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Algebraic degrees of 2-Cayley digraphs over abelian groups*

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Abstract

A digraph Γ is called a 2-Cayley digraph over a group G if there exists a 2-orbit semiregular subgroup of Aut(Γ) isomorphic to G. In this paper, we completely determine the algebraic degrees of 2-Cayley digraphs over abelian groups. This generalizes the main results of Lu and Mönius in 2023. As applications, we consider the algebraic degrees of Cayley digraphs over finite groups admitting an abelian subgroup of index 2. Special attention is paid to the algebraic degrees of Cayley (di)graphs over generalized dihedral groups, generalized dicyclic groups and semi-dihedral groups.

Keywords: Algebraic degree, 2-Cayley digraph, Abelian group.

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1 Introduction

A *digraph* Γ consists of a finite set $V(\Gamma)$ of vertices and a set $E(\Gamma)$ of directed edges, where $E(\Gamma) \subseteq V(\Gamma) \times V(\Gamma)$. If $(u, v) \in E(\Gamma)$ implies $(v, u) \in E(\Gamma)$, then Γ is said to be undirected. For a digraph Γ on n vertices, its *adjacency matrix* $A = (a_{uv})_{n\times n}$ is defined as

$$a_{uv} = \begin{cases} 1, & \text{if } (u, v) \in E(\Gamma), \\ 0, & \text{otherwise.} \end{cases}$$

The characteristic polynomial of Γ is the characteristic polynomial of A. The eigenvalues of A are called the *eigenvalues* of Γ. The collection of eigenvalues of Γ together with their

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multiplicities is called the *spectrum* of Γ, denoted by $\mathrm{Spec}(\Gamma)$. Note that A is not always symmetric, so the eigenvalues of Γ need not be real numbers.

Let G be a finite group and $S \subseteq G \setminus \{e\}$, where e is the identity. The *Cayley digraph* $\Gamma = \mathrm{Cay}(G, S)$ of G with respect to S is defined by $V(\Gamma) = G$ and $E(\Gamma) = \{(g, sg) \mid g \in G, s \in S\}$. If $S = S^{-1}$, then $\Gamma = \mathrm{Cay}(G, S)$ is called a *Cayley graph*. For a digraph Γ, the set of all permutations of $V(\Gamma)$ that preserve the adjacency relation of Γ forms a group, called the *automorphism group* of Γ, and is denoted by $\mathrm{Aut}(\Gamma)$. By a theorem of Sabidussi [15], a digraph Γ is a Cayley digraph over G if and only if there exists a regular subgroup of $\mathrm{Aut}(\Gamma)$ isomorphic to G. As a generalization of Sabidussi's Theorem [1], a digraph Γ is called a 2-*Cayley digraph* over G if there exists a 2-orbit semiregular subgroup of $\mathrm{Aut}(\Gamma)$ isomorphic to G. A 2-Cayley graph is also termed as a semi-Cayley graph in [5, 6]. A special 2-Cayley graph is called a bi-Cayley graph in [19].

For a digraph Γ, its *splitting field* $\mathbb{SF}(\Gamma)$ is the smallest field extension of $\mathbb{Q}$ which contains all eigenvalues of the adjacency matrix of Γ. The extension degree $[\mathbb{SF}(\Gamma) : \mathbb{Q}]$ is called the *algebraic degree* of Γ, denoted by $\deg(\Gamma)$. A digraph Γ is called *integral* if all the eigenvalues of the adjacency matrix of Γ are integers. A digraph Γ is called *algebraically integral* over a number field K if all the eigenvalues of the adjacency matrix of Γ are algebraic integers of K. There is a close connection between the splitting field and the algebraic integrality of a digraph. For example, for any number field K, $\mathbb{SF}(\Gamma) \subseteq K$ if and only if Γ is algebraically integral over K. Integral graphs and algebraically integral graphs have been extensively studied in the literature [2, 3, 4, 8, 9, 11]. In recent years, the splitting field and algebraic degree have attracted much attention. In 2020, Mönius [13] studied the algebraic degrees of circulant graphs $\mathrm{Cay}\,(\mathbb{Z}_p, S)$ for a prime number p. In 2022, Mönius [14] generalized those results in [13] by determining the splitting fields and the algebraic degrees of circulant graphs $\mathrm{Cay}\,(\mathbb{Z}_n, S)$ for arbitrary n. Based on Mönius's work, in 2022, Huang et al. [18] determined the splitting fields and algebraic degrees of mixed Cayley graphs over abelian groups. Lu et al. [12] determined the splitting fields of Cayley graphs over abelian groups and dihedral groups. They also gave bounds for the algebraic degrees of Cayley graphs over dihedral groups. Also in 2022, Sripaisan et al. [16] studied the algebraic degrees of Cayley hypergraphs. For more details, one may refer to the comprehensive survey [10] in this subject.

In this paper, inspired by the above mentioned results, we completely determine the splitting fields and algebraic degrees of 2-Cayley digraphs over abelian groups in Section 3, which generalizes the main results of [12]. From computational viewpoints, we also derive sharp upper and lower bounds for their algebraic degrees. As applications, in Section 4, we consider the algebraic degrees of Cayley digraphs over finite groups admitting an abelian subgroup of index 2. Furthermore, we consider the algebraic degrees of Cayley graphs over generalized dihedral groups and generalized dicyclic groups, and get improved upper bounds. Finally, we determine the algebraic degrees of Cayley digraphs over semi-dihedral groups.

2 Preliminaries

Let G be a finite group. A *representation* of G is a homomorphism $\rho\colon G \to \mathrm{GL}(V)$ for some n-dimensional vector space over the complex field $\mathbb{C}$, where $\mathrm{GL}(V)$ denotes the group of automorphisms of V. The dimension of V is called the *degree* of ρ. Two representations ρ_1 and ρ_2 of G on V_1 and V_2 respectively are *equivalent* if there is an isomorphism $T\colon V_1 \to V_2$ such that $T\rho_1(g) = \rho_2(g)T$ for all $g \in G$.

Let $\rho\colon G \to GL(V)$ be a representation. The *character* $\chi_\rho\colon G \to \mathbb{C}$ of ρ is defined by setting $\chi_\rho(g) = \mathrm{Tr}(\rho(g))$ for $g \in G$, where $\mathrm{Tr}(\rho(g))$ is the trace of the representation matrix of $\rho(g)$ with respect to a specified basis of V. By the degree of χ_ρ we mean the degree of ρ, which is simply $\chi_\rho(1)$. If W is a $\rho(g)$-invariant subspace of V for each $g \in G$, then we call W a $\rho(G)$-invariant subspace of V. If the only $\rho(G)$-invariant subspace of V are $\{0\}$ and V, we call ρ an *irreducible representation* of G, and the corresponding character χ_ρ an *irreducible character* of G. We denote by $\mathrm{IRR}(G)$ and $\mathrm{Irr}(G)$ the complete set of non-equivalent irreducible representations of G and the complete set of non-equivalent irreducible characters of G, respectively.

For any subset $X \subseteq G$, we denote by $\delta_X = (\delta_g)_{g\in G}$ the characteristic vector of X over G, where $\delta_g = 1$ if $g \in X$ and $\delta_g = 0$ if $g \notin X$. For any multi-subset $X \subseteq G$, we denote by $\delta'_X = (\delta'_g)_{g\in G}$ the characteristic vector of X over G, where $\delta'_g = k$ if g appears k times in X and $\delta'_g = 0$ if $g \notin X$. Throughout this paper, we use $X = [x \mid x \in X]$ to denote the multi-set X, and $\varphi(n)$ to denote the Euler totient function of a natural number n (it is the number of the positive integers which are smaller than n and coprime to n). Firstly, we state an equivalent definition of 2-Cayley digraphs.

Lemma 2.1 ([1]). *A digraph Γ is a 2-Cayley digraph over G if and only if there exist subsets T_{ij} of G, where $1 \leqslant i, j \leqslant 2$, such that Γ is isomorphic to a digraph Υ with*

$$V(\Upsilon) = G \times \{1,2\}, \quad E(\Upsilon) = \bigcup_{1\leqslant i,j\leqslant 2} \{((g,i),(tg,j)) \mid g \in G \text{ and } t \in T_{ij}\}.$$

By Lemma 2.1, a 2-Cayley digraph is characterized by a group G and four subsets T_{ij} of G. Thus we denote a 2-Cayley digraph with respect to four subsets T_{ij} by $\Gamma = \mathrm{Cay}\,(G; T_{ij} \mid 1 \leqslant i,j \leqslant 2)$. Note that $V(\Gamma) = G \times \{1,2\}$, $(g,i) \sim (h,j)$ if and only if $hg^{-1} \in T_{ij}$, and Γ is undirected if and only if for all $1 \leqslant i,j \leqslant 2$, $T_{ij} = T_{ji}^{-1}$. Note also that Γ is a digraph without loops if and only if $T_{ii} \subseteq G\backslash\{e\}$, for all $1 \leqslant i \leqslant 2$.

Let $\omega_n = \exp\left(\frac{2\pi \mathrm{i}}{n}\right)$ be the primitive n-th root of unity. We consider an abelian group G of order n. It is well known that

$$G \cong \mathbb{Z}_{n_1} \oplus \cdots \oplus \mathbb{Z}_{n_r},$$

where $n = \prod_{i=1}^{r} n_i$, and n_i is a prime power for $1 \le i \le r$. Without loss of generality, we assume that $G = \mathbb{Z}_{n_1} \oplus \cdots \oplus \mathbb{Z}_{n_r}$ and $\mathbf{0} = (0,\ldots,0) \in G$ is the identity of G.

