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Salami-Slicing Research

Unfortunately, in many places researchers are evaluated by the quantity rather than the quality of the research they produce. Sometimes people up for promotion desperately look for ways to increase the number of the publications in a given period. One such way is a simple practice: instead of publishing a substantial piece of work, of say 20 to 30 pages, authors slice the work into smaller pieces (like salami) and then submit a series of much less substantial papers of 5 to 10 pages each. When assessment systems take only the number of published or accepted papers into account, this allows (and encourages) authors to double or triple their output without producing anything new.

Needless to say, for *Ars Mathematica Contemporanea* we consider salami-slicing research unethical, and we strongly discourage authors to submit only small pieces of their research. We much prefer to publish pieces of work that are substantial and self-contained, whenever possible.

In the same vein, we would like to draw to the attention of readers a statement made in October 2017 by three national academies in Europe about good practice in the evaluation of researchers and research programmes. The Académie des Sciences (in France), the Nationale Akademie der Wissenschaften (‘Leopoldina’, in Germany) and the Royal Society of London (in the United Kingdom) have published some principles of good practice, and some warnings about flawed indicators such as the *h*-index and the use of such indicators as targets for performance. The statement finishes with the following summary, which we feel is important enough to quote:

“Evaluation requires peer review by acknowledged experts working to the highest ethical standards and focusing on intellectual merits and scientific achievements. Bibliometric data cannot be used as a proxy for expert assessment. Well-founded judgment is essential. Over-emphasis on such metrics may seriously damage scientific creativity and originality. Expert peer review should be treated as a valuable resource.”

Dragan Marušič, Tomaž Pisanski
Editors In Chief

and Marston Conder
Editorial Advisor

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Octahedral, dicyclic and special linear solutions of some Hamilton-Waterloo problems

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Abstract

We give a sharply-vertex-transitive solution of each of the nine Hamilton-Waterloo problems left open by Danziger, Quattrocchi and Stevens.

Keywords: Hamilton-Waterloo problem, group action, octahedral binary group, dicyclic group, special linear group.

Math. Subj. Class.: 05C70, 05E18, 05B10

1 Introduction

A cycle decomposition of a simple graph $\Gamma = (V, E)$ is a set $\mathcal{D}$ of cycles whose edges partition E. A partition $\mathcal{F}$ of $\mathcal{D}$ into classes (2-*factors*) each of which covers all V exactly once is said to be a 2-*factorization* of Γ. The *type* of a 2-factor F is the partition $\pi = [\ell_1^{n_1}, \ldots, \ell_t^{n_t}]$ (written in exponential notation) of the integer $|V|$ into the lengths of the cycles of F.

A 2-factorization $\mathcal{F}$ of K_v (the complete graph of order v) or $K_v - I$ (the cocktail party graph of order v) whose 2-factors are all of the same type π is a solution of the so-called Oberwolfach Problem OP$(v; \pi)$. If instead the 2-factors of $\mathcal{F}$ are of two different types π and ψ, then $\mathcal{F}$ is a solution of the so-called Hamilton-Waterloo Problem HWP$(v; \pi, \psi; r, s)$ where r and s denote the number of 2-factors of $\mathcal{F}$ of type π and ψ, respectively.

A complete solution of the OPs whose 2-factors are uniform, namely of the form OP$(\ell n; [\ell^n])$, has been given in [1] and [12]. Other important classes of OPs has been

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solved in [4, 15]. For the time being, to look for a solution to all possible OPs and, above all, HWPs is too ambitious. Anyway it is reasonable to believe that we are not so far from a complete solution of the HWPs whose 2-factors are uniform, namely of the form HWP$(v; [h^{v/h}], [w^{v/w}]; r, s)$. We can say this especially because of the big progress recently done in [10].

Danziger, Quattrocchi and Stevens [11] treated the HWPs whose 2-factors are either triangle-factors or quadrangle-factors, they namely studied HWP$(12n; [3^{4n}], [4^{3n}]; r, s)$. In the following such an HWP will be denoted, more simply, by HWP$(12n; 3, 4; r, s)$. They solved this problem for all possible triples (n, r, s) except the following ones:

(i) $(4, r, 23 - r)$ with $r \in \{5, 7, 9, 13, 15, 17\}$;

(ii) $(2, r, 11 - r)$ with $r \in \{5, 7, 9\}$.

Six of the nine above problems have been recently solved in [14] where it was pointed out that all nine problems were also solved in a work still in preparation [2] by the authors of the present paper. Meanwhile, a solution for each of the remaining three problems not considered in [14] have been given in [16]. Notwithstanding, in the present paper we want to present our solutions to the nine HWPs left open by Danziger, Quattrocchi and Stevens in detail. These solutions, differently from those of [14, 16], are full of symmetries since they are *G-regular* for a suitable group G. We recall that a cycle decomposition (or 2-factorization) of a graph Γ is said to be G-regular when it admits G as an automorphism group acting sharply transitively on all vertices. Here is explicitly our main result:

Theorem 1.1. *There exists a $\overline{O}$-regular 2-factorization of $K_{48} - I$ having r triangle-factors and $23 - r$ quadrangle-factors where $\overline{O}$ is the binary octahedral group and $r \in \{5, 7, 9, 13, 15, 17\}$.*

There exists a Q_{24}-regular 2-factorization of $K_{24} - I$ having r triangle-factors and $11 - r$ quadrangle-factors where Q_{24} is the dicyclic group of order 24 and $r \in \{7, 9\}$.

There exists a $SL_2(3)$-regular 2-factorization of $K_{24} - I$ having six triangle-factors and five quadrangle-factors where $SL_2(3)$ is the 2-dimensional special linear group over $\mathbb{Z}_3$.

2 Some preliminaries

The use of the *classic* method of differences allowed to get cyclic (namely Z_v-regular) solutions of some HWPs in [8, 9, 13]. Now we summarize, in the shortest possible way, the method of *partial differences*. This method, explained in [7] and successfully applied in many papers (see, especially, [6]), has been also useful for the investigation of G-regular 2-factorizations of a complete graph of odd order [9]. The G-regular 2-factorizations of a cocktail party graph can be treated similarly.

Throughout this paper any group G will be assumed to be written multiplicatively and its identity element will be denoted by 1. Let Ω be a *symmetric* subset of a group G; this means that $1 \notin \Omega$ and that $\omega \in \Omega$ if and only if $\omega^{-1} \in \Omega$. The *Cayley graph* on G with *connection-set* Ω, denoted by $\mathrm{Cay}[G : \Omega]$, is the simple graph whose vertices are the elements of G and whose edges are all 2-subsets of G of the form $\{g, \omega g\}$ with $(g, \omega) \in G \times \Omega$.

Remark 2.1. If λ is an involution of a group G, then $\mathrm{Cay}[G : G \setminus \{1, \lambda\}]$ is isomorphic to $K_{|G|} - I$. So, in the following, such a Cayley graph will be always identified with the cocktail party graph of order $|G|$.

Let $Cycle(G)$ be the set of all cycles with vertices in G and consider the natural right action of G on $Cycle(G)$ defined by $(c_1, c_2, \ldots, c_n)^g = (c_1g, c_2g, \ldots, c_ng)$ for every $C = (c_1, c_2, \ldots, c_n) \in Cycle(G)$ and every $g \in G$. The stabilizer and the orbit of any $C \in Cycle(G)$ under this action will be denoted by $Stab(C)$ and $Orb(C)$, respectively. The *list of differences* of $C \in Cycle(G)$ is the multiset ΔC of all possible quotients xy^{-1} with (x, y) an ordered pair of adjacent vertices of C. One can see that the multiplicity $m_{\Delta C}(g)$ of any element $g \in G$ in ΔC is a multiple of the order of $Stab(C)$. Thus it makes sense to speak of the *list of partial differences* of C as the multiset ∂C on G in which the multiplicity of any $g \in G$ is defined by

$$m_{\partial C}(g) := \frac{m_{\Delta C}(g)}{|Stab(C)|}.$$

We underline the fact that ∂C is, in general, a multiset. Note that if ∂C is a set, namely without repeated elements, then it is symmetric so that it makes sense to speak of the Cayley graph $\mathrm{Cay}[G : \partial C]$. The following elementary but crucial result holds.

Lemma 2.2. *If $C \in Cycle(G)$ and ∂C does not have repeated elements, then $Orb(C)$ is a G-regular cycle-decomposition of* $\mathrm{Cay}[G : \partial C]$.

By Remark 2.1, as an immediate consequence of the above lemma we can state the following result.

Theorem 2.3. *Let λ be an involution of a group G. If $\{C_1, \ldots, C_t\}$ is a subset of $Cycle(G)$ such that $\bigcup_{i=1}^{t} \partial C_i = G \setminus \{1, \lambda\}$, then $\bigcup_{i=1}^{t} Orb(C_i)$ is a G-regular cycle-decomposition of $K_{|G|} - I$.*

We need, as last ingredient, the following easy remarks.

Remark 2.4. If $C \in Cycle(G)$ and $V(C)$ is a subgroup of G, then $Orb(C)$ is a 2-factor of the complete graph on G whose stabilizer is the whole G.

If $C_1, \ldots, C_t$ are cycles of $Cycle(G)$ and $\bigcup_{i=1}^{t} V(C_i)$ is a complete system of representatives for the left cosets of a subgroup S of G, then $\bigcup_{i=1}^{t} Orb_S(C_i)$ is a 2-factor of the complete graph on G whose stabilizer is S.

3 Octahedral solutions of six Hamilton-Waterloo problems

Throughout this section G will denote the so-called *binary octahedral group* which is usually denoted by $\overline{O}$. This group, up to isomorphism, can be viewed as a group of units of the skew-field $\mathbb{H}$ of *quaternions* introduced by Hamilton, that is an extension of the complex field $\mathbb{C}$. We recall the basic facts regarding $\mathbb{H}$. Its elements are all real linear combinations of 1, i, j and k. The sum and the product of two quaternions are defined in the natural way under the rules that

$$i^2 = j^2 = k^2 = ijk = -1.$$

If $q = a + bi + cj + dk \neq 0$, then the inverse of q is given by

$$q^{-1} = \frac{a - bi - cj - dk}{a^2 + b^2 + c^2 + d^2}.$$

The 48 elements of the multiplicative group G are the following:

$$\begin{aligned}
&\pm 1, \pm i, \pm j, \pm k;\\
&\tfrac{1}{2}(\pm 1 \pm i \pm j \pm k);\\
&\tfrac{1}{\sqrt{2}}(\pm x \pm y), \quad \{x, y\} \in \tbinom{\{1,i,j,k\}}{2}.
\end{aligned}$$

The use of the octahedral group G was crucial in [3] to get a Steiner triple system of any order $v = 96n + 49$ with an automorphism group acting sharply transitively an all but one point. Here G will be used to get a G-regular solution of each of the six Hamilton-Waterloo problems of order 48 left open in [11]. We will need to consider the following subgroups of G of order 16 and 12, respectively:

- $K = \langle k,\ \frac{1}{\sqrt{2}}(j - k)\rangle$;
- $L = \langle \frac{1}{\sqrt{2}}(j - k),\ \frac{1}{2}(-1 - i + j + k)\rangle$.

3.1 An octahedral solution of HWP$(48; 3, 4; 5, 18)$

Consider the nine cycles of $Cycle(G)$ defined as follows.

$$\begin{aligned}
C_1 &= \big(1,\ -\tfrac{1}{\sqrt{2}}(1-k),\ \tfrac{1}{2}(1-i-j-k)\big)\\
C_2 &= \big(1,\ \tfrac{1}{2}(-1-i+j+k),\ \tfrac{1}{2}(-1+i-j-k)\big)\\
C_3 &= \big(1,\ \tfrac{1}{2}(-1+i+j-k),\ \tfrac{1}{2}(-1-i-j+k)\big)\\
C_4 &= \big(1,\ k,\ -1,\ -k\big)\\
C_5 &= \big(1,\ j,\ -1,\ -j\big)\\
C_6 &= \big(1,\ \tfrac{1}{\sqrt{2}}(-i+k),\ -\tfrac{1}{2}(1+i+j+k),\ -\tfrac{1}{\sqrt{2}}(j+k)\big)\\
C_7 &= \big(1,\ \tfrac{1}{\sqrt{2}}(i-j),\ \tfrac{1}{\sqrt{2}}(1+i),\ \tfrac{1}{2}(1-i-j+k)\big)\\
C_8 &= \big(1,\ \tfrac{1}{2}(1-i+j-k),\ k,\ -\tfrac{1}{\sqrt{2}}(1+j)\big)\\
C_9 &= \big(1,\ \tfrac{1}{\sqrt{2}}(1-i),\ -\tfrac{1}{\sqrt{2}}(1+i),\ \tfrac{1}{2}(-1-i+j-k)\big)
\end{aligned}$$

We note that $Stab(C_i) = V(C_i)$ for $2 \leq i \leq 5$ while all other C_i's have trivial stabilizer. Thus, by Lemma 2.2, one can check that $Orb(C_i)$ is a ℓ_i-cycle decomposition of $\mathrm{Cay}[G : \Omega_i]$ where ℓ_i is the length of C_i and where the Ω_i's are the symmetric subsets of G listed below.

$$\begin{aligned}
\Omega_1 &= \{-\tfrac{1}{\sqrt{2}}(1-k), \tfrac{1}{2}(1-i-j-k), -\tfrac{1}{\sqrt{2}}(1+i)\}^{\pm 1}\\
\Omega_2 &= \{\tfrac{1}{2}(-1-i+j+k)\}^{\pm 1}\\
\Omega_3 &= \{\tfrac{1}{2}(-1+i+j-k)\}^{\pm 1}\\
\Omega_4 &= \{k\}^{\pm 1}\\
\Omega_5 &= \{j\}^{\pm 1}\\
\Omega_6 &= \{\tfrac{1}{\sqrt{2}}(-i+k), \tfrac{1}{\sqrt{2}}(j-k), \tfrac{1}{\sqrt{2}}(1-k), -\tfrac{1}{\sqrt{2}}(j+k)\}^{\pm 1}
\end{aligned}$$

$$\Omega_7 = \{\tfrac{1}{\sqrt{2}}(i-j), \tfrac{1}{2}(1+i-j-k), \tfrac{1}{\sqrt{2}}(i+j), \tfrac{1}{2}(1-i-j+k)\}^{\pm 1}$$
$$\Omega_8 = \{\tfrac{1}{2}(1-i+j-k), -\tfrac{1}{2}(1+i+j+k), -\tfrac{1}{\sqrt{2}}(i+k), -\tfrac{1}{\sqrt{2}}(1+j)\}^{\pm 1}$$
$$\Omega_9 = \{\tfrac{1}{\sqrt{2}}(1-i), i, \tfrac{1}{\sqrt{2}}(1+j), \tfrac{1}{2}(-1-i+j-k)\}^{\pm 1}$$

One can see that the Ω_i's partition $G \setminus \{1,-1\}$. Thus, by Theorem 2.3 we can say that $\mathcal{C} := \bigcup_{i=1}^{9} Orb_G(C_i)$ is a G-regular cycle-decomposition of $K_{48} - I$. Now set $F_i = Orb_{S_i}(C_i)$ where

$$S_i = \begin{cases} K & \text{for } i = 1; \\ G & \text{for } 2 \le i \le 5; \\ L & \text{for } 6 \le i \le 9. \end{cases}$$

By Remark 2.4, each F_i is a 2-factor of $K_{48} - I$ with $Stab(F_i) = S_i$, hence $Orb(F_i)$ has length 3 or 1 or 4 according to whether $i = 1$, or $2 \le i \le 5$, or $6 \le i \le 9$, respectively. The cycles of F_i are triangles or quadrangles according to whether or not $i \le 3$. Thus, recalling that $\mathcal{C}$ is a cycle-decomposition of $K_{48} - I$, we conclude that $\mathcal{F} := \bigcup_{i=1}^{9} Orb(F_i)$ is a G-regular 2-factorization of $K_{48} - I$ with 5 triangle-factors and 18 quadrangle-factors, namely a G-regular solution of HWP$(48; 3, 4; 5, 18)$.

3.2 An octahedral solution of HWP$(48; 3, 4; 7, 16)$

Consider the seven cycles of $Cycle(G)$ defined as follows.

$$C_1 = \big(1,\ -\tfrac{1}{\sqrt{2}}(i+j),\ \tfrac{1}{2}(1-i+j+k)\big)$$
$$C_2 = \big(1,\ \tfrac{1}{2}(-1-i+j+k),\ \tfrac{1}{2}(1-i-j-k)\big)$$
$$C_3 = \big(1,\ \tfrac{1}{2}(-1+i+j-k),\ \tfrac{1}{2}(-1-i-j+k)\big)$$
$$C_4 = \big(1,\ \tfrac{1}{\sqrt{2}}(-i+k),\ \tfrac{1}{2}(1+i+j-k),\ -\tfrac{1}{\sqrt{2}}(j+k)\big)$$
$$C_5 = \big(1,\ \tfrac{1}{\sqrt{2}}(i-j),\ \tfrac{1}{\sqrt{2}}(1-k),\ \tfrac{1}{\sqrt{2}}(1+i)\big)$$
$$C_6 = \big(1,\ \tfrac{1}{\sqrt{2}}(1+k),\ -\tfrac{1}{2}(1+i+j+k),\ \tfrac{1}{\sqrt{2}}(1+j)\big)$$
$$C_7 = \big(1,\ -\tfrac{1}{2}(1+i+j+k),\ \tfrac{1}{2}(1-i+j-k),\ \tfrac{1}{2}(1-i-j+k)\big)$$

We note that $Stab(C_3) = V(C_3)$ while all other C_i's have trivial stabilizer. Thus, by Lemma 2.2, one can check that $Orb(C_i)$ is a ℓ_i-cycle decomposition of Cay$[G : \Omega_i]$ where ℓ_i is the length of C_i and where the Ω_i's are the symmetric subsets of G listed below.

$$\Omega_1 = \{-\tfrac{1}{\sqrt{2}}(i+j), \tfrac{1}{2}(1-i+j+k), \tfrac{1}{\sqrt{2}}(-j+k)\}^{\pm 1}$$
$$\Omega_2 = \{\tfrac{1}{2}(-1-i+j+k), \tfrac{1}{2}(1-i-j-k), \tfrac{1}{2}(-1-i+j-k)\}^{\pm 1}$$
$$\Omega_3 = \{\tfrac{1}{2}(-1+i+j-k)\}^{\pm 1}$$
$$\Omega_4 = \{\tfrac{1}{\sqrt{2}}(-i+k), -\tfrac{1}{\sqrt{2}}(1-k), \tfrac{1}{\sqrt{2}}(i+k), -\tfrac{1}{\sqrt{2}}(j+k)\}^{\pm 1}$$
$$\Omega_5 = \{\tfrac{1}{\sqrt{2}}(i-j), -j, \tfrac{1}{2}(1-i+j-k), \tfrac{1}{\sqrt{2}}(1+i)\}^{\pm 1}$$
$$\Omega_6 = \{\tfrac{1}{\sqrt{2}}(1+k), \tfrac{1}{\sqrt{2}}(-1+j), -\tfrac{1}{\sqrt{2}}(1+i), \tfrac{1}{\sqrt{2}}(1+j)\}^{\pm 1}$$
$$\Omega_7 = \{-\tfrac{1}{2}(1+i+j+k), -i, -k, \tfrac{1}{2}(1-i-j+k)\}^{\pm 1}$$

One can see that the Ω_i's partition $G \setminus \{1, -1\}$. Thus, by Theorem 2.3 we can say that $\mathcal{C} := \bigcup_{i=1}^{7} Orb_G(C_i)$ is a G-regular cycle-decomposition of $K_{48} - I$. Now set $F_i = Orb_{S_i}(C_i)$ where

$$S_i = \begin{cases} K & \text{for } i = 1, 2; \\ G & \text{for } i = 3; \\ L & \text{for } 4 \leq i \leq 7. \end{cases}$$

By Remark 2.4, each F_i is a 2-factor of $K_{48} - I$ with $Stab_G(F_i) = S_i$, hence $Orb_G(F_i)$ has length 3 or 1 or 4 according to whether $i = 1, 2$ or $i = 3$ or $4 \leq i \leq 7$, respectively.

