the canonical epimorphism. Note that $|\pi(X)| = |X|/\lambda$ and hence $|\pi(X)|$ is a p-power. Therefore every basic set of $\mathcal{A}_S$ has a p-power size. Thus, $\mathcal{A}_S$ is a p-S-ring. □

Lemma 6.3 ([13, Proposition 2.13]). *The following statements hold:*

(1) $|\mathbf{O}_\theta(\mathcal{A})| > 1$;

(2) *there exists a chain of $\mathcal{A}$-subgroups $\{e\} = G_0 < G_1 < \cdots < G_s = G$ such that $|G_{i+1} : G_i| = p$ for every $i \in \{0, \ldots, s-1\}$.*

Lemma 6.4. *Let G be an abelian group, $K \in \mathrm{Sup}_2^{\min}(G_{\mathrm{right}})$, and $\mathcal{A} = V(K, G)$. Suppose that H is an $\mathcal{A}$-subgroup of G such that G/H is a p-group for some prime p. Then $\mathcal{A}_{G/H}$ is a p-S-ring.*

Proof. The statement of the lemma follows from [18, Lemma 5.2]. □

7 S-rings over an abelian group of non-powerful order

A number n is called *powerful* if p^2 divides n for every prime divisor p of n. From now throughout this section $G = H \times P$, where H is an abelian group and $P \cong C_p$, where p is a prime coprime to $|H|$. Clearly, $|G|$ is non-powerful. Let $\mathcal{A}$ be an S-ring over G, H_1 a maximal $\mathcal{A}$-subgroup contained in H, and P_1 the least $\mathcal{A}$-subgroup containing P. Note that $H_1 P_1$ is an $\mathcal{A}$-subgroup.

Lemma 7.1 ([20, Lemma 6.3]). *In the above notations, if $H_1 \neq (H_1P_1)_{p'}$, the Hall p'-subgroup of H_1P_1, then $\mathcal{A}_{H_1P_1} = \mathcal{A}_{H_1} \star \mathcal{A}_{P_1}$.*

Lemma 7.2 ([27, Proposition 15]). *In the above notations, if $\mathcal{A}_{H_1P_1/H_1} \cong \mathbb{Z}C_p$ then $\mathcal{A}_{H_1P_1} = \mathcal{A}_{H_1} \star \mathcal{A}_{P_1}$.*

Lemma 7.3 ([11, Lemma 6.2]). *In the above notations, suppose that $H_1 < H$. Then one of the following statements holds:*

(1) *$\mathcal{A} = \mathcal{A}_{H_1} \wr \mathcal{A}_{G/H_1}$ with $\mathrm{rk}(\mathcal{A}_{G/H_1}) = 2$;*

(2) *$\mathcal{A} = \mathcal{A}_{H_1P_1} \wr_S \mathcal{A}_{G/P_1}$, where $S = H_1P_1/P_1$ and $P_1 < G$.*

8 S-rings over C_2^n, $n \leq 5$

All S-rings over the groups C_2^n, where $n \leq 5$, were enumerated with the help of the GAP package COCO2P [16]. The list of all S-rings over these groups is available on the web-page [30] (see also [35]). The next lemma is an immediate consequence of the above computational results (see also [11, Theorem 1.2]).

Lemma 8.1. *Every S-ring over C_2^n, where $n \leq 5$, is schurian.*

To prove Theorem 1.1, we will show that every schurian S-ring over $C_2^5 \times C_p$ is CI. Since the most of schurian S-rings over $C_2^5 \times C_p$ are generalized wreath or star products of S-rings over its proper subgroups, we need to check that all schurian S-rings over proper subgroups of $C_2^5 \times C_p$ are CI. In this section we will do it for $G \cong C_2^n$, where $n \leq 5$. Note that G is a DCI-group by [2, 7] but this does not imply that every S-ring over G is CI (see Remark 4.3). We will describe 2-S-rings over G using computational results and check that all S-rings over G are CI. Until the end of the section G is an elementary abelian 2-group of rank n and $\mathcal{A}$ is a 2-S-ring over G.

Lemma 8.2. *Let $n \leq 3$. Then $\mathcal{A}$ is cyclotomic. Moreover, $\mathcal{A}$ is Cayley minimal except for the case when $n = 3$ and $\mathcal{A} \cong \mathbb{Z}C_2 \wr \mathbb{Z}C_2 \wr \mathbb{Z}C_2$.*

Proof. The first part of the lemma follows from [20, Lemma 5.2]; the second part follows from [20, Lemma 5.3].   □

Analyzing the lists of all S-rings over C_2^4 and C_2^5 available on the web-page [30], we conclude that up to isomorphism there are exactly nineteen 2-S-rings over G if $n = 4$ and there are exactly one hundred 2-S-rings over G if $n = 5$. It can can be established by inspecting the above 2-S-rings one after the other that there are exactly fifteen decomposable and four indecomposable 2-S-rings over G if $n = 4$ and there are exactly ninety six decomposable and four indecomposable 2-S-rings over G if $n = 5$.

