

# Cayley Graphs, Cori Hypermaps, and Dessins d’Enfants

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## Abstract

This paper explains some facts probably known to experts and implicitly contained in the literature about dessins d’enfants but which seem to be nowhere explicitly stated. The 1-skeleton of every regular Cori hypermap is the Cayley graph of its automorphism group, embedded in the underlying orientable surface. Conversely, every Cayley graph of a finite two-generator group has an embedding as the 1-skeleton of a regular hypermap in the Cori representation. For non-regular hypermaps there is an analogous correspondence with Schreier coset diagrams.

*Keywords:* Cayley graph, dessin d’enfant, hypermap.

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## 1 Hypermaps and Belyĭ functions

We recall the definition of *hypermaps*, more precisely compact, oriented topological hypermaps in their Cori representation (see [1], [2], [8]). Let  $X$  be a compact orientable surface. A hypermap  $\mathcal{H}$  on  $X$  is a triple  $(X, S, A)$  where  $S$  and  $A$  denote closed subsets of  $X$  with the following properties.

1.  $B = S \cap A$  is a finite set. Its elements are called the *brins* of  $\mathcal{H}$ .
2.  $S \cup A$  is connected.

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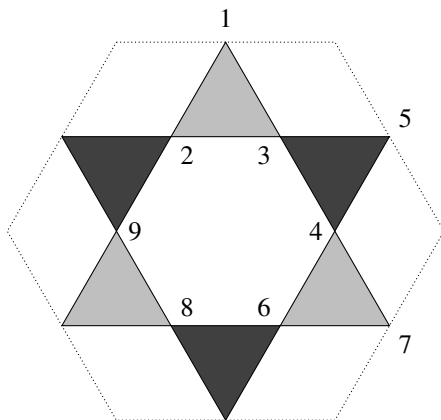


Figure 1: Hypermap on the torus.

3. The components of  $S$  — called the *hypervertices* — and of  $A$  — called the *hyperedges* of  $\mathcal{H}$  — are homeomorphic to closed discs.
4. The components of the complement  $X \setminus S \cup A$  are homeomorphic to open discs, called the *hyperfaces* of  $\mathcal{H}$ .

As a very simple example consider Figure 1. Here the shaded triangles are hyperedges and the black triangles are hypervertices. The white regions are the hyperfaces. This hypermap lies on the torus obtained by identifying opposite edges of the dotted surrounding hexagon.

Hypermaps were first introduced in [1] as tools for computer science. However, nowadays they are a natural ingredient for the *dessins d'enfants* on smooth projective algebraic curves over number fields. (Grothendieck called hypermaps *dessins d'enfants*.) Recall the definition of a *Belyĭ function*  $\beta$  on a compact Riemann surface  $X$  as a meromorphic nonconstant function ramified at most above  $\{0, 1, \infty\}$ , and recall Belyĭ's famous theorem of 1979 that a Belyĭ function exists on a Riemann surface  $X$  if and only if — as an algebraic curve —  $X$  can be defined over a number field, see e.g. [7] or [8]. To every such Belyĭ function  $\beta$  we can associate a hypermap  $\mathcal{H} = (X, S, A)$  in the following way.

On the Riemann sphere  $\overline{\mathbb{C}}$  we define a *universal target hypermap*  $\mathcal{H}_0 = (\overline{\mathbb{C}}, S_0, A_0)$  by taking  $S_0$  and  $A_0$  as closed cells containing 0 and 1 in their respective interiors and intersecting at precisely one point on their boundary. Take e.g. a lemniscate curve around 0 and 1 with node point  $\frac{1}{2}$ . This becomes the unique brin,  $S_0$  and  $A_0$  will be the left and the right part of the curve with their respective inner points, see Figure 2. For any Belyĭ function  $\beta : X \rightarrow \overline{\mathbb{C}}$ , the closed sets  $S = \beta^{-1}(S_0)$  and  $A = \beta^{-1}(A_0)$  satisfy the hypermap conditions, the brins are the preimages  $\beta^{-1}(\frac{1}{2})$  and represent the sheets of the ramified covering  $\beta$ . The first proposition is a variant of a well known central fact in the dessin d'enfant theory.

