Grundy domination and zero forcing in Kneser graphs[*]

Boštjan Brešar [†]

Faculty of Natural Sciences and Mathematics, University of Maribor, Slovenia
Institute of Mathematics, Physics and Mechanics, Ljubljana, Slovenia

Tim Kos [‡]

Institute of Mathematics, Physics and Mechanics, Ljubljana, Slovenia

Pablo Daniel Torres [§]

Depto. de Matemática, Universidad Nacional de Rosario and Consejo Nacional de Investigaciones Científicas y Técnicas, Argentina

Abstract

In this paper, we continue the investigation of different types of (Grundy) dominating sequences. We consider four different types of Grundy domination numbers and the related zero forcing numbers, focusing on these numbers in the well-known class of Kneser graphs $K_{n,r}$. In particular, we establish that the Grundy total domination number $\gamma_{gr}^t(K_{n,r})$ equals $\binom{2r}{r}$ for any $r \geq 2$ and $n \geq 2r + 1$. For the Grundy domination number of Kneser graphs we get $\gamma_{gr}(K_{n,r}) = \alpha(K_{n,r})$ whenever n is sufficiently larger than r. On the other hand, the zero forcing number $Z(K_{n,r})$ is proved to be $\binom{n}{r} - \binom{2r}{r}$ when $n \geq 3r + 1$ and $r \geq 2$, while lower and upper bounds are provided for $Z(K_{n,r})$ when $2r + 1 \leq n \leq 3r$. Some lower bounds for different types of minimum ranks of Kneser graphs are also obtained along the way.

Keywords: Grundy domination number, Grundy total domination number, Kneser graph, zero forcing number, minimum rank.

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1   Introduction

The *Kneser graph*, $K_{n,r}$, where n, r are positive integers such that $n \geq 2r$, has the r-subsets of the n-set as its vertices, and two r-subsets are adjacent in $K_{n,r}$ if they are disjoint. The class of graphs became well-known by the celebrated Erdős-Ko-Rado theorem [11], which determined the independence number $\alpha(K_{n,r})$ of the Kneser graph $K_{n,r}$ to be equal to $\binom{n-1}{r-1}$. Another famous result is Lovász's proof of Kneser's conjecture, which determines the chromatic number of Kneser graphs [22], see also Matoušek for a combinatorial proof of this result [23]. Many other invariants were later considered in Kneser graphs by a number of authors. In particular, the domination number of Kneser graphs was studied in several papers [16, 19, 24], but there is no such complete solution for domination number of Kneser graphs as is the case with the chromatic and the independence number.

Various Grundy domination invariants have been introduced in recent years [5, 7, 8], arising from two standard domination invariants, the domination number and the total domination number. In the context of domination, we say that a vertex x *totally dominates* vertices in its (open) neighborhood, $N(x) = \{y \mid xy \in E(G)\}$, and that x *dominates* vertices in its closed neighborhood, $N[x] = N(x) \cup \{x\}$. A set D in a graph G is a *dominating set* (resp., a *total dominating set*) of G if every vertex in $V(G)$ is dominated (resp., totally dominated) by a vertex in D. (Clearly, G must not have an isolated vertex to have a total dominating set.) The minimum cardinality of a dominating set (resp., total dominating set) is the *domination number* (resp., the *total domination number*) of G, and is denoted by $\gamma(G)$ (resp., $\gamma_t(G)$). Now, Grundy (total) domination number is introduced while applying a greedy algorithm to obtain a (total) dominating set D as follows. Vertices are added to the set D one by one, requiring that a vertex x that was added to D (totally) dominates at least one vertex that was not (totally) dominated before this vertex was added. The longest length of such a sequence in a graph G is the *Grundy (total) domination number*, $\gamma_{gr}(G)$ (resp., $\gamma_{gr}^t(G)$). By imposing an additional condition on such dominating sequences, one gets the so-called Z-dominating sequences and Z-Grundy domination number [4], denoted by $\gamma_{gr}^Z(G)$. See Section 2 for formal definitions, and [6, 9] for more results on Grundy domination and Grundy total domination numbers.

In [5] a strong connection between the Z-Grundy domination number and the zero forcing number of a graph was established. The latter graph invariant has been intensively studied in recent years; it is closely related to another well-known domination concept called power domination, cf. [13, 14]. Moreover, the zero forcing number is very useful in determining the minimum rank of a graph; see the seminal paper about this connection [1] and some further studies [2, 3, 10]. Minimum rank $\mathrm{mr}(G)$ of a graph G is defined as the minimum rank over all symmetric matrices that have non-zero (real) values in the non-zero entries of the adjacency matrix $A(G)$ of G, arbitrary real values in the diagonal, and zero values in all other entries. (See the survey on minimum rank, where many applications and other interesting results on this parameter can be found [12].) In addition, the skew zero forcing number (denoted by $Z_-(G)$) was introduced in [18], and studied in the context of the invariant mr_0, which is a version of the minimum rank in which matrices are in addition required to have empty diagonals. Motivated by the results of [5], Lin [20] noticed a similar connection between the Grundy total domination number and the skew zero forcing number of graphs, and also between the Grundy domination number and another version of the minimum rank parameter, denoted by $\mathrm{mr}_{\hat{\ell}}$. As shown by Lin [20], the following bounds hold:

$$\gamma_{gr}(G) \leq \mathrm{mr}_{\hat{\ell}}(G), \qquad \gamma_{gr}^t(G) \leq \mathrm{mr}_0(G), \qquad \gamma_{gr}^Z(G) \leq \mathrm{mr}(G).$$

