ARS MATHEMATICA CONTEMPORANEA

Volume 9, Number 1, Fall/Winter 2015, Pages 1–144

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The University of Primorska

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The publication is partially supported by the Slovenian Research Agency from the Call for co-financing of scientific periodical publications.

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Dragan Marušič and Tomaž Pisanski

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ARS MATHEMATICA CONTEMPORANEA

Also available at http://amc-journal.eu
ISSN 1855-3966 (printed edn.), ISSN 1855-3974 (electronic edn.)
ARS MATHEMATICA CONTEMPORANEA 9 (2015) 1–18

4-valent graphs of order $6p^2$ admitting a group of automorphisms acting regularly on arcs

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Received 9 May 2012, accepted 5 February 2014, published online 7 May 2014

Abstract

In this paper we classify the 4-valent graphs having $6p^2$ vertices, with p a prime, admitting a group of automorphisms acting regularly on arcs. As a corollary, we obtain the 4-valent one-regular graphs having $6p^2$ vertices.

Keywords: One-regular graphs, 4*-valent, Cayley graphs.*

Math. Subj. Class.: 20B25, 05C25

1 Introduction

For a finite, simple and undirected graph X, we use $V(X)$, $A(X)$ and $\mathrm{Aut}(X)$ to denote its vertex set, arc set and automorphism group, respectively. The graph X is said to be B-vertex-transitive, respectively B-arc-transitive, if B is a subgroup of $\mathrm{Aut}(X)$ acting transitively on $V(X)$, respectively $A(X)$. When $B = \mathrm{Aut}(X)$, the prefix B in the above notation is omitted. Moreover, X is said to be *one-regular* if $\mathrm{Aut}(X)$ acts regularly on $A(X)$, that is, X is arc-transitive and $|\mathrm{Aut}(X)| = |A(X)|$.

Obviously, one-regular graphs are connected, and a graph of valency 2 is one-regular if and only if it is a cycle. The first example of a 3-valent one-regular graph was constructed by Frucht [10], with 432 vertices. Later on, a considerable amount of work has been done on 3-valent one-regular graphs as part of the more general problem dealing with the classification of the 3-valent arc-transitive graphs (see [5, 6, 7, 8, 19, 30]). Marušič and Pisanski [17] have fully classified the 3-valent one-regular Cayley graphs on a dihedral group, and Kwak et al. [14] have similarly classified those of valency 5. Moreover, more

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recently, Feng and Li [9] have classified one-regular graphs of square-free order and of prime valency. Tetravalent one-regular graphs have also received considerable attention (see [1, 2, 15, 16, 24, 25, 28, 29]).

In this paper we are concerned with the classification of the 4-valent one-regular graphs. We recall that such graphs are already classified when their orders are a prime or the product of two (not necessarily distinct) primes [3, 21, 22, 26, 29, 28]. Moreover, for p and q primes, the classification of the 4-valent one-regular graphs of order $4p^2$ or $2pq$ is given in [4, 31]. In this context we prove the following.

Theorem 1.1. *Let p be a prime and let X be a 4-valent graph of order $6p^2$ admitting a group of automorphisms acting regularly on $A(X)$. Then one of the following holds:*

(i) *X is isomorphic to $C(2; 3p^2, 1)$, $C^{\pm 1}(p; 6, 2)$, $Y_{p,\pm 1}$, $Y_{p,\pm\sqrt{3}}$, $Z_{p,\pm\sqrt{-1}}$ or $Z_{p,\pm\sqrt{-3}}$ (see Section 3 for the definition of these graphs);*

(ii) *X is a Cayley graph over G with connection set S where*

- **(a)** $G = \langle x, y \mid x^p = y^{6p} = [x, y] = 1\rangle$ *and* $S = \{y, y^{-1}, xy, (xy)^{-1}\}$, *or*
- **(b)** $G = \langle x, y, z \mid x^p = y^{3p} = z^2 = [x, y] = [x, z] = 1, y^z = y^{-1}\rangle$ *and* $S = \{xz, x^{-1}z, x^{\varepsilon}yz, x^{-\varepsilon}yz\}$ *(here $\varepsilon^2 \equiv -1 \pmod p$ and $p \equiv 1 \pmod 4$), or*
- **(c)** $G = \langle x, y, z \mid x^{p^2} = y^3 = z^2 = [x, y] = [x, z] = 1, y^z = y^{-1}\rangle$ *and* $S = \{xz, x^{-1}z, xyz, x^{-1}yz\}$, *or*
- **(d)** $G = \langle x, y, z \mid x^{p^2} = y^3 = z^2 = [x, y] = [x, z] = 1, y^z = y^{-1}\rangle$ *and* $S = \{xz, x^{-1}z, x^{\varepsilon}yz, x^{-\varepsilon}yz\}$ *(here $\varepsilon^2 \equiv -1 \pmod{p^2}$), or*
- **(e)** $G = \langle x, y, z, t \mid x^p = y^p = z^3 = t^2 = [x, y] = [x, z] = [x, t] = [y, z] = [y, t] = 1, z^t = z^{-1}\rangle$ *and* $S = \{xt, x^{-1}t, yzt, y^{-1}zt\}$;

(iii) *$p \in \{2, 3, 5\}$ and X is described in Section 6.*

The definition of the graphs in part (i) requires a fair amount of notation and terminology, so we do not include their description in this introductory section. Observe that if $p \leq 7$, then $|V(X)| = 24, 54, 150$ or 294. Since a complete census of the 4-valent arc-transitive graphs of order at most 640 has been recently obtained by Potočnik, Spiga and Verret [18, 19], for $p \in \{2, 3, 5, 7\}$, the 4-valent graphs of order $6p^2$ admitting a group of automorphisms acting regularly on $A(X)$ can be downloaded (in `magma` format) from [18].

The proof of Theorem 1.1 is based on "normal quotient" techniques. This method is very powerful and allows to obtain results like Theorem 1.1 when the order of the graph has a prime factorization that is not too complicated (the multiplicity and the number of prime factors are both small). However there are two natural limits to this technique. First, as the order of the graph X becomes more complicated, the local properties of the quotient graph X_N might not be strong enough to be lifted to the graph X we start with (to see a concrete example of this situation see "Case $X_P = O$" in the proof of Theorem 1.1). Second, we believe that results like Theorem 1.1 are useful only when the list of graphs is not too long or too cumbersome to use. Therefore, although some of our arguments apply to graphs having order more complicated than $6p^2$ we do not pursue this classification here because of the natural complications describing each possible family.

A direct application of Theorem 1.1 gives the following.

Corollary 1.2. *Let p be a prime and let X be a 4-valent one-regular graph X of order $6p^2$. Then one of the following holds:*

(i) *X is isomorphic to $Y_{p,\pm 1}$, $Y_{p,\pm\sqrt{3}}$, $Z_{p,\pm\sqrt{-1}}$ or $Z_{p,\pm\sqrt{-3}}$ (see Section 3 for the definition of these graphs);*

(ii) *X is a Cayley graph over G with connection set S where*

- **(a)** $G = \langle x, y \mid x^p = y^{6p} = [x, y] = 1\rangle$ *and* $S = \{y, y^{-1}, xy, (xy)^{-1}\}$, *or*
- **(b)** $G = \langle x, y, z \mid x^p = y^{3p} = z^2 = [x, y] = [x, z] = 1, y^z = y^{-1}\rangle$ *and* $S = \{xz, x^{-1}z, x^{\varepsilon}yz, x^{-\varepsilon}yz\}$ *(here* $\varepsilon^2 \equiv -1 \pmod p$ *and* $p \equiv 1 \pmod 4$*), or*
- **(c)** $G = \langle x, y, z \mid x^{p^2} = y^3 = z^2 = [x, y] = [x, z] = 1, y^z = y^{-1}\rangle$ *and* $\{xz, x^{-1}z, xyz, x^{-1}yz\}$, *or*
- **(d)** $G = \langle x, y, z \mid x^{p^2} = y^3 = z^2 = [x, y] = [x, z] = 1, y^z = y^{-1}\rangle$ *and* $\{xz, x^{-1}z, x^{\varepsilon}yz, x^{-\varepsilon}yz\}$ *(here* $\varepsilon^2 \equiv -1 \pmod{p^2}$*);*

(iii) $p \in \{2, 3, 5\}$ *and X is described in Section 6.*

Observe that we are not claiming that every graph in Corollary 1.2 is one-regular.

The structure of the paper is elementary: in Section 2 we introduce the notation and some basic results that we will need for our proof of Theorem 1.1. Then in Sections 3 and 4 we present some graphs revelant to our investigation. In Section 5 we prove Theorem 1.1 and Corollary 1.2. In Section 6 we give the graphs in part (iii) of Theorem 1.1, and in part (iii) of Corollary 1.2.

Acknowledgements. A toast to our friend Primož Potočnik for being constructively critical.

2 Preliminaries

In this section, we introduce some notations and definitions as well as some preliminary results which will be used later in the paper.

Let X be a connected vertex-transitive graph, and let $B \le \mathrm{Aut}(X)$ be vertex-transitive on X. Suppose that there is some group N such that $1 \ne N \lhd B$, and N is intransitive in its action on $V(X)$. The *normal quotient* X_N is the graph whose vertices are the orbits of N on $V(X)$, with an edge between two distinct vertices v^N and w^N in $V(X_N)$, if and only if there is an edge of X between v_0 and w_0, for some $v_0 \in v^N$ and some $w_0 \in w^N$. Normal quotients were introduced by Praeger in [20] and they turned out to be an invaluable tool for the classification of certain families of vertex-transitive graphs. In fact, our proof of Theorem 1.1 heavily relies on normal quotient techniques.

For a positive integer n, we denote by $\mathbb{Z}_n$ the cyclic group of order n as well as the ring of integers modulo n, by $\mathbb{Z}_n^*$ the invertible elements of $\mathbb{Z}_n$, by D_{2n} the dihedral group of order $2n$, and by C_n and K_n the cycle and the complete graph of order n, respectively. For a group G and a subset S of G with $1 \notin S$ and $S = S^{-1}$, the *Cayley graph* $\mathrm{Cay}(G, S)$ on G with connection set S is defined to have vertex set G and edge set $\{\{g, sg\} \mid g \in G, s \in S\}$.

For the benefit of the reader, we report here [12, Theorems 1.1 and 1.2] and [11, Theorem 1.1], respectively.

Theorem 2.1 ([12, Theorem 1.1])**.** *Let X be a connected B-arc-transitive 4-valent graph, and let N be a minimal normal p-subgroup of B with orbits on $V(X)$ of size p^s. Let K denote the kernel of the action of B on N orbits and let α be a vertex of X. If the quotient X_N is a cycle of length $r \ge 3$, then one of the following holds:*

(a) *$p = 2$ and $X = C(2; r, s)$;*

(b) *p is odd and, if $|K_\alpha| = 2^s$, then $X = C^{\pm 1}(p; st, s)$ or $X = C^{\pm\varepsilon}(p; 2st, s)$ for some $t \geq 1$.*

Theorem 2.2 ([12, Theorem 1.2]). *Let X be a connected B-arc-transitive 4-valent graph, and let $N \cong \mathbb{Z}_p^2$ (p an odd prime) be a minimal normal subgroup of B with orbits on $V(X)$ of size p^s. Let K denote the kernel of the action of B on N-orbits and let α be a vertex of X. If the quotient X_N is a cycle of length $r \geq 3$, then one of the following holds:*

(a) *$s = 1$ or 2, $K_\alpha \cong \mathbb{Z}_2^s$ and $X = C^{\pm 1}(p; st, s)$ or $X = C^{\pm\varepsilon}(p; 2st, s)$ for some $t \geq 1$;*

(b) *$s = 2$, $K_\alpha \cong \mathbb{Z}_2$ and $X = C^{\pm 1}(p; 2t, 2)$, or X belongs to one of two families described in [12, Lemmas 8.4 and 8.7].*

(The graphs $C(2; r, s)$, $C^{\pm 1}(p; st, s)$, $C^{\pm\varepsilon}(p; 2st, s)$ and the graphs in [12, Lemma 8.4 and 8.7] are define in Section 3.)

Theorem 2.3 ([11, Theorem 1.1]). *Let X be a connected 4-valent B-arc-transitive graph. For each normal subgroup N of B, one of the following holds:*

(a) *N is transitive on $V(X)$;*

(b) *X is bipartite and N acts transitively on each part of the bipartition;*

(c) *N has $r \geq 3$ orbits on $V(X)$, the quotient graph X_N is a cycle of length r, and B induces the full automorphism group D_{2r} on X_N;*

(d) *N has $r \geq 5$ orbits on $V(X)$, N acts semiregularly on $V(X)$, the quotient graph X_N is a connected 4-valent B/N-symmetric graph, and X is a B-normal cover of X_N.*

3 Some families of graphs

In this section we present some of the graphs relevant to Theorem 1.1 and Corollary 1.2. All of these graphs were introduced in the pivotal paper [12] of Gardiner and Praeger. (Here we do not include the description of all the graphs in [12], but only those that are closely related to our investigation.)

3.1 The graphs $C(2; r, 1)$

The graph $C(2; r, 1)$ is the lexicographic product of a cycle of length r and an edgeless graph on 2 vertices. In other words, $V(C(2; r, 1)) = \mathbb{Z}_2 \times \mathbb{Z}_r$ with (u, i) being adjacent to (v, j) if and only if $i - j \in \{-1, 1\}$.

From [12, Definition 2.1], it follows that $C(2; r, 1)$ is not one-regular.

3.2 The graphs $C^{\pm 1}(p; st, s)$

(We will be only interested to the case $s \in \{1, 2\}$.) The graph $C^{\pm 1}(p; st, s)$ has vertex set $\mathbb{Z}_p^s \times \mathbb{Z}_{st}$. To describe the adjacencies write every vertex of $C^{\pm 1}(p; st, s)$ as

$$v = ((x_0, \ldots, x_{i-1}, x_i, x_{i+1}, \ldots, x_{s-1}), ms + i),$$

with $x_0, \ldots, x_{s-1} \in \mathbb{Z}_s$, $0 \leq i < s$ and $0 \leq m < t$. Then v is adjacent to

$$((x_0, \ldots, x_{i-1}, x_i \pm 1, x_{i+1}, \ldots, x_{s-1}), ms + i + 1)$$

and
$$((x_0, \dots, x_{i-1} \pm 1, x_i, x_{i+1}, \dots, x_{s-1}), ms + i - 1).$$

The group $\langle a_0\rangle \times \langle a_1\rangle \times \cdots \times \langle a_{s-1}\rangle \cong \mathbb{Z}_p^s$ acts on $V(C^{\pm 1}(p; st, s))$ in the natural way, inducing translations on the first s coordinates and leaving the coordinate in $\mathbb{Z}_{st}$ unchanged. Moreover, $C^{\pm 1}(p; st, s)$ admits the automorphism σ defined by

$$((x_0, x_1, \dots, x_{s-1}), q)^\sigma = ((x_{s-1}, x_0, \dots, x_{s-2}), q + 1).$$

A computation shows that $C^{\pm 1}(p; st, s)$ is a Cayley graph over $G = \langle a_0, \dots, a_{s-1}, \sigma\rangle$. Observe that when $s = 1$ we have $G \cong \mathbb{Z}_p \times \mathbb{Z}_t$.

From [12, Definition 2.2], it follows that $C^{\pm}(p; t, 1)$ is a normal Cayley graph over G with connection set $S = \{a_0\sigma, (a_0\sigma)^{-1}, a_0^{-1}\sigma, (a_0^{-1}\sigma)^{-1}\}$, that is, $C^{\pm 1}(p; t, 1) = \mathrm{Cay}(G, S)$ and $G \trianglelefteq \mathrm{Aut}(C^{\pm 1}(p; t, 1))$. Moreover, if $C^{\pm 1}(p; st, s)$ is one-regular, then $s = 1$.

3.3 The graphs $C^{\pm\varepsilon}(p; 2st, s)$

We introduce this family only for the case that concerns us, that is, $s = 1$. Let p be a prime with $p \equiv 1 \pmod 4$ and let ε be a square root of $-1 \pmod p$. The graph $C^{\pm\varepsilon}(p; 2t, 1)$ has vertex set $\mathbb{Z}_p \times \mathbb{Z}_{2t}$ and the vertex $v = (x, m)$ is adjacent to

$(x \pm \varepsilon, m - 1)$ and $(x \pm 1, m + 1)$	if m is even, and
$(x \pm 1, m - 1)$ and $(x \pm \varepsilon, m + 1)$	if m is odd.

It is easy to check that the mappings a, τ, σ defined by

$$\begin{aligned}(x, m)^a &= (x + 1, m)\\ (x, m)^\tau &= (x, 1 - m)\\ (x, m)^\sigma &= (\varepsilon x, q + 1)\end{aligned}$$

are automorphisms of $C^{\pm\varepsilon}(p; 2t, 1)$. Furthermore, a computation shows that, when t is odd, $C^{\pm\varepsilon}(p; 2t, 1)$ is a Cayley graph over $G = \langle a, \sigma^4, \tau\rangle = \langle a\rangle \times \langle \sigma^4, \tau\rangle \cong \mathbb{Z}_p \times D_{2t}$.

From [12, Definition 2.3], it follows that, for t odd, $C^{\pm\varepsilon}(p; 2t, 1)$ is a normal Cayley graph over G with connection set $S = \{a\tau, (a\tau)^{-1}, a^\varepsilon\sigma^{2t+2}\tau, (a^\varepsilon\sigma^{2t+2}\tau)^{-1}\}$, that is, $C^{\pm\varepsilon}(p; t, 1) = \mathrm{Cay}(G, S)$ and $G \trianglelefteq \mathrm{Aut}(C^{\pm\varepsilon}(p; 2t, 1))$.

For defining the rest of the graphs we recall the concept of *coset graph*. For a group B, a subgroup H and an element $b \in B$, the coset graph $\mathrm{Cos}(B, H, b)$ is the graph with vertex set the set of right cosets $B/H = \{Hg \mid g \in B\}$ and edge set $\{\{Hg, Hbg\} \mid g \in B\}$. The following proposition is due to Sabidussi [23].

Proposition 3.1. *Let H be a core-free subgroup of B and let $b \in B$ with $B = \langle H, b\rangle$ and $b^{-1} \in HbH$. Then $\Gamma = Cos(B, H, b)$ is a connected B-arc-transitive graph of valency $|H : H \cap H^b|$.*

3.4 The graphs arising from [12, Lemma 8.4]: $Y_{p,\pm 1}$ and $Y_{p,\pm\sqrt{3}}$

The graphs described in [12, Section 8 and Lemma 8.4] are very general, here we describe only the graphs relevant to the scope of this article.

Let $N = \langle a_0, a_1\rangle$ be an elementary abelian p-group of order p^2 and let K be the group with presentation

$$K = \langle \sigma, \tau, w \mid \sigma^6 = \tau^2 = w^2 = [\sigma, w] = [\tau, w] = 1, \sigma^\tau = \sigma^{-1}\rangle.$$

Clearly, $K = \langle \sigma, \tau\rangle \times \langle w\rangle \cong D_{12} \times C_2$. Fix $u \in \{-1, 1\}$. We let K act on N via

$$\begin{array}{rcl} a_0^\sigma & = & a_1 \\ a_1^\sigma & = & a_0^{-1}a_1^u \end{array}, \quad \begin{array}{rcl} a_0^\tau & = & a_1 \\ a_1^\tau & = & a_0 \end{array}, \quad \begin{array}{rcl} a_0^w & = & a_0^{-1} \\ a_1^w & = & a_1^{-1} \end{array}.$$

Set $B_u = N \rtimes K$ and $H_u = \langle \tau, w\rangle$. Now the graph $Y_{p,u}$ is defined as $\mathrm{Cos}(B_u, H_u, \sigma^{-1}a_0)$. From Proposition 3.1, it follows that $Y_{p,u}$ is a B_u-arc-transitive graph of valency 4 with $6p^2$ vertices and B_u acts regularly on $A(Y_{p,u})$. Moreover, from [12, Section 8], we have $Y_{p,1} \cong Y_{p,-1}$, and for $p \geq 5$, we have $\mathrm{Aut}(Y_{p,u}) = B_u$.

Next we define $Y_{p,\pm\sqrt{3}}$. Suppose $p \geq 5$ and let $u \in \mathbb{Z}_p$ be a square root of $3 \pmod p$. (Observe that, from the law of quadratic reciprocity, for this example to exist we need $p \equiv \pm 1 \mod 12$.) Let $N = \langle a_0, a_1\rangle$ be an elementary abelian p-group of order p^2 and let K be the group with presentation

$$K = \langle \sigma, \tau, w \mid \sigma^{12} = \tau^2 = w^2 = [\sigma, w] = [\tau, w] = 1, \sigma^6 = w, \sigma^\tau = \sigma^{-1}\rangle.$$

Clearly, $K = \langle \sigma, \tau\rangle \cong D_{24}$. We let K act on N via

$$\begin{array}{rcl} a_0^\sigma & = & a_1 \\ a_1^\sigma & = & a_0^{-1}a_1^u \end{array}, \quad \begin{array}{rcl} a_0^\tau & = & a_1 \\ a_1^\tau & = & a_0 \end{array}, \quad \begin{array}{rcl} a_0^w & = & a_0^{-1} \\ a_1^w & = & a_1^{-1} \end{array}.$$

Set $B_u = N \rtimes K$ and $H_u = \langle w, \tau\rangle$. As for $Y_{p,\pm 1}$, the graph $Y_{p,u}$ is defined by $\mathrm{Cos}(B_u, H_u, \sigma^{-1}a_0)$. From Proposition 3.1, it follows that $Y_{p,u}$ is a B_u-arc-transitive graph of valency 4 with $6p^2$ vertices and B_u acts regularly on $A(Y_{p,u})$. Moreover, from [12, Section 8], we have $Y_{p,\sqrt{3}} \cong Y_{p,-\sqrt{3}}$ and $\mathrm{Aut}(Y_{p,u}) = B_u$.

3.5 The graphs arising from [12, Lemma 8.7]: $Z_{p,\pm\sqrt{-3}}$ and $Z_{p,\pm\sqrt{-1}}$

As for Section 3.4, the graphs described in [12, Section 8 and Lemma 8.7] are very general, here we present only the graphs relevant to the scope of this article.

We start by defining $Z_{p,\pm\sqrt{-3}}$. Suppose $p \geq 5$ and let $u \in \mathbb{Z}_p$ be a square root of $-3 \pmod p$. (Observe that, from the law of quadratic reciprocity, for this example to exist we need $p \equiv 1 \mod 6$.) Let $N = \langle a_0, a_1\rangle$ be an elementary abelian p-group of order p^2 and let K be the group with presentation

$$K = \langle \sigma, \tau, w \mid \sigma^6 = \tau^4 = w^2 = [\sigma, w] = [\tau, w] = 1, \tau^2 = w, \sigma^\tau = \tau^2\sigma^{-1}\rangle.$$

We let K act on N via

$$\begin{array}{rcl} a_0^\sigma & = & a_1 \\ a_1^\sigma & = & a_0 a_1^u \end{array}, \quad \begin{array}{rcl} a_0^\tau & = & a_1 \\ a_1^\tau & = & a_0^{-1} \end{array}, \quad \begin{array}{rcl} a_0^w & = & a_0^{-1} \\ a_1^w & = & a_1^{-1} \end{array}.$$

Set $B_u = N \rtimes K$ and $H_u = \langle \tau\rangle$. Now the graph $Z_{p,u}$ is defined by $\mathrm{Cos}(B_u, H_u, \sigma^{-1}a_0)$. From Proposition 3.1, it follows that $Z_{p,u}$ is a B_u-arc-transitive graph of valency 4 with $6p^2$ vertices and B_u acts regularly on $A(Z_{p,u})$. Moreover, from [12, Section 8], we have $Z_{p,\sqrt{-3}} \cong Z_{p,-\sqrt{-3}}$ and $\mathrm{Aut}(Z_{p,u}) = B_u$.

Finally, we define $Z_{p,\pm\sqrt{-1}}$. Suppose $p \geq 5$ and let $u \in \mathbb{Z}_p$ be a square root of -1 $(\mathrm{mod}\ p)$. (Observe that, from the law of quadratic reciprocity, for this example to exist we need $p \equiv 1 \mod 4$.) Let $N = \langle a_0, a_1 \rangle$ be an elementary abelian p-group of order p^2 and let K be the group with presentation

$$K = \langle \sigma, \tau, w \mid \sigma^{12} = \tau^4 = w^2 = [\sigma, w] = [\tau, w] = 1, \sigma^6 = \tau^2 = w, \sigma^\tau = \sigma^{-1} \rangle.$$

We let K act on N via

$$\begin{array}{ccc} a_0^\sigma & = & a_1 \\ a_1^\sigma & = & a_0 a_1^u \end{array}, \quad \begin{array}{ccc} a_0^\tau & = & a_1 \\ a_1^\tau & = & a_0^{-1} \end{array}, \quad \begin{array}{ccc} a_0^w & = & a_0^{-1} \\ a_1^w & = & a_1^{-1} \end{array}.$$

Set $B_u = N \rtimes K$ and $H_u = \langle \tau \rangle$. As for $Z_{p,\pm\sqrt{-3}}$, the graph $Z_{p,u}$ is defined by $\mathrm{Cos}(B_u, H_u, \sigma^{-1}a_0)$. From Proposition 3.1, it follows that $Z_{p,u}$ is a B_u-arc-transitive graph of valency 4 with $6p^2$ vertices and B_u acts regularly on $A(Z_{p,u})$. Moreover, from [12, Section 8], we have $Z_{p,\sqrt{-1}} \cong Z_{p,-\sqrt{-1}}$ and $\mathrm{Aut}(Z_{p,u}) = B_u$.

4 Graphs for Theorem 1.1 (ii)

In this section we describe the examples introduced in Theorem 1.1 (ii) (and their relation to the graphs in Section 3). For simplicity here we assume that $p \geq 5$. The graphs described in Sections 4.1 and 4.6 were already introduced with much more information and details in [28], here we include yet again their construction for making our note self-contained.

4.1 4-Valent normal Cayley graphs over $\mathbb{Z}_p \times \mathbb{Z}_{6p}$ with $p \geq 5$ prime

We describe the connected normal 4-valent arc-transitive Cayley graphs over the group $\mathbb{Z}_p \times \mathbb{Z}_{6p}$.

Let p be a prime and let G be the group given by generators and relations

$$G = \langle x, y \mid x^p = y^{6p} = [x, y] = 1 \rangle.$$

Let S be a subset of G and assume that $X = \mathrm{Cay}(G, S)$ is connected, normal, 4-valent and arc-transitive. Since X is normal and arc-transitive and since G is abelian of exponent $6p$, we see that S consists of elements of order $6p$. Denote by $\mathcal{S}_{6p}$ the elements of G of order $6p$. Therefore

$$S \subseteq \mathcal{S}_{6p} = \{x^a y^b \mid a \in \mathbb{Z}_p, b \in \mathbb{Z}_{6p}^*\}.$$

It is clear that $\mathrm{Aut}(G)$ acts transitively on $\mathcal{S}_{6p}$ by conjugation. In particular, replacing S by a suitable $\mathrm{Aut}(G)$-conjugate, we may assume that $y \in S$. Therefore

$$S = \{y, y^{-1}, x^u y^v, x^{-u} y^{-v}\},$$

for some $u \in \mathbb{Z}_p^*$ and for some $v \in \mathbb{Z}_{6p}^*$.

Let $B = \{\varphi \in \mathrm{Aut}(G) \mid y^\varphi = y\}$. Given $\varphi \in B$, we have

$$\varphi \quad : \quad \begin{cases} x & \mapsto \quad x^a y^{6b} \\ y & \mapsto \quad y \end{cases}$$

with $a, b \in \mathbb{Z}_p$ and $a \neq 0$. Note that every invertible element of $\mathbb{Z}_{6p}$ is of the form $1 + 6b$ or $-1 + 6b$, for some $b \in \mathbb{Z}_p$. Therefore, we may choose $a, b \in \mathbb{Z}_p$ with $(xy)^\varphi = x^u y^v$ or

$(xy^{-1})^\varphi = x^u y^v$. Thus, replacing S by a suitable B-conjugate, we may assume that either $xy \in S$ or $xy^{-1} \in S$, that is,

$$\begin{array}{rcl} S &=& \{y, y^{-1}, xy, x^{-1}y^{-1}\}, \text{ or} \\ S &=& \{y, y^{-1}, xy^{-1}, x^{-1}y\}. \end{array}$$

Let φ be the automorphism of G with $x^\varphi = x$ and $y^\varphi = y^{-1}$. Clearly, φ maps the first possibility for S onto the second. Therefore, we may assume that

$$S = \{y, y^{-1}, xy, x^{-1}y^{-1}\}.$$

The graph X is in Theorem 1.1 (ii) (a). Also, using [12, Definition 2.2], we see that X is isomorphic to $C^{\pm 1}(p; 6p, 1)$.

4.2 4-Valent normal Cayley graphs over $\mathbb{Z}_p \times D_{6p}$ with $p \geq 5$ prime

We describe the connected normal 4-valent arc-transitive Cayley graphs over the group $\mathbb{Z}_p \times D_{6p}$.

Let p be a prime and let G be the group given by generators and relations

$$G = \langle x, y, z \mid x^p = y^{3p} = z^2 = [x, y] = [x, z] = 1, y^z = y^{-1}\rangle.$$

Let S be a subset of G and assume that $X = \mathrm{Cay}(G, S)$ is connected, normal, 4-valent and arc-transitive. Since X is normal and arc-transitive, every element of S has the same order. The elements of G of odd order lie in $\langle x, y\rangle$ and the involutions of G lie in $\langle y, z\rangle$. Since X is connected, $G = \langle S\rangle$ and hence S consists of elements of order $2p$. Denote by $\mathcal{S}_{2p}$ the elements of G of order $2p$. Therefore

$$S \subseteq \mathcal{S}_{2p} = \{x^a y^b z \mid b \in \mathbb{Z}_{3p}, a \in \mathbb{Z}_p^*\}.$$

We now consider the action of the automorphism group $\mathrm{Aut}(G)$ of G on $\mathcal{S}_{2p}$. Let $\varphi \in \mathrm{Aut}(G)$. Clearly, φ is uniquely determined by the images of x, y and z. By considering the element orders of G, we need to have

$$\varphi \; : \; \left\{ \begin{array}{ccc} x & \mapsto & x^a y^{3b} \\ y & \mapsto & x^c y^d \\ z & \mapsto & y^e z. \end{array} \right.$$

Since $[x, z] = 1$, the element x^φ needs to commute with z^φ and so, with a direct computation, we see that $3b = 0$. Also, as $y^z = y^{-1}$, we have $(y^\varphi)^{z^\varphi} = (y^\varphi)^{-1}$, but this happens only for $c = 0$. Summing up,

$$\varphi : \left\{ \begin{array}{cccl} x & \mapsto & x^a & \text{with } a \in \mathbb{Z}_p^* \\ y & \mapsto & y^d & \text{with } d \in \mathbb{Z}_{3p}^* \\ z & \mapsto & y^e z. & \end{array} \right. \tag{4.1}$$

This shows that all elements of $\mathcal{S}_{2p}$ are $\mathrm{Aut}(G)$-conjugate to xz. Therefore, replacing S by a suitable conjugate under $\mathrm{Aut}(G)$, we may assume that $xz \in S$. In particular, $S = \{xz, x^{-1}z, x^u y^v z, x^{-u} y^v z\}$, for some u, v. As $G = \langle S\rangle \leq \langle x, z, y^v\rangle$, we have that $v \in \mathbb{Z}_{3p}^*$.

Let $B = \{\varphi \in \mathrm{Aut}(G) \mid (xz)^\varphi = xz\}$. From (4.1), we see that $\psi \in B$ only if $x^\psi = x$ and $z^\psi = z$. In particular, replacing S by a suitable B-conjugate, we may assume that $x^u yz \in S$, that is, $v = 1$. Thus $S = \{xz, x^{-1}z, x^u yz, x^{-u}yz\}$.

Let $\varphi \in \mathrm{Aut}(G)$ with $S^\varphi = S$ and $(xz)^\varphi = x^u yz$ (recall that such an automorphism exists because X is a normal arc-transitive Cayley graph over G). From (4.1), we have $x^\varphi = x^u$, $z^\varphi = yz$ and $y^\varphi = y^d$ (for some d coprime to $3p$). Now, $(x^u yz)^\varphi = x^{u^2} y^{d+1} z \in S$. If $x^{u^2} y^{d+1} z \in \{x^u yz, x^{-u}yz\}$, then $d = 0$, contradicting the fact that d is coprime to $3p$. Therefore $x^{u^2} y^{d+1} z \in \{xz, x^{-1}z\}$ and $u^2 = \pm 1$. Thus we have one of the following two possibilities for S:

$$\begin{aligned} S &= \{xz, x^{-1}z, xyz, x^{-1}yz\}, \text{ or} \\ S &= \{xz, x^{-1}z, x^\varepsilon yz, x^{-\varepsilon}yz\}, \text{ where } \varepsilon^2 = -1 \end{aligned}$$

(note that in the second case $p \equiv 1 \pmod 4$). Now we focus our attention to the first possibility for S. Take

$$\psi : \begin{cases} x & \mapsto & x \\ y & \mapsto & y^{-1} \\ z & \mapsto & yz. \end{cases}$$

Since ψ fixes set-wise S, we have $G \rtimes \langle \psi \rangle \le \mathrm{Aut}(X)$. We see that

$$(z\psi)^2 = z\psi z\psi = z(\psi z \psi) = zz^\psi = zyz = y^z = y^{-1}$$

and so $z\psi$ has order $6p$. Moreover $x^{z\psi} = (x^z)^\psi = x^\psi = x$ and x commutes with $z\psi$. Therefore $\langle x, z\psi \rangle \cong \mathbb{Z}_p \times \mathbb{Z}_{6p}$. It is immediate to see that $\langle x, z\psi \rangle$ acts regularly on the vertices of X and so X is a Cayley graph over $\mathbb{Z}_p \times \mathbb{Z}_{6p}$. In particular, from the discussion in Section 4.1 (or with a direct computation) we obtain that X is isomorphic to the graph in Theorem 1.1 (ii) (a).

Therefore we may assume that $p \equiv 1 \pmod 4$ and that $S = \{xz, x^{-1}z, x^\varepsilon yz, x^{-\varepsilon}yz\}$ where $\varepsilon^2 = -1$. The graph X is in Theorem 1.1 (ii) (b). Also, using [12, Definition 2.3] (or Section 3.3), we see that X is isomorphic to $C^{\pm\varepsilon}(p; 6p, 1)$.

4.3 4-Valent normal Cayley graphs over $\mathbb{Z}_p \times \mathbb{Z}_p \times D_6$ with $p \ge 5$ prime

We proceed as in the previous two examples. Let p be a prime and let G be the group given by generators and relations

$$G = \langle x, y, z, t \mid x^p = y^p = z^3 = t^2 = [x, y] = [x, z] = [x, t] = [y, z] = [y, t] = 1, \; z^t = z^{-1} \rangle.$$

Let S be a subset of G and assume that $X = \mathrm{Cay}(G, S)$ is connected, normal, 4-valent and arc-transitive. As the elements of G of odd order lie in $\langle x, y, z \rangle$ and the involutions of G lie in $\langle z, t \rangle$, we must have that S consists of elements of order $2p$. Since $\langle x, y \rangle$ and $\langle z, t \rangle$ are characteristic subgroups of G, we have $\mathrm{Aut}(G) \cong \mathrm{Aut}(\langle x, y \rangle) \times \mathrm{Aut}(\langle z, t \rangle) \cong \mathrm{GL}_2(p) \times D_6$. It follows easily from this description of $\mathrm{Aut}(G)$ and the connectivity of X that we may assume that

$$S = \{xt, x^{-1}t, yzt, y^{-1}zt\}.$$

Thus we obtain the graphs in Theorem 1.1 (ii) (e).

We show that X is not one-regular and hence it is not relevant for Corollary 1.2. Take

$$\psi : \begin{cases} x & \mapsto & x^{-1} \\ y & \mapsto & y \\ z & \mapsto & z \\ t & \mapsto & t. \end{cases}$$

Clearly, ψ defines an automorphism of G that fixes set-wise S. Since ψ fixes the neighbour yzt of 1 and maps xt to $x^{-1}t$, we see that $\mathrm{Aut}(X)$ is not regular on $A(X)$.

4.4 4-Valent normal Cayley graphs over $\mathbb{Z}_{3p} \times D_{2p}$ with $p \geq 5$ prime

Let G be the group given by generators and relations

$$G = \langle x, y, z \mid x^{3p} = y^p = z^2 = [x, y] = [x, z] = 1, y^z = y^{-1}\rangle.$$

Let S be a subset of G and assume that $X = \mathrm{Cay}(G, S)$ is connected, normal, 4-valent and arc-transitive. As the elements of G of odd order lie in $\langle x, y\rangle$, the involutions of G lie in $\langle y, z\rangle$ and the elements of order $2p$ lie in $\langle x^3, y, z\rangle$, we must have that S consists of elements of order $6p$. Denote by $\mathcal{S}_{6p}$ the elements of G of order $6p$. Therefore

$$S \subseteq \mathcal{S}_{6p} = \{x^a y^b z \mid a \in \mathbb{Z}_{3p}^*, b \in \mathbb{Z}_p\}.$$

Arguing as in the previous examples, we see that $\mathrm{Aut}(G)$ acts transitively on $\mathcal{S}_{6p}$ and hence we may assume that $xz \in S$. In particular,

$$S = \{xz, x^{-1}z, x^u y^v z, x^{-u} y^v z\},$$

for some $u \in \mathbb{Z}_{3p}^*$ and some $v \in \mathbb{Z}_p^*$.

Let $B = \{\varphi \in \mathrm{Aut}(G) \mid (xz)^\varphi = xz\}$. Clearly, if $(xz)^\varphi = xz$, then $x^\varphi = x$ and $z^\varphi = z$ because $z = (xz)^{3p}$ and $x^2 = (xz)^2$. Using this observation, it is easy to see that the elements $\varphi \in B$ are of the form

$$\varphi : \begin{cases} x & \mapsto & x \\ y & \mapsto & x^{3a} y^b \\ z & \mapsto & z, \end{cases}$$

for some $a, b \in \mathbb{Z}_p$ with $b \neq 0$. Therefore, we may choose a and b with $(x^u y^v z)^\varphi = xyz$ or $(x^u y^v z)^\varphi = x^{-1}yz$. Therefore (as usual), replacing S by a suitable B-conjugate, we may assume that

$$S = \{xz, x^{-1}z, xyz, x^{-1}yz\}.$$

Take

$$\psi : \begin{cases} x & \mapsto & x \\ y & \mapsto & y^{-1} \\ z & \mapsto & yz. \end{cases}$$

Clearly, ψ defines an automorphism of G. Since ψ fixes set-wise S, we have $G \rtimes \langle\psi\rangle \leq \mathrm{Aut}(X)$. We see that

$$(z\psi)^2 = z\psi z\psi = z(\psi z\psi) = zz^\psi = zyz = y^z = y^{-1}$$

and so $z\psi$ has order $2p$. Moreover $x^{z\psi} = (x^z)^\psi = x^\psi = x$ and x commutes with $z\psi$. Therefore $\langle x, z\psi\rangle \cong \mathbb{Z}_{3p} \times \mathbb{Z}_{2p} \cong \mathbb{Z}_p \times \mathbb{Z}_{6p}$. It is immediate to see that $\langle x, z\psi\rangle$ acts regularly on the vertices of X and so X is a Cayley graph over $\mathbb{Z}_p \times \mathbb{Z}_{6p}$. In particular, from the discussion in Section 4.1 (or with a direct computation) we obtain that X is isomorphic to the graph in Theorem 1.1 (ii) (a).

4.5 4-Valent normal Cayley graphs over $\mathbb{Z}_{p^2} \times D_6$ with $p \geq 5$ prime

Let G be the group given by generators and relations

$$G = \langle x, y, z \mid x^{p^2} = y^3 = z^2 = [x, y] = [x, z] = 1, y^z = y^{-1}\rangle.$$

Let S be a subset of G and assume that $X = \mathrm{Cay}(G, S)$ is connected, normal, 4-valent and arc-transitive. A moment's thought gives that S consists of elements of order $2p^2$. Denote by $\mathcal{S}_{2p^2}$ the elements of G of order $2p^2$. Therefore

$$S \subseteq \mathcal{S}_{2p^2} = \{x^a y^b z \mid a \in \mathbb{Z}_{p^2}^*, b \in \mathbb{Z}_3\}.$$

Since $\langle x\rangle$ and $\langle y, z\rangle$ are characteristic subgroups of G, we have $\mathrm{Aut}(G) = \mathrm{Aut}(\langle x\rangle) \times \mathrm{Aut}(\langle y, z\rangle) \cong \mathbb{Z}_{p^2}^* \times D_6$ and $\mathrm{Aut}(G)$ acts transitively on $\mathcal{S}_{2p^2}$. Hence we may assume that $xz \in S$. In particular, $S = \{xz, x^{-1}z, x^u y^v z, x^{-u} y^v z\}$, for some $u \in \mathbb{Z}_{p^2}^*$ and some $v \in \mathbb{Z}_3^*$. Replacing S by a suitable $\mathrm{Aut}(G)$-conjugate, we may assume that $v = 1$, that is,

$$S = \{xz, x^{-1}z, x^u yz, x^{-u} yz\}.$$

Let $\varphi \in \mathrm{Aut}(G)$ with $S^\varphi = S$ and $(xz)^\varphi = x^u yz$ (recall that such an automorphism exists because X is a normal arc-transitive Cayley graph over G). From the description of $\mathrm{Aut}(G)$, we have $x^\varphi = x^u$ and $z^\varphi = yz$ and $y^\varphi = y^d$ (for some $d \in \mathbb{Z}_3^*$). Now, $(x^u yz)^\varphi = x^{u^2} y^{d+1} z \in S$. As $d \neq 0$, we obtain $d = -1$ and $u^2 = \pm 1$. Thus we have one of the following two possibilities for S:

$$\begin{aligned} S &= \{xz, x^{-1}z, xyz, x^{-1}yz\}, \text{ or} \\ S &= \{xz, x^{-1}z, x^\varepsilon yz, x^{-\varepsilon} yz\}, \text{ where } \varepsilon^2 = -1 \end{aligned}$$

(note that in the second case $p \equiv 1 \pmod 4$). In particular, we obtain that X is isomorphic to the graph in Theorem 1.1 (ii) (c) or (d).

4.6 4-Valent normal Cayley graphs over $\mathbb{Z}_{p^2} \times \mathbb{Z}_6$ with $p \geq 5$

This case is by far the easiest to deal with and we leave it to the conscious reader. There is only one graph arising, namely the graph in Theorem 1.1 (ii) (c).

5 Proof of Theorem 1.1 and Corollary 1.2

We start with some technical preliminary lemmas that will be useful in the proof of Theorem 1.1.

Lemma 5.1. *Let p be a prime with $p \geq 5$ and let N be a normal subgroup of G with $|N| = p$ and $G/N \cong \mathbb{Z}_p \times \mathbb{Z}_6$ or $G/N \cong \mathbb{Z}_p \times D_6$. Then G is isomorphic to one of the following groups*

(i) $\mathbb{Z}_{p^2} \times \mathbb{Z}_6$ *or* $\mathbb{Z}_{p^2} \times D_6$*; or*

(ii) $\mathbb{Z}_p \times (\mathbb{Z}_p \rtimes \mathbb{Z}_6)$ *or* $\mathbb{Z}_p \times (\mathbb{Z}_p \rtimes D_6)$.

Proof. Let P be a Sylow p-subgroup of G. Since G is soluble, G contains a subgroup Q with $|Q| = 6$. As $G/N \cong \mathbb{Z}_p \times \mathbb{Z}_6$ or $G/N \cong \mathbb{Z}_p \times D_6$, we have that $QN/N \cong Q$ (that is, $Q \cong \mathbb{Z}_6$ or $Q \cong D_6$) and that Q centralizes P/N. If P is cyclic, then Q centralizes P by [13, Theorem 1.4]. So $G = P \times Q$ and part (i) follows.

Suppose that P is an elementary abelian p-group. The action of Q by conjugation on P endows P of a structure of an $\mathbb{Z}_pQ$-module. As Q has order coprime to p, the $\mathbb{Z}_pQ$-module N is completely reducible, that is, $P = N \times N'$, for some normal subgroup N' of G of size p. As Q centralizes P/N, we see that Q centralizes N'. Thus $G = N' \times (N \rtimes Q)$ and part (ii) follows. □

Lemma 5.2. *Let p be a prime with $p \geq 7$ and let X be a connected normal 4-valent arc-transitive Cayley graph over one of the groups G in Lemma 5.1 (ii). Let P be a Sylow p-subgroup of G. Assume that $P \lhd G$ and that $X_P = C_6$. Then either $G \cong \mathbb{Z}_p \times \mathbb{Z}_{6p}$, or $G \cong \mathbb{Z}_{3p} \times D_{2p}$, or $G \cong \mathbb{Z}_p \times D_{6p}$, or $G \cong \mathbb{Z}_p \times \mathbb{Z}_p \times D_6$.*

Proof. From Lemma 5.1 (ii), we have $G = \langle x\rangle \times (\langle y\rangle \rtimes \langle z, t\rangle)$ with $\langle x, y\rangle \cong \mathbb{Z}_p \times \mathbb{Z}_p$, $|z| = 3$ and $|t| = 2$. Moreover, $[z, t] = 1$ if $G \cong \mathbb{Z}_p \times (\mathbb{Z}_p \rtimes \mathbb{Z}_6)$ and $z^t = z^{-1}$ if $G \cong \mathbb{Z}_p \times (\mathbb{Z}_p \rtimes D_6)$. If z centralizes y, then $G = \langle x\rangle \times (\langle yz\rangle \rtimes \langle t\rangle)$. In particular, $G \cong \mathbb{Z}_p \times \mathbb{Z}_{6p}$ if t centralizes both y and z, $G \cong \mathbb{Z}_p \times D_{6p}$ if t inverts both y and z, $G \cong \mathbb{Z}_{3p} \times D_{2p}$ if t centralizes z and inverts y, and $G \cong \mathbb{Z}_p \times \mathbb{Z}_p \times D_6$ if t inverts z and centralizes y. In remains to consider the case that z does not centralize y. We show that this case actually does not arise (here we use the fact that G admits a normal Cayley graph).

Note that the automorphism group of $\mathbb{Z}_p$ is cyclic and hence D_6 cannot act faithfully as a group of automorphisms on $\mathbb{Z}_p$. Since we are assuming that z does not centralize y, we must have that $G \cong \mathbb{Z}_p \times (\mathbb{Z}_p \rtimes \mathbb{Z}_6)$, that is, $[z, t] = 1$. Let S be a subset of G with $X = \mathrm{Cay}(G, S)$. Write $A = \mathrm{Aut}(X)$. Since $\langle x\rangle$ is a characteristic subgroup of G and since $G \lhd A$, we obtain that $\langle x\rangle \lhd A$ and $Y = X_{\langle x\rangle}$ is a normal quotient having $6p$ vertices. In particular, Y has valency 2 or 4. Let K be the kernel of the action of A on $V(Y)$ and assume that Y is a cycle. In particular, $A/K \cong D_{12p}$. Now D_{12p} contains exactly two regular subgroups, one isomorphic to $\mathbb{Z}_{6p}$ and the other to D_{6p}. Therefore

$$\frac{GK}{K} \cong \frac{G}{G \cap K} = \frac{G}{\langle x\rangle} \cong \langle y\rangle \rtimes \langle z, t\rangle$$

is isomorphic either to $\mathbb{Z}_{6p}$ or to D_{6p}, contradicting the fact that z does not centralize y. Thus Y is a 4-valent Cayley graph over $GK/K \cong \langle y\rangle \rtimes \langle z, t\rangle$ and $K = \langle x\rangle$.

Now, P/K has order p and is therefore a minimal normal subgroup of A/K. Also, $Y_{P/K} \cong X_P \cong C_6$ and so we are in the position to apply Theorem 2.1 (b) to A/K, P/K and Y. Hence $Y = C^{\pm 1}(p; 6, 1)$ or $Y = C^{\pm\varepsilon}(p; 6, 1)$. From Sections 3.2 and 3.3 we see that Y is a Cayley graph over $\mathbb{Z}_p \times \mathbb{Z}_6$ or over $\mathbb{Z}_p \times D_6$. However, $\langle y\rangle \rtimes \langle z, t\rangle$ is isomorphic to neither of these groups, a contradiction. □

In what follows we denote by O the graph $K_6 - 6K_2$, that is, K_6 with a perfect matching removed. Clearly, O is connected, 4-valent and $|V(O)| = 6$. We refer to O as the *Octahedral* graph.

Proof of Theorem 1.1. We first consider the case that $p \geq 11$. Let B be a subgroup of $\mathrm{Aut}(X)$ acting regularly on $A(X)$, let B_v be the stabilizer in B of the vertex $v \in V(X)$ and let P be a Sylow p-subgroup of B. We show that P is normal in B. Since $|B| = |A(X)| = 24p^2$, Sylow's theorems show that the number of Sylow p-subgroups of B is equal to $|B : \mathbf{N}_B(P)| = 1 + kp$, for some $k \geq 0$. If $k = 0$, then P is normal in B and thus we may assume that $k \geq 1$. Now, $1 + kp$ divides 24 and this is possible if and only if $k = 1$ and $p = 23$, or $k = 1$ and $p = 11$. Suppose that $k = 1$ and $p = 23$. Now $|B : \mathbf{N}_B(P)| = 24$. So $\mathbf{N}_B(P) = P$ and $\mathbf{C}_B(P) = \mathbf{N}_B(P)$. Therefore, by the Burnside's p-complement theorem [27, page 76], we see that B has a normal subgroup N of order 24. In particular, P acts by conjugation as a group of automorphisms on N. As a group of order 24 does not admit non-trivial automorphisms of order 23, we see that P centralizes B. Thus $B \cong N \times P$ and P is normal in B. Finally, suppose that $k = 1$ and $p = 11$. Now, $|B : \mathbf{N}_B(P)| = 12$. Consider the permutation group $\bar{B}$ induced by the action of B on the cosets of $\mathbf{N}_B(P)$. Now, $\bar{B}$ has degree 12 and has order divisible by 11 because (by hypothesis) P is not normal in B. Therefore $\bar{B}$ is a 2-transitive group whose order divides $24 \cdot 11^2$. A quick inspection on the list of 2-transitive groups of degree 12 in `magma` shows that this is impossible. This final contradiction gives that P is normal in B.

As usual, we denote by X_P the normal quotient of X via P. The rest of the proof is a case-by-case analysis depending upon the structure of the normal quotient X_P. At the end of the proof of each case we use the symbol $\blacksquare$ to mean that the theorem is proved in the case under consideration.

As $|B_v| = 4$, we have $B_v \cap P = 1$ and P acts semiregularly on $V(X)$. Thus the orbits of P on $V(X)$ have size p^2 and $|V(X_P)| = 6$. In particular, X_P is either the cycle C_6 or the octahedral graph O (depending on whether X_P has valency 2 or 4).

CASE A: $X_P = O$.

Fix v a vertex of X and let K be the kernel of the action of B on $V(X_P)$. As X_P has valency 4, we obtain $|B/K| = 24$ and $K = P$. Now, the automorphism group W of O is isomorphic to the wreath product $\mathbb{Z}_2 \wr \mathrm{Sym}(3)$, which has order 48. Therefore B/P is isomorphic to a subgroup of index 2 in W. Labelling the vertices of O as $\{1, 2, 3, 4, 5, 6\}$ (so that $\{\{1, 2\}, \{3, 4\}, \{5, 6\}\}$ is the system of imprimitivity for W), we may assume that

$$W = \langle (1, 2), (3, 4), (5, 6), (1, 3)(2, 4), (1, 5)(2, 6) \rangle.$$

The group W has exactly three subgroups of index 2. Namely,

$$\begin{aligned} W_1 &= \langle (1, 2)(3, 4), (1, 2)(5, 6), (1, 3)(2, 4), (1, 5)(2, 6) \rangle, \\ W_2 &= \langle (1, 2), (3, 4), (5, 6), (1, 3, 5)(2, 4, 6) \rangle, \\ W_3 &= \langle (1, 2)(3, 4), (1, 2)(5, 6), (1, 3, 5)(2, 4, 6), (1, 2)(3, 6)(4, 5) \rangle. \end{aligned}$$

The group B/P acts by conjugation as a group of automorphisms on P. So, if $B/P \cong W_i$, then W_i admits an action as a group of automorphisms on P. Assume that B/P acts faithfully on P, that is, $\mathbf{C}_B(P) = P$. In particular, W_i admits a faithful irreducible action on P. Suppose that P is cyclic. Then $\mathrm{Aut}(P)$ is cyclic and so B/P is cyclic. However, W_1, W_2 and W_3 are not cyclic, a contradiction. Thus P is an elementary abelian p-group and $\mathrm{Aut}(P) \cong \mathrm{GL}_2(p)$ (the group of 2×2 invertible matrices). It is clear that the group $\mathrm{SL}_2(p)$ has index 2 in $\mathrm{GL}_2(p)$ and that $\mathrm{SL}_2(p)$ contains a unique element of order 2. This shows that W_i has a normal subgroup T with W_i/T cyclic and with T containing at most one involution. A direct inspection on W_1, W_2 and W_3 shows that such a normal subgroup T does not exist. Therefore B/P does not act faithfully on P and $\mathbf{C}_B(P) > P$.

Another direct inspection on W_1, W_2 and W_3 shows that W_1 and W_3 contain a unique minimal normal subgroup (namely, $\langle(1,2)(3,4),(1,2)(5,6)\rangle$, which has order 4) and W_2 contains exactly two minimal normal subgroups ($\langle(1,2)(3,4)(5,6)\rangle$ having size 2 and $\langle(1,2)(3,4),(1,2)(5,6)\rangle$ having size 4). Therefore, $\mathbf{C}_B(P)$ contains a minimal normal subgroup Q with $|Q| = 2$ or $|Q| = 4$. Suppose that $|Q| = 2$ (and so $B/P \cong W_2$). Now, $W_2/\langle(1,2)(3,4)(5,6)\rangle \cong \mathrm{Alt}(4)$ the alternating group on 4 letters. Arguing as in the previous paragraph, we see that $\mathrm{Alt}(4)$ cannot act faithfully on P. Therefore $|\mathbf{C}_B(P) : P| \geq 4$ and $\mathbf{C}_B(P)$ contains a minimal normal subgroup R with $|R| = 4$. So, replacing Q by R if necessary, we may assume that $|Q| = 4$.

Since Q is characteristic in $\mathbf{C}_B(P)$, we get that Q is normal in B. As 4 does not divide $|V(X)|$, we get that $|Q_v| = 2$ and the Q-orbits have size 2. So, the quotient graph X_Q is a cycle of length $3p^2$. Now from Theorem 2.1 (a), we obtain $X = C(2; 3p^2, 1)$ and X is as in Theorem 1.1 (i). ■

For the remainder of the proof we may assume that $X_P = C_6$.

CASE B: P is a minimal normal subgroup of B.

Fix v a vertex of X and let K be the kernel of the action of B on $V(X_P)$. As $|B| = 24p^2$ and as X_P has valency 2, we have $B/K \cong D_{12}$ and $|K_v| = 2$. So, we see that Theorem 2.2 (b) applies (with $s = 2$), and X is isomorphic to $C^{\pm 1}(p; 6, 2)$ or to one of the graphs defined in [12, Lemmas 8.4 and 8.7]. From Sections 3.2, 3.4 and 3.5, Theorem 1.1 (i) holds. ■

For the remainder of the proof we may assume that B has a minimal normal subgroup N with $N \leq P$ and $|N| = p$. Now by Theorem 2.3 the normal quotient X_N is either C_{6p} or a 4-valent graph.

CASE C: $X_N = C_{6p}$.

