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Viskozno dušene prečne vibracije osno gibajoče se strune

Viscously Damped Transverse Vibrations of an Axially-Moving String

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V tem prispevku predstavljamo analizo delovanja viskoznega dušenja na prečna nihanja osno gibajoče se strune. Analizirani linearni model viskoznega dušenja je zapisan v obliki $b_1 \frac{\partial w}{\partial t} + b_2 v \frac{\partial w}{\partial x}$. *Najprej smo rešili gibalno enačbo lastnega nihanja - linearno parcialno diferencialno enačbo. Nato smo analizirali vplive vrednosti koeficientov viskoznega dušenja* b_1 *in* b_2 *na lastne frekvence in na odziv sistema pri lastnih nihanjih. Pokazali smo, da je potrebno vrednosti koeficientov izbrati previdno, da bi se izognili fizikalno neustreznim odzivom.*

(Ključne besede: enačbe diferencialne, enačbe hiperbolične, nihanja dušena, strune gibajoče)

In this paper the linear viscous-damping mechanism acting on an axially-moving string is analyzed. The analyzed damping model is in the form $b_1 \frac{\partial w}{\partial t} + b_2 v \frac{\partial w}{\partial x}$. *The equation of motion, i.e., the linear partial differential equation, of the free, transverse vibrations of the string's span is solved first. Then the influence of the coefficients* b_1 *and* b_2 *on the natural frequencies and the free responses is studied. It was found that the values of the coefficients should be carefully selected in order to avoid physically unrealistic responses.*

(Keywords: partial differential equations, hyperbolic equations, free damped vibrations, moving string systems)

0 UVOD

Modeliranje osno premikajočih se struktur je deležno nezmanjšane pozornosti že zadnjih 50 let. Pri tem gre za modeliranje verig v verižnih gonilih [1], modeliranje lista pri žagah [2], ali pa modeliranje vej pri jermenskih pogonih [3]. Obsežen pregled modeliranja jermenov in jermenskih pogonov do leta 1992 je podan v prispevku [3]. Problematika osno gibajočega se modela strune je obravnavana v številnih virih, naj izpostavimo le nekatere: [4] do [10].

Prispevek [1] analizira prečna nihanja verig verižnih gonil. Analiza je pokazala, da se amplituda nihanja točke na sredini veje verige poveča, če se poveča osna hitrost potovanja verige, če upoštevamo le $\partial w/\partial t$ del modela viskoznega dušenja. Enak pojav sta odkrila avtorja v prispevku [2]. Prispevek [3] ga na kratko povzame.

Jermeni, ki so v rabi, so v glavnem zahtevne nekovinske strukture, pri katerih je modeliranje disipacije energije praktično neizogibno. Uporabimo

0 INTRODUCTION

The modelling of moving continua has received constant attention over the past 50 years; studies have included the modelling of the vibrations of transmission chains [1], the modelling of band saws [2], and the modelling of belts [3]. A comprehensive review of the modelling of belts and belt drives up until 1992 is presented in [3]. There are many papers dealing with the problem of the moving string: [4] to [10].

The paper of Mahalingam [1] deals with the transverse vibrations of power-transmission chains. It was reported that the mid-span amplitude of the chain span, when kinematically excited at one end of the span, increases when the chain's axial velocity increases if only the $\partial w/\partial t$ part of the chain's velocity is taken into account. The same phenomenon is reported in [2] and recapitulated in [3].

The modelling of energy dissipation in the form of viscous damping, or another more sophisticated energy-dissipation or rheological model, is lahko viskozni model disipacije energije ali pa katerega od bolj izpopolnjenih reoloških modelov ali modelov disipacije energije. Modeli, ki popisujejo viskoelastične lastnosti jermenov, se že uporabljajo pri modeliranju vej jermenov ([9] do [14]).

Viskozni model dušenja lahko služi kot ekvivalentni model disipacije energije še posebej v primerih kompozitnih struktur. Tu se lahko skriva fizikalno ozadje raziskovanega viskoznega modela.

Čeprav so bolj izpopolnjeni materialni modeli že vstopili v domeno modeliranja vej jermenov, pa pojav, opisan v delih [1] in [2], še ni bi deležen večje pozornosti. Slednje je cilj tega prispevka.

practically unavoidable when dealing with modern belts. These are mainly complex non-metal structures, where viscoelasticity plays an important role. The modelling of viscoelastic properties is presented in [9] to [14].

Viscous damping can provide a suitable way of equivalent energy-dissipation modelling, particularly when dealing with composite structures, and this can provide a physical background for the viscous model under consideration.

Although more sophisticated material models have been introduced for belt modelling, the viscous-damping model used in [1] and [2] has not been analyzed in detail. Such an analysis is the aim of this paper.

1 GIBALNA ENAČBA PREČNIH NIHANJ OSNO GIBAJOČE SE STRUNE

Prečni odmik strune od statične ravnovesne lege popisuje koordinata $w = w(x,t)$ (sl. 1).

Predpostavimo, da se struna osno giblje z osno hitrostjo $v = v(t)$. Diferencial koordinate prečnega pomika podaja enačba (1). Uporabimo jo za izpeljavo hitrosti strune v prečni smeri, enačba (2):

1 THE EQUATION OF MOTION OF THE TRANSVERSE VIBRATIONS OF AN AXIALLY-MOVING STRING

The transverse vibrations of a string are represented by the transverse displacement $w = w(x,t)$ (Fig. 1).

