Metodološki zvezki, Vol. 3, No. 1, 2006, 39-48

GD PBIBD(2)s in Incomplete Split-Plot × Split-Block Type Experiments

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Abstract

In this paper we present a method of designing a three-factor experiment with crossed and nested treatment structures. The design considered is called a split-plot × split-block design. A kind of design incomplete with respect to all three factors is examined. Additionally, we consider the usefulness of group divisible partially balanced incomplete block designs with two associate classes in planning such experiments. In modeling data obtained from them, we take into account the structure of experimental material and a four-step randomization scheme for the different kind of units. As regards the analysis of the obtained randomization model with seven strata, we adapt an approach typical of multistratum experiments with orthogonal block structure.

1 Introduction

There are many experimental designs that can be considered for use in a three-or-more-factor experiment. For instance, all experimental designs for one- or two-factor experiments can be applied for this purpose. There are also three-or-more-factor designs specifically developed for the type of such experiments used in agricultural research. These are primarily extensions of either a split-plot or a split-block design (Gomez and Gomez, 1984; LeClerg et al., 1962).

The purpose of this paper is to present a method of designing a three-factor experiment with crossed and nested treatment structures. The design considered is called a split-plot × split-block (SPSB) design (LeClerg et al., 1962). A kind of such design incomplete with respect to the levels of three factors is examined (Ambroży and Mejza, 2003). Additionally, we present the usefulness of group divisible partially balanced incomplete block designs with two associate classes (shortly GD PBIBD(2)s, see Clatworthy, 1973) in planning such experiments.

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The SPSB type design is convenient in field experiments, especially when certain treatments such as types of cultivation, application of irrigation water, varieties etc., may have to be arranged in (crossed and nested) strips across each block.

2 Assumptions and notation

Let us consider a three-factor experiment of SPSB type in which the first factor, say A, has s levels A_1, A_2, …, A_s, the second factor, say B, has t levels B_1, B_2,, …, B_t and the third factor, say C, has w levels C_1, C_2, …, C_w. Thus the number v $(= stw)$ denotes the number of all treatment combinations in the experiment. The experimental material is assumed to be divided into b blocks each with a row-column structure with k_A rows $(k_A < s)$ and k_B columns of the first order, called I-columns for short $(k_B < t)$. So there are $k_A k_B$ intersection plots of the first order within each block, henceforth called whole plots. Then each I-column has to be split into k_C columns of the second order, called II-columns for short $(k_C < w)$. Then there are $k_A k_B k_C$ intersection plots of the second order within each block, henceforth called small plots. So, $n = bk_A k_B k_C$ denotes the number of subplots, which are required in the experiment. Here the rows correspond to the levels of the factor A, also termed row treatments, the I-columns correspond to the levels of factor B, also I-column treatments, and the II-columns are to accommodate the levels of factor C, termed II-column treatments. The arrangement of the factors in the mixed design is very important from the statistical point of view.

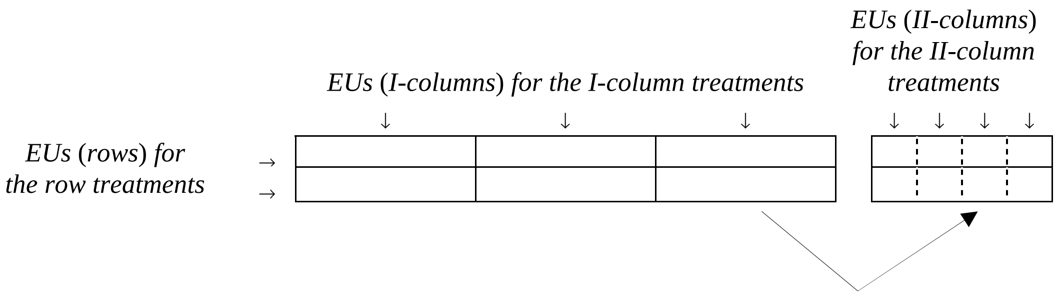

Figure 1: Experimental material of a single block in a complete split-plot × split-block type experiment with the factors A (A_1, A_2), B (B_1, B_2, B_3), C (C_1, C_2, C_3, C_4)

Figure 1 illustrates a sample layout in one block (replication) of a complete SPSB design with two row treatments (A_1, A_2), three I-column treatments (B_1, B_2, B_3) and four II-column treatments (C_1, C_2, C_3, C_4). We consider a situation where the SPSB design is incomplete with respect to three factors and their levels are arranged according to different or identical incidence matrices of some GD PBIBD(2)s.