Let K be the kernel of the action of B on $V(X_N)$. As $|B| = 24p^2$ and X_N is 2-valent, we have $B/K \cong D_{12p}$ and $|K_v| = 2$. So, we are in the position to apply Theorem 2.1 (b) (with $s = 1$) and thus X is isomorphic to either $C^{\pm 1}(p; 6p, 1)$ or to $C^{\pm \varepsilon}(p; 6p, 1)$. From Sections 4.1 and 4.2, we get that in the first case Theorem 1.1 part (ii) (a) holds and in the second case Theorem 1.1 part (ii) (b) holds. ■

For the remainder of the proof we may assume that X_N is a 4-valent graph. So, X is a regular cover of X_N. Denote by K the kernel of the action of B on $V(X_P)$ and recall that $|K| = 2p^2$ because $X_P = C_6$. Also, note that the normal quotient $(X_N)_{P/N}$ is isomorphic to X_P. Now P/N is a minimal normal subgroup of B/N with orbits of size p. The kernel of the action of B/N on X_P is K/N and $|K_vP/P| = 2$. Therefore Theorem 2.1 (b) applies (with $s = 1$ and with B replaced by B/N), and so $X_N = C^{\pm 1}(p; 6, 1)$ or $X_N = C^{\pm \varepsilon}(p; 6, 1)$. From Sections 3.2 and 3.3, we see that $C^{\pm 1}(p; 6, 1)$ is a normal Cayley graph over $\mathbb{Z}_p \times \mathbb{Z}_6$ and that $C^{\pm \varepsilon}(p; 6, 1)$ is a normal Cayley graph over $\mathbb{Z}_p \times D_6$.

Let G/N be the normal subgroup of B/N acting regularly on $V(X_N)$, with $G/N \cong \mathbb{Z}_p \times \mathbb{Z}_6$ or with $G/N \cong \mathbb{Z}_p \times D_6$. Clearly, G acts regularly on $V(X)$ and thus X is a normal Cayley graph over G. From Lemma 5.1, we see that G is isomorphic either to $\mathbb{Z}_{p^2} \times \mathbb{Z}_6$, or $\mathbb{Z}_{p^2} \times D_6$, or $\mathbb{Z}_p \times (\mathbb{Z}_p \rtimes \mathbb{Z}_6)$, or $\mathbb{Z}_p \times (\mathbb{Z}_p \rtimes D_6)$.

If G is isomorphic to $\mathbb{Z}_p \times (\mathbb{Z}_p \rtimes \mathbb{Z}_6)$, or $\mathbb{Z}_p \times (\mathbb{Z}_p \rtimes D_6)$, then from Lemma 5.2 G is isomorphic to $\mathbb{Z}_p \times \mathbb{Z}_{6p}$, or $\mathbb{Z}_p \times D_{6p}$, or $\mathbb{Z}_p \times \mathbb{Z}_p \times D_6$, or $\mathbb{Z}_{3p} \times D_{2p}$. Now the proof follows from Sections 4.1, 4.2, 4.3 and 4.4.

If G is isomorphic either to $\mathbb{Z}_{p^2} \times \mathbb{Z}_6$ or $\mathbb{Z}_{p^2} \times D_6$, then the proof follows from Sections 4.5 and 4.6.

It remains to consider the case that $p \leq 7$. When $p = 7$, we see from [18, 19] that there are seven 4-valent arc-transitive graphs on $6p^2$ vertices. A computer computation shows that six of these graphs are isomorphic to one of the graphs defined in Theorem 1.1 (i) and (ii), and the seventh does not admit a group of automorphisms acting regularly on arcs. When $p = 5$, we see from [18, 19] that there are ten 4-valent arc-transitive graphs on $6p^2$ vertices and they all admit a group of automorphisms acting regularly on arcs. It is a computer computation checking that eight of these graphs are isomorphic to one of the graphs defined in Theorem 1.1 (i) and (ii), and the remaining two are given in Section 6. When $p = 3$, we see from [18, 19] that there are five 4-valent arc-transitive graphs on $6p^2$ vertices and they all admit a group of automorphisms acting regularly on arcs. It is a computer computation checking that three of these graphs are isomorphic to one of the graphs defined in Theorem 1.1 (i) and (ii), and the remaining two are given in Section 6. When $p = 2$, the result follows again with a straightforward computation. $\square$

Proof of Corollary 1.2. Observe that from Sections 3.1, 3.2 and 4.3 the graphs $C(2; 3p^2, 1)$, $C^{\pm 1}(p; 6, 2)$ and the graphs in Theorem 1.1 (ii) (e) are not one-regular. Now the result follows immediately from Theorem 1.1. $\square$

6 Description of the graphs in Theorem 1.1 part (iii) and Corollary 1.2 (iii)

New now describe the exceptional graphs in Theorem 1.1 (iii) and Corollary 1.2 (iii).
CASE $p = 2$.

(i) $X = \mathrm{Cay}(\langle x \rangle, \{x, x^{-1}, x^5, x^{-5}\})$ where $\langle x \rangle$ is a cyclic group of order 24. Now, X is a normal one-regular Cayley graph.

(ii) $X = \mathrm{Cay}(\langle x \rangle, \{x, x^{-1}, x^7, x^{-7}\})$ where $\langle x \rangle$ is a cyclic group of order 24. Now, X is a normal one-regular Cayley graph.

(iii) $X = \mathrm{Cay}(\mathrm{SL}(2,3), S)$ and with connection set

$$S = \left\{ \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \right\}.$$

Now, X is a normal one-regular Cayley graph.

(iv) $X = \mathrm{Cay}(\mathrm{SL}(2,3), S)$ and with connection set

$$S = \left\{ \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} -1 & -1 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 1 & -1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ -1 & -1 \end{pmatrix} \right\}.$$

Now, X is a normal one-regular Cayley graph.

(v) $X = \mathrm{Cay}(G, S)$ with $G = \langle (1,2,3), (1,2), (4,5,6,7) \rangle \cong D_6 \times \mathbb{Z}_4$ and $S = \{(1,3)(4,6)(5,7), (1,2,3)(4,5,6,7), (2,3)(4,6)(5,7), (1,3,2)(4,7,6,5)\}$. Now, X is neither a normal Cayley graph nor one-regular.

CASE $p = 3$.

(i) X is the Cayley graph $\mathrm{Cay}(G, S)$, where G is the subgroup of $\mathrm{GL}(3, 3)$ generated by

$$\begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

and with connection set

$$\left\{ \begin{pmatrix} -1 & -1 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & -1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \right. \\ \left. \begin{pmatrix} -1 & 1 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 1 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \right\}.$$

Now, it is a computation to verify that X is not one-regular.

(ii) X is the Cayley graph $\mathrm{Cay}(G, S)$, where G is the subgroup of $\mathrm{GL}(3, 3)$ generated by

$$\begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

and with connection set

$$\left\{ \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \right. \\ \left. \begin{pmatrix} -1 & -1 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix} \right\}.$$

Now, it is a computation to verify that X is one-regular.

CASE $p = 5$.

(i) X is the coset graph $\mathrm{Cos}(G, H, g)$ where

$$G = \langle (1, 2, 3, 4, 5), (2, 5)(3, 4), (6, 7, 8, 9, 10), (8, 9, 10) \rangle \cong D_5 \times \mathrm{Alt}(5),$$

$H = \langle (1, 5)(2, 4)(6, 9)(8, 10), (6, 10)(8, 9) \rangle$ and $g = (1, 3)(4, 5)(6, 10)(7, 8)$. Now, $|\mathrm{Aut}(X)| = 1200$ and hence X is not one-regular.

(ii) X is the coset graph $\mathrm{Cos}(G, H, g)$ where

$$G = \langle (1, 2, 3, 4, 5), (2, 5)(3, 4), (6, 7, 8, 9, 10), (8, 9, 10) \rangle \cong D_5 \times \mathrm{Alt}(5),$$

$H = \langle (1, 5)(2, 4)(7, 10)(8, 9), (7, 9)(8, 10) \rangle$ and $g = (1, 2)(3, 5)(6, 7)(8, 9)$. Now, $|\mathrm{Aut}(X)| = 600$ and hence X is one-regular.

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ARS MATHEMATICA CONTEMPORANEA

Also available at http://amc-journal.eu
ISSN 1855-3966 (printed edn.), ISSN 1855-3974 (electronic edn.)
ARS MATHEMATICA CONTEMPORANEA 9 (2015) 19–25

Partial product of graphs and Vizing's conjecture

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Received 9 December 2012, accepted 19 December 2013, published online 5 June 2014

Abstract

Let G and H be two graphs with vertex sets $V_1 = \{u_1, ..., u_{n_1}\}$ and $V_2 = \{v_1, ..., v_{n_2}\}$, respectively. If $S \subset V_2$, then the partial Cartesian product of G and H with respect to S is the graph $G \square_S H = (V, E)$, where $V = V_1 \times V_2$ and two vertices (u_i, v_j) and (u_k, v_l) are adjacent in $G \square_S H$ if and only if either ($u_i = u_k$ and $v_j \sim v_l$) or ($u_i \sim u_k$ and $v_j = v_l \in S$). If $A \subset V_1$ and $B \subset V_2$, then the restricted partial strong product of G and H with respect to A and B is the graph $G_A \boxtimes_B H = (V, E)$, where $V = V_1 \times V_2$ and two vertices (u_i, v_j) and (u_k, v_l) are adjacent in $G_A \boxtimes_B H$ if and only if either ($u_i = u_k$ and $v_j \sim v_l$) or ($u_i \sim u_k$ and $v_j = v_l$) or ($u_i \in A$, $u_k \notin A$, $v_j \in B$, $v_l \notin B$, $u_i \sim u_k$ and $v_j \sim v_l$) or ($u_i \notin A$, $u_k \in A$, $v_j \notin B$, $v_l \in B$, $u_i \sim u_k$ and $v_j \sim v_l$). In this article we obtain Vizing-like results for the domination number and the independence domination number of the partial Cartesian product of graphs. Moreover we study the domination number of the restricted partial strong product of graphs.

Keywords: Domination, partial product of graphs, Cartesian product graph, strong product graph, Vizing's conjecture.

Math. Subj. Class.: 05C69, 05C70, 05C76

1 Introduction

Vizing's conjecture [7] is perhaps one of the most popular open problems related to domination in graphs. It states that the domination number of the Cartesian product of two graphs is at least as large as the product of their domination numbers. A high quantity of approaches to that problem have been developed in this sense. The surveys [1, 3] are very complete compilations of the principal results obtained in this topic. Moreover, several Vizing-like results for some other kind of product graphs have been also published [5]. In

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this article we introduce the notion of partial Cartesian product of graphs and obtain some corresponding Vizing-like results for the domination number and independence domination number of partial Cartesian product of graphs.

We establish first the principal terminology and notation which we will use throughout the article. Hereafter $G = (V, E)$ denotes a finite simple graph. The complement of a set D is denoted by $\overline{D}$. A set D is a *dominating set* if every vertex of $\overline{D}$ is adjacent to a vertex of D [4]. A dominating set D is called *minimal* if does not contain any dominating set as a proper subset. The *domination number*, $\gamma(G)$, is the minimum cardinality of any dominating set in G. We say that a set S is a $\gamma(G)$-set if it is a dominating set and $|S| = \gamma(G)$. The following result, due to Ore, can be found in [4].

Lemma 1.1. [4] *A dominating set S is a minimal dominating set of a graph $G = (V, E)$ if and only if for each $u \in S$ one of the following conditions holds:*

- *u is an isolated vertex of S,*
- *there exists $v \in V - S$ for which $N(v) \cap S = \{u\}$.*

2 Results

Let G and H be two graphs with set of vertices $V_1 = \{u_1, ..., u_{n_1}\}$ and $V_2 = \{v_1, ..., v_{n_2}\}$, respectively. We recall that the *Cartesian product* of G and H is the graph $G\Box H = (V, E)$, where $V = V_1 \times V_2$ and two vertices (u_i, v_j) and (u_k, v_l) are adjacent in $G\Box H$ if and only if one of the following conditions is satisfied:

- $u_i = u_k$ and $v_j \sim v_l$, or
- $u_i \sim u_k$ and $v_j = v_l$.

The *strong product* of G and H is the graph $G \boxtimes H = (V, E)$, where $V = V_1 \times V_2$ and two vertices (u_i, v_j) and (u_k, v_l) are adjacent in $G \boxtimes H$ if and only if one of the following conditions is satisfied:

- $u_i = u_k$ and $v_j \sim v_l$, or
- $u_i \sim u_k$ and $v_j = v_l$, or
- $u_i \sim u_k$ and $v_j \sim v_l$.

For a vertex $u_i \in V_1$, the subgraph of $G\Box H$ or $G \boxtimes H$ induced by the set $\{(u_i, v)|v \in V_2\}$ is called an H-layer or an H-fiber. Similarly, for $v_j \in V_2$, the subgraph induced by $\{(u, v_j)|u \in V_1\}$ is a G-layer or a G-fiber.

2.1 Partial Cartesian product of graphs

Let G and H be two graphs with set of vertices $V_1 = \{u_1, ..., u_{n_2}\}$ and $V_2 = \{v_1, ..., v_{n_1}\}$, respectively. If $S \subset V_2$, then the *partial Cartesian product* of G and H *with respect to* S is the graph $G\Box_S H = (V, E)$, where $V = V_1 \times V_2$ and two vertices (u_i, v_j) and (u_k, v_l) are adjacent in $G\Box_S H$ if and only if one of the following conditions is satisfied:

- $u_i = u_k$ and $v_j \sim v_l$, or
- $u_i \sim u_k$ and $v_j = v_l \in S$.

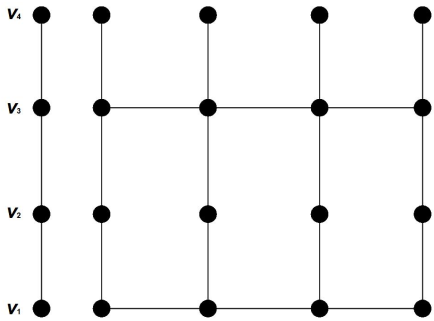

Figure 1: The partial Cartesian product graph $P_4\Box_{\{v_1,v_3\}}P_4$.

An example of a partial Cartesian product graph is shown in Figure 1. Also notice that the partial Cartesian product of G and H is the spanning subgraph of $G\Box H$ obtained by deleting the edges of $G\Box H$ in the H-fibers that correspond to vertices from $V_2 \setminus S$.

Notice that if $S = V_2$, then $G\Box_S H$ is the standard Cartesian product graph $G\Box H$. Also, note that if S is formed by only one vertex v, then $G\Box_S H$ is a particular case of the rooted product graph $G \circ_v H$ defined in [2]. Domination related parameters of rooted product graphs were studied in [6].

Theorem 2.1. *Let G be a graph of order n and let H be any graph. If S is a subset of vertices of H, then*

$$\gamma(G\Box_S H) \geq n\gamma(H) - n|S| + |S|\gamma(G).$$

Proof. Let $V_1 = \{u_1, u_2, ..., u_n\}$ and V_2 be the vertex sets of G and H, respectively. Hence, we consider the set of vertices $S \subset V_2$. Let D be a $\gamma(G\Box_S H)$-set. For every $u_i \in V_1$, let $D_i = \{v \in V_2 : (u_i, v) \in D\}$ and let $X_i \subset \overline{D_i}$ be the set of vertices of H not dominated by D_i. Hence, we have that $D_i \cup X_i$ is a dominating set in H and, as a consequence, $\gamma(H) \leq |D_i| + |X_i|$ for every is $i \in \{1, ..., n\}$. Hence,

$$\begin{aligned} |D| &= \sum_{i=1}^{n} |D_i| \\ &\geq \sum_{i=1}^{n} (\gamma(H) - |X_i|) \\ &= n\gamma(H) - \sum_{i=1}^{n} |X_i|. \end{aligned}$$

Thus, we have that

$$\gamma(G\Box_S H) \geq n\gamma(H) - \sum_{i=1}^{n} |X_i|. \tag{2.1}$$

Notice that if $X_j \neq \emptyset$ for some $j \in \{1, ..., n\}$, then there exists a vertex $v \in V_2$ such that (u_j, v) is dominated by a vertex $(u_l, v) \in D$ with $l \neq j$. Thus, $v \in S$. Hence, for every vertex $w \in S$ let $D_w = \{u_i \in V_1 : (u_i, w) \in (\{u_i\} \times X_i)\}$. Notice that

$\overline{D_w}$ is a dominating set in G. So, $\gamma(G) \leq |\overline{D_w}| = n - |D_w|$. Moreover, we have that $\sum_{i=1}^{n} |X_i| = \sum_{w \in S} |D_w|$.

Now, by using (2.1) we have the following.

$$\begin{aligned}
\gamma(G \Box_S H) &\geq n\gamma(H) - \sum_{i=1}^{n} |X_i| \\
&= n\gamma(H) - \sum_{w \in S} |D_w| \\
&\geq n\gamma(H) - \sum_{w \in S} (n - \gamma(G)) \\
&= n\gamma(H) - n|S| + |S|\gamma(G).
\end{aligned}$$

□

By taking a $\gamma(H)$-set S in the above theorem we obtain a Vizing-like result for the domination number of the partial Cartesian product graph $G \Box_S H$ with respect to S.

Corollary 2.2. *Let G and H be any graphs. Then there exists a set S of vertices of H such that*

$$\gamma(G \Box_S H) \geq \gamma(G)\gamma(H).$$

To see that the above bound is tight we consider the following. We say that a graph $G = (V, E)$ satisfies property $\mathcal{P}$ if it has a $\gamma(G)$-set D such that $D = X \cup Y$, $X \cap Y = \emptyset$, X dominates $V - Y$ and Y dominates $V - X$. Let $\mathcal{F}$ be the family of all graphs satisfying property $\mathcal{P}$.

Proposition 2.3. *Let G be a graph with domination number half its order and let H be a graph of the family $\mathcal{F}$. Then there exists a set S of vertices of H such that*

$$\gamma(G \Box_S H) = \gamma(G)\gamma(H).$$

Proof. Since $H = (V, E)$ is a graph of the family $\mathcal{F}$, there exists a $\gamma(H)$-set S such that $S = X \cup Y$, $X \cap Y = \emptyset$, X dominates $V - Y$ and Y dominates $V - X$. Also, as G has domination number $\frac{n}{2}$, where n is the order of G, we have that there exists a $\gamma(G)$-set D, such that $\gamma(G) = |D| = |\overline{D}| = \frac{n}{2}$. Notice that the set $(D \times X) \cup (\overline{D} \times Y)$ is a dominating set in $G \Box_S H$ and, as a consequence,

$$\gamma(G \Box_S H) \leq |(D \times X) \cup (\overline{D} \times Y)| = |D||X| + |\overline{D}||Y| = \frac{n\gamma(H)}{2} = \gamma(G)\gamma(H).$$

The proof is finished by Corollary 2.2. □

Since for every set S of vertices of a graph H, the partial Cartesian product graph $G \Box_S H$ is a subgraph of the Cartesian product graph $G \Box H$, we have that $\gamma(G \Box_S H) \geq \gamma(G \Box H)$. Hence, notice that if there exists a graph $G \Box_S H$ such that $\gamma(G \Box_S H) = \gamma(G)\gamma(H)$ and $\gamma(G \Box_S H) > \gamma(G \Box H)$, then the Vizing's conjecture is not true. In this sense, the above results lead to the following question.

Question: Are there some graphs G and H such that some set S of vertices of H satisfies that $\gamma(G \Box_S H) = \gamma(G)\gamma(H)$ and $\gamma(G \Box_S H) > \gamma(G \Box H)$?

Next we study the independence domination number of the partial Cartesian product of graphs. A set Y is an *independent dominating set* in a graph G, if Y is a dominating set and there no edge between any two vertices of Y. The minimum cardinality of an independent dominating set of G is the *independence domination number* of G and it is denoted by $i(G)$. The following result from [6] will be useful to present our results.

Lemma 2.4. [6] *Let $G = (V, E)$ be a graph. Then for every set of vertices $A \subset V$,*

$$i(G - A) \geq i(G) - |A|.$$

Theorem 2.5. *Let G be a graph of order n and let H be any graph. If S is a subset of vertices of H, then*

$$i(G\Box_S H) \geq n(i(H) - |S|) + |S|i(G).$$

Proof. Let $V_1 = \{u_1, u_2, ..., u_n\}$ and V_2 be the vertex sets of G and H, respectively, and let $S \subset V_2$. Let Y be an $i(G\Box_S H)$-set. For every $u_i \in V_1$, we define the following subsets of vertices of H:

- $Y_i = \{v \in V_2 \ : \ (u_i, v) \in Y\}$,
- $S_i \subset S$ such that for every $v \in S_i$, (u_i, v) is independently dominated by some vertex $(u_i, v') \in Y$ with $v' \in Y_i$,
- $R_i \subset S$ such that $R_i = S \cap Y_i$.

Thus, we have that for every $i \in \{1, ..., n\}$, Y_i is an independent dominating set in $\langle V_2 - S + R_i + S_i\rangle$. So, by Lemma 2.4 we have that $|Y_i| \geq i(\langle V_2 - S + R_i + S_i\rangle) \geq i(H) - |S| + |R_i| + |S_i|$. Therefore, we obtain that

$$|Y| = \sum_{i=1}^{n} |Y_i| \geq \sum_{i=1}^{n}(i(H)-|S|+|R_i|+|S_i|) = n(i(H)-|S|)+\sum_{i=1}^{n}(|R_i|+|S_i|). \quad (2.2)$$

Now, for every vertex $v \in S$ we define the following subset of vertices of G:

- $A_v = \{u_i \in V_1 \ : \ (u_i, v) \in Y\}$,
- $B_v = \{u_i \in V_1 \ :$ such that for every $u_i \in B_v, \ (u_i, v) \in S_i\}$, *i.e.*, (u_i, v) is independently dominated by some vertex (u_i, v') where $v' \in Y_i$.

Thus, we have that A_v is an independent dominating set in $\langle V_1 - B_v\rangle$ and, by Lemma 2.4, we have that $|A_v| \geq i(\langle V_1 - B_v\rangle) \geq i(G) - |B_v|$. Moreover, notice that $\sum_{i=1}^{n}(|R_i| + |S_i|) = \sum_{v\in S}(|A_v| + |B_v|)$. So, from Equation 2.2 we have the following.

$$\begin{aligned}
|Y| &\geq n(i(H) - |S|) + \sum_{i=1}^{n}(|R_i| + |S_i|) \\
&= n(i(H) - |S|) + \sum_{v\in S}(|A_v| + |B_v|) \\
&\geq n(i(H) - |S|) + \sum_{v\in S}(i(G) - |B_v| + |B_v|) \\
&= n(i(H) - |S|) + |S|i(G).
\end{aligned}$$

Therefore, the result follows. □

Notice that, if S is an $i(H)$-set, then the above result leads to item (i) of the next corollary . Moreover, if S is formed by a single vertex, then the above result leads to a lower bound for the independence domination number of rooted product graphs, which was also obtained in [6].

Corollary 2.6. *Let G be a graph of order n and let H be any graph with vertex set V. Let $S \subset V$.*

- *If S is an $i(H)$-set, then $i(G\Box_S H) \geq i(G)i(H)$.*
- *If $S = \{v\}$, then $i(G\Box_S H) = i(G \circ_v H) \geq n(i(H) - 1) + i(G)$.*

2.2 Partial strong product of graphs

Now, if $A \subset V_1$ and $B \subset V_2$, then the *partial strong product* of G and H *with respect to* A and B is the graph $G \boxtimes_{A,B} H = (V, E)$, where $V = V_1 \times V_2$ and two vertices (u_i, v_j) and (u_k, v_l) are adjacent in $G \boxtimes_{A,B} H$ if and only if one of the following conditions is satisfied:

- $u_i = u_k$ and $v_j \sim v_l$,
- $u_i \sim u_k$ and $v_j = v_l$,
- $u_i \in A$, $v_j \in B$, $u_i \sim u_k$ and $v_j \sim v_l$,
- $u_k \in A$, $v_l \in B$, $u_i \sim u_k$ and $v_j \sim v_l$.

Notice that if $A = V_1$ and $B = V_2$, then $G \boxtimes_{A,B} H$ is the standard strong product graph $G \boxtimes H$. A more restrictive condition on the partial strong product of graphs could be the following one. If $A \subset V_1$ and $B \subset V_2$, then the *restricted partial strong product* of G and H *with respect to* A and B is the graph $G\, {}_A\boxtimes_B\, H = (V, E)$, where $V = V_1 \times V_2$ and two vertices (u_i, v_j) and (u_k, v_l) are adjacent in $G\, {}_A\boxtimes_B\, H$ if and only if one of the following conditions is satisfied:

- $u_i = u_k$ and $v_j \sim v_l$,
- $u_i \sim u_k$ and $v_j = v_l$,
- $u_i \in A$, $u_k \notin A$, $v_j \in B$, $v_l \notin B$, $u_i \sim u_k$ and $v_j \sim v_l$,
- $u_i \notin A$, $u_k \in A$, $v_j \notin B$, $v_l \in B$, $u_i \sim u_k$ and $v_j \sim v_l$.

An example of a restricted partial strong product graph is shown in Figure 2.

Lemma 2.7. *Let G, H be any graphs and let A (respectively B) be a $\gamma(G)$-set (respectively $\gamma(H)$-set). Then $A \times B$ is a minimal dominating set in $G\, {}_A\boxtimes_B\, H$.*

Proof. The proof follows from Lemma 1.1 and the procedure of construction of the restricted partial strong product of graphs. □

We omit the proof of the following result, since it is straightforward.

Theorem 2.8. *Let G, H be any graphs. If A and B are dominating sets in G and H, respectively, then*

$$\gamma(G \boxtimes_{A,B} H) \leq \gamma(G)\gamma(H) \quad \textit{and} \quad \gamma(G\, {}_A\boxtimes_B\, H) \leq \gamma(G)\gamma(H).$$

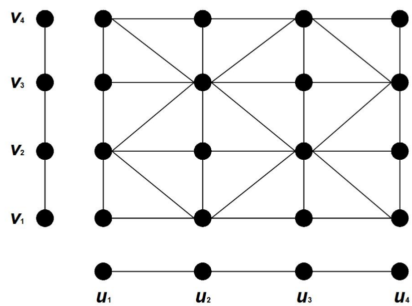

Figure 2: The restricted partial strong product graph $P_4 \,_{\{u_1,u_3\}} \boxtimes_{\{v_2,v_4\}} P_4$.

It is straightforward to observe that the partial Cartesian product graph $G\Box_S H$ with respect to any set of vertices S of G is a subgraph of the restricted partial strong product graph $G\,_A\Box_B H$ for any $\gamma(G)$-set A and any $\gamma(H)$-set B. Thus, according to results of Theorem 2.1, Corollary 2.2 and Theorem 2.8 there would exist a spanning subgraph F of the restricted partial strong product graph $G\,_A\Box_B H$ satisfying that $\gamma(F) = \gamma(G)\gamma(H)$. In this sense, the question is: Is always the Cartesian product graph $G\Box H$ a spanning subgraph of the graph F? If yes, then Vizing's conjecture is true.

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Also available at http://amc-journal.eu
ISSN 1855-3966 (printed edn.), ISSN 1855-3974 (electronic edn.)
ARS MATHEMATICA CONTEMPORANEA 9 (2015) 27–43

Cycle construction and geodesic cycles with application to the hypercube

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Received 2 March 2013, accepted 23 April 2014, published online 9 June 2014

Abstract

Construction of cycles in a graph is investigated, where cycles from particular subsets (such as bases) are added together so that each partial sum is also a cycle or each new cycle intersects the sum of the preceding terms in a nontrivial path. Starting with the geodesic cycles, a hierarchical construction is given. For the hypercube graph, geodesic cycles are characterized, and it is shown how hypercube geodesic cycles can be constructed in two steps from a special basis. Applications are given to inferring commutativity of a diagram in a groupoid from commutativity of some of its cycles.

Keywords: Robust cycle basis, well-arranged sequence, geodesic cycle, Cayley graph, hypercube, forcing commutativity, groupoid diagram.

Math. Subj. Class.: 05C38, 18B40

1 Introduction

Let G be a graph. A cycle-subgraph of G is a connected, 2-regular subgraph. Let $Cyc(G)$ denote the set of all cycle-subgraphs of G. Call $\mathcal{S} \subseteq Cyc(G)$ "weakly robust" if for every $z \in Cyc(G)$, there is an integer $k \geq 1$ and $z_1, \ldots, z_k \in \mathcal{S}$ (not necessarily distinct) such that

$$z = z_1 + z_2 + \cdots + z_k \tag{1.1}$$

and for $2 \leq j \leq k-1$

$$z_1 + z_2 + \cdots + z_j \in Cyc(G); \tag{1.2}$$

that is, $\mathcal{S}$ is weakly robust if it spans the cycle space of G in such a way that every cycle-subgraph has all partial sums remaining cycle-subgraphs (not just Eulerian graphs). Call $\mathcal{S}$ "robust" if it is weakly robust and if, in addition, for $2 \leq j \leq k$,

$$z_j \cap (z_1 + \cdots + z_{j-1}) \text{ is a nontrivial path.} \tag{1.3}$$

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Of course, $\mathcal{S} = Cyc(G)$ has these properties vacuously, taking $k = 1$. Also, (1.3) $\Longrightarrow$ (1.2).

The problem of finding a cycle basis which is weakly robust seems to have first appeared in print in a paper of Dixon and Goodman [2]. Given a spanning tree in a connected graph, one can construct a basis of cycles using the non-tree edges. In [2], it was conjectured that such bases are weakly robust but this was disproved by Sysło [22]. Later, it was shown by Doğrusöz and Krishnamoorthy [3] that the set of region boundaries of a plane graph, excluding the unbounded region, constitute a weakly robust basis.

The problem was rediscovered in [12], including the new question of finding a robust basis, where we showed that for complete graphs, the set of all 3-cycles containing a fixed vertex determines a robust basis. A basis which is not weakly robust (due to A. Vogt) was exhibited. We also found a basis for the bipartite complete graph (the set of all 4-cycles through a fixed edge), which was shown to be weakly robust by Ostermeier et al. [19]. Our motivation was category theory and graph theory, while the work of Stadler and co-authors [9, 11, 15, 16, 19, 8] is also motivated by applications, especially in biochemistry. These latter papers, as well as [12], use notation which is different from that used here.

The properties of cycle bases and related sets discussed below were developed to permit the propagation of commutativity from a basis of cycles to all of the diagram; see [13], [14]. In this paper, it is shown that the commutativity of a d-dimensional hypercube diagram in a groupoid can be *forced* by the commutativity of a carefully chosen subset of the square faces, containing approximately a fraction of $4/d$ of the square faces. In contrast, [13] showed that commutativity cannot be *blocked* by any family with fewer than $d-2$ squares in a d-cube; that is, if all but $d-2$ squares in a d-cube diagram in a groupoid are known to commute, then the entire diagram is commutative.

The paper is organized as follows: Section 2 gives some basic graph-theoretic, algebraic, and topological background. The next section reviews the basic types of well-arranged sequences of cycles, providing the machinery used in the remaining sections. Section 4 covers robustness and weak robustness, including the idea of iterating these constructions, and relates them to Cayley color graphs. In section 5 we give a brief proof that the basis of region boundaries in a plane graph is robust. Section 6 shows how geodesic cycles can build all cycles through an iterative process. Sections 7 and 8 apply only to the hypercube. Geodesic cycles in the d-cube are given a simple characterization and then they are constructed from a recursively defined basis. Section 9 derives the application to commutative diagrams in groupoids and shows that a square which is the sum of two commutative squares can be non-commutative if the two squares added do not intersect in a nontrivial path. Finally, in Section 10, other types of cycle bases are considered.

2 Basic concepts

In this paper, graphs are finite, undirected, and simple (i.e., no loops or parallel edges). For basic graph theory, see, e.g., [7], [1]. A connected graph is *Eulerian* if all of its vertex-degrees are even. For G a graph, $V(G)$ and $E(G)$ denote the sets of vertices and edges, respectively, and we also write $G = (V, E)$. Every graph has a representation as a family of points and closed straight-line segments (corresponding to the vertices and edges, resp.) embedded in 3-dimensional Euclidean space; each segment in the representation joins the two points corresponding to the edge which the segment represents. There is a unique topology which induces the usual topology of a closed interval on the straight-line segments, and this is called the *underlying topology* of the graph. A *cycle* is a graph which is 2-regular and connected. The underlying topology of a cycle is the unique closed connected 1-manifold, i.e., the circle. Let G be a graph and let $Cyc(G)$ denote the set of all subgraphs of G which are cycles.

A *path* is a graph which consists of a single vertex, or a connected graph which contains two vertices of degree 1 with every other vertex of degree 2. A nontrivial path is a path which contains more than one vertex. We write P_r for the path with r vertices.

Let $\mathbf{Z}_2$ denote the field with 2 elements $\{0, 1\}$; $\mathbf{Z}_2^A$ denotes the $\mathbf{Z}_2$-vector space of all functions from a set A to $\mathbf{Z}_2$; $\dim \mathbf{Z}_2^A = |A|$, the cardinality of A. Addition of functions is just coordinate-wise addition (modulo 2), so the zero element is the constant 0 function. To each function in $\mathbf{Z}_2^A$, one can associate the subset of A determined by those elements mapping to 1 and conversely for every subset of A, there is a unique element in $\mathbf{Z}_2^A$ to which it is associated (the "characteristic function" of the set). The symmetric difference of two subsets of A corresponds to the vector space sum of the corresponding elements in $\mathbf{Z}_2^A$ [1, p. 23]. If U is any subset of a vector space V, then $span(U)$ is defined to be the intersection of all linear subspaces of V which contain U. Over the field $\mathbf{Z}_2$, the linear span of U is just the set of all sums of subsets of U; if U is empty, its span is 0.

If $G = (V, E)$ is a graph, there is a linear map

$$\partial_G : \mathbf{Z}_2^{E(G)} \to \mathbf{Z}_2^{V(G)}$$

given by $\partial_G(e) = v + w$ when $e = vw \in E(G)$. The members of $\mathbf{Z}_2^{E(G)}$ are called the 1-*chains* while members of $\mathbf{Z}_2^{V(G)}$ are called 0-*chains*. The kernel (or "null space") $Z(G)$ of the $\mathbf{Z}_2$-linear map ∂_G is the subset of the 1-chains which map to the zero 0-chain. Thus, connected components of subgraphs of G induced by elements of $Z(G)$ correspond to Eulerian subgraphs of G. By a theorem of Euler, H in $Z(G)$ is an edge-disjoint union, hence sum, of cycles [1, p. 24], so $Z(G)$ is the linear span of $Cyc(G)$. So there is a basis for $Z(G)$ consisting entirely of cycles, which is called a *cycle basis*. Addition in $Z(G)$ is denoted $+$. One refers to the vector space $Z(G)$ as the "cycle space" of the graph where the word "cycle" is used in the algebraic, rather than geometric, sense.

The *cyclomatic number* $b(G)$ of a graph G is the dimension of $Z(G)$. It is well-known (and straightforward to show) that

$$b(G) = q - p + k,$$

where q, p, k denote the number of edges, vertices, and connected components of G.

If G is connected, then for every spanning tree T of G, there is a basis $\mathcal{B}(G, T)$ which is the set of cycles z_e formed by adding a single element $e = vw$ in $E(G) \setminus E(T)$. Indeed, as each pair v, w of distinct vertices in G are joined by a unique path in T, $T + e$ has a unique cycle. Each of these cycles has an edge contained by no other cycle, so the set of cycles must be independent. But the cardinality of this independent set is maximal, for T contains $p-1$ edges, so $|\mathcal{B}(G, T)| = q-p+1$, and so it is a basis, the *spanning tree basis*,

3 Well-arranged sequences of cycles

If $z, z' \in Cyc(G)$, we write $z||z'$ and call z, z' *compatible* if $z \cap z' = P$, where P is a nontrivial path. This relation on the set of cycles is symmetric and irreflexive.

For two compatible cycles z, z', the underlying topology of $z + z'$ is the connected sum of the underlying topologies of the components, so $z + z'$ must be a cycle as well. We give a direct argument avoiding the reference to connected sum after introducing a useful notation.

If z is any cycle and P is a nontrivial path contained in z, then there is a unique nontrivial path P' also contained in z such that $z = P \cup P'$ and $P \cap P' = 2K_1$ (i.e., P and P' intersect in two isolated vertices). We write $z - P$ for P'.

Lemma 3.1. *The sum of two compatible cycles is a cycle.*

Proof. Let z, z' be two cycles with $z \cap z' = P$ a nontrivial path. Then

$$z + z' = (z - P) \cup (z' - P).$$

The paths $z - P$ and $z' - P$ intersect in the two endpoints of P, so their union is a cycle. □

Let $z \in Cyc(G)$; suppose $\mathcal{S} \subseteq Cyc(G)$ and $span(\mathcal{S}) = Z(G)$; e.g., $\mathcal{S}$ could be a cycle basis. A *z-sequence in $\mathcal{S}$* is a sequence $\sigma = (s_1, s_2, \ldots, s_k) \in \mathcal{S}^k$ such that

$$z = s_1 + s_2 + \cdots + s_k = \sum_{i=1}^{k} s_i.$$

We do not insist that the entries in σ need all be distinct. Of course, for every cycle z there are z-sequences in $\mathcal{S}$ with all members distinct since $\mathcal{S}$ spans every cycle, but in order to achieve further constraints on the sequence, it may be necessary to allow repetitions of members of the sequence. When there are no repetitions, it will be indicated.

For a sequence $\sigma = (s_1, s_2, \ldots, s_k) \in \mathcal{S}^k$, there are several notions of "well-arranged":
σ is *well-arranged with respect to intersection* (**wai**) if for each j, $2 \le j \le k$, s_j intersects the previous partial sum $\sum_{i=1}^{j-1} s_i$ in a nontrivial path, that is, $s_j || \left(\sum_{i=1}^{j-1} s_i\right)$ for $2 \le j \le k$;
σ is *well-arranged with respect to topology* (**wat**) if each partial sum is a cycle; that is, if the partial sums are topologically constant. These are the only variants of well-arrangedness which we shall need below but for completeness we mention two more.
σ is *well-arranged with respect to connectedness* (**wac**) if each partial sum is connected;
σ is *well-arranged with respect to degree* (**wad**) if each partial sum is regular of degree 2.

By Lemma 3.1, **wai** $\Longrightarrow$ **wat**. However, the converse is false. Noncompatible cycles may sum to a cycle (see, e.g., the example at the end of section 6), so there can be z-sequences which are well-arranged with respect to topology but not with respect to intersection.

Note that **wat** $\Longrightarrow$ **wac** and **wat** $\Longrightarrow$ **wad** trivially. Conversely, if **wad** holds for the z-sequence σ, then σ satisfies a weaker intersection condition that for each j, $2 \le j \le k$, s_j intersects the partial sum $\sum_{i=1}^{j-1} s_i$ in a subgraph with no isolated vertices. Further, this weak intersection condition, in turn, implies **wad**.

4 Robustness and weak robustness

Let G be a graph and let $\mathcal{S} \subseteq Cyc(G)$. We define the *robust closure* $\rho(\mathcal{S})$ (or *weak robust closure* $\rho_w(\mathcal{S})$) of $\mathcal{S}$ with respect to G to be the set of all cycles in G which are the sum of a **wai** (resp., **wat**) sequence of elements in $\mathcal{S}$. A set $\mathcal{S}$ of cycles is called *robust* (*weakly robust*) if $\rho(\mathcal{S}) = Cyc(G)$ (resp. $\rho_w(\mathcal{S}) = Cyc(G)$). Of course, a weakly robust (and so also a robust) set of cycles is necessarily spanning: $span(\mathcal{S}) = Z(G)$. When $\mathcal{S}$ is a basis for $Z(G)$, then every cycle can be written as the sum of a unique subset of the basis, with no element repeated. This non-repetition property of the sequence is not required for the definition of robust and weak robust closure. However, no examples are known where every **wai** (or **wat**) sequence which sums to some cycle z has repeated terms.

Hierarchical iteration is achieved by taking the robust span of the set of cycles produced by the previous stage. Let $\rho^0(\mathcal{S}) = \mathcal{S}$ and, for integer $a \geq 1$, put $\rho^a(\mathcal{S}) := \rho(\rho^{a-1}(\mathcal{S}))$. Then $\rho^a(\mathcal{S}) \subseteq \rho^b(\mathcal{S})$ when $a \leq b$, for a, b nonnegative integers. Also, if $\mathcal{S} \subseteq \mathcal{T}$, then $\rho(\mathcal{S}) \subseteq \rho(\mathcal{T})$.

We note the following for use in Section 9. Let $\mathcal{T} \subseteq Cyc(G)$ for some graph G. Call $\mathcal{T}$ *cooperative* if $(z, z' \in \mathcal{T}$ and $z||z') \implies z + z' \in \mathcal{T}$. According to Lemma 3.1, the set of all cycles is cooperative. From the definition of iterated robust closure and our remarks there, if G is any graph, one has the following.

Lemma 4.1. *If $\mathcal{S} \subseteq \mathcal{T} \subseteq Cyc(G)$, $\mathcal{T}$ is cooperative, and k is a nonnegative integer, then*

$$\rho^k(\mathcal{S}) \subseteq \mathcal{T}.$$

Hierarchically iterated robust closure and the lemma above are applied in Cor. 9.3 to obtain results on the propagation of commutativity in general groupoid diagrams, especially when the diagram has the scheme of a hypercube. (Definitions are given there.)

These results can be expressed in terms of Cayley color graphs [24]; see [16], [19], where the concept is used but not the name. Recall that the *Cayley color graph* of a group $\mathcal{A}$ with respect to a subset $\mathcal{S} \subseteq \mathcal{A}$ is the digraph $\Gamma(\mathcal{A}, \mathcal{S})$ with $\mathcal{A}$ as its vertex-set, where for vertices v, w, there is an arc $a = (v, w)$ if and only if there exists an element $u \in \mathcal{S}$ such that $w = v + u$. (We write "+" as we are only considering the Abelian groups underlying vector spaces.) The arc $a = (v, w)$ is *colored* by assigning to it the unique u satisfying the equation $w = v + u$, and so the Cayley color graph is the ordered pair consisting of the digraph and the corresponding arc colors.

Let $\mathcal{A}$ be the abelian group determined by $Z(G)$. Each element has order 2, so the Cayley color graph is symmetric and can be replaced by an undirected graph. To avoid loops, one assumes that the identity element of the group is not in the set $\mathcal{S}$. Plainly, $\mathcal{S}$ is spanning if and only if $\Gamma(\mathcal{A}, \mathcal{S})$ is connected; see, e.g., [19, Lemma 3.3].

Let G be a graph, let $\mathcal{S} \subseteq Cyc(G)$ with $span(\mathcal{S}) = Z(G)$. Let $\Gamma_{Cyc}(G, \mathcal{S})$ denote the subgraph of $\Gamma(\mathcal{A}, \mathcal{S})$ induced by $Cyc(G) \subseteq \mathcal{A}$. Given two cycles z, z' in G there is a z'-z-path in $\Gamma_{Cyc}(G, \mathcal{S})$ joining them if and only if there is a **wat** z-sequence σ in $\mathcal{S}$ and such that $z' = s_1$. It is convenient to take z' to be the null-set of edges which is 0 in the vector space. The set of all cycles z reachable from 0 by such paths is exactly the weak robust span of $\mathcal{S}$. The graph $\Gamma_{Cyc}(\mathcal{A}, \mathcal{S})$ is connected if and only if there is a $\Gamma_{Cyc}(\mathcal{A}, \mathcal{S})$-path joining 0 to each $z \in Cyc(G)$, that is, if and only if $\mathcal{S}$ is weakly robust.

The robust span of $\mathcal{S}$ is the set of all vertices in the connected component of 0 in the edge-induced subgraph $\Gamma'_{Cyc}(G, \mathcal{S})$ of $\Gamma_{Cyc}(G, \mathcal{S})$ determined by those edges zz' where $z||z'$. In this case, the partial sums are successively modified by replacing a single subpath P with a complementary path $P' = z - P$, where $P \subset z$, for z some member of $\mathcal{S}$.

Instead of robustness or weak robustness, given any family $\mathcal{S}$ of cycles which spans $Z(G)$, one might investigate the least number of connected components, the least maximum degree, or the least number of distinct topologies which occur along the $\Gamma_{Cyc}(\mathcal{A}, \mathcal{S})$ (or

$\Gamma'_{Cyc}(\mathcal{A}, \mathcal{S})$) paths from 0 to z, maximized or totaled over all cycles z in G. One may also focus only on some cycles which need to be built up. See [15] for an application to random networks and also Proposition 10.1 below.

5 Planarity and robustness

For plane graphs, we find a robust basis; cf. [3]. In the following, we do not distinguish between subgraphs of a plane graph and the corresponding subsets of the plane. Let G be a 2-connected plane graph. To prevent pathological embeddings, assume that each edge of G is embedded as a piecewise-linear curve (with a finite number of segments). Then $\mathbf{R}^2 \setminus G$ is a disjoint union of open disks and, using the 2-connectedness of the graph, the closure of each open disk is a closed disk and the boundary of each closed disk is a cycle.

Each plane graph contains exactly one unbounded region, containing points which are arbitrarily far from any point of the graph; all the other regions are called *bounded*. See, e.g., [1, Chap. 4]. Let $\mathcal{R}(G)$ denote the set of all bounded regions. Then

$$\mathcal{B}(G) := \{\partial r : r \in \mathcal{R}(G)\}$$

is a basis of the cycle space of G . It is spanning since if $z \in Cyc(G)$, then $z = \sum \partial r$, summing over all regions r contained within z. By Euler's Formula, the number of all regions is equal to $q - p + 2$, so $|\mathcal{B}(G)| = b(G)$ and hence $\mathcal{B}(G)$ is a cycle basis.

The *interior* $int(P)$ of a nontrivial path P is the path minus its two endpoints. A *topological path* is any curve which is homeomorphic to the closed unit interval $[0, 1]$. A subset of the plane is *topologically path-connected* if any two of its points can be joined by a topological path which is the union of a finite number of straight line segments.

Lemma 5.1. *For any* 2*-connected plane graph* G *and* $z \in Cyc(G)$*, either* $z = \partial r$ *for some* $r \in \mathcal{R}(G)$ *or there exists* $r \in \mathcal{R}(G)$ *such that* $z \cap \partial r$ *is a nontrivial path.*

Proof. Assume $z \neq \partial r$ for all $r \in \mathcal{R}(G)$. Then $\partial r \cap z$ is a disjoint union of paths, some of which can consist of an isolated vertex. Let $\mathcal{R}_1, \mathcal{R}_2$ denote the sets of regions r in $\mathcal{R}(G)$ which are contained within z and for which $H = \partial r \cap z$ has at most 1, resp. at least 2, connected components. If $\mathcal{R}_2$ is empty, then for each edge e in z, the unique region r whose intersection with z includes e must satisfy $\partial r \cap z$ is a nontrivial path. Suppose $\mathcal{R}_2$ is nonempty. If P is any path subgraph of z joining two successive components of $z \cap \partial r$, then r separates the interior of P from the interior of $z - P$ in the sense that any topological path in the plane joining a point in $int(P)$ to a point in $int(z - P)$ must intersect ∂r. See [1, Lemma 4.1.2] for a formal proof. Let P^* be a path subgraph of z joining two successive components and of minimum possible length. Then for any edge e in P^*, there exists a unique region r' for which $e \in E(z \cap \partial r')$. If $z \cap \partial r'$ is not contained in P^*, then the path-connected set r' would allow a topological path between some point in the interior of P^* and some point in the interior of $z - P^*$. But this is impossible by our remark about the separating property of r. Hence, $z \cap \partial r' \subseteq P^*$. By minimality of P^*, r' must intersect z in a single component. Since the intersection contains an edge, it is a nontrivial path. □

Theorem 5.2. *The basis* $\mathcal{R}(G)$ *of a* 2*-connected plane graph* G *is a robust basis; in fact, each cycle is the sum of a* **wai** *sequence with no repeated elements.*

Proof. Let t be the number of regions contained in z; we prove the result by induction on t. For the basis case, $t = 1$ and the **wai** sequence has the single term z. If $t > 1$, then by

Lemma 5.1, there is a region r in $\mathcal{R}(G)$ contained in z such that $\partial r \cap z$ is a nontrivial path. Hence,

$$z' = z + \partial r \in Cyc(G)$$

and z' contains every region contained in z except for r. By the induction hypothesis, there is an ordering of the regions contained in z' so that

$$z' = \partial r_1 + \partial r_2 + \cdots + \partial r_{t-1}$$

and the sequence $(\partial r_1, \ldots, \partial r_{t-1})$ is well-arranged w.r.t. intersection. Therefore, concatenating one more term, ∂r, gives the desired **wai** sequence of members of $\mathcal{R}(G)$. □

6 Geodesic cycles can build all cycles

If z is any cycle (or path), let $\ell(z) := length(z)$ denote the number of edges. The *girth* $g = g(G)$ of a graph G is the length of a shortest cycle [1, p. 8]. A subgraph $H \subseteq G$ is *geodesic* (or *isometric*) if for all v, w vertices of H, $dist_H(v, w) = dist_G(v, w)$, where for $a, b \in V(H)$, $dist_H(a, b)$ is the distance from vertex a to vertex b in H - i.e., the length of a shortest a-b-path in H (and similarly for G). In K_4, any 3-cycle is geodesic but any 4-cycle is not geodesic. It is clear that a cycle of minimum length is necessarily geodesic, so geodesic cycles always exist in any non-forest. The length of a geodesic cycle can't exceed twice the diameter by more than 1 as is shown in [1, proof of Prop. 1.3.2]. Hence, in a bipartite graph, every geodesic cycle has length at most twice the diameter.

Let $Geo(G)$ denote the set of geodesic cycles. We now show that for all graphs G, every cycle belongs to an iterated robust span of the geodesic cycles and bound the number of iterations needed; cf. [4].

Theorem 6.1. *For every graph G and every $z \in Cyc(G)$, one has*

$$z \in \rho^{\ell(z)-g(G)}(Geo(G)) \tag{6.1}$$

Proof. Suppose G is a fixed arbitrary graph and z is any cycle of G. We write $k = k(G, z) = \ell(z) - g(G)$. The statement of the theorem holds when $k = 0$ since then the cycle must be of minimum length, hence geodesic, and so in $\rho^0(Geo(G))$. Suppose the theorem holds when $k(G, z) < n$, where n is some positive integer and let z be a cycle of G with length $n + g(G)$ (i.e., so that $k(G, z) = n$). If z is not a geodesic cycle, there exist non-adjacent vertices v, w in z and a v-w-path P in G, intersecting z in exactly the two vertices v, w (and no edges) such that the length of P is less than the distance between v and w in z. Let P_1, P_2 be the two disjoint v-w-paths in z and put $z_i = P \cup P_i$, $i = 1, 2$. Since (z_1, z_2) is a **wai** sequence with sum z, $z \in \rho(\{z_1, z_2\})$. But both z_1 and z_2 are strictly shorter than z so by the inductive hypothesis, $z_1, z_2 \in \rho^{k'}(Geo(G))$, where

$$k' = \max\{\ell(z_1), \ell(z_2)\} - g(G) \leq k - 1.$$

Therefore, we have z in $\rho^{1+k'}(Geo(G)) \subseteq \rho^k(Geo(G))$ which completes the induction. □

Let $c(G)$ denote the *circumference* of G, which is the length of a longest cycle.

Corollary 6.2. *For every graph G, $Cyc(G) \subseteq \rho^{c(G)-g(G)}(Geo(G))$.*

7 Characterizing geodesic cycles in hypercubes

The *Cartesian product* $G \times G'$ of graphs $G = (V, E)$ and $G' = (V', E')$ is the graph

$$V(G \times G') = V \times V'$$

with $[(v, v'), (w, w')] \in E(G \times G')$ iff ($v = w \,\&\, [v', w'] \in E(G')$) or ($v' = w' \,\&\, [v, w] \in E(G)$) for all v, w in V and all v', w' in V'. The *distance lemma* (e.g., [10, p. 100]) generalizes this

$$d_{G\times G'}((v, v'), (w, w')) = d_G(v, w) + d_{G'}(v', w').$$

The *hypercube* Q_d is the d-fold Cartesian product of K_2 with itself; $V(Q_d) := \{0, 1\}^d$ consist of all binary d-tuples with $vw \in E(Q_d)$ exactly when v and w disagree in a unique coordinate. Thus, each edge is associated with a single *active* coordinate in $[d] := \{1, \ldots, d\}$. We interpret the vertices as vectors or strings as convenient. As Q_d is bipartite with diameter d, the longest geodesic cycle has length not exceeding $2d$.

Write u_j for the vertex with a single 1 in the j-th coordinate so that

$$u_j : [d] \to \mathbf{Z_2}$$

is the characteristic function of the singleton subset $\{j\}$ of $[d]$.

Two vertices are *antipodal* (in the d-cube) if they have distance d (i.e., each coordinate is changed); two *vertices* are *diametrically opposite* (in an even-length cycle) if they are at maximum distance within the subgraph determined by the cycle. Two *edges* are *diametrically opposite* (in an even-length cycle) if they correspond to diametrically opposite vertices in the corresponding line graph. Let $\bar{0} := (0, 0, \ldots, 0)$ be the 0-vector so $\bar{1} := (1, 1, \ldots, 1)$ is its antipode. The vertices determine a group under coordinatewise modulo-2 addition. In fact, Q_d is the Cayley color graph of $\mathbf{Z_2}^d$ w.r.t. $\{u_1, \ldots, u_d\}$.

The graph enjoys a nice property:

Homogeneity If $e = vw$ is any edge of Q_d, there is an automorphism of Q_d carrying vw to $\bar{0}u_1$ and there is a color-preserving automorphism of Q_d carrying vw to $\bar{0}u_r$, where r is the active coordinate in the edge vw.

Recall that a *walk* (of *length* k) in a graph G is a sequence $v_1, v_2, \ldots, v_{k+1}$ of vertices of G such that, for each j, $1 \le j \le k$, v_j and v_{j+1} are adjacent in G. The walk is *closed* if $v_1 = v_{k+1}$. For any length-k v-w-walk $\omega = (v = v_1, v_2, \ldots, v_{k+1} = w)$ in Q_d we define

$$\Phi(\omega) := (r_1, r_2, \ldots, r_k) \in [d]^k,$$

where r_j is the active coordinate for the j-th edge v_jv_{j+1} of the walk, $1 \le j \le k$. Conversely, given any vertex $v \in V(Q_d)$ and any k-tuple s in $[d]^k$, s $= (r_1, \ldots, r_k)$, there is a unique length-k walk ω as above with $v_1 = v$ such that $\Phi(\omega) =$ s. Indeed, take $v_2 = v_1 + u_{r_1}$, $v_3 = v_2 + u_{r_2}$, etc. Also, $\omega^{op} := (v_{k+1}, v_k, \ldots, v_1)$ is a w-v-path and $\Phi(\omega^{op})$ is the reversal of $\Phi(\omega)$. For each vertex v in the hypercube, there is a one-to-one length-preserving correspondence between walks starting at v and sequences in $[d]$.

By the distance lemma, a walk ω, in Q_d, corresponds to a geodesic path if and only if $\Phi(\omega)$ is a permutation of some nonempty subset of $[d]$. Hence, one can characterize walks in the hypercube which correspond to geodesic cycles. Let $a * b$ denote concatenation of strings a, b.

Theorem 7.1. *A walk ω in Q_d is a geodesic cycle if and only if $\exists S \subseteq [d]$, $|S| \geq 2$, such that*

$$\Phi(\omega) = \sigma * \sigma, \tag{7.1}$$

where σ is a string which is a permutation of the elements of S.

Proof. Let z be any geodesic cycle in Q_d. Then for any two diametrically opposite vertices v, w of z, the two distinct v-w-paths P, P' contained in z must both be geodesic paths. Let τ be the closed walk P' followed by P^{op}. By the remark above, both P and P' correspond to permutations of sets S, S', resp. But $S = S'$ as the walk is closed. Hence, any r in $[d]$ appears either twice or not at all in the string $\Phi(\tau)$. If the two appearances of some $r \in [d]$ were not at diametrically opposite edges e, e' of z, then there exist diametrically opposite vertices $v', w' \in V(z)$ such that both e and e' are contained in one of the two geodesic paths in z which is impossible.

Conversely, for a walk of length $2j$ satisfying the permutation condition, any subpath of length $\leq j$ corresponds to a permutation and is geodesic so the cycle is also geodesic. □

8 Building geodesic cycles in the hypercube

Since Q_d is bipartite, shortest cycles have length ≥ 4. For distinct i, j in $[d]$, the closed walk $\omega = (v, v + u_i, v + u_i + u_j, v + u_j, v)$ is a 4-cycle and $\Phi(\omega) = (i, j, i, j)$. Moreover, any 4-cycle, being geodesic, must be of this form. Call 4-cycles *squares* and let $Sq(Q_d)$ be the set of all squares in Q_d. For convenience, during this section we shall write ij for any square s such that $\Phi(s) = (i, j, i, j)$, for distinct $i, j \in [d]$.

We will use the notion of a *strip of squares* in some graph G. This is a sequence

$$\sigma = (s_1, s_2, \ldots, s_k), \tag{8.1}$$

where each s_j is a distinct 4-cycle subgraph of G, no two squares intersect unless one is the successor of the other, and for $2 \leq j \leq k-1$, s_j intersects its predecessor s_{j-1} and its successor s_{j+1} in disjoint, that is, *opposite*, edges. The *length* of a strip of squares is the number of squares in the sequence. If the length of a strip of squares σ as in (8.1) is $k \geq 2$, then we call the strip *nontrivial*.