Lemma 2.2 ([17]). *Let $G = \mathbb{Z}_{n_1} \oplus \cdots \oplus \mathbb{Z}_{n_r}$ be an abelian group of order n. Then $\mathrm{Irr}(G) = \{\chi_l \mid l \in G\}$, where $\chi_l(g) = \prod_{i=1}^{r} \omega_{n_i}^{l_i g_i}$ for all $l = (l_1,\ldots,l_r), g = (g_1,\ldots,g_r) \in G$, and $\omega_{n_i} = \exp(\frac{2\pi \mathrm{i}}{n_i})$.*

For simplicity, for any (multi-)subset S of G, we denote

$$\chi_l(S) = \sum_{s\in S} \chi_l(s).$$

Arezoomand [1] obtained the following result.

Lemma 2.3 ([1]). *Let $\Gamma = \mathrm{Cay}\,(G, T_{ij} \mid 1 \leqslant i,j \leqslant 2)$ be a 2-Cayley digraph over an abelian group $G = \mathbb{Z}_{n_1} \oplus \cdots \oplus \mathbb{Z}_{n_r}$ of order n. Then Γ has eigenvalues*

$$\frac{\chi_l(T_{11}) + \chi_l(T_{22}) \pm \sqrt{\left(\chi_l(T_{11}) - \chi_l(T_{22})\right)^2 + 4\chi_l(T_{21})\chi_l(T_{12})}}{2}, \quad l \in G.$$

Let K be a field. In what follows, we will refer to the subgroup

$$K^{\times 2} = \left\{x^2 : x \in K\right\} \subset K^{\times},$$

where $K^{\times} = K \setminus \{0\}$. More precisely, we shall encounter quite often the quotient $K^{\times}/K^{\times 2}$. The image of $x \in K^{\times}$in $K^{\times}/K^{\times 2}$ will be denoted by $[x]_K$.

Lemma 2.4 ([7, Corollary 1.23]). *Suppose K is a field containing a primitive 2-th root of unity, and let $F = K\left[\sqrt{a_1}, \dots, \sqrt{a_k}\,\right]$, where $a_i \in K$. Then $\mathrm{Gal}(F/K)$ is isomorphic to the subgroup of $K^{\times}/K^{\times 2}$ generated by $[a_1]_K, \dots, [a_k]_K$.*

3 2-Cayley digraphs over abelian groups

In this section, we always assume that $G = \mathbb{Z}_{n_1} \oplus \cdots \oplus \mathbb{Z}_{n_r}$ is an abelian group of order n. Let $\Gamma = \mathrm{Cay}\left(G, T_{ij} \mid 1 \leqslant i, j \leqslant 2\right)$ be a 2-Cayley digraph over G. For any two subsets X and Y of G, we define the multi-set $X + Y = [x + y \mid x \in X, y \in Y]$. For any (multi-)set X and $k \in \mathbb{N}$, $k * X$ denotes the multi-set in which each element of X appears k times. For example, if $X = [1, 1, 2, 2, 2, 3, 4]$ and $k = 2$, then $k * X = 4 * \{1\} \cup 6 * \{2\} \cup 2 * \{3, 4\}$, with duplicate elements allowed in the union.

For two multi-sets $U = k_1 * \{g_1\} \cup k_2 * \{g_2\} \cup \ldots \cup k_s * \{g_s\}$ and $V = q_1 * \{g_1\} \cup q_2 * \{g_2\} \cup \ldots \cup q_t * \{g_t\} \cup k_{s+1} * \{g_{s+1}\} \cup \ldots \cup k_{s+m} * \{g_{s+m}\}$, where $k_i \geq q_i, s > t$ and $g_i, 1 \leq i \leq s + m$ are pairwise distinct, we define $U \setminus V = (k_1 - q_1) * \{g_1\} \cup (k_2 - q_2) * \{g_2\} \cup \ldots \cup (k_t - q_t) * \{g_t\} \cup k_{t+1} * \{g_{t+1}\} \cup \ldots \cup k_s * \{g_s\}$, $V \setminus U = k_{s+1} * \{g_{s+1}\} \cup \ldots \cup k_{s+m} * \{g_{s+m}\}$.

Using the symbols in Lemma 2.3, we let

$$\begin{aligned}
&I_1 = [t \mid t \in T_{11} \text{ or } t \in T_{22}]\,,\\
&I_2 = [t \mid t \in (T_{11} + T_{11}) \text{ or } t \in (T_{22} + T_{22}) \text{ or } t \in 4 * (T_{12} + T_{21})],\\
&I_3 = [t \mid t \in 2 * (T_{11} + T_{22})],
\end{aligned}$$

where I_1, I_2 and I_3 are multi-sets. For example, for the group $G = \mathbb{Z}_4$, if $T_{11} = \{1, 2\}$, $T_{12} = \{1\}$, $T_{21} = \{2\}$ and $T_{22} = \{3\}$, then $I_1 = \{1, 2, 3\}$, $I_2 = 6 * \{3\} \cup 2 * \{2\} \cup \{0\}$ and $I_3 = 2 * \{0, 1\}$.

By Lemma 2.3, we have the following result.

Lemma 3.1. *Let $\Gamma = \mathrm{Cay}\left(G, T_{ij} \mid 1 \leqslant i, j \leqslant 2\right)$ be a 2-Cayley digraph over an abelian group G of order n. Then Γ has eigenvalues*

$$\frac{\chi_l(I_1) \pm \sqrt{\chi_l(I_2 \setminus I_3) - \chi_l(I_3 \setminus I_2)}}{2}, \quad l \in G,$$

where I_1, I_2, I_3 are described as above.

Proof. Firstly, we have

$$\chi_l(T_{11}) + \chi_l(T_{22}) = \chi_l(I_1).$$

In addition,

$$\begin{aligned}
&\left(\chi_l(T_{11}) - \chi_l(T_{22})\right)^2 + 4\chi_l(T_{21})\chi_l(T_{12})\\
=&\chi_l(T_{11} + T_{11}) + \chi_l(T_{22} + T_{22}) + \chi_l\left(4 * (T_{12} + T_{21})\right) - \chi_l\left(2 * (T_{11} + T_{22})\right)\\
=&\chi_l(I_2) - \chi_l(I_3)\\
=&\chi_l(I_2 \setminus I_3) - \chi_l(I_3 \setminus I_2),
\end{aligned}$$

so the result follows from Lemma 2.3. □

Using the symbols in Lemma 3.1, for $l \in G$, let

$$\beta_l = \chi_l(I_1) \text{ and } \gamma_l = \chi_l(I_2 \setminus I_3) - \chi_l(I_3 \setminus I_2). \tag{3.1}$$

As $\beta_l, \gamma_l \in \mathbb{Q}(\omega_n)$, where $n = |G|$, without loss of generality, we assume that K is a field such that $\mathbb{Q} \subseteq K \subseteq \mathbb{Q}(\omega_n)$. Therefore, $\mathrm{Gal}(\mathbb{Q}(\omega_n)/K)) \leq \mathrm{Gal}(\mathbb{Q}(\omega_n)/\mathbb{Q}) \cong \mathbb{Z}_n^* = \{k \in \mathbb{Z}_n \mid \gcd(k, n) = 1\}$. Let

$$\eta\colon\ \mathrm{Gal}(\mathbb{Q}(\omega_n)/\mathbb{Q}) \to \mathbb{Z}_n^*$$

be the isomorphism such that $\sigma(\omega_n) = \omega_n^{\eta(\sigma)}$, where $\sigma \in \mathrm{Gal}(\mathbb{Q}(\omega_n)/\mathbb{Q})$. Let

$$H = \eta(\mathrm{Gal}(\mathbb{Q}(\omega_n)/K)).$$

Then H is a subgroup of $\mathbb{Z}_n^*$. We consider the action of $\mathbb{Z}_n^*$ on $G = \mathbb{Z}_{n_1} \oplus \cdots \oplus \mathbb{Z}_{n_r}$ by setting $kg = k(g_1, \ldots, g_r) = (kg_1, \ldots, kg_r)$ for any $k \in \mathbb{Z}_n^*$ and $g \in G$. Then

$$\sigma\left(\omega_{n_i}^{l_i}\right) = \sigma\left(\omega_n^{nl_i/n_i}\right) = \omega_n^{\eta(\sigma)\cdot nl_i/n_i} = \omega_{n_i}^{\eta(\sigma)l_i},$$

where $l_i \in \mathbb{Z}_{n_i}(1 \leq i \leq r)$. Note that for any $\sigma \in \mathrm{Gal}(\mathbb{Q}(\omega_n)/\mathbb{Q})$, we have

$$\begin{aligned}\sigma(\beta_l) = \sigma(\chi_l(I_1)) = \sigma\left(\sum_{t \in I_1}\prod_{i=1}^{r}\omega_{n_i}^{l_i t_i}\right) = \sum_{t \in I_1}\prod_{i=1}^{r}\sigma\left(\omega_{n_i}^{l_i t_i}\right)\\ = \sum_{t \in I_1}\prod_{i=1}^{r}\omega_{n_i}^{\eta(\sigma)l_i t_i} = \chi_l(\eta(\sigma)I_1),\end{aligned}$$

where $\eta(\sigma)I_1 = \{(\eta(\sigma)t_1, \ldots, \eta(\sigma)t_r) \mid (t_1, \ldots, t_r) \in I_1\}$. Similarly, we have

$$\begin{aligned}\sigma(\gamma_l) &= \sigma(\chi_l(I_2 \setminus I_3) - \chi_l(I_3 \setminus I_2))\\ &= \chi_l(\eta(\sigma)(I_2 \setminus I_3)) - \chi_l(\eta(\sigma)(I_3 \setminus I_2)),\end{aligned}$$

where $I_2 \setminus I_3$ and $I_3 \setminus I_2$ are multi-sets as stated in Lemma 3.1.

We first prove the following results.