The cycles of F_i are triangles or quadrangles according to whether or not $i \leq 3$. Thus, recalling that $\mathcal{C}$ is a cycle-decomposition of $K_{48} - I$, we conclude that $\mathcal{F} := \bigcup_{i=1}^{7} Orb_G(F_i)$ is a G-regular 2-factorization of $K_{48} - I$ with 7 triangle-factors and 16 quadrangle-factors, namely a G-regular solution of HWP$(48; 3, 4; 7, 16)$.

3.3 An octahedral solution of HWP$(48; 3, 4; 9, 14)$

Consider the eight cycles of $Cycle(G)$ defined as follows.

$$\begin{aligned}
C_1 &= \big(1,\ \tfrac{1}{\sqrt{2}}(i+j),\ \tfrac{1}{2}(1-i-j-k)\big) \\
C_2 &= \big(1,\ -\tfrac{1}{\sqrt{2}}(1-k),\ \tfrac{1}{\sqrt{2}}(1+j)\big) \\
C_3 &= \big(1,\ \tfrac{1}{2}(-1-i+j+k),\ \tfrac{1}{2}(1+i-j+k)\big) \\
C_4 &= \big(1,\ \tfrac{1}{\sqrt{2}}(-i+k),\ \tfrac{1}{\sqrt{2}}(1-i),\ \tfrac{1}{2}(-1-i+j-k)\big) \\
C_5 &= \big(1,\ \tfrac{1}{\sqrt{2}}(i-j),\ \tfrac{1}{2}(-1+i+j+k),\ -\tfrac{1}{\sqrt{2}}(j+k)\big) \\
C_6 &= \big(1,\ \tfrac{1}{\sqrt{2}}(1+i),\ \tfrac{1}{\sqrt{2}}(1-i),\ \tfrac{1}{2}(1-i-j+k)\big) \\
C_7 &= \big(1,\ k,\ -1,\ -k\big) \\
C_8 &= \big(1,\ j,\ -1,\ -j\big)
\end{aligned}$$

We note that $Stab(C_i) = V(C_i)$ for $i = 7, 8$ while all other C_i's have trivial stabilizer. By Lemma 2.2, one can check that $Orb(C_i)$ is a ℓ_i-cycle decomposition of Cay$[G : \Omega_i]$ where ℓ_i is the length of C_i and where the Ω_i's are the symmetric subsets of G listed below.

$$\begin{aligned}
\Omega_1 &= \{\tfrac{1}{\sqrt{2}}(i+j), \tfrac{1}{2}(1-i-j-k), \tfrac{1}{\sqrt{2}}(-1+i)\}^{\pm 1} \\
\Omega_2 &= \{-\tfrac{1}{\sqrt{2}}(1-k), \tfrac{1}{\sqrt{2}}(1+j), \tfrac{1}{2}(-1+i+j+k)\}^{\pm 1} \\
\Omega_3 &= \{\tfrac{1}{2}(-1-i+j+k), \tfrac{1}{2}(1+i-j+k), \tfrac{1}{2}(-1-i-j+k)\}^{\pm 1} \\
\Omega_4 &= \{\tfrac{1}{\sqrt{2}}(-i+k), \tfrac{1}{2}(1-i+j+k), \tfrac{1}{\sqrt{2}}(i+k), \tfrac{1}{2}(-1-i+j-k)\}^{\pm 1} \\
\Omega_5 &= \{\tfrac{1}{\sqrt{2}}(i-j), \tfrac{1}{\sqrt{2}}(j-k), -\tfrac{1}{\sqrt{2}}(1+j), -\tfrac{1}{\sqrt{2}}(j+k)\}^{\pm 1} \\
\Omega_6 &= \{\tfrac{1}{\sqrt{2}}(1+i), i, \tfrac{1}{\sqrt{2}}(1-k), \tfrac{1}{2}(1-i-j+k)\}^{\pm 1} \\
\Omega_7 &= \{k\}^{\pm 1} \\
\Omega_8 &= \{j\}^{\pm 1}
\end{aligned}$$

Now note that the Ω_i's partition $G \setminus \{1, -1\}$. Thus, by Theorem 2.3 we can say that $\mathcal{C} := \bigcup_{i=1}^{8} Orb(C_i)$ is a G-regular cycle-decomposition of $K_{48} - I$. Now set $F_i = Orb_{S_i}(C_i)$

where

$$S_i = \begin{cases} K & \text{for } 1 \le i \le 3; \\ L & \text{for } 4 \le i \le 6; \\ G & \text{for } i = 7, 8. \end{cases}$$

By Remark 2.4, each F_i is a 2-factor of $K_{48} - I$ with $Stab_G(F_i) = S_i$, hence $Orb_G(F_i)$ has length 3 or 4 or 1 according to whether $1 \le i \le 3$ or $4 \le i \le 6$ or $i = 7, 8$, respectively. The cycles of F_i are triangles or quadrangles according to whether or not $i \le 3$. Thus, recalling that $\mathcal{C}$ is a cycle-decomposition of $K_{48} - I$, we conclude that $\mathcal{F} := \bigcup_{i=1}^{8} Orb_G(F_i)$ is a G-regular 2-factorization of $K_{48} - I$ with 9 triangle-factors and 14 quadrangle-factors, namely a G-regular solution of HWP$(48; 3, 4; 9, 14)$.

3.4 An octahedral solution of HWP$(48; 3, 4; 13, 10)$

Consider the nine cycles of $Cycle(G)$ defined as follows.

$$\begin{aligned}
C_1 &= \big(1,\ -\tfrac{1}{\sqrt{2}}(i+j),\ -\tfrac{1}{\sqrt{2}}(1+j)\big) \\
C_2 &= \big(1,\ \tfrac{1}{2}(1-i+j-k),\ -\tfrac{1}{\sqrt{2}}(i+k)\big) \\
C_3 &= \big(1,\ \tfrac{1}{\sqrt{2}}(-i+j),\ \tfrac{1}{2}(1-i-j-k)\big) \\
C_4 &= \big(1,\ \tfrac{1}{2}(-1+i-j+k),\ \tfrac{1}{\sqrt{2}}(i-k)\big) \\
C_5 &= \big(1,\ \tfrac{1}{2}(-1-i+j+k),\ \tfrac{1}{2}(-1+i-j-k)\big) \\
C_6 &= \big(1,\ k,\ -1,\ -k\big) \\
C_7 &= \big(1,\ j,\ -1,\ -j\big) \\
C_8 &= \big(1,\ -\tfrac{1}{2}(1+i+j+k),\ \tfrac{1}{2}(-1+i-j+k),\ \tfrac{1}{\sqrt{2}}(1+j)\big) \\
C_9 &= \big(1,\ -\tfrac{1}{\sqrt{2}}(1+k),\ -k,\ \tfrac{1}{2}(-1+i+j-k)\big)
\end{aligned}$$

We note that $Stab(C_i) = V(C_i)$ for $5 \le i \le 7$ while all other C_i's have trivial G-stabilizer. Thus, by Lemma 2.2, one can check that $Orb(C_i)$ is a ℓ_i-cycle decomposition of Cay$[G : \Omega_i]$ where ℓ_i is the length of C_i and where the Ω_i's are the symmetric subsets of G listed below.

$$\begin{aligned}
\Omega_1 &= \{-\tfrac{1}{\sqrt{2}}(i+j), -\tfrac{1}{\sqrt{2}}(1+j), \tfrac{1}{2}(1+i+j-k)\}^{\pm 1} \\
\Omega_2 &= \{\tfrac{1}{2}(1-i+j-k), -\tfrac{1}{\sqrt{2}}(i+k), \tfrac{1}{\sqrt{2}}(1+i)\}^{\pm 1} \\
\Omega_3 &= \{\tfrac{1}{\sqrt{2}}(-i+j), \tfrac{1}{2}(1-i-j-k), \tfrac{1}{\sqrt{2}}(j-k)\}^{\pm 1} \\
\Omega_4 &= \{\tfrac{1}{2}(-1+i-j+k), \tfrac{1}{\sqrt{2}}(i-k), -\tfrac{1}{\sqrt{2}}(j+k)\}^{\pm 1} \\
\Omega_5 &= \{\tfrac{1}{2}(-1-i+j+k)\}^{\pm 1} \\
\Omega_6 &= \{k\}^{\pm 1} \\
\Omega_7 &= \{j\}^{\pm 1} \\
\Omega_8 &= \{-\tfrac{1}{2}(1+i+j+k), i, \tfrac{1}{\sqrt{2}}(-1+i), \tfrac{1}{\sqrt{2}}(1+j)\}^{\pm 1} \\
\Omega_9 &= \{-\tfrac{1}{\sqrt{2}}(1+k), \tfrac{1}{\sqrt{2}}(1-k), \tfrac{1}{2}(1-i+j+k), \tfrac{1}{2}(-1+i+j-k)\}^{\pm 1}
\end{aligned}$$

Now note that the Ω_i's partition $G\backslash\{1,-1\}$. Thus, by Theorem 2.3 we can say that $\mathcal{C} := \bigcup_{i=1}^{9} Orb(C_i)$ is a G-regular cycle-decomposition of $K_{48} - I$. Now set $F_i = Orb_{S_i}(C_i)$ where

$$S_i = \begin{cases} K & \text{for } 1 \leq i \leq 4; \\ G & \text{for } 5 \leq i \leq 7; \\ L & \text{for } i = 8, 9. \end{cases}$$

By Remark 2.4, each F_i is a 2-factor of $K_{48} - I$ with $Stab_G(F_i) = S_i$, hence $Orb_G(F_i)$ has length 3 or 1 or 4 according to whether $1 \leq i \leq 4$ or $5 \leq i \leq 7$ or $i = 8, 9$, respectively.

The cycles of F_i are triangles or quadrangles according to whether or not $i \leq 5$. Thus, recalling that $\mathcal{C}$ is a cycle-decomposition of $K_{48} - I$, we conclude that $\mathcal{F} := \bigcup_{i=1}^{9} Orb_G(F_i)$ is a G-regular 2-factorization of $K_{48} - I$ with 13 triangle-factors and 10 quadrangle-factors, namely a G-regular solution of HWP$(48; 3, 4; 13, 10)$.

3.5 An octahedral solution of HWP$(48; 3, 4; 15, 8)$

Consider the seven cycles of $Cycle(G)$ defined as follows.

$$\begin{aligned}
C_1 &= \big(1,\ \tfrac{1}{2}(-1-i+j+k),\ \tfrac{1}{\sqrt{2}}(i+k)\big) \\
C_2 &= \big(1,\ -\tfrac{1}{\sqrt{2}}(i+j),\ -\tfrac{1}{\sqrt{2}}(1+j)\big) \\
C_3 &= \big(1,\ \tfrac{1}{2}(-1+i+j-k),\ \tfrac{1}{2}(1-i+j+k)\big) \\
C_4 &= \big(1,\ \tfrac{1}{2}(1+i+j+k),\ \tfrac{1}{\sqrt{2}}(1+j)\big) \\
C_5 &= \big(1,\ \tfrac{1}{2}(1-i+j-k),\ \tfrac{1}{\sqrt{2}}(i-k)\big) \\
C_6 &= \big(1,\ -j,\ k,\ -\tfrac{1}{\sqrt{2}}(1-k)\big) \\
C_7 &= \big(1,\ \tfrac{1}{\sqrt{2}}(i-j),\ \tfrac{1}{2}(-1-i+j-k),\ \tfrac{1}{2}(-1+i+j+k)\big)
\end{aligned}$$

Here, every C_i has trivial stabilizer. Thus, by Lemma 2.2, one can check that $Orb(C_i)$ is a ℓ_i-cycle decomposition of $\mathrm{Cay}[G : \Omega_i]$ where ℓ_i is the length of C_i and where the Ω_i's are the symmetric subsets of G listed below.

$$\begin{aligned}
\Omega_1 &= \{\tfrac{1}{2}(-1-i+j+k), \tfrac{1}{\sqrt{2}}(i+k), \tfrac{1}{\sqrt{2}}(-j+k)\}^{\pm 1} \\
\Omega_2 &= \{-\tfrac{1}{\sqrt{2}}(i+j), -\tfrac{1}{\sqrt{2}}(1+j), \tfrac{1}{2}(1+i+j-k)\}^{\pm 1} \\
\Omega_3 &= \{\tfrac{1}{2}(-1+i+j-k), \tfrac{1}{2}(1-i+j+k), \tfrac{1}{2}(-1-i+j-k)\}^{\pm 1} \\
\Omega_4 &= \{\tfrac{1}{2}(1+i+j+k), \tfrac{1}{\sqrt{2}}(1+j), \tfrac{1}{\sqrt{2}}(1+i)\}^{\pm 1} \\
\Omega_5 &= \{\tfrac{1}{2}(1-i+j-k), \tfrac{1}{\sqrt{2}}(i-k), \tfrac{1}{\sqrt{2}}(j+k)\}^{\pm 1} \\
\Omega_6 &= \{-j, +i, \tfrac{1}{\sqrt{2}}(1-k), -\tfrac{1}{\sqrt{2}}(1-k)\}^{\pm 1} \\
\Omega_7 &= \{\tfrac{1}{\sqrt{2}}(i-j), -\tfrac{1}{\sqrt{2}}(1+i), +k, \tfrac{1}{2}(-1+i+j+k)\}^{\pm 1}
\end{aligned}$$

Now note that the Ω_i's partition $G \setminus \{1, -1\}$. Thus, by Theorem 2.3 we can say that $\mathcal{C} := \bigcup_{i=1}^{7} Orb(C_i)$ is a G-regular cycle-decomposition of $K_{48} - I$. Set $F_i = Orb_{S_i}(C_i)$ where

$$S_i = \begin{cases} K & \text{for } 1 \leq i \leq 5; \\ L & \text{for } i = 6, 7. \end{cases}$$

By Remark 2.4, each F_i is a 2-factor of $K_{48} - I$ with $Stab_G(F_i) = S_i$, hence $Orb_G(F_i)$ has length 3 or 4 according to whether $1 \leq i \leq 5$ or $i = 6, 7$, respectively. The cycles of F_i are triangles or quadrangles according to whether or not $i \leq 5$. Thus, recalling that $\mathcal{C}$ is a cycle-decomposition of $K_{48} - I$, we conclude that $\mathcal{F} := \bigcup_{i=1}^{7} Orb_G(F_i)$ is a G-regular 2-factorization of $K_{48} - I$ with 15 triangle-factors and 8 quadrangle-factors, namely a G-regular solution of HWP$(48; 3, 4; 15, 8)$.