Consequently, any lower bound on a Grundy domination parameter gives a lower bound on the corresponding minimum rank parameter.

In this paper, we study four types of Grundy domination parameters (beside the mentioned ones also the L-Grundy domination number) in Kneser graphs, and apply the obtained results to give some bounds or exact results about zero forcing parameters and minimum rank parameters in Kneser graphs. In the next section, we give all the necessary definitions, establish the notation and present some preliminary results. In Section 3 we prove that $\gamma_{gr}(K_{n,2}) = \alpha(K_{n,2}) = n - 1$ for $n \geq 6$, while $\gamma_{gr}(K_{5,2}) = 5$. More generally, for Kneser graphs $K_{n,r}$ when $r > 2$ we establish an asymptotic result, which states that there exists an $n_0 \in \mathbb{N}$ dependent on r such that for all n, $n \geq n_0$, we have $\gamma_{gr}(K_{n,r}) = \alpha(K_{n,r}) = \binom{n-1}{r-1}$. The most complete result is obtained for the Grundy total domination number, where we show that for $r \geq 2$ and $n \geq 2r + 1$, $\gamma_{gr}^t(K_{n,r}) = \binom{2r}{r}$. This result is proven in Section 4, using a set theoretic result due to Gyárfás and Hubenko [15], which is the ordered version of Lovász's result for set-pair collections [21]. Section 5 is about Z-Grundy domination number of Kneser graphs. We prove that for any $r \geq 2$ and $n \geq 3r + 1$, we have $\gamma_{gr}^Z(K_{n,r}) = \binom{2r}{r}$. This immediately implies the result for Kneser graphs with $r \geq 2$ and $n \geq 3r + 1$ about their zero forcing number, notably $Z(K_{n,r}) = \binom{n}{r} - \binom{2r}{r}$; and a lower bound for the minimum rank, which reads as $\mathrm{mr}(K_{n,r}) \geq \binom{2r}{r}$. On the other hand, when $2r + 1 \leq n \leq 3r$ and $r \geq 2$ we only prove the lower bound for Z-Grundy domination number: $\gamma_{gr}^Z(K_{n,r}) \geq \binom{2r}{r} - \binom{4r-1-n}{3r-n}$, which is dependent on both r and n. In Section 6, we investigate the L-Grundy domination number of Kneser graphs $K_{n,r}$, provide the exact result when $r = 2$, and when r is bigger than 2 we prove a lower and an upper bound, which are not far apart. Finally, in the concluding section, we rephrase the Z-Grundy domination number and the Grundy domination number as set-theoretic problems.

2 Preliminaries and notation

Let $[n] = \{1, 2, \ldots, n\}$, where $n \in \mathbb{N}$. In [7] the first type of Grundy dominating sequences was introduced.

Let $S = (v_1, \ldots, v_k)$ be a sequence of distinct vertices of a graph G. The corresponding set $\{v_1, \ldots, v_k\}$ of vertices from the sequence S will be denoted by $\widehat{S}$. The initial segment $(v_1, \ldots, v_i)$ of S will be denoted by S_i. Given a sequence $S' = (u_1, \ldots, u_m)$ of vertices in G such that $\widehat{S} \cap \widehat{S'} = \emptyset$, $S \oplus S'$ is the *concatenation* of S and S', i.e., $S \oplus S' = (v_1, \ldots, v_k, u_1, \ldots, u_m)$. A sequence $S = (v_1, \ldots, v_k)$, where $v_i \in V(G)$, is called a *closed neighborhood sequence* if, for each i

$$N[v_i] \setminus \bigcup_{j=1}^{i-1} N[v_j] \neq \emptyset. \tag{2.1}$$

If for a closed neighborhood sequence S, the set $\widehat{S}$ is a dominating set of G, then S is called a *dominating sequence* of G. We will say that v_i *footprints* the vertices from $N[v_i] \setminus \bigcup_{j=1}^{i-1} N[v_j]$, and that v_i is the *footprinter* of any $u \in N[v_i] \setminus \bigcup_{j=1}^{i-1} N[v_j]$. For a dominating sequence S, any vertex in $V(G)$ has a unique footprinter in $\widehat{S}$. Clearly, the length k of a dominating sequence $S = (v_1, \ldots, v_k)$ is bounded from below by the domination number $\gamma(G)$ of a graph G. We call the maximum length of a dominating sequence in G the *Grundy*

domination number of a graph G and denote it by $\gamma_{gr}(G)$. The corresponding sequence is called a *Grundy dominating sequence* of G or γ_{gr}-*sequence* of G.