Proposition 3.2. *For the symbols in (3.1), we have $\beta_l \in K$ for all $l \in G$ if and only if $hI_1 = I_1$ for all $h \in H$, where $H = \eta(\mathrm{Gal}(\mathbb{Q}(\omega_n)/K))$.*

Proof. Assume that $hI_1 = I_1$ for all $h \in H$. Then for any $\sigma \in \mathrm{Gal}(\mathbb{Q}(\omega_n)/K)$, we have $\eta(\sigma) \in H$. Thus for any $l \in G$, we have

$$\sigma(\beta_l) = \chi_l(\eta(\sigma)I_1) = \chi_l(I_1) = \beta_l.$$

It follows that $\beta_l \in K$ for all $l \in G$.

Conversely, assume that $\beta_l \in K$ for all $l \in G$. For any $h \in H$, there exists some $\sigma \in \mathrm{Gal}(\mathbb{Q}(\omega_n)/K)$ such that $\eta(\sigma) = h$. Then

$$\chi_l(hI_1) = \chi_l(\eta(\sigma)I_1) = \sigma(\beta_l) = \beta_l = \chi_l(I_1).$$

Let $M = (\chi_l(g))_{l,g\in G}$. We get

$$M\delta'_{hI_1} = M\delta'_{I_1}.$$

Note that M is invertible by the orthogonal relations of irreducible characters of G. So we have

$$\delta'_{hI_1} = \delta'_{I_1}.$$

This implies $hI_1 = I_1$. Since h is arbitrary, the result follows. □

Proposition 3.3. *For the symbols in (3.1), we have $\gamma_l \in K$ for all $l \in G$ if and only if $h(I_2 \setminus I_3) = I_2 \setminus I_3$, $h(I_3 \setminus I_2) = I_3 \setminus I_2$ for all $h \in H$, where $H = \eta(\mathrm{Gal}(\mathbb{Q}(\omega_n)/K))$.*

Proof. Assume that $h(I_2 \setminus I_3) = I_2 \setminus I_3$, $h(I_3 \setminus I_2) = I_3 \setminus I_2$ for all $h \in H$. Then for any $\sigma \in \mathrm{Gal}(\mathbb{Q}(\omega_n)/K)$, we have $\eta(\sigma) \in H$. Thus, for any $l \in G$, we have

$$\sigma(\gamma_l) = \chi_l(\eta(\sigma)(I_2 \setminus I_3)) - \chi_l(\eta(\sigma)(I_3 \setminus I_2)) = \chi_l(I_2 \setminus I_3) - \chi_l(I_3 \setminus I_2) = \gamma_l.$$

It follows that $\gamma_l \in K$ for all $l \in G$.

Conversely, assume that $\gamma_l \in K$ for all $l \in G$. For any $h \in H$, there exists some $\sigma \in \mathrm{Gal}(\mathbb{Q}(\omega_n)/K)$ such that $\eta(\sigma) = h$. Then

$$\chi_l(\eta(\sigma)(I_2 \setminus I_3)) - \chi_l(\eta(\sigma)(I_3 \setminus I_2)) = \sigma(\gamma_l) = \gamma_l = \chi_l(I_2 \setminus I_3) - \chi_l(I_3 \setminus I_2).$$

This means that

$$\chi_l(h(I_2 \setminus I_3)) - \chi_l(h(I_3 \setminus I_2)) = \chi_l(I_2 \setminus I_3) - \chi_l(I_3 \setminus I_2).$$

Let $M = (\chi_l(g))_{l,g\in G}$. We get

$$M\delta'_{h(I_2\setminus I_3)} - M\delta'_{h(I_3\setminus I_2)} = M\delta'_{I_2\setminus I_3} - M\delta'_{I_3\setminus I_2}.$$

Note that M is invertible and $I_2 \setminus I_3$ is disjoint with $I_3 \setminus I_2$. So we have $h(I_2 \setminus I_3) = I_2 \setminus I_3$ and $h(I_3 \setminus I_2) = I_3 \setminus I_2$. As h is arbitrary, the result follows. □

Note that $\beta_{\mathbf{0}}, \gamma_{\mathbf{0}} \in \mathbb{Z}$, so we let

$$L = K = \mathbb{Q}(\beta_l, \gamma_l \mid l \in G \setminus \{\mathbf{0}\}) \tag{3.2}$$

and

$$H' = \{h \in \mathbb{Z}_n^* \mid hI_1 = I_1, h(I_2 \setminus I_3) = I_2 \setminus I_3, h(I_3 \setminus I_2) = I_3 \setminus I_2\}. \tag{3.3}$$

Then we have the following result.

Proposition 3.4. *Using the symbols in (3.2) and (3.3), we have $H' = \eta(\mathrm{Gal}(\mathbb{Q}(\omega_n)/L))$.*

Proof. By Propositions 3.2 and 3.3, it is clear that

$$\eta(\mathrm{Gal}(\mathbb{Q}(\omega_n)/L)) \subseteq H'.$$

Now we prove

$$H' \subseteq \eta(\mathrm{Gal}(\mathbb{Q}(\omega_n)/L)).$$

For each $h' \in H'$, let $\sigma = \eta^{-1}(h')$. It follows that $h' = \eta(\sigma)$. For any $l \in G$, we have

$$\sigma(\beta_l) = \chi_l(\eta(\sigma)I_1) = \chi_l(h'I_1) = \chi_l(I_1) = \beta_l.$$

Similarly, for any $l \in G$,

$$\sigma(\gamma_l) = \chi_l(\eta(\sigma)(I_2 \setminus I_3)) - \chi_l(\eta(\sigma)(I_3 \setminus I_2)) = \chi_l(I_2 \setminus I_3) - \chi_l(I_3 \setminus I_2) = \gamma_l.$$

Hence

$$\sigma \in \mathrm{Gal}(\mathbb{Q}(\omega_n)/L) \text{ and } h' = \eta(\sigma) \in \eta(\mathrm{Gal}(\mathbb{Q}(\omega_n)/L)).$$

Thus the result follows. □

Since H' is a subgroup of $\mathbb{Z}_n^*$, by Proposition 3.4, we have

$$L = \mathbb{Q}(\omega_n)^{\eta^{-1}(H')} = \{x \in \mathbb{Q}(\omega_n) \mid \sigma(x) = x \text{ for all } \sigma \in \eta^{-1}(H')\}. \tag{3.4}$$

Considering H' acting on G, assume that $H'g^{(1)}, H'g^{(2)}, \ldots, H'g^{(k)}$ are all distinct orbits of H' on G, where $g^{(i)} \in G$. Let

$$C = \left\{g^{(i)} \mid G \cap H'g^{(i)} \neq \emptyset\right\}. \tag{3.5}$$

Let M be the subgroup of $L^\times/L^{\times 2}$ generated by all $[\gamma_l]_L$ for $l \in C$. Explicitly,

$$M = \langle [\gamma_l]_L \mid l \in C \rangle. \tag{3.6}$$

Now we are ready to prove our main result.

Theorem 3.5. *Let* $\Gamma = \mathrm{Cay}(G, T_{ij} \mid 1 \leqslant i,j \leqslant 2)$ *be a* 2*-Cayley digraph over an abelian group* G *of order* n*. Then the splitting field of* Γ *is* $L\left(\sqrt{\gamma_l} \mid l \in C\right)$ *and the algebraic degree of* Γ *satisfies*

$$\deg(\Gamma) = \frac{\varphi(n)|M|}{|H'|},$$

where γ_l, H', L, C, M *are given in* (3.1) *and* (3.3) – (3.6)*, respectively.*

Proof. If a, b are in the same orbit $H'g^{(i)}$, then there exists $h \in H'$ such that $b = ha$. It follows that

$$\begin{aligned}\gamma_b &= \chi_b(I_2 \setminus I_3) - \chi_b(I_3 \setminus I_2) = \chi_{ha}(I_2 \setminus I_3) - \chi_{ha}(I_3 \setminus I_2) \\ &= \chi_a(h(I_2 \setminus I_3)) - \chi_a(h(I_3 \setminus I_2)) = \chi_a(I_2 \setminus I_3) - \chi_a(I_3 \setminus I_2) \\ &= \gamma_a.\end{aligned}$$

Therefore, there are at most $|C|$ different elements in $\{\gamma_l \mid l \in G\}$.

Set $F = L\left(\sqrt{\gamma_l} \mid l \in C\right)$. Note that $F = \mathbb{Q}\left(\beta_l + \sqrt{\gamma_l}, \beta_l - \sqrt{\gamma_l} \mid l \in G\right)$. So the first assertion follows. By Lemma 2.4,

$$\deg(\Gamma) = [F:\mathbb{Q}] = [F:L][L:\mathbb{Q}] = \frac{[\mathbb{Q}(\omega_n):Q]\,[F:L]}{[\mathbb{Q}(\omega_n):L]} = \frac{\varphi(n)|M|}{|H'|}.$$

This completes the proof. □

It is not easy to calculate $|M|$, but apparently $1 \leq |M| \leq 2^{|C|}$, thus we have

Corollary 3.6. *Let* $\Gamma = \text{Cay}\,(G, T_{ij} \mid 1 \leqslant i, j \leqslant 2)$ *be a* 2*-Cayley digraph over an abelian group* G *of order* n*. Then the algebraic degree of* Γ *satisfies*

$$\frac{\varphi(n)}{|H'|} \leq \deg(\Gamma) \leq \frac{\varphi(n)2^{|C|}}{|H'|},$$

where H', C *are given in* (3.3) *and* (3.5)*, respectively.*

Remark 3.7. Theorem 3.5 and Corollary 3.6 still hold for a 2-Cayley graph. Indeed, we just need to restrict $T_{ij} = T_{ji}^{-1}$ for all $1 \leqslant i, j \leqslant 2$, and modify the associated multi-sets I_1, I_2 and I_3.

The next two examples tell us that both the lower and upper bound in Corollary 3.6 are sharp.