Let us suppose that the string is moving with an axial velocity $v = v(t)$. The differential of the displacement, Eq. (1), is used to deduce the transverse velocity of the string, Eq. (2):

$$\mathrm{d}w = \frac{\partial w}{\partial x}\mathrm{d}x + \frac{\partial w}{\partial t}\mathrm{d}t \tag{1}$$

$$\frac{\mathrm{d}w}{\mathrm{d}t} = \dot{w} = \frac{\partial w}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial w}{\partial t} = \frac{\partial w}{\partial x}v + \frac{\partial w}{\partial t} \tag{2}$$

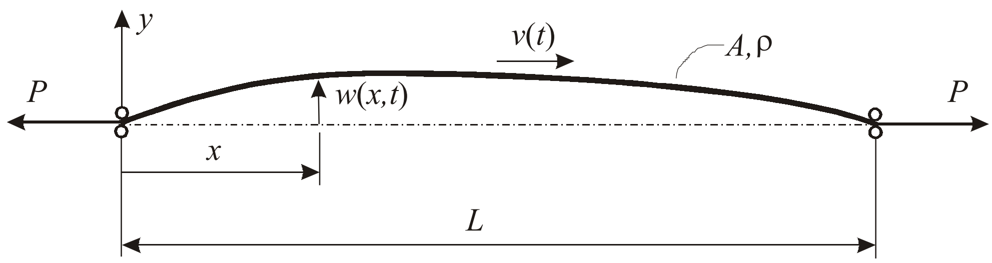
Tako zapišemo diferencialna operatorja:

Hence, the differential operators can be deduced as

$$\frac{\mathrm{D}}{\mathrm{D}t} = \frac{\partial}{\partial x}v + \frac{\partial}{\partial t} \quad \text{in/and} \quad \frac{\mathrm{D}^2}{\mathrm{D}t^2} = v^2\frac{\partial^2}{\partial x^2} + 2v\frac{\partial^2}{\partial x \partial t} + \frac{\partial^2}{\partial t^2} + \dot{v}\frac{\partial}{\partial x} \tag{3}$$

Gibalno enačbo prečnih nihanj (4) osno gibajoče se strune izpeljemo z uporabo 2. Newtonovega zakona:

The equation of motion (4) can be deduced from Newton's second law for a differentially small section of the string, and written as:



Sl. 1. *Osno gibajoča se struna*
Fig. 1. *The axially-moving string*

$$\rho A \frac{D^2 w(x,t)}{Dt^2} = P \frac{\partial^2 w(x,t)}{\partial x^2} - d \frac{D w(x,t)}{Dt} \quad (4),$$

kjer popisujejo: ρ gostoto gradiva strune, A velikost prečnega prereza strune, P natezno obremenitev strune in d koeficient viskoznega dušenja. Leva stran enačbe (4) pomeni vztrajnostno silo v sistemu. Prvi člen na desni strani iste enačbe popisuje aktivno silo zaradi natezanja strune. Drugi člen popisuje silo viskoznega dušenja. Z uporabo diferencialnih operatorjev (3) dobimo gibalno enačbo v obliki:

where ρ is the density of the string material, A is the area of the cross-section of the string, P is the tension force and d is the viscous damping coefficient. The left-hand side of Equation (4) represents the inertial forces of the system. The first part of the right-hand side of the same equation stands for the active force due to the string's tension, and the second part stands for the viscous damping force acting on the string. By considering the differential operators, Eq. (3), the equation of motion can be rewritten as:

$$P \frac{\partial^2 w}{\partial x^2} - \rho A v^2 \frac{\partial^2 w}{\partial x^2} - 2 \rho A v \frac{\partial^2 w}{\partial x \partial t} - \rho A \frac{\partial^2 w}{\partial t^2} - d \frac{\partial w}{\partial t} - \left(\rho A \frac{dv}{dt} + d\, v \right) \frac{\partial w}{\partial x} = 0 \quad (5).$$

Upoštevaje nespremenljivo osno hitrost strune, $dv/dt = 0$, ter z deljenjem enačbe s konstanto ρA, dobimo gibalno enačbo v obliki:

Taking the constant velocity into account, $dv/dt = 0$, and dividing the equation by ρA, the equation becomes:

$$(c^2 - v^2) \frac{\partial^2 w}{\partial x^2} - 2 v \frac{\partial^2 w}{\partial x \partial t} - \frac{\partial^2 w}{\partial t^2} - b \frac{\partial w}{\partial t} - b\, v \frac{\partial w}{\partial x} = 0 \quad ; \quad c = \sqrt{\frac{P}{\rho A}} \quad ; \quad b = \frac{d}{\rho A} \quad (6).$$

Namen tega prispevka je analizirati gibalno enačbo lastnega nihanja osno gibajoče se strune, če imata koeficienta b različne vrednosti. Tako dobimo enačbo, ki nas zanima:

The aim of this paper is to analyze the equation of motion for different values of b. Hence, the equation of interest is:

$$(c^2 - v^2) \frac{\partial^2 w}{\partial x^2} - 2 v \frac{\partial^2 w}{\partial x \partial t} - \frac{\partial^2 w}{\partial t^2} - b_1 \frac{\partial w}{\partial t} - b_2\, v \frac{\partial w}{\partial x} = 0 \quad (7),$$

kjer so $b_1 \geq 0$, $b_2 \geq 0$ in $0 \leq v < c$, kar predpostavlja podkritično osno hitrost strune.

where $b_1 \geq 0$, $b_2 \geq 0$ and $0 \leq v < c$ applying the subcritical string's axial velocity.

