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On the minisymposium problem*

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Abstract

The generalized Oberwolfach problem asks for a factorization of the complete graph K_v into prescribed 2-factors and at most a 1-factor. When all 2-factors are pairwise isomorphic and v is odd, we have the classic Oberwolfach problem, which was originally stated as a seating problem: given v attendees at a conference with t circular tables such that the ith table seats a_i people and $\sum_{i=1}^{t} a_i = v$, find a seating arrangement over the $\frac{v-1}{2}$ days of the conference, so that every person sits next to each other person exactly once.

In this paper we introduce the related *minisymposium problem*, which requires a solution to the generalized Oberwolfach problem on v vertices that contains a subsystem on m vertices. That is, the decomposition restricted to the required m vertices is a solution to the generalized Oberwolfach problem on m vertices. In the seating context above, the larger conference contains a minisymposium of m participants, and we also require that pairs of these m participants be seated next to each other for $\left\lfloor \frac{m-1}{2} \right\rfloor$ of the days.

When the cycles are as long as possible, i.e. v, m and $v-m$, a flexible method of Hilton and Johnson provides a solution. We use this result to provide further solutions when $v \equiv m \equiv 2 \pmod 4$ and all cycle lengths are even. In addition, we provide extensive

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results in the case where all cycle lengths are equal to k, solving all cases when $m \mid v$, except possibly when k is odd and v is even.

Keywords: Minisymposium problem, (generalized) Oberwolfach problem, 2-factorizations, subsystems.

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1 Introduction

We assume that the reader is familiar with the fundamentals of graph theory and of design theory and refer them to [41] and [14], respectively. In particular, a factor is a spanning subgraph and an r-factor is a factor which is r-regular, so in a 1-factor every vertex has degree one and a 2-factor is a disjoint union of cycles. Given a collection of factors, $\mathcal{F}$, an $\mathcal{F}$-factorization of a graph G is a decomposition of the edges of G into subgraphs, each of which is isomorphic to some $F \in \mathcal{F}$. If $\mathcal{F} = \{F\}$ we speak of an F-factorization.

We use K_n to denote the complete graph on n vertices and K_n^* to denote the graph K_n when n is odd and $K_n - I$, where I is a 1-factor, when n is even. Similarly, $K_{m,n}$ denotes the complete bipartite graph with parts of sizes m and n. If the parts are X and Y, respectively, we may also speak of $K_{X,Y}$. A 2-factor is called *uniform* if all of its constituent cycles are of the same length; it is called *Hamiltonian** if its cycles have longest possible lengths given the requirements of the factorization. If a 2-factor, F, consists entirely of cycles of a particular length, k say, we refer to an F-factor and an F-factorization as a C_k-factor and C_k-factorization, respectively. Given a graph G, we denote by $G[n]$ the *lexicographic product* of G with the empty graph on n vertices. Specifically, the vertex set of $G[n]$ is $V(G) \times \mathbb{Z}_n$ (where $\mathbb{Z}_n$ denotes the cyclic group of order n) and $(x, i)(y, j) \in E(G[n])$ if and only if $xy \in E(G)$, $i, j \in \mathbb{Z}_n$.

The well known Oberwolfach problem asks for a 2-factorization of K_n^* into 2-factors all of which are isomorphic to a given 2-factor F. A summary of results up until 2006 can be found in [14, Section VI.12], in particular the case of uniform factors has been solved [1, 2, 26].

Theorem 1.1 ([1, 2, 26])**.** *Given integers $v, k \geq 3$, there is a C_k-factorization of K_v^* if and only if $k \mid v$, except that there is no C_3-factorization of K_6^* or K_{12}^*.*

The case when all cycles in F have even length has been completely solved in [6]. The case with exactly two cycles is solved in [38]. The case of the complete graph $K_\aleph$, where $\aleph$ is any infinite cardinal has been completely solved in [15]. In the related Hamilton-Waterloo problem two 2-factors F_1 and F_2 are specified and we are asked for a factorization of K_n^* into a given number of each of the factors. There has been much recent progress in this problem, see [3, 5, 7, 10, 11, 12, 17, 27, 28, 29, 32, 39, 40]. More generally, in the generalized Oberwolfach problem we are given a set of 2-factors $F_1, F_2, \ldots, F_t$ of K_n and positive integers $\alpha_1, \alpha_2, \ldots, \alpha_t$, where $\sum \alpha_i = \lfloor \frac{n-1}{2} \rfloor$, and are asked for a factorization of K_n^* which contains α_i copies of the 2-factor F_i, see [6, 13, 21]. A major recent development gives a non-constructive asymptotic existence result for the generalized Oberwolfach problem [23].

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Other graphs have also been considered. In particular, Liu has shown the following for the complete multipartite graph.

Theorem 1.2 ([30, 31]). *Let k, t and u be positive integers with $k \geq 3$. There exists a C_k-factorization of $K_t[u]$ if and only if $k \mid tu$, $(t-1)u$ is even, further k is even if $t = 2$, and $(k, t, u) \notin \{(3, 3, 2), (3, 6, 2), (3, 3, 6), (6, 2, 6)\}$.*

Originally the Oberwolfach problem was stated as a seating problem:

> Given an odd number v of attendees at a conference with t circular tables such that the ith table seats a_i people and $\sum_{i=1}^{t} a_i = v$, find a seating arrangement over the $\frac{v-1}{2}$ days of the conference, so that every person sits next to each other person exactly once.

In this paper we introduce the related *minisymposium problem*. In this case we require a solution to the generalized Oberwolfach problem on v vertices such that its restriction to a subset of m vertices constitutes a solution to the generalized Oberwolfach problem on m vertices. Another way of considering the problem asks for a solution to the generalized Oberwolfach problem on v vertices which contains a subsystem on m vertices. In the seating context above, the larger conference contains a minisymposium of m participants, and we also require that pairs of these m participants be seated next to each other for $\left\lfloor \frac{m-1}{2} \right\rfloor$ of the days. A similar problem has been considered, for example, in [9] for whist tournaments.

Section 2 gives the formal definition of a minisymposium factorization and some necessary conditions for its existence, as well as introduces some special cases. In Section 3 we show how to use a flexible theorem by Hilton and Johnson [25] to solve the case of Hamiltonian* 2-factors (where the cycles are as long as possible). The same section considers the case where all cycles are of even length and $v \equiv m \equiv 2 \pmod 4$. Section 4 considers the uniform case, where all cycles have the same length. We completely solve the case where all cycles are of length m when $(v-1)m$ is even. In Section 5 we discuss and give some preliminary results on factorizations that contain more than one subsystem. We provide some concluding remarks in the final section.

2 Preliminaries

We begin by giving a formal definition of a minisymposium factorization. The minisymposium problem is equivalent to the original Oberwolfach problem when $v = m$. Hence we will generally assume that $v > m$.

