The Control of Nonlinear Oscillatory Systems with Delay
Upravljanje nelinearnih nihajočih sistemov z zakasnitvami

Rudolf Pušenjak [1],*, Maks Oblak [1]

[1] Faculty of Industrial Engineering Novo mesto / Šegova ulica 112, SI-8000 Novo mesto

E-Mails: rudi.pusenjak@teleing.com; maks.oblak@fl.uni-mb.si

* Full Prof. Dr. Rudolf Pušenjak, Kidričeva 5, 9240 Ljutomer, Slovenia; Tel.: +386-2-582-14-06

Abstract: This paper treats the adaptation of the Extended Lindstedt-Poincare Method with multiple time scales (EL-PM) for analysis of stationary and nonstationary resonances of harmonically excited Duffing oscillator with time delay feedback control. The fundamental and ⅓ subharmonic resonance, respectively of Duffing oscillator with feedback control are analyzed in details. The comparison of stationary resonances with nonstationary resonances due to the slowly varying excitation frequency is presented by means of examples, which are computed by programming tool Mathematica®.

Key words: Duffing oscillator, time delay feedback control, osnovna in subharmonična resonanca, EL-P method.

Povzetek: Članek obravnava priredbo Razširjene Lindstedt-Poincarejeve Metode z več časovnimi skalami (RL-PM) za analizo stacionarnih in nestacionarnih resonanc harmonično vzbujanega Duffingovega nihala z zakasnitvijo povratno zančnega upravljanja. Analizi osnovne in ⅓ subharmonične resonance Duffingovega nihala z upravljanjem sta prikazani v podrobnostih. V članku je s pomočjo zgledov izvedena primerjava stacionarnih resonanc z nestacionarnimi resonancami, ki jih povzroča počasno spremenljiva vzbujevalna frekvenca, pri čemer so rezultati dobljeni s pomočjo programskega orodja Mathematica®.

Ključne besede: Duffingovo nihalo, upravljanje s časovno zakasnitvijo, osnovna in subharmonična resonanca, RL-P metoda.

1. Introduction

The control of nonlinear dynamical systems, which can exhibit periodic, aperiodic and even chaotic oscillations, represents a great research challenge due to the theoretical achievements and practical applications. Realization of feedback control is connected with inevitable delays, which occur in data acquisition and signal processing. Mathematical modelling of such dynamical systems is described in the form of nonlinear delayed differential equations (DDE's), which cannot be handled by the standard methods. Inspired by the power of the Extended Lindstedt-Poincare Method with multiple time scales (EL-PM) [1], the goal of this paper is to extend its applicability to the problems of time delay nonlinear control in the steady-state of dynamical systems, which are harmonically excited. In the paper, the method is applied to the feedback control of the harmonically excited Duffing oscillator.

2. Nomenclature

w = deflection of the beam
u = displacement
x = dimensionless spatial variable
t = time
c^* = coefficient of viscous damping
Γ = axial force
K = beam stiffness
ω_L = natural angular frequency
ω_0 = linear angular frequency
ω = excitation angular frequency
β = coefficient of cubic nonlinearity
θ = time delay
ε = perturbation parameter
a = gain of the proportional part of the controller
b = gain of the differential part of the controller
p = excitation amplitude
η = subharmonic factor
σ = detuning
τ_1 = fast time scale
τ_2 = slow time scale
$A(\tau_2)$ = amplitude of oscillations as function of the slow time scale
$\Phi_0(\tau_2)$ = phase angle of oscillations as function of the slow time scale
$\gamma(\tau_2)$ = auxiliary variable
r = rate of detuning change

3. Buckling of the hinged-hinged beam: An example of derivation of Duffing equation

Duffing oscillator is a dynamical system with one degree of freedom and a cubic nonlinearity. Examples of Duffing oscillator can be found elsewhere in science and engineering. In mechanics, for example, vibrations of active suspension system, which takes into account the nonlinearity of tires, vibrations of offshore platforms, vibrations of hinged-hinged beams, clamped-hinged beams [2], etc. are modelled by means of Duffing oscillator. In modelling rotordynamics, even coupled nonlinear oscillators of Duffing type are used [3]. In this paper, we consider the buckling phenomenon of the hinged-hinged beam as an example. Deflection of the beam is presented on the Figure 1 and for the sake of simplicity, the beam is analyzed through the dimensionless spatial variable $x=l/L$ on the interval $0 \le x \le 1$, where L is the length of the beam and l is the variable distance, measured from its left end. The beam is at both ends subjected to the compressive axial force Γ and externally forced by a harmonic excitation $P(x,t)$, which depends on the spatial variable x and on the time t. Excitation $P(x,t)$ can be realized in practice, when the beam supports a machine with a rotating imbalance. Viscous damping of the beam is characterized by the coefficient c^* and the beam stiffness is denoted by K. By using Hamilton principle [4], we can derive the governing PDE of small deflections $w(x,t)$, where longitudinal motions of the beam are neglected:

$$\frac{\partial^4 w}{\partial x^4} + \left\{ \Gamma - K \int_0^1 \left[\frac{\partial w}{\partial x}(\xi,t) \right]^2 \mathrm{d}\xi \right\} \frac{\partial^2 w}{\partial x^2} + c^* \frac{\partial w}{\partial t} + \frac{\partial^2 w}{\partial t^2} = P(x,t), \qquad (1)$$

where

$$w(0,t) = w(1,t) = 0, \qquad \frac{\partial^2 w(0,t)}{\partial x^2} = \frac{\partial^2 w(1,t)}{\partial x^2} = 0 \qquad (2)$$

are the corresponding boundary conditions of the hinged-hinged beam at its supports. By simplifying the analysis assuming the symmetry of $P(x,t)$ around the spatial coordinate $x=0.5$, the beam is in the first mode oscillation and both excitation $P(x,t)$ and deflection $w(x,t)$, respectively have a sinusoidal shape in x. However, excitation $P(x,t)$ additionally varies cosinusoidal in dependence on the time. Taking both spatial and temporal variations into account, we seek the solution of Eq. (1) by means of ansatz:

$$P(x,t) = p \cos\omega t \sin\pi x, \qquad w(x,t) = u(t) \sin\pi x, \qquad (3)$$