3 Model of observations

In modeling data obtained from such experiments we take into account the structure of experimental material and a four-step randomization scheme of the different kind of units (Ambroży and Mejza, 2004a). With respect to the analysis of the obtained randomization model with seven strata, we adapt an approach typical of multistratum experiments with orthogonal block structure (Nelder, 1965a, Nelder, 1965b). In this case, we have a zero stratum (0) generated by the vector of ones, an inter-block stratum (1), an inter-row (within the blocks) stratum (2), an inter-I-column (within the blocks) stratum (3), an inter-II-column stratum (4) (within the I-columns), an inter-whole plot (within the blocks) stratum (5), and an inter-subplot (within the whole plots) stratum (6). The statistical analysis of such a model is connected with the algebraic properties of the stratum information matrices for the treatment combinations. The obtained incomplete SPSB design can thus be characterized with respect to a general balance property and stratum efficiency factors of the design for a set of orthogonal contrasts between treatment combination effects (Houtman and Speed, 1983). These contrasts are connected with comparisons among the main effects of the considered factors and interaction effects. When the SPSB design is complete, then all information about these contrasts is contained in a suitable single stratum. In cases where the design is incomplete, the information is split between two or more strata.

Applying the GD PBIBD(2)s to the construction method for SPSB type designs, we obtain different degrees of balance in the strata with respect to the different contrasts. It can be shown that the potential loss in estimating the treatment contrasts can be limited by the proper choice of experimental design.

4 Incomplete SPSB type designs generated by GD PBIBD(2)s

Let $\mathbf{N}_A$ $(s \times b)$, $\mathbf{N}_B$ $(t \times b)$ and $\mathbf{N}_C$ $(w \times b)$ be incidence matrices of GD PBIBD(2)s for the row treatments, the I-column treatments and the II-column treatments with respect to the blocks, respectively. In the present paper the construction method for three-factor experiments is based on the Kronecker product of matrices, denoted by $\otimes$. Then we have

$$\mathbf{N}_1 = \mathbf{N}_A \otimes \mathbf{N}_B \otimes \mathbf{N}_C \ , \qquad (4.1)$$

where $\mathbf{N}_1$ is the treatment combinations vs. blocks incidence matrix of the SPSB design.

Let $\mathbf{C}_*$ ($*$ denotes the letters A, B, C in turn) be information matrices of the subdesigns with the following positive eigenvalues: $\varepsilon_j^{(*)} = 1 - \omega_j^{(*)}/(r_* k_*)$ and their multiplicities $\rho_j^{(*)}$, where r_* is the number of replications of the row treatments, the I-column treatments and the II-column treatments, respectively, k_* is the size of blocks, $\omega_j^{(*)}$ are the eigenvalues of the associate matrices $\mathbf{N}_*\mathbf{N}_*^{'}$, j = 0, 1, 2, and $\sum_{i=1}^{2}\rho_j^{(A)} = s-1$, $\sum_{i=1}^{2}\rho_j^{(B)} = t-1$ and $\sum_{i=1}^{2}\rho_j^{(C)} = w-1$ (Clatworthy, 1973).

To describe properties such as efficiency and balance of the incomplete SPSB design, we will introduce the following abbreviations: let $M_f\{q,\alpha\}$ denote the property that q contrasts among the treatments of factor M (or interaction contrasts) are estimated with efficiency α in the *f*-th stratum. In other words, we say that the design is $M_f\{q,\alpha\}$- balanced. In particular, for $\alpha = 1$, the design is $M_f\{q,1\}$- orthogonal.

First we turn our attention to three-factor interaction contrasts. From the algebraic properties of the information matrices of the incomplete SPSB design and the subdesigns (Ambroży and Mejza, 2004b) we have

Theorem 1. Assume that the row treatments, the I-column treatments and the II-column treatments can be grouped into three different associate schemes. Then the SPSB($\mathrm{GD}^{(A)}, \mathrm{GD}^{(B)}, \mathrm{GD}^{(C)}$) design with the incidence matrix $\mathbf{N}_1 = \mathbf{N}_A \otimes \mathbf{N}_B \otimes \mathbf{N}_C$ and the parameters: $b = b_A b_B b_C$, $k = k_A k_B k_C$, $v = stw$, $r = r_A r_B r_C$ with respect to three-factor interaction contrasts is:

- $(A\times B\times C)_1\{\rho_1^{(A)}\rho_1^{(B)}\rho_1^{(C)}, (1-\varepsilon_1^{(A)})(1-\varepsilon_1^{(B)})(1-\varepsilon_1^{(C)})\}$– balanced,

$(A\times B\times C)_2\{\rho_1^{(A)}\rho_1^{(B)}\rho_1^{(C)}, \varepsilon_1^{(A)}(1-\varepsilon_1^{(B)})(1-\varepsilon_1^{(C)})\}$– balanced,

$(A\times B\times C)_3\{\rho_1^{(A)}\rho_1^{(B)}\rho_1^{(C)}, (1-\varepsilon_1^{(A)})\varepsilon_1^{(B)}(1-\varepsilon_1^{(C)})\}$– balanced,

$(A\times B\times C)_4\{\rho_1^{(A)}\rho_1^{(B)}\rho_1^{(C)}, (1-\varepsilon_1^{(A)})\varepsilon_1^{(C)}\}$–,

$(A\times B\times C)_5\{\rho_1^{(A)}\rho_1^{(B)}\rho_1^{(C)}, \varepsilon_1^{(A)}\varepsilon_1^{(B)}(1-\varepsilon_1^{(C)})\}$–, and

$(A\times B\times C)_6\{\rho_1^{(A)}\rho_1^{(B)}\rho_1^{(C)}, \varepsilon_1^{(A)}\varepsilon_1^{(C)}\}$ – balanced,

- $(A\times B\times C)_1\{\rho_1^{(A)}\rho_1^{(B)}\rho_2^{(C)}, (1-\varepsilon_1^{(A)})(1-\varepsilon_1^{(B)})(1-\varepsilon_2^{(C)})\}$– balanced

$(A\times B\times C)_2\{\rho_1^{(A)}\rho_1^{(B)}\rho_2^{(C)}, \varepsilon_1^{(A)}(1-\varepsilon_1^{(B)})(1-\varepsilon_2^{(C)})\}$– balanced,

$(A\times B\times C)_3\{\rho_1^{(A)}\rho_1^{(B)}\rho_2^{(C)}, (1-\varepsilon_1^{(A)})\varepsilon_1^{(B)}(1-\varepsilon_2^{(C)})\}$– balanced,

$(A\times B\times C)_4\{\rho_1^{(A)}\rho_1^{(B)}\rho_2^{(C)}, (1-\varepsilon_1^{(A)})\varepsilon_2^{(C)}\}$–,

$(A{\times}B{\times}C)_5\{\rho_1^{(A)}\,\rho_1^{(B)}\,\rho_2^{(C)},\ \varepsilon_1^{(A)}\,\varepsilon_1^{(B)}(1-\varepsilon_2^{(C)})\}$–, and

$(A{\times}B{\times}C)_6\{\rho_1^{(A)}\,\rho_1^{(B)}\,\rho_2^{(C)},\ \varepsilon_1^{(A)}\,\varepsilon_2^{(C)}\}$– balanced,

- $(A{\times}B{\times}C)_1\{\rho_1^{(A)}\,\rho_2^{(B)}\,\rho_1^{(C)},\ (1-\varepsilon_1^{(A)})(1-\varepsilon_2^{(B)})(1-\varepsilon_1^{(C)})\}$– balanced,

$(A{\times}B{\times}C)_2\{\rho_1^{(A)}\,\rho_2^{(B)}\,\rho_1^{(C)},\ \varepsilon_1^{(A)}(1-\varepsilon_2^{(B)})(1-\varepsilon_1^{(C)})\}$– balanced,

$(A{\times}B{\times}C)_3\{\rho_1^{(A)}\,\rho_2^{(B)}\,\rho_1^{(C)},\ (1-\varepsilon_1^{(A)})\,\varepsilon_2^{(B)}(1-\varepsilon_1^{(C)})\}$– balanced,

$(A{\times}B{\times}C)_4\{\rho_1^{(A)}\,\rho_2^{(B)}\,\rho_1^{(C)},\ (1-\varepsilon_1^{(A)})\,\varepsilon_1^{(C)}\}$–,

$(A{\times}B{\times}C)_5\{\rho_1^{(A)}\,\rho_2^{(B)}\,\rho_1^{(C)},\ \varepsilon_1^{(A)}\,\varepsilon_2^{(B)}(1-\varepsilon_1^{(C)})\}$–,