Definition 2.1. Given positive integers v and m with $v \geq m$, let

$$\mathcal{F} = \left\{ F_i : \; i = 1, \dots, \left\lfloor \frac{v-1}{2} \right\rfloor - \left\lfloor \frac{m-1}{2} \right\rfloor \right\},$$

be a collection of 2-factors on v vertices and let

$$\mathcal{G} = \left\{ (T_i, U_i) : i = 1, \dots, \left\lfloor \frac{m-1}{2} \right\rfloor \right\},$$

where the T_i are 2-factors on m vertices and the U_i are 2-factors on $v - m$ vertices. We define a *minisymposium factorization* MSF$(\mathcal{F}, \mathcal{G})$ as a factorization of K_v^* into 2-factors $F \in \mathcal{F}$ and $G_i = T_i \cup U_i$, where $(T_i, U_i) \in \mathcal{G}$, such that $\mathcal{T} = \{T_i : \; i = 1, \dots, \lfloor \frac{m-1}{2} \rfloor\}$ is a factorization of a subgraph of K_v^* isomorphic to K_m^*.

Note that with the notation MSF($\mathcal{F},\mathcal{G}$), we assume that the parameters v, m, $\mathcal{T}$ and $\mathcal{U} = \{U_i : i = 1, \ldots, \lfloor \frac{m-1}{2} \rfloor\}$ are defined implicitly. We may also use the notation MSF($\mathcal{F}, (\mathcal{T},\mathcal{U})$), if we wish to explicitly refer to the factorizations $\mathcal{T}$ and $\mathcal{U}$.

An MSF($\mathcal{F},\mathcal{G}$) can be thought of as a 2-factorization of K_v^* with a subsystem of size m. When $m = 1$ or 2, this is just a factorization of K_v^* into 2-factors in $\mathcal{F}$, which is equivalent to a solution of the generalized Oberwolfach problem, and so we will assume $m \geq 3$. Similarly, when $m = v$, this is a factorization into the T_i, so we assume that $m < v$.

Removing the subsystem, we can talk about a 2-factorization of K_v^* with a "hole" of size m. However, care must be taken when either v or m is even as the placement of the various 1-factors must be considered, as noted below.

We note that the size of $\mathcal{F}$ is

$$\left\lfloor \frac{v-1}{2} \right\rfloor - \left\lfloor \frac{m-1}{2} \right\rfloor = \begin{cases} \frac{v-m}{2} & v \equiv m \pmod 2 \\ \frac{v-m+1}{2} & v+1 \equiv m \equiv 0 \pmod 2 \\ \frac{v-m-1}{2} & v \equiv m+1 \equiv 0 \pmod 2. \end{cases}$$

- In the case when both v and m are even, we are considering a factorization of $K_v^* = K_v - I$, where I is a 1-factor, containing a factorization of a subgraph $G - J$ of K_v^* where $G \cong K_m$ and J is a 1-factor of G contained in I.
- When v is even and m is odd, we are considering a factorization of $K_v - I$, where I is a 1-factor, containing a factorization of a subgraph $G \cong K_m$. Note that none of the edges of I are contained in G.
- When v is odd and m is even, we are considering a factorization of $K_v^* = K_v$ containing a factorization of $G - J$, where $G \cong K_m$ and J is a 1-factor of G. We note that the edges of J are not covered by factors in $\mathcal{T}$, and hence must be covered by factors in $\mathcal{F}$.
- When both v and m are odd there is no 1-factor to consider.

Since in all cases $G \cong K_m$, we will henceforth refer to it as *the* K_m. We note that none of the edges of the K_m are covered by $\mathcal{F}$, except in the case of m even and v odd; in this case, the edges of the 1-factor J of the K_m are covered by $\mathcal{F}$. We use this observation in the proofs of the lemmas below, where for a given 2-factor F, we define $a_k(F)$ to be the number of cycles of length k in F.

Lemma 2.2. *For a given v and m, if there is a minisymposium factorization, MSF($\mathcal{F},\mathcal{G}$), then for each factor $F \in \mathcal{F}$ using c_F edges in the K_m,*

$$v - m \geq -c_F + \sum_{i=3}^{v} a_i(F) \left\lceil \frac{i}{2} \right\rceil, \tag{2.1}$$

$$m \leq c_F + \sum_{i=3}^{v} a_i(F) \left\lfloor \frac{i}{2} \right\rfloor. \tag{2.2}$$

In particular, if v is even, or m is odd, all of the $c_F = 0$ and therefore,

$$\frac{\sum_{i=3}^{v} a_i(F) \left\lceil \frac{i}{2} \right\rceil}{\sum_{i=3}^{v} a_i(F) \left\lfloor \frac{i}{2} \right\rfloor} \leq \frac{v-m}{m}. \tag{2.3}$$

Proof. We first deal with the case when v is even, or m is odd. In this case none of the edges of the K_m appear in any $F \in \mathcal{F}$. Therefore, for each $F \in \mathcal{F}$, at most $\lfloor \frac{i}{2} \rfloor$ vertices of any cycle of length i in F are inside the K_m, hence

$$m \leq \sum_{i=3}^{v} a_i(F) \left\lfloor \frac{i}{2} \right\rfloor . \tag{2.4}$$

Similarly, for any cycle of length i in F, at least $\lceil \frac{i}{2} \rceil$ vertices of the cycle are not in the K_m, thus

$$\sum_{i=3}^{v} a_i(F) \left\lceil \frac{i}{2} \right\rceil \leq v - m. \tag{2.5}$$

Thus Inequality (2.3) follows.

Now, if v is odd and m is even the $\frac{m}{2}$ edges in the 1-factor J of the K_m must be used in factors from $\mathcal{F}$. Suppose that $F \in \mathcal{F}$ uses c_F of these edges. Each edge of the K_m used can increase the right hand side of Inequality (2.4) by no more than one and decrease the left hand side of Inequality (2.5) by no more than one.

□

Theorem 2.3. *For a given v and m, if there is a minisymposium factorization, MSF$(\mathcal{F}, \mathcal{G})$, then $v \geq 2m$, unless v is odd and m is even, in which case $v \geq 2m - 1$.*

Proof. When v is even or m is odd, the left hand side of Inequality (2.3) is at least 1 and therefore $v \geq 2m$.