Lemma 8.3. *Let $n \in \{4, 5\}$ and $\mathcal{A}$ indecomposable. Then $\mathcal{A}$ is normal. If in addition $n = 5$ then $\mathcal{A} \cong \mathbb{Z}C_2 \otimes \mathcal{A}'$, where $\mathcal{A}'$ is indecomposable 2-S-ring over C_2^4.*

Proof. Let $n = 4$. One can compute $|\mathrm{Aut}(\mathcal{A})|$ and $|N_{\mathrm{Aut}(\mathcal{A})}(G_{\mathrm{right}})|$ using the GAP package COCO2P [16]. It turns out that for each of the four indecomposable 2-S-rings over G the equality

$$|\mathrm{Aut}(\mathcal{A})| = |N_{\mathrm{Aut}(\mathcal{A})}(G_{\mathrm{right}})|$$

is attained. So every indecomposable 2-S-ring over G is normal whenever $n = 4$.

Let $n = 5$. The straightforward check for each of the four indecomposable 2-S-rings over G yields that $\mathcal{A} = \mathcal{A}_H \otimes \mathbb{Z}L$, where $H \cong C_2^4$, $L \cong C_2$, and $\mathcal{A}_H$ is indecomposable 2-S-ring. Clearly, $\mathbb{Z}L$ is normal. By the above paragraph, $\mathcal{A}_H$ is normal. Since $\mathrm{Aut}(\mathcal{A}) = \mathrm{Aut}(\mathcal{A}_H) \times \mathrm{Aut}(\mathcal{A}_L)$, we obtain that $\mathcal{A}$ is normal. $\qquad\square$

Note that if $p > 2$ then Lemma 8.3 does not hold. In fact, if $p > 2$ then there exists an indecomposable p-S-ring over C_p^5 which is not normal (see [13, Lemma 6.4]).

Lemma 8.4. *Let $n \leq 5$. Then $\mathcal{A}$ is normal whenever one of the following statements holds:*

(1) $\mathcal{A}$ is indecomposable;

(2) $|G : \mathbf{O}_\theta(\mathcal{A})| = 2$;

(3) $n = 4$ and $\mathcal{A} \cong (\mathbb{Z}C_2 \wr \mathbb{Z}C_2) \otimes (\mathbb{Z}C_2 \wr \mathbb{Z}C_2)$.

Proof. If $n \leq 3$ and $\mathcal{A}$ is indecomposable then $\mathcal{A} = \mathbb{Z}G$ by [20, Lemma 5.2]. Clearly, in this case $\mathcal{A}$ is normal. If $n \in \{4, 5\}$ and $\mathcal{A}$ is indecomposable then $\mathcal{A}$ is normal by Lemma 8.3. There are exactly $n - 1$ 2-S-rings over G for which Statement (2) of the lemma holds. For every $\mathcal{A}$ isomorphic to one of these 2-S-rings and for $\mathcal{A} \cong (\mathbb{Z}C_2 \wr \mathbb{Z}C_2) \otimes (\mathbb{Z}C_2 \wr \mathbb{Z}C_2)$ one can compute $|\mathrm{Aut}(\mathcal{A})|$ and $|N_{\mathrm{Aut}(\mathcal{A})}(G_{\mathrm{right}})|$ using the GAP package COCO2P [16]. It turns out that in each case the equality $|\mathrm{Aut}(\mathcal{A})| = |N_{\mathrm{Aut}(\mathcal{A})}(G_{\mathrm{right}})|$ holds and hence $\mathcal{A}$ is normal. $\qquad\square$

Lemma 8.5. *Let $n = 4$. Then $\mathcal{A}$ is cyclotomic.*

Proof. If $\mathcal{A}$ is decomposable then $\mathcal{A}$ is cyclotomic by [20, Lemma 5.6]. If $\mathcal{A}$ is indecomposable then $\mathcal{A}$ is normal by Lemma 8.3. This implies that

$$\mathrm{Aut}(\mathcal{A})_e = (N_{\mathrm{Aut}(\mathcal{A})}(G_{\mathrm{right}}))_e \leq \mathrm{Aut}(G).$$

The S-ring $\mathcal{A}$ is schurian by Lemma 8.1. So from Equation (3.2) it follows that $\mathcal{A} = V(\mathrm{Aut}(\mathcal{A}), G)$ and hence $\mathcal{A} = \mathrm{Cyc}(\mathrm{Aut}(\mathcal{A})_e, G)$. $\qquad\square$

Lemma 8.6. *Let $n = 5$. Suppose that $\mathcal{A}$ is decomposable and $|\mathbf{O}_\theta(\mathcal{A})| = 8$. Then $\mathcal{A}$ is cyclotomic.*

Proof. Let $\mathcal{A}$ be the nontrivial S-wreath product for some $\mathcal{A}$-section $S = U/L$. Note that $|U| \leq 16$, $|G/L| \leq 16$, and $|S| \leq 8$. The S-rings $\mathcal{A}_U$, $\mathcal{A}_{G/L}$, and $\mathcal{A}_S$ are 2-S-rings by Lemma 6.2. So each of these S-rings is cyclotomic by Lemma 8.2 whenever the order of the corresponding group is at most 8 and by Lemma 8.5 otherwise. Since $|\mathbf{O}_\theta(\mathcal{A})| = 8$, we conclude that $|S| \leq 4$ or $|S| = 8$ and $|\mathbf{O}_\theta(\mathcal{A}_S)| \geq 4$. In both cases $\mathcal{A}_S$ is Cayley minimal by Lemma 8.2. This implies that

$$\mathrm{Aut}_U(\mathcal{A}_U)^S = \mathrm{Aut}_{G/L}(\mathcal{A}_{G/L})^S = \mathrm{Aut}_S(\mathcal{A}_S).$$