3.6 An octahedral solution of HWP$(48; 3, 4; 17, 6)$

Consider the ten cycles of $Cycle(G)$ defined as follows.

$$\begin{aligned}
C_1 &= \big(1,\ -\tfrac{1}{\sqrt{2}}(1-k),\ -\tfrac{1}{\sqrt{2}}(i+k)\big)\\
C_2 &= \big(1,\ -\tfrac{1}{\sqrt{2}}(i+j),\ \tfrac{1}{2}(-1+i+j+k)\big)\\
C_3 &= \big(1,\ \tfrac{1}{2}(1+i-j-k),\ -\tfrac{1}{\sqrt{2}}(1+j)\big)\\
C_4 &= \big(1,\ \tfrac{1}{\sqrt{2}}(-i+j),\ \tfrac{1}{\sqrt{2}}(-i+k)\big)\\
C_5 &= \big(1,\ \tfrac{1}{2}(1-i+j-k),\ \tfrac{1}{\sqrt{2}}(1-j)\big)\\
C_6 &= \big(1,\ \tfrac{1}{2}(-1-i+j+k),\ \tfrac{1}{2}(-1+i-j-k)\big)\\
C_7 &= \big(1,\ \tfrac{1}{2}(-1+i+j-k),\ \tfrac{1}{2}(-1-i-j+k)\big)\\
C_8 &= \big(1,\ k,\ -1,\ -k\big)\\
C_9 &= \big(1,\ j,\ -1,\ -j\big)\\
C_{10} &= \big(1,\ \tfrac{1}{\sqrt{2}}(1+i),\ \tfrac{1}{\sqrt{2}}(1-i), \tfrac{1}{2}(1-i-j+k)\big)
\end{aligned}$$

We note that $Stab(C_i) = V(C_i)$ for $6 \leq i \leq 9$ while all other C_i's have trivial stabilizer. Thus, by Lemma 2.2, one can check that $Orb(C_i)$ is a ℓ_i-cycle decomposition of Cay$[G : \Omega_i]$ where ℓ_i is the length of C_i and where the Ω_i's are the symmetric subsets of G listed below.

$$\begin{aligned}
\Omega_1 &= \{-\tfrac{1}{\sqrt{2}}(1-k), -\tfrac{1}{\sqrt{2}}(i+k), \tfrac{1}{2}(-1-i+j-k)\}^{\pm1}\\
\Omega_2 &= \{-\tfrac{1}{\sqrt{2}}(i+j), \tfrac{1}{2}(-1+i+j+k), \tfrac{1}{\sqrt{2}}(-1+i)\}^{\pm1}\\
\Omega_3 &= \{\tfrac{1}{2}(1+i-j-k), -\tfrac{1}{\sqrt{2}}(1+j), \tfrac{1}{\sqrt{2}}(j+k)\}^{\pm1}\\
\Omega_4 &= \{\tfrac{1}{\sqrt{2}}(-i+j), \tfrac{1}{\sqrt{2}}(-i+k), \tfrac{1}{2}(1-i-j-k)\}^{\pm1}\\
\Omega_5 &= \{\tfrac{1}{2}(1-i+j-k), \tfrac{1}{\sqrt{2}}(1-j), \tfrac{1}{\sqrt{2}}(j-k)\}^{\pm1}\\
\Omega_6 &= \{\tfrac{1}{2}(-1-i+j+k)\}^{\pm1}\\
\Omega_7 &= \{\tfrac{1}{2}(-1+i+j-k)\}^{\pm1}\\
\Omega_8 &= \{k\}^{\pm1}\\
\Omega_9 &= \{j\}^{\pm1}\\
\Omega_{10} &= \{\tfrac{1}{\sqrt{2}}(1+i), i, \tfrac{1}{\sqrt{2}}(1-k), \tfrac{1}{2}(1-i-j+k)\}^{\pm1}
\end{aligned}$$

Now note that the Ω_i's partition $G \setminus \{1, -1\}$. Thus, by Lemma 2.2 we can say that $\mathcal{C} := \bigcup_{i=1}^{10} Orb(C_i)$ is a G-regular cycle-decomposition of $K_{48} - I$. Set $F_i = Orb_{S_i}(C_i)$

where

$$S_i = \begin{cases} K & \text{for } 1 \le i \le 5; \\ G & \text{for } 6 \le i \le 9; \\ L & \text{for } i = 10. \end{cases}$$

By Remark 2.4, each F_i is a 2-factor of K_{48} with $Stab_G(F_i) = S_i$, hence $Orb_G(F_i)$ has length 3 or 1 or 4 according to whether $1 \le i \le 5$ or $6 \le i \le 9$ or $i = 10$, respectively. The cycles of F_i are triangles or quadrangles according to whether or not $i \le 7$. Thus, recalling that $\mathcal{C}$ is a cycle-decomposition of $K_{48} - I$, we conclude that $\mathcal{F} := \bigcup_{i=1}^{10} Orb_G(F_i)$ is a G-regular 2-factorization of $K_{48} - I$ with 17 triangle-factors and 6 quadrangle-factors, namely a G-regular solution of HWP$(48; 3, 4; 17, 6)$.

4 Dicyclic solutions of two Hamilton-Waterloo problems

In this section G will denote the dicyclic group of order 24 which is usually denoted by Q_{24}. Thus G has the following presentation:

$$G = \langle a, b \,|\, a^{12} = 1,\ b^2 = a^6,\ b^{-1}ab = a^{-1} \rangle$$

Note that the elements of G can be written in the form $a^i b^j$ with $0 \le i \le 11$ and $j = 0, 1$. The group G has a unique involution which is a^6 and we will need to consider the following subgroups of G:

- $H = \langle b \rangle = \{1, b, a^6, a^6 b\}$;
- $K = \langle a^2 \rangle = \{1, a^2, a^4, a^6, a^8, a^{10}\}$;
- $L = \langle a^2 b, a^3 \rangle = \{1, a^3, a^6, a^9, a^2 b, a^8 b, a^5 b, a^{11} b\}$.

4.1 A dicyclic solution of HWP$(24; 3, 4; 7, 4)$

Consider the four cycles of $Cycle(G)$ defined as follows.

$$\begin{aligned} C_1 &= \big(1,\ a^3 b,\ a^5\big) \\ C_2 &= \big(1,\ a^{10},\ a^7 b\big) \\ C_3 &= \big(1,\ a^4,\ a^8\big) \\ C_4 &= \big(1,\ b,\ a^3 b,\ a\big) \end{aligned}$$

We note that the $Stab(C_3) = V(C_3)$ while all other C_i's have trivial stabilizer. Thus, by Lemma 2.2, one can check that $Orb(C_i)$ is a ℓ_i-cycle decomposition of Cay$[G : \Omega_i]$ where ℓ_i is the length of C_i and where the Ω_i's are the symmetric subsets of G listed below.

$$\begin{aligned} \Omega_1 &= \{a^3 b, a^5, a^2 b\}^{\pm 1} \\ \Omega_2 &= \{a^2, ab, a^5 b\}^{\pm 1} \\ \Omega_3 &= \{a^4\}^{\pm 1} \\ \Omega_4 &= \{b, a^3, a^4 b, a\}^{\pm 1} \end{aligned}$$

Now note that the Ω_i's partition $G \backslash \{1, a^6\}$. Thus, by Theorem 2.3 we can say that $\mathcal{C} := \bigcup_{i=1}^{4} Orb(C_i)$ is a G-regular cycle-decomposition of $K_{24} - I$. Now set $F_i = Orb_{S_i}(C_i)$

where

$$S_i = \begin{cases} L & \text{for } i = 1, 2; \\ G & \text{for } i = 3; \\ K & \text{for } i = 4. \end{cases}$$

By Remark 2.4, each F_i is a 2-factor of $K_{24} - I$ with $Stab_G(F_i) = S_i$, hence $Orb_G(F_i)$ has length 3 or 1 or 4 according to whether $i = 1, 2$ or $i = 3$ or $i = 4$, respectively.

The cycles of F_i are triangles or quadrangles according to whether or not $i \leq 3$. Thus, recalling that $\mathcal{C}$ is a cycle-decomposition of $K_{48} - I$, we conclude that $\mathcal{F} := \bigcup_{i=1}^{4} Orb_G(F_i)$ is a G-regular 2-factorization of $K_{24} - I$ with 7 triangle-factors and 4 quadrangle-factors, namely a G-regular solution of HWP$(24; 3, 4; 7, 4)$.

4.2 A dicyclic solution of HWP$(24; 3, 4; 9, 2)$

Consider the four cycles of $Cycle(G)$ defined as follows.

$$\begin{aligned} C_1 &= (1, b, a^6, a^6 b) \\ C_2 &= (1, a^4 b, a^6, a^{10} b) \\ C_3 &= (1, a^4, a^7 b) \\ C_4 &= (1, a^3 b, a^8 b) \\ C_5 &= (a^4, a^7, a^5) \end{aligned}$$

We note that $Stab(C_i) = V(C_i)$ for $i = 1, 2$ while all other C_i's have trivial stabilizer. By Lemma 2.2, one can check that $Orb(C_i)$ is a ℓ_i-cycle decomposition of Cay$[G : \Omega_i]$ where ℓ_i is the length of C_i and where the Ω_i's are the symmetric subsets of G listed below.

$$\begin{aligned} \Omega_1 &= \{b\}^{\pm 1} \\ \Omega_2 &= \{a^4 b\}^{\pm 1} \\ \Omega_3 &= \{a^4, ab, a^5 b\}^{\pm 1} \\ \Omega_4 &= \{a^3 b, a^2 b, a^5\}^{\pm 1} \\ \Omega_5 &= \{a^1, a^2, a^3\}^{\pm 1} \end{aligned}$$

Also here the Ω_i's partition $G \setminus \{1, a^6\}$, hence $\mathcal{C} := \bigcup_{i=1}^{5} Orb_G(C_i)$ is a G-regular cycle-decomposition of $K_{24} - I$ by Theorem 2.3. Now set:

$$\begin{aligned} F_1 &= Orb_G(C_1), \quad F_2 = Orb_G(C_2), \\ F_3 &= Orb_L(C_3), \quad F_4 = Orb_H(C_4) \cup Orb_H(C_5). \end{aligned}$$

By Remark 2.4, each F_i is a 2-factor of $K_{24} - I$ and we have

$$Stab_G(F_1) = Stab_G(F_2) = G; \quad Stab_G(F_3) = L; \quad Stab_G(F_4) = H$$

so that the lengths of the G-orbits of $F_1, \ldots, F_4$ are 1, 1, 3 and 6, respectively. The cycles of F_i are triangles or quadrangles according to whether or not $i \geq 3$. Thus, recalling that $\mathcal{C}$ is a cycle-decomposition of $K_{48} - I$, we conclude that $\mathcal{F} := \bigcup_{i=1}^{5} Orb_G(F_i)$ is a G-regular 2-factorization of $K_{24} - I$ with 9 triangle-factors and 2 quadrangle-factors, namely a G-regular solution of HWP$(24; 3, 4; 9, 2)$.

5 A special linear solution of HWP$(24; 3, 4; 5, 6)$

In this section G will denote the 2-dimensional special linear group over $\mathbb{Z}_3$, usually denoted by $SL_2(3)$, namely the group of 2×2 matrices with elements in $\mathbb{Z}_3$ and determinant one. The only involution of G is $2E$ where E is the identity matrix of G. The 2-Sylow subgroup Q of G, isomorphic to the group of quaternions, is the following:

$$Q = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}, \begin{bmatrix} 2 & 2 \\ 2 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 2 \\ 2 & 2 \end{bmatrix}, \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \right\}.$$

We will also need to consider the subgroup H of G of order 6 generated by the matrix $\begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix}$. Hence we have:

$$H = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix}, \begin{bmatrix} 2 & 1 \\ 2 & 0 \end{bmatrix}, \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}, \begin{bmatrix} 0 & 2 \\ 1 & 2 \end{bmatrix}, \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix} \right\}.$$

The use of the special linear group G was crucial in [5] to get a Steiner triple system of any order $v = 144n + 25$ with an automorphism group acting sharply transitively an all but one point. Here G will be used to get a G-regular solution of the last Hamilton-Waterloo problem left open in [11].

Consider the six cycles of $Cycle(G)$ defined as follows.

$$C_1 = \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 2 & 0 \\ 2 & 2 \end{bmatrix}, \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix} \right)$$
$$C_2 = \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 2 \\ 1 & 2 \end{bmatrix}, \begin{bmatrix} 2 & 1 \\ 2 & 0 \end{bmatrix} \right)$$
$$C_3 = \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 2 & 2 \end{bmatrix}, \begin{bmatrix} 2 & 2 \\ 1 & 0 \end{bmatrix} \right)$$
$$C_4 = \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix}, \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}, \begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix} \right)$$
$$C_5 = \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}, \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}, \begin{bmatrix} 2 & 2 \\ 2 & 1 \end{bmatrix} \right)$$
$$C_6 = \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \right)$$

Here the stabilizer of C_i is trivial for $i = 1, 6$ while it coincides with $V(C_i)$ for $2 \leq i \leq 5$. By Lemma 2.2, one can check that $Orb(C_i)$ is a ℓ_i-cycle decomposition of $\mathrm{Cay}[G : \Omega_i]$ where ℓ_i is the length of C_i and where the Ω_i's are the symmetric subsets of G listed below.

$$\Omega_1 = \left\{ \begin{bmatrix} 2 & 0 \\ 2 & 2 \end{bmatrix}, \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 2 \\ 1 & 1 \end{bmatrix} \right\}^{\pm 1}$$
$$\Omega_2 = \left\{ \begin{bmatrix} 0 & 2 \\ 1 & 2 \end{bmatrix} \right\}^{\pm 1} \quad \Omega_3 = \left\{ \begin{bmatrix} 0 & 1 \\ 2 & 2 \end{bmatrix} \right\}^{\pm 1}$$
$$\Omega_4 = \left\{ \begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix} \right\}^{\pm 1} \quad \Omega_5 = \left\{ \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \right\}^{\pm 1}$$
$$\Omega_6 = \left\{ \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}, \begin{bmatrix} 2 & 2 \\ 0 & 2 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \right\}^{\pm 1}$$

Once again we see that the Ω_i's partition $G \setminus \{E, 2E\}$, therefore $\mathcal{C} := \bigcup_{i=1}^{6} Orb(C_i)$ is a G-regular cycle-decomposition of $K_{24} - I$. Now set $F_i = Orb_{S_i}(C_i)$ with

$$S_i = \begin{cases} Q & \text{for } i = 1; \\ G & \text{for } 2 \leq i \leq 5; \\ H & \text{for } i = 6. \end{cases}$$

By Remark 2.4, each F_i is a 2-factor of $K_{24} - I$ and we have $Stab_G(F_i) = S_i$ so that the lengths of the G-orbits of $F_1, \ldots, F_6$ are 3, 1, 1, 1, 1 and 4, respectively.

The cycles of F_i have length 3 or 4 according to whether or not $i \leq 3$. Thus, recalling that $\mathcal{C}$ is a cycle-decomposition of $K_{24} - I$, we conclude that $\mathcal{F} := \bigcup_{i=1}^{6} Orb_G(F_i)$ is a G-regular 2-factorization of $K_{24} - I$ with 5 triangle-factors and 6 quadrangle-factors, namely a G-regular solution of HWP$(24; 3, 4; 5, 6)$.

References

[1] B. Alspach, P. J. Schellenberg, D. R. Stinson and D. Wagner, The Oberwolfach problem and factors of uniform odd length cycles, *J. Combin. Theory Ser. A* **52** (1989), 20–43, doi:10.1016/0097-3165(89)90059-9.

[2] S. Bonvicini and M. Buratti, Sharply vertex transitive factorizations of Cayley graphs, in preparation.

[3] S. Bonvicini, M. Buratti, G. Rinaldi and T. Traetta, Some progress on the existence of 1-rotational Steiner triple systems, *Des. Codes Cryptogr.* **62** (2012), 63–78, doi:10.1007/s10623-011-9491-3.

[4] D. Bryant and V. Scharaschkin, Complete solutions to the Oberwolfach problem for an infinite set of orders, *J. Combin. Theory Ser. B* **99** (2009), 904–918, doi:10.1016/j.jctb.2009.03.003.

[5] M. Buratti, 1-rotational Steiner triple systems over arbitrary groups, *J. Combin. Des.* **9** (2001), 215–226, doi:10.1002/jcd.1008.abs.

[6] M. Buratti, Rotational k-cycle systems of order $v < 3k$; another proof of the existence of odd cycle systems, *J. Combin. Des.* **11** (2003), 433–441, doi:10.1002/jcd.10061.

[7] M. Buratti, Cycle decompositions with a sharply vertex transitive automorphism group, *Le Matematiche* **59** (2004), 91–105 (2006), `https://www.dmi.unict.it/ojs/index.php/lematematiche/article/view/164`.

[8] M. Buratti and P. Danziger, A cyclic solution for an infinite class of Hamilton-Waterloo problems, *Graphs Combin.* **32** (2016), 521–531, doi:10.1007/s00373-015-1582-x.

[9] M. Buratti and G. Rinaldi, On sharply vertex transitive 2-factorizations of the complete graph, *J. Combin. Theory Ser. A* **111** (2005), 245–256, doi:10.1016/j.jcta.2004.11.014.

[10] A. Burgess, P. Danziger and T. Traetta, On the Hamilton-Waterloo problem with odd orders, *J. Combin. Des.* (2016), doi:10.1002/jcd.21552.

[11] P. Danziger, G. Quattrocchi and B. Stevens, The Hamilton-Waterloo problem for cycle sizes 3 and 4, *J. Combin. Des.* **17** (2009), 342–352, doi:10.1002/jcd.20219.

[12] D. G. Hoffman and P. J. Schellenberg, The existence of C_k-factorizations of $K_{2n} - F$, *Discrete Math.* **97** (1991), 243–250, doi:10.1016/0012-365x(91)90440-d.

[13] F. Merola and T. Traetta, Infinitely many cyclic solutions to the Hamilton-Waterloo problem with odd length cycles, *Discrete Math.* **339** (2016), 2267–2283, doi:10.1016/j.disc.2016.03.026.

[14] U. Odabaşı and S. Özkan, The Hamilton-Waterloo problem with C_4 and C_m factors, *Discrete Math.* **339** (2016), 263–269, doi:10.1016/j.disc.2015.08.013.

[15] T. Traetta, A complete solution to the two-table Oberwolfach problems, *J. Combin. Theory Ser. A* **120** (2013), 984–997, doi:10.1016/j.jcta.2013.01.003.

[16] L. Wang, F. Chen and H. Cao, The Hamilton-Waterloo problem for C_3-factors and C_n-factors, 2016, `arXiv:1609.00453 [math.CO]`.

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Right quadruple convexity*

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Abstract

A set of four points $w, x, y, z \in \mathbb{R}^d$ (always $d \geq 2$) form a rectangular quadruple if their convex hull is a non-degenerate rectangle. The set M is called rq-convex if for every pair of its points we can find another pair in M, such that the four points form a rectangular quadruple. In this paper we start the investigation of rq-convexity in Euclidean spaces.

Keywords: rq-convex sets, parallelotopes, finite sets, Platonic solids.

Math. Subj. Class.: 53C45, 53C22

1 Introduction

Let $\mathcal{F}$ be a family of sets in $\mathbb{R}^d$. A set $M \subset \mathbb{R}^d$ is called $\mathcal{F}$*-convex* if for any pair of distinct points $x, y \in M$ there is a set $F \in \mathcal{F}$ such that $x, y \in F$ and $F \subset M$.

The third author proposed at the 1974 meeting on Convexity in Oberwolfach the investigation of this very general kind of convexity. Usual convexity, affine linearity, arc-wise connectedness, polygonal connectedness, are just some examples of $\mathcal{F}$-convexity (for suitably chosen families $\mathcal{F}$).

Blind, Valette and the third author [1], and also Böröczky Jr [2], investigated rectangular convexity. Magazanik and Perles dealt with staircase connectedness [5]. The third author

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studied right convexity [8]; then the second and the third author generalized the latter type of convexity and investigated the right triple convexity (see [7] and [6]). All these concepts are particular cases of $\mathcal{F}$-convexity. The rectangular convexity is obtained if $\mathcal{F}$ is the family of all non-degenerate rectangles in $\mathbb{R}^d$.