For a nontrivial strip there are distinguished *start* and *stop* edges for σ: the start edge is the edge of s_1 opposite to the edge $s_1 \cap s_2$, and the stop edge is the edge of s_k opposite to the edge $s_{k-1} \cap s_k$. The *ties* of a nontrivial strip of squares is the set consisting of the start and stop edges, together with all *intersection edges* $s_j \cap s_{j+1}$, $1 \leq j \leq k-1$. Let $G(\sigma) := \bigcup_{j=1}^{k} s_j$ denote the *graph determined by the strip of squares*. A strip of length 1 has graph which is a 4-cycle and so is isomorphic to $Q_1 \times P_2$. The following results can be easily proved by induction on k.

Lemma 8.1. *Let σ be a strip of squares of length $k \geq 2$. Then (i) $G(\sigma)$ is isomorphic to $Q_1 \times P_{k+1}$, (ii) the ties of σ correspond to the edges $Q_1 \times v$ for $v \in V(P_{k+1})$, and (iii) the squares $s_1, \ldots, s_k$ are a robust basis of $G(\sigma)$.*

Lemma 8.2. *In a hypercube, all ties of a strip of squares have the same active coordinate.*

In the following, assume that $d \geq 3$, else everything is trivial. For $1 \leq j \leq d$, define strips σ_1^j in Q_d as follows. Let s_2 be the unique 12 square which contains $\bar{0}$, put $\sigma_1^1 = s_2$, and let $[u_2, u_2 + u_1]$ be the "stop edge" of s_2. For $2 \leq j \leq d$, having defined σ_1^{j-1} let

$$\sigma_1^j := \sigma_1^{j-1} * s_j,$$

where s_j is the unique $1j$ square through the vertex $u_2 + \cdots + u_{j-1}$. Let z_1^j be the sum of the squares in σ_1^j. that is,

$$z_1^j := s_2 + \cdots + s_j.$$

We prove by induction on j that

$$\Phi(z_1^j) = (1, \ldots, j, 1, j, j-1, \ldots, 2). \tag{8.2}$$

For $j = 2$ the result holds and for $2 \leq j \leq d-1$, adding s_{j+1} to z_1^j replaces the stop edge of σ_1^j with a sequence of three edges active consecutively in coordinates $j+1, 1, j+1$. Thus,

$$\Phi(z_1^{j+1}) = \Phi(z_1^j + s_{j+1}) = (1, \ldots, j+1, 1, j+1, j, j-1, \ldots, 2),$$

establishing the induction. Let $\sigma_1 = \sigma_1^d$ and put $z_1 := z_1^d$. Then

$$\Phi(z_1) = (1, 2, \ldots, d, 1, d, d-1, \ldots, 3, 2).$$

Thus, z_1 is the union of two paths P, P_1 from $\bar{0}$ to $\bar{1}$, where $\Phi(P) = (1, 2, \ldots, d)$ and $\Phi(P_1) = (2, \ldots, d, 1)$. Hence, z_1 is a cycle of length $2d$, but it is not geodesic since the intersection edges of σ_1 join non-adjacent vertices of z_1.

We now show that *every* geodesic cycle in Q_d is in the robust span of the set of squares by continuing the above construction.

Theorem 8.3. *Let z be any geodesic cycle of length $2j$ in Q_d, $d \geq 3$. Then there exists a* **wai** *z-sequence of distinct squares $z_1, z_2, \ldots, z_m$, where $m = j(j-1)/2$. So $z \in \rho(Sq(Q_d))$.*

Proof. By the homogeneity of Q_d, it suffices to show that, for each positive integer $d \geq 3$, the unique geodesic cycle z in Q_d through $\bar{0}$ with $\Phi(z) = (1, 2, \ldots, d, 1, 2, \ldots, d)$ is the sum of a sequence σ of squares which is well-arranged w.r.t. intersection.

For $1 \leq j \leq d-1$, let σ_j denote a sequence of squares of length $d - j$ of the form

$$\sigma_j := (j\ j+1, j\ j+2, \ldots, j\ d);$$

put $\sigma := \sigma_1 * \sigma_2 * \cdots * \sigma_{d-1}$, the concatenation. Each σ_j is a strip of squares, the strips are disjoint in the sense that no square belongs to more than one, and every square does belong to a strip. We will show that this sequence of squares is **wai** and has sum z. In the sequel, as we add squares to $z_1 = P \cup P_1$, the path P will be unchanged while the path P_1 will be successively transformed, but without changing its length.

The sequence $\sigma_2 = (s_{23}, s_{24}, \ldots, s_{2d})$ is defined as follows. Let s_{23} be the unique 23 square which includes the vertex $\bar{0}$; note that s_{23} intersects P_1 in the path

$$p_{1,23} = (\bar{0}, u_2, u_2 + u_3)$$

which are the first two edges of P_1. Adding s_{23} to z_1 produces a new cycle $z_{1,23}$ which is the union of P and the path $P_{1,23}$ obtained by replacing $(0, u_2, u_2+u_3)$ by $(0, u_3, u_2+u_3)$ in P_1. Now there is a unique 24 square s_{24} which includes the vertex u_3, and s_{24} intersects $P_{1,23}$ in the path

$$p_{1,24} = (u_3, u_2 + u_3, u_2 + u_3 + u_4)$$

which constitutes the second and third edges of $P_{1,23}$. Put $z_{1,24} = z_{1,23} + s_{24}$. Then $z_{1,24} = P \cup P_{1,24}$, where

$$P_{1,24} = (\bar{0}, u_3, u_3 + u_4, \sum_{j=2}^{4} u_j, \sum_{j=2}^{5} u_j, \ldots, \sum_{j=2}^{d} u_j, \sum_{j=1}^{d} u_j).$$

Now there is a unique 25 square through the vertex $u_3 + u_4$ intersecting $P_{1,24}$ in its third and fourth edges, and so forth up to $2d$. Adding the sum z_2' of the squares in σ_2 to z_1 produces

$$\Phi(z_2) = \Phi(z_1 + z_2') = (1, 2, \ldots, d, 1, 2, d, d-1, \ldots, 3).$$

Proceeding in this fashion, $z = z_d$, $\Phi(z) = (1, \ldots, d, 1, \ldots, d)$, and σ is a **wai** z-sequence as each square after those in the initial subsequence σ_1 intersects the sum of the preceding squares in a path of length 2. □

Thus, the set $Sq(Q_d)$ of all squares in Q_d robustly spans the set $Geo(Q_d)$ of all geodesic cycles in Q_d,

$$Geo(Q_d) \subseteq \rho(Sq(Q_d)).$$

But $Sq(Q_d)$ contains $d(d-1)2^{d-3}$ squares, which is asymptotically $d/4$ times the size of a cycle basis. Indeed, $b(Q_d) = 1 + d2^{d-1} - 2^d = 1 + (d-2)2^{d-1}$; e.g., [7, pp 22–23, 38–39].

In the graph $Q_1 \times G$, as $V(Q_1) = \{0, 1\}$, G is isomorphic to $0 \times G$ under the mapping

$$v \mapsto (0, v),$$

and we write $0 \times H$ for the image under this isomorphism of a subgraph H of G in $0 \times G$ and similarly for $1 \times H$. Also, if $e \in E(G)$, we write e for the corresponding subgraph (which is isomorphic to Q_1), and for $e \in E(G)$, each subgraph $Q_1 \times e$ in $Q_1 \times G$ is a 4-cycle.

We shall use the decomposition

$$E(Q_d) = E(Q_d^0) \cup E(Q_d^1) \cup E_d, \tag{8.3}$$

where

$$Q_d^0 = 0 \times Q_{d-1} \text{ and } Q_d^1 = 1 \times Q_{d-1}$$

denote the "bottom" and "top" subgraphs of the d-cube,

$$E_d := E(Q_d) \setminus (E(Q_d^0) \cup E(Q_d^1))$$

denotes the "side" edges of the d-cube. Similarly, there is a decomposition of the squares

$$Sq(Q_d) = \mathcal{S}_0 \cup \mathcal{S}_1 \cup Sq_d', \tag{8.4}$$

where $\mathcal{S}_j := Sq(Q_d^j)$ for $j = 0, 1$ and $Sq'_d = Sq(Q_d) \setminus (\mathcal{S}_0 \cup \mathcal{S}_1)$ (the set of all "side" squares of Q_d). Each element s in Sq'_d is the product of Q_1 with the unique edge e in $E(Q_{d-1})$ corresponding to the intersection of s with Q_d^0.

Now we can define the *Kainen basis* of Q_d [12]. Let $\mathcal{B}_K(Q_d)$ be the following recursively given collection of squares in Q_d. For $d = 0$ and $d = 1$, the set is empty, and $\mathcal{B}_K(Q_2) = \{Q_2\}$. Having defined $\mathcal{B}_K(Q_{d-1})$ for $d \geq 3$, put

$$\mathcal{B}_K(Q_d) := \mathcal{B}_K(Q_d^0) \cup Sq'_d, \tag{8.5}$$

where $\mathcal{B}_K(Q_d^0)$ is the cycle basis of Q_d^0 corresponding to $\mathcal{B}_K(Q_{d-1})$ under the natural isomorphism. For example, $\mathcal{B}_K(Q_4)$ consists of five squares in one of the 3-cube faces and all twelve side-squares.

Lemma 8.4. *The set $\mathcal{B}_K(Q_d)$ is a cycle basis for Q_d for $d \geq 2$.*

Proof. By induction on d; trivial for $d = 2$. Now let $d \geq 3$. Then $\mathcal{B}_K(Q_d)$ is independent since by induction the squares in $\mathcal{B}_K(Q_d^0)$ in the decomposition (8.5) are independent, while the squares in the sides, Sq'_d, are independent of those in the bottom and also of each other as each contains a unique edge in Q_{d-1}. But $\mathcal{B}_K(Q_d)$ has the cardinality of a basis (and so *is* a basis). Indeed, using (8.5), we have

$$|\mathcal{B}_K(Q_d)| = |\mathcal{B}_K(Q_{d-1}))| + |E(Q_{d-1})| = 1+(d-3)2^{d-2}+(d-1)2^{d-2} = 1+(d-2)2^{d-1},$$

where the second equality again uses the inductive assumption that $\mathcal{B}_K(Q_{d-1})$ is a basis. □

We now show that this basis is sufficient to robustly span every square in Q_d.

Theorem 8.5. *For $d \geq 2$, $Sq(Q_d) \subseteq \rho(\mathcal{B}_K(Q_d))$.*

Proof. As before, the basis case $d = 2$ is trivial. Every square in Q_d which is on the bottom is in $\rho(\mathcal{B}_K(Q_{d-1}))$ by the induction hypothesis and $\rho(\mathcal{B}_K(Q_{d-1})) \subseteq \rho(\mathcal{B}_K(Q_d))$ and the side-squares are in $\mathcal{B}_K(Q_d)$ by definition. Thus, we only need to take care of the squares s on the top; that is, $s \in \mathcal{S}_1 = Sq(Q_d^1)$. Each such square is contained in a unique subgraph Q_3 of Q_d such that $s = Q_3^1$. Let $s' = Q_3^0$. Then $s' \in Sq(Q_d^0)$ so by induction there exists σ' a **wai** s'-sequence in $\mathcal{B}_K(Q_{d-1}) \subseteq \mathcal{B}_K(Q_d)$, and we may append to this sequence the 4 side-squares of Q_3 (in any order) to produce a **wai** sequence in $\mathcal{B}_K(Q_d)$ which sums to s. □

Using Theorem 6.1, we see that for hypercubes every cycle is in the iterated robust closure of $\mathcal{B}_K(Q_d)$.

Corollary 8.6. *For $d \geq 2$ and z any cycle in Q_d, $z \in \rho^{\ell(z)-2}(\mathcal{B}_K(Q_d))$.*

9 Application to commutativity in groupoids

The original motivation for building up cycles using **wai** sequences was to show that *the commutativity of certain diagrams in a groupoid category can be inferred from the commutativity of a small minority of the faces of the diagram* [13]. After defining these terms, we use the results of the previous sections of this paper to prove the above claim and we show that a similar inference *cannot* be made using only **wat** sequences.

For a full discussion of the concept of a category, see, e.g., Mac Lane [17]. Briefly, a *(small) category* $\mathcal{C}$ consists of a set $Obj(\mathcal{C})$ of *objects*, a set $Mor(\mathcal{C})$ of *morphisms*, and two functions ("domain" and "co-domain")

$$dom, cod : Mor(\mathcal{C}) \to Obj(\mathcal{C}).$$

(A morphism thus models the notion of a structure-preserving function between two mathematical objects.) An ordered pair (a, b) of morphisms is called *composable* if $dom(b) = cod(a)$. In addition, $\mathcal{C}$ is required to have a *law of composition* which is a function from the set of composable morphism pairs to the set of morphisms denoted by

$$(a, b) \mapsto a\, b,$$

where $dom(a\, b) = dom(a)$ and $cod(a\, b) = cod(b)$. Often the notation for composition is in reverse order, emulating composition of functions.

It is convenient to write $\mathcal{C}(x, y)$ for the set of all morphisms in $\mathcal{C}$ with domain x and co-domain y, and $a : x \to y$ is another notation for $a \in \mathcal{C}(x, y)$ Typically, the set of morphisms between a given pair of objects has many elements but this set can also be empty or a singleton. For any triple x, y, z of objects of $\mathcal{C}$ the law of composition gives a function $\mathcal{C}(x, y) \times \mathcal{C}(y, z) \to \mathcal{C}(x, z)$.

Composition in a category $\mathcal{C}$ must be *associative*: if (a, b) and (b, c) are composable pairs of morphisms in $\mathcal{C}$, then (ab, c) and (a, bc) are both composable pairs and we require $(ab)c = a(bc)$. It follows by induction that one can define a unique composition $a_1 a_2 \cdots a_k$ for any finite sequence $(a_1, a_2, \ldots, a_k)$ of morphisms for which each successive pair (a_i, a_{i+1}) is composable. The final condition in the definition of a category $\mathcal{C}$ is that for each object x, there is an *identity morphism* $1_x \in \mathcal{C}(x, x)$ such that if x, y are any two objects in $\mathcal{C}$, then for any morphism a in $\mathcal{C}(x, y)$,

$$a 1_y = a = 1_x a.$$

Typical examples of a category are a set of topological spaces and continuous maps, or a set of vector spaces and linear maps, etc., provided that the composition of any two morphisms remains in the set, and of course that the identity morphisms are included.

A directed multigraph $D = (V, A, \Psi)$ is a set V of vertices, a set A of arcs, and a mapping $\Psi : A \to V \times V$ which associates to each arc an ordered pair (v, w), where v is its source and w its target. Arc-pairs (a, a') are *composable* if the target of a is the source of a' and an arc sequence is a *dipath* if each successive pair is composable. A v-w-dipath is a dipath where the first arc has source v and the last arc has target w. A *diagram* δ in a category $\mathcal{C}$ is a directed multigraph $D = (V, A, \Psi)$ (called the *scheme*) and functions f_V, f_A such that

$$f_V : V \to obj(\mathcal{C}), \quad f_A : A \to mor(\mathcal{C})$$

such that if $\Psi(a) = (v, w)$ and $f_V(v) = x, f_V(w) = y$, then $f_A(a) \in \mathcal{C}(x, y)$. Unless necessary, we shall not distinguish between vertices of the diagram and objects of the category, nor between arcs and morphisms. A dipath in the diagram corresponds to a composable sequence of morphisms in the category.

A *face* of the diagram is a distinct pair of v-w-dipaths, for some objects v, w, and this face is *commutative* if the two paths give rise to the same v-w-morphism, using the law of composition in the category. The face is said to "commute." The diagram is commutative if all of its faces commute.

The importance of commutative diagrams is that they can be used to define certain algebraic properties, such as co-associativity, and a basic theory of algebraic syntax can be built up from commutative diagrams - see, e.g., [18] and [5]. For an explicit recent example from computer science, see [21, p. 13, diagram for Remark 1.2.6].

A *groupoid* is a category $\mathcal{G}$ in which every morphism is invertible; i.e., for every morphism a, there is a unique morphism a^{-1} such that composition in either order gives 1. Groupoids are generalizations of ordinary groups and they have become of some interest in a number of fields recently, spreading far from initial applications in topology; see, e.g., [23].

A diagram in a groupoid $\mathcal{G}$ can be extended by including for every arc a in the diagram a reverse arc a^{op} and if the morphism α is assigned to a then in the extended diagram, the inverse morphism α^{-1} is assigned to a^{op}. If the original diagram commutes, then so does the extended diagram and the reverse implication is trivial. Hence, we shall assume that all diagrams in a groupoid category are symmetric.

Any undirected cycle in a groupoid diagram can be parsed into a closed walk by choosing one of the vertices and then proceeding either clockwise or counterclockwise. The cycle is said to *g-commute* if the composition along the walk is the identity; this does not depend on how the cycle is parsed; e.g., if $abc = 1$, then $1 = a^{-1}1a = bca$, etc. A diagram in a groupoid category will be called *g-commutative* if and only if all of its cycles g-commute.

Lemma 9.1. *If a groupoid diagram is g-commutative, then it is commutative.*

Proof. Suppose we have a g-commutative diagram. Let P, Q be two directed paths from v to w in the diagram and let α, β be the morphisms induced by P and Q, resp. It suffices to assume P and Q intersect only in v and w. Let z be the cycle formed by the union of P and Q, parsed to begin at v, follow P, and then Q (in reverse order). Then z induces $\alpha\beta^{-1}$. So the morphism induced by z is 1 if and only if $\alpha = \beta$. Thus, g-commutativity implies ordinary commutativity. □

The converse is false. For example, one can have four objects w, x, y, z with morphisms

$$f : w \to x,\ g : y \to x,\ h : y \to z,\ i : w \to z.$$

The corresponding diagram is automatically commutative because it has no faces but if it is in a groupoid it is g-commutative only if $fg^{-1}hi^{-1} = 1$, i.e., $fg^{-1} = ih^{-1}$.

Proposition 9.2. *The g-commutative cycles in a groupoid diagram form a cooperative set.*

Proof. Let z, z' be commutative cycles in a groupoid diagram, with $P = z \cap z'$ a nontrivial path from v to w. Let α be composition along P, β along $z - P$, and β' along $z' - P$. Then

$$\alpha\beta = 1_v \quad \text{and} \quad \alpha\beta' = 1_v$$

But also $\beta^{-1}\alpha^{-1} = 1_v$. Hence,

$$1_v = \beta^{-1}\alpha^{-1}\alpha\beta' = \beta^{-1}\beta'.$$

Thus, $z + z'$, which is the cycle $(z - P) \cup (z' - P)$ is commutative. □

Using Lemma 4.1 and Proposition 9.2, by Theorem 6.1 and Corollary 8.6, we get:

Corollary 9.3. *A diagram in a groupoid is g-commutative if its geodesic cycles g-commute. A Q_d-diagram in a groupoid is g-commutative if each $z \in \mathcal{B}_K(Q_d)$ g-commutes.*

The sum of two commutative cycles can be a noncommutative cycle when the cycles intersect in more than one nontrivial path. Label the vertices of a complete graph on 4 vertices A, B, C, D in clockwise order, where all vertices correspond to a single object X in the groupoid, and take all morphisms to be 1_X, except for the two diagonal arcs from C to A and from D to B, respectively, which represent some morphism $x : X \to X$, where $x^2 \neq 1_X$. (This is a mild condition on the groupoid, satisfied for instance by the symmetric group S_n for $n \geq 3$.) Consider the two squares A, B, C, D and A, B, D, C; going around the first square gives a 4-fold composition of identities, while the second square gives

$$x \circ 1_X \circ x^{-1} \circ 1_X = 1_X.$$

The two squares intersect in edges AB and CD and the sum of the two squares is the square A, D, B, C which induces the morphism $x^2 \neq 1_X$. Thus, propagation of commutativity to the sum can fail when cycles intersect in two disjoint paths.

10 Remarks

This section is essentially an appendix which compares other types of cycle basis to the one we have constructed. We thank one of the referees for suggesting some of the references given here and for other helpful comments.

A general method for generating cycle bases for Cartesian product graphs, was described by Hammack [6]. It constructs a cycle basis for $G \times G'$ for each pair of graphs G, G' with given spanning trees T, T' and cycle bases $\mathcal{B}, \mathcal{B}'$ for G, G', resp. Then there is a *Hammack basis*

$$\mathcal{B}_H(G, \mathcal{B}, T, G', \mathcal{B}', T') := \mathcal{F} \cup \mathcal{G} \cup \mathcal{G}', \text{ where}$$

$$\mathcal{F} := \{e \times e' : e \in E(T), e' \in E(T')\},$$
$$\mathcal{G} := \{z \times y : z \in \mathcal{B}, y \in V(G')\},$$
$$\mathcal{G}' := \{x \times z' : x \in V(G), z' \in \mathcal{B}'\}.$$

For the hypercube Q_d, it is reasonable to take $G = Q_{d-1}$ and $G' = Q_1$. So $T' = Q_1$ but T could be an arbitrary spanning tree of Q_{d-1}. There is a natural recursive choice for T a spanning tree of Q_{d-1} when $d \geq 3$: just keep adding in all of the "side" edges. For example, the spanning tree constructed for Q_3 looks like a "U" standing on 4 legs. If $d \geq 4$, the Hammack bases (allowing all possible spanning trees) do not include the Kainen basis since the latter uses only bottom squares, but all $(d-1)2^{d-2}$ side-squares, while the former type use both top and bottom squares but only $2^{d-1} - 1$ side squares. For example, $\mathcal{B}_H(Q_4)$ will have 5 squares in both Q_4^0 and Q_4^1 but only 7 side-squares (cf. supra Lemma 8.4). Other iterated Cartesian product graphs have been explicitly studied, e.g., in [9].

Minimum cycle bases (in the sense of having minimum total sum of the lengths of their members) for Cartesian product graphs were constructed by Imrich and Stadler [11]. Like the Hammack bases, these depend on a spanning tree and a cycle basis, and also a vertex, for each factor of a Cartesian product. The *Imrich-Stadler basis* for $G \times G'$ has the form

$$\mathcal{B}_{IS}(G, \mathcal{B}, T, x, G', \mathcal{B}', T', y) := \mathcal{B}_\square \cup \mathcal{G}^y \cup \mathcal{G}'_x,$$

where $\mathcal{B}, T, x$ ($\mathcal{B}', T', y$) are cycle basis, spanning tree, and vertex for G (and for G', resp.), and

$$\mathcal{B}_\square := \{e \times e' : e \in E(T), e' \in E(G')\} \cup \{e \times e' : e \in E(G), e' \in E(T')\},$$

$$\mathcal{G}^y := \{z \times y : z \in \mathcal{B}\},$$
$$\mathcal{G}'_x := \{x \times z' : z' \in \mathcal{B}'\}.$$

If $G' = T'$, then $\mathcal{B}_\square = \{e \times e' : e \in E(G), e' \in E(T')\}$ and $\mathcal{G}'_x$ is empty. So $\mathcal{B}_{IS}(G \times T')$ doesn't depend on T or x. When $G = Q_{d-1}$ and $T = Q_1$, this is just $\mathcal{B}_K(Q_d)$ so $\mathcal{B}_{IS}$ is a generalization of $\mathcal{B}_K$. When the cycle bases of G and G' are minimal and both G and G' have girth at least 4, [11, Thm. 8] shows that $\mathcal{B}_{IS}(G \times G')$ is minimal as well. When G' is a tree, we can get a robustness-related result. In the graph $G \times T, T$ a tree, a cycle z contained in a subgraph $G \times v'$ for some $v' \in V(T)$ is called a *level cycle*.

Proposition 10.1. *Let G be a graph with spanning tree T, cycle-basis $\mathcal{B}$, and $v_0 \in V(G)$. If $\mathcal{B}$ is robust, then every level-cycle in $T \times G$ is in the robust span of the basis $\mathcal{B}'$*

$$\mathcal{B}' := \{v_0 \times z : z \in \mathcal{B}\} \cup \{e' \times e : e' \in E(T),\ e \in E(G)\}.$$

Proof. Suppose that T has t vertices and G has p vertices and q edges. Then it is easy to calculate that $b(T \times G) = qt - p + 1 = |\mathcal{B}'|$ and $\mathcal{B}'$ is a basis by [11, Thm. 4].

We prove the robustness claim by induction on the number t of vertices of T. When $t = 1$, $\mathcal{B}' = \mathcal{B}$ so the result holds. Let T be any tree with t vertices, $t > 1$. Then there is a vertex v in T such that v has degree 1. Let $T' = T - v$. Now consider any level cycle z' in $T \times G$. If z' is in $w \times G$ for some $w \in V(T')$, then by the induction hypothesis, there is a z'-sequence from $\mathcal{B}'$ which is **wai**. Otherwise, if z' is contained in $v \times G$, then $z' = v \times z$ for some $z \in Cyc(G)$. Let v' be the unique vertex of T which is adjacent to v. For the cycle $z'' = v' \times z$, there is a z''-sequence σ' from $\mathcal{B}'$ which is **wai**. If the edges in z are followed in a closed walk, say $e_1, \ldots, e_k$ and if $e = vv'$, then

$$\sigma = \sigma' * (e \times e_1, \ldots, e \times e_k)$$

is a **wai** sequence from $\mathcal{B}'$ which sums to z'. □

Recently, convex cycle bases were studied in [8]. A subgraph H of G is *convex* if for every minimal-length G-path P between vertices $v, w \in V(H)$, P is a subgraph of H. In a hypercube, squares are convex subgraphs. The Imrich-Stadler basis for $G \times H$ is convex when the bases for G and H are convex. They note that the only convex cycles in a hypercube are the squares. Geodesic cycles in the hypercubes are a special case of *planar partial cube* and the latter have been characterized in [20].

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ARS MATHEMATICA CONTEMPORANEA

Also available at http://amc-journal.eu
ISSN 1855-3966 (printed edn.), ISSN 1855-3974 (electronic edn.)
ARS MATHEMATICA CONTEMPORANEA 9 (2015) 45–50

Comparing the irregularity and the total irregularity of graphs

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Received 9 June 2012, accepted 15 August 2013, published online 13 June 2014

Abstract

Albertson [4] has defined the *irregularity* of a simple undirected graph G as $\operatorname{irr}(G) = \sum_{uv \in E(G)} |d_G(u) - d_G(v)|$, where $d_G(u)$ denotes the degree of a vertex $u \in V(G)$. Recently, in [1] a new measure of irregularity of a graph, so-called the *total irregularity*, was defined as $\operatorname{irr}_t(G) = \frac{1}{2} \sum_{u,v \in V(G)} |d_G(u) - d_G(v)|$. Here, we compare the irregularity and the total irregularity of graphs. For a connected graph G with n vertices, we show that $\operatorname{irr}_t(G) \leq n^2 \operatorname{irr}(G)/4$. Moreover, if G is a tree, then $\operatorname{irr}_t(G) \leq (n-2)\operatorname{irr}(G)$.

Keywords: The irregularity of graph, the total irregularity of graph.

Math. Subj. Class.: 05C05, 05C07,05C99

1 Introduction

Let G be a simple undirected graph of order $n = |V(G)|$ and size $m = |E(G)|$. For $v \in V(G)$, the degree of v, denoted by $d_G(v)$, is the number of edges incident to v. Albertson [4] defines the *imbalance* of an edge $e = uv \in E(G)$ as $\operatorname{imb}_G(uv) = |d_G(u) - d_G(v)|$ and the *irregularity* of G as

$$\operatorname{irr}(G) = \sum_{uv \in E(G)} \operatorname{imb}_G(uv). \tag{1.1}$$

Obviously, a connected graph G has irregularity zero if and only if G is regular. In [4] Albertson presented upper bounds on irregularity for bipartite graphs, triangle-free graphs

*Partially supported by Slovenian ARRS Program P1-00383 and Creative Core - FISNM - 3330-13-500033.

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and arbitrary graphs, as well as a sharp upper bound for trees. Some results about the irregularity of bipartite graphs are given in [4, 14]. Related to the work of Albertson is the work of Hansen and Mélot [13], who characterized the graphs with n vertices and m edges with maximal irregularity. Various upper bounds on the irregularity of a graph were given in [19], where K_{r+1}-free graphs, trees and unicyclic graphs with fixed number of vertices of degree one were considered. In [16], relations between the irregularity and the matching number of trees and unicyclic graphs were investigated. More results on irregularity, imbalance and related measures, one can find in [3, 5, 6, 17, 18].

Recently, in [1] a new measure of irregularity of a simple undirected graph, so-called the *total irregularity*, was defined as

$$\mathrm{irr}_t(G) = \frac{1}{2} \sum_{u,v \in V(G)} |d_G(u) - d_G(v)| \,. \tag{1.2}$$

Other approaches, that characterize how irregular a graph is, have been proposed [2, 3, 7, 8, 9, 10, 15]. In this paper, we focus on the relation between the irregularity (1.1) and the total irregularity (1.2) of a graph.

In the sequel we introduce the notation used in the rest of the paper. For $u, v \in V(G)$, we denote by $d_G(u, v)$ the length of a shortest path in G between u and v. In this short paper the notation of the sets, that will be defined next, is always regarding the graph G we consider. By $V_{a,b}$, we denote a set of vertices of a graph with degrees in $[a, b]$, and by $V_{\geq a}$ (resp. $V_{\leq a}$), we denote a set of vertices of a graph with degrees at least a (resp. with degrees at most a). Similarly, by $V^x_{\geq a}$ (resp. $V^x_{\leq a}$), we denote a set of neighboring vertices of a vertex x with degrees at least a (resp. with degrees at most a). The corresponding cardinalities of the above mentioned sets, we denote by small v (e.g., $v_{\leq a} = |V_{\leq a}|$ or $v^x_{\leq a} = |V^x_{\leq a}|$).

A subgraph $T = v_1 v_2 \cdots v_l$ of a graph G, where v_l is a leaf in G, is called a *tread* if $d_G(v_1) = d_G(v_2) = \cdots = d_G(v_{l-1}) = 2$, and v_1 is adjacent to a vertex with degree at least three. Let $T_1 = v_1 v_2 \cdots v_s$ and $T_2 = u_1 u_2 \cdots u_l$ be two threads of a graph G with leaves v_s and u_l, respectively, and let v_0 be the other neighbour of v_1. By $G' = G(T_2 \circ T_1)$ we denote a graph that is obtained from G after a *concatenation* of T_2 to T_1, i.e., after deleting the edge $v_0 v_1$ and adding an edge between u_l and v_1.

2 General graphs

Obviously, $\mathrm{irr}(G) \leq \mathrm{irr}_t(G)$. And, it is not hard to show that equality holds precisely when all non-adjacent vertices have same degree. Such a class of graphs are the complete k-partite graphs. More examples of graphs with equal irregularity and total irregularity can be found in [11]. Now, we give an upper bound on $\mathrm{irr}_t(G)$ in term of $\mathrm{irr}(G)$.

Theorem 2.1. Let G be a connected graph on n-vertices. Then

$$\mathrm{irr}_t(G) \leq \frac{n^2}{4} \mathrm{irr}(G).$$

Moreover, the bound is sharp for infinitely many graphs.

Proof. Let T be a spanning tree of G. Then, any two vertices a, b of G are connected by an unique path $P_{ab} = x_1 x_2 \cdots x_s$ in T, where $x_1 = a$ and $x_s = b$. By the triangle inequality, we have that

$$\begin{aligned} \operatorname{irr}_t(G) &= \frac{1}{2}\sum_{a,b\in V(G)} |d_G(a)-d_G(b)| \\ &\le \frac{1}{2}\sum_{a,b\in V(G)} |d_G(x_1)-d_G(x_2)| + |d_G(x_2)-d_G(x_3)| + \cdots \end{aligned} \tag{2.1}$$

For an edge $uv \in E(T)$, let $n_u = \{x \,|\, x \in V(T) \text{ and } d_T(x,u) < d_T(x,v)\}$. Similarly, let $n_v = \{x \,|\, x \in V(T) \text{ and } d_T(x,u) > d_T(x,v)\}$. Each summand $|d_G(u) - d_G(v)|$ in the last sum of (2.1) occurs in the sum exactly $n_{uv} = n_u n_v$ times. Also, each summand $|d_G(v) - d_G(u)|$ occurs n_{uv} times. Thus,

$$\operatorname{irr}_t(G) \le \sum_{uv\in E(T)} |d_G(u)-d_G(v)| n_{uv}.$$

As $n_{uv} \le (n/2)(n/2) = n^2/4$, and $\sum_{uv\in E(T)} |d_G(u) - d_G(v)| \le \sum_{uv\in E(G)} |d_G(u) - d_G(v)|$, we obtain the desired inequality.

Now, we show that the bound $n^2/4$ is sharp. Let a, b be two distinct integers, say $a < b$. Consider a graph G_a whose all vertices are of degree a, with exception of one vertex u which is of degree $a - 1$. Similarly, consider a graph G_b whose all vertices are of degree b, with exception of one vertex u which is of degree $b - 1$. Let G^* be the graph obtained from G_a and G_b by connecting u and v. Let $n_a = |V(G_a)|$ and $n_b = |V(G_b)|$. Observe that $\operatorname{irr}(G^*) = b - a$ and $\operatorname{irr}_t(G^*) = (b - a)n_a n_b$. Choosing $n_a = n_b = n/2$, we obtain

$$\frac{\operatorname{irr}_t(G^*)}{\operatorname{irr}(G^*)} = n_a n_b = \frac{n^2}{4}.$$

In order to show that such graphs G_a and G_b exist, one may use the theorem of Erdős-Gallai [12] which states that a sequence $d_1 \ge d_2 \ge \cdots \ge d_n$ of non-negative integers with even sum is graphic (i.e., there exist a graph with such a degree sequence) if and only if

$$\sum_{i=1}^{r} d_i \le r(r-1) + \sum_{i=r+1}^{n} \min(r, d_i), \tag{2.2}$$

for all $1 \le r \le n$.

So, fix a, b, and $n_a = n_b$ to be odd numbers with $n_a \gg \max\{a, b\}$. We will show the existence of the graph G_a. In a similar way, one can show the existence of the graph G_b. As $(n_a - 1)a + (a - 1)$ is even, the parity condition of the theorem of Erdős-Gallai is satisfied. So, we need to show only (2.2). For this we consider three cases regarding r and a:

- $r \le a - 1$. Then, (2.2) can be written as $ra \le r(r - 1) + (n_a - r)r$. It obviously holds since $a \ll n_a - r$.
- $r = a$. In this case, (2.2) can be written as $ra \le r(r - 1) + (n_a - r)r - 1$, which holds for a similar reason as the previous case.
- $r \ge a + 1$. Similarly, (2.2) can be written as $ra \le r(r - 1) + (n_a - r)a - 1$, and it holds as $ra \ll r(r - 1)$.

□

3 Trees

In this section, we give an upper bound on $\mathrm{irr}_t(G)$ in term of $\mathrm{irr}(G)$, when G is a tree. To show the bound, we will use the following lemma.

Lemma 3.1. *Let G be a tree, x a vertex of degree $d \geq 3$ incident with threads T_1 and T_2, and let $G' = G(T_2 \circ T_1)$. Then,*

(a) $\mathrm{irr}_t(G) - \mathrm{irr}_t(G') = 2v_{2,d-1}$*;*

(b) $\mathrm{irr}(G) - \mathrm{irr}(G') = 2(d - v^x_{\geq d} - 1)$.

Proof. Let $T_1 = a_1 a_2 \cdots a_s$ and $T_2 = b_1 b_2 \cdots b_l$. We consider the identities separately.

(a) Notice that all other vertices except x and b_l have the same degree in G and G'. Hence, it holds that

$$\begin{aligned}\mathrm{irr}_t(G) - \mathrm{irr}_t(G') &= \sum_{u \neq b_l} (|d_G(x) - d_G(u)| - |d_{G'}(x) - d_{G'}(u)|) \\ &\quad + \sum_{u \neq x} (|d_G(u) - d_G(b_l)| - |d_{G'}(u) - d_{G'}(b_l)|) \\ &\quad + |d_G(x) - d_G(b_l)| - |d_{G'}(x) - d_{G'}(b_l)|.\end{aligned}$$

Since $d_{G'}(x) = d_G(x) - 1 = d - 1$ and $d_{G'}(b_l) = d_G(b_l) + 1 = 2$, further we have

$$\begin{aligned}\mathrm{irr}_t(G) - \mathrm{irr}_t(G') &= \sum_{u \neq b_l} (|d - d_G(u)| - |d - 1 - d_G(u)|) \\ &\quad + \sum_{u \neq x} (|d_G(u) - 1| - |d_G(u) - 2|) + 2. \qquad (3.1)\end{aligned}$$

If $u \in V_{\leq d-1}$, then $|d - d_G(u)| - |d - 1 - d_G(u)| = 1$, otherwise $|d - d_G(u)| - |d - 1 - d_G(u)| = -1$. Hence, the first sum in (3.1) is equal to $v_{\leq d-1} - 1 - v_{\geq d}$. Similarly, if $u \in V_{\geq 2}$, then $|d_G(u) - 1| - |d_G(u) - 2| = 1$, otherwise $|d_G(u) - 1| - |d_G(u) - 2| = -1$. Thus, the second sum in (3.1) is equal to $v_{\geq 2} - 1 - v_1$. Applying these observations, we have

$$\begin{aligned}\mathrm{irr}_t(G) - \mathrm{irr}_t(G') &= v_{\leq d-1} - 1 - v_{\geq d} + v_{\geq 2} - 1 - v_1 + 2 \\ &= v_{\leq d-1} - v_1 + v_{\geq 2} - v_{\geq d} \\ &= 2v_{2,d-1}.\end{aligned}$$

(b) Let $e_1 = xa_1$, $e_2 = xb_1$, $e_3 = b_{l-1}b_l$ and $E_1 = \{e_1, e_2, e_3\}$. Denote by E_2 the set of edges incident to x that are different from e_1 and e_2. Notice that every edge not in $E_1 \cup E_2$ contributes zero to the difference $\mathrm{irr}(G) - \mathrm{irr}(G')$. So, we can infer

$$\begin{aligned}\mathrm{irr}(G) - \mathrm{irr}(G') &= \sum_{uv \in E_2} (\mathrm{imb}_G(uv) - \mathrm{imb}_{G'}(uv)) \\ &\quad + \sum_{uv \in E_1} (\mathrm{imb}_G(uv) - \mathrm{imb}_{G'}(uv)).\end{aligned}$$

Notice that the first sum is equal to $-v^x_{\geq d} + (v^x_{\leq d-1} - 2)$ (we have -2 as the edges e_1 and e_2 are excluded in this sum). In G', the edge $e_1 = xa_1$ does not exist anymore, but there is a new edge $e'_1 = b_l a_1$. Observe that after the concatenation $T_2 \circ T_1$ all other edges preserve their end-vertices. First, we consider the contribution of e_1 and e'_1 in $\mathrm{irr}(G) - \mathrm{irr}(G')$. There are two possibilities regarding the length of T_1:

- $s = 1$: Then, $\mathrm{imb}_G(e_1) = d - 1$ and $\mathrm{imb}_{G'}(e'_1) = 1$;
- $s \geq 2$: In this case, $\mathrm{imb}_G(e_1) = d - 2$ and $\mathrm{imb}_{G'}(e'_1) = 0$.

In both of them, we obtain $\mathrm{imb}_G(e_1) - \mathrm{imb}_{G'}(e'_1) = d - 2$.

Next, we consider the contributions of e_2 and e_3 together. Again, consider two possibilities regarding the length of T_2:

- $l = 1$: Then, $e_2 = e_3$ and $\mathrm{imb}_G(e_2) = d - 1$ and $\mathrm{imb}_{G'}(e_2) = d - 3$;
- $l \geq 2$: In this case, $e_2 \neq e_3$, and $\mathrm{imb}_G(e_2) = d - 2$, $\mathrm{imb}_{G'}(e_2) = d - 3$, $\mathrm{imb}_G(e_3) = 1$ and $\mathrm{imb}_{G'}(e_3) = 0$.

In both cases, we obtain that $\sum_{e \in \{e_2, e_3\}} (\mathrm{imb}_G(e) - \mathrm{imb}_{G'}(e)) = 2$. So finally, we have that

$$\begin{aligned} \mathrm{irr}(G) - \mathrm{irr}(G') &= -v^x_{\geq d} + (v^x_{\leq d-1} - 2) + d - 2 + 2 \\ &= -v^x_{\geq d} + v^x_{\leq d-1} - 2 + d \\ &= 2(d - v^x_{\geq d} - 1). \end{aligned}$$

□

Theorem 3.1. Let G be a tree with n vertices. Then

$$\mathrm{irr}_t(G) \leq (n-2)\mathrm{irr}(G).$$

Moroever, equality holds if and only if G is a path.

Proof. Let $n_1(G)$ be the number of vertices of G with degree one. We will prove the second inequality by induction on $n_1(G)$. If $n_1(G) = 0$, then $G \simeq P_1$, $\mathrm{irr}(G) = \mathrm{irr}_t(G) = 0$, and the equality in the theorem holds. Since G is a tree, $n_1(G) \neq 1$. If $n_1(G) = 2$, then $G \simeq P_n$. In this case $\mathrm{irr}(G) = 2$ and $\mathrm{irr}_t(G) = 2(n-2)$, hence we obtain equality.

Now, assume $n_1(G) > 2$. Then, it is easy to see that G has a vertex x of degree $d \geq 3$, incident with at least two threads T_1 and T_2. Let $G' = G(T_2 \circ T_1)$. Since $n_1(G') = n_1(G) - 1$, we can assume that inequality holds for G', i.e.,

$$\mathrm{irr}_t(G') \leq (n-2)\mathrm{irr}(G'). \tag{3.2}$$

By Lemma 3.1, we have

$$\mathrm{irr}(G') = \mathrm{irr}(G) - 2(d - v^x_{\geq d} - 1) \qquad \text{and} \qquad \mathrm{irr}_t(G') = \mathrm{irr}_t(G) - 2v_{2,d-1}. \tag{3.3}$$

Plugging (3.3) in (3.2), we obtain

$$(n-2)\mathrm{irr}(G) \quad \geq \quad \mathrm{irr}_t(G) - 2v_{2,d-1} + 2(n-2)(d - v^x_{\geq d} - 1). \tag{3.4}$$

As $d(x) = d \geq 3$ and x is incident with two threads, we infer $v^x_{\geq d} + 2 \leq d$, and so $2(d - v^x_{\geq d} - 1) \geq 2$. Observe also that $v_{2,d-1} \leq n - 3$. Hence $2(n-2)(d - v^x_{\geq d} - 1) > 2(n-3) \geq 2v_{2,d-1}$. This together with (3.4) gives $(n-2)\mathrm{irr}(G) > \mathrm{irr}_t(G)$. □

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ARS MATHEMATICA CONTEMPORANEA

Also available at http://amc-journal.eu
ISSN 1855-3966 (printed edn.), ISSN 1855-3974 (electronic edn.)
ARS MATHEMATICA CONTEMPORANEA 9 (2015) 51–61

Maximum genus, connectivity, and Nebeský's Theorem

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Received 19 July 2012, accepted 10 October 2013, published online 20 June 2014

Abstract

We prove lower bounds on the maximum genus of a graph in terms of its connectivity and Betti number (cycle rank). These bounds are tight for all possible values of edge-connectivity and vertex-connectivity and for both simple and non-simple graphs. The use of Nebeský's characterization of maximum genus gives us both shorter proofs and a description of extremal graphs. An additional application of our method shows that the maximum genus is almost additive over the edge cuts.

Keywords: Maximum genus, Nebeský's theorem, Betti number, cycle rank, connectivity.

Math. Subj. Class.: 05C10

1 Introduction

We study cellular embeddings of graphs in orientable surfaces of large genus. Suppose that G is a connected graph with v vertices and e edges embedded with f faces on an orientable surface of genus g, denoted here by S_g. Our goal is to find the *maximum genus* of G,

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$\gamma_M(G)$, which is the maximum value of g over all cellular embeddings of G. The Euler-Poincaré formula asserts that $v - e + f = 2 - 2g$. By combining this formula with the formula $\beta(G) = 1 - v + e$ for the *Betti number* (cycle rank) of G we get $f - 1 = \beta - 2g$. Since the maximum genus g corresponds to the minimum number of faces f and since every embedding has at least one face, $f - 1$ (or equivalently $\beta - 2g$) measures how far the embedding is from the extreme of only one face. We call this quantity the *deficiency* ξ of the embedding. The *deficiency of* G, $\xi(G)$, is the minimum deficiency over all possible embeddings, and so $\xi(G) = \beta(G) - 2\gamma_M(G)$. Note that $\xi(G)$ always has the same parity as $\beta(G)$. In particular, an embedding of a graph with odd Betti number always has at least two faces and hence deficiency at least one. A graph with deficiency 0 or 1 is called *upper embeddable*. For more background on the maximum genus we refer the reader to [3].

In this paper we give lower bounds N on the maximum genus of graphs in various classes $\mathcal{C}$ defined by the graph's Betti number and edge-connectivity. The graphs are allowed to have loops and multiple edges; if these are forbidden we say that the graph is *simple*. Our results can be rephrased as *"Every graph in $\mathcal{C}$ has an embedding in a surface with at least N handles"*. We also get analogous results for Betti number and vertex-connectivity. These classes have been examined before [2, 4, 5, 7, 8, 11], nevertheless, we present a unified approach with new, much shorter proofs, providing additional information about the structure of the extremal graphs.

In addition to the bounds on the maximum genus of a graph in terms of its connectivity, we explore the relationship between the maximum genus of a graph and the maximum genera of components resulting from the removal of an edge-cut. By an *edge-cut* in a graph G we mean a minimal set F of edges that separate two sets of vertices V_1 and V_2 forming a partition $\{V_1, V_2\}$ of the vertex-set of G; every edge from F has precisely one vertex in each of V_1 and V_2. We show that the maximum genus of a graph is almost additive over its edge-cuts (in a sense to be made clear in Section 3). We also investigate a related question asking for the smallest size of an edge-cut that can separate a non-upper-embeddable component.

The primary tool used in this paper is Nebeský's Theorem [12], stated below as Theorem 1.1 (or its equivalent version due to Khomenko and Glukhov [9] stated later in Equation 3.4). Let G be a connected graph and let A be a set of its edges. Let $\mathrm{oc}(G - A)$ denote the number of components of $G - A$ with odd Betti number and $\mathrm{ec}(G - A)$ the number of those with even Betti number. Let

$$\nu(G, A) = \mathrm{ec}(G - A) + 2\mathrm{oc}(G - A) - |A| - 1.$$

Theorem 1.1 (Nebeský [12]). *The deficiency of a graph G is given by*

$$\xi(G) = \max\{\nu(G, A) \ : \ A \subseteq E(G)\}.$$

The idea behind Nebeský's Theorem is easy and its understanding is important to this paper. Consider a maximum-genus embedding of G with f faces. Delete the edges of A one at a time. If deleting an edge increases the number of components, then we embed each new component on its own connected surface. With this convention, deleting an edge can increase the total number of faces, but by at most one. Hence the embedding of $G - A$ has at most $f + |A|$ faces. If a component of $G - A$ has odd Betti number, then its surface has at least two faces. If the Betti number is even, then its surface has at least one face. It follows that $f + |A| \geq \mathrm{ec}(G - A) + 2\mathrm{oc}(G - A)$, and so $\xi(G) = f - 1 \geq \nu(G, A)$. Nebeský's Theorem also asserts that there is a set A of edges in a maximum genus embedding that reverses the steps just described; the proof in this direction is more difficult.

Previous results relating maximum genus, connectivity, and the Betti number mostly used Xuong's Theorem [15] to determine the maximum genus. It is our use of Nebeský's alternative characterization [12] that gives us new insights and shorter proofs; these techniques were first introduced in an earlier unpublished version of this paper (1996).

2 Graphs with given connectivity

In this section we state and prove our results giving upper bounds on the deficiency of graphs with a particular Betti number and connectivity. We then translate these results to give lower bounds on the maximum genus. We need several observations before proceeding.

First, note that if G has a vertex v of degree one, then $\gamma_M(G-v) = \gamma_M(G)$. If v is of degree two, then we can replace its two incident edges with a single edge joining the other two incident vertices. This does not change the maximum genus, but can turn a simple graph into a non-simple one. Since simple and non-simple graphs behave differently, the statements of our results are cleaner if we forbid degree two vertices. Hence we require that our graphs are of minimum degree at least three.

The proofs involve calculating $\nu(G, A)$ and bounding $\beta(G)$ from below. Different types of components in $G - A$ will play different roles. Let $c_0 = c_0(G - A)$ be the number of components in $G - A$ with Betti number zero. Let c_1 denote those components with Betti number one. Let c_2 be the number of components with even Betti number at least 2, and c_3 be the number of components with odd Betti number at least 3. With this notation observe that

$$\nu(G, A) = c_0 + 2c_1 + c_2 + 2c_3 - |A| - 1. \tag{2.1}$$

To bound the Betti number of G, first note that $G - A$ has components whose Betti numbers sum to at least $c_1 + 2c_2 + 3c_3$. Adding in $c_0 + c_1 + c_2 + c_3 - 1$ edges from A, we can build a connected graph whose Betti number is also at least $c_1 + 2c_2 + 3c_3$. Adding the remaining edges in A increases the Betti number by $|A| - (c_0 + c_1 + c_2 + c_3 - 1)$. Hence

$$\beta(G) \geq -c_0 + c_2 + 2c_3 + |A| + 1. \tag{2.2}$$

We are now ready for the first version of our main result.

Theorem 2.1. *Let G be a graph of minimum degree at least three. Then upper bounds on the deficiency $\xi(G)$ are given in following table. The rows correspond to edge-connectivity $k = 1, 2, 3$, or ≥ 4. The same bounds hold where k is the vertex-connectivity and are achieved by graphs of arbitrarily large Betti number.*

k	simple	non-simple
1	$(\beta(G) - 2)/2 \quad (\beta(G) \neq 3)$	$\beta(G)$
2	$(\beta(G) - 4)/3 \quad (\beta(G) \neq 3, 5)$	$\beta(G) - 2$
3	$(\beta(G) - 4)/3 \quad (\beta(G) \neq 3, 5)$	$(\beta(G) - 4)/3 \quad (\beta(G) \neq 3, 5)$
≥ 4	1	1

Proof. We prove the upper bounds for k-edge-connected graphs G. Since k-vertex-connected implies k-edge-connected, the bounds also hold for vertex-connectivity. We achieve the bounds with k-vertex-connected graphs, so the bounds are best possible in both cases. The proof has five main cases.

Case (i). We begin with possibly non-simple graphs having edge connectivity (or vertex connectivity) $k \geq 4$. First observe that $\nu(G, \emptyset)$ is 0 when $\beta(G)$ is even and is 1 when $\beta(G)$ is odd. We will show that no 4-edge-connected G has a non-empty subset A with $\nu(G, A) > 0$. As a consequence, any such graph is upper-embeddable and the bounds in our table are best possible. This result has been noted by many authors, usually using Kundu's Theorem [10] combined with Xuong's Theorem [15].

To establish the inequality, first note that if A is non-empty and $G-A$ is connected, then $\nu(G, A) \leq 0$. If $G - A$ is disconnected, then each component of $G - A$ is incident with at least 4 edges in A. Counting edge ends and dividing by 2 we get $|A| \geq 2c_0+2c_1+2c_2+2c_3$. Substituting in Equation 2.1 gives $\nu(G, A) \leq -c_0 - c_2 - 1 < 0$ as desired.

Case (ii). We next consider graphs that are either 3-edge-connected and non-simple or 2-edge-connected and simple. The bound for simple graphs was first shown in [2].

Let A be a non-empty subset of edges. We need to show that $\nu(G, A) \leq (\beta(G)-4)/3$. Substituting Equation 2.1 for $\nu(G, A)$, Equation 2.2 for $\beta(G)$, and simplifying, it suffices to show that

$$2|A| \geq 2c_0 + 3c_1 + c_2 + 2c_3. \tag{2.3}$$

If G is 3-edge-connected, then every component is incident with at least three edge-ends from A. Hence $2|A| \geq 3(c_0 + c_1 + c_2 + c_3)$ implying Equation 2.3. The bound is achieved if and only if $c_0 = c_2 = c_3 = 0$.

If G is 2-edge-connected and simple, then every component of $G-A$ contributing to c_0 is incident with at least 3 edges (by minimum degree at least 3). A component contributing to c_1 is incident with at least 3 edges by simplicity. The remaining components are incident with 2 edges by connectivity. Hence $2|A| \geq 3c_0 + 3c_1 + 2c_2 + 2c_3$. Again, Equation 2.3 is shown and the bounds follow. The bound is achieved if and only if $c_0 = c_2 = 0$, every component contributing to c_1 is a triangle and every component contributing to c_3 is incident with precisely 2 edges from A.

If A is an empty set of edges, then $\nu(G, A) = 0$ or 1, depending on whether $\beta(G)$ is even or odd respectively. This is less than $(\beta(G) - 4)/3$ unless $\beta(G) = 3$ or 5, hence the two excluded cases.

One way to achieve the above bound is to replace every vertex of a 3-connected simple graph with a triangle so that every vertex is of degree three. Here the set A corresponds to all of the original edges in the graph. This example shows that the bound is best possible in both classes.

Case (iii). The next case is when G is simple and connected, first done in [4]. As before, we need to show that $\nu(G, A) \leq (\beta(G) - 2)/2$ for every pair (G, A). Again substituting in Equations 2.1 and 2.2, it suffices to show that

$$\mu(G, A) := 3|A| - (3c_0 + 4c_1 + c_2 + 2c_3) + 1 \geq 0. \tag{2.4}$$

Our goal is to show that there is a pair (G, A) minimizing μ with a certain structure, allowing us to show that $\mu(G, A) \geq 0$. We start by choosing a pair (G, A) minimizing $\mu(G, A)$ where A is minimal with respect to inclusion, and proceed by proving several claims about (G, A).

Claim 1: *Every edge from A joins two distinct components of $G - A$.*

If there were an edge x such that both ends of x are incident with a single component of $G - A$, then $\mu(G, A - \{x\}) \leq \mu(G, A)$, which contradicts the minimality of A.

Claim 2: *Each component of $G - A$ has odd Betti number.*

To see this, first note that $c_0 = 0$. If not, we could replace an acyclic component with a cycle of suitable length incident with the same edges of A as the original component to obtain a simple graph G' with minimum degree at least 3. For the resulting pair (G', A) we have $\mu(G', A) < \mu(G, A)$, which contradicts the minimality of μ. We further observe that $c_2 = 0$. If not, we could replace a component contributing to c_2 with a suitable subdivision of K_4 contributing to c_3. This can always be done in such a way that the resulting graph G' is simple and has minimum degree at least 3. However, $\mu(G', A) < \mu(G, A)$, again contradicting the minimality of μ.

Claim 3: *No component of $G - A$ is incident with exactly two edges of A.*

Suppose there were such a component C, and let e, f denote the edges from A incident with C. As shown above, $\beta(C)$ is odd, and since G is simple and has minimum degree at least three, $\beta(C) \geq 3$. Consider a pair (G', A') constructed by deleting C from G and joining the other ends of the edges incident with C. It is not difficult to see that G' has minimum degree at least three and that $\mu(G', A') < \mu(G, A)$; however, the graph G' may not be simple. If the graph G' is simple, then $\mu(G', A') < \mu(G, A)$ contradicts the minimality of μ. If G' is not simple, then we distinguish two cases: e and f join C with either one or two components of $G - A$.

If there is one component, it can contribute to μ by at most 4. It follows that $\mu(G, A - \{e, f\}) \leq \mu(G, A)$, contradicting the minimality of A.

If there are two components call them D_1 and D_2. Then joining the ends of e and f in G creates a pair of parallel edges in G'. Moreover, this implies that D_1 and D_2 are joined by some edge in A, say g. As both D_1 and D_2 can contribute to μ by at most 4 and C contributes by 2, it follows that $\mu(G, A - \{e, f, g\}) \leq \mu(G, A)$, again contradicting the minimality of A.

Claim 4: *A component C of $G - A$ has $\beta(C) = 1$ if and only if it is incident with at least three edges of A.*

Let C be a component of $G - A$ incident with at least three edges. If $\beta(C) \geq 3$, then we can replace it with a cycle of suitable length to obtain a simple graph G' with minimum degree at least 3. For the resulting pair (G', A') we again have $\mu(G', A') < \mu(G, A)$, a contradiction. If D is a component incident with just one edge of A, then the simplicity and the minimum degree of G guarantee that $\beta(D) \geq 3$. This proves Claim 4.