**Proposition 1.** *Every hypermap  $\mathcal{H} = (X, S, A)$  determines a unique conformal structure on the compact surface  $X$  such that there is a Belyĭ function  $\beta$  on  $X$  with  $\mathcal{H} = \beta^{-1}\mathcal{H}_0$ .*

In the next section we will recall an easy argument for the proof. As explained above,  $X$  has then not only a Riemann surface structure but can be considered even as a smooth algebraic curve over a number field.

In the following, we will mainly use the *1-skeleton graph*  $G$  defined on  $X$  by the boundary of  $S \cup A$  with the brins as vertices. All vertices of  $G$  have valency 4, and the valencies of the faces reflect ramification properties of the associated Belyĭ function  $\beta$ .

**Proposition 2.** *Every hypervertex  $s$  contains a unique zero of  $\beta$  of order  $\text{val}(s)$ , every hyperedge  $a$  contains a unique zero of  $\beta - 1$  of order  $\text{val}(a)$ , and every hyperface  $f$  contains a unique pole of  $\beta$  of order  $\frac{1}{2}\text{val}(f)$ .*

We mention two other possibilities to represent topological hypermaps:

The *Walsh representation* [10] as a bipartite graph on  $X$  arises from  $\mathcal{H}$  choosing a white vertex in the interior of every component of  $S$  and a black vertex in the interior of every component of  $A$ , and choosing an edge for every brin  $b$  joining the black and the white vertex in the middle of the components of  $S$  and  $A$  meeting in  $b$ . In the context of Belyĭ functions, these bipartite graphs arise as  $\beta$ -preimages of a universal target graph, the real interval  $[0, 1]$  with 0 as white and 1 as black vertex, see Figures 3 and 4(b). In this model, the orders of zero of  $\beta$  and  $\beta - 1$  are the valencies of the white and the black vertices, and the pole orders are half of the valency of the faces.

The *James representation* can be obtained from the Cori representation by blowing up the components of  $S$  and  $A$  so that they meet no longer in isolated points but in 1-cells ending in the common intersection points of hypervertices, hyperedges and the closure of the hyperfaces. In this model, all vertices of the graph defined by the boundary of  $S$  and  $A$  have valency 3, and moreover this has the advantage that all valencies of faces are twice the order of zeros of  $\beta$ ,  $\beta - 1$ ,  $\beta^{-1}$  respectively, hence reflecting the natural symmetry of the critical values  $0, 1, \infty$ . A drawing of the example of Figure 1 in the James representation is given in Figure 4(a). In Figure 4(b) we draw the universal target hypermap.

Note: The Figures 1, 3 and 4(a) are hypermaps for the Fermat cubic  $x^3 + y^3 = z^3$  and its Belyĭ function  $\beta(x : y : z) := x^3/z^3$ , see Example 3 of Section 7 of [7].

## 2 The algebraic hypermap

An *algebraic hypermap* is a quadruple  $(H, B, g_0, g_1)$  consisting of a permutation group  $H$  acting transitively on a finite set  $B$  and generated by two elements  $g_0, g_1$ . The following two facts are well known by [1] and [2].

**Proposition 3.** *There is a one-to-one correspondence between topological and algebraic hypermaps.*

We sketch one direction of the proof. For a topological hypermap  $\mathcal{H}$  let  $B$  the set of brins, and note that from every brin  $b$  starts a unique edge of the graph  $G$  having a component of

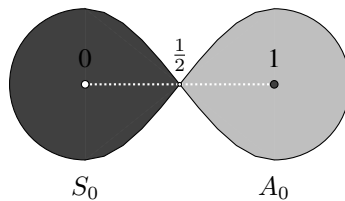


Figure 2: A lemniscate curve.

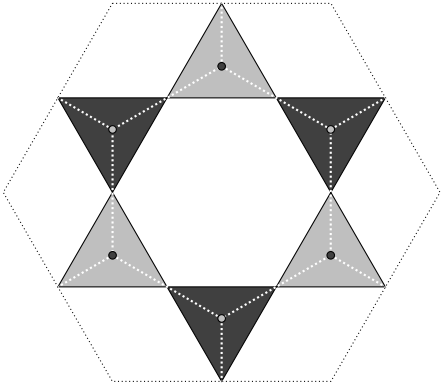


Figure 3: The Walsh representation.