When v is odd and m is even, $\sum_{F \in \mathcal{F}} c_F = \frac{m}{2}$. Since v is odd, each $F \in \mathcal{F}$ must contain at least one odd cycle, therefore $\sum_{i=3}^{v} a_i(F) \lceil \frac{i}{2} \rceil \geq \lceil \frac{v}{2} \rceil = \frac{v+1}{2}$. Also note that the number of factors $F \in \mathcal{F}$ is $\frac{v-m+1}{2}$. Summing Inequality (2.1) over all of the $F \in \mathcal{F}$ twice gives

$$\begin{aligned}
(v - m + 1)(v - m) &\geq 2 \sum_{F \in \mathcal{F}} \left(-c_F + \sum_{i=3}^{v} a_i(F) \left\lceil \frac{i}{2} \right\rceil \right) \\
&= -m + 2 \sum_{F \in \mathcal{F}} \sum_{i=3}^{v} a_i(F) \left\lceil \frac{i}{2} \right\rceil \\
&\geq -m + 2 \sum_{F \in \mathcal{F}} \frac{v+1}{2} \\
&= (v - m + 1) \frac{v+1}{2} - m
\end{aligned}$$

Hence,

$$\begin{aligned}
0 &\leq (v - m + 1)(v - m) - (v - m + 1) \frac{v+1}{2} + m \\
&= \frac{(v - m + 1)(v - 2m - 1) + 2m}{2} \\
&= \frac{(v - (2m - 1))(v - (m + 1))}{2}.
\end{aligned}$$

When $v = m + 1$, the factors in $\mathcal{U}$ are required to be 2-regular graphs on a single vertex, which is not possible, so $v - (m + 1) > 0$. Thus $v - (2m - 1) \geq 0$ and the result follows. □

In the case where v is odd and m is even and $v = 2m - 1$, we have that $\mathcal{U}$ is a factorization of K_{v-m}. So we can interchange both the roles of m and $v - m$, as well as those of $\mathcal{T}$ and $\mathcal{U}$. Thus, without loss of generality, we may assume that $v \geq 2m$ in all cases.

There are two cases of initial special interest. Firstly, the case of *uniform* cycle lengths (when all cycles in a factor are of the same length), which we consider in detail in Section 4. Secondly, the case where the cycles are as long as possible, which in correspondence with the definition of K_n^* and the Hamiltonian-like nature of such factorizations we will call *Hamiltonian** factorizations. We formally define *Hamiltonian** factorizations in Section 3. There we show that a method of Hilton and Johnson completely settles their existence.

An MSF$(\mathcal{F}, \mathcal{G})$ in which all of the factors in $\mathcal{F}$, $\mathcal{T}$ and $\mathcal{U}$ are uniform with the same cycle length k is called *uniform* and we refer to it as a UMSF(v, m, k). In this case we have the following necessary conditions.

Theorem 2.4. *If $v > m$ and a UMSF(v, m, k) exists, then $k \geq 3$, $k \mid m$ and $k \mid v$. Furthermore,*

- *if k is even, then $v \geq 2m$;*
- *if k is odd, then $v \geq \frac{2mk}{k-1}$.*

Proof. Since we are forming 2-factors with cycles of length k, we require $k \geq 3$. The divisibility conditions follow directly from the requirement for a factorization of K_v^* and K_m^* into k-cycles. If k is even, then v is even, since it is a multiple of k, and Theorem 2.3 gives $v \geq 2m$.

If k is odd, we note that for a C_k-factor $F \in \mathcal{F}$, we have $a_i(F) = \frac{v}{k}$ when $i = k$ and 0 otherwise, $\lfloor \frac{k}{2} \rfloor = \frac{k-1}{2}$ and $\lceil \frac{k}{2} \rceil = \frac{k+1}{2}$. Thus, when v is even or m is odd, Inequality (2.3) implies that $v \geq \frac{2mk}{k-1}$ and the result follows.

This leaves the case when v and k are odd and m is even. We sum Inequality (2.1) over all the $F \in \mathcal{F}$ to obtain

$$\begin{aligned} \textstyle\sum_{F \in \mathcal{F}} (v - m) &\geq \textstyle\sum_{F \in \mathcal{F}} \left(-c_F + \sum_{i=3}^{v} a_i(F) \left\lceil \frac{i}{2} \right\rceil \right) \\ \tfrac{1}{2}(v - m + 1)(v - m) &\geq -m/2 + \textstyle\sum_{F \in \mathcal{F}} \frac{v}{k} \frac{k+1}{2} \\ 2k(v - m + 1)(v - m) &\geq v(v - m + 1)(k + 1) - 2mk. \end{aligned}$$

Rearranging and expanding in v gives

$$(k - 1)v^2 + (-3km + m + k - 1)v + 2km^2 \geq 0. \tag{2.6}$$

Let

$$f(v) = (k - 1)v^2 + (-3km + m + k - 1)v + 2km^2.$$

By Theorem 2.3, $v \geq 2m - 1$, but

$$f\,(2m - 1) = m(1 + k - 2m) < 0,$$

so v is at least as large as the larger of the two roots of f. Now

$$f\left(\frac{2mk}{k-1}-2\right) = -2(m+1-k) < 0, \text{ and } f\left(\frac{2mk}{k-1}-1\right) = (k-1)m > 0.$$

Thus f has its larger root between $\frac{2mk}{k-1}-2$ and $\frac{2mk}{k-1}-1$.

It is left to check that $v \neq \lfloor \frac{2mk}{k-1}-1 \rfloor, \lceil \frac{2mk}{k-1}-1 \rceil$. Recalling that $k \mid v$, if $v = \lfloor \frac{2mk}{k-1}-1 \rfloor$ or $v = \lceil \frac{2mk}{k-1}-1 \rceil$, then there exist a rational number $0 \leq \epsilon < 1$ and a positive integer a such that

$$v = \frac{2mk}{k-1} - 1 \pm \epsilon = ak.$$

Multiplying both sides by $\frac{k-1}{2k}$ and rearranging we have that

$$m - a\frac{k-1}{2} = (1 \pm \epsilon)\frac{k-1}{2k}.$$

Since k is odd, the left side is an integer. However, $0 < (1 \pm \epsilon)\frac{k-1}{2k} < 1$, a contradiction. We conclude that $v \geq \frac{2mk}{k-1}$. □

One interesting case in light of these necessary conditions is when $m = k$, i.e. a UMSF(v, k, k). For these parameters, since $k < v$ and $k \mid v$, we must have $v \geq 2k$, so the necessary conditions in Theorem 2.3 are satisfied. We consider these types of factorizations in Section 4.

3 Hamiltonian* and bipartite factors

Considering non-uniform factors, an obvious case to consider is a *Hamiltonian* minisymposium factorization*, which is one in which the cycles have the longest possible lengths. Specifically, the factors in $\mathcal{F}$ are all v-cycles, factors in $\mathcal{T}$ are all m-cycles and the factors in $\mathcal{U}$ are all $(v-m)$-cycles. Such an MSF$(\mathcal{F}, \mathcal{G})$ is denoted by HMSF(v, m). We sometimes refer to the cycles in $\mathcal{T}$ and $\mathcal{U}$ as 'short' cycles. Because of the lengths of these cycles there are no further necessary conditions beyond those of Theorem 2.3.

In a paper on the Oberwolfach problem, Hilton and Johnson prove the following theorem on a flexible construction technique.