Having established some properties of $G - A$, we return to the proof of Case (iii). Form the graph $H = G/(G - A)$ from G by contracting each component of $G - A$ into a vertex. Note that $|E(H)| = |A|$ and that H has no vertices of degree 2. As H is connected, $|E(H)| \geq c_1 + c_3 - 1$. Denote the number of vertices of degree 1 in H by v_1 and the number of vertices of degree at least 3 by v_3. Observe $2|E(H)| \geq 3v_3 + v_1 = 3c_1 + c_3$. By adding the two inequalities we get $3|E(H)| = 3|A| \geq 4c_1 + 2c_3 - 1$. Substituting this into (2.4) yields

$$\mu(G, A) = (4c_1 + 2c_3 - 1) - (4c_1 + 2c_3) + 1 = 0.$$

To achieve the bound $(\beta(G)-2)/2 \geq \nu(G, A)$, start with any tree H having only vertices of degree 1 or 3. Replace each degree three vertex with a triangle and each degree one vertex with a copy of K_4 (possibly with one edge subdivided) so that every vertex has degree at least three.

Case (iv). We next turn to the case where G is non-simple and 2-edge-connected. Every component is incident with at least two edges in A, so $|A| \geq c_0 + c_1 + c_2 + c_3$. This gives $\nu(G, A) \leq c_1 + c_3 - 1$ and $\beta(G) \geq c_1 + 2c_2 + 3c_3 + 1$. To maximize ν with an upper bounded β we must have $c_2 = c_3 = 0$. Hence $\nu(G, A) \leq \beta(G) - 2$, giving another entry in our table. This bound is achieved by the graph formed by replacing every other edge of the cycle C_{2n} with two edges in parallel. These graphs are the *necklaces* of [2].

Case (v). When G is non-simple and 1-edge-connected we have $|A| \geq c_1 + c_2 + c_3 - 1$. This gives $\nu(G, A) \leq \beta(G)$ as desired. The bound is achieved by duplicating every other edge on a path with an even number of edges and contracting one of the two edges incident with a vertex of degree two. Complete characterization of non-simple 1-edge-connected graphs achieving this bound can be found in [13].

We have completed filling in the table and the proof of our theorem. □

Recall that for a graph G embedded in the surface S_g, we have $f - 1 = \beta(G) - 2g$, and hence $\xi(G) = \beta(G) - 2\gamma_M(G)$. We use this formula to translate the upper bounds on ξ to lower bounds on the maximum genus γ_M, giving the following second form of our main result.

Theorem 2.2. *Let G be a graph of minimum degree at least three. Then lower bounds on the maximum $\gamma_M(G)$ are given in following table. The rows correspond to edge-connectivity $k = 1, 2, 3$, or ≥ 4. The same bounds hold where k is the vertex-connectivity and are achieved by graphs of arbitrarily large Betti number.*

k	simple	non-simple
1	$(\beta(G)+2)/4 \quad (\beta(G) \neq 3)$	0
2	$(\beta(G)+2)/3 \quad (\beta(G) \neq 3, 5)$	1
3	$(\beta(G)+2)/3 \quad (\beta(G) \neq 3, 5)$	$(\beta(G)+2)/3 \quad (\beta(G) \neq 3, 5)$
≥ 4	$(\beta(G)-1)/2$	$(\beta(G)-1)/2$

□

We note that our table is slightly different than the one in [2] where, for example, they give the lower bound $\lceil \beta(G)/3 \rceil$ for the maximum genus of 2-edge-connected simple graphs of minimum degree at least 3. Both bounds are tight. They are achieved only for graphs with $\beta(G)$ congruent to one modulo three, when the bounds are the same.

3 Edge-cuts

We now investigate the relationship between the deficiency of a graph and the deficiency of components resulting from the removal of an arbitrary edge-cut. Our motivation is twofold. First, the assumptions concerning arbitrary edge-cuts are much weaker than the connectivity requirements used in Section 2. Second, similar ideas were pursued by Jaeger, Payan, and Xuong [6] who proved that, under natural parity conditions, a graph that can be separated by an edge-cut into two upper-embeddable components is itself upper-embeddable. Our aim is to generalize this result by replacing upper-embeddability with an arbitrary deficiency. The resulting lower bound for deficiency suggests the following question: *"What is the minimum size of an edge-cut that can separate a non-upper-embeddable component in a k-edge-connected graph for $k \geq 4$?"* Our final result, Theorem 3.4, answers this question.

As in the previous section, a crucial role is played by various types of components depending on their Betti numbers. Let F be a fixed edge-cut of a connected graph G with components G_1 and G_2. Let A be an arbitrary subset of edges of G. We distinguish between two types of components of $G - A$: a *pure component* has its vertices entirely in G_1 or entirely in G_2, while a *mixed component* is incident with vertices in both. Let d_0 denote the number of mixed components of $G - A$ with even Betti number. Let d_1 denote the number of pure components of $G - A$ with even Betti number and with vertices from G_1. Define d_2 similarly for pure components with vertices from G_2. Finally, let d_3, d_4, and d_5 denote the odd counterparts of d_0, d_1, and d_2, respectively. Notice that pure components contribute to d_1, d_2, d_4, and d_5 and mixed components contribute to d_0 and d_3.

With this notation, Theorem 1.1 implies the following:

$$\begin{aligned} \xi(G) &\geq \nu(G,A) = \mathrm{ec}(G-A) + 2\mathrm{oc}(G-A) - |A| - 1 \\ &= d_0 + d_1 + d_2 + 2d_3 + 2d_4 + 2d_5 - |A| - 1. \end{aligned} \tag{3.1}$$

We now proceed to the main result of this section, Theorem 3.1. To show that the bounds stated in it are tight we need one more ingredient, Xuong's Theorem [15]: The deficiency of a graph G equals the minimum number of components with odd number of edges (which we call *odd* for short) in the subgraph $G - E(T)$, where the minimum is taken over all spanning trees T of G.

Theorem 3.1. *Let G be a connected graph and let F be an edge-cut such that $G - F$ has precisely two components G_1 and G_2. Then*

$$\xi(G_1) + \xi(G_2) - |F| + 1 \leq \xi(G) \leq \xi(G_1) + \xi(G_2) + 1,$$

both bounds being tight. Equivalently,

$$\gamma_M(G_1) + \gamma_M(G_2) + |F|/2 - 1 \leq \gamma_M(G) \leq \gamma_M(G_1) + \gamma_M(G_2) + |F| - 1.$$

Proof. First we show that $\xi(G) \leq \xi(G_1) + \xi(G_2) + 1$. Let A be some set of edges of G maximal with respect to inclusion such that $\xi(G) = \nu(G, A)$. Set $A_i = A \cap E(G_i)$ for $i \in \{1, 2\}$ and note that

$$A - (A_1 \cup A_2) = A \cap F. \tag{3.2}$$

Our goal is to bound the values of $\nu(G_1, A_1)$ and $\nu(G_2, A_2)$. Pure components are also components of either $G_1 - A_1$ or $G_2 - A_2$. Therefore, a pure component contributing to d_1 or d_2 increases the value of $\nu(G_1, A_1)$ or $\nu(G_2, A_2)$ by precisely 1. Pure components contributing to d_4 or d_5 increase the value of $\nu(G_1, A_1)$ or $\nu(G_2, A_2)$ by precisely 2.

If D is a mixed component, then $D - F$ has at least one component contained in G_1 and at least one component contained in G_2. Therefore D increases $\nu(G_1, A_1) + \nu(G_2, A_2)$ by at least 2. Note that if $\beta(D)$ is even (that is, if D contributes to d_0), then D increases $\nu(G, A)$ only by one. It follows that

$$\begin{aligned} &\mathrm{ec}(G_1, A_1) + 2\mathrm{oc}(G_1, A_1) + \mathrm{ec}(G_2, A_2) + 2\mathrm{oc}(G_2, A_2) \\ \geq\ & 2d_0 + d_1 + d_2 + 2d_3 + 2d_4 + 2d_5. \end{aligned} \tag{3.3}$$

Combining (3.1), (3.2), and (3.3) we get

$$\begin{aligned}
&\xi(G_1)+\xi(G_2)\geq \nu(G_1,A_1)+\nu(G_2,A_2)\\
=\;&\mathrm{ec}(G_1,A_1)+2\mathrm{oc}(G_1,A_1)-|A_1|-1+\\
&\mathrm{ec}(G_2,A_2)+2\mathrm{oc}(G_2,A_2)-|A_2|-1\\
\geq\;&2d_0+d_1+d_2+2d_3+2d_4+2d_5-|A_1|-|A_2|-2\\
=\;&\mathrm{ec}(G-A)+2\mathrm{oc}(G-A)+d_0-|A_1|-|A_2|-2\\
=\;&\mathrm{ec}(G-A)+2\mathrm{oc}(G-A)-|A|-1+d_0+|A-(A_1\cup A_2)|-1\\
=\;&\xi(G)-1+|F\cap A|+d_0\geq \xi(G)-1.
\end{aligned}$$

This establishes the upper bound on $\xi(G)$.

To prove the lower bound, for $i\in\{1,2\}$ let A_i denote a set of edges of G_i maximal with respect to inclusion such that $\nu(G_i,A_i)=\xi(G_i)$. Set $A=A_1\cup A_2\cup F$ and calculate:

$$\begin{aligned}
\xi(G)\;&\geq\;\nu(G,A)=\mathrm{ec}(G-A)+2\mathrm{oc}(G-A)-|A|-1\\
&=\;\mathrm{ec}(G_1-A_1)+\mathrm{ec}(G_2-A_2)+2\mathrm{oc}(G_1-A_1)+\\
&\quad\;2\mathrm{oc}(G_2-A_2)-|A_1|-|A_2|-|F|-1\\
&=\;\nu(G_1,A_1)+\nu(G_2,A_2)-|F|+1=\xi(G_1)+\xi(G_2)-|F|+1.
\end{aligned}$$

This establishes the lower bound on $\xi(G)$.

To see that both bounds are tight, take two copies of the dipole D_n, which has two vertices and n parallel edges, and join the copies by two independent edges e and f. Let G be the resulting graph. Using Xuong's Theorem it is easy to verify that $\xi(D_n)=1$ if n is even, $\xi(D_n)=0$ if n is odd, and that $\xi(G)=1$. Let $F=\{e,f\}$ and let G_1 and G_2 be the components of $G-F$. Then $\xi(G)=\xi(G_1)+\xi(G_2)-|F|+1$ if n is even, and $\xi(G)=\xi(G_1)+\xi(G_2)+1$ if n is odd. □

Theorem 3.1 and the fact that $\xi(G)$ and $\beta(G)$ have the same parity give a shorter proof for the following result of Jaeger, Payan, and Xuong [6].

Corollary 3.2. *Let G be a connected graph and let F be an edge-cut of G such that $G-F$ has precisely two components G_1 and G_2, both upper-embeddable. Then G is upper-embeddable provided that both $\beta(G_1)$ and $\beta(G_2)$ are even, or precisely one of $\beta(G_1)$ and $\beta(G_2)$ is even and $\beta(G)$ is odd.* □

In the final part of the proof of Theorem 3.1 we have shown that the lower bound on $\xi(G)$ is tight in the class of 2-edge-connected graphs. Our next example shows that the lower bound is tight also in the class of 3-edge-connected graphs.

Example 3.3. We construct a rich infinite family of 3-edge-connected graphs G containing an edge-cut F whose removal produces two components G_1 and G_2 such that $\xi(G)=\xi(G_1)+\xi(G_2)-|F|+1$. Recall that the *truncation* of a cubic graph H is a cubic graph $t(H)$ obtained by expanding every vertex of H into a triangle.

Let us start with connected cubic graphs H_1 and H_2 without loops, of order n_1 and n_2, respectively, both greater than 2. Take their truncations $G_1=t(H_1)$ and $G_2=t(H_2)$, and connect G_1 to G_2 by a set $F=\{f_0,f_1,f_2\}$ of three edges arbitrarily. Let G be the resulting graph; it is easy to see that if H_1 and H_2 are 3-edge-connected, so is G. Bouchet [1] proved that $\xi(G_i) = n_i/2 - 1$ (see [14, Theorem 3.3] for a different proof). We now show that $\xi(G) = \xi(G_1) + \xi(G_2) - 2 = \xi(G_1) + \xi(G_2) - |F| + 1 = (n_1 + n_2)/2 - 4$.

For $i \in \{1, 2\}$ let A_i be the set of edges of G_i not lying in a triangle, and let $A = A_1 \cup F \cup A_2$. Since $|A| = 3 + 3(n_1 + n_2)/2$ and $G - A$ consists of $n_1 + n_2$ triangles, Nebeský's Theorem implies that $\xi(G) \geq \nu(G, A) = 2(n_1+n_2) - 3(n_1+n_2)/2 - 3 - 1 = \xi(G_1) + \xi(G_2) - 2$. To prove the reverse inequality, we take in each H_i a spanning tree S_i such that all components of $H_i - E(S_i)$ are paths; it is not difficult to see that this choice is indeed possible [14, Theorem 3.1]. Extend S_i to a spanning tree T_i of G_i by including in T_i two edges from each triangle of G_i. This can be done in such a way that a component of $H_i - E(S_i)$, which is a path of length $m \geq 0$, becomes a path of length $2m+1$ constituting a component of $G_i - E(T_i)$. A straightforward calculation reveals that there are $n_i/2 - 1$ such components in $G_i - E(T_i)$, all other components being isolated vertices. In particular, $G_i - E(T_i)$ has $n_i/2 - 1 = \xi(G_i)$ odd components. Form a spanning tree T of G by adding the edge f_0 to $T_1 \cup T_2$. Clearly, T is also a spanning tree of $G' = G - \{f_1, f_2\}$ and the corresponding cotree has $(n_1 + n_2)/2 - 2 = \xi(G_1) + \xi(G_2) = \xi(G')$ odd components. We now add f_1 and f_2 to G' one by one and modify T, if necessary, each time absorbing one odd component.

Pick f_1 and note that each of its end-vertices belongs to a component P_1 of $G_1 - E(T_1)$ or a component P_2 of $G_2 - E(T_2)$. Clearly, $P_1 \cup \{f_1\} \cup P_2$ is a component of $(G' + f_1) - E(T)$. If P_1 and P_2 have different parity, then $P_1 \cup \{f_1\} \cup P_2$ is even, implying that the cotree $(G' + f_1) - E(T)$ has $\xi(G_1) + \xi(G_2) - 1$ odd components. If both P_1 and P_2 are odd, then $P_1 \cup \{f_1\} \cup P_2$ is an odd component of $(G' + f_1) - E(T)$ and therefore the cotree $(G' + f_1) - E(T)$ has $\xi(G_1) + \xi(G_2) - 2 + 1$ odd components. It remains to consider the case where P_1 and P_2 are both even. Let v be the end-vertex of f_1 in G_1, and let K be the triangle of G_1 containing v. The construction of T_1 implies that P_1 coincides with v and that both edges of K incident with v belong to T_1; take one of them, say k. The third edge of K, denoted by q, belongs to $G_1 - E(T_1)$, and the component P of $G_1 - E(T_1)$ containing it is an odd path. Note that $P - q$ consists of two even paths, possibly trivial. It follows that $T' = T + k - q$ is a spanning tree of $G' + f_1$ such that the component of $(G' + f_1) - E(T')$ containing f_1 is no more odd. Hence, the cotree $(G' + f_1) - E(T')$ has $\xi(G_1) + \xi(G_2) - 1$ odd components, which in turn implies that $\xi(G' + f_1) \leq \xi(G_1) + \xi(G_2) - 1$.

To finish the proof that $\xi(G) \leq \xi(G_1) + \xi(G_2) - 2$ we add f_2 to $G' + f_1$ and proceed similarly, except that we modify T_2 instead of T_1. ■

We next turn our attention to 4-edge-connected graphs. Our next result exhibits a dramatic change in the behavior with regard to the smallest size of an edge-cut that can separate a component with large deficiency: the size of an edge-cut that separates a component with deficiency m must be at least linear in m.

A *leaf* of a graph is a 2-edge-connected subgraph maximal with respect to inclusion. For a graph H, let $\mathrm{ol}(H)$ denote number of leaves of H with odd Betti number. The following equivalent version of Nebeský's theorem holds for every connected graph G (see [9]):

$$\xi(G) = \max\{\mathrm{ol}(G - A) - |A| \ : \ A \subseteq E(G)\}. \tag{3.4}$$

Theorem 3.4. *Let G be a k-edge-connected graph with $k \geq 4$ and let F be an edge-cut of G whose removal produces a component with deficiency at least m. Then*

$$|F| \geq (k-2)m + 2.$$

Proof. Let H be a component of $G - F$ with $\xi(H) \geq m$. Each edge of F has at most one end in H. Choose a subset $A \subseteq E(H)$ such that $\xi(H) = \mathrm{ol}(H - A) - |A|$ and let $l = \mathrm{ol}(H - A)$. As G is k-edge-connected, every leaf from $H - A$ must be incident with at least k edges in G. These edges can be only from F, from A, or they can be bridges of $H - A$. There are $l - 1$ bridges in $H - A$. It follows that

$$kl \leq |F| + 2(l-1) + 2|A|. \tag{3.5}$$

By substituting $l - \xi(H)$ for $|A|$ in (3.5) and manipulating the resulting expression we derive

$$|F| \geq 2\xi(H) + l(k-4) + 2.$$

For $k = 4$ we get

$$|F| \geq 2\xi(H) + 2 = (k-2)\xi(H) + 2 \geq (k-2)m + 2,$$

as required. If $k \geq 5$, then $k - 4 \geq 1$, and using the fact that $l \geq \xi(H)$ we obtain

$$\begin{aligned} |F| &\geq 2\xi(H) + l(k-4) + 2 \geq 2\xi(H) + \xi(H)(k-4) + 2 \\ &\geq (k-2)\xi(H) + 2 \geq (k-2)m + 2. \end{aligned}$$

This again gives the required inequality. □

The following corollary follows directly from Theorem 3.4.

Corollary 3.5. *If G is a k-edge-connected graph with $k \geq 4$ and F is an edge cut of G such that a component of $G - F$ is not upper-embeddable, then $|F| \geq 2k - 2$.* □

Let G be a 4-edge-connected graph and let F be an edge-cut separating it into components G_1 and G_2. Assuming that $m = \xi(G_1) \geq \xi(G_2)$, Theorem 3.4 implies that $|F| \geq (k-2)m + 2 \geq 2m + 2$. Therefore, $\xi(G_1) + \xi(G_2) - |F| + 1$ is always negative and the lower bound in Theorem 3.1 cannot be achieved for 4-edge-connected graphs.

Acknowledgements. The research reported in this paper was supported from the following sources. The second author was partially supported by UK/217/2012 and by VEGA 1/1005/12, the third author was partially supported by APVV-0223-10, and the fourth author was partially supported by APVV project ESF-EC-0009-10 within the EUROCORES Programme EUROGIGA (project GReGAS) of the European Science Foundation. The authors would like to thank Jianer Chen, Yuangqiu Huang, Saroja Kanchi, Deming Li, and Yanpei Liu for fruitful discussions, and an anonymous reviewer for helpful comments.

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ARS MATHEMATICA CONTEMPORANEA

Also available at http://amc-journal.eu
ISSN 1855-3966 (printed edn.), ISSN 1855-3974 (electronic edn.)
ARS MATHEMATICA CONTEMPORANEA 9 (2015) 63–75

The automorphism groups of non-edge-transitive rose window graphs

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Received 2 August 2012, accepted 7 September 2012, published online 20 June 2014

Abstract

In this paper, we will determine the full automorphism groups of rose window graphs that are not edge-transitive. As the full automorphism groups of edge-transitive rose window graphs have been determined, this will complete the problem of calculating the full automorphism group of rose window graphs. As a corollary, we determine which rose window graphs are vertex-transitive. Finally, we determine the isomorphism classes of non-edge-transitive rose window graphs.

Keywords: Rose window graphs, automorphism group, isomorphism problem, vertex-transitive graph.

Math. Subj. Class.: 05E18

1 Introduction

Rose windows graphs are defined as follows (we are using the notation and terminology as in [18]).

Definition 1.1. Let n be a positive integer and $a, r \in \mathbb{Z}_n$ (so arithmetic with a and r is done modulo n). The **rose window graph** $R_n(a, r)$ is defined to be the graph with vertex set $V = \{A_i, B_i : i \in \mathbb{Z}_n\}$ and four kinds of edges:

- $A_i A_{i+1}$ These edges are called **rim** edges.

*Project sponsored by the National Security Agency under Grant Number H98230-11-1-0179.

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- A_iB_i These edges are called **in-spoke** edges.
- $A_{i+a}B_i$ These edges are called **out-spoke** edges.
- B_iB_{i+r} These edges are called **hub** edges.

Rose window graphs were introduced recently by Steve Wilson [18], whose initial motivation was concerned with determining which of these graphs are edge-transitive (and if so what is their full automorphism group), as well as which of these graphs are the underlying graph of a rotary map. He proposed four conjectures concerning questions that he was interested in, and subsequently all of the conjectures have been shown to be true. Edge-transitive rose window graphs were characterized in [5, Theorem 1.2], verifying [18, Conjecture 11] (a graph is edge-transitive if its automorphism group acts transitively on the set of edges). The full automorphism groups of edge-transitive rose window graphs was determined in [6, §3], verifying [18, Conjectures 3 and 5]. The rose window graphs which are the underlying graph of a rotary map were found in [6, Theorem 1.1], answering [18, Question 3], Finally, [18, Conjecture 6] suggesting certain rose window graphs are isomorphic was verified in [6, Theorem 3.6].

Our goal is to essentially complete the work that has already been done regarding calculating the full automorphism groups of rose window graphs, as well determining exactly when two rose window graphs are isomorphic. In this paper, we will calculate the full automorphism groups of rose window graphs that are not edge-transitive (which will finish the problem of determining the full automorphism groups of rose window graphs), see Corollary 3.5 and Corollary 3.9. In Section 4, we will determine the isomorphism classes of rose window graphs that are not edge-transitive. The conclusion of the isomorphism problem for rose window graphs will be given in a sequel to this paper, where the isomorphism classes of edge-transitive rose window graphs will be found.

There are a few additional results in this paper that should be mentioned. First, in Lemma 2.2, we correct a small error in [18, Lemma 2] giving conditions on when a rose window graph has an automorphism that maps every rim edge to a hub edge and vice versa. Also, once the full automorphism groups of rose windows graphs are known, it is relatively straightforward to determine which of these graphs are vertex-transitive, and this is given in Theorem 3.10.

We should point out that our goal is a classical one. Namely, with graphs that have a large amount of symmetry, it is quite standard to ask for their full automorphism groups as well as their isomorphism classes. Perhaps the first family for which this has been done are the generalized Petersen graphs. The automorphism groups of these graphs were obtained by Frucht, Graver, and Watkins [4] in 1971, while the isomorphism classes were found by by Boben, Pisanski, and Žitnik [1]. Using very differrent techniques, Steimle and Staton [16] also determined the isomorphism classes for some, but not all, generalized Petersen graphs, and then used that result to enumerate the generalized Petersen graphs whose isomorphism classes they determined. Using Boben, Pisanski, and Žitnik's determination of the isomorphism classes of all generalized Petersen graphs, Petkovšek and Zakrajšek [9] enumerated generalized Petersen graphs. Determining the isomorphism classes of the rose window graphs should also yield an enumeration of the rose window graphs using techniques similar to those in [9].

Finally, in the last decade or so there has been considerable interest in tetravalent graphs satisfying various properties or in studying certain families of such graphs (for a sample of such work see [3, 7, 10, 11, 12, 13, 17]). See also [14], where a census of all locally imprimitive tetravalent arc-transitive graphs on up to 640 vertices was computed. This work will certainly contribute to the understanding of such graphs.

2 Preliminary results

We first give some obvious automorphisms of rose window graphs. Let $R_n(a,r)$ be a fixed rose window graph and let G be the automorphism group of $R_n(a,r)$. Observe that

$$R_n(a,r) = R_n(a,-r). \tag{2.1}$$

Define $\rho, \mu : V \mapsto V$ by

$$\rho(A_i) = A_{i+1} \quad \text{and} \quad \rho(B_i) = B_{i+1} \qquad (i \in \mathbb{Z}_n), \tag{2.2}$$

$$\mu(A_i) = A_{-i} \quad \text{and} \quad \mu(B_i) = B_{-a-i} \qquad (i \in \mathbb{Z}_n). \tag{2.3}$$

Note that $\rho, \mu \in G$, and therefore $\langle \rho, \mu \rangle \leq G$. The action of $\langle \rho, \mu \rangle$ on the set of edges of $R_n(a,r)$ has three orbits: the set of rim edges, the set of hub edges and the set of spoke edges.

The following result characterizes edge-transitive rose window graphs in terms of rim and spoke edges (we remark that the full automorphism groups of edge-transitive rose window graphs are given in [5], but the following formulation is nonetheless useful).

Lemma 2.1. *The following are equivalent:*

(i) $R_n(a,r)$ *is edge-transitive.*

(ii) *There is an automorphism of* $R_n(a,r)$ *which sends a rim edge to a spoke edge.*

(iii) *There is an automorphism of* $R_n(a,r)$ *which sends a spoke edge to a hub edge.*

Proof. It is clear that (i) implies (ii). To show that (ii) implies (iii), suppose that A_iA_{i+1} is a rim edge mapped to a spoke edge by, say, $\sigma \in G$. Then $\sigma(A_iA_{i+1}) = A_jB_k$ for some $j,k \in \mathbb{Z}_n$, and $\sigma(A_\ell) = B_k$ for $\ell = i$ or $i+1$. Of course, $e_1 = A_\ell B_\ell$ and $e_2 = A_\ell B_{\ell-a}$ are spoke edges, and $\sigma(e_1)$ and $\sigma(e_2)$ are two edges incident with the spoke edge A_jB_k, and all three of these edges are incident with $\sigma(A_\ell) = B_k$. However, B_k is incident with two hub edges and two spoke edges, so at least one of $\sigma(e_1)$ and $\sigma(e_2)$ must be a hub edge.

To show (iii) implies (i), recall that the hub, spoke and rim edges are the edge orbits of $\langle \rho, \mu \rangle$. If σ maps some spoke edge to a hub edge, we have that $H = \langle \rho, \mu, \sigma \rangle$ has at most two edge orbits, and if there are two edge orbits, these consist of spoke and hub edges in one orbit and rim edges being the other orbit. However, if the rim edges form an orbit, then H must map $\{A_i : i \in \mathbb{Z}_n\}$ to itself, and so must map $\{B_i : i \in \mathbb{Z}_n\}$ to itself, and so must map hub edges to themselves. This then implies that H has three edge orbits, a contradiction. So H has one edge orbit and $R_n(r,a)$ is edge-transitive. □

It follows that if $R_n(a,r)$ is not edge-transitive, then it has either two orbits or three orbits on edges. If $R_n(a,r)$ has two orbits on edges, then one orbit consists of rim and hub edges, and the other consists of spoke edges. If $R_n(a,r)$ has three orbits on edges, then the first one consists of rim edges, the second one consists of hub edges, and the third one consists of spoke edges.

Lemma 2 in [18] states that there is an automorphism of $R_n(a,r)$ sending rim edges to hub edges and vice-versa if and only if $r^2 \equiv \pm 1 \pmod{n}$ and $ra \equiv \pm a \pmod{n}$.

However, this is not entirely true. Namely, one can check that the rose window graph $R_{16}(8,3)$ has an automorphism sending rim edges to hub edges and vice-versa via the map $(i,j) \to (i, 11j)$. However it is clear that $r^2 = 9 \not\equiv \pm 1 \pmod{16}$. We now wish to give a correct statement of [18, Lemma 2], and begin with a preliminary lemma.

Lemma 2.2. *Let σ be the automorphism of $R_n(a,r)$, which sends every rim edge to a hub edge and vice versa. Assume also that $\sigma(A_0) = B_0$ and $\sigma(B_0) = A_0$. Then one of the following holds for every $i \in \mathbb{Z}_n$:*

(i) $\sigma(A_i) = B_{ri}$ *and* $\sigma(B_i) = A_{ri}$*;*

(ii) $\sigma(A_i) = B_{ri}$ *and* $\sigma(B_i) = A_{(r+a)i}$*;*

(iii) $\sigma(A_i) = B_{-ri}$ *and* $\sigma(B_i) = A_{-ri}$*;*

(iv) $\sigma(A_i) = B_{-ri}$ *and* $\sigma(B_i) = A_{(-r+a)i}$

Proof. Since $\sigma(A_0) = B_0$ and $\sigma(A_1)$ are adjacent, we have $\sigma(A_1) \in \{B_r, B_{-r}\}$. It is easy to see that if $\sigma(A_1) = B_r$ then $\sigma(A_i) = B_{ri}$ for $i \in \mathbb{Z}_n$, and that if $\sigma(A_1) = B_{-r}$ then $\sigma(A_i) = B_{-ri}$ for $i \in \mathbb{Z}_n$. Now let $s \in \mathbb{Z}_n$ be such that $\sigma(B_1) = A_s$ and note that $\sigma(B_i) = A_{si}$ for $i \in \mathbb{Z}_n$. Moreover, $\sigma(A_1)$ and $\sigma(B_1) = A_s$ are adjacent. Therefore, if $\sigma(A_1) = B_r$, then $s \in \{r, r+a\}$, and if $\sigma(A_1) = B_{-r}$, then $s \in \{-r, -r+a\}$. The result follows. □

Theorem 2.3. *Let $n \geq 3$ be an integer and $a, r \in \mathbb{Z}_n \setminus \{0\}$. Then there is an automorphism of $R_n(a,r)$ sending every rim edge to a hub edge and vice-versa if and only if one of the following holds:*

(i) $a \neq n/2$, $r^2 \equiv 1 \pmod{n}$ *and* $ra \equiv \pm a \pmod{n}$*;*

(ii) $a = n/2$, $r^2 \equiv \pm 1 \pmod{n}$ *and* $ra \equiv \pm a \pmod{n}$*;*

(iii) *n is divisible by* 4, $\gcd(n,r) = 1$, $a = n/2$ *and* $(r^2 + n/2) \equiv \pm 1 \pmod{n}$.

Proof. We first show that if (i), (ii) or (iii) holds, then there is an automorphism of $R_n(a,r)$ sending rim edges to hub edges and vice-versa. By (2.1) we can assume that $ra \equiv -a \pmod{n}$. Observe that if one of conditions (i), (ii) or (iii) holds, then $\gcd(n,r) = 1$. If condition (i) or (ii) holds, then let $\sigma : V \to V$ be defined by $\sigma(A_i) = B_{ri}$ and $\sigma(B_i) = A_{ri}$. As $\gcd(n,r) = 1$, σ is a bijection. It is straightforward to check that σ is also an automorphism of $R_n(a,r)$.

If condition (iii) holds, then let $\sigma : V \to V$ be defined by $\sigma(A_i) = B_{ri}$ and $\sigma(B_i) = A_{ri+(n/2)i}$. Let us show that σ is a bijection. As $\gcd(n,r) = 1$, σ maps $\{A_i \mid i \in \mathbb{Z}_n\}$ to $\{B_i \mid i \in \mathbb{Z}_n\}$ bijectively. As $\gcd(n,r) = 1$, r is odd and $n/2$ is even, σ maps $\{B_i \mid i \in \mathbb{Z}_n\}$ to $\{A_i \mid i \in \mathbb{Z}_n\}$ bijectively. Hence σ is a bijection. It is then straightforward to check that σ is also an automorphism of $R_n(a,r)$.

We now show that if there is an automorphism σ of $R_n(a,r)$ sending rim edges to hub edges and vice-versa, then either (i), (ii) or (iii) holds. Note that in this case it must be the case that $\gcd(n,r) = 1$. Since $\langle \rho, \mu \rangle$ acts transitively on the sets of hub, rim and spoke edges, we may assume (by replacing σ by an appropriate element of $\langle \rho, \mu, \sigma \rangle$) that $\sigma(A_0) = B_0$ and $\sigma(B_0) = A_0$. Using (2.1) we can further assume that $\sigma(A_1) = B_r$. Therefore, by Lemma 2.2, $\sigma(A_i) = B_{ri}$ and $\sigma(B_i) = A_{si}$ for $i \in \mathbb{Z}_n$, where $s \in \{r, r+a\}$. Since σ^2 sends A_0 to A_0 and A_1 to A_{rs}, we have $rs \equiv \pm 1 \pmod{n}$.

Consider an in-spoke A_iB_i and an out-spoke B_iA_{i+a}. The automorphism σ maps the in-spoke A_iB_i to $B_{ri}A_{si}$, and the out-spoke B_iA_{i+a} to $A_{si}B_{ri+ra}$. Hence one of $B_{ri}A_{si}$ and $A_{si}B_{ri+ra}$ is an in-spoke, and the other one is an out-spoke. Therefore, for every $i \in \mathbb{Z}_n$ either

$$ri \equiv si \pmod n \quad \text{and} \quad ri + ra + a \equiv si \pmod n \tag{2.4}$$

or

$$ri + ra \equiv si \pmod n \quad \text{and} \quad ri + a \equiv si \pmod n. \tag{2.5}$$

Note that if (2.5) holds for $i = 0$, then $a = 0$, a contradiction. Therefore (2.4) holds for $i = 0$, implying $ra \equiv -a \pmod n$.

If (2.4) holds for $i = 1$, then we have $r = s$. Since $rs \equiv \pm 1 \pmod n$ this implies $r^2 \equiv \pm 1 \pmod n$. If $a = n/2$, then (ii) holds. If $a \neq n/2$, then multiplying the congruence $ra \equiv -a \pmod n$ by r, we obtain $r^2 a \equiv -ar \equiv a \pmod n$. If $r^2 \equiv -1 \pmod n$, then $-a \equiv a \pmod n$. This implies that $a = n/2$, a contradiction. So if $a \neq n/2$, then $r^2 \equiv 1 \pmod n$. Thus (i) holds.

Suppose now that (2.5) holds for $i = 1$. Then $r + ra \equiv s \pmod n$ and $r + a \equiv s \pmod n$. The two congruences then imply that $ra \equiv a \pmod n$ and $r + a \equiv s \pmod n$. Since also $ra \equiv -a \pmod n$, we have that $a = n/2$ and r is odd. Combining together $r + n/2 \equiv s \pmod n$ and $rs \equiv \pm 1 \pmod n$ gives us $(r^2 + n/2) \equiv \pm 1 \pmod n$.

Observe that $\sigma(B_{n/2}) = A_{s(n/2)} = A_{(r+n/2)(n/2)}$. Suppose n is not divisible by 4. As r and $n/2$ are both odd in this case, we have $\sigma(B_{n/2}) = A_0 = \sigma(B_0)$. But this implies that σ is not a bijection, a contradiction. Therefore, condition (iii) holds. □

It follows from Theorem 2.3 that $R_n(a, r)$ has two orbits of edges if and only if one of conditions (i) or (ii) in Theorem 2.3 is satisfied. We will also use the following result.

Lemma 2.4. *Assume that $R_n(a, r)$ is not edge-transitive and $a = n/2$. Then at least one of*

(i) $r^2 \equiv \pm 1 \pmod n$

(ii) $r^2 + n/2 \equiv \pm 1 \pmod n$

does not hold.

Proof. If both (i) and (ii) above hold, then $n/2$ is congruent to 2, 0 or -2 modulo n. But this is only possible if $n = 4$. If $n = 4$, then $r \in \{1, 3\}$. In both cases $R_n(a, r)$ is edge transitive, a contradiction. □

3 Groups of non edge-transitive rose window graphs

Before proceeding, we will require some additional notation. Let $N = \gcd(n, r)$ denote the number of inner cycles, and let $L = n/N$ denote the length of an inner cycle. Here an inner cycle is a cycle induced by some set of vertices $\{B_i \mid i \in \mathbb{Z}_n\}$. We now define three types of permutations on $V(R_n(r, a))$. To do this we assume that n is even. For $0 \leq \ell \leq n/2 - 1$, we define a permutation on $V(R_n(r, a))$ by

$$\alpha_\ell = (B_\ell, B_{\ell+n/2}).$$

If L is even, then for $0 \leq \ell \leq N - 1$ we let

$$\beta_\ell = (B_\ell, B_{\ell+n/2})(B_{\ell+N}, B_{\ell+N+n/2})(B_{\ell+2N}, B_{\ell+2N+n/2}) \cdots (B_{\ell+n/2-N}, B_{\ell+n-N}).$$

Observe that β_ℓ interchanges every two antipodal vertices of the inner cycle containing B_ℓ. If L is odd, then for $0 \le \ell \le N/2 - 1$ we let

$$\gamma_\ell = (B_{\ell+0}, B_{\ell+n/2})(B_{\ell+N}, B_{\ell+N+n/2})(B_{\ell+2N}, B_{\ell+2N+n/2}) \cdots (B_{\ell+n-N}, B_{\ell+n-N+n/2}).$$

Observe that γ_ℓ interchanges the inner cycle containing B_ℓ and the inner cycle containing $B_{\ell+n/2}$.

Lemma 3.1. *Assume n is even. Then the following hold:*

(i) *For $0 \le \ell \le n/2 - 1$ we have $\alpha_\ell = \rho^\ell \alpha_0 \rho^{-\ell}$.*

(ii) *If L is even, then for $0 \le \ell \le N - 1$ we have $\beta_\ell = \rho^\ell \beta_0 \rho^{-\ell}$.*

(iii) *If L is odd, then for $0 \le \ell \le N/2 - 1$ we have $\gamma_\ell = \rho^\ell \gamma_0 \rho^{-\ell}$.*

Proof. (i) It is straightforward to check that $(\rho^\ell \alpha_0 \rho^{-\ell})(A_i) = A_i$ for every $i \in \mathbb{Z}_n$ and that $(\rho^\ell \alpha_0 \rho^{-\ell})(B_i) = B_i$ for every $i \in \mathbb{Z}_n \setminus \{\ell, \ell + n/2\}$. Similarly we find that $\rho^\ell \alpha_0 \rho^{-\ell}$ interchanges B_ℓ and $B_{\ell+n/2}$. The result follows.

(ii) Since $\beta_0 = \alpha_0 \alpha_N \alpha_{2N} \cdots \alpha_{n/2-N}$ and $\beta_\ell = \alpha_\ell \alpha_{\ell+N} \alpha_{\ell+2N} \cdots \alpha_{\ell+n/2-N}$ the result follows from (i) above.

(iii) Similarly as (ii) above. □

Lemma 3.2. *Assume n is even. Then the following hold:*

(i) *If $L = 4$ then α_ℓ is an automorphism of $R_n(n/2, r)$ for $0 \le \ell \le n/2 - 1$.*

(ii) *If L is even, $L \ne 4$, then β_ℓ is an automorphism of $R_n(n/2, r)$ for $0 \le \ell \le N - 1$.*

(iii) *If L is odd then γ_ℓ is an automorphism of $R_n(n/2, r)$ for $0 \le \ell \le N/2 - 1$.*

Proof. Straightforward. □

Lemma 3.3. *Let $G_\mathcal{A}$ be the point-wise stabiliser of $\{A_0, A_1, \ldots, A_{n-1}\}$ in G. Then the following hold:*

(i) *If $a \ne n/2$ then $G_\mathcal{A}$ is trivial.*

(ii) *If $a = n/2$ and $L = 4$, then $G_\mathcal{A} = \langle \alpha_0, \alpha_1, \ldots, \alpha_{n/2-1} \rangle$.*

(iii) *If $a = n/2$, L is even and $L \ne 4$, then $G_\mathcal{A} = \langle \beta_0, \beta_1, \ldots, \beta_{N-1} \rangle$.*

(iv) *If $a = n/2$ and L is odd, then $G_\mathcal{A} = \langle \gamma_0, \gamma_1, \ldots, \gamma_{N/2-1} \rangle$.*

Proof. Let $\sigma \in G_\mathcal{A}$. Since the outer cycle (that is, the n-cycle induced by the vertices $\{A_i \mid i \in \mathbb{Z}_n\}$) is fixed by σ, for every $i \in \mathbb{Z}_n$ we have either $\sigma(B_i) = B_i$ and $\sigma(B_{i-a}) = B_{i-a}$, or $\sigma(B_i) = B_{i-a}$ and $\sigma(B_{i-a}) = B_i$. If σ is nontrivial, then there exists $j \in \mathbb{Z}_n$ such that $\sigma(B_j) = B_{j-a}$ and $\sigma(B_{j-a}) = B_j$. Applying the above comment to $i = j + a$ we find that $\sigma(B_{j+a}) = B_j$ and $\sigma(B_j) = B_{j+a}$. Therefore $j - a = j + a$, implying $a = n/2$. This proves (i).

Assume $L = 4$. Every α_ℓ $(1 \leq \ell \leq n/2 - 1)$ is clearly in $G_{\mathcal{A}}$ by Lemma 3.2 (i). Therefore $\langle \alpha_0, \alpha_1, \ldots, \alpha_{n/2-1} \rangle \leq G_{\mathcal{A}}$. Pick $\sigma \in G_{\mathcal{A}}$. Since for every i $(i \in \mathbb{Z}_n)$ the automorphism σ either fixes or interchanges B_i and $B_{i+n/2}$, we clearly have $G_{\mathcal{A}} \leq \langle \alpha_0, \alpha_1, \ldots, \alpha_{n/2-1} \rangle$. Therefore $G_{\mathcal{A}} = \langle \alpha_0, \alpha_1, \ldots, \alpha_{n/2-1} \rangle$.

Assume L is even, $L \neq 4$. By Lemma 3.2 (ii), $\langle \beta_0, \beta_1, \ldots, \beta_{N-1} \rangle \leq G_{\mathcal{A}}$. Pick $\sigma \in G_{\mathcal{A}}$. For every i $(i \in \mathbb{Z}_n)$ the automorphism σ either fixes or interchanges B_i and $B_{i+n/2}$. However, since $L \neq 4$, if σ interchanges B_i and $B_{i+n/2}$, then it must interchange every pair of antipodal vertices of the inner cycle containing B_i (and therefore also $B_{i+n/2}$). Hence $G_{\mathcal{A}} \leq \langle \beta_0, \beta_1, \ldots, \beta_{N-1} \rangle$, implying $G_{\mathcal{A}} = \langle \beta_0, \beta_1, \ldots, \beta_{N-1} \rangle$.

Assume L is odd. Again, by Lemma 3.2 (iii), we have $\langle \gamma_0, \gamma_1, \ldots, \gamma_{N/2-1} \rangle \leq G_{\mathcal{A}}$. Pick $\sigma \in G_{\mathcal{A}}$ and assume that σ interchanges B_i and $B_{i+n/2}$. Note that B_i and $B_{i+n/2}$ are now in different inner cycles. Therefore, σ must interchange every B_j of the inner cycle containing B_i with $B_{j+n/2}$ (which is contained in the same inner cycle as $B_{i+n/2}$). It is now clear that $\sigma \in \langle \gamma_0, \gamma_1, \ldots, \gamma_{N/2-1} \rangle$. This implies $G_{\mathcal{A}} = \langle \gamma_0, \gamma_1, \ldots, \gamma_{N/2-1} \rangle$. □

Proposition 3.4. *Let $G_{\{\mathcal{A}\}}$ be the set-wise stabiliser of $\{A_0, A_1, \ldots, A_{n-1}\}$ in G. Then the following hold.*

(i) *If $a \neq n/2$ then $G_{\{\mathcal{A}\}} = \langle \rho, \mu \rangle$.*
(ii) *If $a = n/2$ and $L = 4$, then $G_{\{\mathcal{A}\}} = \langle \rho, \mu, \alpha_0 \rangle$.*
(iii) *If $a = n/2$, L is even and $L \neq 4$, then $G_{\{\mathcal{A}\}} = \langle \rho, \mu, \beta_0 \rangle$.*
(iv) *If $a = n/2$ and L is odd, then $G_{\{\mathcal{A}\}} = \langle \rho, \mu, \gamma_0 \rangle$.*

Proof. Let $\sigma \in G_{\{\mathcal{A}\}}$. Observe that the group induced by $G_{\{\mathcal{A}\}}$ on $\mathcal{A}$ is $\langle \rho, \mu \rangle$, since the subgraph induced by $\mathcal{A}$ is a cycle. Therefore, $\rho^k \mu^\ell \sigma \in G_{\mathcal{A}}$ for appropriate $k \in \mathbb{Z}_n$, $\ell \in \mathbb{Z}_2$. The result now follows from Lemma 3.3 and Lemma 3.1. □

Corollary 3.5. *Assume the automorphism group of $R_n(a, r)$ has three orbits on the edge-set of $R_n(a, r)$ (that is, $R_n(a, r)$ does not satisfy any of the conditions (i) and (ii) of Theorem 2.3). Then the following hold.*

(i) *If $a \neq n/2$ then $G = \langle \rho, \mu \rangle$.*
(ii) *If $a = n/2$ and $L = 4$, then $G = \langle \rho, \mu, \alpha_0 \rangle$.*
(iii) *If $a = n/2$, L is even and $L \neq 4$, then $G = \langle \rho, \mu, \beta_0 \rangle$.*
(iv) *If $a = n/2$ and L is odd, then $G = \langle \rho, \mu, \gamma_0 \rangle$.*

Proof. If $R_n(a, r)$ has three orbits on the edge-set, then one of these three orbits is the set of rim edges. Therefore $G = G_{\{\mathcal{A}\}}$. The result now follows from Proposition 3.4. □

We now turn our attention to the case when $R_n(a, r)$ has two orbits on edges. In this case, in view of Lemma 2.1, the rim edges and the hub edges are in the same orbit, implying that $\gcd(n, r) = 1$. Additionally, every automorphism of such a rose window graph must either fix the rim and hub or interchange them. Now suppose that we have an automorphism ω that interchanges the rim and hub. For any automorphism δ of $R_n(a, r)$, we then have that $\omega\delta$ or δ is contained in $G_{\mathcal{A}}$ (noting that the set-wise stabilizer of $\{A_i : i \in \mathbb{Z}_n\}$ is the same as the set-wise stabilizer of $\{B_i : i \in \mathbb{Z}_n\}$). Thus in order to calculate the automorphism groups of such graphs, we need only find one ω that interchanges the rim and hub, and then $G = \langle G_{\mathcal{A}}, \omega \rangle$. Of course, $G_{\mathcal{A}}$ is given in Lemma 3.3, and we need only consider the parameters listed in Theorem 2.3.

Definition 3.6. (i) Assume $r^2 \equiv \pm 1 \pmod n$ and $ra \equiv -a \pmod n$. Then we define $\delta : V \to V$ by $\delta(A_i) = B_{ri}$ and $\delta(B_i) = A_{ri}$.

(ii) Assume n is divisible by 4, r is odd, $a = n/2$ and $(r^2 + n/2) \equiv \pm 1 \pmod n$. Then we define $\gamma : V \to V$ by $\gamma(A_i) = B_{ri}$ and $\gamma(B_i) = A_{ri+(n/2)i}$.

Lemma 3.7. *Assume $r^2 \equiv \pm 1 \pmod n$ and $ra \equiv -a \pmod n$. Then $\delta \in G$, where δ is as defined in Definition 3.6(i).*

Proof. Note that δ is a bijection since $\gcd(n, r) = 1$. The proof of the fact that δ is an automorphism of $R_n(a, r)$ is straightforward. □

Lemma 3.8. *Assume n is divisible by 4, r is odd, $a = n/2$ and $(r^2 + n/2) \equiv \pm 1 \pmod n$. Then $\gamma \in G$, where γ is as defined in Definition 3.6(ii).*

Proof. We will show that γ is a bijection as once this is established it is straightforward to verify that $\gamma \in G$. Clearly, γ maps $\{A_i \mid i \in \mathbb{Z}_n\}$ bijectively to $\{B_i \mid i \in \mathbb{Z}_n\}$ as r is a unit. Similarly, γ maps $\{B_i \mid i \in \mathbb{Z}_n, i \text{ odd}\}$ bijectively to $\{A_i : i \in \mathbb{Z}_n, i \text{ odd}\}$, and $\{B_i \mid i \in \mathbb{Z}_n, i \text{ even}\}$ to $\{A_i : i \in \mathbb{Z}_n, i \text{ even}\}$. Hence γ is a bijection. □

Corollary 3.9. *Assume the automorphism group of $R_n(a, r)$ has two orbits on the edge-set of $R_n(a, r)$. Then, in view of Theorem 2.3, the following hold.*

(i) *If $a \neq n/2$ and $r^2 \equiv 1 \pmod n$, then $G = \langle \rho, \mu, \delta \rangle$.*

(ii) *If $a = n/2$, $r^2 \equiv \pm 1 \pmod n$ and $ra \equiv -a \pmod n$, then $G = \langle \rho, \mu, \beta_0, \delta \rangle$.*

(iii) *If n is divisible by 4, r is odd, $a = n/2$ and $(r^2 + n/2) \equiv \pm 1 \pmod n$, then $G = \langle \rho, \mu, \beta_0, \gamma \rangle$.*

We remark that in the case (ii) of the previous corollary when $r^2 = -1$, listing β_0 as a generator is redundant as $\delta^2 \mu = \beta_0$. In (iii), β_0 is superfluous as if $r^2 + n/2 \equiv -1 \pmod n$ then $\beta_0 = \gamma^2 \mu$ while if $r^2 + n/2 \equiv 1 \pmod n$ then $\beta_0 = \rho^{-1} \gamma \rho^r \gamma$.

The full automorphism group of all rose window graphs are now known with the previous result. We may then check each case to determine which are vertex-transitive. But given that ρ is always an automorphism of $R_n(a, r)$, $R_n(a, r)$ is vertex-transitive if and only if there is an automorphism of $R_n(a, r)$ which maps a rim vertex to a hub vertex and an automorphism which maps a hub vertex to a rim vertex. Recall that a rose window graph has either three, two or one edge orbit. It has at most two edge orbits if and only if there is an automorphism which maps rim edges (vertices) to hub edges (vertices) and vice versa. Therefore, a rose window graph is vertex-transitive if and only if it has either one or two edge orbits. The edge-transitive rose window graphs are given in [5] and their full automorphism groups were obtained in [6]. Combining this information with Theorem 2.3, we obtain the following result which characterizes exactly which rose window graphs are vertex-transitive.

Theorem 3.10. *Let $n \geq 3$ be an integer and $a, r \in \mathbb{Z}_n \setminus \{0\}$. The rose window graph $R_n(a, r)$ is vertex-transitive if and only if one of the following holds:*

(i) *$r^2 \equiv \pm 1 \pmod n$ and $ra \equiv \pm a \pmod n$;*

(ii) *n is divisible by* 4, *r is odd,* $a = n/2$ *and* $(r^2 + n/2) \equiv \pm 1 \pmod n$*;*

(iii) *n is divisible by* 2, $a = n/2 \pm 2$, *and* $r = n/2 \pm 1$*;*

(iv) *n is divisible by 12,* $a = \pm(n/4+2)$, *and* $r = \pm(n/4-1)$ *or* $a = \pm(n/4-2)$ *and* $r = \pm(n/4+1)$*; or*

(v) *n is divisible by* 2, $a = 2b$, *where* $b^2 \equiv \pm 1 \pmod{n/2}$, *and r is odd such that* $r \equiv \pm 1 \pmod{n/2}$.

4 Isomorphisms of non edge-transitive rose window graphs

Let $R_n(a,r)$ and $R_n(a_1,r_1)$ be non edge-transitive rose window graphs. In this section we consider the problem of finding conditions on a, r, a_1, r_1 to ensure that $R_n(a,r)$ and $R_n(a_1,r_1)$ are isomorphic. For the remainder of this paper, we will, as usual, denote the vertices of $R_n(a,r)$ by $\{A_0, A_1, \ldots, A_{n-1}\} \cup \{B_0, B_1, \ldots, B_{n-1}\}$. The vertices of the rose window graph $R_n(a_1,r_1)$ will be denoted in the natural way by $\{C_0, C_1, \ldots, C_{n-1}\} \cup \{D_0, D_1, \ldots, D_{n-1}\}$. Let ρ and μ denote the automorphisms of $R_n(a,r)$ defined at the beginning of this paper, and ρ_1 and μ_1 the corresponding automorphisms of $R_n(a_1,r_1)$.

Theorem 4.1. *Let $R_n(a,r)$ and $R_n(a_1,r_1)$ be rose window graphs. If one of the following holds, then $R_n(a,r)$ and $R_n(a_1,r_1)$ are isomorphic:*

(i) $r_1 = \pm r$ *and* $a_1 = \pm a$*;*

(ii) $\gcd(n,r) = 1$, $r_1 = \pm r^{-1}$, *and* $a_1 = \pm a r^{-1}$*;*

(iii) *n is even with* $\gcd(n,r) = \gcd(n, n/2+r)$, $a = a_1 = n/2$ *and* $r_1 = \pm(r+n/2)$*;*

(iv) *n is even with* $\gcd(n,r) = \gcd(n, n/2+r) = 1$, $a = a_1 = n/2$ *and* $r_1 = \pm(r+n/2)^{-1}$*;*

(v) $r = \pm 1$, $r_1 = \pm 1$, $\gcd(n,a) = \gcd(n,a_1) = 2$ *and* $aa_1/2 \equiv \pm 2 \pmod n$*;*

(vi) $\gcd(n, n/2-1) = 1$, $r = \pm(n/2-1)$, $r_1 = \pm(n/2-1)$, $\gcd(n,a) = \gcd(n,a_1) = 2$ *and* $aa_1/2 \equiv \pm 2 \pmod n$.

Proof. (i) Note that $R_n(a,r) = R_n(a,-r)$ and that an isomorphism between $R_n(a,r)$ and $R_n(-a,r)$ is given by $\phi(A_i) = C_{-i}$ and $\phi(B_i) = D_{-i}$ for $i \in \mathbb{Z}_n$.

(ii) Assume $\gcd(n,r) = 1$, $r_1 = r^{-1}$, and $a_1 = ar^{-1}$. Then an isomorphism from $R_n(a,r)$ to $R_n(a_1,r_1)$ is given by $\phi(A_i) = D_{-ir^{-1}}$ and $\phi(B_i) = C_{-ir^{-1}}$ for $i \in \mathbb{Z}_n$. The result now follows from (i) above.

(iii) Let $L = \frac{n}{\gcd(n,r)} = \frac{n}{\gcd(n,n/2+r)}$, the length of the inner cycles of $R_n(a,r)$ and $R_n(a_1,r_1)$. We first claim that L is divisible by 4. To this end let $n = 2^i n_o$ and $r = 2^j r_o$, where n_o and r_o are odd positive integers. Since $\gcd(n,r) = \gcd(n, n/2+r)$, we also have that $\gcd(n,r) = \gcd(n/2, r)$, and so $j \le i-1$. Assume now that $j = i-1$. Then $n/2 + r = 2^{i-1}(n_o + r_o) = 2^i(n_o+r_o)/2$ (note that $n_o + r_o$ is even). This shows that $\gcd(n, n/2+r)$ is divisible by 2^i. Since $\gcd(n,r)$ is not divisible by 2^i, we have a contradiction. Therefore, $j \le i-2$, and so L is divisible by 4.

Now define $\phi : V(R_n(n/2,r)) \mapsto V(R_n(n/2, n/2+r))$ by $\phi(A_i) = C_i$ for $i \in \mathbb{Z}_n$ and $\phi(B_{\ell+kr}) = D_{\ell+kr+kn/2}$ for $0 \le \ell \le \gcd(n,r)-1$ and $0 \le k \le L-1$. Choose an inner cycle C of $R_n(n/2,r)$. Note that ϕ maps C to an inner cycle of $R_n(n/2, n/2+r)$, and while doing so, the only change in every other vertex is changing B_i to D_i and on the remaining vertices ϕ interchanges "antipodal vertices" of the cycle. This will produce a bijection if and only if L is divisible by 4, and so ϕ is a bijection. It is now routine to check that ϕ is an isomorphism. The result now follows from (i) above.

(iv) Immediately from (ii) and (iii) above.

(v) Assume $r = 1$, $r_1 = 1$, $\gcd(n, a) = \gcd(n, a_1) = 2$ and $aa_1/2 \equiv 2 \pmod{n}$. Define a mapping from $V(R_n(a, r))$ to $V(R_n(a_1, r_1))$ by $\phi(A_{2i}) = C_{ia_1}$, $\phi(A_{2i+1}) = D_{ia_1}$, $\phi(B_{2i}) = C_{ia_1+1}$, $\phi(B_{2i+1}) = D_{ia_1+1}$ for $0 \le i \le n/2 - 1$. Observe that ϕ is a bijection as $\gcd(n, a_1) = 2$. It is also clear that ϕ is an isomorphism. The result now follows from (i) above.