In this paper we present a discretization of rectangular convexity, the right quadruple convexity, which constitutes a generalization of rectangular convexity. As usual, for $M \subset \mathbb{R}^d$, $\mathrm{bd}M$ denotes its boundary, $\mathrm{int}M$ its interior, $\mathrm{diam}M = \sup_{x,y \in M} \|x - y\|$ its diameter, and $\mathrm{conv}M$ its convex hull. A set of four points $w, x, y, z \in \mathbb{R}^d$ (always $d \geq 2$) form a *rectangular quadruple* if $\mathrm{conv}\{w, x, y, z\}$ is a non-degenerate rectangle. Let $\mathcal{R}$ be the family of all rectangular quadruples. Here, we shall choose $\mathcal{F}$ to be this family $\mathcal{R}$.

Let $M \subset \mathbb{R}^d$. A pair of points $x, y \in M$ is said to enjoy the *rq-property in M* if there exists another pair of points $z,\ w \in M$, such that $\{w, x, y, z\}$ is a rectangular quadruple. The set M is called *rq-convex*, if every pair of its points enjoys the rq-property in M. This property is the *right quadruple convexity*.

Let $A \subset \mathbb{R}^d$. We call A^* an *rq-convex completion* of A, if A^* is rq-convex, $A^* \supset A$ and card($A^* \backslash A$) is minimal (but possibly infinite). Let $\gamma(A) = \mathrm{card}(A^* \backslash A)$, which is called the *$rq$-convex completion number* of A, in case A is finite. For finite n, let $\gamma(n) = \sup\{\gamma(A) :\ \mathrm{card}A = n\}$.

For distinct $x, y \in \mathbb{R}^d$, let $\overline{xy}$ be the line through x, y, xy the line-segment from x to y, H_{xy} the hyperplane through x orthogonal to $\overline{xy}$, and C_{xy} the hypersphere of diameter xy. For $S_1,\ S_2 \subset \mathbb{R}^d$, let $d(S_1, S_2) = \inf\{d(x, y) \mid x \in S_1,\ y \in S_2\}$ denote the *distance* between S_1 and S_2. The d-dimensional unit ball (centred at **0**) is denoted by B_d $(d \geq 2)$. Let us remark that every open set in $\mathbb{R}^d$ is rq-convex.

2 Not simply connected rq-convex sets

In $\mathbb{R}^2$, all compact rectangularly convex sets are conjectured to be extremely circular and symmetric. A planar convex set is *extremely circular* if its set of extreme points lies on a circle. Analogously, it is reasonable to conjecture that all compact rq-convex sets have an extremely circular and symmetric convex hull. Consequently, when investigating a compact connected rq-convexset M, we may reasonably start by assuming that $\mathrm{conv}M$ is extremely circular and symmetric. We shall now take $\mathrm{bdconv}M$ to be a circle. If M is simply connected we get the disc. So, assume $(\mathrm{conv}M) \setminus M \neq \emptyset$.

Theorem 2.1. *If* $\mathrm{conv}M$ *is a disc and* $(\mathrm{conv}M) \setminus M$ *lies in a circular disc of radius r at distance at least* $(\sqrt{3} - 1)r$ *from* $\mathrm{bdconv}M$*, then M is rq-convex.*

This theorem gives a useful sufficient condition for the rq-convexity of a set M which is not simply connected, regardless the shape of $(\mathrm{conv}M) \setminus M$. Notice that it allows both M and its complement to have arbitrarily many components.

Proof. Let Q be a square circumscribed to the disc D of radius r including $(\mathrm{conv}M) \setminus M$. We may suppose that the origin **0** is the centre of D, so $D = rB_2$.

We have $Q \subset \mathrm{conv}M$. Indeed, Q is obviously included in the disc concentric with D of radius $\sqrt{2}r$, which in turn must be included in $\mathrm{conv}M$, since the distance from D to $\mathrm{bdconv}M$ is at least $(\sqrt{3} - 1)r > (\sqrt{2} - 1)r$.

Let $x, y \in M$. We verify the rq-convexity of M at these two points.

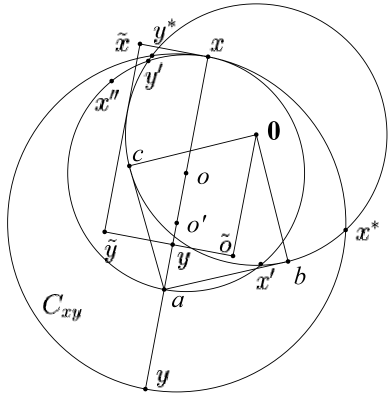

Figure 1: $x \in D, y \notin D$.

Case 1: $x, y \in D$. Choose Q to have a side s parallel to xy. Then x, y and their orthogonal projections on s are vertices of a rectangle. Moreover, these latter vertices lie in $(\text{conv}M) \setminus D$.

Case 2: $x \in D$, $y \notin D$. Let Γ_2, Γ_3 be the two circles concentric with $\text{bd}D$, of radii $r\sqrt{2}$, $r\sqrt{3}$. Let a be the point of $\overline{xy} \cap \Gamma_2$ such that $x \notin ay$. In case $y \in ax$, consider the rectangle $xy\tilde{y}\tilde{x}$, such that $\overline{xy}$ separates $\mathbf{0}$ from $\tilde{x}$, $\tilde{y}$ (if $\mathbf{0} \notin \overline{xy}$) and $\overline{\tilde{x}\tilde{y}}$ is tangent to $\text{bd}D$. See Figure 1. Let $\tilde{o}$ be the orthogonal projection of $\mathbf{0}$ onto $\overline{y\tilde{y}}$. We have

$$\|\tilde{y}\|^2 = \|\tilde{y} - \tilde{o}\|^2 + \|\tilde{o}\|^2 = r^2 + \|\tilde{o}\|^2 \leq r^2 + \|y\|^2 \leq r^2 + \|a\|^2 = 3r^2.$$

Hence, $\tilde{y}$, and of course $\tilde{x}$ too, lie in M, and the rq-property is satisfied in x, y.

We reconsider the case $y = a$. Choose $b, c \in \text{bd}D$ such that $ab\mathbf{0}c$ be a square and $b, \mathbf{0}$ be not separated by $\overline{xy}$. Let $\{x', y'\} = C_{ax} \cap \text{bd}D$, where x' is closer than y' from b. Let x'' be the point of C_{ax} diametrally opposite to x'. For any position of x,

$$\angle \mathbf{0}ax' \leq \angle \mathbf{0}ab.$$

Hence, $\angle \mathbf{0}ax'' \geq \angle \mathbf{0}ac$, whence $x'' \notin D$. (This confirms the rq-property in x, a.) It follows that $\angle x'oy' < \pi$, where o is the centre of C_{ax} and the angle is measured towards $\mathbf{0}$.

In case $a \in xy$, consider the circle C_{xy} with centre o', which cuts $\text{bd}D$ in x^*, y^* (the former being closer than the latter from b). We have

$$\angle xo'x^* < \angle xo'x' < \angle xox'$$

and

$$\angle xo'y^* < \angle xo'y' < \angle xoy',$$

whence

$$\angle x^*o'y^* < \angle x'oy' < \pi,$$

where both angles are taken towards $\mathbf{0}$. Consequently, the points $x^+, y^+ \in C_{xy}$ diametrally opposite to x^*, y^* (respectively) lie outside D. They also lie in different half-circles determined by x, y on C_{xy}. Of these two half-circles, at least one is contained in the disc $\mathrm{conv}M$.

So, either $\{x, y, x^*, x^+\} \subset M$ or $\{x, y, y^*, y^+\} \subset M$, and the rq-property is again satisfied at x, y.

Case 3: $x, y \notin D$. Besides the trivial cases $x, y \in \mathrm{int}M$ and $x, y \in \mathrm{bdconv}M$, we only have the simple situation $x \in \mathrm{int}M$, $y \in \mathrm{bdconv}M$. In that situation, the circle C_{xy} has necessarily two opposite arcs in M starting at x, respectively y. This proves the rq-property at x, y. □

Conjecture 2.2. *Each simply connected rq-convex set in $\mathbb{R}^2$ is convex.*

3 Unbounded rq-convex sets

An infinite family $\mathcal{K}$ of closed convex sets is said to be *uniformly bounded below* if, for some $\lambda > 0$, each of the sets contains a translate of the disc λB_2.

Theorem 3.1. *Let $\mathcal{K}$ be a family of pairwise disjoint closed convex sets in $\mathbb{R}^d$. If $\mathcal{K}$ is finite or uniformly bounded below, then the closure of the complement of $\bigcup \mathcal{K}$ is rq-convex.*

Proof. We may assume that all sets in $\mathcal{K}$ possess interior points, because the case of empty interior is irrelevant. Let M be the closure of $\mathbb{R}^d \setminus \bigcup \mathcal{K}$, and choose $x, y \in M$. Clearly, the only interesting case is when $x, y \in \mathrm{bd}M$.

The condition of uniform boundedness below for infinite $\mathcal{K}$ guarantees that $x \in \mathrm{bd}M$ only if x is a boundary point of some member of $\mathcal{K}$.

Let M' be the intersection of M with an arbitrary 2-dimensional plane $\Pi \ni x, y$. For some $K_x, K_y \in \mathcal{K}$, $x \in \mathrm{bd}K_x$, $y \in \mathrm{bd}K_y$. Consider the supporting hyperplane H_x of K_x at x, the line $H'_x = H_x \cap \Pi$, and analogously H_y and H'_y. If H'_x, H'_y are not orthogonal to xy, there are six different situations in the neighbourhood of x and y, depicted in Figure 2 (subfigures (a)-(f)). (In the figure only the generic case is depicted, when $K_x \cap \Pi$ and $K_y \cap \Pi$ are not degenerate; but the proof works in all cases.)

In the situations of Figure 2 (subfigures (a),(c) and (e)), the circle C_{xy} has two opposite arcs inside M, so M has the rq-property at x, y. In the cases of Figure 2 (subfigures (b),(d) and (f)), a thin rectangle with xy as a side has its short sides in M, so the rq-property is again verified.

The only remaining case is that of at least one of the lines H'_x, H'_y, say the first, being orthogonal to xy. In this case, there is a short line-segment $x(x+v) \subset M$ in any direction v orthogonal to xy. Now, if $y + v \in M$, we found the right quadruple $\{x, y, y+v, x+v\}$. If $y + v \notin M$, i.e. $y + v \in \mathrm{int}K_y$, then $y - v \notin K_y$. Thus, $\{x, y, y-v, x-v\}$ is a suitable rectangular quadruple. □

We can drop the convexity condition if the considered sets are bounded.

Theorem 3.2. *The complement of any bounded set in $\mathbb{R}^d$ is rq-convex.*

The easy proof is left to the reader.

A *plane tiling* $\mathcal{T}$ is a countable family $\{T_1, T_2, \ldots\}$ of closed sets with non-empty interiors, which cover the plane without gaps or overlaps. Every closed set $T_i \in \mathcal{T}$ is called a *tile of* $\mathcal{T}$. We consider the special case in which each tile is a polygon. If the corners and sides of a polygon coincide with the vertices and edges of the tiling, we call the tiling

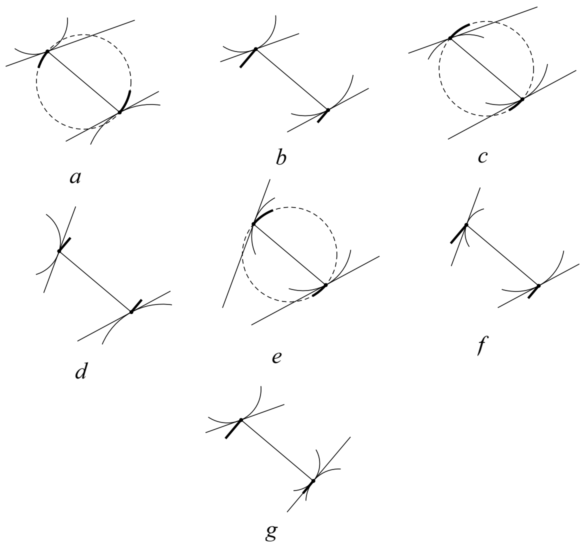

Figure 2: Illustration for the proof of Theorem 3.1.

edge-to-edge. A so-called *type* describes the neighbourhood of any vertex of the tiling. If, for example, in some cyclic order around a vertex there are a triangle, then another triangle, then a square, next a third triangle, and last another square, then its type is $(3^2.4.3.4)$. We consider plane edge-to-edge tilings in which all tiles are regular polygons, and all vertices are of the same type. Thus, the vertex-type defines our tiling up to similarity.

There exist precisely eleven such tilings [3]. These are (3^6), $(3^4.6)$, $(3^3.4^2)$, $(3^2.4.3.4)$, $(3.4.6.4)$, $(3.6.3.6)$, (3.12^2), (4^4), $(4.6.12)$, (4.8^2), and (6^3). They are called *Archimedean tilings.*

Theorem 3.3. *The Archimedean tilings* (4^4), (3^6), (6^3), $(3.6.3.6)$, $(3^4.6)$, $(3.3.4.3.4)$, $(4.8.8)$ *have* rq*-convex vertex sets.*

Theorem 3.4. *The vertex sets of the Archimedean tilings* $(3.3.3.4.4)$, $(3.4.6.4)$, $(4.6.12)$, $(3.12.12)$ *are not* rq*-convex.*

The proofs of Theorems 3.3 and 3.4 are also left to the reader.

4 rq-convex skeleta of parallelotopes

As already remarked in [1], for $d \geq 3$, there is not even any conjectured characterization of rectangularly convex sets in $\mathbb{R}^d$. Among the sets mentioned in [1] as rectangularly convex we find the cylinder $K \times [0, 1]$ with a $(d-1)$-dimensional compact convex set K as basis. In particular, any *right parallelotope*, i.e. the cartesian product of d pairwise orthogonal line-segments, is rectangularly convex and, a fortiori, rq-convex.

Theorem 4.1. *The* 1*-skeleton of any right parallelotope is* rq*-convex.*

Proof. Let $P = I_1 \times I_2 \times \ldots \times I_d$ be our parallelotope, where $I_i = [\mathbf{0}, a_i]$ $(i = 1, \ldots, d)$. We verify the rq-property at the points x, y belonging to the 1-skeleton of P.

Case 1: x, y belong to parallel edges of P. We have without loss of generality

$$x = (x_1, 0, \ldots, 0),$$
$$y = (y_1, a_1, \ldots, a_i, 0, \ldots, 0).$$

Then we choose as third and fourth point

$$u = (y_1, 0, \ldots, 0),$$
$$v = (x_1, a_1, \ldots, a_i, 0, \ldots, 0).$$

Indeed, $xuvy$ is a rectangle.

Case 2: x, y belong to two edges of P having a common endpoint. Say without loss of generality that

$$x = (x_1, 0, \ldots, 0),$$
$$y = (0, y_2, 0, \ldots, 0).$$

Then take

$$u = (x_1, 0, a_3, 0, \ldots, 0),$$
$$v = (0, y_2, a_3, 0, \ldots, 0),$$

completing the vertex set of a rectangle $xuvy$.

Case 3: x, y belong to two non-parallel disjoint edges of P. If

$$x = (x_1, 0, \ldots, 0),$$
$$y = (0, \ldots, 0, y_i, a_{i+1}, \ldots, a_d),$$

then we choose

$$u = (x_1, 0, \ldots, 0, a_{i+1}, \ldots, a_d),$$
$$v = (0, \ldots, 0, y_i, 0, \ldots, 0),$$

and again we get the vertices of a rectangle.

Case 4: x, y belong to the same edge of P. This is immediate. □

Contrary to the case of an arbitrary cylinder, the following is true.

Theorem 4.2. *The boundary of any right parallelotope is rq-convex.*

Proof. Take x, y on the boundary of the parallelotope P. We show that they have the rq-property.

If x, y belong to the same facet F, choose their orthogonal projections onto the opposite facet F'; the four points are vertices of a rectangle.

If $x \in F$, $y \in F'$, choose the projection x' of x onto F' and the projection y' of y onto F; we get the rectangle $xx'yy'$.

If x, y belong to two adjacent facets F, F^*, respectively, take the orthogonal projections x^* and y^* of x and y (respectively) onto $F \cap F^*$. We complete the rectangles $xx^*y^*\tilde{y}$ and $yy^*x^*\tilde{x}$. Then, clearly, $\{x, \tilde{x}, y, \tilde{y}\} \subset \mathrm{bd}P$ is a rectangular quadruple. □

Theorem 4.3. *Not every convex cylinder has an rq-convex boundary.*

Proof. Take a cylinder $Z = E \times [0,1] \subset \mathbb{R}^3$, where $E \subset \mathbb{R}^2$ is convex and $\mathrm{bd}E$ is a long ellipse. Choose x on the long axis of $\mathrm{bd}E \times \{1\}$, close to one of its endpoints $\{e\} = \{e_p\} \times \{1\}$, and let $\{e'\} = \{e'_p\} \times \{1\}$ be the other endpoint. Let $\{y_\varepsilon\} = \{e'_p\} \times \{\varepsilon\}$, where $\varepsilon \geq 0$. See Figure 3.

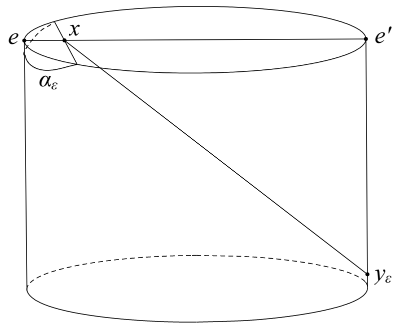

Figure 3: A convex cylinder without rq-convex boundary.

The plane H_{xy_ε} cuts $(\mathrm{bd}E) \times [0,1]$ along an arc α_ε of an ellipse. Let $f(\varepsilon) = d(\{x\}, \alpha_\varepsilon)$. The function f is increasing, and $f(0) > 0$.

The plane $H_{y_\varepsilon x}$ cuts $\mathrm{bd}Z$ along a closed curve (reduced to a single point if $\varepsilon = 0$), of diameter $g(\varepsilon)$. This function g is also increasing, and $g(0) = 0$.

Therefore, for $\varepsilon > 0$ small enough,

$$g(\varepsilon) < f(0) < f(\varepsilon).$$

Choose $y = y_\varepsilon$. The above inequalities show that there is no rectangle $xyy'x'$ with $x', y' \in \mathrm{bd}Z$.