Example 3.8. Let $\Gamma = \text{Cay}\,(\mathbb{Z}_3, T_{ij} \mid 1 \leqslant i, j \leqslant 2)$ be a 2-Cayley graph over $G = \mathbb{Z}_3$. Let $T_{11} = T_{22} = \{1, 2\}$ and $T_{12} = \{1\}$ and $T_{21} = \{2\}$. Then $I_1 = 2 * \{1, 2\}$, $I_2 \setminus I_3 = 4 * \{0\}$ and $I_3 \setminus I_2 = \emptyset$. It follows that $H' = \{1, 2\} = \mathbb{Z}_3^*$, $L = \mathbb{Q}$ and $\gamma_l = 4$ for all $l \in \mathbb{Z}_3$. Thus $|M| = 1$ and $\deg(\Gamma) = \frac{\varphi(3)}{|H'|} = 1$. In fact, $\text{Spec}(\Gamma) = 2 * \{-2, 0\} \cup \{1, 3\}$.

Example 3.9. Let $\Gamma = \text{Cay}\,(\mathbb{Z}_4, T_{ij} \mid 1 \leqslant i, j \leqslant 2)$ be a 2-Cayley digraph over $G = \mathbb{Z}_4$. Let $T_{11} = \{1, 2\}$ and $T_{12} = \{1\}$. Let $T_{21} = \{2\}$ and $T_{22} = \{3\}$. Then $I_1 = \{1, 2, 3\}$, $I_2 \setminus I_3 = 6 * \{3\} \cup 2 * \{2\}$ and $I_3 \setminus I_2 = 2 * \{1\} \cup \{0\}$. It follows that $H' = \{1\}$ and $C = \mathbb{Z}_4$. By Corollary 3.6, $\deg(\Gamma) \leq 2^5 = 32$. In fact, $L = \mathbb{Q}\,(\mathrm{i})$ and $F = L\left(\sqrt{5}, \sqrt{-8\mathrm{i} - 3}, \sqrt{-3}, \sqrt{8\mathrm{i} - 3}\right)$. Obviously, $\deg(\Gamma) = 32$.

Observe that $L = \mathbb{Q}$ if and only if $|H'| = \varphi(n)$, as an application of Theorem 3.5, the next corollary provides a class of integral 2-Cayley digraphs over abelian groups.

Corollary 3.10. *Let* $\Gamma = \text{Cay}\,(G, T_{ij} \mid 1 \leqslant i, j \leqslant 2)$ *be a* 2*-Cayley digraph over an abelian group* G *of order* n*. If* $H' = \mathbb{Z}_n^*$ *and* γ_l *is a square of an integer for each* $l \in C$*, where* γ_l, H', C *are given in* (3.1)*,* (3.3) *and* (3.5)*, respectively, then* Γ *is integral.*

Sometimes, we need not to compute $|M|$ in Theorem 3.5.

Corollary 3.11. *Let* $\Gamma = \text{Cay}\,(G, T_{ij} \mid 1 \leqslant i, j \leqslant 2)$ *be a* 2*-Cayley digraph over an abelian group* G *of order* n*. If* $T_{11} = T_{22}$ *and* $T_{12} = T_{12}^{-1} = T_{21}$*, then the splitting field of* Γ *satisfies*

$$\mathbb{SF}(\Gamma) = \mathbb{Q}\,(\omega_n)^{\eta^{-1}(H'')} = \{x \in \mathbb{Q}\,(\omega_n) \mid \sigma(x) = x \text{ for all } \sigma \in \eta^{-1}(H'')\},$$

the algebraic degree of Γ *satisfies*

$$\deg(\Gamma) = \frac{\varphi(n)}{|H''|},$$

where $H'' = \{h \in \mathbb{Z}_n^* \mid hT_{11} = T_{11}, hT_{12} = T_{12}\}$.

Proof. Since $I_1 = [t \mid t \in 2 * T_{11}]$, $I_2 \setminus I_3 = \left[t \mid t \in 4 * (T_{12} + T_{12}^{-1})\right]$ and $I_3 \setminus I_2 = \emptyset$, we have $\gamma_l = 4\chi_l(T_{12} + T_{12}^{-1}) = 4|\chi_l(T_{12})|^2 = 4\chi_l(T_{12})^2$. Note that $T_{12} = T_{12}^{-1}$. So $\chi_l(T_{12})$ is a real number. It follows that

$$\sqrt{\gamma_l} = 2\chi_l(T_{12}) \text{ or } \sqrt{\gamma_l} = -2\chi_l(T_{12}).$$

The rest of the proof is similar to that of Theorem 3.5. □

Corollary 3.12. *Let* $\Gamma = \mathrm{Cay}\,(G, T_{ij} \mid 1 \leqslant i, j \leqslant 2)$ *be a* 2*-Cayley digraph over an abelian group* G *of order* n*. If* $T_{11} = T_{22}$*,* $T_{12} = T_{12}^{-1} = T_{21}$*, and* $hT_{11} = T_{11}$*,* $hT_{12} = T_{12}$ *for all* $h \in \mathbb{Z}_n^*$*, then* Γ *is integral.*

4 Some applications

4.1 Cayley digraphs over groups admitting an abelian subgroup of index 2

A Cayley digraph over a finite group G with a subgroup of index 2 is a 2-Cayley digraph, as the following result shows.

Lemma 4.1 ([1]). *Let* $\Gamma = \mathrm{Cay}(G, S)$ *be a Cayley (di)graph. Suppose that there exists a subgroup* N *of* G *with index* 2*. If* $\{x_1, x_2\}$ *is a left transversal to* N *in* G*, then* $\Gamma \cong \mathrm{Cay}\,(N, S_{ij} \mid 1 \leqslant i, j \leqslant 2)$*, where* $S_{ij} = \{a \in N \mid x_j^{-1} a x_i \in S\} = N \cap x_j S x_i^{-1}$.

Let A be a finite abelian group of order $n \geq 3$. Let $f \in \mathrm{Aut}(A)$ be of order 2. Let $y \in A$ be such that $f(y) = y$. Let G be a non-abelian finite group admitting an abelian subgroup A of index 2. Then G admits a presentation

$$G = \left\langle A, x \mid x^2 = y, xax^{-1} = f(a), a \in A \right\rangle.$$

Observe that $G = A \cup xA$ and $B = \left\{f(a)a^{-1} \mid a \in A\right\}$ is a subgroup of A. In particular, if $f(a) = a^{-1}$ for $a \in A$, then $B = A^2$ and $y^2 = e$, where e is the identity of A. If $y = e$, then G is the generalized dihedral group $\mathrm{Dih}(A)$, with the presentation

$$\mathrm{Dih}(A) = \left\langle A, x \mid x^2 = e, xax^{-1} = a^{-1}, a \in A \right\rangle.$$

If $y \neq e$ (and so $n = |A|$ is even), then G is the generalized dicyclic group $\mathrm{Dic}(A, y)$, with the presentation

$$\mathrm{Dic}(A, y) = \left\langle A, x \mid x^2 = y, xax^{-1} = a^{-1}, a \in A \right\rangle.$$

As the group operation here is multiplication, we assume that $A = \langle a_1 \rangle_{n_1} \otimes \cdots \otimes \langle a_r \rangle_{n_r}$ and $\mathrm{Irr}(A) = \{\chi_l \mid (a_1^{l_1}, \ldots, a_r^{l_r}) \in A\}$, where $l = (l_1, \ldots, l_r)$. In this subsection, we always assume that G is a group admitting an abelian subgroup A of order n and of index 2. As an application of Theorem 3.5, we consider the algebraic degree of the Cayley digraph $\Gamma = \mathrm{Cay}(G, S)$. Note that $A \cong A' = \mathbb{Z}_{n_1} \oplus \cdots \oplus \mathbb{Z}_{n_r}$. It is worth pointing out that the group operation here should correspond to the addition in Section 3.

By Lemmas 2.3 and 4.1, we get the following result.

Lemma 4.2. *Let* $\Gamma = \mathrm{Cay}(G, S)$ *be a Cayley digraph and* $A = \langle a_1 \rangle_{n_1} \otimes \cdots \otimes \langle a_r \rangle_{n_r}$ *be an abelian subgroup of* G *of order* n *and of index* 2 *with left transversal* $\{x_1, x_2\}$*. Then* Γ *has eigenvalues*

$$\frac{\chi_l(T_{11}) + \chi_l(T_{22}) \pm \sqrt{\left(\chi_l(T_{11}) - \chi_l(T_{22})\right)^2 + 4\chi_l(T_{21})\chi_l(T_{12})}}{2}, \; l = (l_1, \ldots, l_r) \in A',$$

where $n = \prod_{i=1}^{r} n_i$, $T_{ij} = \left\{t = (t_1, \dots, t_r) \mid \left(a_1^{t_1}, \dots, a_r^{t_r}\right) \in x_j S x_i^{-1}\right\}$ *and* $A' = \mathbb{Z}_{n_1} \oplus \cdots \oplus \mathbb{Z}_{n_r}$.

Using the symbols in Lemma 4.2, in a similar way as in Section 3, we define

$$\begin{aligned} I_1 &= [t \mid t \in T_{11} \text{ or } t \in T_{22}], \\ I_2 &= [t \mid t \in (T_{11} + T_{11}) \text{ or } t \in (T_{22} + T_{22}) \text{ or } t \in 4 * (T_{12} + T_{21})], \\ I_3 &= [t \mid t \in 2 * (T_{11} + T_{22})]. \end{aligned}$$

Let

$$\beta_l = \chi_l(I_1) \text{ and } \gamma_l = \chi_l(I_2 \setminus I_3) - \chi_l(I_3 \setminus I_2). \tag{4.1}$$

Let $\eta\colon \operatorname{Gal}(\mathbb{Q}(\omega_n)/\mathbb{Q}) \to \mathbb{Z}_n^*$ be the isomorphism such that $\sigma(\omega_n) = \omega_n^{\eta(\sigma)}$, where $\sigma \in \operatorname{Gal}(\mathbb{Q}(\omega_n)/\mathbb{Q})$. Let

$$L = \mathbb{Q}(\beta_l, \gamma_l \mid l \in A' \setminus \{\mathbf{0}\}) \tag{4.2}$$

and

$$H' = \{h \in \mathbb{Z}_n^* \mid hI_1 = I_1, h(I_2 \setminus I_3) = I_2 \setminus I_3, h(I_3 \setminus I_2) = I_3 \setminus I_2\}. \tag{4.3}$$

Since $\Gamma \cong \operatorname{Cay}(A', T_{ij} \mid 1 \leqslant i, j \leqslant 2)$, by Proposition 3.4, we have the following result.