Theorem 3.1 ([25]). *Let m and n be integers, $1 \leq m < n$. Let $(s_1, \ldots, s_t)$, $s_i \in 1, 2$, $1 \leq i \leq t$, be a composition of $n-1$. Let K_m be edge coloured with t colours $c_1, \ldots, c_t$. Let f_i be the number of edges coloured c_i and $K_m(c_i)$ be the ith colour class. This colouring can be extended to an edge-colouring of K_n in which the colour class $K_n(c_i)$ is an s_i-factor, $1 \leq i \leq t$, and when $s_i = 2$, $K_n(c_i)$ contains exactly one more cycle than $K_m(c_i)$ if and only if for all $i = 1, 2, \ldots, t$:*

$$\begin{aligned} f_i &\geq s_i(m - n/2), \\ s_i n &\text{ is even}, \\ \Delta(K_m(c_i)) &\leq s_i. \end{aligned}$$

This theorem is sufficient to provide a solution to the Hamiltonian* minisymposium factorization.

Corollary 3.2. *An HMSF(v, m) exists if and only if $m \geq 3$, $v \geq 2m-1$ in case m is even, and $v \geq 2m$ in case m is odd.*

Proof. The given conditions are necessary by Theorem 2.3. To prove sufficiency, we will define an edge colouring of the K_m from a decomposition of the K_m into Hamiltonian cycles and possibly a single 1-factor using Theorem 1.1. If m is odd, this defines edge colours c_i, $1 \leq i \leq (m-1)/2$. If v is also odd, extend this to a $(v-1)/2$-edge colouring of the K_m by including $(v-m)/2$ empty colour classes c_i, $(m+1)/2 \leq i \leq (v-1)/2$. Let $s_i = 2$ for all $1 \leq i \leq (v-1)/2$. If v is even, extend the colouring to a $v/2$-edge colouring of the K_m by adding empty colour classes. Let $s_i = 2$ for all $1 \leq i < v/2$ and $s_{v/2} = 1$. In both cases, it can be verified that Theorem 3.1 now gives an HMSF(v, m) as desired.

If m is even, define $m/2$ edge colour classes of the K_m from a decomposition into Hamiltonian cycles and one 1-factor. Let c_1 be the colour class of the 1-factor. If v is also even, extend this to a $v/2$-edge colouring by adding empty colour classes. Let $s_1 = 1$ and $s_i = 2$ for $2 \leq i \leq v/2$. If v is odd, extend this to a $(v-1)/2$-edge colouring by adding empty colour classes. Let $s_i = 2$ for $1 \leq i \leq v/2$. In both cases, it can be verified that Theorem 3.1 now gives an HMSF(v, m) as desired. □

Theorem 3.1 is more than just an existence result; a recursive procedure can be extracted from the proof to algorithmically build the edge decompositions. We have a more direct construction of all HMSF(v, m) which uses difference methods and decompositions of Cayley graphs [16].

Theorem 3.1 can be used much more generally to build minisymposium factorizations. Essentially it shows that it is possible to extend any 2-factorization of the K_m to one of K_v, provided that the necessary conditions hold, where the additional 2-factors are Hamiltonian, with an additional 1-factor when v is even.

When all of the cycles of the factors in $\mathcal{F}, \mathcal{U}$ and $\mathcal{T}$ are bipartite (i.e. contain only even cycles), we apply the Theorem of Häggkvist [24] (given below) to HMSF(v, m) to give us a solution to the minisymposium problem when $v \equiv m \equiv 2 \pmod 4$.

Theorem 3.3 ([24])**.** *If F is a bipartite 2-regular graph of order $2n$, then there is a factorization of $C_n[2]$ into 2 isomorphic copies of F.*

We note that in the case where the factors are bipartite, so all cycle lengths are even, v and m are both even and Theorem 2.3 gives $v \geq 2m$.

Theorem 3.4. *If $v \equiv m \equiv 2 \pmod 4$, and $\mathcal{F} = \{F_i : 1 \leq i \leq (v-m)/2\}$, $\mathcal{U} = \{U_j : 1 \leq j \leq (m-2)/2\}$ and $\mathcal{T} = \{T_j : 1 \leq j \leq (m-2)/2\}$ are sets of bipartite factors with $F_i = F_{i+1}$, $U_j = U_{j+1}$ and $T_j = T_{j+1}$ for every odd i and j, then an MSF$(\mathcal{F}, (\mathcal{T}, \mathcal{U}))$ exists if and only $v \geq 2m$.*

Proof. We note that if $v \equiv m \equiv 2 \pmod 4$, the number of the F_i is $(v-m)/2$ and the number of the U_j and the T_j is $(m-2)/2$, so the number of the F_i, U_j and T_j are all even. We take an HMSF$(v/2, m/2)$, which exists by Corollary 3.2, with factors F'_i of order $v/2$, U'_i of order $(v-m)/2$ and T'_i of order $m/2$. We blow up each vertex by 2 and apply Theorem 3.3 to factor each $F'_i[2]$ into 2 copies of F_i, each $U'_j[2]$ into 2 copies of U_j and each $T'_j[2]$ into 2 copies of T_j. □

If $v \equiv 0 \pmod 4$ or $m \equiv 0 \pmod 4$, then $v/2$ or $m/2$ would be even and the HMSF$(v/2, m/2)$ would contain 1-factors either in $K_{v/2}$ or the $K_{m/2}$. When a 1-factor is blown up as done in Theorem 3.3, it results in a C_4-factor, which prevents constructing the desired MSF unless $\mathcal{F}$, $\mathcal{T}$, and $\mathcal{U}$ already contain this kind of factor.

An immediate consequence of Theorem 3.4 is the following relating to uniform factors.

Corollary 3.5. *If $k > 3$, $k \equiv 2 \pmod 4$, $v \equiv m \equiv k \pmod{2k}$, then a UMSF$(v, m, k)$ exists if and only if $v \geq 2m$.*

4 Uniform factors

In this section we consider the case of uniform factors, i.e. when all cycles are of the same length, k. We recall from Theorem 2.4 that in order for a UMSF(v, m, k) to exist, we require that $k \geq 3$, which we will assume throughout this section. We also require $k \mid m$ and $k \mid v$. Additionally, if k is even, then $v \geq 2m$ and if k is odd, then $v \geq \frac{2mk}{k-1}$.

Corollary 3.5 gives a powerful result in the case when $k \equiv 2 \pmod 4$ and $v \equiv m \equiv k \pmod{2k}$. The case where $k = 3$ has been considered in [34, 35, 36] when v and m are both odd, and [18, 19, 20, 22, 37] when they are both even. However, the case when m and v have opposite parities appears to be completely open. We summarize these results in the following theorem.

Theorem 4.1 ([20, 35])**.** *If $v \equiv m \pmod 2$, there exists a UMSF$(v, m, 3)$ if and only if $v \geq 3m$, $v \equiv m \equiv 0 \pmod 3$, and if v, m are even, then $v, m > 12$.*

We will find the following results useful. A corollary of a result in [4] yields the following.

Theorem 4.2 ([4])**.** *If G is a Hamiltonian decomposable graph, then $G[n]$ is also Hamiltonian decomposable. In particular, $C_m[n]$ has a C_{mn}-factorization for every $m \geq 3$.*

Piotrowski [33] has shown the following result for $m \geq 4$. The case $m = 3$ is covered by Theorem 1.2.