Now from [20, Lemma 4.3] it follows that $\mathcal{A}$ is cyclotomic. $\qquad\square$

In the next two lemmas we establish some properties of decomposable 2-S-rings over $G \cong C_2^5$ whose thin radical is of size 2 or 4. These properties will be used in the proof of Theorem 1.1. The statements of Lemma 8.7 and Lemma 8.8 can be verified by analysis of computational results obtained with the help of the GAP package COCO2P [16]. For every decomposable 2-S-ring $\mathcal{A}$ with $|\mathbf{O}_\theta(\mathcal{A})| \in \{2, 4\}$ over G (see the list [30]), we compute all $\mathcal{A}$-subgroups, automorphism groups, and Cayley automorphism groups of some restrictions and quotients.

Lemma 8.7. *Let $n = 5$. Suppose that $\mathcal{A}$ is decomposable and $|\mathbf{O}_\theta(\mathcal{A})| = 4$. Then one of the following statements holds:*

(1) *there exists an $\mathcal{A}$-subgroup $L \leq \mathbf{O}_\theta(\mathcal{A})$ of order 2 such that $\mathcal{A} = \mathbb{Z}\mathbf{O}_\theta(\mathcal{A}) \wr_S \mathcal{A}_{G/L}$, where $S = \mathbf{O}_\theta(\mathcal{A})/L$;*

(2) *$|\mathrm{Aut}_G(\mathcal{A})| \geq |\mathrm{Aut}_U(\mathcal{A}_U)|$ for every $\mathcal{A}$-subgroup U with $|U| = 16$ and $U \geq \mathbf{O}_\theta(\mathcal{A})$;*

(3) *$\mathcal{A}$ is normal;*

(4) *there exist an $\mathcal{A}$-subgroup $L \leq \mathbf{O}_\theta(\mathcal{A})$ and $X \in \mathcal{S}(\mathcal{A})$ such that $|L| = |X| = 2$, $L \neq \mathrm{rad}(X)$, and $\mathcal{A}_{G/L}$ is normal.*

Lemma 8.8. *Let $n = 5$. Suppose that $\mathcal{A}$ is decomposable, $|\mathbf{O}_\theta(\mathcal{A})| = 2$, and there exists $X \in \mathcal{S}(\mathcal{A})$ with $|X| > 1$ and $|\mathrm{rad}(X)| = 1$. Then $|X| = 4$ and one of the following statements holds:*

(1) *$\mathcal{A} \cong \mathcal{B} \wr \mathbb{Z}C_2$, where $\mathcal{B}$ is a 2-S-ring over C_2^4;*

(2) *$|\mathrm{Aut}_G(\mathcal{A})| \geq |\mathrm{Aut}_U(\mathcal{A}_U)|$ for every $\mathcal{A}$-subgroup U with $|U| = 16$;*

(3) *there exists an $\mathcal{A}$-subgroup L such that $|L| \in \{2, 4\}$ and $\mathcal{A}_{G/L}$ is normal.*

Lemma 8.9. *Let $D \in \mathcal{E}$ such that every S-ring over a proper section of D is CI, $\mathcal{D}$ an S-ring over D, and $S = U/L$ a $\mathcal{D}$-section. Suppose that $\mathcal{D}$ is the nontrivial S-wreath product. Then $\mathcal{D}$ is a CI-S-ring whenever $D/L \cong C_2^k$ for some $k \leq 4$ and $\mathcal{D}_{D/L}$ is a 2-S-ring.*

Proof. The S-ring $\mathcal{D}_{D/L}$ is cyclotomic by Lemma 8.2 whenever $|D/L| \leq 8$ and by Lemma 8.5 whenever $|D/L| = 16$. The S-ring $\mathcal{D}_S$ is a 2-S-ring by Lemma 6.2. If $\mathcal{D}_S \not\cong \mathbb{Z}C_2 \wr \mathbb{Z}C_2 \wr \mathbb{Z}C_2$ then $\mathcal{D}_S$ is Cayley minimal by Lemma 8.2. The S-rings $\mathcal{D}_U$ and $\mathcal{D}_{D/L}$ are CI-S-rings by the assumption of the lemma. So $\mathcal{D}$ is a CI-S-ring by Lemma 5.3.