By substituting Eq. (3) into Eq. (1), by separation of variables and integration involved and by using boundary conditions (2), we can derive the Duffing equation:

$$\ddot{u} + c^* \dot{u} + \pi^2 \left(\pi^2 - \Gamma \right) u + \tfrac{1}{2} K \pi^4 u^3 = p \cos\omega t, \qquad (4)$$

which governs the temporal vibration $u(t)$ of the beam displacement. By using notations:

$$2\varepsilon c = c^*, \quad \omega_L{}^2 = \pi^2 \left(\pi^2 - \Gamma \right), \quad \varepsilon\beta = \tfrac{1}{2} K \pi^4, \quad \sum_{k=0}^{k=1} \varepsilon^k p_k = p, \qquad (5)$$

which can be used to introduce small quantities, depending on the small perturbation parameter ε (with an exception of introducing the natural angular frequency ω_L), Eq. (4) can be rewritten on the standard form:

$$\ddot{u} + 2\varepsilon c \dot{u} + \omega_L{}^2 u + \varepsilon\beta u^3 = \sum_{k=0}^{k=1} \varepsilon^k p_k \cos\omega t, \qquad (6)$$

which is analyzed in the rest of paper in the general case.

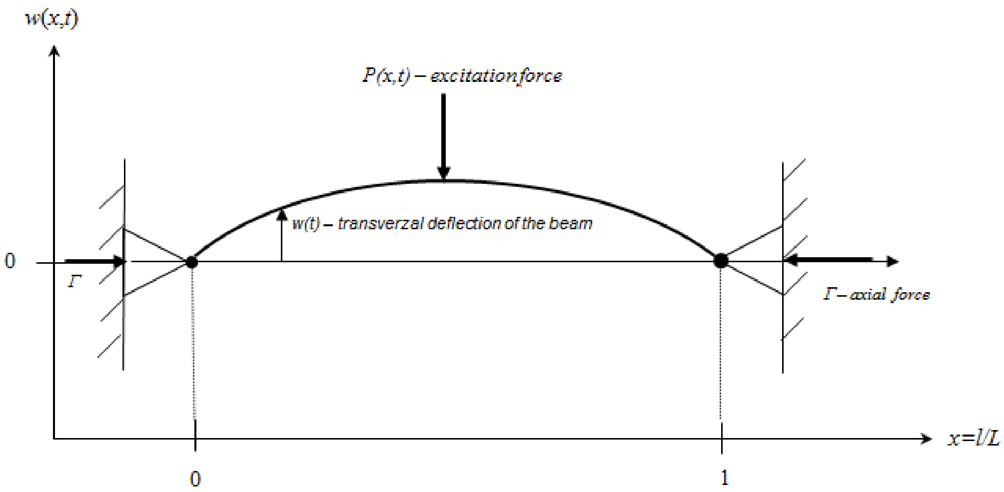


Fig. 1. Deflection w of the hinged-hinged beam

4. Harmonically excited Duffing oscillator with time delay feedback control

Despite of simple harmonically excitation, Duffing oscillator is able to produce various periodic, apperiodic or even chaotic oscillations in dependence on parameter values. Chaotic oscillations are in general unwanted and therefore give rise to the use of active feedback control in order to prevent them. Active control is advantageous over the passive control in many respects, however it opens numerous problems of technical nature. Due to the active control, the dynamics of the system becomes more complicated as the consequence of inevitable time delays in controllers, transducers and actuators such as analogue filters, hydraulic and pneumatic actuators. Time delays can cause instability of the system, which is controlled by the use of feedback loop. In the presence of time delays, the classical control theory has a limited use in linear systems, however it fails in the case of nonlinear systems. In order to facilitate the analysis of the nonlinear dynamical systems with time delay feedback, the Extended Lindstedt-Poincare method with multiple time scales (EL-PM) [1] is proposed in this paper in the case of Duffing oscillator. The analysis of Duffing oscillator with time delay control is performed by assumption of a small modal damping, weak cubic nonlinearity and small gain coefficients of controller. Characteristic for the use of EL-PM is, that all of these parameters are expressed in dependence on a small perturbation parameter ε. In the case of fundamental resonance, the external harmonic excitation is also small and depending on the perturbation parameter ε, however, this restriction is removed in the case of subharmonic resonance. In accordance with this assumptions, the governing equation of Duffing oscillator with harmonic excitation and cubic nonlinearity (6), which applies the time delay feedback control, is rearranged in the form:

$$\frac{\mathrm{d}^2u(t)}{\mathrm{d}t^2}+2\varepsilon c\frac{\mathrm{d}u(t)}{\mathrm{d}t}+\omega_L^2u(t)+\varepsilon\beta u^3(t)=2\varepsilon a u(t-\theta)+2\varepsilon b\frac{\mathrm{d}}{\mathrm{d}t}u(t-\theta)+\sum_{k=0}^{k=1}\varepsilon^k p_k\cos\omega t\,, \tag{7}$$

where $u(t)$ denotes displacements of the dynamical system, $2\varepsilon c$ represents modal damping of the system, ω_L means natural (angular) frequency, β means coefficient of cubic nonlinearity, a and b mean the gain coefficients of the proportional and differential part of the controller, respectively, θ means the time delay, ω denotes the excitation (angular) frequency and where p_k , (k=0,1) are excitation amplitudes, which are adapted in such a way, that $p_0=0$, $p_1=p$ holds in the case of fundamental resonance and $p_0=p$, $p_1=0$ holds in the case of subharmonic resonance. In the sequel, EL-PM with multiple time scales [1] will be applied in the analysis of nonstationary resonances of Duffing oscillator with time delay control. Nonstationary resonances are the phenomenon, which occur at slowly varying excitation frequency or excitation amplitude, respectively. In this paper, only slowly varying excitation frequency will be taken into account. Stationary resonances will be also studied as the special cases, when the excitation frequency has a role of an adjustable, but otherwise constant parameter.