Theorem 4.3 ([33])**.** *There exists a C_m-factorization of $C_m[n]$, except if $n = 2$ and m is odd, or when $(m, n) = (3, 6)$.*

Piotrowski [33] has also shown the following result.

Theorem 4.4 ([33])**.** *Let F be a bipartite 2-regular graph of order $2n$. The complete bipartite graph $K_2[n]$ has an F-factorization if and only if n is even, except when $n = 6$ and F consists of two 6-cycles.*

We now give some recursive constructions for uniform minisymposium factorizations.

Theorem 4.5. *Let $m \geq k \geq 3$ and $t \geq 2$ be integers. If $(t-1)m$ is even and $k \mid m$, then there is a UMSF(mt, m, k), except that there is no UMSF$(6t, 6, 3)$ UMSF$(12t, 12, 3)$, UMSF$(12, 6, 6)$, or UMSF$(2m, m, k)$ when k is odd.*

Proof. The non-existence of a UMSF$(6t, 6, 3)$ and UMSF$(12t, 12, 3)$ are covered by Theorem 4.1. Since a UMSF$(2m, m, k)$ is equivalent to a C_k-factorization of the complete bipartite graph $K_2[m]$, it clearly does not exist when the cycle length k is odd, or when $k = m = 6$ by Theorem 4.4. In all remaining cases, the following conditions simultaneously hold:

1. $(m, k) \notin \{(6, 3), (12, 3)\}$,
2. $(t, m, k) \neq (2, 6, 6)$,
3. if k is odd, then $t > 2$.

The assumptions of Theorems 1.1 and 1.2 are then satisfied. Hence there is a C_k-factorization of $K_t[m]$ and a C_k-factorization of K_m^*, which we use to fill in the parts of size m in $K_t[m]$. This completes the proof. □

Considering the necessary conditions in Theorem 2.4, we get the following corollaries.

Corollary 4.6. *Suppose that either k is even or v is odd, and $m \mid v$. Then there exists a UMSF(v, m, k) if and only if $k \mid m$, $v \geq 2m$ when k is even, and $v \geq 3m$ when k is odd, except that UMSF$(v, 6, 3)$, UMSF$(v, 12, 3)$ and UMSF$(12, 6, 6)$ do not exist.*

Proof. Taking $v = mt$, Theorem 2.4 gives the necessary conditions $k \mid m$ and $v \geq 2m$ when k is even. When k is odd, the necessary condition from Theorem 2.4 is $v \geq \frac{2mk}{k-1}$, but since $m \mid v$, this implies $k \geq 3m$. Given the conditions of k and v, the sufficiency comes from Theorem 4.5. □

We note that if $m \mid v$ this corollary completely solves all cases except when k is odd and v is even. One case of particular interest is when $k = m$, in this case $m \mid v$ is necessary.

Corollary 4.7. *Let $m(t-1)$ be even. Then a UMSF(tm, m, m) exists if and only if $t \geq 2$ when m is even, $t \geq 3$ when m is odd, and $(t, m) \neq (2, 6)$.*

The previous results all require $m \mid v$, however the next theorem allows us to recursively construct solutions to cases where m does not divide v.

Theorem 4.8. *Assume there is a UMSF(v, m, k) and let $t \geq 1$. Then there exists a UMSF(vtk, mtk, ℓ), with $\ell \in \{k, kt\}$, in each of the following cases:*

(1) *v and m have the same parity;*

(2) *v and t are even, $\ell = tk$, and m and k are both odd, except possibly when $(k, t) = (3, 2)$.*

Proof. Letting $w \in \{m, v\}$, we factorize K_{wtk}^* into $\Gamma_w = K_w^*[tk]$ and $\overline{\Gamma}_w = K_{wtk}^* - \Gamma_w$. Note that $\overline{\Gamma}_w$ is the vertex disjoint union of

(1) w copies of K_{tk}^* when w is odd, or

(2) $w/2$ copies of K_{2tk}^* when w is even.

Without loss of generality, we can assume that $\Gamma_m \subseteq \Gamma_v$ and $\overline{\Gamma}_m \subseteq \overline{\Gamma}_v$, except when v is odd and m is even. In this case the components of $\overline{\Gamma}_m$ are copies of K_{2tk}^*, while those of $\overline{\Gamma}_v$ are isomorphic to K_{tk}^*, therefore $\overline{\Gamma}_m \subseteq \overline{\Gamma}_v$ cannot hold. We proceed by constructing

(a) a C_ℓ-factorization of Γ_v containing a C_ℓ-factorization of Γ_m, and

(b) a C_ℓ-factorization of $\overline{\Gamma}_v$ containing a C_ℓ-factorization of $\overline{\Gamma}_m$, which together will provide the desired $UMSF(vtk, mtk, \ell)$.

We blow up each vertex of the UMSF(v, m, k) by tk, to obtain a $C_k[tk]$-factorization of Γ_v containing a $C_k[tk]$-factorization of Γ_m. To construct (a) it is therefore enough to factorize $C_k[tk]$ into C_ℓ-factors, $\ell \in \{k, tk\}$. By Theorem 4.3 there is a C_k-factorization of $C_k[tk]$, except when $(t, k) = (2, 3)$. In this case, the desired UMSF$(6v, 6m, 3)$ exists by Theorem 4.1. Considering that $C_k[tk] = C_k[t][k]$, by Theorem 4.2 there exists a C_{kt}-factorization of $C_k[t]$ which we blow up by k to obtain $C_{kt}[k]$-factorization of $C_k[tk]$. By Theorem 4.3, each $C_{kt}[k]$-factor can be further decomposed into C_{kt}-factors yielding a C_{kt}-factorization of $C_k[tk]$.

It is left to construct (b). If m and v have the same parity, the components of $\overline{\Gamma}_m$ and $\overline{\Gamma}_v$ are pairwise isomorphic: they are copies of K^*_{tk} or K^*_{2tk}. It is then enough to build a C_ℓ-factorization of K^*_{tk} and K^*_{2tk} for $\ell \in \{k, kt\}$. They exist by Theorem 1.1 except when $\ell = k = 3$ and one of the following two conditions hold,

1. mv is odd and $t \in \{2, 4\}$, or

2. m and v are even, and $t \in \{1, 2\}$.

In each of these cases, the existence of the desired $UMSF(3vt, 3mt, 3)$ is guaranteed by Theorem 4.1.

If v and t are even, $\ell = tk$, and both m and k are odd, the components of $\overline{\Gamma}_m$ are isomorphic to K^*_{tk}, while those of $\overline{\Gamma}_v$ are isomorphic to K^*_{2tk}. Since we can factorize K^*_{2tk} into $K_2[tk]$ and two copies of K^*_{tk}, it is enough to decompose both $K_2[tk]$ and K^*_{tk} into C_{tk}-factors. These factorizations exist by Theorem 4.4 and Theorem 1.1, respectively, except possibly when $(k, t) = (3, 2)$. □

We may now use the result on triples (Theorem 4.1) to obtain the following.