Assume that
$$\mathcal{D}_S \cong \mathbb{Z}C_2 \wr \mathbb{Z}C_2 \wr \mathbb{Z}C_2.$$

In this case $|D/L| = 16$, $|S| = 8$, and there exists the least $\mathcal{D}_S$-subgroup A of S of order 2. Every basic set of $\mathcal{D}_{D/L}$ outside S is contained in an S-coset because $\mathcal{D}_{(D/L)/S} \cong \mathbb{Z}C_2$. So $\mathrm{rad}(X)$ is a $\mathcal{D}_S$-subgroup for every $X \in \mathcal{S}(\mathcal{D}_{D/L})$ outside S. If $|\mathrm{rad}(X)| > 1$ for every $X \in \mathcal{S}(\mathcal{D}_{D/L})$ outside S then $\mathcal{D}_{D/L}$ is the S/A-wreath product because A is the least $\mathcal{D}_S$-subgroup. This implies that $\mathcal{D}$ is the $U/\pi^{-1}(A)$-wreath product, where $\pi \colon D \to D/L$ is the canonical epimorphism. One can see that $|D/\pi^{-1}(A)| \leq 8$ and $|U/\pi^{-1}(A)| \leq 4$. The S-rings $\mathcal{D}_{D/\pi^{-1}(A)}$ and $\mathcal{D}_{U/\pi^{-1}(A)}$ are 2-S-rings by Lemma 6.2. The S-ring $\mathcal{D}_{D/\pi^{-1}(A)}$ is cyclotomic by Lemma 8.2 and the S-ring $\mathcal{D}_{U/\pi^{-1}(A)}$ is Cayley minimal by

Lemma 8.2. The S-rings $\mathcal{D}_U$ and $\mathcal{D}_{D/\pi^{-1}(A)}$ are CI-S-rings by the assumption of the lemma. Thus, $\mathcal{D}$ is a CI-S-ring by Lemma 5.3.

Suppose that there exists a basic set X of $\mathcal{D}_{D/L}$ outside S with $|\operatorname{rad}(X)| = 1$. If $\mathcal{D}_{D/L}$ is decomposable then

$$\operatorname{Aut}_{D/L}(\mathcal{D}_{D/L})^S = \operatorname{Aut}_S(\mathcal{D}_S)$$

by [20, Lemma 5.8]. Therefore $\mathcal{D}$ is a CI-S-ring by Lemma 5.1.

If $\mathcal{D}_{D/L}$ is indecomposable then $\mathcal{D}_{D/L}$ is normal by Lemma 8.3. So all conditions of Lemma 5.5 hold for $\mathcal{D}$. Thus, $\mathcal{D}$ is a CI-S-ring. □

Lemma 8.10. *Let $n \leq 5$. Then every S-ring over G is a CI-S-ring.*

Proof. Every S-ring over G is schurian by Lemma 8.1. So to prove the lemma, it is sufficient to prove that $\mathcal{B} = V(K, G)$ is a CI-S-ring for every $K \in \operatorname{Sup}_2^{\min}(G_{\text{right}})$ (see Remark 4.3). The S-ring $\mathcal{B}$ is a 2-S-ring by Lemma 6.4. If $n \leq 4$ then $\mathcal{B}$ is CI by [20, Lemma 5.7]. Thus, if $n = 4$ then the statement of the lemma holds.

Let $n = 5$. Suppose that $\mathcal{B}$ is indecomposable. Then the second part of Lemma 8.3 implies $\mathcal{B} \cong \mathbb{Z}C_2 \otimes \mathcal{B}'$, where $\mathcal{B}'$ is indecomposable 2-S-ring over C_2^4. Since $\mathcal{B}$ is schurian by Lemma 8.1 and every S-ring over an elementary abelian group of rank at most 4 is CI by the above paragraph, we conclude that $\mathcal{B}$ is a CI-S-ring by Lemma 5.6.

Now suppose that $\mathcal{B}$ is decomposable, i.e. $\mathcal{B}$ is the nontrivial $S = U/L$-wreath product for some $\mathcal{B}$-section $S = U/L$. Clearly, $|G/L| \leq 16$. The S-ring $\mathcal{B}_{G/L}$ is a 2-S-ring by Lemma 6.2. Since every S-ring over an elementary abelian group of rank at most 4 is CI, $\mathcal{B}$ is a CI-S-ring by Lemma 8.9. □

9   Proof of Theorem 1.1

Let $G = H \times P$, where $H \cong C_2^5$ and $P \cong C_p$, where p is a prime. These notations are valid until the end of the paper. If $p = 2$ then G is not a DCI-group by [28]. So in view of Lemma 4.2, to prove Theorem 1.1, it is sufficient to prove the following theorem.

Theorem 9.1. *Let p be an odd prime and $K \in \operatorname{Sup}_2^{\min}(G_{\text{right}})$. Then $\mathcal{A} = V(K, G)$ is a CI-S-ring.*

The proof of Proposition 9.1 will be given at the end of the section. We start with the next lemma concerned with proper sections of G.

Lemma 9.2. *Let S be a section of G such that $S \neq G$. Then every schurian S-ring over S is a CI-S-ring.*

Proof. If $S \cong C_2^n$ for some $n \leq 5$ then we are done by Lemma 8.10. Suppose that $S \cong C_2^n \times C_p$ for some $n \leq 4$. Then the statement of the lemma follows from [20, Remark 3.4] whenever $n \leq 3$ and from [20, Remark 3.4, Theorem 7.1] whenever $n = 4$. □

A key step towards the proof of Theorem 9.1 is the following lemma.