2 ANALITIČNA REŠITEV GIBALNE ENAČBE PREČNIH NIHANJ OSNO GIBAJOČE SE STRUNE

2 AN ANALYTICAL SOLUTION OF THE EQUATION OF MOTION FOR THE TRANSVERSE VIBRATIONS OF AN AXIALLY-MOVING STRING

Enačbo (7) preslikamo v drugo kanonično obliko s preslikavo:

Equation (7) can be transformed into the second canonical form by the transformation:

$$\begin{aligned} \zeta &= \alpha x + \beta t \\ \eta &= \gamma x + \delta t \end{aligned} \quad ; \quad \begin{vmatrix} \alpha & \beta \\ \gamma & \delta \end{vmatrix} \neq 0 \quad (8).$$

Če se hočemo znebiti mešanega odvoda v enačbi (7), morajo parametri preslikave zasesti naslednje vrednosti: $\alpha = 1$, $\beta = 0$, $\gamma = v/(c^2 - v^2)$ in $\delta = 1$, preslikavo zapišemo kot:

To get rid of the second mixed derivative in Eq. (7) the values of the transformation parameters should be $\alpha = 1$, $\beta = 0$, $\gamma = v/(c^2 - v^2)$ and $\delta = 1$, and the transformation can be written as:

$$\begin{aligned} \zeta &= x \\ \eta &= x\, v/(c^2 - v^2) + t \end{aligned} \quad (9).$$

Tako lahko gibalno enačbo zapišemo v drugi kanonični obliki kot:

The equation of motion is thus given in the second canonical form as:

$$(c^2 - v^2) \frac{\partial^2 w}{\partial \zeta^2} - \frac{c^2}{c^2 - v^2} \frac{\partial^2 w}{\partial \eta^2} - b_2\, v \frac{\partial w}{\partial \zeta} - \left(\frac{v^2}{c^2 - v^2} b_2 + b_1 \right) \frac{\partial w}{\partial \eta} = 0 \quad (10).$$

Enačbo (10) rešujemo z Euler-Fourierjevim nastavkom, ki loči časovno in krajevno spremenljivko:

Equation (10) can be solved by using the Euler-Fourier approach of variables separation,

$$w(\zeta,\eta)=W(\zeta)T(\eta) \qquad (11).$$

Nastavek (11) vnesemo v enačbo (10) in delimo z nastavkom. Tako lahko enačbo (10) zapišemo kot:

By putting the supposed solution (11) into (10) and dividing the latter by the solution itself, Eq. (10) can be rewritten as:

$$(c^2-v^2)\frac{W''}{W}-b_2 v\frac{W'}{W}=\frac{c^2}{c^2-v^2}\frac{\ddot{T}}{T}+\left(\frac{v^2}{c^2-v^2}b_2+b_1\right)\frac{\dot{T}}{T}=-\omega^2 \qquad (12).$$

In na tej podlagi sestavimo dve navadni diferencialni enačbi:

And thus two ordinary differential equations are formed:

$$(c^2-v^2)W''-b_2 vW'+\omega^2 W=0 \qquad (13)$$

$$\frac{c^2}{c^2-v^2}\ddot{T}+\left(\frac{v^2}{c^2-v^2}b_2+b_1\right)\dot{T}+\omega^2 T=0 \qquad (14).$$

Prvo rešujemo navadno diferencialno enačbo (13), da bi določili izraz za parameter ω. Rešitev predpostavimo v obliki $W(\zeta)=Ce^{\lambda\zeta}$ in dobimo karakteristični polinom:

The ordinary differential equation (13) should be solved first in order to get the expression for the parameter ω. The solution is assumed to be in the form $W(\zeta)=Ce^{\lambda\zeta}$, and the characteristic polynomial is formed as:

$$\lambda^2-\frac{b_2 v}{c^2-v^2}\lambda+\frac{\omega^2}{c^2-v^2}=0 \qquad (15)$$

s koreni

with the roots

$$\lambda_{1,2}=\frac{1}{2}\left[\frac{b_2 v}{c^2-v^2}\pm\sqrt{\left(\frac{b_2 v}{c^2-v^2}\right)^2-4\frac{\omega^2}{c^2-v^2}}\right]=\frac{1}{2}\tilde{\alpha}\pm i\tilde{\gamma} \qquad (16),$$

kjer so $\tilde{\alpha}=b_2 v/(c^2-v^2)$, $\tilde{\beta}=\omega^2/(c^2-v^2)$ in $\tilde{\gamma}=\sqrt{\tilde{\beta}-(\tilde{\alpha}/2)^2}$. Izraz pod korenom enačbe za $\tilde{\gamma}$ je vedno pozitiven, če le velja $v\leq c$. Ta izraz lahko zapišemo tudi kot $\omega^2>\left(\frac{b_2}{2}\right)^2\frac{v^2}{c^2-v^2}$, njegovo veljavo pa ocenimo na podlagi enačbe (20).