(vi) Assume $r = n/2 - 1$, $r_1 = n/2 - 1$, $\gcd(n, a) = \gcd(n, a_1) = 2$ and $aa_1/2 \equiv 2 \pmod{n}$. Note that since $\gcd(n, n/2 - 1) = 1$, we have that $n/2$ is even. Furthermore, since $\gcd(n, a) = \gcd(n, a_1) = 2$, $a/2$ and $a_1/2$ are odd. Define a mapping from $V(R_n(a, r))$ to $V(R_n(a_1, r_1))$ by $\phi(A_{2i}) = C_{ia_1}$, $\phi(A_{2i+1}) = D_{ia_1}$, $\phi(B_{2i}) = C_{1+ia_1}$, $\phi(B_{2i+n/2+1}) = D_{1+ia_1}$ for $0 \le i \le n/2 - 1$. Observe that ϕ is a bijection as $\gcd(n, a_1) = 2$. Furthermore, since $a_1/2$ is odd (and so $(n/4)a_1 = n/2$), ϕ is an isomorphism. The result now follows from (i) above. $\square$

Theorem 4.2. *Let $\phi : R_n(a, r) \to R_n(a_1, r_1)$ be an isomorphism which sends every rim edge of $R_n(a, r)$ to a rim edge of $R_n(a_1, r_1)$. Then one of the following holds:*

(i) $a_1 = \pm a$ *and* $r_1 = \pm r$*;*

(ii) n *is even with* $\gcd(n, r) = \gcd(n, r + n/2)$*,* $a = a_1 = n/2$ *and* $r_1 = \pm(r + n/2)$*.*

Proof. Note that there exist $k \in \mathbb{Z}_n$ and $\ell \in \{0, 1\}$ such that $\mu_1^\ell \rho_1^k \phi$ maps vertex A_i to vertex C_i for each $i \in \mathbb{Z}_n$. Therefore without loss of generality we can assume that ϕ maps vertex A_i to vertex C_i for each $i \in \mathbb{Z}_n$. Observe also that ϕ maps the hub (spoke) edges of $R_n(a, r)$ to the hub (spoke) edges of $R_n(a_1, r_1)$. It follows that $\phi(B_0) \in \{D_0, D_{-a_1}\}$.

Claim 1: If $\phi(B_0) = D_0$ then $a_1 = a$. If, in addition, $a \ne n/2$, then $r_1 = \pm r$. Assume $\phi(B_0) = D_0$. Since vertices B_0 and A_a are adjacent, vertices $\phi(B_0) = D_0$ and $\phi(A_a) = C_a$ are also adjacent. Since $a \ne 0$ this shows that $a_1 = a$. Assume $a \ne n/2$. As A_r and B_r are adjacent, $\phi(A_r) = C_r$ and $\phi(B_r)$ are also adjacent. This shows that $\phi(B_r) \in \{D_r, D_{r-a}\}$. As A_{r+a} and B_r are adjacent, $\phi(A_{r+a}) = C_{r+a}$ and $\phi(B_r)$ are also adjacent. This shows that $\phi(B_r) \in \{D_r, D_{r+a}\}$. Note that since $a \notin \{0, n/2\}$, we have $\{D_r, D_{r-a}\} \cap \{D_r, D_{r+a}\} = \{D_r\}$. Therefore $\phi(B_r) = D_r$ and $r_1 = \pm r$. This proves Claim 1.

Claim 2: If $\phi(B_0) = D_{-a_1}$ then $a_1 = -a$. If, in addition, $a \ne n/2$, then $r_1 = \pm r$. Rearranging the subscripts of the vertices $\{D_0, D_1, \dots, D_{n-1}\}$ according to the rule $x \to x + a_1$ we get the graph $R_n(-a_1, r_1)$ instead of the graph $R_n(a_1, r_1)$. Furthermore, $\phi : R_n(a, r) \to R_n(-a_1, r_1)$ now maps vertex B_0 to (the new) vertex D_0. By Claim 1 we have $-a_1 = a$ and, if $a \ne n/2$, $r_1 = \pm r$. This proves Claim 2.

Assume now $a = a_1 = n/2$ and $r_1 \ne \pm r$. Similarly as above, we find $\phi(B_0) \in \{D_0, D_{n/2}\}$. If $\phi(B_0) = D_0$, then $\phi(B_r) \in \{D_{r_1}, D_{-r_1}\} \cap \{D_r, D_{r+n/2}\}$. Since $r_1 \ne \pm r$, we have $r_1 = \pm(r+n/2)$. It is clear that we have $\gcd(n, r) = \gcd(n, r_1) = \gcd(n, r+n/2)$. The case $\phi(B_0) = D_{n/2}$ is treated similarly. $\square$

Theorem 4.3. *Let $\phi : R_n(a, r) \to R_n(a_1, r_1)$ be an isomorphism which sends every rim edge of $R_n(a, r)$ to a hub edge of $R_n(a_1, r_1)$. Then one of the following holds:*

(i) $a_1 = \pm ar^{-1}$ *and* $r_1 = \pm r^{-1}$*;*

(ii) n *is even with* $\gcd(n,r) = \gcd(n, r+n/2) = 1$*,* $a = a_1 = n/2$ *and* $r_1 = \pm(r + n/2)^{-1}$.

Proof. Since ϕ sends the rim edges of $R_n(a,r)$ to the hub edges of $R_n(a_1,r_1)$, it also sends the hub edges of $R_n(a,r)$ to the rim edges of $R_n(a_1,r_1)$. This shows that $\gcd(n,r) = \gcd(n,r_1) = 1$. Rearranging the vertices of $R_n(a_1,r_1)$ according to the rule $C_i \to D_{ir_1^{-1}}$ and $D_i \to C_{ir_1^{-1}}$ for $i \in \mathbb{Z}_n$ we obtain the graph $R_n(-a_1 r_1^{-1}, r_1^{-1})$ instead of the graph $R_n(a_1,r_1)$. Moreover, ϕ now satisfies the assumptions of Theorem 4.2. If Theorem 4.2 (i) holds, then $r_1 = \pm r^{-1}$ and $a_1 = \pm a r^{-1}$. If Theorem 4.2 (ii) holds, then n is even with $\gcd(n, r+n/2) = \gcd(n,r) = 1$, $a = -a_1 r_1^{-1} = n/2$ and $r_1^{-1} = \pm(r+n/2)$. Since r_1 is odd (recall that $\gcd(n,r_1) = 1$), $-a_1 r_1^{-1} = n/2$ is equivalent to $a_1 = n/2$. The result follows. □

Theorem 4.4. *Let* $\phi : R_n(a,r) \to R_n(a_1,r_1)$ *be an isomorphism which sends every rim edge of* $R_n(a,r)$ *to a spoke edge of* $R_n(a_1,r_1)$*. Then one of the following holds:*

(i) $r = \pm 1$*,* $r_1 = \pm 1$*,* $\gcd(n,a) = \gcd(n,a_1) = 2$ *and* $aa_1/2 \equiv \pm 2 \pmod n$*;*

(ii) $\gcd(n, n/2-1) = 1$*,* $r = \pm(n/2-1)$*,* $r_1 = \pm(n/2-1)$*,* $\gcd(n,a) = \gcd(n,a_1) = 2$ *and* $aa_1/2 \equiv \pm 2 \pmod n$.

Proof. Observe first that as the rim edges of $R_n(a,r)$ are mapped to the spoke edges of $R_n(a_1,r_1)$, the outer cycle of $R_n(a,r)$ is mapped to a cycle of even length in $R_n(a_1,r_1)$. This shows that n is even. Next, as the rim edges of $R_n(a,r)$ are mapped to the spoke edges of $R_n(a_1,r_1)$, the image of a rim edge has endpoints a rim vertex and a hub vertex. As hub edges and rim edges have no endpoints in common, the images of hub edges and rim edges have no endpoints in common. This implies that hub edges cannot be mapped either to the hub edges or the rim edges, and so the hub edges of $R_n(a,r)$ are also mapped to the spoke edges of $R_n(a_1,r_1)$. As the spoke edges of $R_n(a_1,r_1)$ form a single edge orbit, we have that the hub and rim edges of $R_n(a,r)$ also forms a single edge orbit. This shows that $R_n(a,r)$ and $R_n(a_1,r_1)$ have two orbits on edges (and so $\gcd(n,r) = \gcd(n,r_1) = 1$). We may thus assume that $\phi(A_0) = C_i$ and $\phi(A_1) \in \{D_i, D_{i-a_1}\}$ for some $i \in \mathbb{Z}_n$. Multiplying ϕ by appropriate powers of μ_1 and ρ_1 we can further assume that $\phi(A_0) = C_0$ and $\phi(A_1) = D_0$. This implies that $\phi(A_{2i}) = C_{ia_1}$ and $\phi(A_{2i+1}) = D_{ia_1}$ for $0 \le i \le n/2 - 1$. Therefore, the order of a_1 in $\mathbb{Z}_n$ is $n/2$ and thus $\gcd(n,a_1) = 2$. Reversing the role of $R_n(a,r)$ and $R_n(a_1,r_1)$ we also obtain that $\gcd(n,a) = 2$. Note that this also shows that n, a and a_1 are all even. Since $R_4(2,1)$ is edge-transitive, we may assume that $n \ge 6$.

Observe now that $\phi(B_0) \in \{C_1, C_{-1}\}$. We will assume $\phi(B_0) = C_1$; the case $\phi(B_0) = C_{-1}$ is treated similarly. Since B_0 and A_a are adjacent, $\phi(B_0) = C_1$ and $\phi(A_a) = C_{(a/2)a_1}$ are also adjacent. This shows that $aa_1/2 \equiv 2 (\mathrm{mod}\ n)$. Furthermore, $\phi(B_{2ir}) \in \{C_0, C_1, \ldots, C_{n-1}\}$ and $\phi(B_{(2i+1)r}) \in \{D_0, D_1, \ldots, D_{n-1}\}$ for $0 \le i \le n/2 - 1$.

Recall that $\gcd(n,r) = \gcd(n,r_1) = 1$. In particular, r, r_1, r^{-1} and r_1^{-1} are odd. Since $B_1 = B_{r^{-1}r}$ this shows that $\phi(B_1) \in \{D_0, D_1, \ldots, D_{n-1}\}$. Since B_1 and A_1 are adjacent, $\phi(B_1)$ and $\phi(A_1) = D_0$ are also adjacent. Since B_1 and A_{a+1} are adjacent, $\phi(B_1)$ and $\phi(A_{a+1}) = D_{(a/2)a_1} = D_2$ are also adjacent. Therefore $\phi(B_1) \in \{D_{r_1}, D_{-r_1}\} \cap \{D_{2+r_1}, D_{2-r_1}\}$. This shows that $r_1 = \pm 1$ or $r_1 = \pm(n/2-1)$. Reversing the role of $R_n(a,r)$ and $R_n(a_1,r_1)$ we also obtain that $r = \pm 1$ or $r = \pm(n/2-1)$.

Since $R_n(a,r) = R_n(a,-r)$, we need only show $r = 1$ and $r_1 = n/2 - 1$ or $r = n/2 - 1$ and $r_1 = 1$ cannot occur.

Suppose that $r = 1$ and $r_1 = n/2 - 1$. We saw in the previous paragraph that $\phi(B_1) \in \{D_{r_1}, D_{-r_1}\} \cap \{D_{2+r_1}, D_{2-r_1}\}$. Since $n \geq 6$ this implies $\phi(B_1) = D_{n/2+1}$. But B_0 and B_1 are adjacent, and so $\phi(B_0) = C_1$ and $\phi(B_1) = D_{n/2+1}$ are also adjacent. As $1 \neq n/2+1$ this implies $n/2+1+a_1 = 1$ and thus $a_1 = n/2$. It follows that $\gcd(n, a_1) = n/2 \geq 3$, contradicting $\gcd(n, a_1) = 2$.

Finally, if $r = n/2 - 1$ and $r_1 = 1$, then by reversing the roles of $R_n(a,r)$ and $R_n(a_1, r_1)$, this case cannot occur by arguments in the previous case. □

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ARS MATHEMATICA CONTEMPORANEA

Also available at http://amc-journal.eu
ISSN 1855-3966 (printed edn.), ISSN 1855-3974 (electronic edn.)
ARS MATHEMATICA CONTEMPORANEA 9 (2015) 77–91

Minimal covers of equivelar toroidal maps

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Received 8 November 2012, accepted 29 April 2014, published online 11 July 2014

Abstract

Given any equivelar map on the torus, it is natural to consider its covering maps. The most basic of these coverings are finite toroidal maps or infinite tessellations of the Euclidean plane. In this paper, we prove that each equivelar map on the torus has a unique minimal toroidal rotary cover and also a unique minimal toroidal regular cover. That is to say, of all the toroidal rotary (or regular) maps covering a given map, there is a unique smallest. Furthermore, using the Gaussian and Eisenstein integers, we construct these covers explicitly.

Keywords: Minimal covers, Regular and rotary maps, Gaussian and Eisenstein integers.

Math. Subj. Class.: 52B15, 51M20, 52C22

1 Introduction

The classification of regular and chiral maps on the torus was first published in 1948 by Coxeter [3]. Recently, other classifications of toroidal maps have been presented, e.g. [1], [6], [11]. It is known that every equivelar toroidal map has a universal covering, in that it can be covered by a regular tessellation of the Euclidean plane. However, equivelar toroidal maps can be covered by smaller maps that also have high degrees of symmetry. Notably, each equivelar toroidal map has a unique finite minimal regular cover (see [7], [14]).

Recently, there has been much interest in finding such minimal regular covers for different families of maps and abstract polytopes (see for example [8, 9, 17]). Our paper is part of this broader project of understanding minimal covers. However, we focus not only

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on regular covers, where much is known, but also on rotary covers (those that are either regular or chiral).

In this paper we give a proof that any equivelar toroidal map has a unique minimal regular cover on the torus. While this result follows from the previously mentioned work on, our techniques allow for the explicit construction of these covers using the Gaussian and Eisenstein integers. Furthermore, we prove a similar result, that each such map is uniquely covered by a finite minimal rotary toroidal map, and again using number theoretic results, we can construct these covers.

Our main results are the summarized by the following theorem.

Theorem 1.1. *Each equivelar map on the torus has a unique minimal rotary cover on the torus, and a unique minimal regular cover on the torus. Each can be constructed explicitly.*

An unpublished construction of these minimal regular covers was also made by Maksym Skoryk using an application of the theory of linear Diophantine equations. However, in this paper, we use a different approach, which yields the construction of minimal regular covers. Additionally, our construction provides the more general result concerning minimal rotary covers.

The paper is organized as follows. In section 2 we give necessary definitions and state results about maps and covers. Section 3 consists mainly of relevant number theoretic results regarding the Gaussian and the Eisenstein integers. This section concludes with the technical statement of two theorems, which together prove Theorem 1.1. The following two sections provide the proof of these two theorems, where Section 4 deals with minimal rotary covers and Section 5 considers regular covers. Finally, in order to illustrate our results, we construct such minimal covers for some families of equivelar toroidal maps in Section 6.

2 Maps and Covers

In this section we provide some definitions and results that will be of use in the future; most of these ideas, as well as further details, can be found in [2, 13, 18].

A finite graph X embedded on a compact 2-dimensional manifold S such that every connected component of $S\backslash X$ (which is called a *face*) is homeomorphic to an open disc is called a *map* $\mathcal{M}$ (on the surface S). An *automorphism* of $\mathcal{M}$ is an automorphism of the underlying graph X that can be extended to a homeomorphism of the surface S. The automorphisms of a map $\mathcal{M}$ form a group, which is called the *automorphism group*, denoted $\mathrm{Aut}(M)$. Note that although any map is symmetric under uncountably many homeomorphisms for the underlying surface S, we adopt a purely combinatorial (hence finite) point of view. A map $\mathcal{M}$ is *equivelar* of (Schläfli) type $\{p, q\}$ if all its vertices are q-valent and all its faces are topological p-gons.

Definition 2.1. Let $\mathcal{M}$ and $\mathcal{N}$ be maps on the surfaces S and S' respectively. A surjective function $\eta\colon S' \to S$ that preserves adjacency and sends vertices to vertices, edges to edges, and faces to faces of the maps $\mathcal{M}$ and $\mathcal{N}$ is called a *covering* of the map $\mathcal{M}$ by the map $\mathcal{N}$. This is denoted by $\mathcal{N} \searrow \mathcal{M}$, and we say that $\mathcal{N}$ is a cover of $\mathcal{M}$.

It follows from Theorem 4.14 of [15], that any equivelar map $\mathcal{M}$ is covered by the universal map $\mathcal{U}$ of the same type $\{p, q\}$, and that $\mathcal{M}$ is a quotient of $\mathcal{U}$ by some subgroup of the automorphism group of $\mathcal{U}$.

If $\mathcal{M}$ and $\mathcal{N}$ are both of type $\{p, q\}$ then, it can be easily shown from the above definition that there is a constant number K (possibly infinite) of faces of $\mathcal{N}$ that are sent to any face of $\mathcal{M}$. This follows directly from the connectedness of the maps and the adjacency preserving property of a covering. Furthermore, this number K is also the number of edges or vertices of $\mathcal{N}$ that are sent to an edge or vertex of $\mathcal{M}$. In this case, we say that $\mathcal{N} \searrow \mathcal{M}$ is a *K-sheeted covering*.

Hereafter, we consider equivelar maps of type $\{p, q\}$ on a 2-dimensional torus $\mathbb{T}^2$, which are simply called *equivelar toroidal maps*. On the torus the only possible types to consider are $\{4, 4\}$, $\{3, 6\}$, and $\{6, 3\}$. However, since any map of type $\{6, 3\}$ is dual to one of type $\{3, 6\}$, and a cover of the dual is the dual of a cover, we will only consider maps of type $\{4, 4\}$, and $\{3, 6\}$ in this paper.

Equivelar toroidal maps can be seen as quotients of regular tessellations of the Euclidean plane. Given a regular tessellation τ of the plane, we denote the group of translations that preserve the tessellation as T_τ. For each group T_τ, we can pick two shortest linearly independent translations $\mathbf{e_1}$, $\mathbf{e_2}$ which generate T_τ. Furthermore, in this paper $\mathbf{e_1}, \mathbf{e_2}$ are chosen with equal length and with the angle between them equal to $\frac{\pi}{2}$ for $\{4, 4\}$, or $\frac{\pi}{3}$ for $\{3, 6\}$. We call $(\mathbf{e_1}, \mathbf{e_2})$ *the basis connected with the tessellation* τ. Hereafter, for each tessellation we fix the basis connected with it and will consider coordinates of all other vectors in the plane with respect to this basis.

The next theorem gives an essential characterization of all equivelar toroidal maps.

Theorem 2.2. *[16] Let $\mathcal{M}$ be an equivelar map on the torus. Then $\mathcal{M}$ can be obtained as a quotient of a regular tessellation τ of the Euclidean plane by some translation subgroup $G < T_\tau$ generated by two linearly independent vectors. That is to say, $\mathcal{M} = \tau/G$.*

This theorem shows that there is a one-to-one correspondence between equivelar toroidal maps of type $\{p, q\}$ and translation subgroups of T_τ generated by two non-collinear vectors, where τ is the regular tessellation of the plane of type $\{p, q\}$. We also point out that the converse of this theorem is obvious; any map on the torus that is obtained as a quotient of a regular tessellation by a translation subgroup is an equivelar map. We use as a standard notation $\tau_{\mathbf{a},\mathbf{b}} := \tau/\langle \mathbf{a}, \mathbf{b}\rangle$ to indicate a map obtained as a quotient of a plane regular tessellation τ by a subgroup $\langle \mathbf{a}, \mathbf{b}\rangle < T_\tau$.

A *flag* of a planar tessellation τ is a triple of an incident vertex, edge, and face of the tessellation. We then define a flag of a map $\tau_{\mathbf{a},\mathbf{b}}$ as the orbit of a flag under the group $\langle \mathbf{a}, \mathbf{b}\rangle$. When the map is combinatorially equivalent to an abstract polytope (see [13]), this is equivalent to a flag equalling a triple of an incident vertex, edge, and face of the map itself. Two flags of a map on the torus are said to be *adjacent* if they lift to flags in the plane that differ in exactly one element.

Definition 2.3. A map $\mathcal{M}$ is called regular if and only if $\mathrm{Aut}(M)$ acts transitively on the set of flags. A map $\mathcal{M}$ is called chiral if $\mathrm{Aut}(M)$ has two orbits on flags with adjacent flags lying in different orbits.

Definition 2.4. A map $\mathcal{M}$ is called rotary if it is either regular or chiral.

We note here that in the classical work of Coxeter (see [4], originally in [3]) rotary maps are called regular, whereas regular and chiral in our terminology were *reflexible* and *irreflexible* respectively following Coxeter's terminology. All regular and rotary toroidal maps with the type $\{p, q\}$ can be easily described in terms of their corresponding translation subgroups.

Theorem 2.5. *[3, 4] A toroidal map $\mathcal{M}$ of type $\{4,4\}$ is rotary if and only if $\mathcal{M} = \{4,4\}/\langle(s,t),(-t,s)\rangle$ for some $s,t \in \mathbb{Z}$, $s^2+t^2 \neq 0$. A toroidal map of type $\{3,6\}$ is rotary if and only if $\mathcal{M} = \{3,6\}/\langle(s,t),(-t,s+t)\rangle$ for some $s,t \in \mathbb{Z}$, $s^2+t^2 \neq 0$. Additionally, such maps are regular if and only if $st(s-t) = 0$.*

It follows from Theorem 2.5 that in order to obtain a rotary toroidal map of type $\{p,q\}$ we must take a quotient of a regular tessellation of the same type by a subgroup generated by vectors $\mathbf{a}$ and $\mathbf{b}$ of equal length, and with the angle between $\mathbf{a}$ and $\mathbf{b}$ equal to $\pi/2$ for type $\{4,4\}$ or equal to $\pi/3$ for type $\{3,6\}$.

If we let $\mathcal{M}, \mathcal{N}$ be equivelar toroidal maps such that $\mathcal{M}$ is a quotient of a plane regular tessellation τ by its translation subgroup $G < T_\tau$, it follows from this definition that $\mathcal{N} \searrow \mathcal{M}$ if and only if there is a subgroup $H < G$ such that $\mathcal{N} = \tau/H$. Additionally, if H is a subgroup of $G = \langle\mathbf{a},\mathbf{b}\rangle$, then $H = \langle\mathbf{u},\mathbf{v}\rangle$, where

$$\begin{aligned} \mathbf{u} &= n_1\mathbf{a} + m_1\mathbf{b}, \\ \mathbf{v} &= n_2\mathbf{a} + m_2\mathbf{b}, \end{aligned} \tag{2.1}$$

for some integers (not all zero) n_1, n_2, m_1, m_2, with $n_1 m_2 \neq n_2 m_1$. For more details on this, see Sections 3 and 4 of [15]. Thus, for equivelar toroidal maps $\mathcal{N} = \tau/H$, $\mathcal{M} = \tau/G$, $H < G$, it follows that if $\mathcal{N} \searrow \mathcal{M}$ is a K-sheeted covering, then the number K of sheets can be easily found from the representation in (2.1) as follows:

$$K = |n_1 m_2 - n_2 m_1|. \tag{2.2}$$

The expression in (2.2) is simply obtained by comparing the area of the parallelogram spanned by $\mathbf{u}, \mathbf{v}$ to that spanned by $\mathbf{a}, \mathbf{b}$.

Let us also consider the covering from another perspective. For a map $\mathcal{M} = \tau/G$, where $G = \langle\mathbf{a},\mathbf{b}\rangle$, we call the parallelogram spanned by the vectors $\mathbf{a}$, $\mathbf{b}$ a *fundamental region* of $\mathcal{M}$. Then, a fundamental region for the covering map $\mathcal{N} = \tau/H$ can be viewed as K fundamental regions of $\mathcal{M}$ glued together (see figure 1). It is easy to show that the number K of sheets is equal to the index $[G : H]$ of the groups generating the cover $\tau/H \searrow \tau/G$.

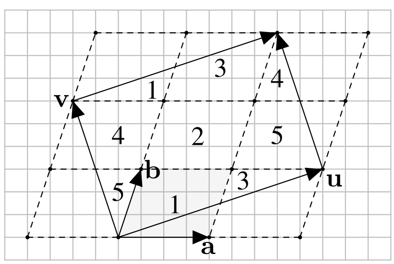

Figure 1: $\{4,4\}_{\mathbf{u},\mathbf{v}} \searrow \{4,4\}_{\mathbf{a},\mathbf{b}}$ is a 5-sheeted covering, and the covering map $\{4,4\}_{\mathbf{u},\mathbf{v}}$ is obtained by gluing together 5 fundamental regions of $\{4,4\}_{\mathbf{a},\mathbf{b}}$

Definition 2.6. [9], [13] An equivelar toroidal map $\mathcal{N}$ is a *minimal rotary (regular) toroidal cover* of an equivelar toroidal map $\mathcal{M}$ if $\mathcal{N}$ is a rotary (regular) map, $\mathcal{N} \searrow \mathcal{M}$, and for any other rotary (regular) equivelar toroid $\mathcal{L}$ such that $\mathcal{N} \searrow \mathcal{L} \searrow \mathcal{M}$ it follows that $\mathcal{L} = \mathcal{N}$ or $\mathcal{L} = \mathcal{M}$.

3 The Gaussian and Eisenstein integers

The Gaussian and Eisenstein integers provide an essential tool for constructing minimal covers for equivelar toroidal maps. Plotting these domains in the complex plane produces precisely the vertex set of a regular tessellation: $\tau = \{4, 4\}$ for the Gaussian integers and $\tau = \{3, 6\}$ for the Eisenstein integers. Since we use the Gaussian and Eisenstein integers to construct minimal covering maps, we must recall some of their properties in this section. We will follow [12].

The *Gaussian integers* $\mathbb{Z}[i]$ are defined as $\{a + bi \,|\, a, b \in \mathbb{Z}\}$, where $i = \sqrt{-1}$. Similarly, the *Eisenstein integers* $\mathbb{Z}[\omega]$ are defined as $\{a + b\omega \,|\, a, b \in \mathbb{Z}\}$, where $\omega = (1 + i\sqrt{3})/2$. Hereafter, we write $\mathbb{Z}[\sigma]$, $\sigma = i, \omega$, to denote either of these two sets.

In this paper we use the following notation: given $\alpha = a + b\sigma \in \mathbb{Z}[\sigma]$, we call its *conjugate* the number $\overline{\alpha} := a + b\overline{\sigma}$, where $\overline{\sigma}$ is the conjugate complex number to $\sigma \in \mathbb{C}$. Also we call $\operatorname{Re}\alpha := a$ and $\operatorname{Im}\alpha := b$ the *real* and *imaginary* parts respectively. Note here that if $\sigma = i$, then this is the traditional notion of 'real part', and 'imaginary part' of a complex number. However, if $\sigma = \omega$, then the 'traditional' real and imaginary parts of $a + b\omega$ are $a + b/2$ and $b\sqrt{3}/2$, respectively.

It is straightforward to prove the following technical lemma, which we will need later.

Lemma 3.1. *For every $\alpha, \beta \in \mathbb{Z}[\sigma]$ the following holds:*

(1) $\operatorname{Im}(\overline{\alpha}\beta) = \operatorname{Re}\alpha \operatorname{Im}\beta - \operatorname{Im}\alpha \operatorname{Re}\beta$*;*

(2) $\operatorname{Im}(\overline{\alpha}\beta) = -\operatorname{Im}(\alpha\overline{\beta})$.

For every $\alpha = a + b\sigma \in \mathbb{Z}[\sigma]$, we assign the *norm* $N(\alpha) := \alpha\overline{\alpha}$. We recall that this is a multiplicative function; namely, for $\alpha, \beta \in \mathbb{Z}[\sigma]$, $N(\alpha\beta) = N(\alpha)N(\beta)$. A number $\alpha \in \mathbb{Z}[\sigma]\backslash\{0\}$ *divides* $\beta \in \mathbb{Z}[\sigma]$ if and only if there is $\gamma \in \mathbb{Z}[\sigma]$ such that $\beta = \alpha\gamma$. In this case we will use the notation $\alpha|\beta$. Recall that, in the ring of Gaussian integers, the units are only $\pm 1, \pm i$, while in the ring of Eisenstein integers the units are only $\pm 1, \pm\omega, \pm\overline{\omega}$.

Two numbers $\alpha, \beta \in \mathbb{Z}[\sigma]$ are *associated* ($\alpha \simeq \beta$) iff $\alpha = \beta\varepsilon$ for some unit ε. It is easy to show that $\simeq$ is an equivalence relation. A number $\pi \in \mathbb{Z}[\sigma]$ is a *prime* in the ring $\mathbb{Z}[\sigma]$ if and only if it is not a unit and $\alpha|\pi$ implies that α is either a unit or associated with π. Equivalently, $\pi \in \mathbb{Z}[\sigma]$ is a prime if and only if it is not a unit and from $\pi|(\alpha\beta)$ it follows that $\pi|\alpha$ or $\pi|\beta$.

The Gaussian and the Eisenstein integers are unique factorization domains and thus there is a theorem analogous to the fundamental theorem of arithmetic for these rings.

Theorem 3.2 (Unique Factorization Theorem, [12])**.** *Let $\mathbb{Z}[\sigma]$ be the ring of Gaussian or Eisenstein integers and let $\mathbb{P}_\sigma \subset \mathbb{Z}[\sigma]$ be a set of primes in $\mathbb{Z}[\sigma]$ such that every prime number $\pi \in \mathbb{Z}[\sigma]$ is associated with a unique prime $\pi' \in \mathbb{P}_\sigma$, so that none of the numbers in $\mathbb{P}_\sigma$ is associated with any other. Then for every $\alpha \in \mathbb{Z}[\sigma]$, $\alpha \neq 0$ there exists a unique representation:*

$$\alpha = \varepsilon\pi_1^{e_1} \dots \pi_n^{e_n}$$

where ε is a unit, $\pi_1, \dots, \pi_n \in \mathbb{P}_\sigma$ are pairwise distinct prime numbers, and $e_1, \dots, e_n \in \mathbb{N}$ are natural numbers.

Additionally, it is possible to describe all the primes in $\mathbb{Z}[\sigma]$ in terms of their norms. For the Gaussian integers $\mathbb{Z}[i]$ there are the following possible types of primes π:

(1) $\pi \simeq 1 + i$, $N(\pi) = 2$. Since $\overline{1 + i} = \overline{\imath}(1 + i)$, $\overline{\pi}$ is associated with π;

(2) rational prime numbers $q \equiv 3 \pmod 4$. As $q \in \mathbb{Z}$ its conjugate $\overline{q} = q$, $\overline{q}$ is associated with q;

(3) $\pi \in \mathbb{Z}[i]$ such that $N(\pi) = p$ is an rational prime $p \equiv 1 \pmod 4$. Note that in this case $\overline{\pi}$ is a prime which is not associated with π.

For the Eisenstein ring $\mathbb{Z}[\omega]$ there are similar types of primes:

(1) $\pi \simeq 1 + \omega$; $N(\pi) = 3, \overline{\pi} \simeq \pi$;

(2) rational prime numbers $q \equiv 2 \pmod 3$, $\overline{q} = q$;

(3) $\pi \in \mathbb{Z}[\omega]$ such that $N(\pi) = p$ is an rational prime $p \equiv 1 \pmod 3$. $\overline{\pi}$ is a prime not associated with π.

The following two lemmas follow from the Unique Factorization Theorem and the classification of the primes. The proofs for the both of this lemmas are straightforward and, therefore, omitted.

Lemma 3.3. *Let $\alpha \in \mathbb{Z}[\sigma]\backslash\{0\}$ and $\alpha = \pi_0^{a_0} q_1^{a_1} \ldots q_s^{a_s} \pi_1^{b_1} \ldots \pi_t^{b_t}$ be its prime decomposition, where $\pi_0 \simeq 1 + \sigma$, for $j \in \{1, \ldots s\}$ the numbers q_j are rational primes of type (2), for $k \in \{1, \ldots t\}$ the numbers π_k are primes of type (3), and all exponents are nonnegative integers. Then the following are equivalent:*

(1) $\operatorname{GCD}(\operatorname{Re}\alpha, \operatorname{Im}\alpha) = 1$*;*

(2) $a_0 \in \{0, 1\}$, $a_1 = \ldots = a_s = 0$ *and* $\pi_j \not\simeq \overline{\pi}_k$ *for all* $k, j \in \{1, \ldots, t\}$.

Lemma 3.4. *Let $\alpha \in \mathbb{Z}[\sigma]\backslash\{0\}$. Then $(1+\sigma)|\alpha$ if and only if $\operatorname{Re}\alpha \equiv \operatorname{Im}\alpha \pmod{N(1+\sigma)}$.*

A complex number $\gamma \in \mathbb{Z}[\sigma]$ is a *greatest common divisor (GCD)* of $\alpha, \beta \in \mathbb{Z}[\sigma]$, $N(\alpha) + N(\beta) \neq 0$, if $\gamma|\alpha$ and $\gamma|\beta$ and for every $\gamma' \in \mathbb{Z}[\sigma]$ such that $\gamma'|\alpha$ and $\gamma'|\beta$ it follows $\gamma'|\gamma$. From the Unique Factorization Theorem it follows that a greatest common divisor is well defined *up to associates*. Thus, we can consider *the GCD equivalence class* $[\operatorname{GCD}(\alpha, \beta)]$.

Further, we use the common notation $\gamma = \operatorname{GCD}(\alpha, \beta)$ implying that $\gamma \in [\operatorname{GCD}(\alpha, \beta)]$. If there is a rational integer $n \in [\operatorname{GCD}(\alpha, \beta)]$, then we specifically take $\operatorname{GCD}(\alpha, \beta) := |n|$. For example, $[\operatorname{GCD}(3, 6i)] = \{3, -3, 3i, -3i\}$, and thus $\operatorname{GCD}(3, 6i) = 3$.

As we have seen, either the Gaussian or the Eisenstein integers can be linked with a regular tessellation τ. Then the numbers 1 and σ from $\mathbb{Z}[\sigma]$ correspond to the basis $(\mathbf{e_1}, \mathbf{e_2})$ of translations connected with τ, and the operations of addition and multiplication by an integer in $\mathbb{Z}[\sigma]$ can be seen as the same operations in T_τ treated as a vector space over $\mathbb{Z}$ with the basis $(\mathbf{e_1}, \mathbf{e_2})$. From this point of view, we will identify every two vectors $\mathbf{a} = (a_1, a_2)$, $\mathbf{b} = (b_1, b_2)$ with the complex numbers $\alpha := a_1 + a_2\sigma \in \mathbb{Z}[\sigma]$ and $\beta := b_1 + b_2\sigma \in \mathbb{Z}[\sigma]$. Finally, using this idea, we will denote $\tau_{\alpha,\beta} := \tau_{\mathbf{a},\mathbf{b}}$. Furthermore we let $\tau_\eta := \tau_{\eta,\sigma\eta}$.

It follows from Theorem 2.5 that, if $\eta \in \mathbb{Z}[\sigma]\backslash\{0\}$, then τ_η is a rotary toroidal map, and all rotary maps can be obtained in this way. Note that if $\eta \simeq \eta'$, then $\tau_\eta = \tau_{\eta'}$. This implies that we need only look for a generator η of a rotary map τ_η up to associates. Additionally, a rotary map τ_η is regular iff $\eta \simeq n$ or $\eta \simeq n(1+\sigma)$, where $n \in \mathbb{Z}\backslash\{0\}$.

Using the notation developed in this section, we are now ready to state the formal versions of the two theorems which together yield the main theorem of this paper. The proofs of these two theorems are the content of the next two sections. We remind the reader of our convenient but non-standard definition for $\operatorname{Im}\gamma$ when $\gamma \in \mathbb{Z}[\omega]$.

Theorem 3.5. *Let $\tau_{\alpha,\beta}$ be an equivelar toroidal map represented as a quotient of a regular planar tessellation τ by a translation subgroup $\langle\alpha,\beta\rangle < T_\tau$ generated by two non-collinear vectors, corresponding to the complex numbers $\alpha,\beta \in \mathbb{Z}[\sigma]$. Let $\gamma = \mathrm{GCD}(\alpha,\beta)$. There exists a unique minimal rotary covering map equal to $\tau_{\eta_{min}}$ with*

$$\eta_{min} = \frac{\mathrm{Im}\,(\overline{\alpha}\beta)}{N(\gamma)}\gamma.$$

Moreover, the number K_{min} of fundamental regions of $\tau_{\alpha,\beta}$ glued together in order to obtain $\tau_{\eta_{min}}$ is equal to

$$K_{min} = \frac{|\mathrm{Im}(\overline{\alpha}\beta)|}{N(\gamma)}.$$

Theorem 3.6. *Let $\tau_{\alpha,\beta}$ be an equivelar toroidal map represented as a quotient of a regular planar tessellation τ by a translation subgroup $\langle\alpha,\beta\rangle < T_\tau$ generated by two non-collinear vectors, corresponding to the complex numbers $\alpha,\beta \in \mathbb{Z}[\sigma]$. Let $c = \mathrm{GCD}(\mathrm{Re}\,\alpha, \mathrm{Im}\,\alpha, \mathrm{Re}\,\beta, \mathrm{Im}\,\beta)$. Then for $\tau_{\alpha,\beta}$ there exists a unique minimal regular covering map $\tau_{\eta_{min}}$ with*

$$\eta_{min} = \begin{cases} \frac{\mathrm{Im}(\overline{\alpha}\beta)}{N(1+\sigma)c}(1+\sigma), & \textit{if } \frac{\mathrm{Re}\,\alpha}{c} \equiv \frac{\mathrm{Im}\,\alpha}{c} \textit{ and } \frac{\mathrm{Re}\,\beta}{c} \equiv \frac{\mathrm{Im}\,\beta}{c} \bmod N(1+\sigma); \\ \frac{\mathrm{Im}(\overline{\alpha}\beta)}{c}, & \textit{otherwise.} \end{cases}$$

Moreover, the number K_{min} of fundamental regions of $\tau_{\alpha,\beta}$ glued together in order to obtain $\tau_{\eta_{min}}$ is equal to

$$K_{min} = \begin{cases} \frac{|\mathrm{Im}(\overline{\alpha}\beta)|}{N(1+\sigma)c^2}, & \textit{if } \frac{\mathrm{Re}\,\alpha}{c} \equiv \frac{\mathrm{Im}\,\alpha}{c} \textit{ and } \frac{\mathrm{Re}\,\beta}{c} \equiv \frac{\mathrm{Im}\,\beta}{c} \bmod N(1+\sigma); \\ \frac{|\mathrm{Im}(\overline{\alpha}\beta)|}{c^2}, & \textit{otherwise.} \end{cases}$$

4 Proof of Theorem 3.5

In this section we give the proof of our first main theorem, which states that every equivelar map on the torus has a unique minimal rotary cover on the torus. To accomplish this proof, we translate the concepts of covers from Section 2 into the terminology developed in Section 3, and construct all possible toroidal rotary covers for a given map. By doing so, we can explicitly find a cover that is minimal.

Let $\tau_{\alpha,\beta}$ be an equivelar toroidal map represented as a quotient of a regular planar tessellation τ by a translation subgroup $\langle\alpha,\beta\rangle < T_\tau$ generated by two non-collinear vectors, corresponding to complex numbers $\alpha,\beta \in \mathbb{Z}[\sigma]$.

We have seen that each rotary map can be described as τ_η for some $\eta \in \mathbb{Z}[\sigma]$. Additionally, we have seen that covering maps satisfy the conditions of (2.1). Thus, in order to obtain a rotary map τ_η which covers $\tau_{\alpha,\beta}$, we need to find a number $\eta \in \mathbb{Z}[\sigma]\backslash\{0\}$ such that:

$$\begin{cases} n_1\alpha + m_1\beta = \sigma\eta, \\ n_2\alpha + m_2\beta = \eta, \end{cases} \tag{4.1}$$

for some integers n_1, m_1, n_2, m_2, with $n_1m_2 \neq n_2m_1$. Let $\nu = -n_1 + n_2\sigma$ and $\mu = m_1 - m_2\sigma$. we must find $\nu,\mu \in \mathbb{Z}[\sigma]$, $N(\nu) + N(\mu) \neq 0$, where (using the notation of Section 3):

$$\begin{cases} -\alpha\ \mathrm{Re}\,\nu + \beta\ \mathrm{Re}\,\mu = \sigma\eta, \\ \ \ \alpha\ \mathrm{Im}\,\nu - \beta\ \mathrm{Im}\,\mu = \eta. \end{cases} \tag{4.2}$$

We proceed by solving system (4.2) considering η as a parameter. From the possible solutions to this system, we can determine the exact value of η which generates the minimal cover $\tau_{\eta_{min}} \searrow \tau_{\alpha,\beta}$.

Multiplying the second equation by σ, and subtracting the first from it; we get the following equation

$$\alpha\nu - \beta\mu = 0. \tag{4.3}$$

Let $\gamma := \mathrm{GCD}(\alpha, \beta)$. Then $\alpha/\gamma, \beta/\gamma \in \mathbb{Z}[\sigma]$, and $\mathrm{GCD}(\alpha/\gamma, \beta/\gamma) = 1$. From this and (4.3) it follows that

$$\nu = \frac{\beta}{\gamma}\delta,\ \mu = \frac{\alpha}{\gamma}\delta, \tag{4.4}$$

for some $\delta \in \mathbb{Z}[\sigma]\backslash\{0\}$. In order to also be a solution to system (4.2) it is sufficient that $\gamma|\eta$, and thus $\eta/\gamma \in \mathbb{Z}[\sigma]$. To solve system (4.2) we simply substitute (4.4) into the second equation from (4.2). This yields:

$$\frac{\alpha}{\gamma}\,\mathrm{Im}\left(\frac{\beta}{\gamma}\delta\right) - \frac{\beta}{\gamma}\,\mathrm{Im}\left(\frac{\alpha}{\gamma}\delta\right) = \frac{\eta}{\gamma}.$$

After multiplying by δ and taking the imaginary and real parts of both sides, we can use both parts of Lemma 3.1 to get that:

$$\begin{aligned} \mathrm{Im}\left(\frac{\eta}{\gamma}\delta\right) &= 0, \\ \mathrm{Re}\left(\frac{\eta}{\gamma}\delta\right) &= N(\delta)\left(\frac{\mathrm{Im}\,(\overline{\alpha}\beta)}{N(\gamma)}\right). \end{aligned}$$

Therefore,

$$\eta = \left(\frac{\mathrm{Im}\,(\overline{\alpha}\beta)}{N(\gamma)}\right)\overline{\delta}\gamma. \tag{4.5}$$

Thus, $\nu = \beta/\gamma \cdot \delta$ and $\mu = \alpha/\gamma \cdot \delta$ is a solution to system (4.2), and a rotary map τ_η covers $\tau_{\alpha,\beta}$ if and only if the parameter η satisfies (4.5). Notice here that $\frac{\mathrm{Im}(\overline{\alpha}\beta)}{N(\gamma)} \in \mathbb{Z}$.

Let us show that the unique minimal rotary cover is generated by the complex number (up to associates)

$$\eta_{min} = \frac{\mathrm{Im}\,(\overline{\alpha}\beta)}{N(\gamma)}\gamma.$$

Indeed, observe that for any $\delta_1, \delta_2 \in \mathbb{Z}[\sigma]\backslash\{0\}$ we have $\tau_{\delta_1\delta_2} \searrow \tau_{\delta_1}$. Therefore, for any $\delta \in \mathbb{Z}[\sigma]\backslash\{0\}$ and the corresponding complex number η satisfying (4.5), the map τ_η covers $\tau_{\eta_{min}}$. Then from definition (2.6) it follows that $\tau_{\eta_{min}}$ is a minimal cover of $\tau_{\alpha,\beta}$. Even more, if δ is not a unit, then the corresponding rotary map $\tau_\eta \neq \tau_{\eta_{min}}$. Thus, $\tau_{\eta_{min}}$ is a unique minimal cover. The first part of Theorem 3.5 is proved.

Let us find the number K_{min} of fundamental regions of $\tau_{\alpha,\beta}$ that we should glue together in order to obtain $\tau_{\eta_{min}}$. From (2.2), for an arbitrary cover τ_η substituting our solutions for μ and ν, this number is equal to

$$\begin{aligned} K &= |m_1 n_2 - m_2 n_1| = |\operatorname{Re}\mu \operatorname{Im}\nu - \operatorname{Re}\nu \operatorname{Im}\mu| = |\operatorname{Im}(\overline{\mu}\nu)| \\ &= \left|\operatorname{Im}\left(\frac{\overline{\alpha}}{\overline{\gamma}}\overline{\delta}\frac{\beta}{\gamma}\delta\right)\right| = \frac{|\operatorname{Im}(\overline{\alpha}\beta)|}{N(\gamma)} N(\delta). \end{aligned} \tag{4.6}$$

Finally, since for the minimal cover $\delta \simeq 1$, we obtain

$$K_{min} = \frac{|\operatorname{Im}(\overline{\alpha}\beta)|}{N(\gamma)}.$$

This completes the proof of Theorem 3.5.

5 Proof of Theorem 3.6

In this section we give the proof of our second main theorem, which describes how every equivelar map on the torus has a unique minimal regular cover on the torus. Let $\tau_{\alpha,\beta}$ be an equivelar toroidal map represented as a quotient of a regular planar tessellation τ by a translation subgroup $\langle \alpha, \beta \rangle < T_\tau$ generated by two non-collinear vectors, corresponding to complex numbers $\alpha, \beta \in \mathbb{Z}[\sigma]$.

It follows from Theorem 2.5 that in order to produce any regular cover on the torus $\tau_\eta \searrow \tau_{\alpha,\beta}$ we need to take a quotient of a regular tessellation τ by a translation subgroup $\langle \eta, \sigma\eta \rangle < T_\sigma$ with $\eta \simeq n$ or $\eta \simeq n(1+\sigma)$, $n \in \mathbb{Z}\backslash\{0\}$.

To proceed, we separately construct the families of regular covers of each type (depending on η), and determine which type produces the minimal cover.

I. Let $\eta \simeq n$, $n \in \mathbb{Z}\backslash\{0\}$. Since we are looking for a value of η up to associates, we may assume $\eta \in \mathbb{Z}$. Then as $\frac{\operatorname{Im}(\overline{\alpha}\beta)}{N(\gamma)} \in \mathbb{Z}$, from (4.5) it follows that $\overline{\delta}\gamma \in \mathbb{Z}$. Writing $\gamma = c\gamma_1$, where $c = \operatorname{GCD}(\operatorname{Re}\gamma, \operatorname{Im}\gamma)$, we have $\operatorname{GCD}(\operatorname{Re}\gamma_1, \operatorname{Im}\gamma_1) = 1$, and from Lemma 3.3 it follows that δ is equal $k\gamma_1$ for some $k \in \mathbb{Z}\backslash\{0\}$. Thus, regular covers of this type have the form τ_{η_I} for

$$\eta_I = \frac{\operatorname{Im}(\overline{\alpha}\beta)}{c} k, \tag{5.1}$$

where $k \in \mathbb{Z}\backslash\{0\}$ is an arbitrary non-zero integer.

II. Let $\eta \simeq n(1+\sigma)$, $n \in \mathbb{Z}\backslash\{0\}$. As in the previous case, (4.5) implies that $\eta \simeq n(1+\sigma)$ if and only if $\overline{\delta}\gamma \simeq n_1(1+\sigma)$ for some $n_1 \in \mathbb{Z}\backslash\{0\}$. Since $\overline{1+\sigma} \simeq 1+\sigma$, we have $(1+\sigma)^{2k} \simeq (N(1+\sigma))^k$.

Thus, we should consider two subcases: $\gamma = (1+\sigma)^{2k+1}\gamma'$ and $\gamma = (1+\sigma)^{2k}\gamma'$, where k is a positive integer and $(1+\sigma) \not|\, \gamma'$.

(IIa) Let $\gamma = (1+\sigma)^{2k+1}\gamma'$. Then $\overline{\delta}\gamma \simeq n_1(1+\sigma)$ if and only if $\overline{\delta}(1+\sigma)^{2k}\gamma'$ is associated with a non-zero integer. As noted above, $(1+\sigma)^{2k} \simeq (N(1+\sigma))^k$. Additionally, if $\gamma' = d\gamma''$ and $d = \operatorname{GCD}(\operatorname{Re}\gamma', \operatorname{Im}\gamma')$, then $\overline{\delta}(1+\sigma)^{2k}\gamma'$ is associated with a non-zero integer if and only if $\delta \simeq \gamma'' l$ for some $l \in \mathbb{Z}\backslash\{0\}$. Note that if $c = \operatorname{GCD}(\operatorname{Re}\gamma, \operatorname{Im}\gamma)$, then $c = (N(1+\sigma))^k d$.

Therefore, regular covers of this subcase have the form $\tau_{\eta_{IIa}}$ for

$$\eta_{IIa} = \frac{\operatorname{Im}(\overline{\alpha}\beta)}{N(1+\sigma)c}(1+\sigma)l, \tag{5.2}$$

where $l \in \mathbb{Z}\backslash\{0\}$ is arbitrary.

(IIb) Let $\gamma = (1+\sigma)^{2k}\gamma'$. Then $\overline{\delta}\gamma \simeq n_1(1+\sigma)$, $n_1 \in \mathbb{Z}\backslash\{0\}$, if and only if $\overline{\left(\frac{\delta}{1+\overline{\sigma}}\right)\gamma}$ is associated with a non-zero integer. As above, if $\gamma = c\gamma_1$, $c = \mathrm{GCD}(\mathrm{Re}\,\gamma, \mathrm{Im}\,\gamma)$, then K is minimal if and only if $\frac{\delta}{1+\overline{\sigma}} \simeq \gamma_1$. Thus using (4.5), we find that regular covering maps $\tau_{\eta_{IIb}}$ for this subcase are provided by

$$\eta_{IIb} = \frac{\mathrm{Im}(\overline{\alpha}\beta)}{c}(1+\sigma)l, \tag{5.3}$$

for an arbitrary non-zero integer l.

Using the observation at the end of Section 4, we now can identify a minimal regular cover in families (5.1) – (5.3). Note that we should consider either the cases I and IIa, or I and IIb.

We claim in subcase IIa that the minimal regular cover $\tau_{\eta_{min}}$ is generated by

$$\eta_{min} = \frac{\mathrm{Im}(\overline{\alpha}\beta)}{N(1+\sigma)c}(1+\sigma).$$

Indeed, it is obvious that $\tau_{\eta_{IIa}} \searrow \tau_{\eta_{min}}$ for any l in η_{IIa} and $\tau_{\eta_{IIa}} \neq \tau_{\eta_{min}}$ unless $l = 1$. Furthermore, since $\eta_I = \eta_{min} \cdot \left(\overline{(1+\sigma)}k\right)$ for any non-zero integer k corresponding to (5.1), we have $\tau_{\eta_I} \searrow \tau_{\eta_{min}}$. Thus, the claim holds.

Turning to subcase IIb, the unique minimal regular cover cover $\tau_{\eta_{min}}$ is likewise generated by

$$\eta_{min} = \frac{\mathrm{Im}(\overline{\alpha}\beta)}{c}.$$

The last thing we should notice here is that

$$c = \mathrm{GCD}(\mathrm{Re}\,\gamma, \mathrm{Im}\,\gamma) = \mathrm{GCD}(\mathrm{Re}\,\alpha, \mathrm{Im}\,\alpha, \mathrm{Re}\,\beta, \mathrm{Im}\,\beta).$$

This follows from the observation that if $k \in \mathbb{Z}, k \neq 0$ and $\alpha \in \mathbb{Z}[\sigma]\backslash\{0\}$, then $k|\alpha$ if and only if $k|\,\mathrm{Re}\,\alpha$ and $k|\,\mathrm{Im}\,\alpha$.

Finally, applying Lemma 3.4 to distinguish between the subcases and computing the corresponding numbers K_{min} of fundamental regions using (4.6), we finish the proof of Theorem 3.6.

6 Examples

To illustrate our theorems, we now build the minimal rotary and minimal regular covering maps for different families of equivelar toroidal maps. Our examples are motivated by the classification of equivelar toroidal maps by symmetry type (found in [11]; a detailed classification of 2- and 4-orbit toroids was first studied in [10]).

6.1 Maps of type $\{4, 4\}$

It is clear that the minimal rotary and minimal regular covering map of a regular map is just the map itself. For a chiral map, the minimal rotary cover also coincides with the initial map; however the minimal regular cover differs. Furthermore, given a map in any of the other families of 2-orbit toroidal maps, the minimal regular cover is equal to the minimal rotary cover. In particular, for integers $a > b > 0$, we have the following four cases.

Example 6.1. For a map $\{4,4\}_{(a,0),(0,b)}$, we get $\alpha = a$, $\beta = bi$, $\gamma = \mathrm{GCD}(a, bi) = \mathrm{GCD}(a,b) = c$, $|\mathrm{Im}(\overline{\alpha}\beta)| = ab$. Thus the minimal regular (and rotary) cover is $\{4,4\}_{\frac{ab}{c}}$, which can be obtained by gluing $\frac{ab}{c^2}$ fundamental regions of the initial map.

Example 6.2. For a map $\{4,4\}_{(a,b),(a,-b)}$, we get $\alpha = a+bi$, $\beta = a-bi$, $c = \mathrm{GCD}(a,b)$, $|\mathrm{Im}(\overline{\alpha}\beta)| = 2ab$. Since $\overline{\alpha} = \beta$, then $\gamma = c$ if $\frac{a}{c} \not\equiv \frac{b}{c} \pmod 2$, or $\gamma = c(1+i)$ if $\frac{a}{c} \equiv \frac{b}{c} \pmod 2$. In the first case the minimal regular (and rotary) cover is $\{4,4\}_{\frac{2ab}{c}}$, and the number of fundamental regions is equal to $\frac{2ab}{c^2}$. In the second case the minimal regular (and rotary) cover is $\{4,4\}_{\frac{ab}{c}(1+i)}$, where the number of fundamental regions of the initial toroidal map we glued together in order to obtain this cover is equal to $\frac{ab}{c^2}$.

Example 6.3. For a map $\{4,4\}_{(a,b),(b,a)}$ the situation is similar to Example 6.2. We have $\alpha = a+bi$, $\beta = b+ai$, $c = \mathrm{GCD}(a,b)$, and $|\mathrm{Im}(\overline{\alpha}\beta)| = a^2 - b^2$. Since $i\overline{\alpha} = \beta$, we get that $\gamma = c$ if $\frac{a}{c} \not\equiv \frac{b}{c} \pmod 2$, or $\gamma = c(1+i)$ otherwise. Thus, the minimal regular (and rotary) cover is built in the same manner as in Example 6.2.

Example 6.4. For the final family of 2-orbit equivelar toroidal maps $\{4,4\}_{(a,a),(b,-b)}$, we get $\alpha = a+ai$, $\beta = b-bi$, $c = \mathrm{GCD}(a,b)$, $\gamma = c(1-i)$, and $|\mathrm{Im}(\overline{\alpha}\beta)| = 2ab$. Thus, the minimal regular (and rotary) cover is $\{4,4\}_{\frac{ab}{c}(1+i)}$, which can be obtained by gluing $\frac{ab}{c^2}$ fundamental regions of the initial map (see Figure 2 (b)).

Given a 4-orbit toroidal map, the minimal regular cover and the minimal rotary cover may or may not coincide; this depends on γ. For example, the minimal regular and minimal rotary covers of the map $\{4,4\}_{(2,1),(7,0)}$ coincide. On the other hand, the covers of $\{4,4\}_{(4,3),(5,0)} = \{4,4\}_{(4,3),(-1,3)}$ are distinct (see Figure 2 (a)).

6.2 Maps of type $\{3,6\}$

The situation for regular and chiral toroidal maps of type $\{3,6\}$ is similar to the $\{4,4\}$ case. Let us instead consider the families of 3-orbit maps of type $\{3,6\}$. Let $a, b \in \mathbb{N}$.

Example 6.5. For a 3-orbit toroidal map $\{3,6\}_{(a,0),(-b,2b)}$, we have that $\alpha = a$, $\beta = -b+2b\omega$, $c = \mathrm{GCD}(a,b)$, and $|\mathrm{Im}(\overline{\alpha}\beta)| = 2ab$. Since $\beta = b\omega(1+\omega)$, we know that $\gamma = c$ if $\frac{a}{c} \not\equiv 0 \pmod 3$ and $\gamma = c(1+\omega)$ otherwise. This implies that the minimal regular (and rotary) cover is equal to $\{3,6\}_{\frac{2ab}{c}}$, where the number of fundamental regions glued to obtain the cover is $\frac{2ab}{c^2}$ in the first case; and $\{3,6\}_{\frac{2ab}{3c}(1+\omega)}$, where the number of fundamental regions is equal to $\frac{2ab}{3c^2}$, in the second case.