In a similar way total dominating sequences were introduced in [8], when G is a graph without isolated vertices. Using the same notation as in the previous paragraph, we say that a sequence $S = (v_1, \ldots, v_k)$, where $v_i \in V(G)$, is called an *open neighborhood sequence* if, for each i

$$N(v_i) \setminus \bigcup_{j=1}^{i-1} N(v_j) \neq \emptyset. \tag{2.2}$$

We also speak of *total footprinters* of which meaning should be clear. It is easy to see that an open neighborhood sequence S in G of maximum length yields $\widehat{S}$ to be a total dominating set; the sequence S is then called a *Grundy total dominating sequence* or γ_{gr}^t-*sequence*, and the corresponding invariant the *Grundy total domination number* of G, denoted $\gamma_{gr}^t(G)$. Any open neighborhood sequence S, where $\widehat{S}$ is a total dominating set is called a *total dominating sequence*.

Two additional variants of the Grundy (total) domination number were introduced in [5]. Let G be a graph without isolated vertices. A sequence $S = (v_1, \ldots, v_k)$, where $v_i \in V(G)$, is called a *Z-sequence* if, for each i

$$N(v_i) \setminus \bigcup_{j=1}^{i-1} N[v_j] \neq \emptyset. \tag{2.3}$$

Then the *Z-Grundy domination number* $\gamma_{gr}^Z(G)$ of the graph G is the length of a longest Z-sequence. Note that such a sequence is also a closed neighborhood sequence and hence $\gamma_{gr}^Z(G) \leq \gamma_{gr}(G)$. Given a Z-sequence S, the corresponding set $\widehat{S}$ of vertices will be called a *Z-set*. Note that $\gamma_{gr}^Z(G) = \gamma_{gr}(G)$ if and only if there exists a Grundy dominating sequence for G each vertex of which footprints some of its neighbors.
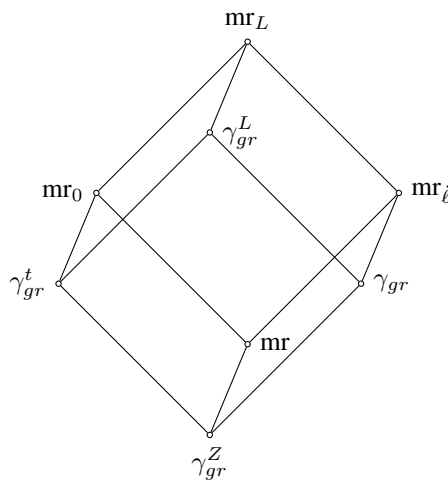

Figure 1: Connections between different Grundy domination and minimum rank parameters.

While observing the defining conditions of the three concepts in (2.1), (2.2) and (2.3), it is natural to consider also the fourth concept. It gives the longest sequences among all four versions, and we call it L-Grundy domination. Given a graph G, a sequence $S = (v_1, \ldots, v_k)$, where v_i are distinct vertices from G, is called an *L-sequence* if, for each i

$$N[v_i] \setminus \bigcup_{j=1}^{i-1} N(v_j) \neq \emptyset. \tag{2.4}$$

Then the *L-Grundy domination number* $\gamma_{gr}^L(G)$ of the graph G is the length of a longest L-sequence. Given an L-sequence S, the corresponding set $\widehat{S}$ of vertices will be called an *L-set* (the requirement that all vertices in S are distinct, indeed makes $\widehat{S}$ to be a set, and prevents the creation of an infinite sequence by repetition of one and the same vertex). Note that it is possible that v_i L-footprints only itself. We will make use of the following result.

Proposition 2.1 ([5, Proposition 3.1]). *If G is a graph with no isolated vertices, then*

(i) $\gamma_{gr}^Z(G) \leq \gamma_{gr}(G) \leq \gamma_{gr}^L(G) - 1$,

(ii) $\gamma_{gr}^Z(G) \leq \gamma_{gr}^t(G) \leq \gamma_{gr}^L(G)$

and all the bounds are sharp.

As mentioned earlier, Lin established the connections between different Grundy domination numbers, zero forcing numbers and minimum rank invariants. (For definitions of different zero forcing and minimum rank parameters we refer to [20].)

Theorem 2.2 ([20]). *If G is a graph without isolated vertices, then*

(1) $|V(G)| - Z_{\hat{\ell}}(G) = \gamma_{gr}(G) \leq \mathrm{mr}_{\hat{\ell}}(G)$,

(2) $|V(G)| - Z_-(G) = \gamma_{gr}^t(G) \leq \mathrm{mr}_0(G)$

(3) $|V(G)| - Z(G) = \gamma_{gr}^Z(G) \leq \mathrm{mr}(G)$.