Rešitev enačbe (13) je linearna kombinacija rešitve obeh korenov:

where $\tilde{\alpha}=b_2 v/(c^2-v^2)$, $\tilde{\beta}=\omega^2/(c^2-v^2)$ and $\tilde{\gamma}=\sqrt{\tilde{\beta}-(\tilde{\alpha}/2)^2}$. The expression under the square root in $\tilde{\gamma}$ is non-negative as long as $v\leq c$. This expression can also be rewritten as $\omega^2>\left(\frac{b_2}{2}\right)^2\frac{v^2}{c^2-v^2}$. Its validity can be verified on the basis of Eq. (20).

The solution of (13) is a linear combination of both roots:

$$W(\zeta)=e^{\tilde{\alpha}L/2}(C_1\cos\tilde{\gamma}\zeta+C_2\sin\tilde{\gamma}\zeta) \qquad (17),$$

kjer sta C_1 in C_2 konstanti. Funkcija (17) mora zadostiti robnim pogojem:

where C_1 and C_2 are constants. The function (17) must satisfy the boundary conditions

$$\begin{aligned} w(0,t)=0 \quad&\Rightarrow\quad W(0)T(t)=0 \quad&\Rightarrow\quad W(0)=0=C_1\\ w(L,t)=0 \quad&\Rightarrow\quad W(L)T(L\,v/(c^2-v^2)+t)=0 \quad&\Rightarrow\quad W(L)=0=C_2 e^{\tilde{\alpha}L/2}\sin\tilde{\gamma}L\end{aligned} \qquad (18),$$

iz česar izhaja:

which yields

$$\sin\tilde{\gamma}L=0 \quad\Rightarrow\quad \tilde{\gamma}_k=\frac{k\pi}{L}=\sqrt{\tilde{\beta}-\tilde{\alpha}^2/2} \quad ; \quad k=1,2,3,... \qquad (19).$$

Iz enačbe (19) dobimo izraz za ω_k; $k=1,2,3,\ldots$

It is straightforward to derive an expression for ω_k; $k=1,2,3,\ldots$ from Eq. (19).

$$\omega_k^2 = (c^2 - v^2)\frac{k^2\pi^2}{L^2} + \frac{v^2}{c^2 - v^2}\left(\frac{b_2}{2}\right)^2 \qquad (20).$$

Nabor navadnih diferencialnih enačb (14) lahko, glede na funkcijo $T_k(t)$, zapišemo kot:

With respect to the time function $T_k(t)$, the set of the ordinary differential equations (14) can be rewritten as:

$$\ddot{T}_k + \left(\frac{v^2}{c^2}(b_2 - b_1) + b_1\right)\dot{T}_k + \frac{c^2 - v^2}{c^2}\omega_k^2 T_k = 0 \qquad (21).$$

Pri reševanju diferencialne enačbe (21) uporabimo nastavek v obliki $T_k(\eta) = C_k\, e^{\lambda_k \eta}$ in dobimo karakteristični polinom:

The solution is assumed to be in the form $T_k(\eta) = C_k\, e^{\lambda_k \eta}$, and the characteristic polynomial is formed as:

$$\lambda_k^2 + \left(\frac{v^2}{c^2}(b_2 - b_1) + b_1\right)\lambda_k + \frac{c^2 - v^2}{c^2}\omega_k^2 = 0 \qquad (22)$$

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with the roots

$$\lambda_{k_{1,2}} = \frac{1}{2}\left[-\left(\frac{v^2}{c^2}(b_2 - b_1) + b_1\right) \pm \sqrt{\left(\frac{v^2}{c^2}(b_2 - b_1) + b_1\right)^2 - 4\,\omega_k^2\,\frac{c^2 - v^2}{c^2}}\right] = \frac{1}{2}\left[-\hat{\alpha} \pm \sqrt{\hat{\alpha}^2 - 4\,\hat{\beta}_k}\right] \qquad (23),$$

kjer sta $\hat{\alpha} = \frac{v^2}{c^2}(b_2 - b_1) + b_1$ in $\hat{\beta}_k = \omega_k^2\,\frac{c^2 - v^2}{c^2}$.
Če velja:

where $\hat{\alpha} = \frac{v^2}{c^2}(b_2 - b_1) + b_1$ and $\hat{\beta}_k = \omega_k^2\,\frac{c^2 - v^2}{c^2}$.
If the relation:

$$\hat{\beta}_k > \frac{1}{4}\hat{\alpha}^2 \quad \Rightarrow \quad \omega_k^2 > \frac{1}{4}\frac{c^2 - v^2}{c^2}\left(\frac{v^2}{c^2 - v^2}b_2 + b_1\right)^2, \qquad (24),$$

potem imamo opravka z nihanjem, torej s podkritičnim dušenjem. Če pa neenačbi (24) ni zadoščeno, nihanja ni in opredelimo dušenje kot kritično ali nadkritično.

is satisfied, then vibrations exist and the system is underdamped. Otherwise, the system is overdamped or critically damped.