Example 6.6. For a 3-orbit map $\{3,6\}_{(a,b),(a+b,-b)}$, we write that $\alpha = a+b\omega$, $\beta = a+b-b\omega$, $c = \mathrm{GCD}(a,b)$, and $|\mathrm{Im}(\overline{\alpha}\beta)| = 2ab+b^2$. Since $\overline{\alpha} = \beta$, we get that $\gamma = c$ if $\frac{a}{c} \not\equiv \frac{b}{c} \pmod 3$ and $\gamma = c(1+\omega)$ otherwise. Thus, the minimal regular (and rotary) cover is $\{3,6\}_{\frac{2ab+b^2}{c}}$, where the number of fundamental regions is equal to $\frac{2ab+b^2}{c^2}$ in the first case; and $\{3,6\}_{\frac{2ab+b^2}{3c}(1+\omega)}$, where the number of fundamental regions is $\frac{2ab+b^2}{3c^2}$ in the second case (see Figure 3(b)).

As in the last subsection, the 6-orbit toroidal maps of type $\{3,6\}$ have minimal regular and minimal rotary covers which may or may not coincide. Again this depends on γ (see Figure 3).

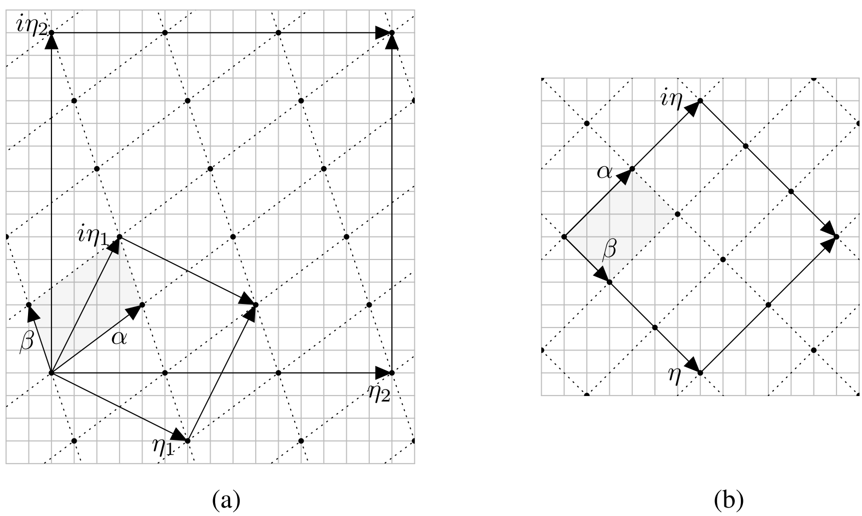

Figure 2: **(a)** For the 4-orbit toroidal map $\mathcal{M} = \{4,4\}_{\alpha,\beta}$, $\alpha = 4 + 3i$, $\beta = -1 + 3i$ the complex number $\eta_1 = 6 - 3i$ generates the unique minimal rotary covering toroidal map $\{4,4\}_{6-3i}$, which can be obtained by gluing 3 fundamental regions of the initial map. Meanwhile, the unique minimal regular covering toroidal map for $\mathcal{M}$ is $\{4,4\}_{15}$, generated by $\eta_2 = 15$, can be obtained by gluing 15 fundamental regions. **(b)** For the 2-orbit toroidal map $\{4,4\}_{\alpha,\beta}$, $\alpha = 3 + 3i$, $\beta = 2 - 2i$ the minimal rotary and regular covering toroidal maps coincide with the regular toroidal map $\{4,4\}_\eta$, $\eta = 6 - 6i$, which can be obtained by gluing together 6 fundamental regions of the initial map.

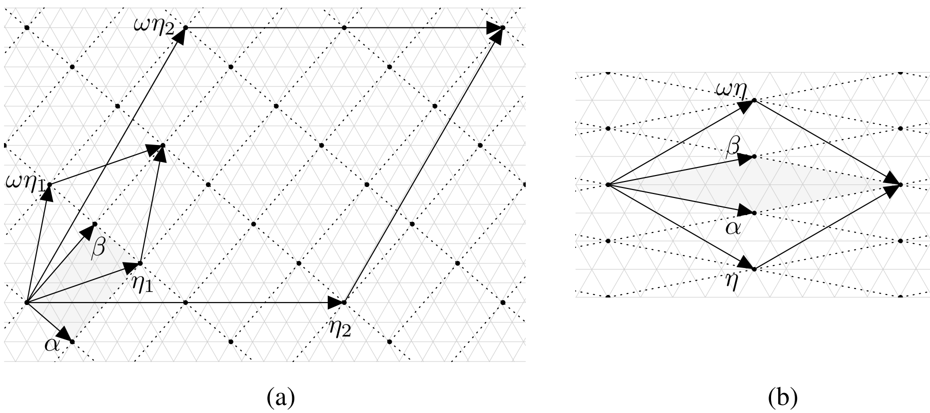

Figure 3: **(a)** For the 6-orbit toroidal map $\mathcal{M} = \{3,6\}_{\alpha,\beta}$, $\alpha = 3 - 2\omega$, $\beta = 1 + 4\omega$ the unique minimal rotary covering toroidal map is $\{3,6\}_{\eta_1}$ with $\eta_1 = 4 + 2\omega$, which can be obtained by gluing 2 fundamental regions of the initial map. Meanwhile, the unique minimal regular covering map for $\mathcal{M}$ is $\{3,6\}_{\eta_2}$ with $\eta_2 = 14$, obtained by gluing 14 fundamental regions. **(b)** For the 3-orbit toroidal map $\{3,6\}_{\alpha,\beta}$, $\alpha = 5 - \omega$, $\beta = 4 + \omega$ the minimal rotary and regular covers coincide and are equal to $\{3,6\}_{\eta}$, $\eta = 6 - 3\omega$, which can be obtained by gluing 3 fundamental regions.

7 Concluding remarks

As noted in Section 1, this paper is part of a larger project which seeks to understand the structure of highly symmetric covers of various maps and polytopes. While many of the results in this area have concerned minimal regular covers, there are other natural questions to consider.

For example, if instead we are given a map on the torus which is not equivelar, the minimal regular and rotary covers will not be maps on the torus. Instead, it is natural to try and construct toroidal covers of these maps which have the greatest possible symmetry. Such a question is being considered in a natural successor of this paper (see [5]).

Acknowledgments

Much of this work was done while the authors were involved in the Fields-MITACS 2011 Summer Undergraduate Research Program. We would like to express our deep gratitude to the Fields Institute for the hospitality and the creative atmosphere. We also would like to thank Daniel Pellicer, Isabel Hubard, and Asia Weiss for their constant support, and for providing the question which has been solved in this paper. The first named author also would like to thank Maksym Skoryk and Alexander Borisenko for fruitful discussions. Both authors thank the anonymous referees for many valuable remarks and improvements.

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ARS MATHEMATICA CONTEMPORANEA

Also available at http://amc-journal.eu
ISSN 1855-3966 (printed edn.), ISSN 1855-3974 (electronic edn.)
ARS MATHEMATICA CONTEMPORANEA 9 (2015) 93–107

Group distance magic labeling of direct product of graphs

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Received 16 January 2013, accepted 28 August 2013, published online 11 July 2014

Abstract

Let $G = (V, E)$ be a graph and Γ an Abelian group, both of order n. A group distance magic labeling of G is a bijection $\ell\colon V \to \Gamma$ for which there exists $\mu \in \Gamma$ such that $\sum_{x\in N(v)} \ell(x) = \mu$ for all $v \in V$, where $N(v)$ is the neighborhood of v. In this paper we consider group distance magic labelings of direct product of graphs. We show that if G is an r-regular graph of order n and $m = 4$ or $m = 8$ and r is even, then the direct product $C_m \times G$ is Γ-distance magic for every Abelian group of order mn. We also prove that $C_m \times C_n$ is $\mathbb{Z}_{mn}$-distance magic if and only if $m \in \{4, 8\}$ or $n \in \{4, 8\}$ or $m, n \equiv 0 \pmod 4$. It is also shown that if $m, n \not\equiv 0 \pmod 4$ then $C_m \times C_n$ is not Γ -distance magic for any Abelian group Γ of order mn.

Keywords: Abelian group, distance magic labeling, group labeling, direct product of graphs.

Math. Subj. Class.: 05C15, 05C22, 05C25, 05C76

*The author was partially supported by National Science Centre grant nr 2011/01/D/ST1/04104.

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1 Introduction and preliminaries

All graphs considered in this paper are simple finite graphs. We use $V(G)$ for the vertex set and $E(G)$ for the edge set of a graph G. The *neighborhood* $N(x)$ or more precisely $N_G(x)$, when needed, of a vertex x is the set of vertices adjacent to x, and the *degree* $d(x)$ of x is $|N(x)|$, the size of the neighborhood of x. By C_n we denote a cycle on n vertices.

We recall two out of four standard graph products (see [8]). Let G and H be two graphs. Both, the *Cartesian product* $G\Box H$ and the *direct product* $G \times H$ are graphs with the vertex set $V(G) \times V(H)$. Two vertices (g, h) and (g', h') are adjacent in:

- $G\Box H$ if $g = g'$ and h is adjacent to h' in H, or $h = h'$ and g is adjacent to g' in G;
- $G \times H$ if g is adjacent to g' in G and h is adjacent to h' in H.

Distance magic labeling (also called *sigma labeling*) of a graph $G = (V(G), E(G))$ of order n is a bijection $\ell\colon V \to \{1, \dots, n\}$ with the property that there is a positive integer k (called *magic constant*) such that $w(x) = \sum_{y\in N_G(x)} \ell(y) = k$ for every $x \in V(G)$, where $w(x)$ is the *weight* of vertex x. If a graph G admits a distance magic labeling, then we say that G is a *distance magic graph*. See [2] (and also [7]) for the survey on distance magic graphs.

The idea of distance magic labeling of graphs has been motivated by the constructions of magic squares. However, finding an r-regular distance magic graph is equivalent to finding equalized incomplete tournament $\mathrm{EIT}(n, r)$ [6]. In an *equalized incomplete tournament* $\mathrm{EIT}(n, r)$ of n teams with r rounds, every team plays exactly r other teams and the total strength of the opponents that team i plays is k.

Some graphs which are distance magic among (some) products can be seen in [1, 3, 4, 5, 9, 10]. Recently a subclass of distance magic graphs was introduced in [1] that behave nicely among products. A distance magic graph G is called *balanced* if there exists a bijection $\ell\colon V(G) \to \{1, \dots, |V(G)|\}$ such that for every $w \in V(G)$ the following holds: if $u \in N(w)$ with $\ell(u) = i$, then there exists $v \in N(w)$, $v \neq u$, with $\ell(v) = |V(G)|+1-i$. We say that u is the *twin vertex* of v (and vice versa) and ℓ is called a *balanced distance magic labeling*. It is easy to see that a balanced distance magic graph has an even number of vertices and that it is an r-regular graph for an even r. Simple examples are empty graph on an even number of vertices, cycle C_4, and $K_{2n} - M$, for a perfect matching M. The following theorem was proved in [1] and will be used in the second section.

Theorem 1.1 ([1])**.** The direct product $G \times H$ is a balanced distance magic graph if and only if one of the graphs is balanced distance magic and the other one is regular.

Group distance magic labeling of graphs was recently introduced by Froncek in [5] as in some sense a generalization of distance magic labeling. Let Γ be a finite Abelian group of order n. A Γ-*distance magic labeling* of a graph G with $|V(G)| = n$ is an injection from V to Γ such that the weight of every vertex $x \in V$ is equal to the same element $\mu \in \Gamma$, called the *magic constant*. If there exists a Γ-distance magic labeling of G, we say that G is a Γ-*distance magic graph*. A graph G is called a *group distance magic graph* if there exists a Γ-distance magic labeling for every Abelian group Γ of order $|V(G)|$. The connection between distance magic graphs and Γ-distance magic graphs is as follows. Let

G be a distance magic graph of order n. If we replace n in $\{1,\dots,n\}$ by 0, we obtain a $\mathbb{Z}_n$-distance magic labeling. Hence every distance magic graph is a $\mathbb{Z}_n$-distance magic graph. The question remains what happens when we replace $\mathbb{Z}_n$ by some other Abelian group, and which graphs are Γ-distance magic but not distance magic. The following theorem was proved in [5]:

Theorem 1.2 ([5])**.** The Cartesian product $C_m \Box C_k$, $m,k \geq 3$, is a $\mathbb{Z}_{mk}$-distance magic graph if and only if km is even.

Froncek also showed that the graph $C_{2k} \Box C_{2k}$ has a $(\mathbb{Z}_2)^{2k}$-distance magic labeling for $k \geq 2$ and $\mu = (0,0,\dots,0)$ ([5]).

It seems that the direct product is the natural choice among (standard) products to deal with Γ-distance magic graphs and group distance magic graphs in general. The reason for this is that the direct product is suitable product if we observe graphs as categories. Hence it should perform well with the product of (Abelian) groups. The confirmation of this will be illustrated in the first theorem of each forthcoming section. This fact also makes the direct product the most natural among graph products, but on the other hand the most difficult to handle. Namely, $G \times H$ does not need to be connected, even if both factors are. More precisely, $G \times H$ is connected if and only if both G and H are connected and at least one of them is non-bipartite [11]. The direct product is commutative, associative, and has attracted a lot of attention in the research community in last 50 years. Probably the biggest challenge (among all products) is the famous Hedetniemi's conjecture:

$$\chi(G \times H) = \min\{\chi(G), \chi(H)\}.$$

This conjecture suggests that the chromatic number of the direct product depends only on the properties of one factor and not both. This is not so rare and also in this work we show that it is enough for one factor to be a balanced distance magic graph and then the direct product with any regular graph will result in a group distance magic graph. For more about the direct product and products in general we recommend the book [8].

For $V(G) = \{x_0, x_1, \dots, x_{|V(G)|-1}\}$ and $V(H) = \{y_0, y_1, \dots, y_{|V(H)|-1}\}$ we use

$$V(G \times H) = \{v_{i,j} : i \in \{0,1,\dots,|V(G)|-1\}, j \in \{0,1,\dots,|V(H)|-1\}\}.$$

The fundamental theorem of finite Abelian groups states that the finite Abelian group Γ of order n can be expressed as the direct sum of cyclic subgroups of prime-power order. This implies that $\Gamma \cong \mathbb{Z}_{p_1^{\alpha_1}} \times \dots \times \mathbb{Z}_{p_m^{\alpha_m}}$, where $n = \prod_{i=1}^m p_i^{\alpha_i}$ and p_i for $i \in \{1,\dots,m\}$ are not necessarily distinct primes. In particular, if $n \equiv 0 \pmod 4$, then we have the following possibilities: $\Gamma \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_{p_1^{\alpha_1}} \times \dots \times \mathbb{Z}_{p_m^{\alpha_m}}$ and $n = 4\prod_{i=1}^m p_i^{\alpha_i}$ or $\Gamma \cong \mathbb{Z}_4 \times \mathbb{Z}_{p_1^{\alpha_1}} \times \dots \times \mathbb{Z}_{p_m^{\alpha_m}}$ and $n = 4\prod_{i=1}^m p_i^{\alpha_i}$ or $\Gamma \cong \mathbb{Z}_{2^{\alpha_0}} \times \mathbb{Z}_{p_1^{\alpha_1}} \times \dots \times \mathbb{Z}_{p_m^{\alpha_m}}$ and $n = 2^{\alpha_0}\prod_{i=1}^m p_i^{\alpha_i}$, $\alpha_0 \geq 3$. This fact will be used often in what follows. Recall that any group element $g \in \Gamma$ of order 2 (i.e., $g \neq 0$, $2g = 0$) is called an *involution*, and that a non-trivial finite group has elements of order 2 if and only if the order of the group is even. Moreover every cyclic group of even order has exactly one involution. We will use the notation a_0 for the identity element of an Abelian group $\mathcal{A}$.

In the next section we present some general results about group distance magic labelings on the direct products. In the last section we concentrate on direct product of cycles. We will prove also that a graph $C_m \times C_n$ is $\mathbb{Z}_{mn}$-distance magic if and only if $m \in \{4,8\}$ or $n \in \{4,8\}$ or $m,n \equiv 0 \pmod 4$. Moreover, we will show that if $m,n \not\equiv 0 \pmod 4$ then $C_m \times C_n$ is not Γ-distance magic for any Abelian group Γ of order mn.

2 General results

We start with the following general theorem for direct product of graphs:

Observation 2.1. If an r_1-regular graph G_1 is Γ_1-distance magic and an r_2 -regular graph G_2 is Γ_2-distance magic, then the direct product $G_1 \times G_2$ is $\Gamma_1 \times \Gamma_2$-distance magic.

Proof. Let $\ell_i \colon V(G_i) \to \Gamma_i$ be a Γ_i-distance magic labeling, and μ_i the magic constant for the graph G_i, $i \in \{1,2\}$. Define the labeling $\ell : V(G_1 \times G_2) \to \Gamma_1 \times \Gamma_2$ for $G_1 \times G_2$, as:

$$\ell((x,y)) = (\ell_1(x), \ell_2(y)).$$

Obviously, ℓ is a bijection and moreover, for any $(u,w) \in V(G_1 \times G_2)$:

$$\begin{aligned} w(u,w) &= \sum_{(x,y)\in N_{G_1\times G_2}(u,w)} \ell(x,y) = \left(r_2 \sum_{x\in N_{G_1}(u)} \ell_1(x), r_1 \sum_{y \in N_{G_2}(w)} \ell_2(y) \right), \\ w(u,w) &= (r_2\mu_1, r_1\mu_2) = \mu, \end{aligned}$$

which settles the proof. □

Theorem 2.2. If G is a balanced distance magic graph, then G is a group distance magic graph.

Proof. Let G be a balanced distance magic graph of order n. Recall that n is an even number and G is an r-regular graph for an even r. For any Abelian group Γ of order n holds $\Gamma \cong \mathbb{Z}_{2t} \times \mathcal{A}$ for some natural number $t > 0$ and some Abelian group $\mathcal{A}$ of order $\frac{n}{2t}$. If $g \in \Gamma$, then we can write $g = (j, a_i)$ where $j \in \mathbb{Z}_{2t}$ and $a_i \in \mathcal{A}$ for $i \in \{0,1,\dots,\frac{n}{2t}-1\}$. Let $V(G) = \{u_1, u_2, \dots, u_{\frac{n}{2}}, u'_1, u'_2, \dots, u'_{\frac{n}{2}}\}$. For $i \in \{1,\dots,\frac{n}{2}\}$ we define the following labeling ℓ for a vertex u_i and its twin vertex u'_i:

$$\ell(u_i) = \left((i-1)(\operatorname{mod} t), a_{\lfloor \frac{i-1}{t} \rfloor}\right) \text{ and } \ell(u'_i) = (2t-1, a_0) - \ell(u_i).$$

Since $\ell(u_i) + \ell(u'_i) = (2t-1, a_0)$ for every $i \in \{1,\dots,\frac{n}{2}\}$, we get

$$\begin{aligned} w(v) &= \sum_{u\in N(v)} \ell(u) = \sum_{u_i \in N(v)} (\ell(u_i) + \ell(u'_i)) = \\ &= \sum_{u_i\in N(v)} (2t-1, a_0) = \frac{r}{2}(2t-1, a_0), \end{aligned}$$

where $\frac{r}{2}$ is an integer since r is even. Moreover, every element of Γ is used exactly once and so G is Γ-distance magic. □

Since for any graph G of order m, the graph $\overline{K}_n \times G$ is isomorphic to $\overline{K}_{nm}$, by Theorems 1.1 and 2.2 the next result immediately follows.

Theorem 2.3. If G is a balanced distance magic graph and H an r-regular graph for $r \geq 1$, then $G \times H$ is a group distance magic graph. □

Notice that by the above Theorem 2.3, if G is an r-regular graph ($r \geq 1$), then the graph $C_4 \times G$ is a group distance magic graph. We cannot generalize this result to other cycles than C_4. Namely, $C_n \times K_2$ is isomorphic to $2C_n$ (i.e., the union of two cycles C_n) when n is even and to C_{2n} when n is odd. It is easy to see that both $2C_n$ for $n \neq 4$ and C_{2n} for $n \neq 2$ are not Γ- distance magic for any Abelian group Γ of order $2n$ (for $n \geq 5$, in both cases under the assumption that there is some group distance magic labeling ℓ, we obtain $\ell(v_i) = \ell(v_{i+4})$ for all the vertices v_i, and we easily derive a contradiction also for $n = 3$). Nevertheless, for many regular graphs the result still holds. For C_8 as we will see next.

Theorem 2.4. If G is an r-regular graph of order n for some even r, then direct product $C_8 \times G$ is a group distance magic graph.

Proof. Let $V(G) = \{x_0, x_1, \ldots, x_{n-1}\}$ be the vertex set of G, let $C_8 = u_0 u_1 \ldots u_7 u_0$, and $H = C_8 \times G$. Notice that if $x_p x_q \in E(G)$, then $v_{j,q} \in N_H(v_{i,p})$ if and only if $j \in \{i-1, i+1\}$ (where the sum on the first suffix is taken modulo 8). We are going to consider three cases, depending on the structure of Γ.

Case 1: $\Gamma \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathcal{A}$ for some Abelian group of order $2n$.

We can write $g \in \Gamma$ as (j_1, j_2, a_k) for $j_1, j_2 \in \mathbb{Z}_2$ and $a_k \in \mathcal{A}$ for $k \in \{0, 1, \ldots, 2n-1\}$. For $j \in \{0, 1, \ldots, n-1\}$ we set

$$\ell(v_{i,j}) = \begin{cases} (0, 0, a_{2j+i}), & \text{if} \quad i \in \{0, 1\}, \\ (0, 1, a_{2j+i-4}), & \text{if} \quad i \in \{4, 5\}, \\ (1, 1, a_0) - \ell(v_{i-2,j}), & \text{if} \quad i \in \{2, 3, 6, 7\}. \end{cases}$$

Clearly, $\ell : V(C_8 \times G) \to \Gamma$ is a bijection and $\ell(v_{i,j}) + \ell(v_{i+2,j}) = (y_i, a_0)$, where $y_i \in \{(1, 1), (1, 0)\}$, and so $2y_i = (0, 0)$. Hence for every $i \in \{0, 1, \ldots, 7\}$ and $j \in \{0, 1, \ldots, n-1\}$ we get

$$\begin{aligned} w(v_{i,j}) &= \sum_{x_p \in N_G(x_j)} (\ell(v_{i-1,p}) + \ell(v_{i+1,p})) = \sum_{x_p \in N_G(x_j)} (y_{i-1}, a_0) = \\ &= \frac{r}{2}(0, 0, a_0) = (0, 0, a_0) \end{aligned}$$

and $C_8 \times G$ is Γ-distance magic since r is even.

Case 2: $\Gamma \cong \mathbb{Z}_4 \times \mathcal{A}$ for some Abelian group $\mathcal{A}$ of order $2n$.

If $g \in \Gamma$, then we can write $g = (j, a_k)$ for $j \in \mathbb{Z}_4$ and $a_k \in \mathcal{A}$ for $k \in \{0, 1, \ldots, 2n-1\}$. For $j \in \{0, 1, \ldots, n-1\}$ we define

$$\ell(v_{i,j}) = \begin{cases} (0, a_{2j+i}), & \text{if} \quad i \in \{0, 1\}, \\ (2, a_{2j+i-4}), & \text{if} \quad i \in \{4, 5\}, \\ (3, a_0) - \ell(v_{i-2,j}), & \text{if} \quad i \in \{2, 3, 6, 7\}. \end{cases}$$

Again $\ell : V(C_8 \times G) \to \Gamma$ is obviously a bijection and $\ell(v_{i,j}) + \ell(v_{i+2,j}) = (y_i, a_0)$, where $y_i \in \{1, 3\}$, and thus $2y_i = 2$. Hence for every $i \in \{0, 1, \ldots, 7\}$ and $j \in \{0, 1, \ldots, n-1\}$ we get

$$w(v_{i,j}) = \sum_{x_p \in N_G(x_j)} (\ell(v_{i-1,p}) + \ell(v_{i+1,p})) = \sum_{x_p \in N_G(x_j)} (y_{i-1}, a_0) = \frac{r}{2}(2, a_0) = (r, a_0)$$

and $C_8 \times G$ is Γ-distance magic.

Case 3: $\Gamma \cong \mathbb{Z}_{2^\alpha} \times \mathcal{A}$ for $\alpha > 2$ and some Abelian group $\mathcal{A}$ of order $\frac{n}{2^{\alpha-3}}$.

If $g \in \Gamma$, we can write $g = (p, a_k)$ for $p \in \mathbb{Z}_{2^\alpha}$ and $a_k \in \mathcal{A}$ for $k \in \{0, 1, \ldots, \frac{n}{2^{\alpha-3}} - 1\}$. For $j \in \{0, 1, \ldots, \frac{n}{2^{\alpha-3}} - 1\}$ define the following labeling ℓ:

$$\ell(v_{i,j}) = \begin{cases} \left((2j+i)(\operatorname{mod} 2^{\alpha-2}), a_{\lfloor j \cdot 2^{-\alpha+3} \rfloor}\right), & \text{if } \ i \in \{0,1\}, \\ \left(2^{\alpha-1}, a_0\right) + \ell(v_{i-4,j}), & \text{if } \ i \in \{4,5\}, \\ (2^\alpha - 1, a_0) - \ell(v_{i-2,j}), & \text{if } \ i \in \{2,3,6,7\}. \end{cases}$$

As in previous cases $\ell : V(C_8 \times G) \to \Gamma$ is a bijection and $\ell(v_{i,j}) + \ell(v_{i+2,j}) = (y_i, a_0)$ for some $y_i \in \{2^{\alpha-1} - 1, 2^\alpha - 1\}$. Thus $2(\ell(v_{i,j}) + \ell(v_{i+2,j})) = (2y_i, a_0) = (-2, a_0)$. For every $i \in \{0, 1, \ldots, 7\}$ and $j \in \{0, 1, \ldots, n-1\}$ we get

$$\begin{aligned} w(v_{i,j}) &= \sum_{x_p \in N_G(x_j)} (\ell(v_{i-1,p}) + \ell(v_{i+1,p})) = \sum_{x_p \in N_G(x_j)} (y_{i-1}, a_0) = \\ &= \frac{r}{2}(-2, a_0) = (-r, a_0) \end{aligned}$$

and $C_8 \times G$ is $\mathbb{Z}_{2^\alpha} \times \mathcal{A}$-distance magic. □

The natural question arises whether we can prove similar results for every cycle C_n where $n \equiv 0 \pmod 4$. The answer to this question is negative as we will see in the next section. It will also be clear from the following section why we cannot expect similar results for $n \not\equiv 0 \pmod 4$. (For both claims see Theorem 3.5.) However, below we give some groups Γ such that for G being an r-regular graph of order n for some even r, the direct product $C_{2^p} \times G$ admits a Γ-distance magic labeling.

Proposition 2.5. If G is an r-regular graph of order n for some even r, then the direct product $C_{2^p} \times G$, $p \geq 2$, admits an $\mathcal{A} \times \mathcal{B}$-distance magic labeling for any Abelian group $\mathcal{B}$ of order n and an Abelian group $\mathcal{A}$ such that:

- $\mathcal{A} \cong (\mathbb{Z}_2)^p$,
- $\mathcal{A} \cong \mathbb{Z}_4 \times (\mathbb{Z}_2)^{p-2}$,
- $\mathcal{A} \cong \mathbb{Z}_8 \times (\mathbb{Z}_2)^{p-3}$,
- $\mathcal{A} \cong (\mathbb{Z}_4)^2 \times (\mathbb{Z}_2)^{p-4}$.

Proof. Let $V(G) = \{x_0, x_1, \ldots, x_{n-1}\}$ be the vertex set of G, let $C_{2^p} = u_0 u_1 \ldots u_{2^p-1} u_0$, and $H = C_{2^p} \times G$. Notice that if $x_p x_q \in E(G)$, then $v_{j,q} \in N_H(v_{i,p})$ if and only if $j \in \{i-1, i+1\}$ (where the sum on the first suffix is taken modulo 2^p). Let the elements of $\mathcal{B}$ be $b_0, b_1, \ldots, b_{n-1}$. Recall that for any element $r \in (\mathbb{Z}_2)^p$ we have $2r = 0$.

Case 1: $\mathcal{A} \cong (\mathbb{Z}_2)^p$. Let the elements of $(\mathbb{Z}_2)^p$ be $r_0, \ldots, r_{2^p-1}$. Each element of $\Gamma \cong (\mathbb{Z}_2)^p \times \mathcal{B}$ can be thus expressed as (r_i, b_j), where $r_i \in (\mathbb{Z}_2)^p$ and $b_j \in \mathcal{B}$. We define the labeling ℓ as follows:

$$\ell(v_{i,j}) = \begin{cases} (r_i, b_j), & \text{if } \ i(\operatorname{mod} 4) \in \{0,1\}, \\ (r_i, -b_j), & \text{if } \ i(\operatorname{mod} 4) \in \{2,3\}. \end{cases}$$

It is straightforward to check that ℓ is bijective and $\ell(v_{i,j})+\ell(v_{i+2,j})=(r_i+r_{i+2},b_0)$. Hence for every $i\in\{0,1,\dots,2^p-1\}$ and $j\in\{0,1,\dots,n-1\}$ we get

$$\begin{aligned} w(v_{i,j}) &= \sum_{x_d\in N_G(x_j)}(\ell(v_{i-1,d})+\ell(v_{i+1,d})) = \sum_{x_d\in N_G(x_j)}(r_{i-1}+r_{i+1},b_0) = \\ &= \frac{r}{2}(r_0,b_0)=(r_0,b_0) \end{aligned}$$

and $C_{2^p}\times G$ is $(\mathbb{Z}_2)^p\times\mathcal{B}$-distance magic since r is even.

Case 2: $\mathcal{A}\cong\mathbb{Z}_4\times(\mathbb{Z}_2)^{p-2}$. Let the elements of $(\mathbb{Z}_2)^{p-2}$ be $r_0,\dots,r_{2^{p-2}-1}$. Each element of $\Gamma\cong\mathbb{Z}_4\times(\mathbb{Z}_2)^{p-2}\times\mathcal{B}$ can be thus expressed as (q,r_i,b_j), where $q\in\mathbb{Z}_4$, $r_i\in(\mathbb{Z}_2)^{p-2}$, and $b_j\in\mathcal{B}$. We define the labeling ℓ in the following way

$$\ell(v_{4i+q,j})=\begin{cases}(q,r_i,b_j), & \text{if } \ q\in\{0,1\},\\ (q,r_i,-b_j), & \text{if } \ q\in\{2,3\}.\end{cases}$$

Again it is easy to check that ℓ is bijective and $\ell(v_{i,j})+\ell(v_{i+2,j})=(y_i,z_i,b_0)$, where $y_i\in\{0,2\}$ and $z_i=r_{\lfloor i/4\rfloor}+r_{\lfloor (i+2)/4\rfloor}$, and so $2(y_i,z_i)=(0,r_0)$. Hence for every $i\in\{0,1,\dots,2^p-1\}$ and $j\in\{0,1,\dots,n-1\}$ we get

$$\begin{aligned} w(v_{i,j}) &= \sum_{x_d\in N_G(x_j)}(\ell(v_{i-1,d})+\ell(v_{i+1,d})) = \sum_{x_d\in N_G(j)}(y_{i-1},z_{i-1},b_0) = \\ &= \frac{r}{2}(0,r_0,b_0)=(0,r_0,b_0) \end{aligned}$$

and $C_{2^p}\times G$ is $\mathbb{Z}_4\times(\mathbb{Z}_2)^{p-2}\times\mathcal{B}$-distance magic since r is even.

Case 3: $\mathcal{A}\cong\mathbb{Z}_8\times(\mathbb{Z}_2)^{p-3}$. Let the elements of $(\mathbb{Z}_2)^{p-3}$ be $r_0,\dots,r_{2^{p-3}-1}$. Each element of $\Gamma\cong\mathbb{Z}_8\times(\mathbb{Z}_2)^{p-3}\times\mathcal{B}$ can be thus expressed as $(\sigma(q),r_i,b_j)$, where $q\in\mathbb{Z}_8$, $r_i\in(\mathbb{Z}_2)^{p-3}$, $b_j\in\mathcal{B}$, and the function $\sigma:\mathbb{Z}_8\to\mathbb{Z}_8$ is defined as $\sigma(2)=3$, $\sigma(3)=2$, $\sigma(6)=7$, $\sigma(7)=6$, and $\sigma(j)=j$ for remaining $j\in\mathbb{Z}_8$. We define the labeling ℓ in the following way.

$$\ell(v_{8i+q,j})=\begin{cases}(\sigma(q),r_i,b_j), & \text{if } \ q(\text{mod}\,4)\in\{0,1\},\\ (\sigma(q),r_i,-b_j), & \text{if } \ q(\text{mod}\,4)\in\{2,3\}.\end{cases}$$

It is easy to see that ℓ is bijective and $\ell(v_{i,j})+\ell(v_{i+2,j})=(y_i,z_i,b_0)$, where $(y_i,z_i)\in\{(3,2r_{\lfloor i/8\rfloor}),(7,r_{\lfloor i/8\rfloor}+r_{\lfloor (i+2)/8\rfloor})\}$, so $2(y_i,z_i)=(6,r_0)$. Hence for every $i\in\{0,1,\dots,2^p-1\}$ and $j\in\{0,1,\dots,n-1\}$ we get

$$w(v_{i,j})=\sum_{x_d\in N_G(x_j)}(\ell(v_{i-1,d})+\ell(v_{i+1,d}))=\sum_{x_d\in N_G(x_j)}(y_{i-1},z_{i-1},b_0)=\frac{r}{2}(6,r_0,b_0)$$

and $C_{2^p}\times G$ is $\mathbb{Z}_8\times(\mathbb{Z}_2)^{p-3}\times\mathcal{B}$-distance magic since r is even.

Case 4: $\mathcal{A}\cong(\mathbb{Z}_4)^2\times(\mathbb{Z}_2)^{p-4}$. Let the elements of $(\mathbb{Z}_2)^{p-4}$ be $r_0,\dots,r_{2^{p-4}-1}$. Each element of $\Gamma\cong(\mathbb{Z}_4)^2\times(\mathbb{Z}_2)^{p-4}\times\mathcal{B}$ can be thus expressed as $(\sigma(q),r_i,b_j)$, where $q\in\mathbb{Z}_{16}$, $r_i\in(\mathbb{Z}_2)^{p-4}$, $b_j\in\mathcal{B}$, and the function $\sigma:\mathbb{Z}_{16}\to(\mathbb{Z}_4)^2$ is defined as

i	0	1	2	3	4	5	6	7
$\sigma(i)$	(0,0)	(0,3)	(1,1)	(1,2)	(2,2)	(2,1)	(3,3)	(3,0)
i	8	9	10	11	12	13	14	15
$\sigma(i)$	(0,2)	(0,1)	(1,3)	(1,0)	(2,0)	(2,3)	(3,1)	(3,2)

We define the labeling ℓ in the following way:

$$\ell(v_{16i+q,j}) = \begin{cases} (\sigma(q), r_i, b_j), & \text{if} \quad q(\text{mod}\, 4) \in \{0,1\}, \\ (\sigma(q), r_i, -b_j), & \text{if} \quad q(\text{mod}\, 4) \in \{2,3\}. \end{cases}$$

It is easy to see that ℓ is bijective and $\ell(v_{i,j}) + \ell(v_{i+2,j}) = (y_i, z_i, b_0)$, where $y_i \in \{(1,1),(1,3),(3,1),(3,3)\}$ and $z_i = r_{\lfloor i/16 \rfloor} + r_{\lfloor (i+2)/16 \rfloor}$, so $2(y_i, z_i) = (2, 2, r_0)$. Hence for every $i \in \{0, 1, \dots, 2^p - 1\}$ and $j \in \{0, 1, \dots, n-1\}$ we get

$$w(v_{i,j}) = \sum_{x_d \in N_G(x_j)} (\ell(v_{i-1,d}) + \ell(v_{i+1,d})) = \sum_{x_d \in N_G(x_j)} (y_{i-1}, z_{i-1}, b_0) = \frac{r}{2}(2, 2, r_0, b_0)$$

and $C_{2^p} \times G$ is $(\mathbb{Z}_4)^2 \times (\mathbb{Z}_2)^{p-4} \times \mathcal{B}$-distance magic since r is even. □

Proposition 2.6. If G is an r-regular graph of order n for some even r and n, then the direct product $C_{2^p} \times G$, $p \geq 2$, admits an $\mathcal{A} \times \mathcal{B}$-distance magic labeling for any Abelian group $\mathcal{B}$ of order $\frac{n}{2}$ and an Abelian group $\mathcal{A}$ such that:

- $\mathcal{A} \cong \mathbb{Z}_8 \times \mathbb{Z}_4 \times (\mathbb{Z}_2)^{p-4}$,
- $\mathcal{A} \cong \mathbb{Z}_{16} \times (\mathbb{Z}_2)^{p-3}$.

Proof. Let $V(G) = \{x_0, x_1, \dots, x_{n-1}\}$ be the vertex set of G, let $C_{2^p} = u_0 u_1 \dots u_{2^p-1} u_0$, and $H = C_{2^p} \times G$. Notice that if $x_p x_q \in E(G)$, then $v_{j,q} \in N_H(v_{i,p})$ if and only if $j \in \{i-1, i+1\}$ (where the sum on the first suffix is taken modulo 2^p). Let the elements of $\mathcal{B}$ be $b_0, b_1, \dots, b_{n/2-1}$.

Case 1: $\mathcal{A} \cong \mathbb{Z}_8 \times \mathbb{Z}_4 \times (\mathbb{Z}_2)^{p-4}$. Each element of $\Gamma \cong \mathbb{Z}_8 \times \mathbb{Z}_4 \times (\mathbb{Z}_2)^{p-4} \times \mathcal{B}$ can be expressed as $(\sigma(q), r_i, b_j)$, where $q \in \mathbb{Z}_{32}$, $r_i \in (\mathbb{Z}_2)^{p-4}$, $b_j \in \mathcal{B}$, and the function $\sigma : \mathbb{Z}_{32} \to \mathbb{Z}_8 \times \mathbb{Z}_4$ is defined as:

i	0	1	2	3	4	5	6	7
$\sigma(i)$	(0,0)	(1,0)	(3,1)	(2,1)	(4,2)	(5,2)	(7,3)	(6,3)
i	8	9	10	11	12	13	14	15
$\sigma(i)$	(0,2)	(1,2)	(3,3)	(2,3)	(4,0)	(5,0)	(7,1)	(6,1)
i	16	17	18	19	20	21	22	23
$\sigma(i)$	(0,3)	(1,3)	(3,2)	(2,2)	(4,1)	(5,1)	(7,0)	(6,0)
i	24	25	26	27	28	29	30	31
$\sigma(i)$	(0,1)	(1,1)	(3,0)	(2,0)	(4,3)	(5,3)	(7,2)	(6,2)

For $j \equiv 0 \pmod 2$ we define the labeling ℓ in the following way:

$$\ell(v_{16i+t,j}) = \begin{cases} (\sigma(t), r_i, b_{\lfloor \frac{j}{2} \rfloor}), & \text{if} \quad t(\text{mod}\, 4) \in \{0,1\}, \\ (\sigma(t), r_i, -b_{\lfloor \frac{j}{2} \rfloor}), & \text{if} \quad t(\text{mod}\, 4) \in \{2,3\}; \end{cases}$$

and for $j \equiv 1 \pmod 2$ we set

$$\ell(v_{16i+t,j}) = \begin{cases} (\sigma(16+t), r_i, b_{\lfloor \frac{j}{2} \rfloor}), & \text{if} \quad t(\bmod 4) \in \{0,1\}, \\ (\sigma(16+t), r_i, -b_{\lfloor \frac{j}{2} \rfloor}), & \text{if} \quad t(\bmod 4) \in \{2,3\}. \end{cases}$$

It is straightforward to check that ℓ is bijective and $\ell(v_{i,j}) + \ell(v_{i+2,j}) = (y_i, z_i, b_0)$, where $y_i \in \{(3,1),(7,3),(7,1)\}$ and $z_i = r_{\lfloor i/16 \rfloor} + r_{\lfloor (i+2)/16 \rfloor + 1}$, so $2(y_i, z_i) = (6, 2, r_0)$. Hence for every $i \in \{0, 1, \ldots, 2^p - 1\}$ and $j \in \{0, 1, \ldots, n-1\}$ we get

$$w(v_{i,j}) = \sum_{x_d \in N_G(x_j)} (\ell(v_{i-1,d}) + \ell(v_{i+1,d})) = \sum_{x_d \in N_G(x_j)} (y_{i-1}, z_{i-1}, b_0) = \frac{r}{2}(6, 2, r_0, b_0)$$

and $C_{2^p} \times G$ is $\mathbb{Z}_8 \times \mathbb{Z}_4 \times (\mathbb{Z}_2)^{p-4} \times \mathcal{B}$-distance magic since r is even.

Case 2: $\mathcal{A} \cong \mathbb{Z}_{16} \times (\mathbb{Z}_2)^{p-3}$. Each element of $\Gamma \cong \mathbb{Z}_{16} \times (\mathbb{Z}_2)^{p-3} \times \mathcal{B}$ as $(\sigma(q), r_i, b_j)$, where $q \in \mathbb{Z}_{16}$, $r_i \in (\mathbb{Z}_2)^{p-3}$, $b_j \in \mathcal{B}$, and the function $\sigma : \mathbb{Z}_{16} \to \mathbb{Z}_{16}$ is defined as:

i	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
$\sigma(i)$	0	2	1	15	8	10	9	7	4	6	13	11	12	14	5	3

We define the labeling ℓ for $j \equiv 0 \pmod 2$ as

$$\ell(v_{8i+t,j}) = \begin{cases} (\sigma(t), r_i, b_{\lfloor \frac{j}{2} \rfloor}), & \text{if} \quad t(\bmod 4) \in \{0,1\}, \\ (\sigma(t), r_i, -b_{\lfloor \frac{j}{2} \rfloor}), & \text{if} \quad t(\bmod 4) \in \{2,3\}, \end{cases}$$

and for $j \equiv 1 \pmod 2$ by

$$\ell(v_{8i+t,j}) = \begin{cases} (\sigma(8+t), r_i, b_{\lfloor \frac{j}{2} \rfloor}), & \text{if} \quad t(\bmod 4) \in \{0,1\}, \\ (\sigma(8+t), r_i, -b_{\lfloor \frac{j}{2} \rfloor}), & \text{if} \quad t(\bmod 4) \in \{2,3\}. \end{cases}$$

Again it is easy to see that ℓ is bijective and $\ell(v_{i,j}) + \ell(v_{i+2,j}) = (y_i, z_i, b_0)$, where $y_i \in \{1, 9\}$ and $z_i = r_{\lfloor i/8 \rfloor} + r_{\lfloor (i+2)/8 \rfloor}$, so $2(y_i, z_i) = (2, r_0)$. Hence for every $i \in \{0, 1, \ldots, 2^p - 1\}$ and $j \in \{0, 1, \ldots, n-1\}$ we get

$$w(v_{i,j}) = \sum_{x_d \in N_G(x_j)} (\ell(v_{i-1,d}) + \ell(v_{i+1,d})) = \sum_{x_d \in N_G(x_j)} (y_{i-1}, z_{i-1}, b_0) = \frac{r}{2}(2, r_0, b_0)$$

and $C_{2^p} \times G$ is $\mathbb{Z}_{16} \times (\mathbb{Z}_2)^{p-3} \times \mathcal{B}$-distance magic since r is even. □

3 Γ-distance magic labeling of $C_m \times C_n$

In this section we concentrate on the direct product of two cycles.

Theorem 3.1. If $m, n \equiv 0 \pmod 4$, then the direct product $C_m \times C_n$ is $\mathcal{A} \times \mathcal{B}$-distance magic for any Abelian groups $\mathcal{A}$ and $\mathcal{B}$ of order m and n, respectively.

Proof. Let $\Gamma = \mathcal{A} \times \mathcal{B}$ and let b_0 be the identity of $\mathcal{B}$. We consider three cases, depending on the factorization of Γ.

Case 1: $\Gamma \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathcal{G} \times \mathcal{B}$ for some Abelian group $\mathcal{G}$ of order $\frac{m}{4}$.

If $g \in \Gamma$, then we can write $g = (j_1, j_2, a_k, b_j)$ for $j_1, j_2 \in \mathbb{Z}_2$, $a_k \in \mathcal{G}$ for $k \in \{0, 1, \dots, \frac{m}{4} - 1\}$, and $b_j \in \mathcal{B}$ for $j \in \{0, 1, \dots, n-1\}$. For $i \in \{0, 1, \dots, m-1\}$ we set:

$$\ell(v_{i,j}) = \begin{cases} (0, i, a_{\lfloor \frac{i}{4} \rfloor}, b_j), & \text{if} \quad i (\operatorname{mod} 4) \in \{0, 1\}, \\ (1, 1, a_0, b_0) - \ell(v_{i-2,j}), & \text{if} \quad i (\operatorname{mod} 4) \in \{2, 3\}, \end{cases}$$

for $j (\operatorname{mod} 4) \in \{0, 1\}$ and

$$\ell(v_{i,j}) = \begin{cases} (0, i, -a_{\lfloor \frac{i}{4} \rfloor}, b_j), & \text{if} \quad i (\operatorname{mod} 4) \in \{0, 1\}, \\ (1, 1, a_0, b_0) - \ell(v_{i-2,j}), & \text{if} \quad i (\operatorname{mod} 4) \in \{2, 3\}, \end{cases}$$

for $j (\operatorname{mod} 4) \in \{2, 3\}$.

It is easy to see that $\ell : V(C_m \times C_n) \to \Gamma$ is a bijection and $w(v_{i,j}) = \ell(v_{i-1,j-1}) + \ell(v_{i-1,j+1}) + \ell(v_{i+1,j-1}) + \ell(v_{i+1,j+1}) = (0, 0, a_0, b_0)$.

Case 2: $\Gamma \cong \mathbb{Z}_4 \times \mathcal{G} \times \mathcal{B}$ for some Abelian group $\mathcal{G}$ of order $\frac{m}{4}$.

If $g \in \Gamma$, then we can write $g = (q, a_k, b_j)$ for $q \in \mathbb{Z}_4$, $a_k \in \mathcal{G}$ for $k \in \{0, 1, \dots, \frac{m}{4} - 1\}$, and $b_j \in \mathcal{B}$ for $j \in \{0, 1, \dots, n-1\}$. For $i \in \{0, \dots, m-1\}$ we define the following labeling ℓ:

$$\ell(v_{i,j}) = \begin{cases} (i, a_{\lfloor \frac{i}{4} \rfloor}, b_j), & \text{if} \quad i (\operatorname{mod} 4) \in \{0, 1\}, \\ (3, a_0, b_0) - \ell(v_{i-2,j}), & \text{if} \quad i (\operatorname{mod} 4) \in \{2, 3\}, \end{cases}$$

for $j (\operatorname{mod} 4) \in \{0, 1\}$ and

$$\ell(v_{i,j}) = \begin{cases} (i, -a_{\lfloor \frac{i}{4} \rfloor}, b_j), & \text{if} \quad i (\operatorname{mod} 4) \in \{0, 1\}, \\ (3, a_0, b_0) - \ell(v_{i-2,j}), & \text{if} \quad i (\operatorname{mod} 4) \in \{2, 3\}, \end{cases}$$

for $j (\operatorname{mod} 4) \in \{2, 3\}$.

Again $\ell : V(C_m \times C_n) \to \Gamma$ is a bijection and $w(v_{i,j}) = \ell(v_{i-1,j-1}) + \ell(v_{i-1,j+1}) + \ell(v_{i+1,j-1}) + \ell(v_{i+1,j+1}) = (2, a_0, b_0)$.

Case 3: $\Gamma \cong \mathbb{Z}_{2^\alpha} \times \mathcal{G} \times \mathcal{B}$ for $\alpha > 2$ and some Abelian group $\mathcal{G}$ of order $\frac{m}{2^\alpha}$.

Notice that this case is meaningful only when $m \equiv 0 \pmod 8$. If $g \in \Gamma$, then we can write that $g = (q, a_k, b_j)$ for $q \in \mathbb{Z}_{2^\alpha}$, $a_k \in \mathcal{G}$ for $k \in \{0, 1, \dots, \frac{m}{2^\alpha} - 1\}$ and $b_j \in \mathcal{B}$ for $j \in \{0, 1, \dots, n-1\}$. For $i \in \{0, \dots, m-1\}$ we define the following labeling ℓ:

$$\ell(v_{i,j}) = \begin{cases} \left((i (\operatorname{mod} 2^\alpha)) 2^{\alpha-2}, a_{\lfloor i/2^\alpha \rfloor}, b_j\right), & \text{if} \quad i (\operatorname{mod} 2^\alpha) \in \{0, 1\}, \\ \left((1 - i (\operatorname{mod} 2^\alpha)) 2^{\alpha-2}, -a_{\lfloor i/2^\alpha \rfloor}, -b_j\right), & \text{if} \quad i (\operatorname{mod} 2^\alpha) \in \{2, 3\}, \\ (1, a_0, b_0) + \ell(v_{i-4,j}), & \text{if} \quad i (\operatorname{mod} 2^\alpha) \notin \{0, 1, 2, 3\}, \end{cases}$$

for $j (\operatorname{mod} 4) \in \{0, 1\}$ and

$$\ell(v_{i,j}) = \begin{cases} \left(-(i (\operatorname{mod} 2^\alpha)) 2^{\alpha-2} - 1, -a_{\lfloor i/2^\alpha \rfloor}, -b_j\right), & \text{if} \quad i (\operatorname{mod} 2^\alpha) \in \{0, 1\}, \\ \left((i (\operatorname{mod} 2^\alpha - 1)) 2^{\alpha-2} - 1, a_{\lfloor i/2^\alpha \rfloor}, b_j\right), & \text{if} \quad i (\operatorname{mod} 2^\alpha) \in \{2, 3\}, \\ (-1, a_0, b_0) + \ell(v_{i-4,j}), & \text{if} \quad i (\operatorname{mod} 2^\alpha) \notin \{0, 1, 2, 3\}, \end{cases}$$

for $j (\operatorname{mod} 4) \in \{2, 3\}$.

It is easy to see that $\ell : V(C_m \times C_n) \to \Gamma$ is a bijection and moreover, $w(v_{i,j}) = \ell(v_{i-1,j-1}) + \ell(v_{i+1,j-1}) + \ell(v_{i-1,j+1}) + \ell(v_{i+1,j+1}) = (-2, a_0, b_0)$. □

One can ask if it is possible to find an $\mathcal{A} \times \mathcal{B}$ -distance magic labeling of $C_m \times C_n$, if $|\mathcal{A}| > m$. A partial answer is given by the following observation.

Proposition 3.2. If $m, n \equiv 0 \pmod 4$, then the direct product $C_m \times C_n$ is $\mathbb{Z}_t \times \mathcal{A}$ -distance magic for $m|t$ and any Abelian group $\mathcal{A}$ order $\frac{mn}{t}$.

Proof. Let $\Gamma = \mathbb{Z}_t \times \mathcal{A}$ where $m|t$ and $\mathcal{A}$ is an Abelian group of order $\frac{mn}{t}$. If $g \in \Gamma$, then we can write that $g = (j, a_k)$ for $j \in \mathbb{Z}_t$ and $a_k \in \mathcal{A}$ for $k \in \{0, 1, \dots, \frac{mn}{t} - 1\}$. For $i \in \{0, 1, \dots, m-1\}$ let

$$\ell(v_{i,j}) = \begin{cases} \left(\frac{jm (\operatorname{mod} t)}{2}, a_{\lfloor \frac{jm}{t} \rfloor}\right), & \text{if} \quad i = 0 \\ \left(\frac{jm (\operatorname{mod} t)}{2} + \frac{m}{4}, a_{\lfloor \frac{jm}{t} \rfloor}\right), & \text{if} \quad i = 1 \\ \left(-\frac{jm (\operatorname{mod} t)}{2} - \frac{m}{4}, -a_{\lfloor \frac{jm}{t} \rfloor}\right), & \text{if} \quad i = 2 \\ \left(-\frac{jm (\operatorname{mod} t)}{2} - \frac{m}{2}, -a_{\lfloor \frac{jm}{t} \rfloor}\right), & \text{if} \quad i = 3 \\ \ell(v_{i-4,j}) + (1, a_0), & \text{if} \quad i > 3, \end{cases}$$

for $j (\operatorname{mod} 4) \in \{0, 1\}$ and

$$\ell(v_{i,j}) = \begin{cases} \left(-\frac{jm (\operatorname{mod} t)}{2} - 1, -a_{\lfloor \frac{jm}{t} \rfloor}\right), & \text{if} \quad i = 0 \\ \left(-\frac{jm (\operatorname{mod} t)}{2} - \frac{m}{4} - 1, -a_{\lfloor \frac{jm}{t} \rfloor}\right), & \text{if} \quad i = 1 \\ \left(\frac{jm (\operatorname{mod} t)}{2} + \frac{m}{4} - 1, a_{\lfloor \frac{jm}{t} \rfloor}\right), & \text{if} \quad i = 2 \\ \left(\frac{jm (\operatorname{mod} t)}{2} + \frac{m}{2} - 1, a_{\lfloor \frac{jm}{t} \rfloor}\right), & \text{if} \quad i = 3 \\ \ell(v_{i-4,j}) + (-1, a_0), & \text{if} \quad i > 3, \end{cases}$$

for $j (\operatorname{mod} 4) \in \{2, 3\}$.

Notice that we obtain mn/t blocks such that in every block we have all elements from $\mathbb{Z}_t$ as the first coordinate. Moreover for $i \in \{0, 1, \dots, mn/t - 1\}$ in i-th block we have labels (j, a_i), where $j \in \{0, 1, \dots, t/2 - 1\}$. Therefore ℓ is bijective and $\mu = (-2, a_0)$ is the magic constant. □

The above results encourage us to post the following conjecture.

Conjecture 3.3. If $m, n \equiv 0 \pmod 4$, then $C_m \times C_n$ is a group distance magic graph.

Now we are going to present some sufficient conditions for a graph G not to be group distance magic.

Theorem 3.4. Assume that $m, n \geq 3$, $m, n \notin \{4, 8\}$, $m = 4b + d$ and $n = 4a + c$ for some integers $a, b \geq 0$ where $c \in \{0, 1, 2, 3\}$ and $d \in \{1, 2, 3\}$. If an Abelian group Γ of order mn has less than $\max\{2, a-1\}$ involutions, then $C_m \times C_n$ is not Γ-distance magic.

Proof. Let m, n, a, b, c, d be as in the statement of the theorem. Thus $m \not\equiv 0 \pmod 4$. Let $G = C_m \times C_n$. Assume that there exists a group Γ of order mn such that G is Γ-distance magic, i.e., there is a bijection $\ell : V(G) \to \Gamma$ such that for every $x \in V(G)$, $w(x) = \mu$ for some constant $\mu \in \Gamma$. Furthermore, let g_0 be the identity of Γ.