In order to master both fundamental as well as subharmonic resonances, one introduces the so called subharmonic factor η, which has the value η=1 at the fundamental (or even at the superharmonic) resonance, but it becomes an adequate natural number, such as η=3 at the ⅓ subharmonic resonance, for example. Deviation of the excitation frequency ω from the linear frequency ω_0 can be expressed by equation:

$$\omega=\eta\omega_0+\varepsilon\sigma\,, \tag{8}$$

where the parameter σ stands for the detuning. On the basis of Eq. (8), the fast time scale τ_1 and the slow time scale τ_2, respectively, are introduced:

$$\tau_1 = \omega_0 t \qquad (9.a)$$

$$\tau_2 = \varepsilon t\,, \qquad (9.b)$$

which give rise to the replacement of time derivatives $\mathrm{d}/\mathrm{d}t$ in $\mathrm{d}^2/\mathrm{d}t^2$ with differential operators:

$$\frac{\mathrm{d}}{\mathrm{d}t} = \omega_0 \frac{\partial}{\partial \tau_1} + \varepsilon \frac{\partial}{\partial \tau_2} \qquad (10.a)$$

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2} = \omega_0^2 \frac{\partial^2}{\partial \tau_1^2} + 2\varepsilon\omega_0 \frac{\partial^2}{\partial \tau_1 \partial \tau_2} + \varepsilon^2 \frac{\partial^2}{\partial \tau_2^{\,2}}\,. \qquad (10.b)$$

Using partial derivatives (10.a,b), Eq. (7) is rewritten into nonlinear partial differential equation of the form:

$$\begin{aligned}&\omega_0^2 \frac{\partial^2 u}{\partial \tau_1^2} + 2\varepsilon\omega_0 \frac{\partial^2 u}{\partial \tau_1 \partial \tau_2} + \varepsilon^2 \frac{\partial^2 u}{\partial \tau_2^{\,2}} + 2\varepsilon c\left(\omega_0 \frac{\partial u}{\partial \tau_1} + \varepsilon \frac{\partial u}{\partial \tau_2}\right) + \omega_L^2 u + \varepsilon\beta u^3 \\ &= 2\varepsilon a u\left(\tau_1 - \omega_0\theta, \tau_2\right) + 2\varepsilon\beta\omega_0 \frac{\partial}{\partial \tau_1} u\left(\tau_1 - \omega_0\theta, \tau_2\right) + 2\varepsilon^2 b \frac{\partial}{\partial \tau_2} u\left(\tau_1 - \omega_0\theta, \tau_2\right) + \varepsilon p \cos\left(\tau_1 + \sigma\tau_2\right)\end{aligned} \qquad (11.a)$$

in the case of fundamental resonance and into equation

$$\begin{aligned}&\omega_0^2 \frac{\partial^2 u}{\partial \tau_1^2} + 2\varepsilon\omega_0 \frac{\partial^2 u}{\partial \tau_1 \partial \tau_2} + \varepsilon^2 \frac{\partial^2 u}{\partial \tau_2^{\,2}} + 2\varepsilon c\left(\omega_0 \frac{\partial u}{\partial \tau_1} + \varepsilon \frac{\partial u}{\partial \tau_2}\right) + \omega_L^2 u + \varepsilon\beta u^3 \\ &= 2\varepsilon a u\left(\tau_1 - \omega_0\theta, \tau_2\right) + 2\varepsilon b\omega_0 \frac{\partial}{\partial \tau_1} u\left(\tau_1 - \omega_0\theta, \tau_2\right) + 2\varepsilon^2 b \frac{\partial}{\partial \tau_2} u\left(\tau_1 - \omega_0\theta, \tau_2\right) + p \cos\left(\eta\tau_1 + \sigma\tau_2\right)\end{aligned} \qquad (11.b)$$

in the case of subharmonic resonance. The approximate solution of Eqs. (11.a,b) in dependence of time scales τ_1 and τ_2 can be found by applying the perturbation procedure, when one expresses displacements $u(t)$ in the form of the power series:

$$u = \sum_{k=0}^{\infty} \varepsilon^k u_k\left(\tau_1, \tau_2\right). \qquad (12)$$

5. Analysis of nonstationary fundamental resonance of Duffing oscillator with delay feedback control

When power series (12) is introduced into Eq. (11.a) and terms of like powers of ε on both sides of equation are equated, we get the following set of partial differential equations, that are linear, if are successively solved:

$$\varepsilon^0: \qquad \omega_0^2 \frac{\partial^2 u_0}{\partial \tau_1^2} + \omega_L^2 u_0 = 0 \qquad (13)$$

$$\begin{aligned}\varepsilon^1: \qquad &\omega_0^2 \frac{\partial^2 u_1}{\partial \tau_1^2} + \omega_L^2 u_1 = p\cos\left(\tau_1 + \sigma\tau_2\right) - 2\omega_0\left(\frac{\partial^2 u_0}{\partial \tau_1 \partial \tau_2} + c\frac{\partial u_0}{\partial \tau_1}\right) - \beta u_0^{\,3} \\ &+ 2a u_0\left(\tau_1 - \omega_0\theta, \tau_2\right) + 2b\omega_0 \frac{\partial}{\partial \tau_1} u_0\left(\tau_1 - \omega_0\theta, \tau_2\right)\end{aligned} \qquad (14)$$

$\vdots$

The general solution of Eq. (13) is sought in the form:

$$u_0\left(\tau_1, \tau_2\right) = A\left(\tau_2\right)\cos\left[\tau_1 - \Phi_0\left(\tau_2\right)\right], \qquad (15)$$

where both the amplitude $A(\tau_2)$ and phase $\Phi_0(\tau_2)$, respectively, are modulated in dependence on the slow time scale and are currently undetermined. They will be determined in the next step of the perturbation procedure by elimination of so called secular terms in the solution $u_1(\tau_1,\tau_2)$ of Eq. (14). Due to the modulated amplitude $A(\tau_2)$ and phase $\Phi_0(\tau_2)$ in Eq. (15), oscillations of Duffing oscillator are aperiodic. By substitution of solution (15) into Eq. (13) and by considering the fact, that frequencies ω_0, ω_L can take only positive values, the following important relation is obtained:

$$\omega_0 = \omega_L \qquad (16)$$

By substitution of the general solution (15) into Eq. (14) and by considering relation (16) one has:

$$\omega_L^2\left(\frac{\partial^2 u_1}{\partial\tau_1^2}+u_1\right)=p\cos\left[\tau_1-\Phi_0(\tau_2)\right]\cos\left[\sigma\tau_2+\Phi_0(\tau_2)\right]-p\sin\left[\tau_1-\Phi_0(\tau_2)\right]\sin\left[\sigma\tau_2+\Phi_0(\tau_2)\right]+$$
$$2\omega_L\left[\left[\frac{dA(\tau_2)}{d\tau_2}+cA(\tau_2)\right]\sin\left[\tau_1-\Phi_0(\tau_2)\right]-A(\tau_2)\cos\left[\tau_1-\Phi_0(\tau_2)\right]\frac{d\Phi_0(\tau_2)}{d\tau_2}\right] \tag{17}$$
$$-\frac{1}{4}\beta A^3(\tau_2)\left[3\cos\left[\tau_1-\Phi_0(\tau_2)\right]+\cos\left(3\left[\tau_1-\Phi_0(\tau_2)\right]\right)\right]+2aA(\tau_2)\cos\left[\tau_1-\Phi_0(\tau_2)\right]\cos(\omega_L\theta)$$
$$+2aA(\tau_2)\sin\left[\tau_1-\Phi_0(\tau_2)\right]\sin(\omega_L\theta)-2b\omega_L A(\tau_2)\sin\left[\tau_1-\Phi_0(\tau_2)\right]\cos(\omega_L\theta)+2b\omega_L A(\tau_2)\cos\left[\tau_1-\Phi_0(\tau_2)\right]\sin(\omega_L\theta)$$

The right hand side of Eq. (17) contains secular terms, which cause the unlimited growth of the solution. In order to obtain an uniform solution, secular terms must be eliminated by assembling terms, which appear at $\cos[\tau_1\text{-}\Phi_0(\tau_2)]$ and $\sin[\tau_1\text{-}\Phi_0(\tau_2)]$, respectively and by equating the obtained expressions with zero:

$$p\cos\left[\sigma\tau_2+\Phi_0(\tau_2)\right]-2\omega_L A(\tau_2)\frac{d\Phi_0(\tau_2)}{d\tau_2}-\frac{3}{4}\beta A^3(\tau_2)+2aA(\tau_2)\cos(\omega_L\theta)+2b\omega_L A(\tau_2)\sin(\omega_L\theta)=0, \tag{18}$$

$$-p\sin\left[\sigma\tau_2+\Phi_0(\tau_2)\right]+2\omega_L\left[\frac{dA(\tau_2)}{d\tau_2}+cA(\tau_2)\right]+2aA(\tau_2)\sin(\omega_L\theta)-2b\omega_L A(\tau_2)\cos(\omega_L\theta)=0. \tag{19}$$

By introduction of the new variable:

$$\gamma(\tau_2)=\sigma\tau_2+\Phi_0(\tau_2), \tag{20}$$

Eqs. (18) and (19) are rewritten on the form:

$$p\cos\left[\gamma(\tau_2)\right]+2\omega_L A(\tau_2)\left[\sigma+\frac{\mathrm{d}\sigma}{\mathrm{d}\tau_2}\tau_2-\frac{d\gamma(\tau_2)}{d\tau_2}\right]-\frac{3}{4}\beta A^3(\tau_2)+2aA(\tau_2)\cos(\omega_L\theta)+2b\omega_L A(\tau_2)\sin(\omega_L\theta)=0, \tag{21}$$

$$-p\sin\left[\gamma(\tau_2)\right]+2\omega_L\left[\frac{dA(\tau_2)}{d\tau_2}+cA(\tau_2)\right]+2aA(\tau_2)\sin(\omega_L\theta)-2b\omega_L A(\tau_2)\cos(\omega_L\theta)=0. \tag{22}$$

Equations (21) and (22) represent the system of ordinary nonlinear differential equations. Solution of both equations give us the course of amplitude and phase, respectively of nonstationary fundamental resonance of Duffing oscillator with time delay feedback control. Solution of the system of Eqs. (21) and (22) can be obtained by various methods of numerical integration, where often the Runge-Kutta method is used.

5.1. Stationary fundamental resonance of Duffing oscillator with delay feedback control

Stationary oscillations of Duffing oscillator appear, when $A(\tau_2)=A=const$, $\Phi_0(\tau_2)=\Phi_0=const$, $\gamma(\tau_2)=\gamma=const$. Consequently, it holds $\frac{dA(\tau_2)}{d\tau_2}=0$, $\frac{d\gamma(\tau_2)}{d\tau_2}=0$, $\frac{\mathrm{d}\sigma(\tau_2)}{\mathrm{d}\tau_2}=0$ and Eqs. (21), (22) are simplified to the nonlinear algebraic equations:

$$p\cos\gamma+2\omega_L\sigma A-\frac{3}{4}\beta A^3+2aA\cos(\omega_L\theta)+2b\omega_L A\sin(\omega_L\theta)=0, \tag{23}$$

$$-p\sin\gamma+2c\omega_L A+2aA\sin(\omega_L\theta)-2b\omega_L A\cos(\omega_L\theta)=0. \tag{24}$$

By squaring Eqs. (23) and (24) and adding, we get the following equation of the amplitude-detuning response of Duffing oscillator with time delay feedback control:

$$\left\{\left[\omega_L\sigma-\frac{3}{8}\beta A^2+a\cos(\omega_L\theta)+b\omega_L\sin(\omega_L\theta)\right]^2+\left[c\omega_L+a\sin(\omega_L\theta)-b\omega_L\cos(\omega_L\theta)\right]^2\right\}A^2=\frac{p^2}{4}. \tag{25}$$

The corresponding course of the phase of Duffing oscillator is obtained, when Eq. (24) is divided by Eq. (23):

$$\tan\gamma = \frac{p\sin\gamma}{p\cos\gamma} = \frac{b\omega_L\cos(\omega_L\theta) - c\omega_L - a\sin(\omega_L\theta)}{\omega_L\sigma - \frac{3}{8}\beta A^2 + a\cos(\omega_L\theta) + b\omega_L\sin(\omega_L\theta)}. \qquad (26)$$

When the amplitude A and phase angle γ, respectively are determined, the steady-state solution of the zeroth order for displacements $u(t)$ can be computed in the form:

$$u(t) = A\cos(\omega t - \gamma) + O(\varepsilon). \qquad (27)$$

From this solution we can see, that oscillations of Duffing oscillator are nearly harmonic (a more accurate analysis would be show, that oscillations are in fact periodic due to the contribution of higher harmonics in the solution $u_1(\tau_1,\tau_2)$). When the control of Duffing oscillator is not applied, that is, when $a = b = 0$, then Eqs. (25), (26) reduce on relationships:

$$\left[\left(\omega_L\sigma - \frac{3}{8}\beta A^2\right)^2 + (\omega_L c)^2\right]A^2 = \frac{p^2}{4}, \qquad (28)$$

$$\tan\gamma = \frac{c\omega_L}{\frac{3}{8}\beta A^2 - \omega_L\sigma}. \qquad (29)$$

The obtained result describes the stationary fundamental resonance of Duffing oscillator that is in accordance with result, obtained by Nayfeh in Mook [5], which use multiple scales as well as with result of Leung in Fung [6], which apply the incremental harmonic balance method. On the basis of this comparison we can conclude that zeroth order approximation of nonlinear oscillations of Duffing oscillator by EL-PM with two time scales offers results, which are accurate enough.