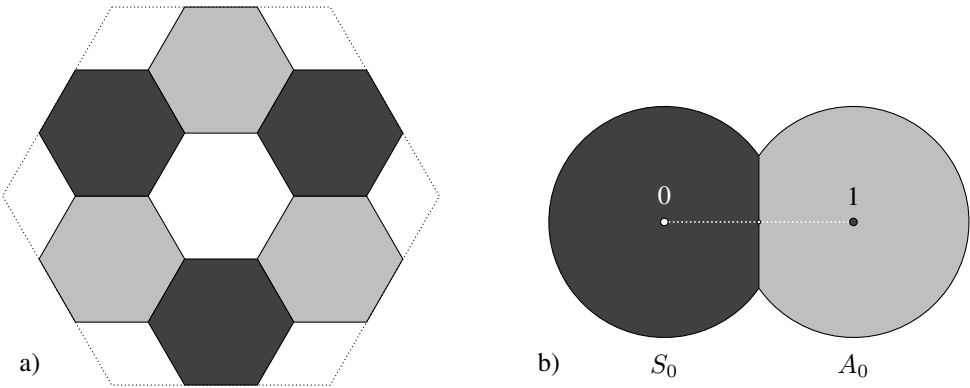


Figure 4: a) Same example as in Figure 1 and 3, but in James representation. b) The universal target hypermap in James representation.

$S$  to the left; call its end vertex  $g_0(b)$ . Similarly, in every  $b \in B$  starts a unique edge of  $G$  having a component of  $A$  to the left; call its end vertex  $g_1(b)$ . These  $g_0, g_1$  generate a permutation group acting on  $B$ , the *hypercartographic group*  $H$ . Since  $G$  is connected, the action is transitive.

The other direction of the proof, i.e. the construction of a topological hypermap from an algebraic one, could be performed in a similar way, but it will be easier using the universal hypermap; see the end of this section.

- Remarks.* 1) For the covering defined by the Belyĭ function,  $H$  is the monodromy group.  
 2) In the Walsh representation,  $H$  is a permutation group acting on the set of edges, and  $g_0, g_1$  generate the cyclic permutations of the edges around the white resp. black vertices in counterclockwise order.  
 3) Similarly, in the James representation we may consider permutations of the set of boundary edges between  $S$  and  $A$ . However we will concentrate on the Cori representation for reasons which will become clear in the next section.  
 4) According to Proposition 3, we will use the same letter  $\mathcal{H}$  for both the topological and the algebraic hypermap.

*Definition.* A permutation  $\alpha$  of  $B$  is called an *automorphism* of the hypermap  $\mathcal{H} = (H, B, g_0, g_1)$  if it is  $H$ -equivariant, i.e. if

$$\alpha(g(b)) = g(\alpha(b)) \quad \text{for all } b \in B \quad \text{and for all } g \in H.$$

A hypermap is called *regular* if  $\text{Aut } \mathcal{H}$  acts transitively on  $B$ .

Note that this last condition is equivalent to saying that  $\text{Aut } \mathcal{H}$  is of order  $|B|$  since a nontrivial automorphism cannot have fixed points on  $B$ . By the transitivity of  $H$ , any automorphism  $\alpha$  is uniquely determined by the image of one single brin  $b$ . Other ways to reformulate the regularity condition are given in Theorem 2 of [2]:

**Proposition 4.** *For a hypermap  $\mathcal{H}$  the following conditions are equivalent.*

1.  $\mathcal{H}$  is regular;
2.  $\text{Aut } \mathcal{H}$  is isomorphic to  $H$ ;
3.  $|H| = |B|$ .

For the consequences of regularity on the associated Belyĭ functions and the underlying *quasiplatonic curves* see [11, Theorems 4 and 5]. We can add a further equivalence taking into account that  $H$  acts transitively and without fixed points on the set of brins.