Proposition 4.3. *Using the symbols in* (4.2) *and* (4.3)*, we have* $H' = \eta(\operatorname{Gal}(\mathbb{Q}(\omega_n)/L))$.

Now we consider H' acting on $A' = \mathbb{Z}_{n_1} \oplus \cdots \oplus \mathbb{Z}_{n_r}$. Assume that $H'a^{(1)}, H'a^{(2)}, \dots, H'a^{(k)}$ are all distinct orbits of H' on A', where $a^{(i)} \in A'$. Let

$$C = \left\{a^{(i)} \mid A' \cap H'a^{(i)} \neq \emptyset\right\} \tag{4.4}$$

and

$$M = \langle [\gamma_l]_L \mid l \in C \rangle. \tag{4.5}$$

Since $\Gamma \cong \operatorname{Cay}(A', T_{ij} \mid 1 \leqslant i, j \leqslant 2)$, using the conclusions in Section 3, we immediately get the following results.

Theorem 4.4. *Let* $\Gamma = \operatorname{Cay}(G, S)$ *be a Cayley digraph, and* $A = \langle a_1 \rangle_{n_1} \otimes \cdots \otimes \langle a_r \rangle_{n_r}$ *be an abelian subgroup of* G *of order* n *and of index 2 with left transversal* $\{x_1, x_2\}$*. Then the splitting field of* Γ *is* $L\left(\sqrt{\gamma_l} \mid l \in C\right)$ *and the algebraic degree of* Γ *satisfies*

$$\deg(\Gamma) = \frac{\varphi(n)|M|}{|H'|},$$

where $L = \mathbb{Q}(\omega_n)^{\eta^{-1}(H')} = \{x \in \mathbb{Q}(\omega_n) \mid \sigma(x) = x \text{ for all } \sigma \in \eta^{-1}(H')\}$ *and* γ_l, H', C, M *are given in* (4.1) *and* (4.3) – (4.5)*, respectively.*

Corollary 4.5. *Let* $\Gamma = \mathrm{Cay}(G, S)$ *be a Cayley digraph, and* $A = \langle a_1\rangle_{n_1} \otimes \cdots \otimes \langle a_r\rangle_{n_r}$ *be an abelian subgroup of* G *of order* n *and of index* 2 *with left transversal* $\{x_1, x_2\}$*. Then the algebraic degree of* Γ *satisfies*

$$\frac{\varphi(n)}{|H'|} \le \deg(\Gamma) \le \frac{\varphi(n)2^{|C|}}{|H'|},$$

where H', C *are given in* (4.3) *and* (4.4)*, respectively.*

Corollary 4.6. *Let* $\Gamma = \mathrm{Cay}(G, S)$ *be a Cayley digraph, and* $A = \langle a_1\rangle_{n_1} \otimes \cdots \otimes \langle a_r\rangle_{n_r}$ *be an abelian subgroup of* G *of order* n *and of index* 2 *with left transversal* $\{x_1, x_2\}$*. If* $H' = \mathbb{Z}_n^*$ *and* γ_l *is a square of an integer for each* $l \in C$*, where* γ_l, H', C *are given in* (4.1)*,* (4.3) *and* (4.4)*, respectively, then* Γ *is integral.*

Corollary 4.7. *Let* $\Gamma = \mathrm{Cay}(G, S)$ *be a Cayley digraph, and* $A = \langle a_1\rangle_{n_1} \otimes \cdots \otimes \langle a_r\rangle_{n_r}$ *be an abelian subgroup of* G *of order* n *and of index* 2 *with left transversal* $\{x_1, x_2\}$*. Let* $T_{ij} = \left\{(t_1, \ldots, t_r) \mid \left(a_1^{t_1}, \ldots, a_r^{t_r}\right) \in x_j S x_i^{-1}\right\}$*. If* $T_{11} = T_{22}$ *and* $T_{12} = T_{12}^{-1} = T_{21}$*, then the splitting field of* Γ *satisfies*

$$\mathbb{SF}(\Gamma) = \mathbb{Q}(\omega_n)^{\eta^{-1}(H'')} = \{x \in \mathbb{Q}(\omega_n) \mid \sigma(x) = x \text{ for all } \sigma \in \eta^{-1}(H'')\},$$

the algebraic degree of Γ *satisfies*

$$\deg(\Gamma) = \frac{\varphi(n)}{|H''|},$$

where $H'' = \{h \in \mathbb{Z}_n^* \mid hT_{11} = T_{11}, hT_{12} = T_{12}\}$*.*

Corollary 4.8. *Let* $\Gamma = \mathrm{Cay}(G, S)$ *be a Cayley digraph, and* $A = \langle a_1\rangle_{n_1} \otimes \cdots \otimes \langle a_r\rangle_{n_r}$ *be an abelian subgroup of* G *of order* n *and of index* 2 *with left transversal* $\{x_1, x_2\}$*. Let* $T_{ij} = \left\{(t_1, \ldots, t_r) \mid \left(a_1^{t_1}, \ldots, a_r^{t_r}\right) \in x_j S x_i^{-1}\right\}$*. If* $T_{11} = T_{22}$*,* $T_{12} = T_{12}^{-1} = T_{21}$*, and* $hT_{11} = T_{11}$*,* $hT_{12} = T_{12}$ *for all* $h \in \mathbb{Z}_n^*$*, then* Γ *is integral.*

4.2 Cayley graphs over generalized dihedral groups

In the following two subsections, we consider *Cayley graphs* but not digraphs. The generalized dihedral group $\mathrm{Dih}(A)$ is given by the following presentation

$$\mathrm{Dih}(A) = \left\langle A, x \mid x^2 = e, xax^{-1} = a^{-1}, a \in A\right\rangle.$$

Let $\Gamma = \mathrm{Cay}(\mathrm{Dih}(A), S)$ be a Cayley digraph. Using the symbols in Subsection 4.1, note that $A = \langle a_1\rangle_{n_1} \otimes \cdots \otimes \langle a_r\rangle_{n_r}$, and $|A| = n$, so $|\mathrm{Dih}(A)| = 2n$. Without loss of generality, let $x_1 = e$ and $x_2 = x$. Then

$$\begin{aligned}
T_{11} &= \left\{(t_1, \ldots, t_r) \mid \left(a_1^{t_1}, \ldots, a_r^{t_r}\right) \in S\right\}, \\
T_{12} &= \left\{(t_1, \ldots, t_r) \mid \left(a_1^{t_1}, \ldots, a_r^{t_r}\right) \in xS\right\}, \\
T_{21} &= \left\{(t_1, \ldots, t_r) \mid \left(a_1^{t_1}, \ldots, a_r^{t_r}\right) \in Sx\right\}, \\
T_{22} &= \left\{(t_1, \ldots, t_r) \mid \left(a_1^{t_1}, \ldots, a_r^{t_r}\right) \in xSx\right\}.
\end{aligned} \tag{4.6}$$

For the algebraic degree of the digraph Γ, we just need to replace T_{ij} given in Subsection 4.1 with T_{ij} given in (4.6), so we omit the details here.

We are now interested in the algebraic degree of the undirected Cayley graph $\Gamma = \mathrm{Cay}(\mathrm{Dih}(A), S)$. Using the symbols in (4.6), as $S = S^{-1}$, we have $T_{22}^{-1} = T_{22} = T_{11} = T_{11}^{-1}$ and $T_{12}^{-1} = T_{21}$. Let $t = (t_1, \dots, t_r)$. In a similar way as in Subsection 4.1, we define

$$I_1 = [t \mid t \in T_{11} \text{ or } t \in T_{22}],$$
$$I_2 = [t \mid t \in (T_{11} + T_{11}) \text{ or } t \in (T_{22} + T_{22}) \text{ or } t \in 4 * (T_{12} + T_{21})],$$
$$I_3 = [t \mid t \in 2 * (T_{11} + T_{22})].$$

It follows that $I_1 = [t \mid t \in 2 * T_{11}]$, $I_2 \setminus I_3 = \left[t \mid t \in 4 * (T_{12} + T_{12}^{-1})\right]$ and $I_3 \setminus I_2 = \emptyset$. In fact, by Lemma 4.2, the eigenvalues of the Cayley graph $\Gamma = \mathrm{Cay}(\mathrm{Dih}(A), S)$ are

$$\chi_l(T_{11}) \pm |\chi_l(T_{12})|, \; l \in A',$$

where $A' = \mathbb{Z}_{n_1} \oplus \cdots \oplus \mathbb{Z}_{n_r}$. Note that $|\chi_l(T_{12})| = \sqrt{\chi_l(T_{12})\chi_l(T_{12}^{-1})} = \sqrt{\chi_l(T_{12} + T_{12}^{-1})}$, the multi-sets I_1 and $I_2 \setminus I_3$ can be reduced to $I_1' = T_{11}$ and $(I_2 \setminus I_3)' = \left[t \mid t \in T_{12} + T_{12}^{-1}\right]$.