6. Analysis of nonstationary subharmonic resonance of Duffing oscillator with time delay feedback control

Analysis of subharmonic resonances is performed by the similar procedure as in the case of fundamental resonance starting from Eq. (11.b). In this paper, we restrict oneself to the ⅓ subharmonic resonance, where the variable excitation frequency ω can be expressed by means of detuning σ as follows:

$$\omega = 3\omega_0 + \varepsilon\sigma. \qquad (30)$$

In the case of ⅓ subharmonic resonance, Eq. (11.b) is rewritten on the form:

$$\begin{aligned}&\omega_0^2\frac{\partial^2 u}{\partial\tau_1^2} + 2\varepsilon\omega_0\frac{\partial^2 u}{\partial\tau_1\partial\tau_2} + \varepsilon^2\frac{\partial^2 u}{\partial\tau_2^2} + 2\varepsilon c\left(\omega_0\frac{\partial u}{\partial\tau_1} + \varepsilon\frac{\partial u}{\partial\tau_2}\right) + \omega_L^2 u + \varepsilon\beta u^3 = p\cos(3\tau_1 + \sigma\tau_2)\\ &+2\varepsilon a u(\tau_1 - \omega_0\theta, \tau_2) + 2\varepsilon b\omega_0\frac{\partial}{\partial\tau_1}u(\tau_1 - \omega_0\theta, \tau_2) + 2\varepsilon^2 b\frac{\partial}{\partial\tau_2}u(\tau_1 - \omega_0\theta, \tau_2)\end{aligned} \qquad (31)$$

By substituting the power series (12) into Eq. (31) and by equating terms of like powers of ε on both sides of equation, one obtains the following system of linear partial differential equations (PDE's):

$$\varepsilon^0: \qquad \omega_0^2\frac{\partial^2 u_0}{\partial\tau_1^2} + \omega_L^2 u_0 = p\cos(3\tau_1 + \sigma\tau_2), \qquad (32)$$

$$\varepsilon^1: \quad \omega_0^2\frac{\partial^2 u_1}{\partial\tau_1^2} + \omega_L^2 u_1 = -2\omega_0\left(\frac{\partial^2 u_0}{\partial\tau_1\partial\tau_2} + c\frac{\partial u_0}{\partial\tau_1}\right) - \beta u_0^3 + 2a u_0(\tau_1 - \omega_0\theta, \tau_2) + 2b\omega_0\frac{\partial}{\partial\tau_1}u_0(\tau_1 - \omega_0\theta, \tau_2), \qquad (33)$$

⋮

The general solution of Eq. (32) is equal to the sum of the general solution of the corresponding homogenous PDE and the particular solution

$$u_0(\tau_1,\tau_2) = A(\tau_2)\cos[\tau_1 - \Phi_0(\tau_2)] + E_0(\tau_2)\cos\left(\frac{\omega}{\omega_0}\tau_1\right), \qquad (34)$$

where $A(\tau_2)\cos[\tau_1-\Phi_0(\tau_2)]$ describes the solution of the homogeneous equation and $E_0(\tau_2)\cos\left(\frac{\omega}{\omega_0}\tau_1\right)$ is the corresponding particular solution. When the solution $A(\tau_2)\cos[\tau_1-\Phi_0(\tau_2)]$ is substituted into homogeneous equation, relation (16) is obtained again. Because the particular solution satisfies Eq. (32), too, we can compute the unknown amplitude $E_0(\tau_2)$ by substituting the particular solution into Eq. (32):

$$E_0(\tau_2)=\frac{p}{{\omega_L}^2-\omega^2}=\frac{p}{{\omega_L}^2-(3\omega_L+\varepsilon\sigma)^2}=-\frac{p}{8{\omega_L}^2+\varepsilon\sigma(6\omega_L+\varepsilon\sigma)}. \quad (35)$$

By means of Eq. (34) we ascertain that the amplitude of particular solution $E_0(\tau_2)$ is modulated in the case of nonstationary resonance (due to the variability of excitation frequency ω), while it is constant in the case of stationary resonance. Solution (34) can be rewritten by considering Eq. (35) on the form:

$$u_0(\tau_1,\tau_2)=A(\tau_2)\cos[\tau_1-\Phi_0(\tau_2)]-\frac{p}{8{\omega_L}^2+\varepsilon\sigma(6\omega_L+\varepsilon\sigma)}\cos(3\tau_1+\sigma\tau_2). \quad (36)$$

By substituting Eq. (36) into Eq. (33) and by considering relation (16) we get:

$$\begin{aligned}
\omega_L^2\frac{\partial^2 u_1}{\partial\tau_1^2}+\omega_L^2u_1=2\omega_L\Bigg[&\frac{\mathrm{d}A(\tau_2)}{\mathrm{d}\tau_2}\sin[\tau_1-\Phi_0(\tau_2)]-A(\tau_2)\cos[\tau_1-\Phi_0(\tau_2)]\frac{\mathrm{d}\Phi_0(\tau_2)}{\mathrm{d}\tau_2}\\
&-\frac{3p\sigma}{\left[8{\omega_L}^2+\varepsilon\sigma(6\omega_L+\varepsilon\sigma)\right]}\cos(3\tau_1+\sigma\tau_2)+\frac{12\varepsilon(3\omega_0+\varepsilon\sigma)p}{\left[8{\omega_L}^2+\varepsilon\sigma(6\omega_L+\varepsilon\sigma)\right]^2}\frac{\mathrm{d}\sigma}{\mathrm{d}\tau_2}\sin(3\tau_1+\sigma\tau_2)\Bigg]+2\omega_Lc\Bigg[A(\tau_2)\sin[\tau_1-\Phi_0(\tau_2)]\\
&-\frac{3p}{\left[8{\omega_L}^2+\varepsilon\sigma(6\omega_L+\varepsilon\sigma)\right]}\sin(3\tau_1+\sigma\tau_2)\Bigg]-\beta\left[A(\tau_2)\cos[\tau_1-\Phi_0(\tau_2)]-\frac{p}{\left[8{\omega_L}^2+\varepsilon\sigma(6\omega_L+\varepsilon\sigma)\right]}\cos(3\tau_1+\sigma\tau_2)\right]^3\\
&+2a\left[A(\tau_2)\cos[\tau_1-\omega_L\theta-\Phi_0(\tau_2)]-\frac{p}{\left[8{\omega_L}^2+\varepsilon\sigma(6\omega_L+\varepsilon\sigma)\right]}\cos(3\tau_1-3\omega_L\theta+\sigma\tau_2)\right]\\
&-2b\omega_L\left[A(\tau_2)\sin[\tau_1-\omega_L\theta-\Phi_0(\tau_2)]-\frac{3p}{\left[8{\omega_L}^2+\varepsilon\sigma(6\omega_L+\varepsilon\sigma)\right]}\sin(3\tau_1-3\omega_L\theta+\sigma\tau_2)\right]
\end{aligned} \quad (37)$$

Equation (37) contains secular terms on its right hand side. By using a similar procedure as in the case of fundamental resonance, secular terms are eliminated, when terms, which appear at $\cos[\tau_1-\Phi_0(\tau_2)]$ and $\sin[\tau_1-\Phi_0(\tau_2)]$, respectively, are collected and equated by zero:

$$2\omega_L\frac{\mathrm{d}A(\tau_2)}{\mathrm{d}\tau_2}+2\omega_LcA(\tau_2)-\frac{3\beta A^2(\tau_2)p}{4\left[8{\omega_L}^2+\varepsilon\sigma(6\omega_L+\varepsilon\sigma)\right]}\sin[\sigma\tau_2+3\Phi_0(\tau_2)]+2aA(\tau_2)\sin(\omega_L\theta)-2b\omega_LA(\tau_2)\cos(\omega_L\theta)=0 \quad , \quad (38)$$

$$\begin{aligned}
&-2\omega_L\frac{\mathrm{d}\Phi_0(\tau_2)}{\mathrm{d}\tau_2}-\frac{3}{4}\beta A^2(\tau_2)-\frac{3\beta pA(\tau_2)}{4\left[8{\omega_L}^2+\varepsilon\sigma(6\omega_L+\varepsilon\sigma)\right]}\cos[\sigma\tau_2+3\Phi_0(\tau_2)]-\frac{3\beta p^2}{2\left[8{\omega_L}^2+\varepsilon\sigma(6\omega_L+\varepsilon\sigma)\right]^2}.\\
&+2a\cos(\omega_L\theta)+2b\omega_L\sin(\omega_L\theta)=0
\end{aligned} \quad (39)$$

By introducing a new variable

$$\gamma(\tau_2)=\sigma\tau_2+3\Phi_0(\tau_2), \quad (40)$$

Eqs. (38) and (39) can be rewritten on the form:

$$\frac{\mathrm{d}A(\tau_2)}{\mathrm{d}\tau_2}=-cA(\tau_2)+\frac{3\beta A^2(\tau_2)p}{8\omega_L\left[8{\omega_L}^2+\varepsilon\sigma(6\omega_L+\varepsilon\sigma)\right]}\sin[\gamma(\tau_2)]-\frac{a}{\omega_L}A(\tau_2)\sin(\omega_L\theta)+bA(\tau_2)\cos(\omega_L\theta), \quad (41)$$

$$\frac{\mathrm{d}\gamma(\tau_2)}{\mathrm{d}\tau_2}=\sigma+\frac{\mathrm{d}\sigma}{\mathrm{d}\tau_2}\tau_2-\frac{9}{8}\frac{\beta}{\omega_L}A^2(\tau_2)-\frac{9\beta pA(\tau_2)}{8\omega_L\left[8\omega_L^{\,2}+\varepsilon\sigma(6\omega_L+\varepsilon\sigma)\right]}\cos\left[\gamma(\tau_2)\right]-\frac{9\beta p^2}{4\omega_L\left[8\omega_L^{\,2}+\varepsilon\sigma(6\omega_L+\varepsilon\sigma)\right]^2}+\frac{3a}{\omega_L}\cos(\omega_L\theta)+3b\sin(\omega_L\theta)\,. \tag{42}$$

Equations (41) and (42) represents the amplitude response and phase response equation, respectively, of ⅓ nonstationary subharmonic resonance of Duffing oscillator with time delay control. The corresponding amplitude response and phase response equation, respectively, of ⅓ stationary subharmonic resonance is obtained, when amplitude and phase does not change in dependence on slow time scale τ_2, that is, when $A(\tau_2)=A=const$, $\Phi_0(\tau_2)=\Phi_0=const$, $\gamma(\tau_2)=\gamma=const$. Consequently, $\mathrm{d}A(\tau_2)/\mathrm{d}\tau_2=0,\ \mathrm{d}\gamma(\tau_2)/\mathrm{d}\tau_2=0,\ \mathrm{d}\sigma(\tau_2)/\mathrm{d}\tau_2=0$ and Eqs. (41) and (42) are reduced on the form:

$$cA-\frac{3\beta A^2 p}{8\omega_L\left[8\omega_L^{\,2}+\varepsilon\sigma(6\omega_L+\varepsilon\sigma)\right]}\sin(\gamma)+\frac{a}{\omega_L}A\sin(\omega_L\theta)-bA\cos(\omega_L\theta)=0\,, \tag{43}$$

$$\sigma-\frac{9}{8}\frac{\beta}{\omega_L}A^2-\frac{9\beta p^2}{4\omega_L\left[8\omega_L^{\,2}+\varepsilon\sigma(6\omega_L+\varepsilon\sigma)\right]^2}-\frac{9\beta pA}{8\omega_L\left[8\omega_L^{\,2}+\varepsilon\sigma(6\omega_L+\varepsilon\sigma)\right]}\cos(\gamma)+\frac{3a}{\omega_L}\cos(\omega_L\theta)+3b\sin(\omega_L\theta)=0\,\cdot \tag{44}$$