$(A{\times}B{\times}C)_6\{\rho_1^{(A)}\,\rho_2^{(B)}\,\rho_1^{(C)},\ \varepsilon_1^{(A)}\,\varepsilon_1^{(C)}\}$ – balanced,

- $(A{\times}B{\times}C)_1\{\rho_1^{(A)}\,\rho_2^{(B)}\,\rho_2^{(C)},\ (1-\varepsilon_1^{(A)})(1-\varepsilon_2^{(B)})(1-\varepsilon_2^{(C)})\}$– balanced,

$(A{\times}B{\times}C)_2\{\rho_1^{(A)}\,\rho_2^{(B)}\,\rho_2^{(C)},\ \varepsilon_1^{(A)}\ (1-\varepsilon_2^{(B)})(1-\varepsilon_2^{(C)})\}$– balanced,

$(A{\times}B{\times}C)_3\{\rho_1^{(A)}\,\rho_2^{(B)}\,\rho_2^{(C)},\ (1-\varepsilon_1^{(A)})\,\varepsilon_2^{(B)}(1-\varepsilon_2^{(C)})\}$– balanced,

$(A{\times}B{\times}C)_4\{\rho_1^{(A)}\,\rho_2^{(B)}\,\rho_2^{(C)},\ (1-\varepsilon_1^{(A)})\,\varepsilon_2^{(C)}\}$–,

$(A{\times}B{\times}C)_5\{\rho_1^{(A)}\,\rho_2^{(B)}\,\rho_2^{(C)},\ \varepsilon_1^{(A)}\,\varepsilon_2^{(B)}(1-\varepsilon_2^{(C)})\}$–, and

$(A{\times}B{\times}C)_6\{\rho_1^{(A)}\,\rho_2^{(B)}\,\rho_2^{(C)},\ \varepsilon_1^{(A)}\,\varepsilon_2^{(C)}\}$ – balanced,

- $(A{\times}B{\times}C)_1\{\rho_2^{(A)}\,\rho_1^{(B)}\,\rho_1^{(C)},\ (1-\varepsilon_2^{(A)})(1-\varepsilon_1^{(B)})(1-\varepsilon_1^{(C)})\}$– balanced,

$(A{\times}B{\times}C)_2\{\rho_2^{(A)}\,\rho_1^{(B)}\,\rho_1^{(C)},\ \varepsilon_2^{(A)}\ (1-\varepsilon_1^{(B)})(1-\varepsilon_1^{(C)})\}$– balanced,

$(A{\times}B{\times}C)_3\{\rho_2^{(A)}\,\rho_1^{(B)}\,\rho_1^{(C)},\ (1-\varepsilon_2^{(A)})\,\varepsilon_1^{(B)}(1-\varepsilon_1^{(C)})\}$– balanced,

$(A{\times}B{\times}C)_4\{\rho_2^{(A)}\,\rho_1^{(B)}\,\rho_1^{(C)},\ (1-\varepsilon_2^{(A)})\,\varepsilon_1^{(C)}\}$– ,

$(A{\times}B{\times}C)_5\{\rho_2^{(A)}\,\rho_1^{(B)}\,\rho_1^{(C)},\ \varepsilon_2^{(A)}\,\varepsilon_1^{(B)}(1-\varepsilon_1^{(C)})\}$–, and

$(A{\times}B{\times}C)_6\{\rho_2^{(A)}\,\rho_1^{(B)}\,\rho_1^{(C)},\varepsilon_2^{(A)}\,\varepsilon_1^{(C)}\}$ – balanced,

- $(A{\times}B{\times}C)_1\{\rho_2^{(A)}\,\rho_1^{(B)}\,\rho_2^{(C)},\ (1-\varepsilon_2^{(A)})(1-\varepsilon_1^{(B)})(1-\varepsilon_2^{(C)})\}$– balanced,

$(A{\times}B{\times}C)_2\{\rho_2^{(A)}\,\rho_1^{(B)}\,\rho_2^{(C)},\ \varepsilon_2^{(A)}\ (1-\varepsilon_1^{(B)})(1-\varepsilon_2^{(C)})\}$– balanced

$(A{\times}B{\times}C)_3\{\rho_2^{(A)}\,\rho_1^{(B)}\,\rho_2^{(C)},\ (1-\varepsilon_2^{(A)})\,\varepsilon_1^{(B)}(1-\varepsilon_2^{(C)})\}$– balanced