(4) $|V(G)| - Z_L(G) = \gamma_{gr}^L(G) \leq \mathrm{mr}_L(G)$.

The connections between different Grundy domination and minimum rank parameters that follow from Proposition 2.1 and Theorem 2.2 are presented in the Hasse diagram in Figure 1.

3 Grundy domination number

It is easy to check (we did this by computer) that the Grundy domination number of the Petersen graph, $K_{5,2}$, equals 5. Since this is bigger (by 1) than the independence number of the Petersen graph, it is somewhat surprising that for all n bigger than 5, the Grundy domination number equals the independence number of $K_{n,2}$.

Proposition 3.1. *For $n \geq 6$, $\gamma_{gr}(K_{n,2}) = \alpha(K_{n,2}) = n - 1$.*

Proof. Clearly, as a sequence of vertices forming a maximum independent set is a Grundy dominating sequence, we have $\gamma_{gr}(K_{n,2}) \geq \alpha(K_{n,2}) = n - 1$. In addition, we have checked by computer small cases, establishing $\gamma_{gr}(K_{n,2}) = n - 1$, for $6 \leq n \leq 8$.

For the proof of the reversed inequality for $n > 8$, let $S = (v_1, \ldots, v_k)$ be a Grundy dominating sequence of $K_{n,2}$. Suppose $k > n - 1$. Hence, $\widehat{S}$ is not a stable set. We may assume that $\widehat{S}_i$ is a stable set for some $i \geq 1$, while v_{i+1} is adjacent to some $v_j \in \widehat{S}_i$. Note that, since $n \geq 6$, we may assume without loss of generality that $v_j = \{1, j+1\}$ for $1 \leq j \leq i$ (if $i = 3$ and $\widehat{S}_i = \{\{1,2\}, \{1,3\}, \{2,3\}\}$, then all vertices are already dominated by $\widehat{S}_i$, a contradiction).

First, if $i \leq 2$, then after $S_3 = (v_1, v_2, v_3)$, or $S_2 = (v_1, v_2)$ if $i = 1$, at most four vertices remain undominated, notably vertices from $\{\{1,3\}, \{1,4\}, \{2,3\}, \{2,4\}\}$. Hence, counting the largest possible number of steps, we get $k \leq 7$. As $n \geq 8$, this is only possible when $n = 8$, in which case we have $\gamma_{gr}(K_{8,2}) = 7$, already noticed above.

Hence, we may assume that $i > 2$. Therefore, after the segment S_i is included, only the vertices in $\{\{1, i+2\}, \ldots, \{1, n\}\}$ remain undominated. Since each vertex in the sequence S has to dominate some undominated vertex $k \leq i + n - (i + 1) = n - 1$, a contradiction. $\qquad\square$

While we could not find exact values of Grundy domination number for all Kneser graphs $K_{n,r}$, we prove in some sense similar result as we have for $K_{n,2}$. Namely, as soon as n is large enough with respect to a given r (e.g., when $r = 2$, this was $n \geq 6$), we can prove that $\gamma_{gr}(K_{n,r}) = \alpha(K_{n,r})$.

Theorem 3.2. *For any $r \geq 2$, there exists $n_0 \in \mathbb{N}$ such that for all n, $n \geq n_0$, we have*

$$\gamma_{gr}(K_{n,r}) = \alpha(K_{n,r}) = \binom{n-1}{r-1}.$$

Proof. The result is proven for $r = 2$ by Proposition 3.1. Fix $r \geq 3$, and consider a dominating sequence $S = (v_1, \ldots, v_m)$ of $K_{n,r}$, assuming that S is not a stable set.

In the first case, suppose that already v_1 and v_2 are adjacent, and without loss of generality we may assume that $v_1 = \{1, \ldots, r\}$ and $v_2 = \{r + 1, \ldots, 2r\}$. Let V' be the set of vertices, not dominated by $\{v_1, v_2\}$. Note that every element in V' is an r-set that contains at least one element from v_1 and at least one element from v_2. Denoting by $c = \{2r+1, \ldots, n\}$, any element in V' consists of i elements from c, where $0 \leq i \leq r-2$, j elements from v_1, where $1 \leq j \leq r - i - 1$, and consequently, $r - i - j$ elements from v_2 (note that $r - i - j \geq 1$).