**Proposition 5.** *The hypermap  $\mathcal{H}$  is regular if and only if for some brin  $b \in B$  the mapping*

$$H \rightarrow B : g \mapsto g(b)$$

*is injective, hence defines an identification of  $H$  with  $B$ . In this case, every  $b \in B$  defines such an identification. In this setting,  $H$  acts on  $H = B$  by left multiplication.*

We call  $\mathcal{H}$  a *hypermap of type*  $(p, q, r)$  if the generators of the hypercartographic group  $H$  are of order

$$p = \text{ord } g_0, \quad q = \text{ord } g_1, \quad r = \text{ord } g_\infty \quad \text{for } g_\infty := (g_0 g_1)^{-1}.$$

For regular hypermaps of type  $(p, q, r)$  we can now give the following easy argument to prove Proposition 1 and to construct a topological hypermap from an algebraic one. Let  $\Delta$  be the triangle group of signature  $(p, q, r)$ , that is with presentation

$$\langle \gamma_0, \gamma_1, \gamma_\infty \mid \gamma_0^p = \gamma_1^q = \gamma_\infty^r = \gamma_0\gamma_1\gamma_\infty = 1 \rangle$$

acting discontinuously on the simply connected domain  $\mathcal{U}$  — in the most common case  $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} < 1$  that is the upper half plane equipped with the hyperbolic metric. On  $\mathcal{U}$  we can construct a *universal* Walsh hypermap (see [2]) by taking the fixed point  $s_0$  of  $\gamma_0$  and all its  $\Delta$ -images as white vertices, taking the  $\Delta$ -orbit of the fixed point  $s_1$  of  $\gamma_1$  as the set of black vertices, and taking the geodesic path between  $s_0$  and  $s_1$  and all its  $\Delta$ -images as edges. (As an exercise we leave to the reader the question how to modify this construction in the case  $\mathcal{U} = \overline{\mathbb{C}}$  where the  $\gamma_i$  have two fixed points.) Consider the kernel  $K$  of the obvious homomorphism

$$\Delta \rightarrow H \cong \text{Aut } \mathcal{H}, \quad \gamma_i \mapsto g_i \quad \text{for } i = 0, 1, \infty.$$

The quotient  $X := K \backslash \mathcal{U}$  has a unique complex structure, and we find the original hypermap (in its Walsh representation) as the quotient of the universal hypermap by  $K$ . At the same time, this construction gives the Belyĭ function for  $\mathcal{H}$  as quotient map

$$K \backslash \mathcal{U} \rightarrow \Delta \backslash \mathcal{U} \cong \overline{\mathbb{C}}$$

if we identify the  $\Delta$ -orbits of the fixed points of  $\gamma_0, \gamma_1, \gamma_\infty$  with  $0, 1, \infty$ . Moreover, we see that  $\text{Aut } \mathcal{H} \cong \Delta/K$  acts as a group of holomorphic automorphisms on the Riemann surface  $X$  (caution: this action on the edges or the brins is in general very different from the action of the isomorphic group  $H$  introduced after Proposition 3).

The general case of non-regular hypermaps can be derived from the regular case: Let  $F$  be the subgroup of  $H$  fixing the brin  $b$ . Then there is a  $H$ -equivariant bijection between the set  $B$  of brins and the left cosets  $hF \in H/F$ . If  $\Gamma$  is the preimage of  $F$  under the canonical group homomorphism  $\Delta \rightarrow \Delta/K \cong H$  used above, the surface  $X$  is now the quotient  $\Gamma \backslash \mathcal{U}$ , again having a canonical conformal structure and a canonical Belyĭ function. We note in particular

**Proposition 6.** *Every hypermap is isomorphic to the quotient of a regular hypermap by some subgroup  $F$  of its automorphism group. As algebraic hypermaps, the regular hypermap*

$$(H, H, g_0, g_1) \quad \text{and its quotient} \quad (H, H/F, g_0, g_1)$$

*have the same hypercartographic group and the same generators, acting on  $H$  and on  $H/F$  by left multiplication. For the topological hypermaps, the underlying surfaces are  $X$  and  $F \backslash X$  where  $F$  acts as a subgroup of the automorphism group of  $X$ .*

### 3 Cayley graphs

Using Propositions 3 and 5 and an identification  $B = H$  of the set of brins with the hypercartographic group we can state now the main result.