Lemma 9.3. *Let $\mathcal{A}$ be an S-ring over G and U an $\mathcal{A}$-subgroup with $U \geq P$. Suppose that P is an $\mathcal{A}$-subgroup, $\mathcal{A}$ is the nontrivial S-wreath product, where $S = U/P$, $|S| = 16$, and $\mathcal{A}_{G/P}$ is a 2-S-ring. Then $\mathcal{A}$ is a CI-S-ring.*

Proof. Firstly we prove two lemmas concerned with some special cases of Lemma 9.3.

Lemma 9.4. *Suppose that S has a gwr-complement with respect to $\mathcal{A}_{G/P}$. Then $\mathcal{A}$ is a CI-S-ring.*

Proof. The condition of the lemma implies that there exists an $\mathcal{A}_{G/P}$-subgroup A such that $\mathcal{A}_{G/P}$ is the nontrivial S/A-wreath product. This means that $\mathcal{A}$ is the nontrivial $U/\pi^{-1}(A)$-wreath product, where $\pi \colon G \to G/P$ is the canonical epimorphism. Note that $|G/\pi^{-1}(A)| \leq 16$ and $\mathcal{A}_{G/\pi^{-1}(A)} \cong \mathcal{A}_{(G/P)/A}$ is a 2-S-ring by Lemma 6.2. Therefore $\mathcal{A}$ is a CI-S-ring by Lemma 9.2 and Lemma 8.9.   □

Lemma 9.5. *Suppose that S does not have a gwr-complement with respect to $\mathcal{A}_{G/P}$. Then*

$$|\operatorname{Aut}_{G/P}(\mathcal{A}_{G/P})^S| = |\operatorname{Aut}_{G/P}(\mathcal{A}_{G/P})|.$$

Proof. To prove the lemma it is sufficient to prove that the group

$$(\operatorname{Aut}_{G/P}(\mathcal{A}_{G/P}))_S = \{\varphi \in \operatorname{Aut}_{G/P}(\mathcal{A}_{G/P}) : \ \varphi^S = \operatorname{id}_S\}$$

is trivial. Let $\varphi \in (\operatorname{Aut}_{G/P}(\mathcal{A}_{G/P}))_S$. Put $\mathcal{C} = \operatorname{Cyc}(\langle\varphi\rangle, G/P)$. Clearly, $\langle\varphi\rangle \leq \operatorname{Aut}(\mathcal{A}_{G/P})$. So from Equations (3.1) and (3.2) it follows that $\mathcal{C} \geq \mathcal{A}_{G/P}$. Lemma 6.1 yields that $\mathcal{C}$ is a 2-S-ring. Since $\varphi^S = \operatorname{id}_S$, we conclude that $\mathbf{O}_\theta(\mathcal{C}) \geq S$.

If $\mathcal{C} \neq \mathbb{Z}(G/P)$ then $\mathbf{O}_\theta(\mathcal{C}) = S$. Therefore $\mathcal{C} = \mathbb{Z}S \wr_{S/A} \mathbb{Z}((G/P)/A)$ for some $\mathcal{C}$-subgroup A by Statement (i) of [19, Proposition 4.3]. This implies that $\mathcal{A}_{G/P} = \mathcal{A}_S \wr_{S/A} \mathcal{A}_{((G/P)/A)}$ because $\mathcal{C} \geq \mathcal{A}_{G/P}$ and S is both $\mathcal{A}_{G/P}$, $\mathcal{C}$-subgroup. We obtain a contradiction with the assumption of the lemma. Thus, $\mathcal{C} = \mathbb{Z}(G/P)$ and hence φ is trivial. So the group $(\operatorname{Aut}_{G/P}(\mathcal{A}_{G/P}))_S$ is trivial.   □

If $\mathcal{A}_{G/P}$ is indecomposable then $\mathcal{A}_{G/P}$ is normal by Lemma 8.3. So $\mathcal{A}$ is a CI-S-ring by Lemma 9.2 and Lemma 5.5. Further we assume that $\mathcal{A}_{G/P}$ is decomposable. Due to Lemma 9.4, we may assume also that

$$S \text{ does not have a gwr-complement with respect to } \mathcal{A}_{G/P}. \tag{9.1}$$

If there exists $X \in \mathcal{S}(\mathcal{A}_{G/P})$ outside S with $|X| = 1$ then $\mathcal{A}$ is a CI-S-ring by Lemma 9.2 and Lemma 5.8. So we may assume that

$$\mathbf{O}_\theta(\mathcal{A}_{G/P}) \leq S. \tag{9.2}$$

Note that $|\mathbf{O}_\theta(\mathcal{A}_{G/P})| > 1$ by Statement (1) of Lemma 6.3 and $|\mathbf{O}_\theta(\mathcal{A}_{G/P})| \leq 16$ by Equation (9.2). So $|\mathbf{O}_\theta(\mathcal{A}_{G/P})| \in \{2, 4, 8, 16\}$. We divide the rest of the proof into four cases depending on $|\mathbf{O}_\theta(\mathcal{A}_{G/P})|$.

Case 1: $|\mathbf{O}_\theta(\mathcal{A}_{G/P})| = 16$**.**

Due to Equation (9.2), we conclude that $\mathcal{A}_S = \mathbb{Z}S$. So $\mathcal{A}$ is a CI-S-ring by Lemma 9.2 and Lemma 5.2.