Corollary 4.9. *Let* $v \equiv m \equiv 0, 3 \pmod 6$*, with* $v \geq 3m$ *and* $m \notin \{0, 6, 12\}$*. Then there exists a UMSF*$(3tv, 3tm, 3t)$ *for all* $t > 0$.

Additionally, we may use Theorem 3.5 to obtain the following result.

Corollary 4.10. *Let* $3 < k$*,* $k \equiv 2 \pmod 4$*,* $v \equiv m \equiv k \pmod{2k}$ *and* $v \geq 2m$*. Then there exists a UMSF*(vtk, mtk, k) *and a UMSF*(vtk, mtk, tk) *for all* $t > 0$.

We note that the above result can be used to obtain UMSF's with cycle length, subsystem size or number of vertices congruent to $0 \pmod 4$ by taking t even. However, in all cases, the number of vertices and subsystem size will be divisible by k^2.

5 Multiple subsystems

A natural question to ask is if a system can have multiple subsystems. In general, it seems likely to be hard to navigate through the lattice of subsystems and all the possible ways the subsystems can be distributed across the main system. However, when the subsystems are disjoint, have small common intersections or are nested, the problem is more tractable. We give some preliminary results in the next three subsections.

5.1 Disjoint subsystems

In the uniform case, the flexibility of Theorem 1.2 allows us to create a large number of disjoint subsystems. We refer to a factorization of K_v^* into k-cycles with subsystems on disjoint vertex sets of sizes m_j for $1 \le j \le n$ as a UMSF$(v, \{m_j\}, k)$.

Lemma 5.1. *Let $k \mid m_j$ for $1 \le j \le n$. Let m be an integer such that there is a UMSF(m, m_j, k) for each $1 \le j \le n$. Then there exists a UMSF$(ms, \{m_j\}, k)$ for all $s \ge \max\{2, n\}$ if k is even, and for all $s \ge \max\{3, n\}$ such that $(s-1)m$ is even if k is odd, except when $(k, s, m) = (6, 2, 6)$.*

Proof. Theorem 1.2 guarantees the existence of a C_k-factorization of $K_s[m]$. For each $1 \le j \le n$, place a copy of the ingredient UMSF(m, m_j, k) on the jth part of size m of $K_s[m]$, and any C_k-factorization of K_m^* on each of the remaining parts. The definite exception $(k, s, m) = (6, 2, 6)$ follows from the non-existence of a UMSF$(12, 6, 6)$ (see Theorem 4.5). □

As with the uniform factorizations containing a single subsystem in this paper, the easiest case is when $m_j \mid m$ for all $1 \le j \le n$ and either k is even or m is odd.

Corollary 5.2. *Let $m = \text{lcm}\{m_j : j = 1, \ldots, n\}$, and assume the following conditions are all satisfied:*

(1) $k \mid m_j$ *for all* $j \in \{1, \ldots, n\}$ *and* $m \mid v$*;*

(2) $k(m-1)$ *is even;*

(3) *if* $k = 3$*, then* $m_j \notin \{6, 12\}$ *for all* j*;*

(4) *if* $(k, m) = (6, 12)$*, then* $m_j \ne 6$ *for all* j*;*

(5) $(v, m, k) \ne (12, 6, 6)$*;*

(6) $v/m \ge \max\{2, n\}$ *if* k *is even;*

(7) $v/m \ge \max\{3, n\}$ *and* v *is odd if* k *is odd.*

Then there exists a UMSF$(v, \{m_j\}, k)$.

Proof. Since each m_j is a divisor of m and conditions (1)–(4) hold, we can apply either Theorem 1.1 or Corollary 4.6, as needed, to ensure the existence of a UMSF(m, m_j, k) for every $j \in \{1, \ldots, n\}$. These are the ingredient designs needed in Lemma 5.1, which can be applied in view of conditions (5)–(7). □

We note that for any fixed multiset of m_j, this corollary constructs UMSF$(v, \{m_j\}, k)$ for all but a finite number of v permitted by the necessary conditions when $m_j \mid v$ and either k is even or m is odd.

5.2 Scattered subsystems

The proof of Lemma 5.1 builds systems whose factors intersect either all of the subsystems or none of them. A balancing of the sizes of these intersections could be an interesting property. For instance, we could ask for systems whose factors do not intersect more than one subsystem. In other words, we ask for a C_k-factorization $\mathcal{F}$ of K_v^* that contains n subsystems of sizes $m_1, m_2, \ldots, m_n$, such that no two factors of any of the subsystems are contained in the same factor of $\mathcal{F}$. We denote such a factorization by UMSF$(v, [m_j], k)$ and say that the subsystems are *scattered*.

Partial results in this direction can be easily obtained by making use of cycle frames. We recall that a *k-cycle frame* (k-CF) of $K_s[m]$ is a decomposition of $K_s[m]$ into holey C_k-factors; a *holey C_k-factor* is a vertex-disjoint union of k-cycles covering all vertices $K_s[m]$ except those belonging to one part. The following result, proven in [8], provides necessary and sufficient conditions for the existence of k-cycle frames.

Theorem 5.3 ([8])**.** *Let $m \geq 2$ and $k, s \geq 3$. There exists a k-cycle frame of $K_s[m]$ if and only if m is even, $m(s-1) \equiv 0 \pmod{k}$, k is even when $s = 3$, and $(k, m, s) \neq (6, 6, 3)$.*

By making use of Theorem 5.3, we obtain the following.

Lemma 5.4. *Let $u \in \{1, 2\}$. If there exists a UMSF$(2m+u, m_j, k)$ for each $1 \leq j \leq n$, then there exists a UMSF$(2ms+u, [m_j], k)$ whenever $2s \equiv 2 \pmod{k}$ and $s \geq n$.*

Proof. Let $n \geq 1$, $2s \equiv 2 \pmod{k}$ and $s \geq n$. It follows that $2m(s-1) \equiv 0 \pmod{k}$, $k = 4$ when $s = 3$, and $(k, 2m, s) \neq (6, 6, 3)$. Therefore, Theorem 5.3 guarantees the existence of a k-cycle frame $\mathcal{F}$ of $K_s[2m]$. Let P_i denote the i-th part of $K_s[2m]$, for $1 \leq i \leq s$. Also, let

$$\mathcal{F} = \{F_{ij} : \ 1 \leq i \leq s, 1 \leq j \leq m\},$$

where the F_{ij}s are the holey C_k-factors of $\mathcal{F}$ missing P_i, for $1 \leq i \leq s$. By assumption, there is a UMSF$(2m+u, m_j, k)$ on $P_i \cup \{\infty_1, \infty_u\}$, say $\mathcal{H}_i = \{H_{ij} : \ 1 \leq j \leq m\}$. It follows that $\mathcal{F}^* = \{F_{ij} \cup H_{ij} : \ 1 \leq i \leq s, 1 \leq j \leq m\}$ is a C_k-factorization of K_{2ms+u} with scattered subsystems of sizes $m_1, m_2, \ldots, m_n$. Indeed, the factors of the subsystems belong to the H_{ij}s, each of which belongs to exactly one factor of $\mathcal{F}^*$. □

In the UMSF$(2ms+u, [m_j], k)$ constructed in the proof of Lemma 5.4, two subsystems may intersect in 0, 1, or 2 vertices, which are necessarily in the set $\{\infty_1, \infty_u\}$.