Hence

$$|V'| = \sum_{i=0}^{r-2} \binom{n-2r}{i} \sum_{j=1}^{r-i-1} \binom{r}{j}\binom{r}{r-j-i}.$$

Note that $\sum_{j=1}^{r-i-1} \binom{r}{j}\binom{r}{r-j-i}$ is not dependent on n, hence for fixed r this is a constant, while $\sum_{i=0}^{r-2} \binom{n-2r}{i}$ is a polynomial in n of degree $r - 2$. Hence $|V'| = O(n^{r-2})$. On the other hand, $\alpha(K_{n,r}) = \binom{n-1}{r-1}$, hence the resulting dominating sequence is of size $\Omega(n^{r-1})$. Note that the length of the sequence S is at most $2 + |V'|$. Hence, if n is big enough, S is not a Grundy dominating sequence, because its length is less than $\binom{n-1}{r-1}$.

In the second case, the initial segment of S, S_k, is a stable set, for some $k > 1$, while v_{k+1} is adjacent to some vertex in S_k. Without loss of generality, we may assume that

$v_k = \{1, \ldots, r\}$ and $v_{k+1} = \{r + 1, \ldots, 2r\}$. Note as above that the size of the set V' of vertices in $K_{n,r}$, which are not in $N(v_k) \cup N(v_{k+1})$ is $O(n^{r-2})$. To complete the proof of this case, consider the subgraph $G' = K_{n,r} - (\cup_{v \in S_k} N[v])$. Note that G' must have an edge, for otherwise the proof is already done (indeed, no edges in this graph means, that an optimal sequence S consists of the stable set $\widehat{S_k} \cup V(G')$). Let ab be an edge in $V(G')$. Clearly, for all vertices v in S_k, v is not adjacent to the endvertices of ab, i.e., we have $v \cap a \neq \emptyset$ and $v \cap b \neq \emptyset$. In the same way as above we get that the longest such sequence S_k has at most $\sum_{i=0}^{r-2} \binom{n-2r}{i} \sum_{j=1}^{r-i-1} \binom{r}{j}\binom{r}{r-j-i}$ vertices, and so $|S_k| = k$ is of the order $O(n^{r-2})$. Note that the set of vertices not footprinted by vertices in S_{k+1} is contained in V'. Then $|S_{k+1}| + |V'| = k + 1 + |V'|$ is an upper bound for $|S|$ and remains of the order $O(n^{r-2})$. Therefore, for n big enough, $|S|$ will not be greater that $\binom{n-1}{r-1}$, which is of order $\Omega(n^{r-1})$. $\qquad \square$

For the zero forcing parameter $Z_{\dot{\ell}}$ and minimum rank parameter $\mathrm{mr}_{\dot{\ell}}$ Theorem 3.2 gives the following observation.

Corollary 3.3. *For any $r \geq 2$, there exists $n_0 \in \mathbb{N}$ such that for all n, $n \geq n_0$, we have*

$$Z_{\dot{\ell}}(K_{n,r}) = \binom{n-1}{r} \quad and$$

$$\mathrm{mr}_{\dot{\ell}}(K_{n,r}) \geq \alpha(K_{n,r}) = \binom{n-1}{r-1}.$$

Note that for $2r + 1 \leq n < n_0$ a lower bound could be improved by improving the values of $\gamma_{gr}(K_{n,r})$. It would be even more interesting to find an upper bound for $\mathrm{mr}_{\dot{\ell}}(K_{n,r})$, which might require tools from linear algebra.

Unfortunately, we do not know how large must be n_0 in Theorem 3.2 and Corollary 3.3 when $r \geq 3$. This is an interesting issue yet to be investigated. We could only check by computer that $\gamma_{gr}(K_{7,3}) = 20$, while clearly $\alpha(K_{7,3}) = 15$.

4 Grundy total domination number

Unlike the Grundy domination number, we prove that the Grundy total domination number of $K_{n,r}$ does not depend on n. Moreover, we provide the exact value of $\gamma_{gr}^t(K_{n,r})$ for all cases. To this end, we use an ordered version of Lovász's result for set-pair collections [21] provided by Gyárfás and Hubenko [15].

Lemma 4.1 ([15, Lemma 1]). *Let $T = \{(A_i, B_i) \mid 1 \leq i \leq k\}$ be a set-pair collection with $|A_i| = a$, $|B_i| = b$ satisfying the following conditions:*

1. $A_i \cap B_i = \emptyset$ for $1 \leq i \leq k$;

2. $A_i \cap B_j \neq \emptyset$ for $1 \leq i < j \leq k$.

Then $k \leq \binom{a+b}{a}$.

Proposition 4.2. *For $r \geq 2$ and $n \geq 2r + 1$, $\gamma_{gr}^t(K_{n,r}) = \binom{2r}{r}$.*

Proof. In order to obtain the lower bound, it is enough to note that any sequence S such that $\widehat{S} = \{A \subset [2r] \mid |A| = r\}$ is a total dominating sequence of $K_{n,r}$.