Case 2: $|\mathbf{O}_\theta(\mathcal{A}_{G/P})| = 8$**.**

Since $\mathcal{A}_{G/P}$ is decomposable, Lemma 8.6 implies that $\mathcal{A}_{G/P}$ is cyclotomic. The S-ring $\mathcal{A}_S$ is a 2-S-ring by Lemma 6.2. In view of Equation (9.2), we obtain that $|\mathbf{O}_\theta(\mathcal{A}_S)| = 8$. So Statement (ii) of [19, Proposition 4.3] yields that the S-ring $\mathcal{A}_S$ is Cayley minimal. Thus, $\mathcal{A}$ is a CI-S-ring by Lemma 9.2 and Lemma 5.3.

Case 3: $|\mathbf{O}_\theta(\mathcal{A}_{G/P})| = 4$.

In this case one of the statements of Lemma 8.7 holds for $\mathcal{A}_{G/P}$. If Statement (1) of Lemma 8.7 holds for $\mathcal{A}_{G/P}$ then we obtain a contradiction with Equation (9.1).

If Statement (2) of Lemma 8.7 holds for $\mathcal{A}_{G/P}$ then $|\operatorname{Aut}_{G/P}(\mathcal{A}_{G/P})| \geq |\operatorname{Aut}_S(\mathcal{A}_S)|$. From Lemma 9.5 it follows that $|\operatorname{Aut}_{G/P}(\mathcal{A}_{G/P})^S| = |\operatorname{Aut}_{G/P}(\mathcal{A}_{G/P})|$ and hence

$$|\operatorname{Aut}_{G/P}(\mathcal{A}_{G/P})^S| \geq |\operatorname{Aut}_S(\mathcal{A}_S)|.$$

Since $\operatorname{Aut}_{G/P}(\mathcal{A}_{G/P})^S \leq \operatorname{Aut}_S(\mathcal{A}_S)$, we conclude that $\operatorname{Aut}_{G/P}(\mathcal{A}_{G/P})^S = \operatorname{Aut}_S(\mathcal{A}_S)$. Thus, $\mathcal{A}$ is a CI-S-ring by Lemma 9.2 and Lemma 5.1.

If Statement (3) of Lemma 8.7 holds for $\mathcal{A}_{G/P}$ then $\mathcal{A}_{G/P}$ is normal. In this case $\mathcal{A}$ is a CI-S-ring by Lemma 9.2 and Lemma 5.5.

Suppose that Statement (4) of Lemma 8.7 holds for $\mathcal{A}_{G/P}$, i.e. there exists an $\mathcal{A}_{G/P}$-subgroup $A \leq \mathbf{O}_\theta(\mathcal{A}_{G/P})$ of order 2 and $X = \{x_1, x_2\} \in \mathcal{S}(\mathcal{A}_{G/P})$ such that $\mathcal{A}_{(G/P)/A}$ is normal and $A \neq \operatorname{rad}(X)$. Let $L = \pi^{-1}(A)$, where $\pi \colon G \to G/P$ is the canonical epimorphism, and $\mathcal{B} = V(N, G)$, where $N = \operatorname{Aut}(\mathcal{A})_{G/L} G_{\text{right}}$.

Prove that $\mathcal{B}$ is a CI-S-ring. Lemma 5.4 implies that $\mathcal{B}$ is the S-wreath product. From Equations (3.1) and (3.2) it follows that $\mathcal{B} \geq \mathcal{A}$. So $\mathcal{B}_{G/P} \geq \mathcal{A}_{G/P}$ and hence $\mathcal{B}_{G/P}$ is a 2-S-ring by Lemma 6.1. We obtain that $\mathcal{B}$ and U satisfy the conditions of Lemma 9.3.

One can see that X is a $\mathcal{B}_{G/P}$-set and

$$\mathbf{O}_\theta(\mathcal{B}_{G/P}) \geq \mathbf{O}_\theta(\mathcal{A}_{G/P}) \tag{9.3}$$

because $\mathcal{B}_{G/P} \geq \mathcal{A}_{G/P}$. The definition of $\mathcal{B}$ yields that every basic set of $\mathcal{B}$ is contained in an L-coset and hence every basic set of $\mathcal{B}_{G/P}$ is contained in an A-coset. Therefore

$$\{x_1\}, \{x_2\} \in \mathcal{S}(\mathcal{B}_{G/P}) \tag{9.4}$$

because X is a $\mathcal{B}_{G/P}$-set and $A \neq \operatorname{rad}(X)$. Now from Equations (9.3) and (9.4) it follows that

$$|\mathbf{O}_\theta(\mathcal{B}_{G/P})| \geq 8. \tag{9.5}$$

If $\mathcal{B}_{G/P}$ is indecomposable then $\mathcal{B}_{G/P}$ is normal by Lemma 8.3 and hence $\mathcal{B}$ is CI by Lemma 9.2 and Lemma 5.5. If S has a gwr-complement with respect to $\mathcal{B}_{G/P}$ then $\mathcal{B}$ is CI by Lemma 9.4. If $\mathbf{O}_\theta(\mathcal{B}_{G/P}) \not\leq S$ then $\mathcal{B}$ is CI by Lemma 9.2 and Lemma 5.8. Suppose that none of the above conditions does not hold for $\mathcal{B}$. Then, in view of Equation (9.5), $\mathcal{B}$ satisfies all conditions from one of the Cases 1 or 2. Therefore, $\mathcal{B}$ is CI.