Theorem 4.5 provides sufficient conditions for the existence of a UMSF(v, m, k) if m is a divisor of v. From that, we easily obtain the following corollary.

Corollary 5.5. *Let $u \in \{1, 2\}$, $k \geq 3$, and let $k \mid m_j \mid (2m+u)$ for each $1 \leq j \leq n$. Then there exists a UMSF$(2ms+u, [m_j], k)$ whenever the following conditions hold:*

(1) *m_j is even or $(2m+u)/m_j$ is odd,*

(2) *$2n \leq 2s$ and $2s \equiv 2 \pmod{k}$,*

except when $(m_j, k) \in \{(6, 3), (12, 3)\}$, and except possibly when $(2m+u, m_j, k) = (12, 6, 6)$, or k is odd and $2m+u = 2m_j$, for some $j \in \{1, \ldots, n\}$.

Note that for values of the triple $(2m+u, m_j, k)$ determining a possible exception in Corollary 5.5 it is possible for a UMSF$(2ms+u, [m_j], k)$ to exist. However, our method cannot construct them because the UMSF$(2m+u, m_j, k)$ to use in the construction does not exist. It is possible that other construction methods would build a UMSF$(2ms+u, [m_j], k)$.

5.3 Nested subsystems

A scenario complementary to the susbsystems being all disjoint is when the subsystems are completely nested, on vertex sets $M_1 \supset M_2 \supset \cdots \supset M_{n-1}$. We modify our notation slightly for this section to make it less cumbersome in this specific context.

Definition 5.6. Let $v = m_0 > m_1 > \cdots > m_{n-1} > m_n = 0$ be non-negative integers. For $1 \le i \le \lfloor \frac{m_0-1}{2} \rfloor$ and $0 \le j < n$, let $U_{i,j}$ be a 2-regular graph of order

$$|V(U_{i,j})| = \begin{cases} m_j - m_{j+1}, & \text{if } 1 \le i \le \lfloor \frac{m_{j+1}-1}{2} \rfloor, \\ m_j, & \text{if } \lfloor \frac{m_{j+1}+1}{2} \rfloor \le i \le \lfloor \frac{m_j-1}{2} \rfloor, \\ 0, & \text{if } \lfloor \frac{m_j+1}{2} \rfloor \le i \le \lfloor \frac{m_0-1}{2} \rfloor. \end{cases}$$

A *nested minisymposium factorization* nMSF$(\{U_{i,j} :\ 1 \le i \le \lfloor \frac{m_0-1}{2} \rfloor,\ 0 \le j < n\})$ is a 2-factorization $\mathcal{F} = \{F_i : 1 \le i \le \lfloor \frac{m_0-1}{2} \rfloor\}$ of K_v^* such that

- $V(K_v^*) = M_0 \supset M_1 \supset \cdots \supset M_{n-1} \supset M_n = \varnothing$ are nested sets with $|M_j| = m_j$;
- each 2-factor $F_i = \bigcup_{j=0}^{n-1} F_{i,j}$, where

$$V(F_{i,j}) = \begin{cases} M_j \setminus M_{j+1} & \text{if } 1 \le i \le \lfloor \frac{m_{j+1}-1}{2} \rfloor \\ M_j & \text{if } \lfloor \frac{m_{j+1}+1}{2} \rfloor \le i \le \lfloor \frac{m_j-1}{2} \rfloor \\ \varnothing & \text{otherwise,} \end{cases}$$

  and each $F_{i,j}$ is isomorphic to $U_{i,j}$;
- for every $0 \le \ell < n$, $\{\bigcup_{j=\ell}^{n-1} F_{i,j} : 1 \le i \le \lfloor \frac{m_\ell - 1}{2} \rfloor\}$ is a 2-factorization of a graph isomorphic to $K_{m_\ell}^*$.

In other words, the factorization $\mathcal{F}$ of K_v^* restricted to vertex set M_ℓ factorizes a graph isomorphic to $K_{m_\ell}^*$ into 2-factors whose structure is determined by the $U_{i,j}$s.

Our construction of nested minisymposium factorizations is most tidily expressed by defining holey factorizations.

Definition 5.7. Given positive integers v and m with $v \ge m$, let

$$\mathcal{U} = \left\{ U_i :\ 1 \le i \le \left\lfloor \frac{v-1}{2} \right\rfloor \right\},$$

be a collection of 2-regular graphs on $v - m$ vertices for $1 \le i \le \lfloor \frac{m-1}{2} \rfloor$, and on v vertices for $\lfloor \frac{m+1}{2} \rfloor \le i \le \lfloor \frac{v-1}{2} \rfloor$. A *holey factorization* HF$(\mathcal{U})$ is a decomposition $\mathcal{F} = \{F_i :\ 1 \le i \le \lfloor \frac{v-1}{2} \rfloor\}$ of $K_v^* - G$ (i.e., K_v^* minus the edges of G) where each $F_i \cong U_i$ and $G \cong K_m^*$.

If v is even, then there is a 1-factor, I_v, on the vertices of K_v^* whose edges are not present in $K_v^* - G$. If v and m are both even there is a 1-factor, J_m, on the vertices of G which is a subgraph of I_v. If v is even and m odd, then no edges of I_v are induced on the vertices of G. If v is odd and m is even, then there is a 1-factor J_m on the vertices of G whose edges are present in $K_v^* - G$.

By removing the 2-factors of a subsystem or "filling the hole" with them (making the J_m in the hole coincide with the I_m of the subsystem as required by the parities of v and m) we have an equivalence between the existence of minisymposium factorizations and holey factorizations.

Theorem 5.8. *Let $\mathcal{T} = \{T_i : 1 \le i \le \lfloor \frac{m-1}{2} \rfloor\}$ be a 2-factorization of K_m^* and*

$$\mathcal{U} = \left\{ U_i : 1 \le i \le \left\lfloor \frac{v-1}{2} \right\rfloor \right\},$$

be a collection of 2-regular graphs on $v-m$ vertices for $1 \le i \le \lfloor \frac{m-1}{2} \rfloor$ and on v vertices for $\lfloor \frac{m+1}{2} \rfloor \le i \le \lfloor \frac{v-1}{2} \rfloor$. Then a HF($\mathcal{U}$) exists if and only if a

$$MSF(\{U_i : \lfloor \tfrac{m+1}{2} \rfloor \le i \le \lfloor \tfrac{v-1}{2} \rfloor\}, \{(T_i, U_i) : 1 \le i \le \lfloor \tfrac{m-1}{2} \rfloor\})$$

exists.