To prove the upper bound, let $S = (v_1, \ldots, v_k)$ be a total dominating sequence of $K_{n,r}$. For $1 \leq i \leq k$, let u_i be a vertex totally footprinted by v_i. It is not hard to see that the set-pair collection $T = \{(v_i, u_i) \mid 1 \leq i \leq k\}$ satisfies both conditions in Lemma 4.1. Since $|v_i| = |u_i| = r$ for $1 \leq i \leq k$, then $k \leq \binom{2r}{r}$ and the result follows. $\qquad \square$

For the skew zero forcing number $Z_-(G)$ and the minimum rank parameter $\mathrm{mr}_0(G)$ we get the following consequences.

Corollary 4.3. *If $r \geq 2, n \geq 2r+1$, then $Z_-(K_{n,r}) = \binom{n}{r} - \binom{2r}{r}$ and $\mathrm{mr}_0(K_{n,r}) \geq \binom{2r}{r}$.*

5   Z-Grundy domination number

It is easy to check that the Z-Grundy domination number of the Petersen graph $K_{5,2}$ and of $K_{6,2}$ equal to 5. Note that Proposition 4.2 provided a general upper bound, i.e., for $r \geq 2$ and $n \geq 2r+1$, $\gamma_{gr}^Z(K_{n,r}) \leq \binom{2r}{r}$. We can prove that this bound is reached in the following cases.

Proposition 5.1. *For $r \geq 2$ and $n \geq 3r+1$, $\gamma_{gr}^Z(K_{n,r}) = \binom{2r}{r}$.*

Proof. Let us consider the following sets

$$\widehat{S_1} = \{A \mid A \subset [2r], 1 \in A, |A| = r\},$$
$$\widehat{S_2} = \{A \mid A \subset \{r+2, r+3, \ldots, 3r\}, |A| = r\} - \{2r+1, \ldots, 3r\}.$$

Let S_1 be any sequence of $\widehat{S_1}$ and let S_2 be any sequence of $\widehat{S_2}$. We claim that $S = S_1 \oplus (\{2r+1, \ldots, 3r\}) \oplus S_2$ is a Z-dominating sequence. Indeed, each $u \in \widehat{S_1}$ Z-footprints at least $f = [2r] - u$, $\{2r+1, \ldots, 3r\}$ Z-footprints $[r-1] \cup \{3r+1\}$ and each $v \in \widehat{S_2}$ Z-footprints at least $f = \{1\} \cup (\{r+2, r+3, \ldots, 3r\} - v)$. Hence, $\gamma_{gr}^Z(K_{n,r}) \geq |S| = |S_1| + 1 + |S_2| = 2\binom{2r-1}{r} = \binom{2r}{r}$.

As we have mentioned, the equality follows directly from Proposition 2.1 and 4.2. $\qquad \square$

Proposition 5.2. *For $r \geq 2$ and $2r+1 \leq n \leq 3r$, $\gamma_{gr}^Z(K_{n,r}) \geq \binom{2r}{r} - \binom{4r-1-n}{3r-n}$.*

Proof. Let us consider the following sets

$$\widehat{S_1} = \{A \mid A \subset [2r], 1 \in A, |A| = r\},$$
$$\widehat{S_2} = \{A \mid A \subset \{n-2r+2, n-2r+3, \ldots, n\}, \{2r+1, \ldots, n\} \not\subset A, |A| = r\}.$$

Note, that

$$|\widehat{S_2}| = \binom{2r-1}{r} - \binom{2r-1-(n-2r)}{r-(n-2r)} = \binom{2r-1}{r} - \binom{4r-1-n}{3r-n}.$$

Let S_1 be any sequence of $\widehat{S_1}$ and let S_2 be any sequence of $\widehat{S_2}$. We claim that $S = S_1 \oplus S_2$ is a Z-dominating sequence. Indeed, each $u \in \widehat{S_1}$ Z-footprints at least $f = [2r] - u$ and each $v \in \widehat{S_2}$ Z-footprints at least $f = \{1\} \cup (\{n-2r+2, n-2r+3, \ldots, n\} - v)$. Hence, $\gamma_{gr}^Z(K_{n,r}) \geq |S| = |S_1| + |S_2| = 2\binom{2r-1}{r} - \binom{4r-1-n}{3r-n} = \binom{2r}{r} - \binom{4r-1-n}{3r-n}$. $\qquad \square$

Perhaps the most interesting case is that of odd graphs $K(2r+1, r)$:

Corollary 5.3. *For $r \geq 2$, $\gamma_{gr}^Z(K_{2r+1,r}) \geq \binom{2r}{r} - \binom{2r-2}{r-1} = \frac{3r-2}{r}\binom{2r-2}{r-1}$.*

For the zero forcing numbers of Kneser graphs we get the following.

Corollary 5.4.

(i) *For $r \geq 2$ and $2r + 1 \leq n \leq 3r$, $Z(K_{n,r}) \leq \binom{n}{r} - \binom{2r}{r} + \binom{4r-1-n}{3r-n}$.*

(ii) *For $r \geq 2$ and $n \geq 3r + 1$, $Z(K_{n,r}) = \binom{n}{r} - \binom{2r}{r}$.*

Propositions 5.1 and 5.2 yield the following consequences for the minimum ranks, $\mathrm{mr}(K_{n,r})$.