Consider now the sphere C_{xy}. The set $C_{xy} \cap Z$ has four components: a component Z_1 containing x, another one Z_2 containing y, a third Z_3 containing the point e^* diametrically opposite to e' in C_{xy}, and a fourth, $\{e'\}$. It is easily seen that the only pairs of diametrically opposite points in $Z_1 \cup Z_2 \cup Z_3 \cup \{e'\}$ are (x, y) and (e', e^*). But $e^* \in \mathrm{int}Z$, so $\mathrm{bd}Z$ is not rq-convex. □

5 rq-convexity of finite sets

In these last two sections, we shall use the following notation. For $x, y \in \mathbb{R}^d$, we set $W_{xy} = H_{xy} \cup H_{yx} \cup C_{xy}$. Let $\mathcal{A}$ be the family of all finite point sets in $\mathbb{R}^2$.

Theorem 5.1. *For any set $A \in \mathcal{A}$ with $\mathrm{card}A = n \geq 3$, we have $\gamma(A) \leq n^2 - 2n$.*

Proof. If A is included in a line L, consider a line L' parallel to L and the orthogonal projection A' of A onto L'. Then obviously $A \cup A'$ is rq-convex and $\mathrm{card}A' = n \leq n^2 - 2n$, since $n \geq 3$.

If A is not included in any line, let $A = \{a_1, a_2, \ldots, a_n\}$, and assume that a_1a_2 is a side of the polygon $\mathrm{conv}A$. Obviously there are at most $n-2$ lines $L_1, \ldots, L_{n-2}$ passing through the remaining points of A and parallel to $\overline{a_1a_2}$. Also, there are at most n lines $L'_1, \ldots, L'_n$ passing through the points of A orthogonally onto $L_0 = \overline{a_1a_2}$.

The set

$$A' = \bigcup_{0\le i\le n-2;1\le j\le n} (L_i \cap L'_j)$$

is obviously rq-convex and has at most $n(n-1)$ points, whence $\gamma(A) \le n^2 - 2n$. □

Thus, for any $n \ge 3$, $\gamma(n) \le n^2 - 2n$. In particular, $\gamma(3) = 3$.

Theorem 5.2. *There are precisely two kinds of* 6*-point* rq*-convex sets in* $\mathcal{A}$*, shown in Figure 4.*

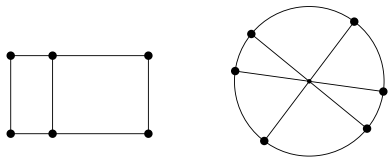

Figure 4: 6-point rq-convex sets.

Proof. Let $F = \{a, b, c, d, e, f\}$ be a 6-point rq-convex set. We assume without loss of generality that $\{a, b, c, d\} \in \mathcal{R}$, where $\angle abc = \frac{\pi}{2}$. By the definition of rq-convexity, e, f must meet one of the following seven conditions.

C_1. $\{\{e, f, a, b\}, \{e, f, c, d\}\} \subset \mathcal{R}$;

C_2. $\{\{e, f, a, c\}, \{e, f, b, d\}\} \subset \mathcal{R}$;

C_3. $\{\{e, f, a, d\}, \{e, f, b, c\}\} \subset \mathcal{R}$;

C_4. $\{\{e, f, a, b\}, \{e, f, a, c\}, \{e, f, a, d\}\} \subset \mathcal{R}$;

C_5. $\{\{e, f, b, a\}, \{e, f, b, c\}, \{e, f, b, d\}\} \subset \mathcal{R}$;

C_6. $\{\{e, f, c, a\}, \{e, f, c, b\}, \{e, f, c, d\}\} \subset \mathcal{R}$;

C_7. $\{\{e, f, d, a\}, \{e, f, d, b\}, \{e, f, d, c\}\} \subset \mathcal{R}$.

Clearly, C_1 and C_3 generate the same kind of set F, and so do C_4, C_5, C_6 and C_7.

Case 1: e, f satisfy C_1. By the definition of rq-convexity, $e, f \in W_{ab} \cap W_{cd}$, and so $e, f \in (H_{ab} \cup H_{ba}) \setminus \{a, b, c, d\}$ and ef, ab are parallel. Without loss of generality, we may suppose $e \in H_{ab}$, $f \in H_{ba}$, which leads to the three solutions depicted in Figure 5, all of them providing a 6-point set of the first type.

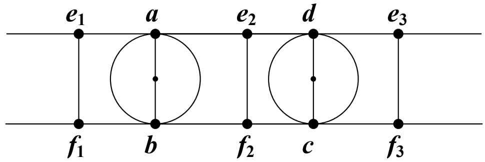

Figure 5: e, f satisfy C_1.

Case 2: e, f satisfy C_2. By the definition of rq-convexity, we have $e, f \in W_{ac} \cap W_{bd}$, so e, f are antipodal points of C_{ac}; see Figure 6.

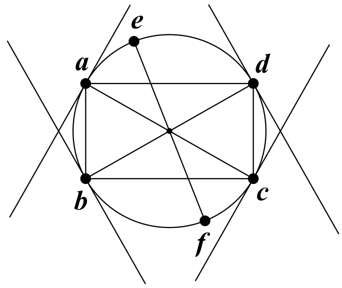

Figure 6: e, f satisfy C_2.

Case 3: e, f satisfy C_4. By the definition of rq-convexity, $(e,\ f) \in W_{ab} \cap W_{ac} \cap W_{ad}$. But $W_{ab} \cap W_{ac} \cap W_{ad} = \{a, b, d\}$, so we obtain no solution in this case. See Figure 7. □

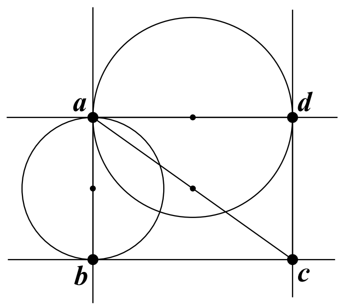

Figure 7: e, f satisfy C_3.

Theorem 5.3. *There are precisely three kinds of* 8*-point* rq*-convex sets, shown in Figure 8.*

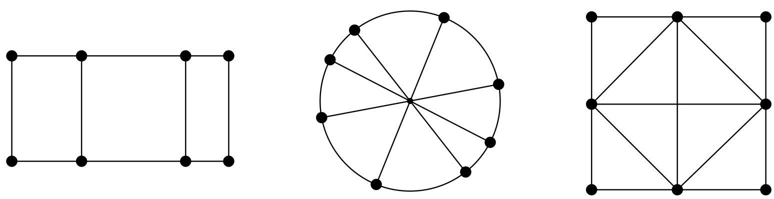

Figure 8: 8-point rq-convex sets.

The proof is ten pages long, so we decided not to include it into the paper. It is a case-by-case examination, treating separately those sets which contain a 6-point rq-convex subset and those which do not. It can be read in [4].

Theorem 5.4. *The smallest odd cardinality of an* rq*-convex set in* $\mathbb{R}^2$ *is* 9.

Proof. It is obvious that every rq-convex set contains a rectangular quadruple and quickly seen that no fifth point can be added to a rectangular quadruple keeping rq-convexity. Similarly, knowing what a 6-point rq-convex set looks like (Theorem 5.2), it is an easy exercise to establish that there is no 7-point rq-convex set containing a 6-point rq-convex set.

Next, we will consider the case that a 7-point rq-convex set does not contain any 6-point rq-convex subset. Let $F = \{a, b, c, d, e, f, g\}$ be such a set. Suppose a, c realise the diameter of F. Since F is rq-convex, there is another pair of antipodal points of C_{ac} in F, say $\{b, d\}$. Hence $\{a, b, c, d\} \in \mathcal{R}$, and put $\operatorname{conv}\{a, b, c, d\} = R$.

For the set of points $\{x, y\} \subset \{e, f, g\}$, if there exist $z, w \in \{a, b, c, d\}$ such that $\{w, x, y, z\} \in \mathcal{R}$, then we say that $\{x, y\}$ is *rq-good*. Next we will prove that for any two points $x, y \in \{e, f, g\}$, $\{x, y\}$ is not rq-good. Suppose $\{e, f\}$ is rq-good. Then there exist $z, w \in \{a, b, c, d\}$, such that $\{e, f, z, w\} \in \mathcal{R}$.

Case 1: zw is a diagonal of R. Without loss of generality, we assume $zw = ac$, so $e, f \in W_{ac}$. Since $\{a, c\}$ realise the diameter of F, e, f are antipodal on C_{ac}. But so we obtain a 6-point rq-convex subset of F, which contradicts our assumption about F.

Case 2: zw is an edge of R. We assume without loss of generality that $zw = ab$. Then $e, f \in W_{ab}$. Clearly, e, f must be antipodal points of C_{ab}. Take a diameter a_0b_0 of C_{ab} orthogonal to $\overline{ab}$, such that $\overline{ab}$ separates a_0 from cd. We may suppose that e belongs to the (smaller) arc of C_{ab} from a to a_0; see Figure 9. If $\|a-b\| \leq \|b-c\|$, then $\|c-e\| > \|c-a\|$,

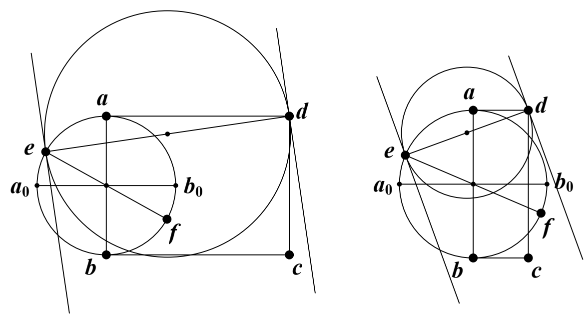

Figure 9: zw is an edge of R.

which is impossible. So, $\|a - b\| > \|b - c\|$.

As F is rq-convex, $\{e, d\}$ enjoys the rq-property and there exist $p, q \in \{a, b, c, f, g\}$ such that $p, q \in W_{ed}$. Clearly, $a, b, c \notin H_{de}$. Since $d \notin C_{ab}$, we have $f \notin H_{de}$. Further, we easily verify that $a, b, c, f \notin H_{ed} \cup H_{de}$. It follows that p, q are two antipodal points of C_{ed}. Since $\angle dae > \frac{\pi}{2}$, $\angle dce < \frac{\pi}{2}$, we get $a, c \notin C_{ed}$. Hence, $p = b \in C_{ed}$ or $p = f \in C_{ed}$, and $q = g$.

(i) $p = b \in C_{ed}$. We only can choose g such that b, g are antipodal points of C_{ed}. As $\angle ebf = \frac{\pi}{2}$, b, f, d are collinear. But then we get a 6-point rq-convex set $\{e, b, a, f, g, d\} \subset F$, contradicting our choice of F; see Figure 10.

(ii) $p = f \in C_{ed}$. Now, f, g are antipodal points of C_{ed}. Hence, $efdg$ is a rectangle. The points g and b are separated by $\overline{ad}$, or $a \in dg$ if $e = a_0$. Also,

$$\|d - g\| = \|e - f\| = \|a - b\| > \|b - c\| = \|a - d\|.$$

It follows that $\|c - g\| > \|c - a\|$, and a contradiction is obtained. See Figure 11. Hence,

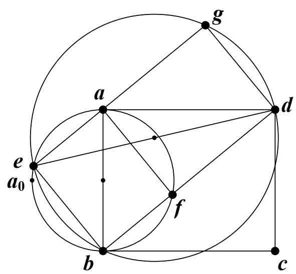

Figure 10: $p = b \in C_{ed}$.

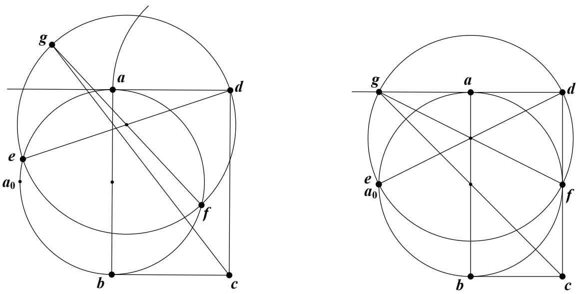

Figure 11: $p = f \in C_{ed}$.

$\{e, f\}$ is not rq-good.

Since F is rq-convex, we must have $\{e, f, g, a\}$, $\{e, f, g, b\}$, $\{e, f, g, c\}$, $\{e, f, g, d\} \in \mathcal{R}$, which is true only for $a = b = c = d$, which is impossible. Thus, there is no 7-point rq-convex set.

On the other hand, a 9-point rq-convex set is easily produced: just take the intersection $L \cap L'$, where L is the union of three horizontal lines and L' the union of three vertical lines. □

Consider now the square lattice $\mathbb{Z}^2$, and the usual norm

$$\|(x, y)\|_m = \max\{|x|, |y|\},$$

defining in $\mathbb{Z}^2$ the discs of radius $n \in \mathbb{Z}$

$$Q(n) = \{(x, y) : \|(x, y)\|_m \leq n\},$$

centred at the origin $\mathbf{0}$. The subset $Q(n) \setminus Q(n-1)$ will be called *boundary of* $Q(n)$, and $Q(n-1)$ its *interior*. Obviously, $Q(n)$ is rq-convex, for any $n \geq 1$, and so is its boundary too. Does $Q(n)$ remain rq-convex if one deletes parts of its interior (but not all of it)?

Theorem 5.5. *The set $Q(n) \setminus \{\mathbf{0}\}$ is rq-convex.*

Proof. For any pair of points $x = (x_1, x_2)$, $y = (y_1, y_2)$ in $Q(n) \setminus \{\mathbf{0}\}$, consider the points $(x_1, y_2), (y_1, x_2) \in \mathbb{Z}^2$. If none of them is $\mathbf{0}$, the rq-propertyis verified at (x, y), as $(x_1, y_2), (y_1, x_2) \in Q(n)$.

Otherwise, assume without loss of generality that $x_1 = y_2 = 0$. We can also assume that both x_2, y_1 are positive, the other cases being symmetrical. Consider the points $x' = (-x_2, x_2 - y_1)$ and $y' = (y_1 - x_2, -y_1)$. Then $x', y' \in Q(n) \setminus \{\mathbf{0}\}$ and x, y, y', x' are the vertices of a square. □

Perhaps removing several layers of the boundary, thereby giving a set $Q(n) \setminus Q(m)$ for $m < n - 1$, will provide an rq-convex set?

Theorem 5.6. *The set $Q(n) \setminus Q(n-2)$ is not rq-convex, for any $n \geq 3$.*

Proof. Assume first that $n > 3$. Consider the points $x = (n, 2-n)$ and $y = (n-3, n-1)$. The point $(n-3, 2-n)$ does not belong to $Q(n) \setminus Q(n-2)$. The line H_{yx} meets $Q(n)$ again at $y' = (-n, n-4)$. But the fourth vertex $x' = (3-n, -n-1)$ of the square $xyy'x'$ lies outside $Q(n)$, whence $Q(n) \setminus Q(n-2)$ is not rq-convex.

Now, consider the set $Q(3) \setminus Q(1)$. In this case take the points $x = (3, -1)$, $y = (-1, 2)$. As $(-1, -1) \notin Q(3) \setminus Q(1)$ and $H_{yx} \cap Q(3) \setminus Q(1) = \emptyset$, the result is proven. □

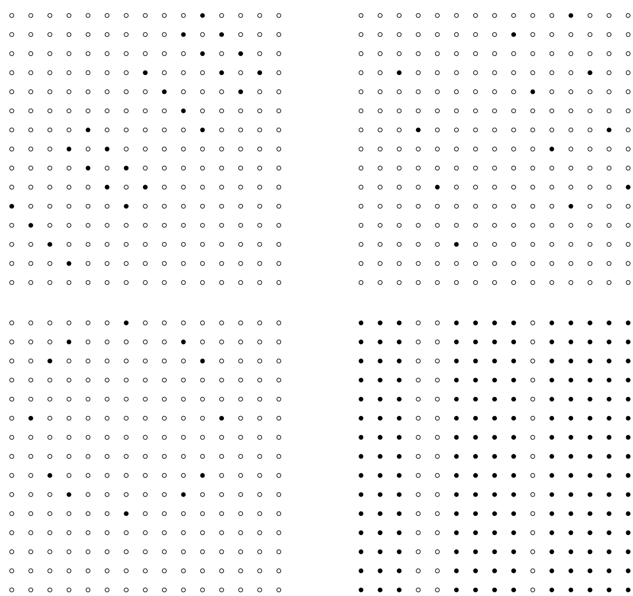

Figure 12: rq-convex proper subsets of $Q(n)$.

It seems that $Q(n) \setminus \{\mathbf{0}\}$ is the only rq-convex proper subset of $Q(n)$ containing the boundary of $Q(n)$ but different from it. However, this is not proven. Other proper subsets of $Q(n)$ which are rq-convex abound. For some examples, see Figure 12, where the solid black dots form rq-convex proper subsets of $Q(n)$.

6 rq-convexity of the vertex sets of Platonic solids

Due to their symmetry, the vertex sets of the cube, regular octahedron, regular dodecahedron, and regular icosahedron are all rq-convex. Among the Platonic solids, only the regular tetrahedron lacks this property. But what is the rq-convex completion number of the vertex set of the regular tetrahedron?

Theorem 6.1. *The rq-convex completion number of the vertex set of the regular tetrahedron is* 3.

Proof. Let $T = \{a, b, c, d\}$ denote the vertex set of a regular tetrahedron in $\mathbb{R}^3$. Obviously, for any $x,\ y \in T$, we have $T \cap W_{xy} = \{x, y\}$. Also, it is easily seen that there is no 5-point rq-convex set containing T. Suppose there is a 6-point rq-convex set $\{a, b, c, d, x, y\}$. The only suitable pair of points $x,\ y \in W_{ab} \cap W_{cd}$ is obtained when $\{x, y\} = (H_{ab} \cap C_{cd}) \cup (H_{ba} \cap C_{cd})$. But then b, c do not enjoy the rq-property in $\{a, b, c, d, x, y\}$. Hence $\gamma(T) \geq 3$.

Next, we only need to prove $\gamma(T) \leq 3$. Let $a_1,\ b_1,\ c_1$ denote the midpoints of $ad,\ bd,\ cd$, respectively. See Figure 13. The line $L_a \ni a_1$ parallel to $\overline{bc}$ and the anal-

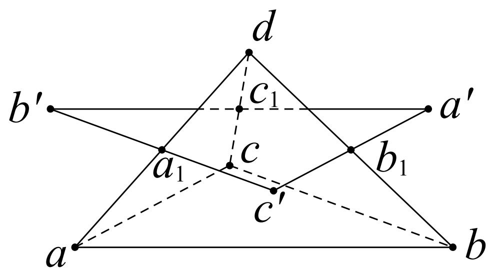

Figure 13: A 7-point rq-convex set containing the vertices of a regular tetrahedron.

ogous lines L_b and L_c are coplanar. Put

$$\{a'\} = L_b \cap L_c, \quad \{b'\} = L_c \cap L_a, \quad \{c'\} = L_a \cap L_b.$$

Obviously, $ab'dc', bc'da', ca'db'$ are squares, while $a'b'ab, b'c'bc, c'a'ca$ are rectangles. Thus, $\{a, b, c, d, a', b', c'\}$ is a 7-point rq-convex set, and $\gamma(T) = 3$. $\square$

Theorem 6.1 reveals the existence of 7-point rq-convex sets in $\mathbb{R}^3$, in contrast with the inexistence of such sets in $\mathbb{R}^2$. What happens in higher dimensions?