By squaring Eqs. (43) and (44) and adding, we get the equation of the amplitude response of stationary subharmonic resonance with time delay control

$$\left\{9\left[c+\frac{a}{\omega_L}\sin(\omega_L\theta)-b\cos(\omega_L\theta)\right]^2+\left[\sigma-\frac{9\beta}{8\omega_L}A^2-\frac{9\beta p^2}{4\omega_L\left[8\omega_L^{\,2}+\varepsilon\sigma(6\omega_L+\varepsilon\sigma)\right]^2}+\frac{3a}{\omega_L}\cos(\omega_L\theta)+3b\sin(\omega_L\theta)\right]^2\right\}A^2 =\frac{81\beta^2p^2A^4}{64\omega_L^{\,2}\left[8\omega_L^{\,2}+\varepsilon\sigma(6\omega_L+\varepsilon\sigma)\right]^2}\,. \tag{45}$$

Equation (45) possesses a trivial solution A=0 and a nontrivial solution, which is obtained by solving the equation:

$$\left[\sigma-\frac{9\beta}{8\omega_L}A^2-\frac{9\beta p^2}{4\omega_L\left[8\omega_L^{\,2}+\varepsilon\sigma(6\omega_L+\varepsilon\sigma)\right]^2}+\frac{3a}{\omega_L}\cos(\omega_L\theta)+3b\sin(\omega_L\theta)\right]^2 -\frac{81\beta^2p^2A^2}{64\omega_L^{\,2}\left[8\omega_L^{\,2}+\varepsilon\sigma(6\omega_L+\varepsilon\sigma)\right]^2}+\left[3c+\frac{3a}{\omega_L}\sin(\omega_L\theta)-3b\cos(\omega_L\theta)\right]^2=0\,. \tag{46}$$

When Duffing oscillator is not controlled, that is, when $a=b=0$, then amplitude response of stationary subharmonic resonance is reduced on the form:

$$\left[\sigma-\frac{9\beta}{8\omega_L}A^2-\frac{9\beta p^2}{4\omega_L\left[8\omega_L^{\,2}+\varepsilon\sigma(6\omega_L+\varepsilon\sigma)\right]^2}\right]^2-\frac{81\beta^2p^2A^2}{64\omega_L^{\,2}\left[8\omega_L^{\,2}+\varepsilon\sigma(6\omega_L+\varepsilon\sigma)\right]^2}+9c^2=0\,\cdot \tag{47}$$

7. Computational aspects

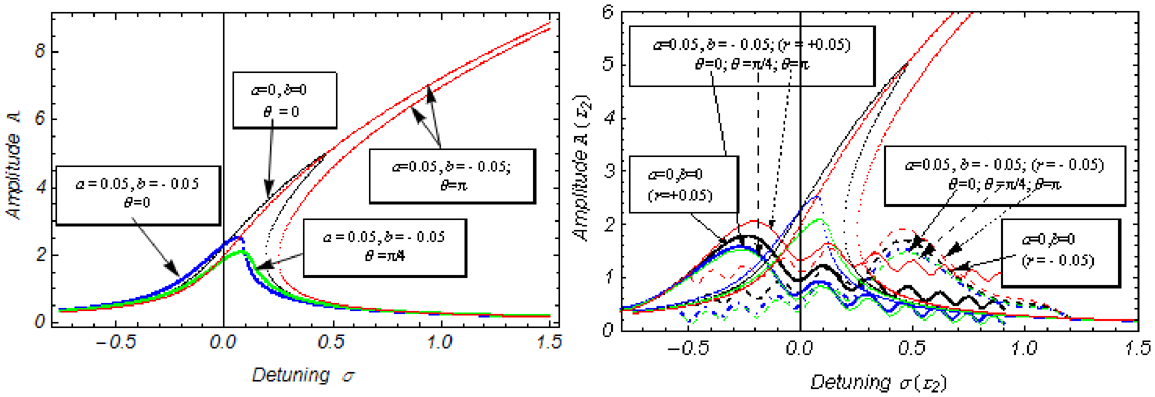
The present method for both fundamental and ⅓ subharmonic resonance of Duffing oscillator with time delay control is implemented in programming environment Mathematica® to apply symbolic as well as numeric computational capabilities. For example, stationary resonances (Eqs. (25), (28) for fundamental and (46), (47) for ⅓ subharmonic resonance, respectively) are computed algebraically. On contrary, the corresponding nonstationary resonances (Eq. (21), (22) for fundamental and Eq. (41), (42) for ⅓ subharmonic resonance, respectively) are computed by numerical integration, using embedded Runge-Kutta method. The question is, why not use the numerical integration directly on Duffing equation (7), however there are many reasons, why we not use this approach. The first reason is that we are looking for the steady-state oscillations and thus we must integrate Eq. (7) each time until the transient phenomenon dies. Because we must repeat this long lasting computation for each particular value of detuning (and time delay as well), the approach is evidently impractical. This inefficiency becomes even more apparent in computation of nonstationary resonances, where detuning is slowly varied and a special strategy must be applied to achieve the elimination of the transient phenomenon. The second reason is, that numerical integration of Eq. (7) is able to compute stable solutions only. Therefore, unstable branches of resonance curves cannot be obtained by numerical integration of Eq. (7), however they are easily computed by the present method. If we limit only on stationary resonances with delay control, we can alternatively apply the incremental harmonic balance (IHB) method [3], which will be presented by authors for a wide kinds of dynamic structures in a forthcoming paper. The preliminary results shows a good agreement between EL-P and IHB method in the case of Duffing oscillator with delay control.