Corollary 5.5.

(i) *For $r \geq 2$ and $2r + 1 \leq n \leq 3r$, $\mathrm{mr}(K_{n,r}) \geq \binom{2r}{r} - \binom{4r-1-n}{3r-n}$.*

(ii) *For $r \geq 2$ and $n \geq 3r + 1$, $\mathrm{mr}(K_{n,r}) \geq \binom{2r}{r}$.*

6   L-Grundy domination number

Proposition 6.1. *For $n \geq 5$, $\gamma_{gr}^L(K_{n,2}) = n + 2$.*

Proof. First, we prove that $S = (\{1, 2\}, \{1, 3\}, \{1, 4\}, \ldots, \{1, n\}, \{2, 3\}, \{2, 4\}, \{3, 4\})$ is a L-sequence of $K_{n,2}$, where $n \geq 5$. Note that S is an L-sequence, since each $\{1, i\}$, where $i \in [n] - \{1\}$, L-footprints itself, $\{2, 3\}$ L-footprints $\{1, 4\}$, $\{2, 4\}$ L-footprints $\{1, 3\}$, and $\{3, 4\}$ L-footprints $\{1, 2\}$. Hence, $\gamma_{gr}^L(K_{n,2}) \geq n + 2$.

For the proof of the reversed inequality for $n \geq 5$, let $S = (v_1, \ldots, v_k)$ be a maximal L-sequence of $K_{n,2}$. We may assume without loss of generality that $v_1 = \{1, 2\}$. Clearly, v_1 L-footprints itself and vertices of the form $\{i, j\}$, where $i, j \in [n] - \{1, 2\}$ and $i \neq j$. Next, assume v_2 is a neighbor of v_1. Without loss of generality we may assume $v_2 = \{3, 4\}$. Thus, v_2 L-footprints itself and vertices of the form $\{i, j\}$, where $i \in \{1, 2\}, j \in [n] - \{3, 4\}$ and $i \neq j$. Hence, just the vertices $\{1, 3\}, \{1, 4\}, \{2, 3\}$ and $\{2, 4\}$ remain not L-dominated by S_2. The rest can be L-dominated with at most 4 vertices. Follows, $|S| \leq 6 < n + 2$.

Next, suppose v_2 is not a neighbor of v_1. Without loss of generality let $v_2 = \{1, 3\}$. In this case, the vertex $\{2, 3\}$ and the vertices of the form $\{1, i\}$, where $i \in [n] - \{1\}$, remain not L-dominated by S_2. Next, we distinguish again 4 cases:

(i) v_3 is a neighbor of v_1 and v_2 (v_3 is of the form $\{i, j\}$, where $i, j \in [n] - \{1, 2, 3\}$ and $i \neq j$),

(ii) v_3 is a neighbor of v_1 and not of v_2 (v_3 is of the form $\{3, i\}$, where $i \in [n] - \{1, 2, 3\}$),

(iii) v_3 is a neighbor of v_2 and not of v_1 (v_3 is of the form $\{2, i\}$, where $i \in [n] - \{1, 2, 3\}$), and

(iv) v_3 is not a neighbor of v_1 or v_2 (v_3 is $\{2, 3\}$ or is of the form $\{1, i\}$, where $i \in [n] - \{1, 2, 3\}$).

In the cases (i), (ii), (iii) or if $v_3 = \{2, 3\}$ in the case (iv), there are at most 3 vertices left, that are not L-dominated by $\widehat{S_3}$. In all cases we can L-dominate the rest with at most 4 vertices. Hence, $|S| \leq 7 \leq n + 2$.

In the case (iv), where $v_3 = \{1, i\}$ ($i \in [n] - \{1, 2, 3\}$), the vertex v_3 L-footprints just itself and $\{2, 3\}$. Hence, just the vertices of the form $\{1, i\}$, where $i \in [n] - \{1\}$, remain not L-dominated by $\widehat{S_3}$. Note, that it is possible that v_i is first L-footprinted by itself and then later again by another vertex from S. Next, if $v_4 = \{1, 4\}$, then v_4 just L-footprints itself and the same vertices stay not L-dominated. Hence, to make the sequence S as large as possible, next in the sequence can be vertices $v_{i-1} = \{1, i\}$ for $i = 4, \ldots, n$.