2.1 Podkritično dušenje

2.1 The underdamped system

V tem primeru je izrazu (24) zadovoljeno. Rešitev karakterističnega polinoma (22) zapišemo kot:

In this case Expression (24) is satisfied. The roots of the characteristic polynomial (22) are:

$$\lambda_{k_{1,2}} = -\frac{1}{2}\hat{\alpha} \pm i\hat{\gamma}_k \qquad (25),$$

kjer je $\hat{\gamma}_k = \sqrt{\hat{\beta}_k - (\hat{\alpha}/2)^2}$. Rešitev diferencialne enačbe (21) zapišemo kot:

where $\hat{\gamma}_k = \sqrt{\hat{\beta}_k - (\hat{\alpha}/2)^2}$. The solution of the ordinary differential equation (21) can be written as:

$$T_k(\eta) = e^{-\hat{\alpha}\eta/2}\left(A\cos(\hat{\gamma}_k\eta) + B\sin(\hat{\gamma}_k\eta)\right) \qquad (26).$$

Iz enačbe (26) vidimo, da je parameter $\hat{\gamma}_k$ enak lastni frekvenci dušenega nihanja:

It can be seen in Eq. (26) that the parameter $\hat{\gamma}_k$ is equal to the damped natural frequency:

$$\omega_{d_k}^2 = \hat{\gamma}_k^2 = \left(\frac{c^2 - v^2}{c}\right)^2\frac{k^2\pi^2}{L^2} + \frac{1}{4}\frac{c^2 - v^2}{c^2}\left[\frac{v^2}{c^2}(b_2 - b_1)^2 - b_1^2\right] \quad ; \quad k = 1, 2, 3, \ldots \qquad (27).$$

Odziv k-tega načina izpeljemo kot:

The k-th mode is as follows:

$$w_k(x,t) = e^{\frac{1}{2}\frac{v}{c^2}(b_2 - b_1)x}\,\sin\frac{k\pi x}{L}\,e^{-\frac{1}{2}\frac{1}{c^2}[v^2(b_2 - b_1) + c^2 b_1]t}\left[A_k\cos\left(\omega_{d_k}\left(\frac{xv}{c^2 - v^2} + t\right)\right) + B_k\sin\left(\omega_{d_k}\left(\frac{xv}{c^2 - v^2} + t\right)\right)\right] \qquad (28).$$

2.2 Kritično dušenje

V primeru kritičnega dušenja velja $\hat{\beta} = \hat{\alpha}^2/4$ oziroma $\omega^2 = \frac{1}{4}\frac{c^2-v^2}{c^2}\left(\frac{v^2}{c^2-v^2}b_2 + b_1\right)^2$, kar pomeni, da ima karakteristični polinom (22) dve enaki realni rešitvi:

2.2 The critically damped system

In this case the equality $\hat{\beta} = \hat{\alpha}^2/4$ or $\omega^2 = \frac{1}{4}\frac{c^2-v^2}{c^2}\left(\frac{v^2}{c^2-v^2}b_2 + b_1\right)^2$ is established. This means that the characteristic polynomial (22) has two identical roots.

$$\lambda_{1,2} = -\frac{1}{2}\left(\frac{v^2}{c^2}(b_2 - b_1) + b_1\right) = -\frac{1}{2}\hat{\alpha} \qquad (29).$$

Rešitev diferencialne enačbe (21) zapišemo kot:

The solution of the ordinary differential equation (21) can be written as:

$$T(\eta) = \left(A + B\eta\right)e^{-\frac{1}{2}\hat{\alpha}\eta} \qquad (30)$$

in odziv *k*-tega načina kot:

and the expression for the *k*-th mode as:

$$\begin{aligned} w_k(x,t) \;=\; & e^{\frac{1}{2}\frac{v}{c^2}(b_2-b_1)x}\sin\frac{k\pi x}{L}\, e^{-\frac{1}{2}\frac{1}{c^2}[v^2(b_2-b_1)+c^2 b_1]t} \\ & \left[A_k + B_k\left(\frac{xv}{c^2-v^2}+t\right)\right] \end{aligned} \qquad (31).$$

2.3 Nadkritično dušenje

V primeru nadkritičnega dušenja se nihanja ne pojavijo ker velja $\hat{\beta}_k < \hat{\alpha}^2/4$ oziroma $\omega_k^2 < \frac{1}{4}\frac{c^2-v^2}{c^2}\left(\frac{v^2}{c^2-v^2}b_2 + b_1\right)^2$, kar tudi pomeni, da ima karakteristični polinom (22) dve različni realni rešitvi:

2.3. The overdamped system

In the case of the overdamped system the vibrations cannot appear and $\hat{\beta}_k < \hat{\alpha}^2/4$ or $\omega_k^2 < \frac{1}{4}\frac{c^2-v^2}{c^2}\left(\frac{v^2}{c^2-v^2}b_2 + b_1\right)^2$, which means that the characteristic polynomial (22) has two different real roots:

$$\lambda_{k_{1,2}} = \frac{1}{2}\hat{\alpha} \pm \hat{\varepsilon}_k \qquad (32),$$

kjer je $\hat{\varepsilon}_k = \sqrt{(\hat{\alpha}/2)^2 - \hat{\beta}_k} = i\hat{\gamma}_k$. Rešitev diferencialne enačbe (21) zapišemo kot:

where $\hat{\varepsilon}_k = \sqrt{(\hat{\alpha}/2)^2 - \hat{\beta}_k} = i\hat{\gamma}_k$. The solution of the ordinary differential equation (21) can be written as:

$$T_k(\eta) = e^{-\hat{\alpha}\eta/2}\left(A_k\cosh(\hat{\varepsilon}_k\eta) + B_k\sinh(\hat{\varepsilon}_k\eta)\right) \qquad (33)$$

in odziv *k*-tega načina kot:

and the expression for the *k*-th mode as:

$$w_k(x,t) = e^{\frac{1}{2}\frac{v}{c^2}(b_2-b_1)x}\sin\frac{k\pi x}{L}\, e^{-\frac{1}{2}\frac{1}{c^2}[v^2(b_2-b_1)+c^2 b_1]t}\left[A_k\cosh\left(\hat{\varepsilon}_k\left(\frac{xv}{c^2-v^2}+t\right)\right) + B_k\sinh\left(\hat{\varepsilon}_k\left(\frac{xv}{c^2-v^2}+t\right)\right)\right] \qquad (34).$$

3 REZULTATI IN RAZPRAVA

Pod predpostavko oziroma pri pogojih $b_1 \geq 0$, $b_2 \geq 0$ in $0 \leq v < c$ pričakujemo, da bo model viskoznega dušenja, kot model disipacije energije v sistemu, vplival na lastne frekvence dušenega nihanja strune na način, da se bodo omenjene lastne frekvence zmanjšale ob povečanju vrednosti koeficientov viskoznega dušenja. Druga domneva govori o tem, da se amplituda odziva sistema ne bi smela zmanjševati počasneje s povečanjem osne hitrosti strune.

3 RESULTS AND DISCUSSION

Assuming that $b_1 \geq 0$, $b_2 \geq 0$ and $0 \leq v < c$, one can expect that the viscous-damping energy-dissipation model would influence the natural frequencies of the vibrations in such a way that the frequencies would not increase with the increasing value of the coefficient of the viscous damping model. The second assumption is that the response's amplitude should not decrease more rapidly with the string's increasing axial velocity.

3.1 Vpliv koeficientov viskoznega dušenja na lastno frekvenco prečnega nihanja strune

Nihanje strune s svojo lastno frekvenco se pojavi le pri podkritičnem dušenju. Enačba (27) lastne frekvence dušenega nihanja je sestavljena iz dveh delov. Prvi del je enak lastni frekvenci nedušenega nihanja osno gibajoče se strune. Na drugi del pomembno vpliva viskozno dušenje. Pričakujemo, da se bo lastna frekvenca dušenega nihanja zmanjšala glede na lastno frekvenco nedušenega nihanja, iz česar izhaja matematično formuliran pogoj:

3.1 The influence of the viscous damping coefficients on the natural frequency of the damped transverse string vibrations

Let us focus first on the natural frequencies whose values should not increase if the damping also increases. It is clear that the expression for the natural frequency (27) is made up of two parts. The first part is actually the undamped natural frequency of the moving string, and the second part is influenced by the damping mechanism. Since the damping should decrease the natural frequency the following relation must be satisfied:

$$\frac{v^2}{c^2}(b_2-b_1)^2-b_1^2\leq 0 \qquad (35).$$

Rešitev tega pogoja je ploskev, ki jo podaja neenačba:

The solution of relation (35) is a half-plane defined by:

$$b_1\geq\frac{v}{c+v}b_2 \qquad (36),$$

le ta pa je omejena s premico:

and bounded by the straight line:

$$b_1=\frac{v}{c+v}b_2=k_1\,b_2 \qquad (37).$$

Koeficient strmine premice k_1 doseže najmanjšo vrednost pri $v=0$, $k_1(v=0)=0$, in največjo vrednost v mejnem primeru $v\to c$, $k_1(v\to c)=0{,}5$. Mejo med nihanjem strune in nadkritičnim obnašanjem slednje za vsak način posebej izpeljemo iz enačbe $\omega_k^2=\frac{1}{4}\frac{c^2-v^2}{c^2}\left(\frac{v^2}{c^2-v^2}b_2+b_1\right)^2$ (gl. 2.2).:

The coefficient k_1 reaches its minimum value at $v=0$, $k_1(v=0)=0$, and its maximum value in the limit case of $v\to c$, $k_1(v\to c)=0.5$. The border between the underdamped and overdamped systems' response for each mode can be deduced from $\omega_k^2=\frac{1}{4}\frac{c^2-v^2}{c^2}\left(\frac{v^2}{c^2-v^2}b_2+b_1\right)^2$, see subsection 2.2.:

$$b_{k_1}=f_k(b_2)=\frac{1}{c^2-v^2}\left[-b_2\,v^2+c\sqrt{b_2^2\,v^2+\left(2\frac{k\pi}{L}(c^2-v^2)\right)^2}\right] \qquad (38).$$

Enačba (38) doseže svojo najmanjšo vrednost v točki $b_{k_1}=b_{k_2}=2\frac{k\pi}{L}\sqrt{c^2-v^2}$ in se asimptotično približuje enačbi (37).

Equation (38) reaches its minimum at point $b_{k_1}=b_{k_2}=2\frac{k\pi}{L}\sqrt{c^2-v^2}$. It also asymptotically approaches Eq. (37).