For any integers i, p, and s we have

$$w(v_{i+p+1,s+1}) = \ell(v_{i+p,s}) + \ell(v_{i+p,s+2}) + \ell(v_{i+p+2,s}) + \ell(v_{i+p+2,s+2}) = \mu,$$

where the first suffix is taken modulo m, and the second one modulo n. Comparing the above equality for $p = 0$ and $p = 2$, we obtain

$$\ell(v_{i,s}) + \ell(v_{i,s+2}) = \ell(v_{i+4,s}) + \ell(v_{i+4,s+2}).$$

More generally, if we consider the equality for $p = j$ and $p = j + 2$ for some integer j, we obtain that

$$\ell(v_{i+j,s}) + \ell(v_{i+j,s+2}) = \ell(v_{i+j+4,s}) + \ell(v_{i+j+4,s+2})$$

for every j. In consequence,

$$\ell(v_{i,s}) + \ell(v_{i,s+2}) = \ell(v_{i+4j,s}) + \ell(v_{i+4j,s+2}) \tag{3.1}$$

for every j. As $m \not\equiv 0 \pmod 4$, there exists such a j that $i + 4j \equiv i + 2 \pmod m$. This way we obtain from (3.1)

$$\ell(v_{i,s}) + \ell(v_{i,s+2}) = \ell(v_{i+2,s}) + \ell(v_{i+2,s+2}). \tag{3.2}$$

Substituting s with $s + 2$ in (3.2) we obtain

$$\ell(v_{i,s+2}) + \ell(v_{i,s+4}) = \ell(v_{i+2,s+2}) + \ell(v_{i+2,s+4}) \tag{3.3}$$

and finally by subtracting (3.3) from (3.2)

$$\ell(v_{i,s}) - \ell(v_{i,s+4}) = \ell(v_{i+2,s}) - \ell(v_{i+2,s+4}). \tag{3.4}$$

In a similar way as (3.1) we can prove that for any i, j, and s

$$\ell(v_{i,s}) + \ell(v_{i+2,s}) = \ell(v_{i,s+4j}) + \ell(v_{i+2,s+4j}). \tag{3.5}$$

In particular from (3.5) for $j = 1$ we get

$$\ell(v_{i,s}) - \ell(v_{i,s+4}) = \ell(v_{i+2,s+4}) - \ell(v_{i+2,s}). \tag{3.6}$$

This leads, if we add together (3.4) and (3.6), to

$$2(\ell(v_{i,s}) - \ell(v_{i,s+4})) = g_0.$$

By comparing the last equality for $s = 4p$ and $s = 4(p + 1)$, where p is any nonnegative integer, we can observe that

$$2(\ell(v_{i,0}) - \ell(v_{i,4p})) = g_0$$

at least for every $1 \le p \le a - 1$. This bound is sharp when $c = 0$. Let first $a \ge 2$. We have at least

$$2(\ell(v_{i,0}) - \ell(v_{i,4})) = g_0 \text{ and } 2(\ell(v_{i,0}) - \ell(v_{i,8})) = g_0. \tag{3.7}$$

On the other hand, if $a < 2$, then $c \neq 0$, since $n \notin \{4, 8\}$. Hence $v_{i,0}, v_{i,4}$, and $v_{i,8}$ are again different vertices and (3.7) holds as well.

Fix $i = 0$. Since ℓ is bijection, $\ell(v_{0,0}) - \ell(v_{0,4p}) \neq 0$ for every p such that $p \in \{1, \dots, a-1\}$, therefore $\ell(v_{0,0}) - \ell(v_{0,4p})$ has to be an involution for every $p \in \{1, \dots, a-1\}$ and we have at least two involutions when $n = 3$. Moreover, bijectivity of ℓ implies that

$$\ell(v_{0,0}) - \ell(v_{0,4p_1}) \neq \ell(v_{0,0}) - \ell(v_{0,4p_2})$$

for every $p_1 \neq p_2$ such that $p_1, p_2 \in \{1, \dots, a-1\}$. Thus the set of involutions of Γ has to consists of at least $\max\{2, a-1\}$ distinct elements. □

Theorem 3.5. If $m, n \not\equiv 0 \pmod 4$ then $C_m \times C_n$ is not Γ-distance magic for any Abelian group Γ of order mn.

Proof. If m, n are odd, then any group Γ has odd order mn and we are done by Theorem 3.4, as there are no involutions in Γ. If $m \equiv 2 \pmod 4$ and n is odd (or $n \equiv 2 \pmod 4$ and m is odd, resp.), then $mn \equiv 2 \pmod 4$. Thus by the fundamental theorem of finite Abelian groups $\Gamma \cong \mathbb{Z}_2 \times \mathbb{Z}_{p_1^{\alpha_1}} \times \ldots \times \mathbb{Z}_{p_k^{\alpha_k}}$ where $mn = 2\prod_{i=1}^{k} p_i^{\alpha_i}$ and $p_i > 2$ for $i \in \{1, \dots, k\}$ are not necessarily distinct primes. Therefore there exists exactly one involution i in Γ ($i = (1, 0, \dots, 0)$) and we are done by Theorem 3.4.

Suppose now that $m, n \equiv 2 \pmod 4$, then $m = 2 + 4a$, $n = 2 + 4b$ and $mn = 4(1 + 2a)(1 + 2b)$ for some integers a, b. Thus by the fundamental theorem of finite Abelian groups $\Gamma \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathcal{A}$ or $\Gamma \cong \mathbb{Z}_4 \times \mathcal{A}$ for some Abelian group $\mathcal{A}$ of order $(1 + 2a)(1 + 2b)$. Since $(1 + 2a)(1 + 2b)$ is an odd number, then, if $\Gamma \cong \mathbb{Z}_4 \times \mathcal{A}$, there exists only one involution $i = (2, 0)$ in Γ and $C_m \times C_n$ is not $\mathbb{Z}_4 \times \mathcal{A}$-distance magic by Theorem 3.4.

In the case $\Gamma \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathcal{A}$, there exist exactly three involutions $i_1 = (1, 0, 0)$, $i_2 = (0, 1, 0)$ and $i_3 = (1, 1, 0)$ in Γ. Let $G = C_m \times C_n$ and assume that G is Γ-distance magic, i.e., there is a bijection $\ell : V(G) \to \Gamma$ such that for every $x \in V(G)$, $w(x) = \mu$ for some constant μ. Using the same arguments as in the proof of Theorem 3.4, since $m \equiv 2 \pmod 4$, we obtain that $2(\ell(v_{0,0}) + \ell(v_{0,2})) = 2(\ell(v_{0,2}) + \ell(v_{0,4})) = \mu$ and $2(\ell(v_{0,4}) + \ell(v_{0,6})) = 2(\ell(v_{0,6}) + \ell(v_{0,8})) = \mu$. On the other hand since $n \equiv 2 \pmod 4$ we have $\gcd(2, n) = \gcd(4, n) = 2$ and there exists α' such that $4\alpha' \equiv 2 \pmod n$. By repeating the above arguments we get $2(\ell(v_{0,4}) + \ell(v_{2,4})) = 2(\ell(v_{2,4}) + \ell(v_{4,4})) = \mu$ and $2(\ell(v_{0,4}) + \ell(v_{m-2,4})) = 2(\ell(v_{m-2,4}) + \ell(v_{m-4,4})) = \mu$. Therefore:

$$\begin{aligned}
2(\ell(v_{0,0}) - \ell(v_{0,4})) &= 2g_1 = 0,\\
2(\ell(v_{0,8}) - \ell(v_{0,4})) &= 2g_2 = 0,\\
2(\ell(v_{4,4}) - \ell(v_{0,4})) &= 2g_3 = 0\\
2(\ell(v_{m-4,4}) - \ell(v_{0,4})) &= 2g_4 = 0.
\end{aligned}$$

If any $g_i = 0$ (for $i \in \{1, 2, 3, 4\}$), then the labeling ℓ is not a bijection as $m, n \not\equiv 0 \pmod 4$. Thus we can assume that all g_i are involutions and by the Pigeonhole Principle there exist $j \neq i$ such that $g_i = g_j$ (since there are only three involutions in Γ) what implies that the labeling ℓ is not a bijection $m, n \not\equiv 0 \pmod 4$ (e.g., if $g_2 = g_3$, then $\ell(v_{0,8}) = \ell(v_{0,4})$), a contradiction. □

The immediate corollary follows.

Corollary 3.6. Assume that $m, n \geq 3$ and $\{m, n\} = \{4a, 4b + c\}$ for some integers $a \geq 3$ and $b \geq 0$, $c \in \{1, 2, 3\}$. Then $C_m \times C_n$ can be Γ-distance magic only in the following cases:

- $c \in \{1, 3\}$ and $\Gamma \cong \mathcal{A} \times (\mathbb{Z}_2)^{p+2}$ for some Abelian group $\mathcal{A}$ of odd order, where $a = 2^p$,
- $c \in \{1, 3\}$ and $\Gamma \cong \mathcal{A} \times \mathbb{Z}_3 \times (\mathbb{Z}_2)^p$ for some Abelian group $\mathcal{A}$ of odd order, where $a = 3 \cdot 2^{p-2}$,
- $c \in \{1, 3\}$ and $\Gamma \cong \mathcal{A} \times (\mathbb{Z}_2)^p \times \mathbb{Z}_4$ for some Abelian group $\mathcal{A}$ of odd order, where $a = 2^p$,
- $c \in \{1, 3\}$ and $\Gamma \cong \mathcal{A} \times (\mathbb{Z}_2)^{p-2} \times (\mathbb{Z}_4)^2$ for some Abelian group $\mathcal{A}$ of odd order, where $a = 2^p$,
- $c \in \{1, 3\}$ and $\Gamma \cong \mathcal{A} \times (\mathbb{Z}_2)^{p-1} \times \mathbb{Z}_8$ for some Abelian group $\mathcal{A}$ of odd order, where $a = 2^p$,
- $c = 2$ and $\Gamma \cong \mathcal{A} \times (\mathbb{Z}_2)^{p+3}$ for some Abelian group $\mathcal{A}$ of odd order, where $a = 2^p$,
- $c = 2$ and $\Gamma \cong \mathcal{A} \times \mathbb{Z}_3 \times (\mathbb{Z}_2)^{p+1}$ for some Abelian group $\mathcal{A}$ of odd order, where $a = 3 \cdot 2^{p-2}$,
- $c = 2$ and $\Gamma \cong \mathcal{A} \times \mathbb{Z}_3 \times (\mathbb{Z}_2)^{p-1} \times \mathbb{Z}_4$ for some Abelian group $\mathcal{A}$ of odd order, where $a = 3 \cdot 2^{p-2}$,
- $c = 2$ and $\Gamma \cong \mathcal{A} \times \mathbb{Z}_5 \times (\mathbb{Z}_2)^p$ for some Abelian group $\mathcal{A}$ of odd order, where $a = 5 \cdot 2^{p-3}$,
- $c = 2$ and $\Gamma \cong \mathcal{A} \times \mathbb{Z}_7 \times (\mathbb{Z}_2)^p$ for some Abelian group $\mathcal{A}$ of odd order, where $a = 7 \cdot 2^{p-3}$,
- $c = 2$ and $\Gamma \cong \mathcal{A} \times (\mathbb{Z}_2)^{p+1} \times \mathbb{Z}_4$ for some Abelian group $\mathcal{A}$ of odd order, where $a = 2^p$,
- $c = 2$ and $\Gamma \cong \mathcal{A} \times (\mathbb{Z}_2)^{p-1} \times (\mathbb{Z}_4)^2$ for some Abelian group $\mathcal{A}$ of odd order, where $a = 2^p$,
- $c = 2$ and $\Gamma \cong \mathcal{A} \times (\mathbb{Z}_2)^p \times \mathbb{Z}_8$ for some Abelian group $\mathcal{A}$ of odd order, where $a = 2^p$,
- $c = 2$ and $\Gamma \cong \mathcal{A} \times (\mathbb{Z}_2)^{p-2} \times \mathbb{Z}_4 \times \mathbb{Z}_8$ for some Abelian group $\mathcal{A}$ of odd order, where $a = 2^p$,
- $c = 2$ and $\Gamma \cong \mathcal{A} \times (\mathbb{Z}_2)^{p-1} \times \mathbb{Z}_{16}$ for some Abelian group $\mathcal{A}$ of odd order, where $a = 2^p$.

Proof. Let $a = \alpha 2^p$ for some odd number α. Observe that the number of involutions is equal to $2^\beta - 1$, where β is the number of the factors $\mathbb{Z}_{2k}$ of Γ. By Theorem 3.4 we have $\alpha 2^p - 1 \leq 2^\beta - 1$ and hence $\alpha \leq 2^{\beta - p}$. It is straightforward to see that if $c \in \{1, 3\}$, then the maximum number of such factors is $p + 2$ and $\alpha \leq 4$, while in the case when $c = 2$ it is $p + 3$ and $\alpha \leq 8$. Moreover $\mathbb{Z}_6 \cong \mathbb{Z}_3 \times \mathbb{Z}_2$, $\mathbb{Z}_{10} \cong \mathbb{Z}_5 \times \mathbb{Z}_2$, $\mathbb{Z}_{12} \cong \mathbb{Z}_4 \times \mathbb{Z}_3$ and $\mathbb{Z}_{14} \cong \mathbb{Z}_7 \times \mathbb{Z}_2$ are the only groups of the respective order, so the listed groups are the only ones that consist of at least p factors $\mathbb{Z}_{2k}$. $\square$

In the previous section in Propositions 2.5 and 2.6 we presented constructions for all the cases from Corollary 3.6, where $m = 2^p$ or $n = 2^p$ for some integer p. However we think that whole Corollary 3.6 gives not only necessary but also sufficient conditions for a graph to be group distance magic so we post the following conjecture.

Conjecture 3.7. Assume that $m, n \geq 3$ and $\{m, n\} = \{4a, 4b + c\}$ for some integers $a \geq 3$ and $b \geq 0$, $c \in \{1, 2, 3\}$. Then $C_m \times C_n$ is Γ-distance magic in the following cases:

- $c \in \{1, 3\}$ and $\Gamma \cong \mathcal{A} \times \mathbb{Z}_3 \times (\mathbb{Z}_2)^p$ for some Abelian group $\mathcal{A}$ of odd order, where $a = 3 \cdot 2^{p-2}$,
- $c = 2$ and $\Gamma \cong \mathcal{A} \times \mathbb{Z}_3 \times (\mathbb{Z}_2)^{p+1}$ for some Abelian group $\mathcal{A}$ of odd order, where $a = 3 \cdot 2^{p-2}$,
- $c = 2$ and $\Gamma \cong \mathcal{A} \times \mathbb{Z}_3 \times (\mathbb{Z}_2)^{p-1} \times \mathbb{Z}_4$ for some Abelian group $\mathcal{A}$ of odd order, where $a = 3 \cdot 2^{p-2}$,
- $c = 2$ and $\Gamma \cong \mathcal{A} \times \mathbb{Z}_5 \times (\mathbb{Z}_2)^p$ for some Abelian group $\mathcal{A}$ of odd order, where $a = 5 \cdot 2^{p-3}$,
- $c = 2$ and $\Gamma \cong \mathcal{A} \times \mathbb{Z}_7 \times (\mathbb{Z}_2)^p$ for some Abelian group $\mathcal{A}$ of odd order, where $a = 7 \cdot 2^{p-3}$,

We finish with the following result.

Theorem 3.8. A graph $C_m \times C_n$ is $\mathbb{Z}_{mn}$-distance magic if and only if $m \in \{4, 8\}$ or $n \in \{4, 8\}$ or $m, n \equiv 0 \pmod 4$.

Proof. If $m \not\equiv 0 \pmod 4$ and $n \notin \{4, 8\}$, or $n \not\equiv 0 \pmod 4$ and $m \notin \{4, 8\}$, then the group $\mathbb{Z}_{mn}$ has at most one involution i (namely $i = \frac{mn}{2}$, if mn is even) and so $C_m \times C_n$ is not $\mathbb{Z}_{mn}$-distance magic by Theorem 3.4. If $n = 4$ or $m = 4$ then $C_m \times C_n$ is $\mathbb{Z}_{mn}$-distance magic by Theorem 2.3 and if $m, n \equiv 0 (\mathrm{mod}\, 4)$, then $C_m \times C_n$ is $\mathbb{Z}_{mn}$-distance magic by Proposition 3.2. If $n = 8$ or $m = 8$, then the graph $C_m \times C_n$ is $\mathbb{Z}_{mn}$-distance magic by Theorem 2.4. $\square$

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ARS MATHEMATICA CONTEMPORANEA

Also available at http://amc-journal.eu
ISSN 1855-3966 (printed edn.), ISSN 1855-3974 (electronic edn.)
ARS MATHEMATICA CONTEMPORANEA 9 (2015) 109–113

On spectral radius and energy of complete multipartite graphs

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Received 18 June 2013, accepted 8 March 2014, published online 8 August 2014

Abstract

Let $K_{n_1,n_2,\dots,n_p}$ denote the complete p-partite graph, $p > 1$, on $n = n_1 + n_2 + \cdots + n_p$ vertices and let $n_1 \geq n_2 \geq \cdots \geq n_p > 0$. We show that for a fixed value of n, both the spectral radius and the energy of complete p-partite graphs are minimal for complete split graph $CS(n, p-1)$ and are maximal for Turán graph $T(n, p)$.

Keywords: Spectral radius of graph, graph energy, complete multipartite graph, complete split graph, Turán graph.

Math. Subj. Class.: 05C50

1 Introduction

Let G be a simple graph on n vertices, and $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ be its eigenvalues (i.e., the eigenvalues of the (0,1)-adjacency matrix of G) [2, 4]. Then $\lambda_1 = \lambda_1(G)$ is said to be the *spectral radius* of the graph G whereas

$$\mathcal{E} = \mathcal{E}(G) = \sum_{i=1}^{n} |\lambda_i| \tag{1.1}$$

*Corresponding author. The work of DS was supported by Research Programme P1-0285 and Research Project J1-4021 of the Slovenian Research Agency and Research Grant ON174033 of the Ministry of Education and Science of Serbia.

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is its *energy*. The spectral radius is one of the most thoroughly investigated graph-spectral parameters [2, 4, 3]. Also the graph energy has recently attracted much attention [6, 7]. In spite of this, not much is known on λ_1 and $\mathcal{E}$ of complete multipartite graphs [5].

Let $p > 1$. Denote by $K_{n_1,n_2,\dots,n_p}$ the complete p-partite graph on $n = n_1+n_2+\cdots+n_p$ vertices. For convenience, we assume that $n_1 \geq n_2 \geq \cdots \geq n_p > 0$. Two particular types of complete multipartite graphs are:

- Complete split graph $CS(n, p-1) = K_{n-p+1,1,\dots,1}$ consisting of an independent set of $n-p+1$ vertices and a clique of $p-1$ vertices, such that each vertex of the independent set is adjacent to each vertex of the clique, and
- Turán graph $T(n,p) \cong K_{\lceil n/p\rceil,\dots,\lceil n/p\rceil,\lfloor n/p\rfloor,\dots,\lfloor n/p\rfloor}$, the $(p+1)$-clique-free graph with maximum number of edges [10].

A fundamental result on the spectrum of complete multipartite graphs is:

Theorem 1.1 ([9, 8]). *A connected graph has exactly one positive eigenvalue of its adjacency matrix if and only if it is a complete multipartite graph.*

An immediate consequence of Theorem 1.1 is

Lemma 1.2. *If λ_1 is the spectral radius of the complete multipartite graph $K_{n_1,n_2,\dots,n_p}$, then $\mathcal{E}(K_{n_1,n_2,\dots,n_p}) = 2\,\lambda_1$.*

Proof. Since all graph eigenvalues are real numbers and their sum is zero, from Eq. (1.1) follows that $\mathcal{E}(G)$ is equal to twice the sum of positive eigenvalues. $\square$

It is known that the characteristic polynomial of $K_{n_1,n_2,\dots,n_p}$ is given by [2, 5]

$$\phi(K_{n_1,n_2,\dots,n_p}, \lambda) = \lambda^{n-p}\left(1 - \sum_{i=1}^{p} \frac{n_i}{\lambda + n_i}\right)\prod_{j=1}^{p}(\lambda + n_j)\,. \tag{1.2}$$

The spectrum of $K_{n_1,n_2,\dots,n_p}$ consists of the spectral radius λ_1 determined from the equation $\sum_{i=1}^{p} \frac{n_i}{\lambda+n_i} = 1$, eigenvalue 0 with multiplicity $n-p$ and $p-1$ eigenvalues situated in the intervals $[-n_p, -n_{p-1}], \dots, [-n_2, -n_1]$. In the special case $n_1 = n_2 = \cdots = n_p = t$, the spectrum of $K_{t,t,\dots,t}$ consists of the spectral radius $t(p-1)$ with unit multiplicity, eigenvalue 0 with multiplicity $p(t-1)$, and eigenvalue $-t$ with multiplicity $p-1$, so that

$$\mathcal{E}(K_{t,t,\dots,t}) = 2(p-1)t\,.$$

Remark 1.3. The 1-partite complete graph (when $p = 1$ and $t = n$) is the edgeless graph $\overline{K_n}$ for which, consistently, $\lambda_1 = \mathcal{E} = 0$. The n-partite complete graph (when $p = n$ and $t = 1$) is the ordinary complete graph K_n for which, consistently, $\lambda_1 = n-1$ and $\mathcal{E} = 2(n-1)$.

If $p = 2$, then the special case of Eq. (1.2) is

$$\phi(K_{n_1,n_2}, \lambda) = \lambda^{n-2}\left(\lambda^2 - n_1\,n_2\right),$$

from which the (well known) expressions for spectral radius and energy follow:

$$\lambda_1(K_{n_1,n_2}) = \sqrt{n_1\,n_2}, \qquad \mathcal{E}(K_{n_1,n_2}) = 2\sqrt{n_1\,n_2}\,.$$

Because $n_1 + n_2 = n$, for a fixed number of vertices n, we arrive at:

Claim 1.4. 1° *$\lambda_1(K_{n_1,n_2})$ and $\mathcal{E}(K_{n_1,n_2})$ are minimal if $n_1 = n-1$ and $n_2 = 1$.*
2° *$\lambda_1(K_{n_1,n_2})$ and $\mathcal{E}(K_{n_1,n_2})$ are maximal if $n_1 - n_2 \leq 1$.*

If $p = 3$, then the special case of Eq. (1.2) is

$$\phi(K_{t_1,t_2,t_3}, \lambda) = \lambda^{p-3}\Big(\lambda^3 - (t_1\,t_2 + t_2\,t_3 + t_3\,t_1)\,\lambda - 2\,t_1\,t_2\,t_3\Big)\,. \tag{1.3}$$

Using (1.3), it is relatively easy to show the following:

Claim 1.5. 1° *Let $n_1 + n_2 + n_3$ be equal to a fixed integer n. Then the spectral radius and the energy of K_{n_1,n_2,n_3} are minimal if $n_1 = n-2$ and $n_2 = n_3 = 1$.*
2° *Let $n_1 + n_2 + n_3$ be equal to a fixed integer n. Then the spectral radius and the energy of K_{n_1,n_2,n_3} are maximal if $n_1 - n_3 \leq 1$.*

The above claims were the motivation for establishing our main results:

Theorem 1.6. *Let $p \geq 2$ and $n_1 + n_2 + \cdots + n_p$ be equal to a fixed integer n. Then the spectral radius and the energy of $K_{n_1,n_2,\ldots,n_p}$ are minimal if $K_{n_1,n_2,\ldots,n_p} \cong CS(n, p-1)$.*

Theorem 1.7. *Let $p \geq 2$ and $n_1 + n_2 + \cdots + n_p$ be equal to a fixed integer n. Then the spectral radius and the energy of $K_{n_1,n_2,\ldots,n_p}$ are maximal if $K_{n_1,n_2,\ldots,n_p} \cong T(n,p)$.*

2 Proofs of Theorems 1.6 and 1.7

Let λ_1 and x be, respectively, the spectral radius and the corresponding unit eigenvector of the adjacency matrix of $K_{n_1,\ldots,n_p}$. Since λ_1 is a simple eigenvalue of $K_{n_1,\ldots,n_p}$, similar vertices have equal x-components. Hence, we may denote by x_i the common x-component of vertices in the part of $K_{n_1,\ldots,n_p}$ having n_i vertices for $i = 1,\ldots,p$. From the eigenvalue equation, we have:

$$\lambda_1\,x_i = \sum_{\substack{k=1\\k\neq i}}^{p} n_k\,x_k = X - n_i\,x_i$$

where $X = \sum_{k=1}^{p} n_k\,x_k$. Then

$$x_i = \frac{X}{\lambda_1 + n_i}\,. \tag{2.1}$$

Lemma 2.1. *If $n_i - n_j \geq 2$, then*

$$\lambda_1(K_{n_1,\ldots,n_i-1,\ldots,n_j+1,\ldots,n_p}) > \lambda_1(K_{n_1,\ldots,n_i,\ldots,n_j,\ldots,n_p})\,.$$

Proof. Let λ_1, x and E denote, respectively, the spectral radius, the corresponding eigenvector, and the edge set of $K_{n_1,\ldots,n_i,\ldots,n_j,\ldots,n_p}$, and let λ_1^*, A^* and E^* denote, respectively, the spectral radius, the adjacency matrix, and the edge set of $K_{n_1,\ldots,n_i-1,\ldots,n_j+1,\ldots,n_p}$. From the Variational theorem we have

$$\begin{aligned}
\lambda_1^* \geq x^T A^* x &= \sum_{uv\in E^*} 2x_u x_v \\
&= \sum_{uv\in E} 2x_u x_v + \sum_{uv\in E^*\setminus E} 2x_u x_v - \sum_{uv\in E\setminus E^*} 2x_u x_v \\
&= \lambda_1 + 2x_i(n_i-1)x_i - 2x_i n_j x_j \\
&= \lambda_1 + 2x_i X\left(\frac{n_i-1}{\lambda_1+n_i} - \frac{n_j}{\lambda_1+n_j}\right)
\end{aligned} \tag{2.2}$$

by Eq. (2.1). Next, note that $K_{n_1,\dots,n_i,\dots,n_j,\dots,n_p}$ has K_{n_i,n_j} as an induced subgraph, so that, by the Interlacing theorem [2],

$$\lambda_1 \geq \sqrt{n_i\, n_j} > n_j\ .$$

Therefore,

$$\frac{n_i-1}{\lambda_1+n_i} - \frac{n_j}{\lambda_1+n_j} = \frac{(n_i-n_j-1)\lambda_1 - n_j}{(\lambda_1+n_i)(\lambda_1+n_j)} \geq \frac{\lambda_1-n_j}{(\lambda_1+n_i)(\lambda_1+n_j)} > 0$$

so that $\lambda_1^* > \lambda_1$ follows from Eq. (2.2). □

Proof of Theorem 1.6 Let $K_{m_1,\dots,m_p}$ be the complete multipartite graph with the smallest spectral radius. If there are two parameters $m_i \geq m_j \geq 2$, then $(m_i+1)-(m_j-1) \geq 2$ and from Lemma 2.1

$$\lambda_1(K_{m_1,\dots,m_i,\dots,m_j,\dots,m_p}) > \lambda_1(K_{m_1,\dots,m_i+1,\dots,m_j-1,\dots,m_p})$$

contradicting the choice of $K_{m_1,\dots,m_i,\dots,m_j,\dots,m_p}$. Hence, all parameters $m_1,\dots,m_p$ are equal to one, except for one parameter equal to $n-p+1$, so that $K_{m_1,\dots,m_p} \cong CS(n,p-1)$. □

Proof of Theorem 1.7 Let $K_{m_1,\dots,m_p}$ be the complete multipartite graph with the largest spectral radius. It is apparent from Lemma 2.1, that $|m_i - m_j| \leq 1$ holds for all $i \neq j$, as otherwise, assuming $m_i - m_j \geq 2$,

$$\lambda_1(K_{m_1,\dots,m_i-1,\dots,m_j+1,\dots,m_p}) > \lambda_1(K_{m_1,\dots,m_i,\dots,m_j,\dots,m_p})$$

contradicting the choice of $K_{m_1,\dots,m_i,\dots,m_j,\dots,m_p}$. The condition $|m_i - m_j| \leq 1$ for $i \neq j$ implies that each parameter m_i is equal to either $\lfloor n/p \rfloor$ or $\lceil n/p \rceil$, so that $K_{m_1,\dots,m_p} \cong T(n,p)$. □

Remark 2.2. Delorme [5] proved (a bit too concisely) that changing the arbitrary e parameters of a complete multipartite graph by their average value increases the spectral radius by relying on the characteristic polynomial. While this result can be substituted for Lemma 2.1 in the proofs of Theorems 1.6 and 1.7, Lemma 2.1 is an independent result based on the principal eigenvector and inspired by the rotation lemma from [1].

Remark 2.3. Delorme also asked in [5] whether the spectral radius of complete multipartite graph $K_{n_1,\dots,n_p}$ is a concave function on the $(p-1)$-dimensional simplex Y : $\sum_{i=1}^p n_i = n \wedge (\forall i \in \{1,\dots,p\})(n_i \geq 0)$, i.e., whether

$$\lambda_1(K_{t(n_1,\dots,n_p)+(1-t)(m_1,\dots,m_p)}) \geq t\,\lambda_1(K_{n_1,\dots,n_p}) + (1-t)\,\lambda_1(K_{m_1,\dots,m_p})$$

for any two points $(n_1,\dots,n_p), (m_1,\dots,m_p) \in Y$ and each $t \in [0,1]$ such that

$$t(n_1,\dots,n_p) + (1-t)(m_1,\dots,m_p) \in Y?$$

Delorme proved this affirmatively for $p \leq 3$ in [5]. We tested it computationally for $t = 0.5$, $p \in \{4,\dots,10\}$ and $n \leq 33$ and found no counterexamples to the above question on these simplices.

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ARS MATHEMATICA CONTEMPORANEA

Also available at http://amc-journal.eu
ISSN 1855-3966 (printed edn.), ISSN 1855-3974 (electronic edn.)
ARS MATHEMATICA CONTEMPORANEA 9 (2015) 115–124

Laceable knights

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Received 12 December 2012, accepted 17 August 2013, published online 26 August 2014

Abstract

A bipartite graph is *Hamilton-laceable* if for any two vertices in different parts there is a Hamiltonian path from one to the other. Using two main ideas (an algorithm for finding Hamiltonian paths and a decomposition lemma to move from smaller cases to larger) we show that the graph of knight's moves on an $m \times n$ board is Hamilton-laceable iff $m \geq 6$, $n \geq 6$, and one of m, n is even. We show how the algorithm leads to new conjectures about Hamiltonian paths for various families, such as generalized Petersen graphs, I-graphs, and cubic symmetric graphs.

Keywords: Hamilton-laceable, generalized Petersen graphs, Hamilton-connected, Hamiltonian paths, knight graph, traceable.

Math. Subj. Class.: 05C45, 05C85

1 Introduction

Let $N_{m,n}$ be the graph of knight's moves on an $m \times n$ chessboard. Knight's tours, which are Hamiltonian cycles in these graphs, have been considered for over 1000 years and in 1991 Allen Schwenk ([13]; see also [5]) characterized the knight graphs having a Hamiltonian cycle. Assuming $m \leq n$, he proved that $N_{m,n}$ is Hamiltonian iff at least one of m, n is even, m is not 1, 2, or 4, and (m, n) is not $(3, 4)$, $(3, 6)$, or $(3, 8)$.

Two related questions arise: Which knight's graphs are (a) Hamilton-laceable (HL); (b) traceable? A bipartite graph is *HL* if for any two vertices in different parts there is a Hamiltonian path from one to the other. And a graph is *traceable* if it has at least one Hamiltonian path.

An HL graph with at least one edge is necessarily Hamiltonian. Here we give a complete characterization of the HL knight's graphs: $N_{m,n}$ is HL iff m and n are at least 6 and

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at least one of m, n is even. The problem has been investigated for square boards; Conrad et al. [5] showed that $N_{n,n}$ is HL iff n is even and $n \geq 6$.

For traceability, the problem has been solved, though unpublished. Mark Krusemeyer settled it in 1972 (see also [8]). A paper in 1978 by Cull and DeCurtins [6] settles all cases except $N_{3,n}$ and $N_{4,n}$. The failure of the 4×4 case appears in [16, p. 51]. We include here a simple approach to the $3 \times n$ and $4 \times n$ cases. The complete characterization is that the only nontraceable, connected knight graphs are $N_{3,5}$, $N_{3,6}$, and $N_{4,4}$.

Our approach uses computation and the algorithms used have broad application to Hamiltonian properties. We discuss some conjectures inspired by connectivity and laceability computations on several thousand graphs.

2 Knight graph laceability

Our method is similar to that of [5] in that we use computation for small cases and then settle the general case by decomposition into small cases. Using *Mathematica*'s `FindHamiltonianCycle` function for the computer search means that the amount of programming is quite small. We use HP to abbreviate *Hamiltonian path*.

Theorem 2.1. *$K_{m,n}$ is not HL if $m \leq 5$ or both m, n are odd.*

Proof. By Schwenk's Theorem we need only consider $K_{3,n}$ and $K_{5,n}$. Those cases follow from the following lemma since the corner vertices satisfy the hypothesis. □

Lemma 2.2. *If G is bipartite and Hamiltonian and has two vertices of degree 2 that have exactly one common neighbor, then G is not HL.*

Proof. Let u and v be the degree-2 vertices with common neighbor w. Let x be the neighbor of u that is not w and let a be a neighbor of x that is not u. Such must exist because the graph is Hamiltonian, and so has no degree-1 vertices. Then a is neither w (because graph is bipartite) nor v (w is the only common neighbor of u and v). There can be no Hamiltonian path from a to w because once a path strikes one of u, v it must go to w and therefore end without visiting both u and v. □

The main tool for the general result is a decomposition lemma, but there is a complication. Let S and F denote the start and finish points of an HP. There is one case — where S and F occupy the rightmost corners of a component board as shown in Figure 1 — where it is not clear that combining two HL boards leads to an HL board. The lemma that follows avoids this case. We say that an $m \times n$ board is HL if $K_{m,n}$ is HL; the *width* of an $m \times n$ board is n, the horizontal dimension.

Lemma 2.3 (Decomposition Lemma)**.** *Suppose the $m \times n$ board G splits into two HL boards by a vertical line. Then G is HL provided*

- *m is odd; or*
- *m is even, n is even, and the two component boards have different widths; or*
- *m is even, n is odd, and the two component boards have widths that differ by 2 or more.*

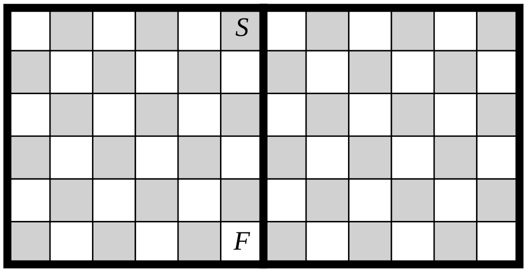

Figure 1: A difficult case on a 6×12 board.

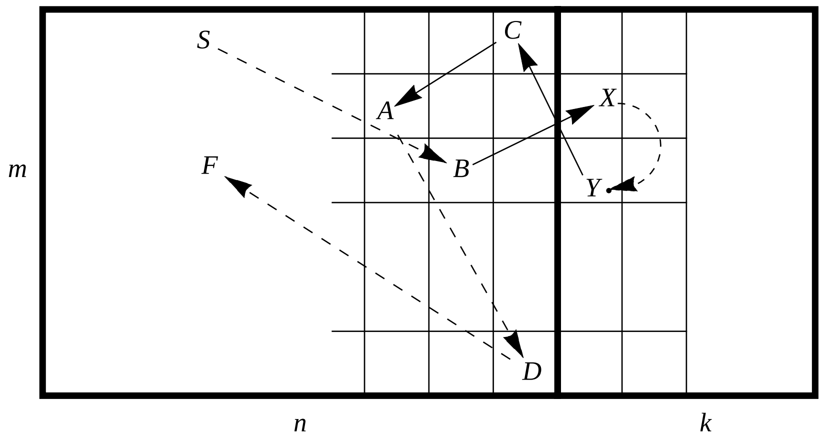

Figure 2: Building an HP from S to F. Dashed arrows stand for a longer part of a Hamiltonian path.

Proof. Note that $m \geq 6$ by Theorem 2.1, so there are at least three white and three black squares in any column. Call the two smaller boards (called *halves*), each with m rows, *left* and *right*. If S and F are in different halves, just take an HP in S's half to an end point near the border, cross over to the F's half, and finish by using an HP in that half to reach F.

Suppose S and F are in the same half, say the left, and C and D are the rightmost corners of the left half. If one of C or D is neither S nor F, say C, find an HP in the left from S to F. The path must, just before reaching C, visit A or B, the two neighbors of C. Assume, by switching S and F if necessary, that it first passes through B, the rightmost of A and B. Then break at B, go into the right side to X (see Figure 2), and traverse the right side by an HP from X to Y. Finish by jumping to C and continuing to A and F by the original path.

This leaves the troublesome case that S and F occupy the corners C and D; here it is not clear how to build an HP in the large board. Because S and F have opposite colors, m is even. So we can simply divide G vertically but in the other order, leaving S and F where they are. Because of the conditions on the widths of the halves, S and F no longer occupy the two rightmost corner positions of the left half, and the method of the preceding

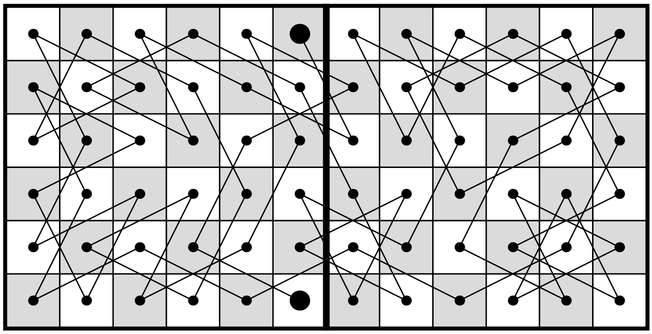

Figure 3: A resolution of the single troublesome case for $K_{6,12}$.

paragraph applies. $\square$

The next result is based on an algorithm for finding HPs. Several of these cases were first done in [5], and even before that Schwenk worked out the $K_{8,8}$ case by hand. Using *Mathematica*, the verification of all 18 cases took about 24 minutes.

Theorem 2.4. *$K_{m,n}$ is HL whenever $(m, n) \in \{(7, 8), (7, 10), (9, 10)\}$ or m is 6, 8, 10, or 12 and* $6 \leq m \leq n \leq 13$.

Proof. Assume (proof by computation follows) that $K_{6,6}$, $K_{6,8}$, and $K_{6,10}$ are HL. Then their doubles (transposing as necessary) — 6×12, 8×12, 10×12 — can be handled by dividing the board in half vertically. Laceability follows from the proof of the decomposition lemma except in the case that S and F occupy the two right-hand corners of the left half. But those cases are easily handled computationally by a search for a single HP, using the algorithm described shortly; the case of 6×12 is shown in Figure 3.

And having $K_{6,6}$, we can resolve $K_{12,12}$. This is because the 12×12 board can be bisected vertically into 12×6s, and the decomposition proof takes care of all cases except S and F occupying the rightmost corners of the left half. But then we can divide the board horizontally into two 6×12s, which places S and F in different halves and allows the proof of the decomposition lemma to be applied. Also, having 6×13 (proof below) yields 12×13 by direct application of decomposition.

The remaining 18 cases are 6×6, 6×7, 6×8, 6×9, 6×10, 6×11, 6×13, 7×8, 7×10, 8×8, 8×9, 8×10, 8×11, 8×13, 9×10, 10×10, 10×11, and 10×13. These are handled by finding a complete set of HPs. The square cases were done in [5], but we can take care of all cases with about 18 minutes of computation. Here is an algorithm based on *Mathematica*'s fast `FindHamiltonianCycle` function.

Hamiltonian path algorithm

Input: Graph G and two vertices S, F.
Output: TRUE if there is a Hamiltonian path in G from S to F; FALSE otherwise.

Step 1. Form graph H from G by adding a new vertex v and the edges $v \leftrightarrow S$ and $v \leftrightarrow F$.

Step 2. Use a Hamiltonian-cycle finder to check if such a cycle exists in H.

This algorithm is quite simple and settles the 18 cases of Theorem 2.4 in 24 minutes. There is no need to apply it to all pairs (S, F) from the two parts. One can restrict S to orbit representatives under the automorphism group. Using a fast algorithm for graph isomorphism, one can add leaves to each of two vertices to determine if they are in the same orbit. When working with larger graphs having a lot of symmetry, this use of orbit representatives cuts down the computation time dramatically. □

The algorithm above was suggested by a referee. Our original algorithm was a little more complicated and can provide more speed in some cases. One can add edge $S \leftrightarrow F$ to the graph if it is not there already, and then consider, one at a time, the other edges leaving S, checking to see if there is a Hamiltonian cycle that uses that edge. By putting a time constraint on this search one avoids possible dead ends without examining all the possibilities. For many HC or HL graphs this leads to a faster algorithm; in essence, it is a way of tweaking whatever Hamiltonian cycle finder one is using.

Another idea that is sometimes useful when the time limit runs out without a path being found is to permute the graph and try again. The idea of using permutations to speed up graph algorithms has been fruitful in other areas, such as the 4-coloring of planar graphs efficiently [8]. The Hamilton cycle algorithm used by *Mathematica* has several cases, but a main one is the Angluin–Valiant heuristic. For larger problems one can use traveling salesman problem software to search for Hamiltonian cycles (see §5).

Theorem 2.5. *$N_{m,n}$ is Hamilton-laceable iff $6 \leq m$, $6 \leq n$, and one of m, n is even.*

Proof. The forward direction follows from Theorem 2.1. For the other direction, assume $m \leq n$.

Case 1. m is odd and $m \leq 13$. In this case n is even. If $n \leq 10$, we have the conclusion by computation (Theorem 2.4), so assume $n \geq 12$. Then n splits into the even numbers 6 and $n - 6$, and the conclusion follows from the decomposition lemma by induction and Theorem 2.4.

Case 2. m is even and $m \leq 12$. If $n \leq 13$, we have the conclusion by Theorem 2, so assume $n \geq 14$. Then n splits into 6 and $n-6$; because $(n-6)-6 \geq 2$, the decomposition lemma and induction yield the result.

Case 3. m is odd and $m \geq 15$. In this case n is even. Use induction on m; the cases $7 \times n$, $9 \times n$, $11 \times n$, and $13 \times n$ follow from Case 1. For $m \geq 15$, an $m \times n$ board splits into $6 \times n$ and $(m - 6) \times n$ boards. The first is HL by case 2 and the second is HL by the inductive hypothesis. Because $(m - 6) - 6 \geq 2$, the decomposition lemma applies.

Case 4. m is even and $m \geq 14$. An $m \times n$ board splits into $6 \times n$ and $(m - 6) \times n$ boards; because $(m - 6) - 6 \geq 2$, the result follows inductively from case 2 (for the base case) and the decomposition lemma. □

A dynamic demonstration at [15] shows all 32^2 paths in the case of the traditional 8×8 board; Figure 4 shows a typical case. As noted, this classic chessboard case was resolved in [1]; the work here settles all rectangular boards.

3 Traceable knights

A graph is *traceable* if it has a Hamiltonian path. Cull and De Curtins [6] proved that $N_{m,n}$ is traceable if $m \geq 5$. Mark Krusemeyer reports having solved the complete problem in

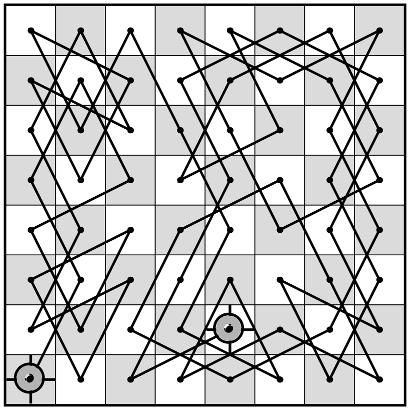

Figure 4: A snapshot (start at lower left, finish in center of second row) of a demonstration [15] that shows all 1024 knight's paths on a chessboard.

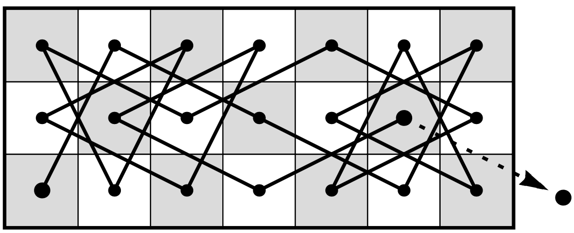

Figure 5: A knight's path from $(1, 1)$ to $(6, 2)$ on a 3×7 board. Similar paths exist with 7 replaced by 8 through 13, and hence for widths past 7.

1972 in an addendum to his doctoral thesis. We present here an approach to the $m = 3$ and $m = 4$ cases. For $m = 3$, $N_{3,3}$ is disconnected, while the HP algorithm shows that $N_{3,5}$ and $N_{3,6}$ are not traceable and $N_{3,n}$ is traceable when $n \in \{4, 7, 8, 9, 10, 11, 12\}$. Moreover, for the last six cases, there is an HP that goes from the lower-left corner to the next-to-last rightmost point of the center row (Figure 5). Therefore it is a simple matter to chain two such paths together to get an HP in the case of $n \geq 13$. This settles the $m = 3$ case.

The case of $N_{4,n}$ is harder. Using the HP algorithm with some edges deleted led to the gadget in Figure 6, which is the key to a proof that HPs exist for all $N_{4,n}$

Theorem 3.1 (Krusemeyer, Cull, De Curtins)**.** *The only connected knight graphs that are not traceable are* $N_{3,5}$, $N_{3,6}$, $N_{4,4}$.

Proof. Cull and De Curtins [6], using methods similar to the methods of the present paper (decomposition and induction), showed that $N_{m,n}$ is traceable when $5 \leq m \leq n$. The case $m = 3$ was discussed above. A computer search shows that $N_{4,4}$ is not traceable; this can also be proved by hand [16, p. 51]. Figure 6(b) shows $N_{4,5}$, $N_{4,6}$, and $N_{4,7}$ with HPs from the lower left corner to the upper left corner. The gadget of Figure 6(a) can be used to extend any such path leftward to one with the same property, but on a board with three

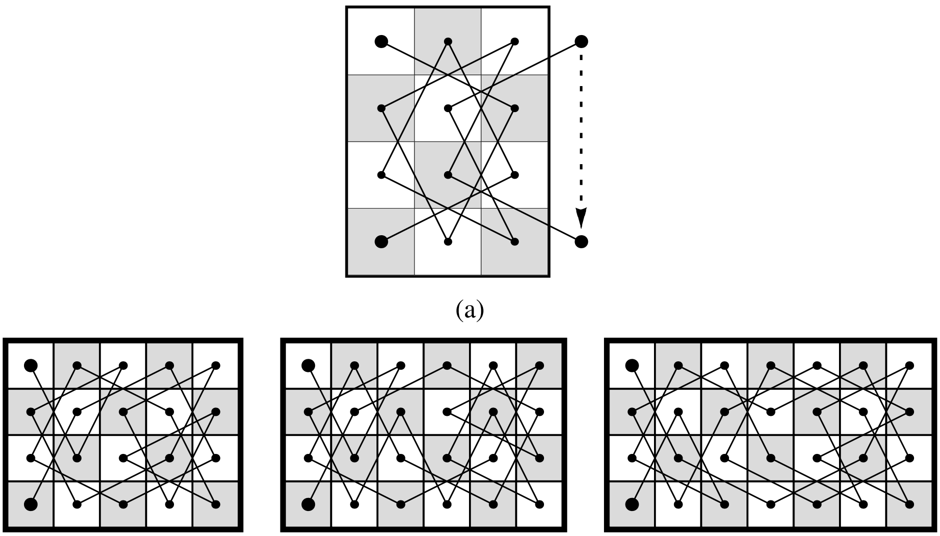

Figure 6: (a) The two disjoint paths in the 4×3 board allow one to construct Hamiltonian paths in any $4 \times n$ board with $n \geq 5$. (b) Corner-to-corner paths for 4×5, 4×6, 4×7.

more columns. This gives corner-to-corner HPs for all $(4, n)$ with $n \geq 5$. The gadget was found by using the HP algorithm on a larger graph with various edges deleted so to force the condition of one edge out and one edge in. $\square$

4 Further applications of the algorithm

The HP algorithm can be used on many families of bipartite graphs and works as well to study whether a nonbipartite graph is *Hamilton-connected* (HC), which means that there is a Hamiltonian path between any two vertices. *Mathematica*'s `GraphData` database has 572 graphs that are bipartite, Hamiltonian, vertex-transitive, and not a cycle; the largest has 2048 vertices. Using the HP algorithm we have shown that all 572 are HL. In particular, the HP algorithm shows that all the Foster graphs (connected, cubic, edge-transitive, and vertex-transitive) with 768 or fewer vertices (as well as the known Foster graphs up to 1000 vertices) are HL when bipartite and HC otherwise. Recall the folklore conjecture (based on a question by Lovász about traceabiliy of vertex-transitive graphs; see [10]) that connected vertex-transitive graphs are Hamiltonian except for five small examples. Computations show that one almost always gets stronger Hamiltonian properties than just the existence of a Hamiltonian cycle.

Question 4.1. Are even cycles the only bipartite, Hamiltonian, vertex-transitive graphs that are not Hamilton-laceable?

And the same behavior has been observed for edge-transitive graphs.

Question 4.2. Are even cycles the only bipartite, Hamiltonian, edge-transitive graphs that are not Hamilton-laceable?

One can ask similar question for Hamilton-connected graphs. Of the 1557 nonbipartite, vertex-transitive, Hamiltonian graphs in the database (cycles excluded), the HP algorithm resolved the Hamilton-connected status of all of them. Only one — the dodecahedral graph — failed to be Hamilton-connected. Several of these graphs had over 1000 vertices; the largest was $\text{Kneser}_{14,6}$ with 3003 vertices.

Question 4.3. Are the odd cycles and the dodecahedral graph the only nonbipartite, Hamiltonian, vertex-transitive graphs that are not Hamilton-connected? And we have the same question with edge-transitivity replacing vertex-transitivity.

Here are more detailed summaries for some interesting families:

Generalized Petersen graphs GP(n, k). These cubic graphs, generalizations of the Petersen graph, are defined when $n \geq 3$ and $k < n/2$. They are all Hamiltonian except GP$(6q + 5, 2)$ and isomorphs (Alspach [1]). These are bipartite when n is even and k is odd. It is easy to see that GP$(n, 1)$ is HL when bipartite, HC otherwise. Alspach and Liu [2] have shown that GP$(2m, 3)$ is HL; some authors include cases such as GP$(5, 3)$, which is isomorphic to the non-Hamiltonian Petersen graph GP$(5, 2)$, but here we restrict to $k < n/2$. A recent paper [12] by Richter contains further results on Hamiltonian paths in this family. The HP algorithm shows that GP(n, k) is HL when $n \leq 100$ and so it is reasonable to conjecture that all bipartite graphs in the family are HL. Alspach and Qin [3] proved that GP$(4m, 2m - 1)$ is HL.

The situation regarding HC and HL for $k \leq 3$ is:

GP$(n, 1)$ is HL when n is even, HC otherwise (Alspach and Liu [2]).
GP$(n, 2)$ is HC iff $n \equiv 1$ or $3 (\text{mod } 6)$ [2, 12]
GP$(n, 3)$ is HL when n is even, HC otherwise [2].

Pensaert [11] found that GP$(6, 2)$ and GP$(12, 4)$ are not HC and conjectured that GP(n, k) is HC or HL whenever $n \geq 3k + 1$. The case GP$(3k + 1, k)$ is isomorphic to GP$(3k + 1, 3)$, a settled case. The HP algorithm extends the Pensaert examples by showing that GP$(6q, 2q)$ is not HC when $q \leq 12$. The algorithm also shows that, with the exceptions mentioned for bipartite, non-Hamiltonian (GP$(6q + 5, 2)$), and GP$(6q, 2q)$, the graphs GP(n, k) are HC for $k \leq 16$ and $2k + 1 \leq n \leq 100$. So we have the conjectures that these last two results, positive and negative, hold in all cases.

The I-graph computations discussed next show that GP(n, k) is HL in all the bipartite cases with $n \leq 100$.

I-graphs $I(n, j, k)$. These cubic graphs, where $1 < j, k < n$ and $j, k \neq n/2$, are generalizations of the GP family; GP$(n, k) = I(n, 1, k)$ when $k < n/2$. An I-graph is bipartite and connected iff n is even, j and k are odd, and $\gcd(n, j, k) = 1$. Horvat et al [7] give a simple condition for two I-graphs to be isomorphic. Computation suggests that all connected I-graphs, except the generalized Petersen exceptions, are Hamiltonian; that has been recently proved by Bonvicini and Pisanski [4]. Even stronger Hamiltonian properties appear to hold: the HP algorithm can be used to check for laceability: all bipartite, connected $I(n, j, k)$ for $n \leq 100$ are HL.

Bipartite Kneser graphs $H_{n,k}$. It is a well-known conjecture that the connected ones ($k \neq n/2$) are Hamiltonian; this has been proved for $n \leq 27$ [14]. These graphs are vertex- and edge-transitive, so it suffices to restrict the start point for a Hamiltonian path to a single vertex, v. Further, using permutations on the underlying n-set shows that one need only consider $k+1$ types of pairs (v, F), the type being the size of the intersection of the set represented by v with that represented by F. These considerations speed up the HP algorithm, and the results are that $H_{n,k}$ is HL for all $4 \leq n \leq 14$ and $k < n/2$, and also for $(n, 2)$ with $n \leq 43$ and $(n, 3)$ with $n \leq 25$. Note that $H_{3,1}$ is a 6-cycle and so is not HL.

5 Conclusion

The work presented here shows how one can call on fast algorithms even for NP-complete problems as a way of learning about families of graphs, so long as the graphs are not terribly large. There are 853 bipartite Hamiltonian graphs in *Mathematica*'s database and the algorithms presented here have resolved the laceability of all of them.

Since finding Hamiltonian cycles or paths is NP-complete, it is not surprising that our methods fail for some large graphs. An idea that has proved useful on the larger examples (such as $\text{Kneser}_{13,6}$, $\text{Kneser}_{14,5}$, $\text{Kneser}_{14,6}$, and an alternating group Cayley graph with 2520 vertices) is to use the cutting-edge technology of traveling salesman problem software, such as *Concorde* or *LKH*. One can turn a graph into a weighted complete graph with weights 0 on the edges and 1 on the nonedges; then if a shortest traveling salesman tour has weight 0, one has a Hamltonian cycle; if not, then no such cycle exists. This idea was used to determine Hamilton-connectivity in the large examples.

Acknowledgments

We are grateful to Brian Alspach and Bruce Richter for valuable comments, to Rasmus Kamper for helping with the setup of *Concorde* and *LKH*, and to referees for several helpful comments.

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ARS MATHEMATICA CONTEMPORANEA

Also available at http://amc-journal.eu
ISSN 1855-3966 (printed edn.), ISSN 1855-3974 (electronic edn.)
ARS MATHEMATICA CONTEMPORANEA 9 (2015) 125–144

Matrices and their Kirchhoff graphs

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Received 11 October 2012, accepted 21 July 2014, published online 28 September 2014

Abstract

The fundamental relationship between matrices over the rational numbers and a newly defined type of graph, a Kirchhoff graph, is established. For a given matrix, a Kirchhoff graph represents the orthogonal complementarity of the null and row spaces of that matrix. A number of basic results are proven, and then a relatively complicated Kirchhoff graph is constructed for a matrix that is the transpose of the stoichiometric matrix for a reaction network for the production of sodium hydroxide from salt. A Kirchhoff graph for a reaction network is a circuit diagram for that reaction network. Finally it is conjectured that there is at least one Kirchhoff graph for any matrix with rational elements, and a process for constructing an incidence matrix for a Kirchhoff graph from a given matrix is discussed.

Keywords: Kirchhoff graphs, fundamental theorem of linear algebra, reaction networks.

Math. Subj. Class.: 05C20, 05C50, 05C90

1 Introduction

To understand the relationship between matrices and a newly-defined type of graph, a Kirchhoff or fundamental graph, consider the simple directed graph in Figure 1 and the incidence matrix for this digraph:

$$\begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & 0 \\ 0 & -1 & -1 \end{bmatrix} \tag{1.1}$$

In this matrix element $a_{ij} = 1$ if the j-th edge $\boldsymbol{s}_j$ exits the i-th vertex $\boldsymbol{\nu}_i$, $a_{ij} = -1$ if $\boldsymbol{s}_j$ enters $\boldsymbol{\nu}_i$, and $a_{ij} = 0$ if $\boldsymbol{s}_j$ is not incident to $\boldsymbol{\nu}_i$. This use of 1 and -1 is the opposite of what many authors use in graph theory, but it matches the forward direction for steps

*The support of the National Science Foundation under grant DMS-0707692 is gratefully acknowledged.
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in chemical reaction networks and hence is preferred here. The edges of the graph in Figure 1 and indeed of all the graphs considered here are vectors (have specified length and direction). Notice that the rows of this incidence matrix are linearly dependent, and that the vector $[1, 1, -1]^{\mathrm{T}}$ is in its null space. But this null-space vector also represents the cycle[1] in this digraph since

$$(1)\boldsymbol{s}_1 + (1)\boldsymbol{s}_2 + (-1)\boldsymbol{s}_3 = \mathbf{0}$$

is the cycle. Also the cuts for each vertex[2] of this graph are (as always) represented by the rows of the incidence matrix. So the graph in Figure 1 is a representation of the orthogonality of the null and row spaces of this incidence matrix, and this orthogonality is essentially

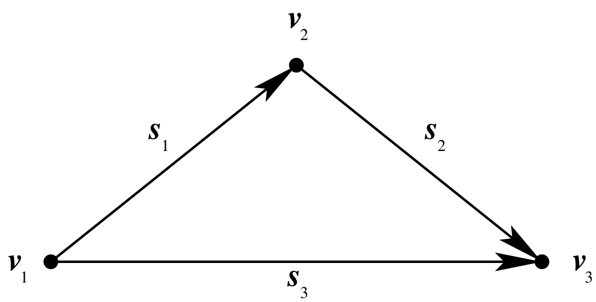

Figure 1: A simple example to illustrate the concept of Kirchhoff graph.

the classic result that the cycle space and the cut space of a (standard) graph are orthogonal complements (*cf. e.g.* Diestel(1997) [5, p. 22] or Bollobás(1998) [1, p. 53]).