Without loss of generality let $v_n = \{2, 3\}$ (until now in the sequence are all vertices that contain 1). Hence, just the vertices $\{1, 2\}$ and $\{1, 3\}$ remain not L-dominated by $\widehat{S_n}$. Follows, $|S| \leq n + 2$. $\qquad \square$

Proposition 6.2. *For $r \geq 2$ and $n \geq 2r + 1$, $\gamma_{gr}^L(K_{n,r}) \geq \binom{n-1}{r-1} + \binom{2r-1}{r}$.*

Proof. Let $\widehat{S_1} = \{A \mid A \subset [n], 1 \in A, |A| = r\}$ and let $\widehat{S_2} = \{A \mid A \subset [2r] - \{1\}, |A| = r\}$. Let S_1 be any sequence of $\widehat{S_1}$ and let S_2 be any sequence of $\widehat{S_2}$. We claim that $S = S_1 \oplus S_2$ is an L-dominating sequence.

Indeed, each $u \in \widehat{S_1}$ L-footprints at least itself and each $v \in \widehat{S_2}$ L-footprints at least $f = [2r] - v$. Hence, $\gamma_{gr}^L(K_{n,r}) \geq |S| = |S_1| + |S_2| = \binom{n-1}{r-1} + \binom{2r-1}{r}$. $\qquad \square$

For the upper bound, we first present a general result bounding the L-Grundy domination number of a graph G with no isolated vertices by using the independence number $\alpha(G)$ and the Grundy total domination number $\gamma_{gr}^t(G)$.

Proposition 6.3. *For a graph G with no isolated vertices, $\gamma_{gr}^L(G) \leq \alpha(G) + \gamma_{gr}^t(G) - 1$.*

Proof. Let $S = (v_1, \ldots, v_k)$ be an L-sequence of G. Let A, B and C be the sets of vertices of $\widehat{S}$ such that every vertex in A only L-footprints itself, every vertex in B L-footprints itself and (at least) one more vertex and every vertex in C does not L-footprint itself. Note that $\{A, B, C\}$ is a partition of $\widehat{S}$. Besides, $A \cup B$ is a stable set of G. Hence, $|A| + |B| \leq \alpha(G)$. Let $S' = (w_1, \ldots, w_m)$ be the subsequence of S (respecting the order in S) such that $\widehat{S'} = B \cup C$. Clearly, S' is an open neighborhood sequence in G, thus $m \leq \gamma_{gr}^t(G)$.

Therefore, $|S| = |A| + |B| + |C| \leq \alpha(G) + \gamma_{gr}^t(G) - |B| \leq \alpha(G) + \gamma_{gr}^t(G) - 1$, since $v_1 \in B$. $\qquad \square$

Corollary 6.4. *For $r \geq 2$ and $n \geq 2r + 1$, $\gamma_{gr}^L(K_{n,r}) \leq \binom{n-1}{r-1} + \binom{2r}{r} - 1$.*

Note that the gap between the lower and the upper bound in Proposition 6.2 and Corollary 6.4 is $\binom{2r-1}{r} - 1$, which is fixed with respect to n.

7   Set-theoretic connections

Following the set-theoretic connections as in the case of Grundy total domination number (see Lemma 4.1), we ask the following.

Problem 7.1. Let $T = \{(A_i, B_i) \mid (A_i \cup B_i) \subseteq [n], |A_i| = |B_i| = r, \text{ for all } i \in [k]\}$ be a set-pair collection satisfying the following conditions:

1. $A_i \cap B_i = \emptyset$ or $A_i = B_i$ for $1 \leq i \leq k$;

2. $A_i \cap B_j \neq \emptyset$ and $A_i \neq B_j$ for $1 \leq i < j \leq k$.

Note that $|T| = k$.

Determine the smallest value $f(n, r)$ for which $k \leq f(n, r)$ for all such set-pair collections T.

From Theorem 3.2, for any $r \geq 2$ there exists $n_0 \in \mathbb{N}$ such that for all n, $n \geq n_0$, we have $f(n, r) = \gamma_{gr}(K_{n,r}) = \alpha(K_{n,r}) = \binom{n-1}{r-1}$. Note that in this case, the cardinality n of the universal set in which A_i and B_i are contained plays an important role in determining $f(n, r)$. As we see in the next case, allowing condition $A_i = B_i$ is also relevant to the problem.

In a similar way, we propose the corresponding question for the Z-Grundy domination number.

Problem 7.2. Let $T = \{(A_i, B_i) \mid (A_i \cup B_i) \subseteq [n], |A_i| = |B_i| = r, \text{ for all } i \in [k]\}$ be a set-pair collection satisfying the following conditions:

1. $A_i \cap B_i = \emptyset$ for $1 \leq i \leq k$;

2. $A_i \cap B_j \neq \emptyset$ and $A_i \neq B_j$ for $1 \leq i < j \leq k$.

Note that $|T| = k$.

Determine the smallest value $f_Z(n, r)$ for which $k \leq f_Z(n, r)$ for all such set-pair collections T.

Note that $f_Z(n, r) = \gamma_{gr}^Z(K_{n,r}) = \binom{2r}{r}$ for $n \geq 3r + 1$. In this case $f_Z(n, r)$ is independent of n, but it is unclear whether this function is dependent on n when $2r + 1 \leq n \leq 3r$.

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