References

[1] R. Blind, G. Valette and T. Zamfirescu, Rectangular convexity, *Geom. Dedicata* **9** (1980), 317–327, doi:10.1007/bf00181177.

[2] K. Böröczky, Jr., Rectangular convexity of convex domains of constant width, *Geom. Dedicata* **34** (1990), 13–18, doi:10.1007/bf00150684.

[3] B. Grünbaum and G. C. Shephard, *Tilings and Patterns*, Dover Publications, New York, 2nd edition, 2016.

[4] D. Li, L. Yuan and T. Zamfirescu, 8-point rq-convex sets, 2016, `http://sxxy.hebtu.edu.cn/upload/fckeditor/2015-10/30112750.pdf`.

[5] E. Magazanik and M. A. Perles, Staircase connected sets, *Discrete Comput. Geom.* **37** (2007), 587–599, doi:10.1007/s00454-007-1308-9.

[6] L. Yuan and T. Zamfirescu, Right triple convex completion, *J. Convex Anal.* **22** (2015), 291–301, http://www.heldermann.de/JCA/JCA22/JCA221/jca22014.htm.

[7] L. Yuan and T. Zamfirescu, Right triple convexity, *J. Convex Anal.* **23** (2016), 1219–1246, http://www.heldermann.de/JCA/JCA23/JCA234/jca23046.htm.

[8] T. Zamfirescu, Right convexity, *J. Convex Anal.* **21** (2014), 253–260, http://www.heldermann.de/JCA/JCA21/JCA211/jca21014.htm.

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Game distinguishing numbers of Cartesian products

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Abstract

The distinguishing number of a graph H is a symmetry related graph invariant whose study started two decades ago. The distinguishing number $D(H)$ is the least integer d such that H has a distinguishing d-coloring. A distinguishing d-coloring is a coloring $c : V(H) \to \{1, \ldots, d\}$ invariant only under the trivial automorphism. In this paper, we continue the study of a game variant of this parameter, recently introduced. The distinguishing game is a game with two players, Gentle and Rascal, with antagonistic goals. This game is played on a graph H with a fixed set of $d \in \mathbb{N}$ colors. Alternately, the two players choose a vertex of H and color it with one of the d colors. The game ends when all the vertices have been colored. Then Gentle wins if the d-coloring is distinguishing and Rascal wins otherwise. This game defines two new invariants, which are the minimum numbers of colors needed to ensure that Gentle has a winning strategy, depending who starts the game. The invariant could be infinite. In this paper, we focus on the Cartesian product, a graph operation well studied in the classical case. We give sufficient conditions on the order of two connected factors H and F that are relatively prime, which ensure that one of the game distinguishing numbers of the Cartesian product $H\Box F$ is finite. If H is a so-called involutive graph, we give an upper bound of order $D^2(H)$ for one of the game distinguishing numbers of $H\Box F$. Finally, using in part the previous result, we compute the exact value of these invariants for Cartesian products of relatively prime cycles. It turns out that the value is either infinite or equal to 2, depending on the parity of the product order.

Keywords: Distinguishing number, graph automorphism, combinatorial game.

Math. Subj. Class.: 05C57, 05C69, 91A43

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1 Introduction

In this paper, we consider only simple graphs. For a graph H, the sets $V(H)$ and $E(H)$ respectively denote the vertex set and the edge set of H. For an integer $n \geq 3$, we denote by C_n the cycle of order n. For $n \geq 2$, we respectively write K_n and P_n for the clique and the path of order n. The distinguishing number $D(H)$ of a graph H is a symmetry related graph invariant introduced two decades ago [1]. More precisely, $D(H)$ is the least integer d such that H has a distinguishing d-coloring. A distinguishing d-coloring is a vertex-coloring $c : V(H) \to \{1, \dots, d\}$, invariant only under the trivial automorphism. More generally, we say that an automorphism σ of H preserves the coloring c, or is a color preserving automorphism, if for all $u \in V(H)$, we have $c(u) = c(\sigma(u))$. The automorphism group of H will be denoted by $\mathrm{Aut}(H)$. Clearly, for each coloring c of the vertex set of H, the set $\mathrm{Aut}_c(H) = \{\sigma \in \mathrm{Aut}(H) : c \circ \sigma = c\}$ is a subgroup of $\mathrm{Aut}(H)$. A coloring c is distinguishing if $\mathrm{Aut}_c(H)$ is trivial. The ten last years have seen a flourishing number of works on this subject, and Cartesian products of graphs were thoroughly investigated in [2, 5, 7, 8, 12, 14, 15, 16]. In particular, the exact value of $D(K_n\Box K_m)$ is given in [7, 14]. Another interesting result for our purpose is that if $k \geq 2$, then $D(C_{n_1}\Box \cdots \Box C_{n_k}) = 2$, except for $C_3\Box C_3$. In that case, $D(C_3\Box C_3) = 3$. This result is an easy consequence of more general results in [14]. Cartesian products have also been investigated in the context of distinguishing edge-coloring [4, 10]. In this paper, we are interested in a game variant of the distinguishing number, introduced in [11]. Defining game invariants for graphs is not a new idea. The two most known game invariants are the game chromatic number, introduced by Brahms [6] in 1981, and the game domination numbers introduced more recently by Brešar, Klavžar and Rall [3].

The distinguishing game is a game with two players, Gentle and Rascal, with antagonistic goals. This game is played on a graph H with a fixed set of d colors, $d \in \mathbb{N}$. Alternately, the two players choose a vertex of H and color it with one of the d colors. The game ends when all the vertices have been colored. If the d-coloring is distinguishing then Gentle wins. Otherwise Rascal wins.

This game defines two invariants for a graph H. The *$\mathcal{G}$-game distinguishing number* $D_{\mathcal{G}}(H)$ is the minimum number of colors needed to ensure that Gentle has a winning strategy for the game on H, assuming he is playing first. If Rascal is sure to win whatever the number of colors we allow, then $D_{\mathcal{G}}(H) = \infty$. Similarly, the *$\mathcal{R}$-game distinguishing*

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number $D_{\mathcal{R}}(H)$ is the minimum number of colors needed to ensure that Gentle has a winning strategy, assuming Rascal is playing first. Characterizing graphs with infinite game distinguishing number seems to be a challenging open question. In [11], the authors give sufficient conditions for a graph to have one infinite game distinguishing number.

Proposition 1.1. *[11] Let H be a graph and σ a nontrivial automorphism of H such that $\sigma \circ \sigma = \mathrm{id}_H$.*

1. *If $|V(H)|$ is even, then $D_{\mathcal{G}}(H) = \infty$.*
2. *If $|V(H)|$ is odd, then $D_{\mathcal{R}}(H) = \infty$.*

Also in [11], the exact values of those invariants have been computed for many cycles and hypercubes. For a large class of graphs, the so-called involutive graphs, a quadratic upper bound involving the classical distinguishing number has been provided (see Section 4 for the definition). We give here the precise statement of the results used in this paper.

Theorem 1.2. *[11] Let C_n be a cycle of order $n \geq 3$.*

1. *If n is even (resp. odd), then $D_{\mathcal{G}}(C_n) = \infty$ (resp. $D_{\mathcal{R}}(C_n) = \infty$).*
2. *If n is even and $n \geq 8$, then $D_{\mathcal{R}}(C_n) = 2$.*
3. *If n is odd, not prime and $n \geq 9$, then $D_{\mathcal{G}}(C_n) = 2$.*
4. *If n is prime and $n \geq 5$, then $D_{\mathcal{G}}(C_n) \leq 3$.*

Moreover $D_{\mathcal{R}}(C_4) = D_{\mathcal{R}}(C_6) = 3$, $D_{\mathcal{G}}(C_3) = \infty$ and $D_{\mathcal{G}}(C_5) = D_{\mathcal{G}}(C_7) = 3$.

Theorem 1.3. *[11] If H is an involutive graph with $D(H) \geq 2$, then $D_{\mathcal{R}}(H) \leq D^2(H) + D(H) - 2$.*

In this paper, we deal with Cartesian products of connected relatively prime graphs. In Section 3, we prove the following theorem which gives sufficient conditions on the order of the two factors to have a finite distinguishing number.

Theorem 1.4. *Let H and F be two nontrivial connected relatively prime graphs of order n and m, respectively.*

1. *If n is even and $m \geq n-1$, then $D_{\mathcal{R}}(H \Box F) \leq m+1$.*
2. *If n is odd, m is even and $m \geq 2n-2$, then $D_{\mathcal{R}}(H \Box F) \leq m+1$.*
3. *If n and m are odd and $m \geq 2n-1$, then $D_{\mathcal{G}}(H \Box F) \leq m+1$.*

In Section 4, we investigate the case where one factor is an involutive graph. In that case, if the classical distinguishing number of the other factor is not too big, we have a quadratic upper bound involving the classical distinguishing number of the involutive factor.

Theorem 1.5. *Let H be a connected involutive graph of order n, with $D(H) \geq 2$ and F a connected graph, relatively prime to H. If $D(F) \leq \begin{pmatrix} \frac{n+d^2+d}{2} - 1 \\ \frac{d^2+d}{2} - 1 \end{pmatrix}$, then $D_{\mathcal{R}}(H \Box F) \leq d^2 + d - 2$, where $d = D(H)$.*

Finally, in Section 5, we compute the exact value of the two invariants for Cartesian products of relatively prime cycles. Since even cycles are involutive graphs, a part of this result arises as a corollary of the above theorem.

Theorem 1.6. *Let $n_1, \dots, n_k$, with $k \geq 2$, be k distinct natural numbers greater or equal to* 3.

1. *If $\prod_{i=1}^{k} n_i$ is even, then $D_{\mathcal{G}}(C_{n_1} \Box \cdots \Box C_{n_k}) = \infty$ and $D_{\mathcal{R}}(C_{n_1} \Box \cdots \Box C_{n_k}) = 2$.*

2. *If $\prod_{i=1}^{k} n_i$ is odd, then $D_{\mathcal{R}}(C_{n_1} \Box \cdots \Box C_{n_k}) = \infty$ and $D_{\mathcal{G}}(C_{n_1} \Box \cdots \Box C_{n_k}) = 2$.*

All these three results highly involve the so-called fiber-strategy for Gentle. Section 2 is devoted to the definition and the properties of this strategy.

2 Cartesian products of graphs and the fiber-strategy

In this section, we give the minimal background needed on Cartesian products and define an efficient strategy for Gentle, the so-called fiber-strategy, based on the fibers structure of Cartesian products of graphs. For more information on Cartesian product see [13].

2.1 Cartesian products of graphs

Let H and F be two connected relatively prime graphs. The vertices of $H \Box F$ will be denoted by (u, v), where $u \in V(H)$ and $v \in V(F)$. An H-fiber is a subgraph of $H \Box F$ induced by all the vertices having the same second coordinate. We write H^v, where $v \in V(F)$, for the H-fiber induced by $\{(u, v) | u \in V(H)\}$. Similarly, we define F^u, with $u \in V(H)$. The H-fibers and the F-fibers are respectively isomorphic to H and F. The automorphism group of $H \Box F$ is isomorphic to $\mathrm{Aut}(H) \times \mathrm{Aut}(F)$. If σ is an automorphism of $H \Box F$, it can be seen as a couple (ψ, ϕ), where $\psi \in \mathrm{Aut}(H)$ and $\phi \in \mathrm{Aut}(F)$. In that case, $\sigma((u, v)) = (\psi(u), \phi(v))$. Another important fact is that σ must send an H-fiber to another H-fiber and the same for the F-fibers. More precisely, $\sigma(H^v) = H^{\phi(v)}$ and $\sigma(F^u) = F^{\psi(u)}$. To show that a color preserving automorphism has to be the identity of $\mathrm{Aut}(H \Box F)$, we will mostly proceed as follows. First, we show that an H-fiber cannot be sent to another one, which means that ϕ is the identity and σ fixes the H-fibers setwise. Using these informations, we prove that ψ is also the identity.

2.2 Fiber-strategy

Now, we state some technical results about the fiber-strategy, a strategy that Gentle will follow in almost all the proofs of our main results. In a game on $H \Box F$, with H nontrivial, we say that Gentle follows the *H-fiber-strategy* if we are in one of the following two cases.

Case 0:

- *$|V(H)|$ is even and Rascal starts.*
- *When Rascal plays in an H-fiber, Gentle plays in the same H-fiber.*

Case 1:

- *$|V(H)|$ is odd.*
- *$|V(F)|$ is even and Rascal starts or $|V(F)|$ is odd and Gentle starts.*
- *When Rascal colors the first vertex of a totally uncolored H-fiber, Gentle colors the first vertex of another such H-fiber.*
- *When Rascal colors a vertex in an H-fiber which already has a colored vertex, Gentle colors a vertex in the same H-fiber.*

The H-fiber strategy is always valid. In Case 0, the parity of each H-fiber ensures that Rascal will always be the first to run out of moves in an H-fiber. Hence, Rascal is always the first to play in each H-fiber. In Case 1, after Gentle's move, there is always an even number of remaining totally uncolored H-fibers. Hence, Rascal will always be the first to run out of new totally uncolored H-fibers to play in.

The following properties are easy and given without proof. They are however fundamental to prove the results in the further sections.

Proposition 2.1. *Assume that Gentle plays according to the H-fiber-strategy.*

1. *He will color the last vertex of each H-fiber.*
2. *In Case 0, Rascal will be the first to play in all the H-fibers. Then, Gentle will play all the second moves in each H-fiber, Rascal will play all the third moves and so on.*
3. *In Case 1, Gentle will play the first in exactly $\left\lceil \frac{|V(H)|}{2} \right\rceil$ different H-fibers. Then, Rascal will play all the second moves in each H-fiber, Gentle will play all the third moves and so on.*

In Case 0, the moves in an H-fiber alternate exactly as in the game played only on H, when Rascal starts (see Fig. 1, where R_i and G_i respectively denote the i-th move of Rascal and Gentle). This property will be often used by Gentle to play in an H-fiber following a winning strategy for the game on H. In Case 1, in an H-fiber where Gentle plays first, the moves alternate as in the game played only on H, when Gentle starts. In an H-fiber where Rascal plays first, the only difference is that he is going to play the two first moves in a row (see Fig. 2, where R_i and G_i have the same meaning as before). In that case the Lemma 2.2 could be useful. It says that for vertex transitive graphs, if you can win the game playing first, then you can be a real gentleman and let your opponent play this first move.

Lemma 2.2. *Assume H is vertex transitive. Then either all the first moves are winning for the first player or they are all losing.*

Proof. Assume there is $u_0 \in V(H)$ such that coloring u_0 with 1 is a winning move for the first player. We have to prove that for any $v \in V(H)$, coloring v with 1 is also a winning move. Let G be the game in which the first player has played the winning move u_0 and let G' be the game in which his first move has been to color another vertex v_0 with 1. Let c and c' be respectively the coloring built during the game G and G'. Since H is vertex transitive, there exists $\sigma \in \mathrm{Aut}(H)$ such that $\sigma(v_0) = u_0$. The winning strategy for the first player in G' is defined by his winning strategy in G. When his opponent colors a vertex w in the game G', the first player imagines that his opponent has colored $\sigma(w)$ in G with the same color. There is a vertex w' such that coloring w' is a winning answer for the

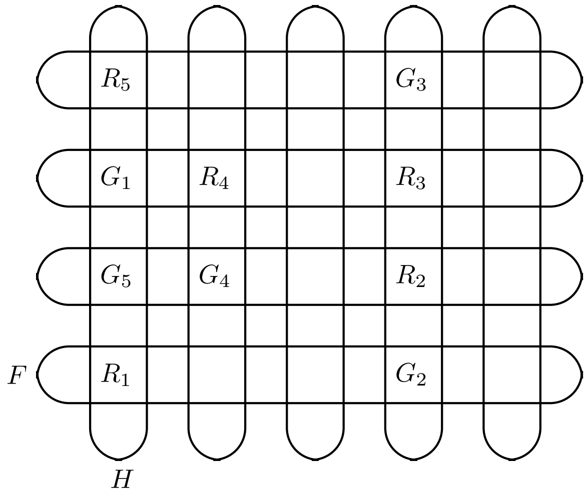

Figure 1: How moves alternate in Case 0 of the H-fiber strategy (Rascal starts).

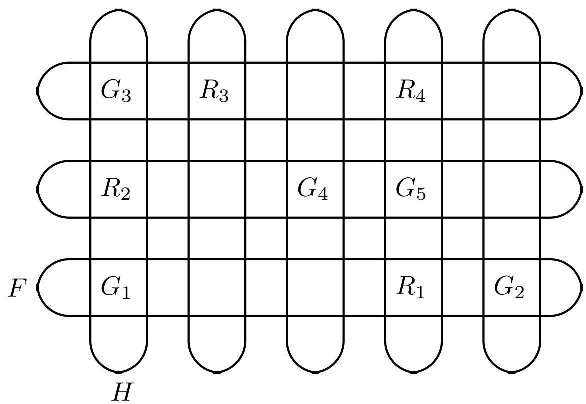

Figure 2: How moves alternate in Case 1 of the H-fiber strategy (Gentle starts).

first player in G. In the game G', the first player's answer will be to color $\sigma^{-1}(w')$ such that $c'(\sigma^{-1}(w')) = c(w')$.