Let

$$\beta_l = \chi_l(I_1') \text{ and } \gamma_l = \chi_l((I_2 \setminus I_3)'). \tag{4.7}$$

Let $\eta\colon \mathrm{Gal}\left(\mathbb{Q}\left(\omega_n\right)/\mathbb{Q}\right) \to \mathbb{Z}_n^*$ be the isomorphism such that $\sigma\left(\omega_n\right) = \omega_n^{\eta(\sigma)}$, where $\sigma \in \mathrm{Gal}\left(\mathbb{Q}\left(\omega_n\right)/\mathbb{Q}\right)$. Let

$$L = \mathbb{Q}\left(\beta_l, \gamma_l \mid l \in A' \setminus \{\mathbf{0}\}\right) \tag{4.8}$$

and

$$H' = \{h \in \mathbb{Z}_n^* \mid hI_1' = I_1', h(I_2 \setminus I_3)' = (I_2 \setminus I_3)'\}. \tag{4.9}$$

Note that $I_1 = 2 * I_1'$ and $I_2 \setminus I_3 = 4 * (I_2 \setminus I_3)'$. So we have the following result by Proposition 4.3.

Proposition 4.9. *Using the symbols in* (4.8) *and* (4.9)*, we have* $H' = \eta\left(\mathrm{Gal}\left(\mathbb{Q}\left(\omega_n\right)/L\right)\right)$.

Similarly, we consider H' acting on A'. Assume that $H'a^{(1)}, H'a^{(2)}, \dots H'a^{(k)}$ are all distinct orbits of H' on A', where $a^{(i)} \in A'$. Let $B = \{x \in A' \mid 2x = \mathbf{0}\}$ and $A' = B \cup E \cup E^{-1}$, where B, E, E^{-1} are disjoint. Let

$$C' = \left\{a^{(i)} \mid (B \cup E) \cap H'a^{(i)} \neq \emptyset\right\} \tag{4.10}$$

and

$$M = \langle [\gamma_l]_L \mid l \in C' \rangle. \tag{4.11}$$

Then the following results hold.

Theorem 4.10. *Let* $\Gamma = \mathrm{Cay}(\mathrm{Dih}(A), S)$ *be a Cayley graph over the generalized dihedral group* $\mathrm{Dih}(A)$ *of order* $2n$*. Then the splitting field of* Γ *is* $L\left(\sqrt{\gamma_l} \mid l \in C'\right)$ *and the algebraic degree of* Γ *satisfies*

$$\deg(\Gamma) = \frac{\varphi(n)|M|}{|H'|},$$

where $L = \mathbb{Q}(\omega_n)^{\eta^{-1}(H')} = \{x \in \mathbb{Q}(\omega_n) \mid \sigma(x) = x \text{ for all } \sigma \in \eta^{-1}(H')\}$ *and* γ_l, H', C', M *are given in* (4.7) *and* (4.9) – (4.11)*, respectively.*

Proof. Since $((I_2 \setminus I_3)')^{-1} = (I_2 \setminus I_3)'$, it follows that

$$\gamma_l = \chi_l((I_2 \setminus I_3)') = \chi_l(((I_2 \setminus I_3)')^{-1}) = \chi_{-l}((I_2 \setminus I_3)') = \gamma_{-l}.$$

Then the result follows from Theorem 4.4. □

Corollary 4.11. *Let* $\Gamma = \mathrm{Cay}(\mathrm{Dih}(A), S)$ *be a Cayley graph over the generalized dihedral group* $\mathrm{Dih}(A)$ *of order* $2n$*. Then the algebraic degree of* Γ *satisfies*

$$\frac{\varphi(n)}{|H'|} \leq \deg(\Gamma) \leq \frac{\varphi(n)2^{|C'|}}{|H'|},$$

where H', C' *are given in* (4.9) *and* (4.10)*, respectively.*

Corollary 4.12. *Let* $\Gamma = \mathrm{Cay}(\mathrm{Dih}(A), S)$ *be a Cayley graph over the generalized dihedral group* $\mathrm{Dih}(A)$ *of order* $2n$*. If* $H' = \mathbb{Z}_n^*$ *and* γ_l *is a square of an integer for each* $l \in C'$*, where* γ_l, H', C' *are given in* (4.7)*,* (4.9) *and* (4.10)*, respectively, then* Γ *is integral.*

Corollary 4.13. *Let* $\Gamma = \mathrm{Cay}(\mathrm{Dih}(A), S)$ *be a Cayley graph over the generalized dihedral group* $\mathrm{Dih}(A)$ *of order* $2n$*. Let*

$$T_{11} = \left\{(t_1, \dots, t_r) \mid \left(a_1^{t_1}, \dots, a_r^{t_r}\right) \in S\right\} \text{ and } T_{12} = \left\{(t_1, \dots, t_r) \mid \left(a_1^{t_1}, \dots, a_r^{t_r}\right) \in xS\right\}.$$

If $T_{12} = T_{12}^{-1}$*, then the splitting field of* Γ *satisfies*

$$\mathbb{SF}(\Gamma) = \mathbb{Q}(\omega_n)^{\eta^{-1}(H'')} = \{x \in \mathbb{Q}(\omega_n) \mid \sigma(x) = x \text{ for all } \sigma \in \eta^{-1}(H'')\},$$

the algebraic degree of Γ *satisfies*

$$\deg(\Gamma) = \frac{\varphi(n)}{|H''|},$$

where $H'' = \{h \in \mathbb{Z}_n^* \mid hT_{11} = T_{11}, hT_{12} = T_{12}\}$*.*

Corollary 4.14. *Let* $\Gamma = \mathrm{Cay}(\mathrm{Dih}(A), S)$ *be a Cayley graph over the generalized dihedral group* $\mathrm{Dih}(A)$ *of order* $2n$*. Let*

$$T_{11} = \left\{(t_1, \dots, t_r) \mid \left(a_1^{t_1}, \dots, a_r^{t_r}\right) \in S\right\} \text{ and } T_{12} = \left\{(t_1, \dots, t_r) \mid \left(a_1^{t_1}, \dots, a_r^{t_r}\right) \in xS\right\}.$$

If $T_{12} = T_{12}^{-1}$ *and for all* $h \in \mathbb{Z}_n^*$*,* $hT_{11} = T_{11}$ *and* $hT_{12} = T_{12}$*, then* Γ *is integral.*

In particular, let $\mathrm{Dih}(A) = D_{2n} = \langle a, b \mid a^n = b^2 = e, bab = a^{-1}\rangle$ be the dihedral group of order $2n$. Then $A' = \mathbb{Z}_n$, $I_1' = T_{11} = \{t \mid a^t \in S\}$, $T_{12} = \{t \mid ba^t \in S\}$ and $(I_2 \setminus I_3)' = \left[t \mid t \in T_{12} + T_{12}^{-1}\right]$. Note that

$$\beta_l = \chi_l(I_1') \text{ and } \gamma_l = \chi_l((I_2 \setminus I_3)'), \tag{4.12}$$

where $\chi_l(t) = \omega_n^{lt}$ and $0 \leq l \leq n-1$. Furthermore,

$$L = \mathbb{Q}(\beta_l, \gamma_l \mid 1 \leq l \leq n-1).$$

We first try to simplify the expression of L.

Lemma 4.15. *Let K be a field such that $\mathbb{Q} \subseteq K \subseteq \mathbb{Q}(\omega_n)$. If $\beta_1, \gamma_1 \in K$, then $\beta_l, \gamma_l \in K$ for $1 \leq l \leq n-1$.*

Proof. For $1 \leq l \leq n-1$, let $\sigma_l : \mathbb{Q}(\omega_n) \to \mathbb{Q}(\omega_n)$ be defined by $\sigma_l(\omega_n) = \omega_n^l$. It is clear that σ_l is a homomorphism and $\beta_l = \sigma_l(\beta_1), \gamma_l = \sigma_l(\gamma_1)$. Thus, for any $\sigma \in \mathrm{Gal}(\mathbb{Q}(\omega_n)/K)$, we have

$$\sigma(\beta_l) = \sigma(\sigma_l(\beta_1)) = \sigma\left(\sigma_l\left(\sum_{t\in I_1'} \omega_n^t\right)\right) = \sum_{t\in I_1'} \omega_n^{\eta(\sigma)tl} = \sigma_l(\sigma(\beta_1)) = \sigma_l(\beta_1) = \beta_l.$$

Similarly, $\sigma(\gamma_l) = \gamma_l$. Therefore, $\beta_l, \gamma_l \in K$. □

By Lemma 4.15, we have

$$L = \mathbb{Q}(\beta_1, \gamma_1).$$

Let

$$H' = \{h \in \mathbb{Z}_n^* \mid hI_1' = I_1', h(I_2 \setminus I_3)' = (I_2 \setminus I_3)'\}. \tag{4.13}$$

Then $H' = \eta(\mathrm{Gal}(\mathbb{Q}(\omega_n)/L))$. Note that

$$C' = \left\{a^{(i)} \mid \{0, 1, \ldots, \lfloor n/2 \rfloor\} \cap H' a^{(i)} \neq \emptyset\right\} \tag{4.14}$$

and

$$M = \langle [\gamma_l]_L \mid l \in C' \rangle, \tag{4.15}$$

where $H'a^{(1)}, H'a^{(2)}, \ldots H'a^{(k)}$ are all distinct orbits of H' on $\mathbb{Z}_n$. Consequently, we have the following corollaries.

Corollary 4.16. *Let $\Gamma = \mathrm{Cay}(D_{2n}, S)$ be a Cayley graph over the dihedral group D_{2n}. Then the splitting field of Γ is $L\left(\sqrt{\gamma_l} \mid l \in C'\right)$ and the algebraic degree of Γ satisfies*

$$\deg(\Gamma) = \frac{\varphi(n)|M|}{|H'|},$$

where $L = \mathbb{Q}(\omega_n)^{\eta^{-1}(H')} = \{x \in \mathbb{Q}(\omega_n) \mid \sigma(x) = x \text{ for all } \sigma \in \eta^{-1}(H')\}$ and γ_l, H', C', M are given in (4.12) – (4.15), respectively.