Because in a nested minisymposium factorization the holes are nested and emptying or filling them does not affect the edges outside the hole, this equivalence extends to nested minisymposium factorizations and shows that they can be constructed exactly when the various holey factorizations of $K_{m_j}^*$ with holes of size m_{j+1} exist.

Theorem 5.9. *Let $v = m_0 > m_1 > \cdots > m_{n-1} > m_n = 0$ be positive integers. For $1 \le i \le \lfloor \frac{m_0-1}{2} \rfloor$ and $0 \le j < n$, let $U_{i,j}$ be a 2-regular graph of order*

$$|V(U_{i,j})| = \begin{cases} m_j - m_{j+1}, & \text{if } 1 \le i \le \lfloor \frac{m_{j+1}-1}{2} \rfloor, \\ m_j, & \text{if } \lfloor \frac{m_{j+1}+1}{2} \rfloor \le i \le \lfloor \frac{m_j-1}{2} \rfloor, \\ 0, & \text{if } \lfloor \frac{m_j+1}{2} \rfloor \le i \le \lfloor \frac{m_0-1}{2} \rfloor. \end{cases}$$

A nested minisymposium factorization nMSF($\{U_{i,j} : 1 \le i \le \lfloor \frac{m_0-1}{2} \rfloor, 0 \le j < n\}$) exists if and only if a HF($\{U_{i,j} : 1 \le i \le \lfloor \frac{m_j-1}{2} \rfloor\}$) exists for each $0 \le j < n$.

Proof. The forward direction is proved simply by restricting the system to M_j and removing the subsystem on M_{j+1}. The converse is proved by a recursive construction starting with $j = n-1$: in this case, a HF($\{U_{i,n-1} : 1 \le i \le \lfloor \frac{m_{n-1}-1}{2} \rfloor\}$) is an nMSF($\{U_{i,n-1} : 1 \le i \le \lfloor \frac{m_{n-1}-1}{2} \rfloor\}$), say $\mathcal{F}_{n-1}$.

At stage $j < n-1$, use Theorem 5.8 to construct an nMSF($\{U_{i,\ell} : 1 \le i \le \lfloor \frac{m_j-1}{2} \rfloor, j \le \ell < n\}$), say $\mathcal{F}_j$, by filling the hole in the HF($\{U_{i,j} : 1 \le i \le \lfloor \frac{m_j-1}{2} \rfloor\}$) with the nMSF($\{U_{i,\ell} : 1 \le i \le \lfloor \frac{m_{j+1}-1}{2} \rfloor, j+1 \le \ell < n\}$), denoted by $\mathcal{F}_{j+1}$, built at stage $j+1$. □

Between the extremes of disjoint and nested subsystems, there are factorizations with multiple subsystems with arbitrary intersections. Some structured instances of this much more general problem may be amenable to solution but we leave this to future work.

6 Conclusions and further work

We have introduced the minisymposium problem: a subsystem variant of the generalized Oberwolfach problem. This variant asks for a solution to a generalized Oberwolfach problem that contains a subsystem of a given size. When v, the number of vertices, is even, it is traditional in 2-factor decomposition problems to ask for decompositions of $K_v - I$ where I is a 1-factor. When the number of vertices in the system and the subsystem are both even, then we require that the 1-factor in the subsystem be a subgraph of the 1-factor in the full system. Therefore when the parities of the system and the subsystem agree, the problem becomes more tractable. When the parities are opposite, either the 1-factor of the full system must avoid the subsystem, or the edges of the 1-factor in the subsystem must be in 2-factors of the whole system.

Clearly, this is a very broad statement and we identify some particularly interesting cases, Hamiltonian* and uniform. In the Hamiltonian* minisymposium problem there are as few cycles as possible and in the uniform minisymposium problem all cycles are of the same length. We have shown that the work of Hilton and Johnson provides a complete solution for the Hamiltonian* minisymposium problem in Corollary 3.2. In the case when $v \equiv m \equiv 2 \pmod 4$, Theorem 3.4 uses this Hamiltonian* construction to provide a wide range of solutions when the resulting factors are all bipartite. In particular, a uniform factorization with $k \equiv 2 \pmod 4$, $v \geq 2m$ and $v \equiv m \equiv k \pmod{2k}$ always exists. Corollary 4.10 can be used to extend this to uniform factorizations where $k \equiv 0 \pmod 4$ or $v \equiv m \equiv 0 \pmod{2k}$.

In Section 4 we considered the uniform case. We have solved a large part of the spectrum. In particular, when k is even or v is odd, Corollary 4.6 gives all cases when $m \mid v$ and Corollary 4.7 completely solves all cases when $k = m$ has the same parity as v. Theorem 4.8 gives a powerful recursive construction which is applicable in cases where m does not divide v. By applying it to the case when $k = 3$, we obtain uniform factorizations with cycle lengths divisible by 3. The case when m is odd and v is even seems to be the hardest. Even in the simplest case when $k = 3$, which has been well studied otherwise [18, 19, 20, 22, 34, 35, 36, 37], the case with v and m having opposite parities has not been previously considered and remains open.

While the Hamiltonian* problem is solved and we have made significant inroads into the uniform case, the general problem remains wide open. We expect that when m and v have the same parity solutions will be easier to find. When m and v have opposite parity we expect that v odd with m even is more tractable than the reverse. Considering 2-factorizations where a solution to the Oberwolfach problem is known might be a good starting point. A natural case to consider is the case when all factors are isomorphic i.e. $F_i \cong T_j \cup U_j$ for all i and j. Uniform factorizations are an example of this, but other variations are possible, for example, requiring all factors to be isomorphic to $C_{v-m} \cup C_m$. Indeed, Theorem 3.4 solves all these cases when the factors are bipartite and $v \equiv m \equiv 2 \pmod 4$, but this broader variant remains open.

More complex variants can also be considered. We have briefly considered systems with multiple subsystems. When these subsystems are completely nested the problem essentially reduces to the existence of the necessary ingredients as described in Theorem 5.9. Let $\{m_j\}$ be a multiset of subsystem sizes. When the subsystems are pairwise disjoint, v is divisible by each m_j and either v is odd or at least one subsystem is even, then Lemma 5.1 and Corollary 5.2 use Theorem 1.2 to construct a UMSF$(v, \{m_j\}, k)$ for all but a finite number of admissible v. Even in the seemingly simple case when the subsystems are all disjoint the problem remains generally open even for the uniform case. Further partial results are obtained when the subsystems are scattered, that is, when no two minisymposia have meetings taking place on the same day. Cycle frames in Theorem 5.3 allow us to construct uniform factorizations as described in Lemma 5.4 and Theorem 5.5. The more general case when the intersections of multiple subsystems are arbitrary is completely open.

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