By assumption, the coloring c is a winning one for the first player. Moreover, for any $v \in V(H)$, $c(v) = c'(\sigma(v))$. Hence, an automorphism ψ preserves the coloring c if and only if $\sigma \circ \psi \circ \sigma^{-1}$ preserves the coloring c'. This shows that c' is also a winning coloring for the first player. In conclusion, if there is a winning move for the first player, then any first move is a winning move for him. $\square$

3 Cartesian products of complete graphs.

Our goal in this section is to prove Theorem 1.4 which asserts, under certain conditions on their orders, that for two nontrivial connected graphs H and F that are relatively prime, and of respective order n and m, at least one game distinguishing number of $H\Box F$ is finite. Except when both cardinalities are equal, it comes directly from the following theorem involving Cartesian products of complete graphs. Indeed, a distinguishing coloring of $K_n\Box K_m$ is always a distinguishing coloring of $H\Box F$. In the first item of Theorem 1.4, when both factors have the same even cardinality, the corresponding product of complete graphs is not covered by the below result. But, we are in fact going to prove that Gentle's strategy breaks all automorphisms of the subgroup of $\mathrm{Aut}(K_n\Box K_m)$ isomorphic to $\mathrm{Aut}(K_n) \times \mathrm{Aut}(K_m)$. A coloring which distinguishes this subgroup will always be a distinguishing coloring of $H\Box F$, when the factors are relatively prime.

Theorem 3.1. *Let n and m be two distinct natural numbers greater or equal to 2.*

1. *If $n \times m$ is even (resp. odd), then $D_{\mathcal{G}}(K_n\Box K_m) = \infty$ (resp. $D_{\mathcal{R}}(K_n\Box K_m) = \infty$).*
2. *If n is even, $m \neq n$ and $m \geq n-1$, then $D_{\mathcal{R}}(K_n\Box K_m) \leq m+1$.*
3. *If n is odd, m is even and $m \geq 2n-2$, then $D_{\mathcal{R}}(K_n\Box K_m) \leq m+1$.*
4. *If n and m are odd and $m \geq 2n-1$, then $D_{\mathcal{G}}(K_n\Box K_m) \leq m+1$.*

Proof. The first item is a straightforward application of Proposition 1.1. For the last items, note that $n \neq m$. Hence, the two factors K_n and K_m are relatively prime. The vertices of $K_n\Box K_m$ are denoted by (i,j), with $i \in \{1,\ldots,n\}$ and $j \in \{1,\ldots,m\}$. The meta-color of a K_n-fiber is the list $(c_1,\ldots,c_{m+1})$, where c_l, with $l \in \{1,\ldots,m+1\}$, is the number of vertices in this fiber which are colored with the color l at the end of the game. An important remark is that a color preserving automorphism also preserves the meta-colors of the K_n-fibers.

We are going to prove the second statement. We have to give a winning strategy for Gentle playing second with $m+1$ colors. A proper edge coloring of K_n, with $n-1$ colors gives $n-1$ perfect matchings, whose union covers all the edges of K_n. We denote these matchings by $M_1,\ldots,M_{n-1}$. Gentle's winning strategy is as follows. First of all, he plays according to the K_n-fiber-strategy. If Rascal plays in one of the fibers K_n^j, with $j \in \{1,\ldots,n-1\}$, Gentle plays with respect to the matching M_j. It means that if Rascal colors the vertex (i,j), then Gentle colors the unique vertex (k,j) in K_n^j, such that ik is an edge in the matching M_j. Otherwise, he plays as he wants with respect to the K_n-fiber-strategy. See Fig. 3, where R_i and G_i respectively denote the i-th move of Rascal and Gentle. Gentle chooses the colors as follows. First, he always plays a color different from the one used by Rascal just before. Second, if he has to color the last vertex of a K_n-fiber, he chooses the color in a way that the meta-color of this fiber is distinct from all the meta-colors of the already totally colored K_n-fibers. He has at most $m-1$ meta-colors to avoid. It is always possible because he can choose among the m colors not used by Rascal just before.

Let us prove now that this strategy yields a distinguishing coloring c. Applying the above strategy, Gentle will color the last vertex of each K_n-fiber (see Proposition 2.1). Hence, he controls the meta-color of all the K_n-fibers. They will all have a distinct meta-color at the end of the game. Therefore, a color preserving automorphism σ cannot switch

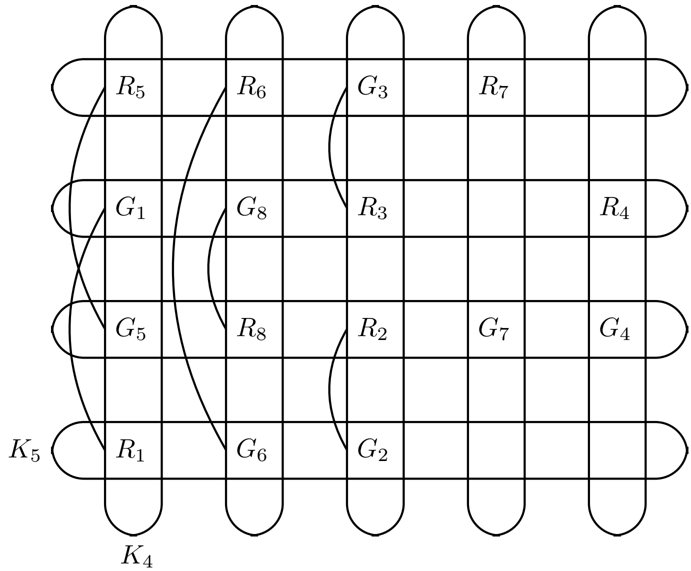

Figure 3: Playing according to the matchings in a K_4-fiber strategy.

these fibers. It means that $\sigma = (\psi, \mathrm{id}_{K_m})$, where $\psi \in \mathrm{Aut}(K_n)$. Assume that ψ is not the identity. There exists $i \in \{1, \dots, n\}$, such that $\psi(i) \neq i$. The edge $\psi(i)i$ belongs to a matching M_j, with $j \in \{1, \dots, n-1\}$. Since Gentle has colored either (i, j) or $(\psi(i), j)$, these two vertices have not the same color. It shows that the automorphism $(\psi, \mathrm{id}_{K_m})$ does not preserve the coloring c. In conclusion, c is a distinguishing coloring.

We prove now the third item. The general ideas are similar as above, but since the K_n-fibers have odd order, a matching does not cover all the vertices of K_n. Hence, if Rascal is the first to play in a K_n-fiber, Gentle cannot immediately play with respect to a matching. Since n is odd, we need n matchings to cover all the edges of K_n. Each matching does not cover exactly one vertex and this uncovered vertex is distinct for each of them. We denote by M_j, with $j \in \{1, \dots, n\}$, the matching which does not cover the vertex j. Without lost of generality, we assume that Rascal's first move is to color the vertex $(1, 1)$ in K_n^1. Gentle will again follow a K_n-fiber strategy. Hence, he will be the first to color a vertex in exactly $\frac{m}{2}$ different K_n-fibers, say $K_n^2, \dots, K_n^{\frac{m}{2}+1}$. When Gentle colors a vertex of K_n^j, with $j \in \{1, \dots, \frac{m}{2} + 1\}$, if it is the first vertex of K_n^j to be colored, he chooses the vertex (j, j). Otherwise, he plays with respect to the matching M_j. When the fiber-strategy leads him to play in other K_n-fibers, he plays wherever he wants with respect to the K_n-fiber strategy. For the choice of the colors, he plays as in the previous strategy. The proof that the coloring built during the game is distinguishing is exactly the same. Just note that by hypothesis, $\frac{m}{2} + 1 \geq n$. Hence, Gentle has enough K_n-fibers to use the n matchings needed to cover all the edges of K_n.

The proof of the last item is the same as for the previous one. Because Gentle starts, he will be the first to play in $\frac{m+1}{2}$ different K_n-fibers, which is by hypothesis greater or equal to n. $\square$

For $K_2\Box K_m$, we can compute the exact value of $D_{\mathcal{R}}$. For $m \in \{2,3,4\}$, it is shown in [11] that $D_{\mathcal{R}}(K_2\Box K_m) = 3$. For $m \geq 5$, we are going to prove that we need exactly m colors. Then, the bound obtained above is close to be tight.

Let c be a coloring of $K_2\Box K_m$, with $m \geq 5$. We say that two distinct K_2-fibers, K_2^i and K_2^j are colored the same if $c(K_2^i) = c(K_2^j)$. If we have also $c((i,1)) = c((j,1))$, we say that the two fibers are strictly colored the same.

Proposition 3.2. *If $m \geq 5$, then $D_{\mathcal{R}}(K_2\Box K_m) = m$.*

Proof. First, we show that with m distinct colors Gentle has a winning strategy. Recall that Rascal starts. When Rascal plays in a K_2-fiber, Gentle answers by coloring the second vertex of this K_2-fiber. That means he plays according to a K_2-fiber strategy. He colors in a way that the new colored K_2-fiber is not colored the same as another K_2-fiber already colored before. This is always possible, because there are at most $m-1$ different K_2-fibers colored before and Gentle can use m colors. Moreover, he can ensure that at least one K_2-fiber is not monochromatic. Let us prove now that this strategy yields a distinguishing coloring. Assume σ is a color preserving automorphism. Then $\sigma(K_2^i) = K_2^i$, for all $i \in \{1,\dots,m\}$. But, there is at least one bi-chromatic K_2-fiber. Hence, σ must also fix this K_2-fiber pointwise. Therefore, σ has to fix all the K_2-fibers pointwise. In conclusion, σ is the identity.

It remains to prove that Rascal has a winning strategy, if strictly less than m colors are allowed during the game. Remark that, if two distinct K_2-fibers, K_2^i and K_2^j are strictly colored the same at any moment in the game, then Rascal wins. Indeed, there is an automorphism σ such that $\sigma((i,1)) = (j,1)$, $\sigma((i,2)) = (j,2)$ and σ fixes all the other vertices.

Rascal starts by coloring $(1,1)$ with 1. There are two cases.

Case 1: Gentle colors the vertex $(1,2)$.

Rascal answers by coloring $(2,1)$ with 1. If Gentle colors a vertex different than $(2,2)$, Rascal wins at his next turn by coloring $(2,2)$ with the same color as $(1,2)$. So, we can suppose that Gentle colors $(2,2)$. Turn by turn, this shows that Rascal can color all the vertices of the form $(i,1)$ with the color 1, and that Gentle is forced to color only the vertices of the form $(i,2)$. Since Gentle has strictly less than m colors at his disposal, there are two vertices $(i_0,2)$, $(j_0,2)$, which receive the same color. Hence, the two K_2-fibers, $K_2^{i_0}$ and $K_2^{j_0}$ will be strictly colored the same and Rascal will win.

Case 2: Gentle first move is to color the vertex $(2,x)$, with $x \in \{1,2\}$.

Rascal answers by coloring with 1 the vertex $(3,1)$. Now, if Gentle plays in K_2^1 or K_2^3, Rascal wins because he can play such that K_2^1 and K_2^3 are strictly colored the same. Suppose that Gentle plays a vertex which is not in K_2^1 or K_2^3. Since $m \geq 5$, at least one vertex in the fiber K_m^1 is still uncolored, say $(5,1)$. Rascal replies by coloring this vertex with 1. The vertices $(1,2)$, $(3,2)$ and $(5,2)$ are still uncolored. Rascal can ensure that at least two of the three K_2-fibers, K_2^1, K_2^3 and K_2^5 are strictly colored the same. Indeed, if Gentle is the first to color one of these three uncolored vertices, Rascal copies this color in one of the two remaining vertices. Otherwise, he will be able to decide the coloring of two of them. In conclusion, Rascal will also win in this second case. □

Of course, Theorem 3.1 does not cover all possibilities. We did not manage to prove that in the remaining cases the invariants are finite. But we know that the K_n-fiber strategy used above by Gentle will fail in these cases. More precisely, we have the following proposition.

Proposition 3.3. *Let n and m be two distinct natural numbers at least equal to 2. Whatever the number of colors is allowed, if Gentle follows a K_n-fiber strategy on $K_n \square K_m$, he looses in both of the following cases:*

- *Rascal starts, n is odd, m is even and $m < 2n-2$,*
- *Gentle starts, n and m are odd and $m < 2n-1$.*

Proof. We prove the first statement. The second can be proved in exactly the same way. Rascal's winning strategy is to create two K_m-fibers, say K_m^1 and K_m^n, which are strictly colored the same. More precisely, if $u \in K_m^1$ and $v \in K_m^n$ are in the same K_n-fiber then they share the same color. In that case, the automorphism which only permutes K_m^1 and K_m^n is a color preserving automorphism.

Rascal proceeds as follows. He plays his $\frac{m}{2}$ first moves in the same K_m-fiber, say K_m^1. Since Gentle plays according to a K_n-fiber strategy, at the end of the $(\frac{m}{2})^{\text{th}}$ turn of Gentle each K_n-fiber has exactly one colored vertex. These m first moves are called the first phase of the game. Let k be the number of uncolored vertices in K_m^1 at the end of this first phase. We have: $0 \leq k \leq \frac{m}{2}$ (k could be equal to 0, if Gentle has only played in K_m^1 during the first phase). The forthcoming k moves of Rascal and k moves of Gentle will be called the second phase of the game. In this phase, when Rascal plays in a K_n-fiber, Gentle has to answer by a move in this same K_n-fiber. Hence, Rascal can play all the k remaining uncolored vertices of K_m^1. Let u be such a vertex. There is a unique vertex v in the same K_n-fiber than u, which is already colored (this vertex has been colored by Gentle during the first phase). Rascal copies the color of v to color u. During this second part of the game, Gentle has played in at most k distinct K_m-fibers. Since $m < 2n-2$, then $k < n-1$. Hence, there exists a K_m-fiber, say K_m^n, in which Gentle has not played during this second phase. In K_m^n, there is at most one colored vertex, say w. In that case, w has been colored by Gentle during the first phase of the game. This vertex w shares the same color as the vertex of K_m^1, which is in the same K_n-fiber (this vertex has been colored by Rascal during the second phase). Therefore, Rascal can now color all the uncolored vertices of K_m^n, such that K_m^1 and K_m^n are strictly colored the same. □

4 Cartesian products of involutive graphs.

In this section, we study the game distinguishing numbers of Cartesian products of involutive graphs and prove Theorem 1.5. The class of involutive graphs has been introduced in [11]. It contains graphs like even cycles, hypercubes or more generally diametrical graphs and even graphs (see [9]). Let us recall the definition. An *involutive graph* H is a graph such that there exists an involution, $bar : V(H) \to V(H)$, which commutes with all automorphisms and has no fixed point. In other words:

- for every $u \in V(H)$, we have $\overline{\overline{u}} = u$ and $\overline{u} \neq u$,
- for every $\sigma \in \mathrm{Aut}(H)$ and every $u \in V(H)$, we have $\sigma(\overline{u}) = \overline{\sigma(u)}$.

The set $\{u, \overline{u}\}$ will be called a *block*. An important remark is that an automorphism of an involutive graph has to map a block to a block. In other words, there is a natural action of the automorphism group on the set of blocks. We introduce the following concepts, which are going to play a similar role as the meta-colors used in Section 3.

If H is an involutive graph and c is a vertex-coloring with d colors, then the *type* t of a block $\{u, \overline{u}\}$ is defined by:

$$t(\{u,\overline{u}\}) = \begin{cases} c(u) - c(\overline{u}) \bmod d & \text{if } c(u) - c(\overline{u}) \bmod d \in \{0,\dots,\lfloor \frac{d}{2} \rfloor\} \\ c(\overline{u}) - c(u) \bmod d & \text{otherwise.} \end{cases}$$

The *block-list* of H, $L_c(H)$ is the list $(n_0,\dots,n_{\lfloor \frac{d}{2} \rfloor})$ of length $\lfloor \frac{d}{2} \rfloor + 1$, where n_i, with $i \in \{0,\dots,\lfloor \frac{d}{2} \rfloor\}$, is the number of blocks of type i, according to the coloring c. Note that, if σ is an automorphism of H, then $t(\sigma(\{u,\bar{u}\})) = t(\{u,\bar{u}\})$ and $L_c(\sigma(H)) = L_c(H)$.

Assume now that H is a connected involutive graph and F is a connected graph relatively prime to H. The following proposition asserts that if the classical distinguishing number of F is not too big, then $D_{\mathcal{R}}(H\Box F)$ is bounded above by $D_{\mathcal{R}}(H)$. Theorem 1.5 will be a straightforward application of this result.

Theorem 4.1. *Let H be a connected involutive graph. Assume that Gentle has a winning strategy playing second on H, with $d \geq D_{\mathcal{R}}(H)$ colors. Moreover, this strategy yields colorings, whose block-list is in a fixed set $\mathcal{L}$.*

If F is a connected graph relatively prime to H, with $D(F) \leq \binom{\frac{|V(H)|}{2} + \lfloor \frac{d}{2} \rfloor}{\lfloor \frac{d}{2} \rfloor} - |\mathcal{L}| + 1,$ *then $D_{\mathcal{R}}(H\Box F) \leq d$.*

Proof. We have to give a Gentle winning strategy with d colors, assuming Rascal starts. The coloring obtained at the end of the game will be denoted by c.

First of all, Gentle will follow an H-fiber strategy. Note that $|V(H)|$ is even. Hence, we are in Case 0 of this strategy. Let (u_1,v_1) be the first vertex of $H\Box F$ colored by Rascal. Gentle imagines a distinguishing coloring c' of F with $D(F)$ colors. When Gentle has to play in the fiber H^{v_1}, he chooses the vertex and the color according to a winning strategy in H. This is possible, because Gentle's moves and Rascal's moves in H^{v_1} alternate like the moves in a game played only on H, when Rascal starts (see Proposition 2.1). Moreover, $d \geq D_{\mathcal{R}}(H)$ by hypothesis. In the other H-fibers, when Rascal plays the vertex (u,v), Gentle answers by coloring the vertex $(\overline{u},v)$. In this way, Gentle will be able to control the block-list of these fibers. More precisely, he chooses the colors such that:

$$(\dagger)\ \text{ for every } v,w \in V(F),\ L_c(H^v) = L_c(H^w) \text{ if } c'(v) = c'(w).$$

This is possible if there exists at least $(D(F)-1) + |\mathcal{L}|$ distinct possible block-lists. Indeed, Gentle cannot control in advance the block-list of the fiber H^{v_1}. By hypothesis, we only know that this block-list will belong to $\mathcal{L}$. Hence, $|\mathcal{L}|$ kinds of block-list are used to stand for the imaginary color $c'(v_1)$. Finally, with $(D(F)-1) + |\mathcal{L}|$ possible block-lists, Gentle has enough possibilities to associate distinct block-lists to distinct colors of the coloring c'. The number of block-lists is the number of weak compositions of $\frac{|V(H)|}{2}$ (the number of blocks) into $\lfloor \frac{d}{2} \rfloor + 1$ natural numbers (the number of block types). So, there are $\binom{\frac{|V(H)|}{2} + \lfloor \frac{d}{2} \rfloor}{\lfloor \frac{d}{2} \rfloor}$ kinds of block-lists, which is by hypothesis greater or equal to $(D(F)-1) + |\mathcal{L}|$.