$(A{\times}B{\times}C)_4\{\rho_2^{(A)}\,\rho_1^{(B)}\,\rho_2^{(C)},\ (1-\varepsilon_2^{(A)})\,\varepsilon_2^{(C)}\}$–,

$(A{\times}B{\times}C)_5\{\rho_2^{(A)}\,\rho_1^{(B)}\,\rho_2^{(C)},\ \varepsilon_2^{(A)}\,\varepsilon_1^{(B)}(1-\varepsilon_2^{(C)})\}$–, and

$(A{\times}B{\times}C)_6\{\rho_2^{(A)}\,\rho_1^{(B)}\,\rho_2^{(C)},\ \varepsilon_2^{(A)}\,\varepsilon_2^{(C)}\}$– balanced,

- $(A{\times}B{\times}C)_1\{\rho_2^{(A)}\,\rho_2^{(B)}\,\rho_1^{(C)},\ (1-\varepsilon_2^{(A)})(1-\varepsilon_2^{(B)})(1-\varepsilon_1^{(C)})\}$– balanced,

$(A{\times}B{\times}C)_2\{\rho_2^{(A)}\,\rho_2^{(B)}\,\rho_1^{(C)},\ \varepsilon_2^{(A)}\ (1-\varepsilon_2^{(B)})(1-\varepsilon_1^{(C)})\}$– balanced

$(A{\times}B{\times}C)_3\{\rho_2^{(A)}\,\rho_2^{(B)}\,\rho_1^{(C)},\ (1-\varepsilon_2^{(A)})\,\varepsilon_2^{(B)}(1-\varepsilon_1^{(C)})\}$– balanced

$(A{\times}B{\times}C)_4\{\rho_2^{(A)}\,\rho_2^{(B)}\,\rho_1^{(C)},\ (1-\varepsilon_2^{(A)})\,\varepsilon_1^{(C)}\}$–,

$(A{\times}B{\times}C)_5\{\rho_2^{(A)}\ \rho_2^{(B)}\ \rho_1^{(C)},\ \varepsilon_2^{(A)}\ \varepsilon_2^{(B)}(1-\varepsilon_1^{(C)})\}$–,

$(A{\times}B{\times}C)_6\{\rho_2^{(A)}\ \rho_2^{(B)}\ \rho_1^{(C)},\ \varepsilon_2^{(A)}\ \varepsilon_1^{(C)}\}$ – balanced.

- $(A{\times}B{\times}C)_1\{\rho_2^{(A)}\ \rho_2^{(B)}\ \rho_2^{(C)},\ (1-\varepsilon_2^{(A)})(1-\varepsilon_2^{(B)})(1-\varepsilon_2^{(C)})\}$– balanced,

$(A{\times}B{\times}C)_2\{\rho_2^{(A)}\ \rho_2^{(B)}\ \rho_2^{(C)},\ \varepsilon_2^{(A)}(1-\varepsilon_2^{(B)})(1-\varepsilon_2^{(C)})\}$– balanced,

$(A{\times}B{\times}C)_3\{\rho_2^{(A)}\ \rho_2^{(B)}\ \rho_2^{(C)},\ (1-\varepsilon_2^{(A)})\varepsilon_2^{(B)}(1-\varepsilon_2^{(C)})\}$– balanced,

$(A{\times}B{\times}C)_4\{\rho_2^{(A)}\ \rho_2^{(B)}\ \rho_2^{(C)},\ (1-\varepsilon_2^{(A)})\varepsilon_2^{(C)}\}$–,

$(A{\times}B{\times}C)_5\{\rho_2^{(A)}\ \rho_2^{(B)}\ \rho_2^{(C)},\ \varepsilon_2^{(A)}\ \varepsilon_2^{(B)}(1-\varepsilon_2^{(C)})\}$–, and

$(A{\times}B{\times}C)_6\{\rho_2^{(A)}\ \rho_2^{(B)}\ \rho_2^{(C)},\ \varepsilon_2^{(A)}\ \varepsilon_2^{(C)}\}$ – balanced.

Table 1: Stratum efficiency factors of the incomplete SPSB design with respect to some types of orthogonal contrasts ($\eta_i^{(*)} = 1-\varepsilon_i^{(*)}$, where i = 1, 2 and ($*$) denotes the letters (A), (B), (C) in turn.).