Corollary 4.17. *Let $\Gamma = \mathrm{Cay}(D_{2n}, S)$ be a Cayley graph over the dihedral group D_{2n}. Then the algebraic degree of Γ satisfies*

$$\frac{\varphi(n)}{|H'|} \leq \deg(\Gamma) \leq \frac{\varphi(n)2^{|C'|}}{|H'|},$$

where H', C' are given in (4.13) and (4.14), respectively.

Corollary 4.18. *Let $\Gamma = \mathrm{Cay}(D_{2n}, S)$ be a Cayley graph over the dihedral group D_{2n}. Let $T_{11} = \{t \mid a^t \in S\}$ and $T_{12} = \{t \mid ba^t \in S\}$. If $T_{12} = T_{12}^{-1}$, then the splitting field of Γ satisfies*

$$\mathbb{SF}(\Gamma) = \mathbb{Q}(\omega_n)^{\eta^{-1}(H'')} = \{x \in \mathbb{Q}(\omega_n) \mid \sigma(x) = x \text{ for all } \sigma \in \eta^{-1}(H'')\},$$

the algebraic degree of Γ *satisfies*

$$\deg(\Gamma) = \frac{\varphi(n)}{|H''|},$$

where $H'' = \{h \in \mathbb{Z}_n^* \mid hT_{11} = T_{11}, hT_{12} = T_{12}\}$.

There are results similar to Corollaries 4.12 and 4.14 as well, we omit them here. We end this subsection with the following example.

Example 4.19. Let $D_{16} = \langle a, b \mid a^8 = b^2 = e, bab = a^{-1}\rangle$ be the dihedral group of order 16 and $S = \left\{a, a^7, b\right\}$. We consider the algebraic degree of $\Gamma = \mathrm{Cay}(D_{16}, S)$. Then $T_{11} = \{1, -1\}$ and $T_{12} = \{0\} = T_{12}^{-1}$. It follows that $H'' = \{1, -1\} \leq \mathbb{Z}_8^*$. By Corollary 4.18, $\mathbb{SF}(\Gamma) = \mathbb{Q}\left(\omega_8\right)^{\eta^{-1}(H'')} = \mathbb{Q}(\sqrt{2})$ and $\deg(\Gamma) = \frac{\varphi(8)}{|H''|}$=2. In fact,

$$\mathrm{Spec}(\Gamma) = 2 * \{\sqrt{2}+1, \sqrt{2}-1, -\sqrt{2}+1, -\sqrt{2}-1\} \cup 3 * \{1, -1\} \cup \{-3, 3\}.$$

4.3 Cayley graphs over generalized dicyclic groups

For the generalized dicyclic group $\mathrm{Dic}(A, y)$, it has the following presentation

$$\mathrm{Dic}(A, y) = \left\langle A, x \mid x^2 = y, xax^{-1} = a^{-1}, a \in A \right\rangle.$$

We put our focus on the algebraic degree of the *Cayley graph* $\Gamma = \mathrm{Cay}(\mathrm{Dic}(A, y), S)$. Using the symbols in Subsection 4.1, since $A = \langle a_1\rangle_{n_1} \otimes \cdots \otimes \langle a_r\rangle_{n_r}$, and $|A| = n$ is even, say $n = 2m$, then $|\mathrm{Dic}(A, y)| = 4m$. Let $x_1 = e$ and $x_2 = x$. Then

$$\begin{aligned}
T_{11} &= \left\{(t_1, \ldots, t_r) \mid \left(a_1^{t_1}, \ldots, a_r^{t_r}\right) \in S\right\}, \\
T_{12} &= \left\{(t_1, \ldots, t_r) \mid \left(a_1^{t_1}, \ldots, a_r^{t_r}\right) \in xS\right\}, \\
T_{21} &= \left\{(t_1, \ldots, t_r) \mid \left(a_1^{t_1}, \ldots, a_r^{t_r}\right) \in Sx^{-1}\right\}, \\
T_{22} &= \left\{(t_1, \ldots, t_r) \mid \left(a_1^{t_1}, \ldots, a_r^{t_r}\right) \in xSx^{-1}\right\}.
\end{aligned}$$

Since $S = S^{-1}$, we have $T_{22}^{-1} = T_{22} = T_{11} = T_{11}^{-1}$ and $T_{12}^{-1} = T_{21}$. Let $t = (t_1, \ldots, t_r)$. By similar arguments as those in Subsection 4.2, we just need to consider $I_1' = T_{11}$ and $(I_2 \setminus I_3)' = \left[t \mid t \in T_{12} + T_{12}^{-1}\right]$. Let

$$\beta_l = \chi_l(I_1') \text{ and } \gamma_l = \chi_l((I_2 \setminus I_3)'). \tag{4.16}$$

Let $\eta\colon \mathrm{Gal}\left(\mathbb{Q}\left(\omega_{2m}\right)/\mathbb{Q}\right) \to \mathbb{Z}_{2m}^*$ be the isomorphism such that $\sigma\left(\omega_{2m}\right) = \omega_{2m}^{\eta(\sigma)}$, where $\sigma \in \mathrm{Gal}\left(\mathbb{Q}\left(\omega_{2m}\right)/\mathbb{Q}\right)$.

Let $L = \mathbb{Q}\left(\beta_l, \gamma_l \mid l \in A' \setminus \{\mathbf{0}\}\right)$ and

$$H' = \{h \in \mathbb{Z}_{2m}^* \mid hI_1' = I_1', h(I_2 \setminus I_3)' = (I_2 \setminus I_3)'\}. \tag{4.17}$$

By Proposition 4.3, we have $H' = \eta\left(\mathrm{Gal}\left(\mathbb{Q}\left(\omega_{2m}\right)/L\right)\right)$.

Also, we consider H' acting on $A' = \mathbb{Z}_{n_1} \oplus \cdots \oplus \mathbb{Z}_{n_r}$. Assume that $H'a^{(1)}, H'a^{(2)}, \ldots$ $H'a^{(k)}$ are all distinct orbits of H' on A', where $a^{(i)} \in A'$. Let $B = \{x \in A' \mid 2x = \mathbf{0}\}$ and $A' = B \cup E \cup E^{-1}$, where B, E, E^{-1} are disjoint. Let

$$C' = \left\{a^{(i)} \mid (B \cup E) \cap H'a^{(i)} \neq \emptyset\right\} \tag{4.18}$$

and

$$M = \langle [\gamma_l]_L \mid l \in C' \rangle . \tag{4.19}$$

In a similar way as in Theorem 4.10, we get the following result.

Theorem 4.20. *Let $\Gamma = \mathrm{Cay}(\mathrm{Dic}(A, y), S)$ be a Cayley graph over $\mathrm{Dic}(A, y)$ of order $4m$. Then the splitting field of Γ is $L\left(\sqrt{\gamma_l} \mid l \in C'\right)$, and the algebraic degree of Γ satisfies*

$$\deg(\Gamma) = \frac{\varphi(2m)|M|}{|H'|},$$

where $L = \mathbb{Q}(\omega_{2m})^{\eta^{-1}(H')} = \{x \in \mathbb{Q}(\omega_{2m}) \mid \sigma(x) = x \text{ for all } \sigma \in \eta^{-1}(H')\}$ and γ_l, H', C', M are given in (4.16) – (4.19), respectively.

Corollary 4.21. *Let $\Gamma = \mathrm{Cay}(\mathrm{Dic}(A, y), S)$ be a Cayley graph over $\mathrm{Dic}(A, y)$ of order $4m$. Then the algebraic degree of Γ satisfies*

$$\frac{\varphi(2m)}{|H'|} \le \deg(\Gamma) \le \frac{\varphi(2m)2^{|C'|}}{|H'|},$$

where H', C' are given in (4.17) and (4.18), respectively.

Corollary 4.22. *Let $\Gamma = \mathrm{Cay}(\mathrm{Dic}(A, y), S)$ be a Cayley graph over $\mathrm{Dic}(A, y)$ of order $4m$. If $H' = \mathbb{Z}_{2m}^*$ and γ_l is a square of an integer for each $l \in C'$, where γ_l, H', C' are given in (4.16) – (4.18), respectively, then Γ is integral.*

Corollary 4.23. *Let $\Gamma = \mathrm{Cay}(\mathrm{Dic}(A, y), S)$ be a Cayley graph over $\mathrm{Dic}(A, y)$ of order $4m$. Let*

$$T_{11} = \left\{(t_1, \ldots, t_r) \mid \left(a_1^{t_1}, \ldots, a_r^{t_r}\right) \in S\right\} \text{ and } T_{12} = \left\{(t_1, \ldots, t_r) \mid \left(a_1^{t_1}, \ldots, a_r^{t_r}\right) \in xS\right\}.$$

If $T_{12} = T_{12}^{-1}$, then the splitting field of Γ satisfies

$$\mathbb{SF}(\Gamma) = \mathbb{Q}(\omega_{2m})^{\eta^{-1}(H'')} = \{x \in \mathbb{Q}(\omega_{2m}) \mid \sigma(x) = x \text{ for all } \sigma \in \eta^{-1}(H'')\},$$

the algebraic degree of Γ satisfies

$$\deg(\Gamma) = \frac{\varphi(2m)}{|H''|},$$

where $H'' = \{h \in \mathbb{Z}_{2m}^* \mid hT_{11} = T_{11}, hT_{12} = T_{12}\}$.