3.2 Vpliv koeficientov viskoznega dušenja na amplitudo odziva strune

Druga domneva govori o tem, da se amplituda odziva sistema ne bi smela zmanjševati počasneje s povečanjem osne hitrosti strune v. Izrazi k-tega načina so podani v enačbah (28), (31) in (34). Prvo moramo najti vrh lastne oblike z zanemaritvijo faznega zaostajanja točk vzdolž dolžine strune, ki je skrito v koordinati η, kot primer naj bo enačba (28). Lastno obliko okarakterizira izraz:

3.2 The influence of the viscous damping coefficients on the response amplitude

In contrast, we expect that the response's amplitudes would not decrease more rapidly with the string's increasing axial velocity, v. The expressions for the k-th mode are given in Equations (28), (31) and (34). The maximum of the mode shape should be found first. By neglecting the different phase lags of the different points along the string's length, which are hidden in the coordinate η, e.g., see Eq. (28), the mode shapes are governed primarily by the following expression:

$$\bar{W}_k(x) = e^{\frac{-v}{2c^2}(b_2-b_1)x} \sin\frac{k\pi x}{L} \quad ; \quad k = 1,2,3,... \tag{39}$$

iz katerega lahko sklepamo o legi največje amplitude nihanja strune zaradi koordinate lege v odvisnosti od predznaka izraza $b_2 - b_1$. Če je $b_2 - b_1 \geq 0$, sledi, da se vrh pojavi na zadnjem polvalu lastne oblike, ko je $x_k = (2k-1)L/(2k)$, ter če je $b_2 - b_1 < 0$, sledi, da se vrh pojavi na prvem polvalu lastne oblike, ko je $x_k = L/(2k)$.

Amplitudo odziva, glede na enačbe (28), (31) in (34), dominantno opredeljuje enačba:

which applies a different shape-maximum position, as the expression $b_2 - b_1$ can have different signs. In the case of $b_2 - b_1 \geq 0$ the shape maximum is found at $x_k = (2k-1)L/(2k)$, and in the case of $b_2 - b_1 < 0$ the shape maximum is found at $x_k = L/(2k)$.

The response's amplitude is, according to Equations (28), (31) and (34), governed by the expression:

$$Amp(w(x_k,t)) = \exp\left[\frac{1}{2c^2}\left(v[b_2-b_1]x - [v^2(b_2-b_1) + c^2 b_1]t\right)\right] \tag{40}$$

kjer x lahko zavzame dve vrednosti $x_k = (2k-1)L/(2k)$ ali $x_k = L/(2k)$, glede na predznak izraza $b_2 - b_1$. Eksponentna funkcija je monotona, zato zadošča analiza eksponenta. Zahtevamo, da se eksponent ne povečuje s povečevanjem hitrosti strune, kar matematično zapišemo kot:

where x can take only two values, $x_k = (2k-1)L/(2k)$ or $x_k = L/(2k)$, depending on the sign of $b_2 - b_1$. The exponential function is monotonous, so an analysis of the exponent is sufficient. The demand for decreasing amplitudes with the string's increasing axial velocity can be mathematically formulated as:

$$\frac{\partial}{\partial v}\left(v[b_2-b_1]x_k - [v^2(b_2-b_1) + c^2 b_1]t\right) = (x_k - 2vt)(b_2-b_1) \leq 0 \tag{41}$$

Poglejmo si oba primera vrednosti $b_2 - b_1$. Če je $b_2 - b_1 \geq 0$, sledi $x_k = (2k-1)L/(2k)$ in $x_k - 2vt \leq 0$, kar velja za poljuben, dovolj velik t. Dovolj velik čas, ki zadovolji vse načine, dobimo, ko gre $k \to \infty$, je $t_{cr} = L/(2v)$, kar zadovolji enačbo (41). Če pa je $b_2 - b_1 < 0$, sledita $x_k = L/(2k)$ in $x_k - 2vt > 0$, kar pa je mogoče zagotoviti le za nekatere načine in za omejen čas. To pa v splošnem ne velja, saj je lastnih oblik neskončno veliko in ker lahko t zasede poljubne vrednosti. Sledi, da rešitev $b_2 - b_1 < 0$ ni fizikalno sprejemljiva.

Če povzamemo, dobimo sistem treh neenačb, od katerih sta pomembni le prvi dve:

Two different cases are possible again. In the first case, when $b_2 - b_1 \geq 0$, then $x_k = (2k-1)L/(2k)$ and $x_k - 2vt \leq 0$, which is true for sufficiently large t. The largest mode, $k \to \infty$, would give a time large enough, $t_{cr} = L/(2v)$, to satisfy the condition in Eq. (41). In the second case, when $b_2 - b_1 < 0$, then $x_k = L/(2k)$ and $x_k - 2vt > 0$, which can only be met for some nodes and for a limited amount of time, and cannot be met for all of the nodes. For this reason, the second case is considered to be unrealistic.

When summing together all of the constraints:

$$\begin{aligned} b_1 &\geq k_1 b_2 \quad ; \quad k_1 = \frac{v}{c+v} < k_2 \\ b_1 &\leq k_2 b_2 \quad ; \quad k_2 = 1 \\ t &\geq t_{cr} \quad ; \quad t_{cr} = \frac{L}{2v} \end{aligned}$$

in tako je mogoče opredeliti sprejemljivo območje koeficientov b_1 in b_2. Predstavljeno je na sliki 2 kot senčeno področje.