Type of contrast	Number of contrasts	Stratum (1)	Stratum (2)	Stratum (3)	Stratum (4)	Stratum (5)	Stratum (6)
A	$\rho_1^{(A)}$ $\rho_2^{(A)}$	$\eta_1^{(A)}$ $\eta_2^{(A)}$	$\varepsilon_1^{(A)}$ $\varepsilon_2^{(A)}$				
B	$\rho_1^{(B)}$ $\rho_2^{(B)}$	$\eta_1^{(B)}$ $\eta_2^{(B)}$		$\varepsilon_1^{(B)}$ $\varepsilon_2^{(B)}$			
C	$\rho_1^{(C)}$ $\rho_2^{(C)}$	$\eta_1^{(C)}$ $\eta_2^{(C)}$			$\varepsilon_1^{(C)}$ $\varepsilon_2^{(C)}$		
$A{\times}B$	$\rho_1^{(A)}\ \rho_1^{(B)}$ $\rho_1^{(A)}\ \rho_2^{(B)}$ $\rho_2^{(A)}\ \rho_1^{(B)}$ $\rho_2^{(A)}\ \rho_2^{(B)}$	$\eta_1^{(A)}\ \eta_1^{(B)}$ $\eta_1^{(A)}\ \eta_2^{(B)}$ $\eta_2^{(A)}\ \eta_1^{(B)}$ $\eta_2^{(A)}\ \eta_2^{(B)}$	$\varepsilon_1^{(A)}\ \eta_1^{(B)}$ $\varepsilon_1^{(A)}\ \eta_2^{(B)}$ $\varepsilon_2^{(A)}\ \eta_1^{(B)}$ $\varepsilon_2^{(A)}\ \eta_2^{(B)}$	$\eta_1^{(A)}\ \varepsilon_1^{(B)}$ $\eta_1^{(A)}\ \varepsilon_2^{(B)}$ $\eta_2^{(A)}\ \varepsilon_1^{(B)}$ $\eta_2^{(A)}\ \varepsilon_2^{(B)}$		$\varepsilon_1^{(A)}\ \varepsilon_1^{(B)}$ $\varepsilon_1^{(A)}\ \varepsilon_2^{(B)}$ $\varepsilon_2^{(A)}\ \varepsilon_1^{(B)}$ $\varepsilon_2^{(A)}\ \varepsilon_2^{(B)}$	
$A{\times}C$	$\rho_1^{(A)}\ \rho_1^{(C)}$ $\rho_1^{(A)}\ \rho_2^{(C)}$ $\rho_2^{(A)}\ \rho_1^{(C)}$ $\rho_2^{(A)}\ \rho_2^{(C)}$	$\eta_1^{(A)}\ \eta_1^{(C)}$ $\eta_1^{(A)}\ \eta_2^{(C)}$ $\eta_2^{(A)}\ \eta_1^{(C)}$ $\eta_2^{(A)}\ \eta_2^{(C)}$	$\varepsilon_1^{(A)}\ \eta_1^{(C)}$ $\varepsilon_1^{(A)}\ \eta_2^{(C)}$ $\varepsilon_2^{(A)}\ \eta_1^{(C)}$ $\varepsilon_2^{(A)}\ \eta_2^{(C)}$		$\eta_1^{(A)}\ \varepsilon_1^{(C)}$ $\eta_1^{(A)}\ \varepsilon_2^{(C)}$ $\eta_2^{(A)}\ \varepsilon_1^{(C)}$ $\eta_2^{(A)}\ \varepsilon_2^{(C)}$		$\varepsilon_1^{(A)}\ \varepsilon_1^{(C)}$ $\varepsilon_1^{(A)}\ \varepsilon_2^{(C)}$ $\varepsilon_2^{(A)}\ \varepsilon_1^{(C)}$ $\varepsilon_2^{(A)}\ \varepsilon_2^{(C)}$
$B{\times}C$	$\rho_1^{(B)}\ \rho_1^{(C)}$ $\rho_1^{(B)}\ \rho_2^{(C)}$ $\rho_2^{(B)}\ \rho_1^{(C)}$ $\rho_2^{(B)}\ \rho_2^{(C)}$	$\eta_1^{(B)}\ \eta_1^{(C)}$ $\eta_1^{(B)}\ \eta_2^{(C)}$ $\eta_2^{(B)}\ \eta_1^{(C)}$ $\eta_2^{(B)}\ \eta_2^{(C)}$		$\varepsilon_1^{(B)}\ \eta_1^{(C)}$ $\varepsilon_1^{(B)}\ \eta_2^{(C)}$ $\varepsilon_2^{(B)}\ \eta_1^{(C)}$ $\varepsilon_2^{(B)}\ \eta_2^{(C)}$	$\varepsilon_1^{(C)}$ $\varepsilon_2^{(C)}$ $\varepsilon_1^{(C)}$ $\varepsilon_2^{(C)}$		

Conclusion 1. In the incomplete SPSB design generated by three GD designs, information about the basic interaction contrasts of type $A\times B\times C$ is contained in all strata if the GD designs are of R types. When the generating designs are of type SR or S, then some of the contrasts are estimable in one stratum only.