Corollary 4.24. *Let $\Gamma = \mathrm{Cay}(\mathrm{Dic}(A, y), S)$ be a Cayley graph over $\mathrm{Dic}(A, y)$ of order $4m$. Let*

$$T_{11} = \left\{(t_1, \ldots, t_r) \mid \left(a_1^{t_1}, \ldots, a_r^{t_r}\right) \in S\right\} \text{ and } T_{12} = \left\{(t_1, \ldots, t_r) \mid \left(a_1^{t_1}, \ldots, a_r^{t_r}\right) \in xS\right\}.$$

If $T_{12} = T_{12}^{-1}$ and for all $h \in \mathbb{Z}_{2m}^*$, $hT_{11} = T_{11}$ and $hT_{12} = T_{12}$, then Γ is integral.

Furthermore, for the dicyclic group $\mathrm{Dic}_{4m} = \left\langle a, b \mid a^{2m} = e, a^m = b^2, b^{-1}ab = a^{-1}\right\rangle$, as direct consequences of Theorem 4.20 and Corollaries 4.21 – 4.24, we have similar results, so we omit the details here.

Example 4.25. Let $\mathrm{Dic}_{12} = \langle a, b \mid a^6 = e, a^3 = b^2, b^{-1}ab = a^{-1}\rangle$ be the dicyclic group of order 12 and $S = \left\{a, a^5, ab, a^2b, a^4b, a^5b\right\}$. We consider the algebraic degree of $\Gamma = \mathrm{Cay}(\mathrm{Dic}_{12}, S)$. Then $T_{11} = \{1, 5\}$ and $T_{12} = \{1, 2, 4, 5\} = T_{12}^{-1}$. It follows that $H'' = \{1, 5\} = \mathbb{Z}_6^*$. By Corollary 4.24, $\deg(\Gamma) = 1$. In fact, $\mathrm{Spec}(\Gamma) = 4 * \{-1, 1\} \cup 3 * \{-2\} \cup \{6\}$.

4.4 Cayley digraphs over semi-dihedral groups

For the semi-dihedral group SD_{8m}, it has the following presentation

$$\mathrm{SD}_{8m} = \langle a, b \mid a^{4m} = b^2 = e, bab = a^{2m-1} \rangle.$$

We now consider the algebraic degree of the *Cayley digraph* $\Gamma = \mathrm{Cay}(\mathrm{SD}_{8m}, S)$. Using the symbols in Subsection 4.1, it follows that $A = \langle a \rangle_{4m}$ and $A' = \mathbb{Z}_{4m}$. Let $x_1 = e$ and $x_2 = b$. Then

$$\begin{aligned} T_{11} &= \left\{ t \mid a^t \in S \right\}, \\ T_{12} &= \left\{ t \mid ba^t \in S \right\}, \\ T_{21} &= \left\{ t \mid a^t b \in S \right\}, \\ T_{22} &= \left\{ t \mid a^{(2m-1)t} \in S \right\}. \end{aligned}$$

In a similar way as in Subsection 4.1, we define

$$\begin{aligned} I_1 &= [t \mid t \in T_{11} \text{ or } t \in T_{22}], \\ I_2 &= [t \mid t \in (T_{11} + T_{11}) \text{ or } t \in (T_{22} + T_{22}) \text{ or } t \in 4 * (T_{12} + T_{21})], \\ I_3 &= [t \mid t \in 2 * (T_{11} + T_{22})]. \end{aligned}$$

Let

$$\beta_l = \chi_l(I_1) \text{ and } \gamma_l = \chi_l(I_2 \setminus I_3) - \chi_l(I_3 \setminus I_2), \tag{4.20}$$

where $\chi_l(t) = \omega_{4m}^{lt}$ and $0 \le l \le 4m - 1$. Let K be a field such that $\mathbb{Q} \subseteq K \subseteq \mathbb{Q}(\omega_{4m})$. Then $\mathrm{Gal}(\mathbb{Q}(\omega_{4m})/K)) \le \mathrm{Gal}(\mathbb{Q}(\omega_{4m})/\mathbb{Q}) \cong \mathbb{Z}_{4m}^*$. Let $\eta \colon \mathrm{Gal}(\mathbb{Q}(\omega_{4m})/\mathbb{Q}) \to \mathbb{Z}_{4m}^*$ be the isomorphism such that $\sigma(\omega_{4m}) = \omega_{4m}^{\eta(\sigma)}$, where $\sigma \in \mathrm{Gal}(\mathbb{Q}(\omega_{4m})/\mathbb{Q})$. Let

$$L = \mathbb{Q}(\beta_l, \gamma_l \mid 1 \le l \le 4m - 1).$$

The following lemma helps to simplify the expression of L.

Lemma 4.26. *If $\beta_1, \gamma_1 \in K$, then $\beta_l, \gamma_l \in K$ for $1 \le l \le 4m - 1$.*

Proof. The proof is similar to that of Lemma 4.15. □

By Lemma 4.26, we have

$$L = \mathbb{Q}(\beta_1, \gamma_1).$$

Let

$$H' = \{h \in \mathbb{Z}_{4m}^* \mid hI_1 = I_1, h(I_2 \setminus I_3) = I_2 \setminus I_3, h(I_3 \setminus I_2) = I_3 \setminus I_2\}. \tag{4.21}$$

By Proposition 4.3, we have $H' = \eta(\mathrm{Gal}(\mathbb{Q}(\omega_{4m})/L))$.

Assume that $H'a^{(1)}, H'a^{(2)}, \ldots H'a^{(k)}$ are all distinct orbits of H' on $\mathbb{Z}_{4m}$. Let

$$C = \{a^{(i)} \mid \mathbb{Z}_{4m} \cap H'a^{(i)} \ne \emptyset\} \tag{4.22}$$

and

$$M = \langle [\gamma_l]_L \mid l \in C \rangle. \tag{4.23}$$

By Theorem 4.4 and Corollaries 4.5 – 4.8, we have

Theorem 4.27. *Let $\Gamma = \mathrm{Cay}(\mathrm{SD}_{8m}, S)$ be a Cayley digraph over the semi-dihedral group SD_{8m}. Then the splitting field of Γ is $L\left(\sqrt{\gamma_l} \mid l \in C\right)$ and the algebraic degree of Γ satisfies*

$$\deg(\Gamma) = \frac{\varphi(4m)|M|}{|H'|},$$

where $L = \mathbb{Q}\left(\omega_{4m}\right)^{\eta^{-1}(H')} = \{x \in \mathbb{Q}\left(\omega_{4m}\right) \mid \sigma(x) = x \text{ for all } \sigma \in \eta^{-1}(H')\}$ and γ_l, H', C, M are given in (4.20) – (4.23), respectively.

Corollary 4.28. *Let $\Gamma = \mathrm{Cay}(\mathrm{SD}_{8m}, S)$ be a Cayley digraph over the semi-dihedral group SD_{8m}. Then the algebraic degree of Γ satisfies*

$$\frac{\varphi(4m)}{|H'|} \leq \deg(\Gamma) \leq \frac{\varphi(4m)2^{|C|}}{|H'|},$$

where H', C are given in (4.21) and (4.22), respectively.

Corollary 4.29. *Let $\Gamma = \mathrm{Cay}(\mathrm{SD}_{8m}, S)$ be a Cayley digraph over the semi-dihedral group SD_{8m}. If $H' = \mathbb{Z}_{4m}^*$ and γ_l is a square of an integer for each $l \in C$, where γ_l, H', C are given in (4.20) – (4.22), respectively, then Γ is integral.*

Corollary 4.30. *Let $\Gamma = \mathrm{Cay}(\mathrm{SD}_{8m}, S)$ be a Cayley digraph over the semi-dihedral group SD_{8m}. If $T_{11} = T_{22}$ and $T_{12} = T_{12}^{-1} = T_{21}$, then the splitting field of Γ satisfies*

$$\mathbb{SF}(\Gamma) = \mathbb{Q}\left(\omega_{4m}\right)^{\eta^{-1}(H'')} = \{x \in \mathbb{Q}\left(\omega_{4m}\right) \mid \sigma(x) = x \text{ for all } \sigma \in \eta^{-1}(H'')\},$$

the algebraic degree of Γ satisfies

$$\deg(\Gamma) = \frac{\varphi(4m)}{|H''|},$$

where $H'' = \{h \in \mathbb{Z}_{4m}^* \mid hT_{11} = T_{11}, hT_{12} = T_{12}\}$.

Corollary 4.31. *Let $\Gamma = \mathrm{Cay}(\mathrm{SD}_{8m}, S)$ be a Cayley digraph over the semi-dihedral group SD_{8m}. If $T_{11} = T_{22}$, $T_{12} = T_{12}^{-1} = T_{21}$ and for all $h \in \mathbb{Z}_{4m}^*$, $hT_{11} = T_{11}$ and $hT_{12} = T_{12}$, then Γ is integral.*

We end this paper with the following example.

Example 4.32. Let $\mathrm{SD}_{16} = \langle a, b \mid a^8 = b^2 = e, b^{-1}ab = a^3\rangle$ be the semi-dihedral group of order 16 and $S = \left\{a^2, a^6, ba, ba^5\right\}$. We consider the algebraic degree of $\Gamma = \mathrm{Cay}(\mathrm{SD}_{16}, S)$. Then $T_{11} = T_{22} = \{2, 6\}$ and $T_{12} = \{1, 5\}, T_{12}^{-1} = T_{21} = \{3, 7\}$. Thus, $I_1 = 2 * \{2, 6\}$, $I_2 \setminus I_3 = 8 * \{0, 4\}$ and $I_3 \setminus I_2 = \emptyset$. It follows that $H' = \{1, 3, 5, 7\} = \mathbb{Z}_8^*$. Since $\gamma_1 = 8[1 + (-1)^l]$ is a square of integer for each $l \in \mathbb{Z}_8$, by Corollary 4.29, $\deg(\Gamma) = 1$. In fact, $\mathrm{Spec}(\Gamma) = 10 * \{0\} \cup 2 * \{-2, 1, 4\}$.

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