**Theorem 7.** *Let  $\mathcal{H} = (X, S, A) = (H, H, g_0, g_1)$  be a regular hypermap of type  $(p, q, r)$ . Then the 1-skeleton of the hypermap in its Cori representation is the Cayley graph of  $H$  with generators  $g_0$  and  $g_1$ .*

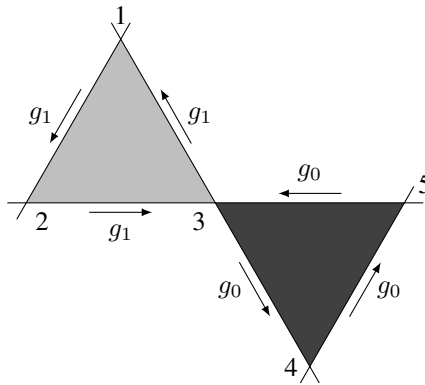


Figure 5: An enlargement of part of Figure 1.

*Proof.* Suppose for simplicity that the orders of the generators are  $> 2$ , the other cases will be treated separately in Section 5. Proposition 5 gives already a labelling of the brins (= vertices) with the group elements. It remains to give a direction to the edges and to label them with the generators  $g_0$  or  $g_1$ . By construction of the hypercartographic group, every edge leads from a vertex  $h$  to a vertex  $g_i(h) = g_i \cdot h$ , so the label  $g_i$  and the direction are obvious. (In other words, in the direction of the edge the complement of  $S \cup A$  is at its right, it has either a component of  $S$  or a component of  $A$  at its left, and according to these two cases we have  $i = 0$  or  $1$ , respectively.) From Figures 1, 3, 4(a) we get Figure 5.

With these labels,  $G$  becomes the Cayley graph of  $H$ . Note that this Cayley graph is embedded in  $X$  in a special way: the edges incident with any vertex  $h$  are  $g_0, g_0^{-1}, g_1, g_1^{-1}$  in counterclockwise order around  $h$ , see Figure 5. Here we assume the obvious convention to label a  $g_i$ -edge ending in  $h$  locally by  $g_i^{-1}$ . For example in Figure 5 we get

$$g_0 = (9, 2, 7)(6, 8, 1)(3, 4, 5) \quad , \quad g_1 = (2, 3, 1)(4, 6, 7)(8, 9, 5)$$

and then

$$g_1 g_0 = (1, 7, 5)(2, 4, 8)(3, 6, 9)$$

( $g_0$  followed by  $g_1$ ).

The group  $H$  generated by  $g_0$  and  $g_1$  is  $C_3 \times C_3$  and so the graph  $G$  can be regarded as a way of drawing the Cayley graph of  $C_3 \times C_3$  on a torus.

*Remark 5*) Similarly to the definitions given in Section 1 one may introduce also *infinite hypermaps*, e.g. the universal hypermaps used in Section 2, if we omit the compactness of  $X$  and replace finiteness of  $B$  with discreteness. Counting arguments do not work anymore, therefore our version of Proposition 4 does not apply to infinite regular hypermaps, for theory and examples see [6]. However, Proposition 5 remains valid, and therefore Theorem 7 generalises to infinite regular hypermaps as well.

We come back to the finite case and think of  $H$  as a permutation group acting on  $H$  by left multiplication. With the same local counterclockwise ordering  $g_0, g_0^{-1}, g_1, g_1^{-1}$  of the edges in the Cayley graph as above, we use the direction of Proposition 3 from the algebraic hypermap  $(H, H, g_0, g_1)$  to the topological hypermap  $\mathcal{H} = (X, S, A)$  and get the following converse of Theorem 7.

**Theorem 8.** *Let  $H$  be a finite group with two generators  $g_0, g_1$ . Its Cayley graph  $G$  has an embedding into a compact oriented surface  $X$  as the 1-skeleton of a hypermap  $\mathcal{H}$  in its Cori representation.*

## 4 Schreier coset diagrams

The generalisation of Theorem 7 to the case of not necessarily regular hypermaps is obvious.