The relationship between the graph in Figure 1 and its incidence matrix easily extends to multi-digraphs where the edges are again vectors. As a simple example of this extension, consider the multi-digraph in Figure 2 and its incidence matrix[3]:

$$\begin{bmatrix} 2 & 0 & 1 \\ -2 & 1 & 0 \\ 0 & -1 & -1 \\ 0 & 1-1 & 0 \\ 0 & 0 & 1-1 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 1 \\ -2 & 1 & 0 \\ 0 & -1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{1.2}$$

Here "$1 - 1$" means that the same vector enters and exits a given vertex. Again the graph in Figure 2 represents the orthogonality of the row and null spaces of this incidence matrix since the cycle is

$$(1)\boldsymbol{s}_1 + (2)\boldsymbol{s}_2 + (-2)\boldsymbol{s}_3 = \mathbf{0}$$

and the vertex cuts "lie" in this row space.

What is really interesting is that the relationship can be extended even further: For any matrix with rational elements, it would seem possible to construct a multi-digraph whose

[1]For the current discussion, a *cycle* in a graph is a closed walk, *i.e.* an alternating sequence of vertices and edges incident to adjacent vertices $\{\boldsymbol{\nu}_1, \boldsymbol{s}_1, \boldsymbol{\nu}_2, ...\boldsymbol{\nu}_n, \boldsymbol{s}_n, \boldsymbol{\nu}_1\}$ beginning and ending with the same vertex, with no assumption that any vertices or edges are distinct.

[2]A *vertex cut* in a graph is the set of edges incident to the vertex (the minimum set of edges that would need to be removed to isolate the vertex from the rest of the graph).

[3]The term *incidence matrix* as used here is a generalization of the standard definition. Here each column corresponds to an edge vector, even though that vector may appear multiple times in the graph

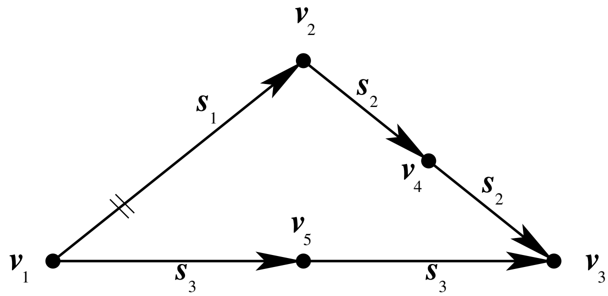

Figure 2: A multi-digraph that is a somewhat more complicated Kirchhoff graph. The double hash marks crossing s_1 indicate that two copies of this edge vector connect the first two vertices.

cycle space corresponds to the null space of the matrix and whose cut space lies in the row space of the matrix. Indeed the concept of a Kirchhoff graph is a sort of inverse of the relationship between the cycle space and the cut space for the standard graph in that one starts with an arbitrary matrix and then attempts to construct the graph. Thus each Kirchhoff graph for a matrix is a graph-theoretic representation of the fundamental theorem of linear algebra which states that the null space and the row space of any matrix are orthogonal complements.[4] Because of the relationship between Kirchhoff graphs for a given matrix and the fundamental spaces for that matrix, a Kirchhoff graph can also be termed a *fundamental graph*. Also from the simple examples above, one sees that the exact length and orientation of the edge vectors are not important. The key issue, rather, is the multiplicities of any given vector in a cycle and the multiplicities of that vector between given vertices. The graph in Figure 1 is a Kirchhoff graph for any matrix with the same row space and null space as the matrix in (1.1), and the graph in Figure 2 is a Kirchhoff graph for any matrix with the same row space and null space as the matrix in (1.2).

The concept of a Kirchhoff graph comes out of chemical reaction network theory. As the name implies, when based on a reaction network, a Kirchhoff graph satisfies both the Kirchhoff current law and the Kirchhoff potential law, and is therefore a circuit diagram for that reaction network. Their role in this context was discussed by Fehribach(2009) [9], and also by Fishtik, Datta *et al.* [11, 12, 13, 10, 14, 25] and in some of their references. In the latter works, Kirchhoff graphs are referred to as reaction route graphs. The concept of a Kirchhoff graph thus connects the fundamental theorem of linear algebra with the fundamental conservation principle in the Kirchhoff laws. There is also the important and distinct concept of a Kirchhoff or Laplacian matrix and the well-known Kirchhoff theorem which relates the eigenvalues of the Kirchhoff matrix for a graph to the number of spanning trees of that graph.

A variety of other graphs have been used to discuss reaction networks. For a general review, particularly of early uses, see Fehribach(2007) [8]. An important recent sequence of work on graphs and reaction networks began with the work of Horn(1973) [15, 16], followed by that of Perelson & Oster(1974) [22], Clarke(1980) [3], Schlosser & Feinberg(1994) [23], and the work of Craciun & Feinberg(2006) [4]. The graphs studied in

[4]Not all authors/texts use this terminology, but it has become more widely used in recent years; *cf.* Strang(2003) [24].

these articles are termed species-complex-linkage (SCL) graphs and species-complex (SC) graphs, and are useful in studying reaction kinetics and the stability of equilibria, but they are not connected to the fundamental spaces of any matrix, nor do they give a representation of the fundamental theorem of linear algebra. Also in general the above articles specifically discuss graphs; see their references for other related articles that discuss the reaction networks themselves.

Throughout this work, we consider matrices with rational elements, but then consider basis vectors for the null and row spaces with integer elements. This is possible since for any matrix over the rationals, one can multiply each element by the least common multiple of all the denominators to obtain a matrix with integral elements, but the same null and row space as the original rational matrix. Also it is important to realize that the correspondences between the null and row spaces on the one hand and the cycle and cut spaces on the other are *not* equivalences. The null and row spaces are vector spaces over the rationals, while the cycle and cut spaces are modules over the integers (it is easy to interpret integral multiples of a cycle, but not fractional multiples). For a Kirchhoff graph G of a matrix A, there are natural embeddings of the cycle space of G into the null space of A, and of the cut space of G into the row space of A.

The next section contains the formal definition of a Kirchhoff graph and several basic results; these make rigorous the ideas introduced in the examples above. Since the impetus for the concept of Kirchhoff graph comes from the theory of chemical reaction networks, Section 3 presents the development of a Kirchhoff graph for such a network. The final two sections discuss the construction of Kirchhoff graphs, as well as open problems and future work.

2 Kirchhoff Graphs—Definition and Basic Results

Consider an $m \times n$ matrix over the rationals: $A \in \mathcal{M}_{m,n}(\mathbb{Q})$. The main question to be considered here is "Which graph or graphs best represent this matrix in terms of its fundamental spaces and the fundamental theorem of linear algebra?" That is, which graph(s) best reflect the duality that $\mathrm{Row}(A)$ and $\mathrm{Null}(A)$ are orthogonal complements:

$$\mathrm{Row}(A) \perp \mathrm{Null}(A)\,, \qquad \mathrm{Row}^{\mathrm{T}}(A) \oplus \mathrm{Null}(A) = \mathbb{Q}^n$$

where the superscript $^{\mathrm{T}}$ indicates the transpose of all vectors in the space. The proposed answer to this question is the Kirchhoff graph:

Definition 2.1. For a given matrix $A \in \mathcal{M}_{m,n}(\mathbb{Q})$, a geometric cyclic multi-digraph G is a **Kirchhoff graph** for A if and only if the following two conditions are satisfied:

1. For $u_j \in \mathbb{Z}$, $\boldsymbol{u} = [u_1, u_2, ..., u_n]^{\mathrm{T}} \in \mathrm{Null}(A)$ if and only if there is a cycle in G where, for each j, $1 \le j \le n$, the j^{th} directed edge appears with multiplicity $|u_j|$. The sign of u_j gives the relative direction that the j^{th} edge is crossed.
2. For a given vertex of G, if the j^{th} edge exits with multiplicity $v_j \in \mathbb{Z}$, then $\boldsymbol{v} = [v_1, v_2, ..., v_n] \in \mathrm{Row}(A)$. When v_j is negative, the edge enters the vertex (exits in the negative sense).

Remark 2.2.

1. While in general $A \in \mathcal{M}_{m,n}(\mathbb{Q})$, there is no loss of generality in assuming that $A \in \mathcal{M}_{m,n}(\mathbb{Z})$ since one can multiply A by the least common multiple of all the denominators of the elements of A without affecting the null and row spaces. Similarly one can take the elements of the basis vectors for the null space or row space to be integers.
2. According to this definition, two matrices with the same null and row spaces have the same Kirchhoff graph(s).
3. For the present discussion, a digraph G is *cyclic* if and only if $\boldsymbol{s}_i$ is an edge vector of G implies $\boldsymbol{s}_i$ is an edge vector of some cycle $C \subset G$. The trivial graph with one vertex and no edges is then vacuously cyclic.
4. These graphs are "geometric" in that edges are vectors. Two edges with the same length and direction are the same vector and are therefore identified. For discussions of the more-general topic of geometric graphs, see Pach, *et al.* [19, 20, 21].
5. The second condition in Definition 2.1 assures that all of the vertex cuts lie in the row space of the matrix, but not all dimensions of the row space necessarily need to be represented explicitly as vertex cuts. As is discussed below, if A is nonsingular, the simplest Kirchhoff graph is a single vertex with no edges.
6. Making the outward direction positive matches the tradition in analysis and is what is needed for discussing reaction networks; unfortunately it is the opposite what is generally used in graph theory.
7. In the extreme cases where either $\mathrm{Null}(A) = \{0\}$ or $\mathrm{Row}(A) = \{0\}$, the Kirchhoff graph can be defined as a single vertex with no edges. When $\mathrm{Null}(A) = \{0\}$, this graph results from the only allowed cycle being a null cycle where all edge vectors appear an even number of times and cancel as one moves around the cycle; the simplest null cycle is a single vertex. When $\mathrm{Row}(A) = \{0\}$, then the second condition in the definition requires that all edges vectors begin and end at the same vertex and therefore have length zero.
8. As will be discussed in Section 3 below, if the matrix A is the transpose of the stoichiometric matrix for a reaction network, the first condition in Definition 2.1 is a form of the Kirchhoff potential (or voltage) law, while the second condition is a form of the Kirchhoff current law.

The rest of this section is devoted to some basic properties and results associated with Kirchhoff graphs.

2.1 Some Basic Results

Proposition 2.3. *Suppose that* $A \in \mathcal{M}_{m,n}(\mathbb{Q})$ *and* $\mathrm{Null}(A) = \mathrm{Span}\{[a_1, a_2, ..., a_n]^{\mathrm{T}}\}$, $a_j \in \mathbb{Z}$, *with at least one* $a_j \neq 0$. *Then* $\dim(\mathrm{Null}(A)) = 1$ *and* $\dim(\mathrm{Row}(A)) = n - 1$, *and a Kirchhoff graph for* A *can be given as a single cycle with* $|a_1| + |a_2| + ... + |a_n|$ *vertices.*

Proof. Suppose first that $\mathrm{Null}(A) = \mathrm{Span}\{[a_1, a_2, ..., a_n]^{\mathrm{T}}\}$ with $a_j \neq 0\ \forall j$. Then $\mathrm{Row}(A) = \mathrm{Span}\{[-a_2, a_1, 0, ..., 0], [0, -a_3, a_2, ..., 0], ..., [0, ..., -a_n, a_{n-1}]\}$, and these $n-1$ vectors give $n-1$ cut (vertex balance) conditions for vertices where differing edge vectors meet. The final condition is a linear combination these row-space basis vectors: $[-a_n, 0, 0, ..., a_1]$. Unless $|a_j| = 1\,\forall j$, there are also null vertices where the same edges enter and exit.

If $a_j = 0$ for some j, then the unit vector with 1 as the j-th element and all other elements being 0 lies in Row(A). The edge $\boldsymbol{s}_j$ does not appear in the graph, and the vertex balance conditions are formed as before leaving out this unit vector. (Recall that the definition of a Kirchhoff graph does not require that all the basis vectors for the row space be represented in a graph.) □

Example 2.4. A simple example is helpful in understanding the above proposition: Suppose that Null$(A) = $ Span$\{[0, 1, -3, 2]^{\mathrm{T}}\}$. Then

$$\text{Row}(A) = \text{Span}\{[1, 0, 0, 0], [0, 3, 1, 0], [0, 0, 2, 3]\}$$

and $[0, -2, 0, 1]$ is the additional vector from Row(A) needed for the vertex where the $\boldsymbol{s}_4$ edges join $\boldsymbol{s}_2$. Figure 3 shows a one-cycle Kirchhoff graph for this example.

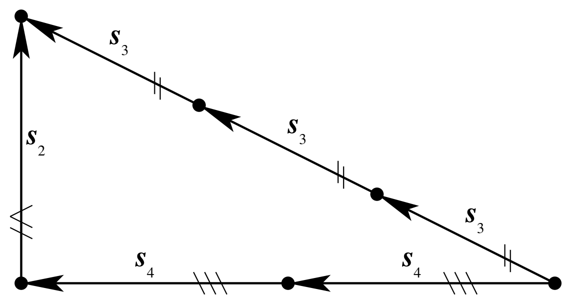

Figure 3: A simple cycle to illustrate Proposition 2.3. The hash marks give the multiplicity of each edge indicating how the cut space of the graph corresponds to Row(A).

Remark 2.5. If fewer than three of the elements a_j are nonzero, then the Kirchhoff graph is degenerate, as when the null space or row space is trivial. When only one a_j is nonzero, there is one vertex and an edge of length zero (the edge must begin and end at the same vertex); when two a_j are nonzero, there are two vertices and two overlapping edges, as in the degenerate case shown in Figure 5 below.

The previous proposition suggests that matrices can have multiple Kirchhoff graphs since the edges can appear in any order in the cycle. For example, one could split the two $\boldsymbol{s}_4$ vectors so that the order moving around the cycle is $\boldsymbol{s}_2$, $\boldsymbol{s}_4$, $-3\boldsymbol{s}_3$, and finally the second $\boldsymbol{s}_4$. In fact, a given matrix may have two or more Kirchhoff graphs which differ even more than just the order that edges appear in a cycle.

Proposition 2.6. *A given matrix $A \in \mathcal{M}_{m,n}(\mathbb{Q})$ may have multiple, distinct Kirchhoff graphs, i.e., a matrix does not necessarily have a unique Kirchhoff graph.*

Proof. Consider the matrix

$$A = \begin{bmatrix} 1 & 0 & 1 & 1 \\ -1 & 1 & 0 & 1 \end{bmatrix} \tag{2.1}$$

Here $\mathrm{Row}(A)$ is just the span of the two rows of A, while

$$\mathrm{Null}(A) = \mathrm{Span}\left\{ \begin{bmatrix} 1 \\ 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ -1 \end{bmatrix} \right\} \tag{2.2}$$

At least[5] two Kirchhoff graphs exist for this matrix (and any other matrix with the same row and null spaces), as seen in Figure 4. □

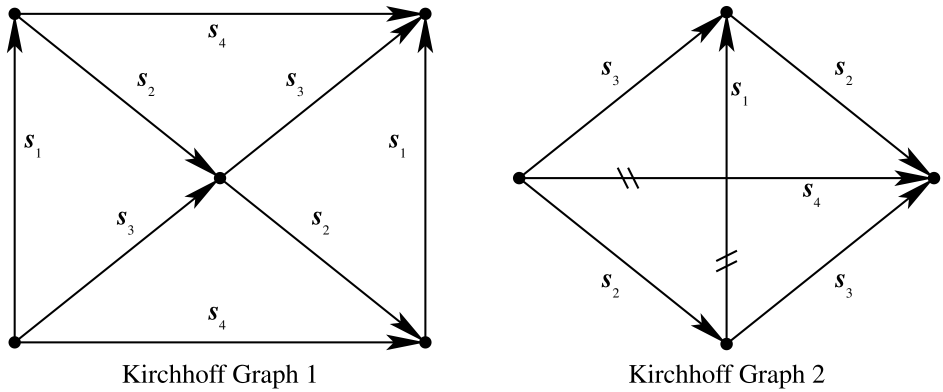

Figure 4: Two Kirchhoff graphs for A in (2.1). Again the hash marks in Graph 2 indicate that two copies of that edge are needed in that position.

Now suppose that $\dim(\mathrm{Row}(A)) = 1$ and thus $\dim(\mathrm{Null}(A)) = n - 1$. This is again a degenerate case, and a Kirchhoff graph in this case again depends on what one is willing to accept as a cycle. For our purposes here, we again allow for a degenerate cycle where the edge vectors double back on themselves.

Proposition 2.7. *Suppose that* $A \in \mathcal{M}_{m,n}(\mathbb{Q})$ *and that* $\dim(\mathrm{Row}(A)) = 1$. *Then* $\dim(\mathrm{Null}(A)) = n - 1$*, and a Kirchhoff graph for A can be given as a degenerate one-dimensional set of cycles whose edge vectors lie on top of each other.*

Proof. Suppose first that $\mathrm{Row}(A) = \mathrm{Span}\{[a_1, a_2, ..., a_n]\}$ with $a_j \neq 0$. Then $\mathrm{Null}(A) = \mathrm{Span}\{[-a_2, a_1, 0, ..., 0]^{\mathrm{T}}, [0, -a_3, a_2, ..., 0]^{\mathrm{T}}, ..., [0, 0, ..., -a_n, a_{n-1}]^{\mathrm{T}}\}$, and these $n-1$ vectors represent $n-1$ degenerate cycles. Each of the vertices in the middle of the cycle is a null vertex—a vertex where exactly the same edges enter and exit. On the other hand, the edges entering/leaving the two end vertices satisfies the row space condition, *i.e.*, at each end $|a_j|$ copies of $\boldsymbol{s}_j$ enter or exit and the sign of a_j determines which end the edge $\boldsymbol{s}_j$ enters and which end it exits. If $a_j = 0$, then $\boldsymbol{s}_j = \mathbf{0}$ since this is the only edge vector that ends where it begins. □

Example 2.8. Reversing the situation in the previous example, suppose that $\mathrm{Row}(A) = \mathrm{Span}\{[0, 1, -3, 2]\}$. Then $\mathrm{Null}(A) = \mathrm{Span}\{[1, 0, 0, 0]^{\mathrm{T}}, [0, 3, 1, 0]^{\mathrm{T}}, [0, 0, 2, 3]^{\mathrm{T}}\}$. The first vector in the span ($[1, 0, 0, 0]^{\mathrm{T}}$) implies that $\boldsymbol{s}_1 = \mathbf{0}$. Figure 5 shows a degenerate Kirchhoff graph for this example.

[5]Clearly the union of any two Kirchhoff graphs is also a Kirchhoff graph, but the author believes that these are the only two minimal or prime Kirchhoff graphs for this matrix.

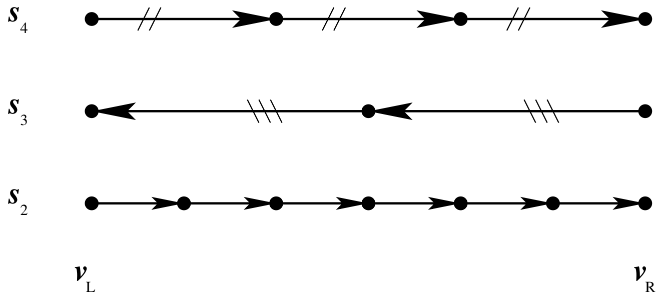

Figure 5: A degenerate Kirchhoff graph to illustrate Proposition 2.7. The three sets of vectors must be overlaid to form the Kirchhoff graph so that all edge vector sets begin or end at the two end vertices $\boldsymbol{\nu}_L$ and $\boldsymbol{\nu}_R$. The middle vertices are null vertices where copies of an edge both begin and end.

Another basic case that should be considered is the one mentioned in the Introduction:

Proposition 2.9. *Suppose that $A \in \mathcal{M}_{m,n}(\mathbb{Q})$ is row equivalent to the incidence matrix for a standard cyclic digraph (having at most one edge vector between any two vertices, and no edge vector appearing multiple times in the digraph). Then this digraph is a Kirchhoff graph for A.*

Remark 2.10. A matrix is the incidence matrix for a standard cyclic digraph if and only if (1) all elements are 0 or ± 1, (2) each column has exactly one 1 and one -1, (3) no two columns have their nonzero elements in the same rows, and (4) each row has at least two nonzero elements.

Proof. In this case the standard result that the cycle space and the cut space of a graph are orthogonal complements yields the desired result, since $\mathrm{Null}(A)$ and $\mathrm{Row}(A)$ are preserved under row operations (*cf. e.g.* Diestel [5, p. 22] or Bollobás [1, p. 53]). □

One might think that every geometric cyclic multi-digraph is the Kirchhoff graph for some matrix; this is not the case, as the following counterexample shows. Consider the

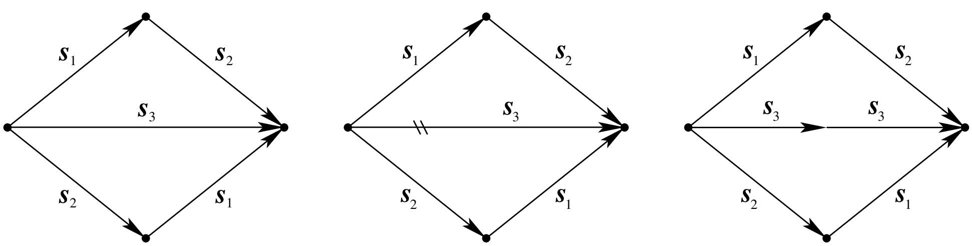

Figure 6: Three similar geometric cyclic multi-digraphs: the leftmost is not the Kirchhoff graph of any matrix; the center and rightmost are Kirchhoff graphs. Note that the Kirchhoff graph in the center is not minimal; it is simply the union of two triangular Kirchhoff graphs, each triangular graph having one copy of each edge vector.

leftmost geometric cyclic multi-digraph in Figure 6. If this graph is a Kirchhoff graph for some matrix A, then $[1, 1, -1]^{\mathrm{T}}$ must be in $\mathrm{Null}(A)$ and $[1, 1, 1]$ must be in $\mathrm{Row}(A)$ which of course is impossible since these vectors are not orthogonal. So the cycle space and the cut space for this graph are not orthogonal. There are two possible Kirchhoff graphs similar to this non-Kirchhoff graph, and they are shown in the center and on the right in Figure 6.

Finally the following proposition is an immediate consequence of the definition of Kirchhoff graph:

Proposition 2.11. *If G is a Kirchhoff graph for a matrix A, and if B is the same as A except that columns j_1 and j_2 are interchanged, then a Kirchhoff graph for B is obtained by interchanging the labeling of edge vectors $\boldsymbol{s}_{j_1}$ and $\boldsymbol{s}_{j_2}$ in G.*

3 Kirchhoff Graphs and Reaction Networks

Let us now consider connections between Kirchhoff graphs and chemical reaction networks. As was mentioned in the Introduction, the study of such reaction networks led to the definition of a Kirchhoff graph. The reaction network discussed below describes the production of sodium hydroxide from a brine (salt) solution through electrolysis. There are a number of processes to accomplish this production (Castner-Kellner, diaphragm, membrane), and a number of variations of these processes [2, 17]. The steps presented here do not all necessarily occur in all processes, but considering all the steps together allows one to compare various processes.

The reaction steps for the network to be studied here are as follows:

$$\begin{array}{rrcl}
\boldsymbol{s}_1: & \mathrm{NaCl} & \rightleftharpoons & \mathrm{Na}^+ + \mathrm{Cl}^- \\
\boldsymbol{s}_2: & 2\mathrm{Cl}^- & \rightleftharpoons & \mathrm{Cl}_2 + 2\mathrm{e}^- \\
\boldsymbol{s}_3: & 2\mathrm{Na}^+ + 2\mathrm{e}^- + 2\mathrm{H}_2\mathrm{O} & \rightleftharpoons & 2\mathrm{NaOH} + \mathrm{H}_2 \\
\boldsymbol{s}_4: & \mathrm{H}^+ + \mathrm{Cl}^- & \rightleftharpoons & \mathrm{HCl} \\
\boldsymbol{s}_5: & \mathrm{Na}^+ + \mathrm{OH}^- & \rightleftharpoons & \mathrm{NaOH} \\
\boldsymbol{s}_6: & 2\mathrm{H}^+ + 2\mathrm{e}^- & \rightleftharpoons & \mathrm{H}_2 \\
\boldsymbol{s}_7: & \mathrm{H}_2\mathrm{O} & \rightleftharpoons & \mathrm{H}^+ + \mathrm{OH}^- \\
\boldsymbol{s}_8: & 2\mathrm{H}_2\mathrm{O} + 2\mathrm{e}^- & \rightleftharpoons & \mathrm{H}_2 + 2\mathrm{OH}^-
\end{array} \tag{3.1}$$

For this network, the charged species are viewed as *intermediate*, while the uncharged species are viewed as *terminal*. The net concentrations of the intermediate species are constant as the reaction steps proceed, while the terminal species are being produced or consumed by the reaction network. Based on these steps (3.1) and this definition of intermediate and terminal species, the achievable overall reactions for this reaction network are determined using basic linear algebra [9]. In this case the two achievable overall reactions are

$$\begin{array}{rrcl}
\boldsymbol{b}_1: & 2\mathrm{NaCl} + 2\mathrm{H}_2\mathrm{O} & \rightleftharpoons & \mathrm{Cl}_2 + \mathrm{H}_2 + 2\mathrm{NaOH} \\
\boldsymbol{b}_2: & \mathrm{NaCl} + \mathrm{H}_2\mathrm{O} & \rightleftharpoons & \mathrm{HCl} + \mathrm{NaOH}
\end{array} \tag{3.2}$$

Combining these two overall reactions (3.2) with the reaction steps (3.1) above, one obtains the entire reaction network.

Now consider the stoichiometric matrix for this network:

$$A^{\mathrm{T}} = \begin{bmatrix} 0 & 1 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 2 & -2 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ -2 & 0 & 0 & -2 & 0 & 0 & -2 & 0 & 1 & 0 & 2 \\ 0 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ -2 & 0 & 0 & 0 & -2 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 \\ -2 & 0 & 2 & 0 & 0 & 0 & -2 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -2 & -2 & 1 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & -1 & -1 & 0 & 0 & 1 & 1 \end{bmatrix} \tag{3.3}$$

In a stoichiometric matrix, the entries are the stoichiometric coefficients for each chemical equation with positive entries for products and negative entries for reactants (species that are consumed in a reaction step). The rows of A^{T} correspond to the eight steps plus the two overall reactions; the columns correspond to the eleven chemical species, namely, in order, e^-, Cl^-, OH^-, Na^+, H^+, NaCl, H_2O, Cl_2, H_2, HCl and NaOH.

To construct a Kirchhoff graph for this reaction network, we must compute both $\mathrm{Null}(A)$ and $\mathrm{Row}(A)$ where A is the transpose of the stoichiometric matrix in (3.3); one finds that $\mathrm{Null}(A) = \mathrm{Span}\{\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3, \boldsymbol{u}_4\}$ where

$$\boldsymbol{u}_1 := \begin{bmatrix} 0 \\ 0 \\ -1 \\ 0 \\ 2 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad \boldsymbol{u}_2 := \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 2 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \quad \boldsymbol{u}_3 := \begin{bmatrix} 2 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -1 \\ 0 \end{bmatrix}, \quad \boldsymbol{u}_4 := \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 1 \\ 0 \\ 1 \\ 0 \\ 0 \\ -1 \end{bmatrix}, \tag{3.4}$$

and that $\mathrm{Row}(A) = \mathrm{Span}\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3, \boldsymbol{v}_4, \boldsymbol{v}_5, \boldsymbol{v}_6\}$ where

$$\begin{array}{lcl} \boldsymbol{v}_1 & := & [\,1 \;\; 0 \;\; -2 \;\; 0 \;\; -1 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0\,], \\ \boldsymbol{v}_2 & := & [\,0 \;\; 0 \;\; 0 \;\; 0 \;\; 1 \;\; 0 \;\; -1 \;\; -2 \;\; 0 \;\; 0\,], \\ \boldsymbol{v}_3 & := & [\,0 \;\; 0 \;\; 0 \;\; 1 \;\; 0 \;\; 2 \;\; -1 \;\; 0 \;\; 0 \;\; 0\,], \\ \boldsymbol{v}_4 & := & [\,1 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 2 \;\; 1\,], \\ \boldsymbol{v}_5 & := & [\,0 \;\; 1 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 1 \;\; 0\,], \\ \boldsymbol{v}_6 & := & [\,0 \;\; 0 \;\; 0 \;\; 1 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 1\,]. \end{array} \tag{3.5}$$

Null space vectors $\boldsymbol{u}_3$ and $\boldsymbol{u}_4$ give the combinations of reaction steps that yield the two overall reactions; vectors $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$ give two linearly independent null cycles among these steps—combinations of steps that exactly cancel. The six vectors in (3.5) give six linearly independent reaction-rate balance conditions which guarantee the rates for all ten steps in the reaction network balance so that all of the species concentrations change only through the overall reactions (see the specific example for r_{NaCl} below). By inspection, the columns and rows in (3.4) and (3.5) form a basis for $\mathrm{Null}(A)$ and $\mathrm{Row}(A)$, respectively.

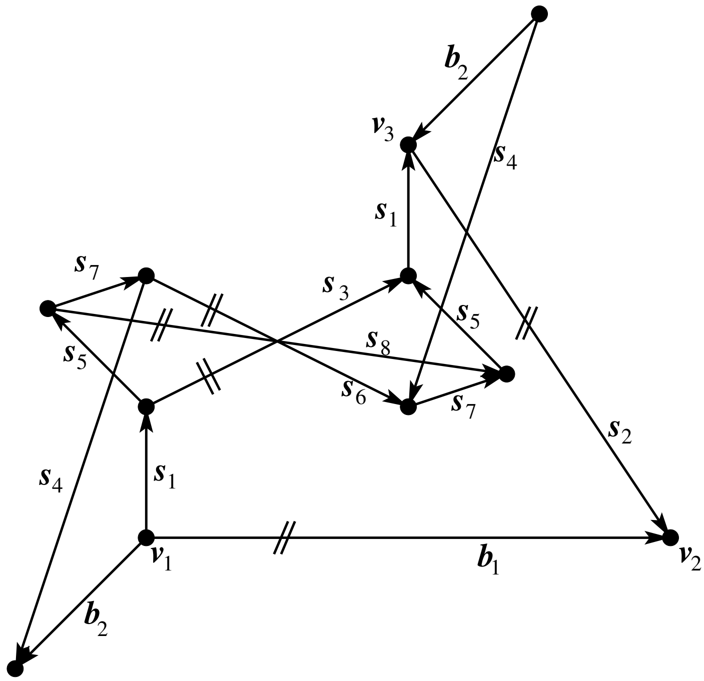

Figure 7: Kirchhoff graph for NaCl-NaOH network. Hash marks again indicate the multiplicity of edge vectors between vertices.

One Kirchhoff graph for this network is shown in Figure 7. This Kirchhoff graph is in fact the two dimensional projection of vectors which actually lie in an eleven dimensional phase space for the eleven chemical species in this reaction network. All of the nodes of Figure 7 lie in Row(A), while the cycles of Figure 7 correspond to vectors that span Null(A); these are of course the two defining properties that make the graph in Figure 7 a Kirchhoff graph for this reaction network, (3.1) and (3.2). Indeed one can confirm that the graph in Figure 7 is Kirchhoff by observing that the cut or incidence vectors of edges for each vertex are linear combinations of the vectors in (3.5), while the cycles in this graph are linear combinations of the vectors in (3.4).

To see how the Kirchhoff graph in Figure 7 satisfies the Kirchhoff laws and is thus a circuit diagram for the NaCl-NaOH reaction network, the concepts of electrochemical potential and component potential must be introduced. An electrochemical potential μ_{X} can be defined for each species X in a reaction network, and as potentials, they can be combined linearly. The linear combinations of electrochemical potentials that are based on the stoichiometry of each reaction step $\boldsymbol{s}_j$ and each overall reaction $\boldsymbol{b}_k$ are the component potentials for the reaction network. Up to an arbitrary reference potential, these component potentials are the potentials for the vertices of the Kirchhoff graph. So for example, if the potential of vertex $\boldsymbol{\nu}_2$ in Figure 7 is set to a reference potential ($\mu_{\boldsymbol{\nu}_2} = \mu_{\mathrm{ref}}$), then the electrochemical potential for the vertex $\boldsymbol{\nu}_3$ at the opposite end of $\boldsymbol{s}_2$ is given by the stoichiometry of the second reaction step:

$$\mu_{\boldsymbol{\nu}_3} = \mu_{\mathrm{ref}} + 2\mu_{\mathrm{Cl}^-} - \mu_{\mathrm{Cl}_2} - 2\mu_{\mathrm{e}^-}$$

The signs in the linear combination are determined by the direction of the reaction step. The difference in potential between two vertices connected by a reaction step vector is the *affinity* of that reaction step and is a property of that reaction step. Similarly a component potential can be determined for each vertex, and because of the stoichiometry, the net change in potential around any cycle in this Kirchhoff graph must be zero. This is of course the Kirchhoff potential law. For a more thorough discussion of electrochemical potentials and component potentials in the context of reaction networks, *cf. e.g.* Newman(2004) [18] and Fehribach [7, 6].

To see that the Kirchhoff graph also satisfies the Kirchhoff current law, consider the reaction steps that are incident on vertex $\boldsymbol{\nu}_1$. Because the total amounts of each species must be conserved when the overall reactions are taken into account, the reaction rates for each reaction step ($r_{\boldsymbol{s}_j}$ and $r_{\boldsymbol{b}_k}$) are not independent, but rather must satisfy a set of rate-balance conditions: the rate of production/consumption of species X must be zero ($r_{\mathrm{X}} = 0$). So for example, the balance for $\boldsymbol{\nu}_1$ represents the rate (or current) balance for the consumption of NaCl. Since NaCl occurs in reactions $\boldsymbol{s}_1$, $\boldsymbol{b}_1$ and $\boldsymbol{b}_2$, the stoichiometry again yields the needed rate balance condition:

$$r_{\mathrm{NaCl}} = r_{\boldsymbol{s}_1} + 2r_{\boldsymbol{b}_1} + r_{\boldsymbol{b}_2} = 0$$

This is the Kirchhoff current law at vertex $\boldsymbol{\nu}_1$ and corresponds precisely to $\boldsymbol{v}_4$ in (3.5). Similar rate balance conditions for other species or linear combinations of species hold for each vertex in the Kirchhoff graph. Thus both Kirchhoff laws are satisfied by this Kirchhoff graph, and the result is a fundamental connection between the fundamental theorem of linear algebra for the transpose of the stoichiometric matrix for a reaction network and the conservation principles of the Kirchhoff laws. Since the Kirchhoff laws are satisfied, a chemical engineer can use a Kirchhoff graph as a circuit diagram for a chemical reaction network in the same way an electrical engineer can use a traditional circuit diagram for an electrical network.

4 Existence and Construction of Kirchhoff Graphs

As the NaCl-NaOH network makes clear, it is relatively easy to verify that a given geometric cyclic multi-digraph is a Kirchhoff graph for a given reaction network or a given matrix. One needs only to check that the two conditions in the definition for a Kirchhoff graph are satisfied. The two real issues are those of existence and construction of Kirchhoff graph(s) for a given reaction network or matrix. The question of existence is open, but the author offers the following conjecture:

Conjecture 4.1. *Every matrix over $\mathbb{Q}$ has a Kirchhoff graph.*

Proving this conjecture seems to be a difficult and interesting issue. While no proof is given here, a process for constructing Kirchhoff graphs offers one possible approach for a constructive proof—showing that the process always converges to a Kirchhoff graph would establish existence. The following two subsections demonstrate this process for two of the matrices above; the first is relatively simple, the second, somewhat more complicated. The final subsection below gives a general summary of the process.

4.1 Construction of First Kirchhoff Graph in Figure 4

One method to systematically construct a Kirchhoff graph for A in (2.1) is to "weave" the bases vectors of the null space through the row space to form the incidence matrix for a graph. To begin, consider two rows of A and the first basis vector for Null(A) in (2.2). Since the first elements in these rows are 1 and -1, these rows can correspond to the first two vertices in the graph with edge $\boldsymbol{s}_1$ connecting them. Now since the second element of the first row is a zero, one should consider the negative of the second row as a new third row of the developing incidence matrix; this new row corresponds to a new third vertex, with $\boldsymbol{s}_2$ now connecting the second and third vertices. But there would seem to be a problem in that the zero in the third position of the third row does not seem to allow an edge to connect the first and third vertices or to complete the first cycle. This problem can be overcome, however, by adding the negative of the first row to the original choice for the third row; this allows the 1 in the third position of the first row and the -1 in the third position of the new third row to correspond to $-\boldsymbol{s}_3$, completing the cycle corresponding to the first basis vector for Null(A). The partial three-vertex incidence matrix constructed so far is

$$\begin{bmatrix} \mathbf{1} & 0 & \mathbf{1} & 1 \\ -\mathbf{1} & \mathbf{1} & 0 & 1 \\ 0 & -\mathbf{1} & -\mathbf{1} & -2 \end{bmatrix}. \tag{4.1}$$

Here bold face numbers correspond to the vertices of this first cycle; the non-bold entries in the fourth column correspond to "loose ends" that must be "tied up" as the incidence matrix is developed. Only when all of the loose ends are dealt with in some way can one hope to have the incidence matrix for a complete Kirchhoff graph.

Now to tie up at least some of the loose ends and incorporate the next basis vector from Null(A), additional rows from Row(A) corresponding to new vertices must be appended to the developing incidence matrix. Since this next null space basis vector corresponds to the cycle $\boldsymbol{s}_2 + \boldsymbol{s}_3 - \boldsymbol{s}_4$, it would seem to make sense to start with the $\boldsymbol{s}_2$ edge represented in the second column of the partial incidence matrix (4.1). Moving across this edge in the forward direction, one needs a forward copy of $\boldsymbol{s}_3$; this can be created by adding the negative of the third row in (4.1) to that third row (thus the vertex corresponding to the new third row becomes a null vertex), then appending as a fourth row the negative of the first row. The resulting (still) partial incidence matrix is now

$$\begin{bmatrix} \mathit{1} & 0 & \mathit{1} & 1 \\ -\mathit{1} & \mathbf{1} & 0 & \mathbf{1} \\ 0 & 1-\mathbf{1} & \mathbf{1}-\mathit{1} & 0 \\ -1 & 0 & -\mathbf{1} & -\mathbf{1} \end{bmatrix}. \tag{4.2}$$

The cycle corresponding to the second basis vector from Null(A) is now shown in bold in (4.2), while the entries corresponding to the remaining two edges of the first basis vector are now in italics, and the remaining loose ends are in standard typeface.

The partial incidence matrix in (4.2) still has three loose ends and thus is still incomplete. But these loose ends themselves now lie in Row(A) and thus can be tied up by appending one more row which is the negative of the current second row. To make the final incidence matrix symmetric, this final new row should be the fourth, and the previous fourth row should become the fifth. The now complete incidence matrix is

$$\begin{bmatrix} \mathit{1} & 0 & \mathit{1} & 1 \\ -\mathit{1} & \mathbf{1} & 0 & \mathbf{1} \\ 0 & 1-\mathbf{1} & \mathbf{1}-\mathit{1} & 0 \\ 1 & -1 & 0 & -1 \\ -1 & 0 & -\mathbf{1} & -\mathbf{1} \end{bmatrix}. \tag{4.3}$$

Again the typeface indicates which column-element pairs represent vertices which are connected by an edge. All of the vertices satisfy the second Kirchhoff graph condition because all of the rows of (4.3) are in Row(A), and it is easy to check that all of the cycles represented in (4.3) are in Null(A). Keeping in mind that an edge vector that appears multiple times in the graph must always have a fixed length and direction, one can use (4.3) to draw the left Kirchhoff graph in Figure 4, or an equivalent version of this graph.

Of course there were many free choices in the above construction where one could have added new vertices and/or edges in alternate ways. One set of alternate choices would lead to the other graph in Figure 4. Still other choices could lead to a diverging construction of a larger and larger graph which never satisfies both of the Kirchhoff conditions at all of its vertices and for all of its cycles. A full algorithm for a construction would need a measure of how close/far the process is from converging to determine whether to proceed with an ongoing construction, or to go back to some earlier point in the construction and make a different choice for rows to add or cycles to weave.

4.2 Construction of Kirchhoff Graph in Figure 7

Now we turn our attention to the construction of a Kirchhoff graph for a somewhat larger matrix—A defined by (3.3). To begin, consider vectors $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$ from Null(A) in (3.4) and vectors $\boldsymbol{v}_1$, $\boldsymbol{v}_2$ and $\boldsymbol{v}_3$ from Row(A) in (3.5). These cycles/cuts do not involve either of the overall reactions $\boldsymbol{b}_1$ or $\boldsymbol{b}_2$ and therefore constitute an important subgraph to the Kirchhoff graph we seek. Because the third element of $\boldsymbol{u}_1$ is -1, let us start with row vector $\boldsymbol{v}_1$, and take this as the first row of a partial incidence matrix. Since this is in fact the only row space vector with a nonzero third entry, the only choice for the second row of the incidence matrix is the negative of the first row, and the -2 and 2 in the third column of these first two rows of a partial incidence matrix represent $\boldsymbol{s}_3$ from $\boldsymbol{u}_1$. The next row in the incidence must be connected to the second row by either $\boldsymbol{s}_5$ or $\boldsymbol{s}_8$ (the other two edge vectors in $\boldsymbol{u}_1$). Since the only nonzero entries in the second incidence matrix row are in the first and fifth columns, one should hope to use $\boldsymbol{s}_5$ to connect the second and third rows of the incidence matrix. This third row cannot be a multiple of the first or second since then $\boldsymbol{s}_3$ and $\boldsymbol{s}_5$ would need to be multiples of each other. So the only possible choice among the given row space vectors that $\boldsymbol{s}_5$ can connect to is the negative of row vector $\boldsymbol{v}_2$ in (3.5). To complete this first cycle and represent the rest of the entries $\boldsymbol{v}_1$, the fourth row on the incidence matrix can be the negative of the third. Two copies of $\boldsymbol{s}_8$ then connect the 2 and -2 in the eight columns of the third and fourth rows of the incidence matrix, and finally an $\boldsymbol{s}_5$ then connects the 1 and -1 in the fifth column of the fourth and first rows, thus completing the cycle and weaving $\boldsymbol{v}_1$, through several row vectors from (3.5). This weave is shown in (4.4) where the bold face numbers correspond to vertices that are in the first cycle. In (4.4), note that each row corresponds to a vertex, and each column corresponds to an edge vector. Since there are two $\boldsymbol{s}_5$ edge vectors in this cycle, the entries corresponding to each vector are realigned to the left or right to indicate which vertices are connected:

$$\begin{bmatrix} 1 & 0 & \mathbf{-2} & 0 & \mathbf{-1} & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & \mathbf{2} & 0 & \mathbf{1} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \mathbf{-1} & 0 & 1 & \mathbf{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \mathbf{1} & 0 & -1 & \mathbf{-2} & 0 & 0 \end{bmatrix}. \tag{4.4}$$

Again, the convention is that edge vectors go from positive entries to negative ones, and these entries must always have the same absolute value. Also the non-bold elements of (4.4) represent edge vectors that are currently open (not connected) and must be connected in the eventual full incidence matrix.

Now we must try to incorporate the second cycle (*i.e.*, the second vector from (3.4)) into the partial incidence matrix. Although there is no guarantee, one can hope that the s_8 edge vector from the first cycle is common to this second cycle and therefore start with it. Because the (3,7) element of (4.4) is 1, it makes sense to try to have s_7 leave ν_3. This edge cannot go to ν_4 since it would then connect the same vertices as s_8. Since one also needs s_6 in this cycle, it makes sense to add v_3 to the partial incidence matrix, and thereby to add a new vertex to the graph. Adding $-v_3$ as the sixth row of the partial incidence matrix, one connects these rows and their vertices using edge vectors s_6 and s_7 to complete the second cycle. The rows of the partial incidence matrix for the first six vertices are now given in (4.5); the first cycle is now in italics, while the new cycle is in bold:

$$\begin{bmatrix} 1 & 0 & -\mathit{2} & 0 & -\mathit{1} & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & \mathit{2} & 0 & \mathit{1} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\mathit{1} & 0 & \mathbf{1} & \mathbf{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \mathit{1} & 0 & \mathbf{-1} & \mathbf{-2} & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & \mathbf{2} & \mathbf{-1} & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & \mathbf{-2} & \mathbf{1} & 0 & 0 & 0 \end{bmatrix}. \tag{4.5}$$

Next one must include u_3 and u_4, the latter two null space vectors from (3.4), and do this in such a way so as to tie up the non-boldface, non-italicized entries in (4.5). Considering u_3, one must have two copies of the s_1 edge vector and one s_3 in this cycle. The first two vertices (the first two rows of (4.5)) are already connected by two copies of s_3 and have loose ends for s_1 in the proper direction, so this would seem a natural place to tie in the third cycle. To complete this cycle, one must add a vertex corresponding to v_4 and add an s_1 edge vector to connect this seventh row of the partial incidence matrix to its second row. Continuing this cycle, one must add an eighth row to the partial incidence matrix which is two times v_5, and two copies of b_1 to connect the vertices that correspond to these seventh and eighth rows. Finally two copies of s_2 are needed to complete the cycle, but upon checking each of the rows in (3.5), one finds that none have the correct requirements for a final vertex. This means that linear combinations of the rows in (3.5) must be considered. Taking into account all of the requirements for this final vertex, one finds that the correct linear combination is $v_4 - 2v_5$. This linear combination then becomes the ninth row of the partial incidence matrix, and the corresponding vertex is connected to ν_1 by one copy of s_1, and to ν_8 by two copies of s_2, all with the proper orientation to complete the third cycle. The partial incidence matrix for the first nine vertices is thus

$$\begin{bmatrix}
\mathbf{1} & 0 & \mathbf{-2} & 0 & \mathit{-1} & 0 & 0 & 0 & 0 & 0 \\
\mathbf{-1} & 0 & \mathbf{2} & 0 & \mathit{1} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \mathit{-1} & 0 & \mathit{1} & \mathit{2} & 0 & 0 \\
0 & 0 & 0 & 0 & \mathit{1} & 0 & \mathit{-1} & \mathit{-2} & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & \mathit{2} & \mathit{-1} & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & \mathit{-2} & \mathit{1} & 0 & 0 & 0 \\
\mathbf{1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{2} & 1 \\
0 & \mathbf{-2} & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{-2} & 0 \\
\mathbf{-1} & \mathbf{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1
\end{bmatrix}. \tag{4.6}$$

Again the new cycle in (4.6) is in bold, while the two previous cycles are now in italics.

In this case, the final cycle corresponding to $\boldsymbol{u}_4$ is easy to include, but it must occur twice. The partial incidence matrix (4.6) has only four loose connections (nonzero entries that are neither in italics or bold), and happily these can be connected pairwise by copies of $\boldsymbol{s}_4$ and $\boldsymbol{b}_2$ through the addition of two new vertices, one corresponding to $\boldsymbol{v}_6$, and the other corresponding to its negative. The full incidence matrix is then

$$\begin{bmatrix}
1 & 0 & -2 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 2 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & 1 & 2 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & -1 & -2 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 2 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & -2 & 1 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 1 \\
0 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & 0 \\
-1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\
0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & -1 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}. \tag{4.7}$$

For simplicity, no entries in (4.7) are in bold or italics, but entries corresponding to different copies of the same edge are still shifted to the right or left side of their column.

Remark 4.2. Again when the edge vectors that satisfy the definition of Kirchhoff graph are actually drawn, (Definition 2.1), their exact length and direction are not important. One needs only to ensure that a given edge vector $\boldsymbol{s}_i$ has the same length and orientation each time it appears in the graph, and to avoid any degeneracies (accidentally placing two distinct vertices in the same position). Because a Kirchhoff graph is actually a two dimensional projection of a much higher dimensional geometric structure, the incidence matrix is sufficient to produce some projection of this structure. For example, if the Kirchhoff graph comes from a reaction network with N species, then the vector structure lies in the stoichiometric space $\mathbb{Q}^N$.

Dividing the process into two parts as was done above, first constructing the part of the graph that does not involve the overall reactions (or some other group of reaction steps), then adding on these remaining steps (edges) as the row and null space vectors require, is often a very effective way of constructing a full Kirchhoff graph. Of course the above construction is not guaranteed to produce a Kirchhoff graph, and in particular there are points in the process where one might choose the wrong row and not be able to neatly complete a given cycle. One would then have to go back and consider different rows or linear combinations of rows from (3.5) to form possible rows in a partial incidence matrix. Fortunately this laborious process should be computerizable, so that a computer can sort through the possible weaves to produce a Kirchhoff graph, assuming that one actually exists.

4.3 Summary of Kirchhoff Graph Construction

While the construction process used above contains too many free choices to be called an algorithm, it does have a certain structure which it is useful to summarize. Given a matrix A over $\mathbb{Q}$, the process can be summarized in the following general steps:

Step 1: Construct bases with integral elements for both Null(A) and Row(A). Even for moderately large matrices, this construction can be accomplished effectively with any of a number of software packages/applications.

Step 2: Refine the bases found in Step 1 to find ones that satisfy minimal total absolute sum norms: If $\{\boldsymbol{x}_i\}$ is an integral basis for one of these two spaces, with $\boldsymbol{x}_i = (x_{ij})$ and $x_{ij} \in \mathbb{Z}$, refine this basis until $\sum_{ij} |x_{ij}|$ minimal.

Step 3: Starting with the minimal basis for Row(A), begin to construct a first attempt at an incidence matrix for the first vector in Null(A). The construction may require linear combinations of the row vectors. This first attempt should have column pairs which represent each edge in the cycle corresponding to this first vector in Null(A), but the order of the edges may or may not match what is eventually needed, and there will in general be "loose ends," entries in the matrix which represent one end of an edge for which there is not yet an entry for the other end. As in the earlier examples, it may also be necessary to introduce one or more null vertices (a rows of zeros) to form an incidence matrix consistent with this first null space vector.

Step 4: Next, attempt to include the next vector from Null(A) in the growing incidence matrix, using as many of the loose ends as possible, but also adding new rows from Row(A) as needed. Note that each additional row in the incidence matrix corresponds to an additional vertex in the developing Kirchhoff graph.

Step 5: If *all* vectors from Null(A) have been included and *all* loose ends have been elliminated, then the construction of the incidence matrix and the Kirchhoff graph are complete. If there remain vectors from Null(A) to be included, but the number of loose ends is in some sense declining, return to Step 4 to add another null space vector. If the number of loose ends is in some increasing, then return to Step 3 and/or Step 4 to change the order in which the edges appear in one or more cycles, or change the row vectors that determine which edges are incident to each vertex.

5 Conclusion

A Kirchhoff graph for a rational matrix is a graph-theoretic representation of the fundamental theorem of linear algebra for that matrix. When the matrix is based on a reaction network, a Kirchhoff graph is a circuit diagram for that reaction network, and it represents the connection between the Kirchhoff laws and the fundamental theorem of linear algebra.

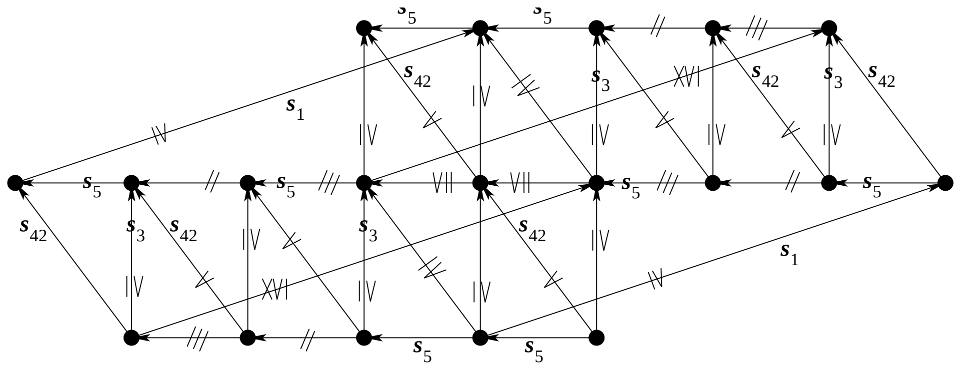

Figure 8: An example of a relatively simple matrix with a fairly intricate Kirchhoff graph. Edge vector s_{42} is the vector sum of s_2 and s_4 which always occur in sequence. The multiplicity of an edge vector between given vertices is given by hash marks which should be read as Roman numerals. To reduce crowding, not all edge vectors are labeled.

Certainly the most interesting open question unresolved in this work is the actual existence of a Kirchhoff graph for any given matrix. While it is not obvious that the conjecture is true, it is clear that there is no guarantee that a Kirchhoff graph for a relatively simple matrix will itself be in any sense simple. Consider the matrix

$$A = \begin{bmatrix} 1 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ -4 & 0 & 0 & 1 & 1 \end{bmatrix}. \tag{5.1}$$

Then

$$\text{Null}(A) = \text{Span}\left\{[\,0,\ 1,\ 1, -1,\ 1]^{\mathrm{T}}, [\,1,\ 0, -1,\ 0,\ 4]^{\mathrm{T}}\right\} \tag{5.2}$$

and $\mathrm{Row}(A)$ is the span of the three rows of A. Therefore a Kirchhoff graph for this matrix must have two linearly independent cycles, and all of its vertices must lie in the row space, but this really does not indicate how intricate a Kirchhoff graph for A is. The author believes that the Kirchhoff graph given in Figure 8 is the simplest one corresponding to A in (5.1). It is difficult to image that anyone could guess how complicated a Kirchhoff graph for A needs to be just looking at A, $\mathrm{Null}(A)$ and $\mathrm{Row}(A)$. Other matrices that would appear similar to A have much simpler Kirchhoff graphs, but there are also other relatively simple matrices (or relatively simple reaction networks) where to the author's knowledge no Kirchhoff graph has yet been constructed and any possible Kirchhoff graph must be large and intricate. For example, consider the matrix

$$A = \begin{bmatrix} 1003 & 1829 & -17 & 0 \\ 0 & 59 & 1411 & 1003 \end{bmatrix}. \tag{5.3}$$

The rows of A in (5.3) clearly form a basis for $\mathrm{Row}(A)$, and no linear combination of these two rows will lead to a simpler set of vertex conditions for any possible Kirchhoff graph.

One also finds that

$$\mathrm{Null}(A) = \mathrm{Span}\left\{ \begin{bmatrix} 31 \\ -17 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 59 \\ -83 \end{bmatrix} \right\}, \tag{5.4}$$

so any possible Kirchhoff graph for this matrix must be a complicated weave of large cycles based on these null space vectors. Again considering linear combinations of these null space vectors would not simplify the graph. The rows of (5.3) and the vectors of (5.4) satisfy the minimal total absolute sum norm condition discussed in Step 2 above. If the matrix in (5.3) is the stoichiometric matrix of some reaction network, even if one succeeds in constructing a Kirchhoff graph, it would seem to be of little use since it would be far too complicated. But since stoichiometric coefficients are typically 0, ± 1 or occasionally ± 2, elements like those in (5.3) are not realistic as stoichiometric coefficients.

From an applications point of view, the key point is that when the Kirchhoff graph for a reaction network is relatively simple, it can be used to study the reaction network in the same way that a circuit diagram is used to study an electrical network. One can construct equivalent circuits, study how temperature, pH or other changes which reaction pathway is dominant, or compute the rate of an overall reaction from the rates of the individual steps, to name a few.

In terms of future work, the process for constructing Kirchhoff graphs needs to be computerized. This will make it possible to find Kirchhoff graphs for matrices and reaction networks where construction by hand is too time consuming to be practical.

Finally it seems worth noting that there is a certain aesthetic beauty in the intricacy of the more-complicated Kirchhoff graphs such as the one in Figure 8.

6 Acknowledgments

The author wishes to thank Manasi Vartek for originally bringing to his attention the NaCl-NaOH reaction network, and also wishes to thank Ravi Datta and Ilie Fishtik for introducing him to the concept of reaction route graphs, and Pete Christopher, Willy Hereman, Bill Martin, Brigitte Servatius, Gil Strang and Lubos Thoma for many illuminating discussions on graph theory and linear algebra. Finally, the author wishes to thank the referees for numerous helpful suggestions and corrections.

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