8. Results

Now look on results of EL-P analysis of fundamental and ⅓ subharmonic resonance of Duffing oscillator with time delay control. Both analysis are performed by using programming envinronment Mathematica®, where the following values of parameters c=0.05, ω_L=1, β=0.05 are choosen in both cases. The excitation amplitude in the case of fundamental resonance is choosen to be equal p=0.5, while in the case of subharmonic resonance it is p=30. Analysis of stationary resonances in both cases are computed at first, while the corresponding analysis of nonstationary resonances follows to bring out important differences. Analysis of stationary resonance without control, that is, with values a=0, b=0 of gain parameters is performed in accordance with Eqs. (28),(29) in the case of fundamental resonance and by using Eq. (47) in the case of ⅓ subharmonic resonance. The stationary resonance of Duffing oscillator with time delay control is computed with values a=0.05, b= -0.05 of gain parameters and various time delays taking values θ=0, θ=π/4 and θ=π, respectively, where Eq. (25) is used in the case of fundamental and Eq. (46), respectively in the case of ⅓ subharmonic resonance. Courses of stationary fundamental resonance of Duffing oscillator without control and with applied control are plotted on the Fig.2.a and corresponding nonstationary resonances on the Fig. 2.b. The variable excitation frequency ω and corresponding detuning σ are assumed to be linear functions of the slow time scale:

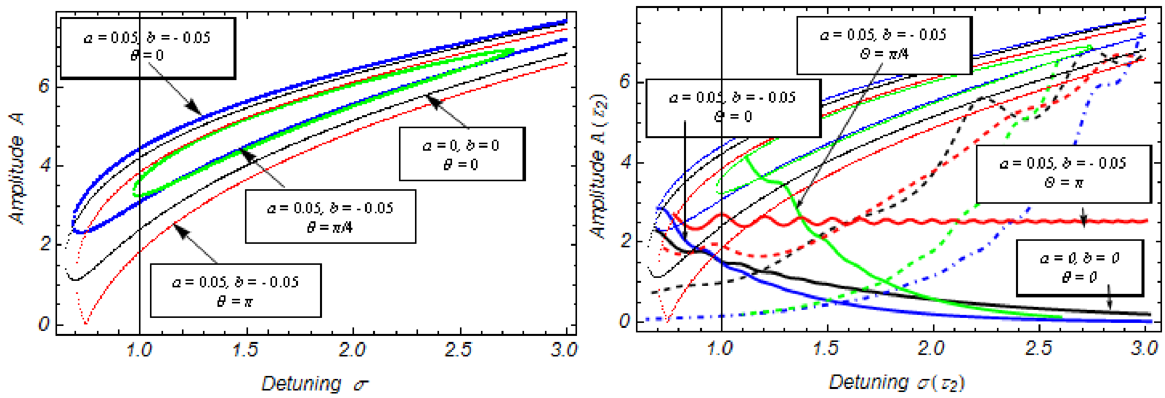
$$\sigma(\tau_2) = \sigma_0 + r\tau_2 , \qquad (48)$$

where σ_0 denotes the detuning in the time τ_2=0 and r denotes the rate of detuning change. In the analysis of both fundamental and ⅓ subharmonic resonance, respectively, the rate of detuning change r=±0.05 is used (the sign + corresponds to the passage with increasing of detuning (or increasing of excitation frequency) and the sign – corresponds to the passage with decreasing of detuning (or decreasing of the excitation frequency). From both Figures 2.a,b is apparent, that a small time delay (such as θ=π/4, for example) causes the reduction of the peak amplitude of the fundamental resonance curve, while a great time delay (θ=π in the study) causes the magnification of the peak amplitude. Thus for the case of fundamental resonance we can conclude, that the desired peak amplitude can be controlled by the proper choice of delay. However, the courses of nonstationary fundamental resonances show a drastic difference, which is characterized in change of amplitude and appearance of many amplitude peaks. After the first amplitude peak, heights of subsequent amplitude peaks are smaller and smaller. Stationary fundamental resonances of Duffing oscillator are ploted in Fig. 2.b for comparison. Response curves obtained for stationary and nonstationary ⅓ subharmonic resonances are ploted on Figures 3.a,b. Fig. 3.a shows, that delays in the case of stationary ⅓ subharmonic resonance of Duffing oscillator cause a different behavior of the controlled system in respect to the fundamental resonance. When the feedback control of Duffing oscillator is applied and the delay is equal zero, the response curve has a slightly larger amplitude as in the uncontrolled case. On contrary, when the feedback control of Duffing oscillator is applied and the delay has a great value, such as θ=π, the response curve has a slightly smaller amplitude as in the uncontrolled case. Therefore, changing of delay cannot be so effective in reduction of amplitude as in the case of fundamental resonance. Nonstationary ⅓ subharmonic resonances of Duffing oscillator show a drastic change in respect to corresponding stationary resonances, which are ploted in Fig. 3.b for comparison.



a) b)

Fig 2. Fundamental resonance of Duffing oscillator with time delay control. a) stationary resonance without control (a=b=0) and with applied control at different values of delay θ. b) the corresponding nonstationary resonances with rate of detuning change $r = \pm 0.05$.



a) b)

Fig 3. ⅓ subharmonic resonance of Duffing oscillator with time delay control. a) stationary resonance without control (a=b=0) and with applied control at different values of delay θ. b) the corresponding nonstationary resonances with rate of detuning change $r = \pm 0.05$.

In both Figures 2.b and 3.b, nonstationary resonances are drawn with continuous lines when detuning increases and by broken lines, when detuning decreases.

9. Conclusion

In the paper it was shown, that EL-PM can be successfuly applied to the control of Duffing oscillator with time delay and therefore can be extended to problems of nonlinear theory of control. Moreover, it comes out, that EL-PM proves as useful in solving delayed nonlinear differential equations. The practical outcome of present analysis is that delays, which are troublesome in general, in fundamental resonance can be used with benefit, when the peak amplitude of resonance should be maintained in the prescribed range.

References

1. R. R. Pušenjak, *Extended Lindstedt-Poincare method for non-stationary resonances of dynamical systems with cubic nonlinearities*, J. Sound and Vib. **314**, 194-216, 2008.
2. R. Pušenjak, *Razvejitve pri Van der Pol-Duffingovem nihalu*, Stroj. vestnik **49**, št. 7/8, 370-384, 2003.
3. R. R. Pušenjak, M. M. Oblak. *Incremental harmonic balance method with multiple time variables for dynamical systems with cubic non-linearities***,** Int. j. numer. Methods eng., vol. **59**, iss. 2, 255-292, 2004.
4. A. Preymount, *Twelve Lectures on Structural Dynamics*, Springer, Dordrecht, 2013.
5. A. H. Nayfeh, D. T. Mook, *Nonlinear Oscillations*, Wiley, New York, 1979.
6. A. Y. T. Leung, T. C. Fung, *Phase increment analysis of damped Duffing Oscillators*, Int. J. Numer. Meth. Eng. **28**, 193—209, 1989.