**Theorem 9.** *Let  $\mathcal{H}$  be a hypermap whose hypercartographic group  $H$  is generated by elements  $g_0, g_1$ . Then the 1-skeleton graph  $G$  of its Cori representation is an embedding of the Schreier coset diagram of the left residue classes  $H/F$  into the surface  $X$  where  $F$  is the subgroup of  $H$  fixing a brin  $b$  of  $\mathcal{H}$ .*

Note that, according to Proposition 6, we use here the multiplication of  $H$  on  $H/F$  from the left, not the convention of [3]. (For the case of generators of order 1 or 2 see again the next section.) The choice of the brin is irrelevant because  $H$  acts transitively on the set of brins, so all fixing subgroups are conjugate in  $H$  and lead to isomorphic Schreier diagrams. There is also a converse to Theorem 9, but for the precise statement we have to recall that by construction as a permutation group the hypercartographic group acts effectively on  $B$ . In other words, the trivial subgroup  $\{1\}$  is the only normal subgroup of  $H$  contained in  $F$ . To pass from an arbitrary coset diagram to an algebraic hypermap we have therefore to divide out a certain normal subgroup.

**Theorem 10.** *Let  $\Phi$  be a finite group with two generators  $\phi_0, \phi_1$  and  $U$  a subgroup of  $\Phi$ , and let  $N$  be the intersection of all subgroups conjugate to  $U$ . Suppose that in*

$$H := \Phi/N, \quad F := U/N$$

*$H$  has generators  $g_i := \phi_i \bmod N$ ,  $i = 0, 1$ . Then the Schreier coset diagram of  $\Phi/U$  is the 1-skeleton  $G$  of the hypermap  $\mathcal{H} = (X, S, A) = (H, H/F, g_0, g_1)$ , embedded into the compact surface  $X$ .*

## 5 Remarks about maps and involutions

For the proofs in the last two sections we supposed always the generators to be of order  $> 2$ . How to modify definitions and proofs in the very common cases where one of the generators is an involution and the other is of order  $> 2$ ? Usually in the case  $g_1^2 = 1$  hypermaps are replaced with *maps* obtained by the Walsh representation of the hypermap — where all black vertices have valency 2 — simply by omitting these black vertices. Then the action of the *cartographic group*  $H := \langle g_0, g_1 \rangle$  has to be defined on the set of *darts*, i.e. on the directed edges of the resulting graph. But this graph is not the Cayley graph of  $H$ , of course.

One possibility to overcome this difficulty is to apply the concept of hypermaps as described above literally also to the case of a generating involution  $g_1$ , accepting that the components of  $A$  are only digons whose boundary edges are labelled both with  $g_1 = g_1^{-1}$ . This leads in Theorems 7 and 8 not to the usual concept of Cayley graphs (see [3]) in which only one edge for a generator of order 2 starts from every vertex. These usual Cayley graphs are obtained with another possibility already proposed in [2]: collapse the components of  $A$  to one-dimensional edges, so replace the respective condition 3. in the hypermap definition with

3a. The components of  $A$  are homeomorphic to closed intervals.



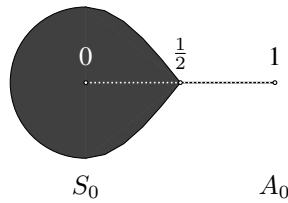


Figure 6: Universal target for a modified Cori representation of maps of type  $(p, 2, r)$ .

See [2], Figure 6, for an example of type  $(6, 2, 3)$ . Clearly for the corresponding Belyĭ functions there is a universal target hypermap of this type pictured in Figure 6.  $S_0$  is the left half of the lemniscate curve together with its interior points, and  $A_0$  is the closed interval between  $\frac{1}{2}$  and 1, the only brin is  $\frac{1}{2}$ . Note that all brins in the so-defined hypermaps have valency 3 as vertices in its 1-skeleton graph  $G$ . With these conventions, all proofs above extend without difficulty to the case of order 2 generators.

We leave it to the reader to extend this consideration to the very special situations that both generators are involutions or one generator is the identity.

Of course, many authors have explored the connections between Cayley or Schreier graphs and Map Theory. For example A. Vince [9] has constructed Schreier maps from the Coxeter group that contains the group generated by  $g_0$  and  $g_1$  with index 2. In our context this Schreier graph is the 1-skeleton of the James representation of the dessin, see Figure 4 and [5]. The Vince and James approach also applies to non-orientable hypermaps, see also Izquierdo and Singerman [4]. However, no one has previously used the Cori representation or stressed the connection with dessin d'enfants.

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