Conclusion 2. From the above theorem we can easily obtain information about the remaining basic contrasts. For instance, if we want information about $s-1$ (= $\rho_1^{(A)}+\rho_2^{(A)}$) contrasts among the main effects of the factor A we have to assume that $\varepsilon_1^{(B)}=\varepsilon_2^{(B)}=\varepsilon_0^{(B)}$(= 0) with $\rho_1^{(B)}=\rho_2^{(B)}=\rho_0^{(B)}$(= 1) and $\varepsilon_1^{(C)}=\varepsilon_2^{(C)}=\varepsilon_0^{(C)}$(= 0) with $\rho_1^{(C)}=\rho_2^{(C)}=\rho_0^{(C)}$(= 1). If we are interested in the interaction contrasts of type $B\times C$, then in the theorem we have to assume that $\varepsilon_1^{(A)}=\varepsilon_2^{(A)}=\varepsilon_0^{(A)}$(= 0) with $\rho_1^{(A)}=\rho_2^{(A)}=\rho_0^{(A)}$(= 1), only. We present the conclusions about these contrasts we present in Table 1.

5 Example

In this example we illustrate how orthogonal contrasts can be estimated in a $(4\times 6\times 4)$ factor experiment. Assume that four row treatments can be grouped into an associate scheme with two groups ($m^{(A)}$ = 2) and two treatments within each group ($p^{(A)}$ = 2), so $s = m^{(A)}p^{(A)} = 4$. Similarly, six I-column treatments and four II-column treatments can be grouped in two different associates schemes, therefore $t = m^{(B)}p^{(B)} = 6$, $w = m^{(C)}p^{(C)} = 4$. Then the experiment is set up in an incomplete SPSB($\mathrm{GD}^{(A)}$, $\mathrm{GD}^{(B)}$, $\mathrm{GD}^{(C)}$) design, in which the $\mathrm{GD}^{(A)}$ design and $\mathrm{GD}^{(C)}$ design are of type SR and $\mathrm{GD}^{(B)}$ is of type S (plans: SR1 and S1, respectively, cf. Clatworthy, 1973). Thus the parameters of the generating designs are as follows:

$$\begin{aligned}
&\mathrm{GD}^{(A)}:\ b_A = 4,\ k_A = 2,\ s = m^{(A)}p^{(A)} = 4,\ r_A = 2,\ \lambda_1^{(A)} = 0,\ \lambda_2^{(A)} = 1,\\
&\mathrm{GD}^{(B)}:\ b_B = 3,\ k_B = 4,\ t = m^{(B)}p^{(B)} = 6,\ r_B = 2,\ \lambda_1^{(B)} = 2,\ \lambda_2^{(B)} = 1,\\
&\mathrm{GD}^{(C)}:\ b_C = 4,\ k_C = 2,\ w = m^{(C)}p^{(C)} = 4,\ r_C = 2,\ \lambda_1^{(C)} = 0,\ \lambda_2^{(C)} = 1.
\end{aligned} \tag{5.1}$$

Then the incidence matrix with respect to the blocks of the SPSB design is

$$\mathbf{N}_1 = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix}. \tag{5.2}$$

The parameters of the SPSB(SR1, S1, SR1) design are equal to: $b = 48$, $k = 16$, $v = 96$, $r = 8$. The eigenvalues of the matrices $\mathbf{C}_A$, $\mathbf{C}_B$ and $\mathbf{C}_C$ are: $\varepsilon_1^{(A)} = \varepsilon_1^{(C)} = 1/2$ with multiplicities $\rho_1^{(A)} = \rho_1^{(C)} = 2$, $\varepsilon_2^{(A)} = \varepsilon_2^{(C)} = 1$ with multiplicities $\rho_2^{(A)} = \rho_2^{(C)} = 1$, $\varepsilon_1^{(B)} = 1$ with multiplicity $\rho_1^{(B)} = 3$, and $\varepsilon_2^{(B)} = 3/4$ with $\rho_2^{(B)} = 2$.