Clearly, $\mathcal{A}_{G/L} \cong \mathcal{A}_{(G/P)/A}$ and hence $\mathcal{A}_{G/L}$ is normal. Also $\mathcal{A}_{G/L}$ is CI by Lemma 9.2. The S-ring $\mathcal{B}$ is CI by the above paragraph. Thus, $\mathcal{A}$ is CI by Lemma 4.4.

Case 4: $|\mathbf{O}_\theta(\mathcal{A}_{G/P})| = 2$.

Let $A = \mathbf{O}_\theta(\mathcal{A}_{G/P})$. Clearly, A is the least $\mathcal{A}_{G/P}$-subgroup. If $|\operatorname{rad}(X)| > 1$ for every $X \in \mathcal{S}(\mathcal{A}_{G/P})$ outside S then $A \leq \operatorname{rad}(X)$ for every $X \in \mathcal{S}(\mathcal{A}_{G/P})$ outside S and we obtain a contradiction with Equation (9.1). So there exists $X \in \mathcal{S}(\mathcal{A}_{G/P})$ outside S with $|\operatorname{rad}(X)| = 1$. From Equation (9.2) it follows that $|X| > 1$. Lemma 8.8 implies that $|X| = 4$. The number $\lambda = |X \cap Ax|$ does not depend on $x \in X$ by Lemma 2.1. If $\lambda = 2$ then $A \leq \operatorname{rad}(X)$, a contradiction. Therefore

$$\lambda = 1. \tag{9.6}$$

One of the statements of Lemma 8.8 holds for $\mathcal{A}_{G/P}$. If Statement (1) of Lemma 8.8 holds for $\mathcal{A}_{G/P}$ then there exists $Y \in \mathcal{S}(\mathcal{A}_{G/P})$ with $|Y| = 16$ and $|\operatorname{rad}(Y)| = 16$. Since $|S| = 16$, we conclude that Y lies outside S and hence $Y = (G/P) \setminus S$. This means that S is a gwr-complement to S with respect to $\mathcal{A}_{G/P}$. However, this contradicts Equation (9.1).

If Statement (2) of Lemma 8.8 holds for $\mathcal{A}_{G/P}$ then $|\operatorname{Aut}_{G/P}(\mathcal{A}_{G/P})| \geq |\operatorname{Aut}_S(\mathcal{A}_S)|$. So Lemma 9.5 implies that $\operatorname{Aut}_{G/P}(\mathcal{A}_{G/P})^S = \operatorname{Aut}_S(\mathcal{A}_S)$. Therefore, $\mathcal{A}$ is CI by Lemma 9.2 and Lemma 5.1

Suppose that Statement (3) of Lemma 8.8 holds for $\mathcal{A}_{G/P}$, i.e. there exists an $\mathcal{A}_{G/P}$-subgroup B such that $|B| \in \{2,4\}$ and $\mathcal{A}_{(G/P)/B}$ is normal. Let $L = \pi^{-1}(B)$, where $\pi \colon G \to G/P$ is the canonical epimorphism, and $\mathcal{B} = V(N,G)$, where $N = \operatorname{Aut}(\mathcal{A})_{G/L} G_{\mathrm{right}}$.

We prove that $\mathcal{B}$ is a CI-S-ring. As in Case 3, $\mathcal{B}$ is the S-wreath product by Lemma 5.4 and $\mathcal{B} \geq \mathcal{A}$ by Equations (3.1) and (3.2). So $\mathcal{B}_{G/P} \geq \mathcal{A}_{G/P}$ and hence $\mathcal{B}_{G/P}$ is a 2-S-ring by Lemma 6.1. Therefore $\mathcal{B}$ and U satisfy the conditions of Lemma 9.3.

Note that X is a $\mathcal{B}_{G/P}$-set and Equation (9.3) holds because $\mathcal{B}_{G/P} \geq \mathcal{A}_{G/P}$. By the definition of $\mathcal{B}$, every basic set of $\mathcal{B}$ is contained in an L-coset and hence every basic set of $\mathcal{B}_{G/P}$ is contained in a B-coset. The set X is a $\mathcal{B}_{G/P}$-set with $|X| = 4$ and $|\operatorname{rad}(X)| = 1$. So there exists $X_1 \in \mathcal{S}(\mathcal{B}_{G/P})$ such that

$$X_1 \subset X \text{ and } |X_1| \in \{1,2\}.$$

If $|X_1| = 1$ then $X_1 \subseteq \mathbf{O}_\theta(\mathcal{B}_{G/P})$. If $|X_1| = 2$ then X_1 is a coset by a $\mathcal{B}_{G/P}$-subgroup A_1 of order 2. Clearly, $A_1 \subseteq \mathbf{O}_\theta(\mathcal{B}_{G/P})$. In view of Equation (9.6), we have $A_1 \neq A$. Thus, in both cases $\mathbf{O}_\theta(\mathcal{B}_{G/P}) \not\leq A$. Together with Equation (9.3) this implies that