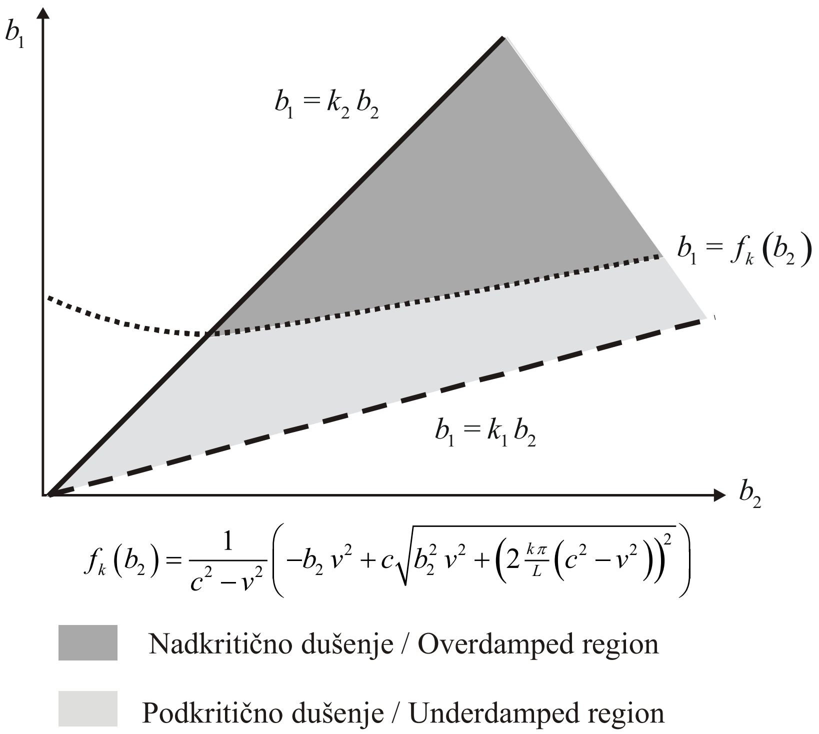
the region of acceptable values for b_1 and b_2 can be deduced. It is presented in Figure 2 as the shaded region between the solid and the dashed lines.

4 SKLEPI

V prispevku smo analizirali vplive viskoznega mehanizma dušenja na lastna prečna nihanja osno premikajoče se strune. Pokazali smo, da vrednosti parametrov b_1 in b_2 ne moremo povsem poljubno izbirati, če želimo ohraniti fizikalno korektno

4 CONCLUSIONS

The viscous-damping mechanism acting on the transverse vibrations of an axially-moving string was analyzed. It was shown that the values of the parameters b_1 and b_2 cannot be chosen completely freely if physically meaningless results are



Sl. 2. *Fizikalno sprejemljivo območje vrednosti koeficientov b_1 in b_2 leži med premicama v senčenem območju. Svetlejše senčeno območje pomeni nihanje strune, temnejše pa odziv strune z nadkritičnim dušenjem.*

Fig. 2. T*he acceptable-values region (shaded) for the viscous-damping coefficients b_1 and b_2. The lighter shaded region represents the underdamped system, and the darker shaded region represents the overdamped system.*

obnašanje sistema. Senčeno področje na sliki 2 pomeni območje dvojic vrednosti parametrov b_1 in b_2, ki daje fizikalno sprejemljive rezultate.

Kot rešitev problema predlagamo, da če je le mogoče, ne razlikujemo med vrednostmi b_1 in b_2, torej $b_1 = b_2 = b$. Tako pravilo ima močno fizikalno ozadje, saj je sila viskoznega dušenja enaka zmnožku koeficienta viskoznega dušenja in hitrosti delca strune. Slednja pa je definirana z enačbo (2) in uporabljena v zapisu gibalne enačbe (4), kjer pa ne obstaja matematični formalizem, ki bi lahko pripeljal do različnih vrednosti parametra b, kakor to srečamo v enačbi (7). Zatorej ni jasno, kaj je avtorje prispevkov [1] in [2] spodbudilo k taki uporabi modela viskoznega dušenja. Vendar, če uporabimo viskozno dušenje kot ekvivalentni mehanizem disipacije energije, na primer, v jermenih, se lahko pojavi potreba po različnih vrednostih parametra b, če želimo zajeti odtekajočo energijo vseh različnih mehanizmov disipacije energije v jermenu.

to be avoided. They are within the shaded region in Figure 2.

The solution would be not to distinguish between the values of b_1 and b_2 and to use only $b_1 = b_2 = b$, if possible. This might have a better physical explanation as the viscous damping force is proportional to the product of the viscous damping coefficient and the velocity. The latter is defined by (2) and used in (4), where there is no mathematical mechanism for obtaining different values for the coefficients b_1 and b_2, as in Eq. (7). But, if the viscous-damping model is used as an equivalent mechanism of energy dissipation in belts, as an example, the different values of b might be necessary for the viscous-damping model in order to encompass all the dissipated energy from the different mechanisms of energy dissipation in belts. The exact reason for using the notation b_1 and b_2 by the authors of [1] and [2] is unknown.

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