By Theorem 1 above, the incomplete SPSB(SR1, S1, SR1) design is:

$A_1\{2, 1/2\}$– balanced and $A_2\{2, 1/2\}$– balanced, $A_2\{1, 1\}$– orthogonal,

$B_1\{2, 1/4\}$– balanced and $B_3\{2, 3/4\}$– balanced, $B_3\{3, 1\}$– orthogonal,

$C_1\{2, 1/2\}$– balanced and $C_4\{2, 1/2\}$– balanced, $C_4\{1, 1\}$– orthogonal,

$(A{\times}B)_1\{4, 1/8\}$–, $(A{\times}B)_2\{4, 1/8\}$–, $(A{\times}B)_3\{4, 3/8\}$–, and $(A{\times}B)_5\{4, 3/8\}$– balanced,
$(A{\times}B)_3\{6, 1/2\}$–, and $(A{\times}B)_5\{6, 1/2\}$– balanced,
$(A{\times}B)_2\{2, 1/4\}$– , and $(A{\times}B)_5\{2, 3/4\}$– balanced,
$(A{\times}B)_5\{3, 1\}$– orthogonal;

$(A{\times}C)_1\{4, 1/4\}$–, $(A{\times}C)_2\{4, 1/4\}$–, $(A{\times}C)_4\{4, 1/4\}$–, and $(A{\times}C)_6\{4, 1/4\}$– balanced,
$(A{\times}C)_2\{2, 1/2\}$–, and $(A{\times}C)_6\{2, 1/2\}$– balanced,
$(A{\times}C)_4\{2, 1/2\}$– , and $(A{\times}C)_6\{2, 1/2\}$– balanced,
$(A{\times}C)_6\{1, 1\}$– orthogonal;

$(B{\times}C)_1\{4, 1/8\}$–, $(B{\times}C)_3\{4, 3/8\}$–, and $(B{\times}C)_4\{4, 1/2\}$– balanced,
$(B{\times}C)_3\{6, 1/2\}$–, and $(B{\times}C)_4\{6, 1/2\}$– balanced,
$(B{\times}C)_4\{5, 1\}$– orthogonal;

$(A{\times}B{\times}C)_3\{12, 1/4\}$–, $(A{\times}B{\times}C)_4\{12, 1/4\}$–, $(A{\times}B{\times}C)_5\{12, 1/4\}$–, and $(A{\times}B{\times}C)_6\{12, 1/4\}$– balanced;

$(A{\times}B{\times}C)_4\{10, 1/2\}$–, and $(A{\times}B{\times}C)_6\{10, 1/2\}$– balanced;

$(A{\times}B{\times}C)_1\{8, 1/16\}$–, $(A{\times}B{\times}C)_2\{8, 1/16\}$–, $(A{\times}B{\times}C)_3\{8, 3/16\}$–, $(A{\times}B{\times}C)_4\{8, 1/4\}$–, $(A{\times}B{\times}C)_5\{8, 3/16\}$–, and $(A{\times}B{\times}C)_6\{8, 1/4\}$– balanced;

$(A{\times}B{\times}C)_5\{6, 1/2\}$–, and $(A{\times}B{\times}C)_6\{6, 1/2\}$– balanced;

$(A{\times}B{\times}C)_6\{5, 1\}$– orthogonal;

$(A{\times}B{\times}C)_2\{4, 1/8\}$–, $(A{\times}B{\times}C)_5\{4, 3/8\}$–, and $(A{\times}B{\times}C)_6\{4, 1/2\}$– balanced.

Summing up, it can be noticed that in the generated SPSB design, some of the basic contrasts are estimated with full efficiency equal to 1. This results from the generating designs of types SR and S. These contrasts are estimable in one stratum only, as in a complete SPSB design. Information about other basic contrasts can be split into two or more strata. Statistical inferences (estimates and tests) about them can be obtained using the information from one stratum separately or by performing for them the combined estimation and testing based on information from all the strata in which they are estimable (Caliński and Kageyama, 2000).

Additional remarks. No standard software package known to the authors can be used to implement the designs under consideration. At the moment this paper is only theoretical. The authors would be very grateful for suggestions concerning the possible use of software for planning the generated designs and statistical inferences.

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