$$|\mathbf{O}_\theta(\mathcal{B}_{G/P})| \geq 4. \tag{9.7}$$

If $\mathcal{B}_{G/P}$ is indecomposable then $\mathcal{B}_{G/P}$ is normal by Lemma 8.3 and hence $\mathcal{B}$ is CI by Lemma 9.2 and Lemma 5.5. If S has a gwr-complement with respect to $\mathcal{B}_{G/P}$ then $\mathcal{B}$ is CI by Lemma 9.4. If $\mathbf{O}_\theta(\mathcal{B}_{G/P}) \not\leq S$ then $\mathcal{B}$ is CI by Lemma 9.2 and Lemma 5.8. Suppose that none of the above conditions does not hold for $\mathcal{B}$. Then, in view of Equation (9.7), $\mathcal{B}$ satisfies all conditions from one of the Cases 1, 2 or 3. Therefore, $\mathcal{B}$ is CI.

The S-ring $\mathcal{A}_{G/L}$ is normal because it is isomorphic to $\mathcal{A}_{(G/P)/B}$. The S-rings $\mathcal{A}_{G/L}$ and $\mathcal{B}$ are CI by Lemma 9.2 and the above paragraph respectively. Thus, $\mathcal{A}$ is CI by Lemma 4.4.

All cases were considered. $\square$

Proof of Theorem 9.1. Let H_1 be a maximal $\mathcal{A}$-subgroup contained in H and P_1 the least $\mathcal{A}$-subgroup containing P.

Lemma 9.6. *If $H_1 = H$ then $\mathcal{A}$ is a CI-S-ring.*

Proof. The S-ring $\mathcal{A}_{G/H}$ is a p-S-ring over $G/H \cong C_p$ by Lemma 6.4. So $\mathcal{A}_{G/H} \cong \mathbb{Z}C_p$. Clearly, $G = HP_1$. Therefore $\mathcal{A} = \mathcal{A}_H \star \mathcal{A}_{P_1}$ by Lemma 7.2. Since H and $P_1/(H \cap P_1)$ are proper sections of G, the S-rings $\mathcal{A}_H$ and $\mathcal{A}_{P_1/(H \cap P_1)}$ are CI by Lemma 9.2. Thus, $\mathcal{A}$ is CI by Lemma 5.6. $\square$

Lemma 9.7. *If $H_1 < H$ and $H_1 P_1 = G$ then $\mathcal{A}$ is a CI-S-ring.*

Proof. Since $H_1 \neq (H_1 P_1)_{p'} = H$, Lemma 7.1 implies that $\mathcal{A} = \mathcal{A}_{H_1} \star \mathcal{A}_{P_1}$. The S-rings $\mathcal{A}_{H_1}$ and $\mathcal{A}_{P_1/(H \cap P_1)}$ are CI by Lemma 9.2 because H_1 and $P_1/(H_1 \cap P_1)$ are proper sections of G. Therefore $\mathcal{A}$ is CI by Lemma 5.6.   □

In view of Lemma 9.6, we may assume that $H_1 < H$. Then one of the statements of Lemma 7.3 holds for $\mathcal{A}$. If Statement (1) of Lemma 7.3 holds for $\mathcal{A}$ then

$$\mathcal{A} = \mathcal{A}_{H_1} \wr \mathcal{A}_{G/H_1},$$

where $\mathrm{rk}(\mathcal{A}_{G/H_1}) = 2$. If H_1 is trivial then $\mathrm{rk}(\mathcal{A}) = 2$. Obviously, $\mathcal{A}$ is CI in this case. If H_1 is nontrivial then $\mathcal{A}$ is CI by Lemma 9.2 and Lemma 5.2.

Assume that Statement (2) of Lemma 7.3 holds for $\mathcal{A}$, i.e.

$$\mathcal{A} = \mathcal{A}_U \wr_S \mathcal{A}_{G/P_1},$$

where $U = H_1 P_1$, $S = U/P_1$, and $P_1 < G$. In view of Lemma 9.7, we may assume that $H_1 P_1 < G$, i.e. $\mathcal{A}$ is the nontrivial S-wreath product. The group G/P_1 is a 2-group of order at most 32 because $P_1 \geq P$. Lemma 6.4 implies that $\mathcal{A}_{G/P_1}$ is a 2-S-ring. If $|G/P_1| \leq 16$ then $\mathcal{A}$ is CI by Lemma 9.2 and Lemma 8.9. So we may assume that $|G/P_1| = 32$. Clearly, in this case

$$P_1 = P.$$

In view of Statement (2) of Lemma 6.3, we may assume that

$$|S| = 16.$$

Indeed, if $|S| < 16$ then S is contained in an $\mathcal{A}_{G/P}$-subgroup S' of order 16 by Statement (2) of Lemma 6.3. Clearly, $\mathcal{A} = \mathcal{A}_{U'} \wr_{S'} \mathcal{A}_{G/P}$, where $U' = \pi^{-1}(S')$ and $\pi \colon G \to G/P$ is the canonical epimorphism. Replacing S by S', we obtain the required.

Now all conditions of Lemma 9.3 hold for $\mathcal{A}$ and U. Thus, $\mathcal{A}$ is CI by Lemma 9.3.   □

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