Now, we prove that the coloring obtained with this strategy is distinguishing. Assume σ is a color preserving automorphism. We have $\sigma = (\psi,\phi)$, where $\psi \in \mathrm{Aut}(H)$ and $\phi \in \mathrm{Aut}(F)$. This automorphism maps blocks in H^v to blocks in $H^{\phi(v)}$, for any $v \in V(F)$. Hence, the automorphism ϕ preserves the block-lists of the H-fibers: $L_c(H^v) = L_c(H^{\phi(v)})$, for all $v \in V(F)$. By condition $(\dagger)$, this automorphism preserves also the distinguishing coloring c' of F. Hence, ϕ is the identity of $\mathrm{Aut}(F)$. This implies that $\sigma(H^{v_1}) = H^{v_1}$. But the coloring of this H-fiber is obtained by following a winning strategy for Gentle in the game on H. Therefore, ψ is the identity of H. In conclusion, σ is trivial and the coloring c is distinguishing. □

Theorem 1.5 is a straightforward application of the above result for two reasons. First, we know that for an involutive graph H, we have $D_{\mathcal{R}}(H) \leq D^2(H) + D(H) - 2$ (see Theorem 1.3). Moreover, with $D^2(H) + D(H) - 2$ colors, Gentle has a winning strategy such that he knows exactly the block-list he will get at the end of the game (see the proof of Theorem 1.6 in [11]). It means, with the notation of the above theorem, that $\mathcal{L}$ is just a singleton.

5 Cartesian products of cycles

In this final section, we give a proof of Theorem 1.6. Note that proving the statement about the infinity of the invariants is a straightforward application of Proposition 1.1. For the cycle C_n of order $n \geq 3$, we set $V(C_n) = \{1, \ldots, n\}$ and $E(C_n) = \{ij \mid |i - j| = 1 \bmod n,\ i, j \in V(C_n)\}$. We begin with toroidal grids of even order. Since even cycles are involutive graphs the first proposition is a direct corollary of Theorem 4.1.

Proposition 5.1. *Let n and m be two distinct natural numbers greater or equal to 3. If n is even and $n \geq 8$, then $D_{\mathcal{R}}(C_n \square C_m) = 2$.*

Proof. In [11, Proposition 4.1], the winning strategy used by Gentle with two colors leads to exactly three bi-chromatic blocks, when $n \geq 12$. To show that $D_{\mathcal{R}}(C_8) = D_{\mathcal{R}}(C_{10}) = 2$, they used an exhaustive computer check. This computing also gives that there is a Gentle's winning strategy which leads to one or three bi-chromatic blocks if $n = 8$, and to one or four bi-chromatic blocks if $n = 10$. Therefore, with the same notation as in Theorem 4.1, we have $|\mathcal{L}| \leq 2$. For all $m \geq 3$, we have $D(C_m) \leq 3$. Thus, we have $D(C_m) \leq \begin{pmatrix} \frac{n+2}{2} \\ 1 \end{pmatrix} - 2 + 1$, and we can directly apply Theorem 4.1 to get $D_{\mathcal{R}}(C_n \square C_m) = 2$. □

Proposition 5.2. *Let n be in $\{4, 6\}$ and F a connected graph relatively prime to C_n, with at least three vertices. If $D(F) \leq 3$, then $D_{\mathcal{R}}(C_n \square F) = 2$.*

Proof. Let $n \in \{4, 6\}$. We denote by c the coloring built during the game. We have to give a winning strategy for Gentle with 2 colors. Gentle plays according to a C_n-fiber strategy and uses the block-lists as meta-colors. Here, the problem is that $D_{\mathcal{R}}(C_n) = 3$. Gentle fancies a distinguishing coloring c' of F, where the three colors are really used. As in Theorem 4.1, Gentle can control the block-list of the C_n-fibers such that:

$$\text{for every } v \in V(F),\ L_c(C_n^v) = \begin{cases} (n, 0) & \text{if } c'(v) = 1, \\ (n-1, 1) & \text{if } c'(v) = 2, \\ (n-2, 2) & \text{if } c'(v) = 3. \end{cases}$$

Moreover, for $v \in V(F)$, if $L_c(C_n^v)$ must be equal to $(n-1, 1)$ or $(n-2, 2)$, he plays such that the block $\{(1, v), (n/2, v\}$ is of type 1.

Now, we prove that the coloring c is distinguishing. Assume σ is a color preserving automorphism. We have $\sigma = (\psi, \phi)$, where $\psi \in \mathrm{Aut}(C_n)$ and $\phi \in \mathrm{Aut}(F)$. For all

$v \in V(F)$, $L_c(C_n^v) = L_c(C_n^{\phi(v)})$. Hence, for all $v \in V(F)$, $c'(v) = c'(\phi(v))$. Since c' is a distinguishing coloring of F, we get that ϕ is trivial. Hence, σ fixes the C_n-fibers setwise. Since there is at least one C_n-fiber with block-list $(n-1, 1)$, ψ must be the symmetry Δ of axes $(1, \frac{n}{2})$ or the identity. But Δ does not preserve the coloring in the C_n-fibers, whose block-list is $(n-2, 2)$. Indeed, in such a fiber, one of the block of type 1 is stable under Δ. The other block of type 1 is sent by Δ to a block of type 0 or switched to itself. In both cases, it breaks the coloring. In conclusion, ψ is the identity and so is σ. $\square$

This result directly implies the following corollary.

Corollary 5.3. *Let m be an integer greater or equal to 3.*

1. *If $m \neq 6$, then $D_{\mathcal{R}}(C_6 \square C_m) = 2$.*
2. *If $m \neq 4$, then $D_{\mathcal{R}}(C_4 \square C_m) = 2$.*

The following proposition gives a general upper bound, when one factor has distinguishing number less or equal to 2. A corollary of this result is the case where both factors have odd cardinality and at least one of them is not prime.

Proposition 5.4. *Let H and F be two connected relatively prime graphs. Assume H is vertex transitive, $D(H) \geq 2$ and $D(F) \leq 2$.*

1. *If $|V(H)|$ and $|V(F)|$ are odd, then $D_{\mathcal{G}}(H \square F) \leq D_{\mathcal{G}}(H)$.*
2. *If $|V(H)|$ is odd and $|V(F)|$ is even, then $D_{\mathcal{R}}(H \square F) \leq D_{\mathcal{G}}(H)$.*
3. *If $|V(H)|$ is even, then $D_{\mathcal{R}}(H \square F) \leq D_{\mathcal{R}}(H)$.*

Proof. We prove the first statement. Let c be the coloring built during the game. For each H-fiber H^v, with $v \in V(F)$, we define:

$$p(H^v) = \begin{cases} 1 & \text{if } |\{u \in H^v | c((u, v)) = 1\}| \text{ is odd} \\ 2 & \text{otherwise.} \end{cases}$$

We have to define a Gentle's winning strategy with $D_{\mathcal{G}}(H)$ colors. Gentle is going to play according to an H-fiber strategy. Note that we are in Case 1 of this strategy. In the H-fibers, where Gentle is the first to play, the moves alternate exactly as in a game played only on H, with Gentle playing first (see Proposition 2.1). In the other H-fibers, it is also the case, except for the first move which is played by Rascal. In other words, Rascal will play the two first moves in a row in these H-fibers. Since H is vertex transitive, we assume, by Lemma 2.2, that Gentle has played first also in these H-fibers. Therefore, Gentle can play following a winning strategy for H in each H-fiber. He plays like this as long as one H-fiber is totally colored, say H^{v_0}. Now, he fancies a distinguishing coloring c' of F such that $c'(v_0) = p(H^{v_0})$. For the later moves, he plays such that:

$$(\ddagger) \text{ for every } v \in V(F),\ c'(v) = p(H^v).$$

Since he follows an H-fiber strategy, we recall that he is going to play the last move in each H-fiber. Hence, he is able to decide the parity of the number of vertices colored with 1 in each of them.

Let us prove now that the coloring c is distinguishing. Let σ be a color preserving automorphism. For all $v \in V(F)$, we have $p(\sigma(H^v)) = p(H^v)$. Since c' is a distinguishing coloring of F, it implies, by (‡), that σ fixes the H-fibers setwise. Therefore, $\sigma(H^{v_0}) = H^{v_0}$. But the coloring on this H-fiber is obtained by following a winning strategy on H. Then, H^{v_0} must be fixed pointwise by σ. In conclusion, σ is the identity.

For the two remaining statements, the proof is almost the same. The only difference is that for the third item, we are in Case 0 of the H-fiber strategy. □

Corollary 5.5. *Let n and m be two distinct odd natural numbers greater or equal to 3. If n is not prime and $m \geq 7$, then $D_{\mathcal{G}}(C_n \square C_m) = 2$.*

Proof. Under the hypothesis of the corollary, we have $D_{\mathcal{G}}(C_n) = 2$ and $D(C_m) = 2$. Thus, this is a straightforward application of Proposition 5.4. □

With the previous results, we are able to compute the distinguishing numbers of the toroidal grid $C_n \square C_m$, except for the following cases:

- $C_3 \square C_m$, with $m \neq 3$ and m odd,
- $C_5 \square C_m$, with $m \neq 5$ and m odd,
- $C_n \square C_m$, with $n \neq m$, n and m odd and prime.

To settle this remaining cases, we state the following proposition.

Proposition 5.6. *Let n and m be two two distinct odd natural numbers greater or equal to 3. If n is prime and $m \geq 7$, then $D_{\mathcal{G}}(C_n \square C_m) = 2$.*

Proof. Let c be the coloring built during the game. For each C_n-fiber C_n^j, with $j \in \{1, \ldots, m\}$, we define:

$$p(C_n^j) = \begin{cases} 1 & \text{if } |\{i \in C_n^j | c((i,j)) = 1\}| \text{ is odd} \\ 2 & \text{otherwise.} \end{cases}$$

Let M_1, M_2 and M_3 be three maximum matchings of C_n, whose union covers $E(C_n)$. Let v_1, v_2 and v_3 be the only vertices of C_n, which are respectively not covered by M_1, M_2 and M_3. Let c' be a distinguishing coloring of C_m, with 2 colors. Such a coloring exists because $m > 5$.

We have to outline a Gentle's winning strategy with 2 colors. He is going to follow a C_n-fiber strategy. Since $m \geq 7$, there are at least three distinct C_n-fibers, $C_n^{j_1}, C_n^{j_2}, C_n^{j_3}$, with $j_1, j_2, j_3 \in \{1, \ldots, m\}$, where Gentle is the first to play. The first vertex that Gentle is going to color in $C_n^{j_k}$, with $k \in \{1, 2, 3\}$ is (v_k, j_k). He colors it such that $c((v_k, j_k)) = c'(j_k)$. In the other C_n-fibers, where he is the first to play, the first vertex he chooses and the color he uses do not matter. For the later moves in $C_n^{j_k}$, he will choose the vertices with respect to the matching M_k. Moreover, he uses a color distinct from the one used by Rascal just before. In this way, the parity of the number of vertices in $C_n^{j_k}$ colored with 1 only depends on $c((v_k, j_k))$. Hence $p(C_n^{j_k}) = c((v_k, j_k)) = c'(j_k)$, for $k \in \{1, 2, 3\}$. For the moves in C_n^j, with $j \notin \{j_1, j_2, j_3\}$, Gentle plays whatever he wants, except when he colors the last vertex of the C_n-fiber. In that case, he chooses the color such that $p(C_n^j) = c'(j)$.

We prove now that c is distinguishing. Let σ be a color preserving automorphism. For all the C_n-fibers, we have $p(C_n^j) = c'(j)$. Since c' is a distinguishing coloring of

C_m, the automorphism σ fixes the C_n-fibers setwise. Thus, we have $\sigma = (\psi, \mathrm{id})$, with $\psi \in \mathrm{Aut}(C_n)$. Since n is prime, any nontrivial rotation acts transitively on a C_n-fiber. As at least one such fiber is not monochromatic, ψ could not be a nontrivial rotation. On the other hand, if ψ is an axial symmetry, since n is odd, there is an edge $e \in E(C_n)$ such that $\psi(e) = e$. This edge belongs to one of the three matchings, say M_1. Gentle has played such that in $C_n^{j_1}$, the edge corresponding to e is not monochromatic. Therefore ψ cannot preserve the coloring. In conclusion, ψ must be the identity and so is σ. □

We are now ready to prove Theorem 1.6.

Proof of Theorem 1.6. Let $n_1, \ldots, n_k$, with $k \geq 2$, be k distinct numbers greater or equal to 3. If $\prod_{i=1}^{k} n_i$ is even (resp. odd), we have to prove that $D_{\mathcal{R}}(C_{n_1} \square \cdots \square C_{n_k}) = 2$ (resp. $D_{\mathcal{G}}(C_{n_1} \square \cdots \square C_{n_k}) = 2$). If $k = 2$, this is a consequence of Propositions 5.1 and 5.6 and Corollaries 5.3 and 5.5, except for $C_3 \square C_5$. An exhaustive computer check proves that in that case two colors are also enough. For $k \geq 3$, we proceed by induction. If we are not dealing with $C_3 \square C_4 \square C_5$, we can assume that $n_k \geq 6$. Hence, $D(C_{n_k}) = 2$ and by induction $D_{\mathcal{R}}(C_{n_1} \square \cdots \square C_{n_{k-1}}) = 2$ or $D_{\mathcal{G}}(C_{n_1} \square \cdots \square C_{n_{k-1}}) = 2$, depending on the parity. Finally, we apply Proposition 5.4, with $H = C_{n_1} \square \cdots \square C_{n_{k-1}}$ and $F = C_{n_k}$, to get the expected results. For $C_3 \square C_4 \square C_5$, we apply Proposition 5.2 to show that $D_{\mathcal{R}}(C_3 \square C_4 \square C_5) = 2$. □

Corollary 5.7. *Let $n_1, \ldots, n_k$ be k distinct natural numbers greater or equal to* 2.

1. *If* $\prod_{i=1}^{k} n_i$ *is even, then* $D_{\mathcal{G}}(P_{n_1} \square \cdots \square P_{n_k}) = \infty$ *and* $D_{\mathcal{R}}(P_{n_1} \square \cdots \square P_{n_k}) = 2$.

2. *If* $\prod_{i=1}^{k} n_i$ *is odd, then* $D_{\mathcal{R}}(P_{n_1} \square \cdots \square P_{n_k}) = \infty$ *and* $D_{\mathcal{G}}(P_{n_1} \square \cdots \square P_{n_k}) = 2$.

Proof. If $k = 1$, we easily have $D_{\mathcal{R}}(P_n) = 2$, when n is even and $D_{\mathcal{G}}(P_n) = 2$, when n is odd (see [11]). If $k \geq 2$ and $n_i \geq 3$, for all $i \in \{1, \ldots, k\}$, then it is a straightforward consequence of Theorem 1.6. Indeed, in this case, a distinguishing coloring of $C_{n_1} \square \cdots \square C_{n_k}$ is also a distinguishing coloring of $P_{n_1} \square \cdots \square P_{n_k}$. If one factor, say P_{n_1} is isomorphic to P_2, then we can apply Proposition 5.4, with $H = P_{n_1}$ and $F = P_{n_2} \square \cdots \square P_{n_k}$. Hence, $D(P_{n_2} \square \cdots \square P_{n_k}) = 2$ (the only Cartesian products of paths for which it is not true are $P_2 \square P_2$ and $P_2 \square P_2 \square P_2$, with $D(P_2 \square P_2) = D(P_2 \square P_2 \square P_2) = 3$). □

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References

[1] M. O. Albertson and K. L. Collins, Symmetry breaking in graphs, *Electron. J. Combin.* **3** (1996), #R18, `http://www.combinatorics.org/ojs/index.php/eljc/article/view/v3i1r18`.

[2] B. Bogstad and L. J. Cowen, The distinguishing number of the hypercube, *Discrete Math.* **283** (2004), 29–35, doi:10.1016/j.disc.2003.11.018.

[3] B. Brešar, S. Klavžar and D. F. Rall, Domination game and an imagination strategy, *SIAM J. Discrete Math.* **24** (2010), 979–991, doi:10.1137/100786800.

[4] I. Broere and M. Pilśniak, The distinguishing index of the Cartesian product of countable graphs, *Ars Math. Contemp.* **13** (2017), 15–21, `https://amc-journal.eu/index.php/amc/article/view/792`.

[5] E. Estaji, W. Imrich, R. Kalinowski, M. Pilśniak and T. Tucker, Distinguishing Cartesian products of countable graphs, *Discuss. Math. Graph Theory* **37** (2017), 155–164, doi:10.7151/dmgt.1902.

[6] U. Faigle, U. Kern, H. Kierstead and W. T. Trotter, On the game chromatic number of some classes of graphs, *Ars Combinatoria* **35** (1993), 143–150.

[7] M. J. Fisher and G. Isaak, Distinguishing colorings of Cartesian products of complete graphs, *Discrete Math.* **308** (2008), 2240–2246, doi:10.1016/j.disc.2007.04.070.

[8] M. J. Fisher and G. Isaak, Distinguishing numbers of Cartesian products of multiple complete graphs, *Ars Math. Contemp.* **5** (2012), 159–173, `https://amc-journal.eu/index.php/amc/article/view/245`.

[9] F. Göbel and H. J. Veldman, Even graphs, *J. Graph Theory* **10** (1986), 225–239, doi:10.1002/jgt.3190100212.

[10] A. Gorzkowska, R. Kalinowski and M. Pilśniak, The distinguishing index of the Cartesian product of finite graphs, *Ars Math. Contemp.* **12** (2017), 77–87, `https://amc-journal.eu/index.php/amc/article/view/909`.

[11] S. Gravier, K. Meslem, S. Schmidt and S. Slimani, A new game invariant of graphs: the game distinguishing number, *Discrete Math. Theor. Comput. Sci.* **19** (2017), doi:10.23638/dmtcs-19-1-2.

[12] W. Imrich, J. Jerebic and S. Klavžar, The distinguishing number of Cartesian products of complete graphs, *European J. Combin.* **29** (2008), 922–929, doi:10.1016/j.ejc.2007.11.018.

[13] W. Imrich and S. Klavžar, *Product Graphs: Structure and Recognition*, Wiley-Interscience Series in Discrete Mathematics and Optimization, Wiley-Interscience, New York, 2000.

[14] W. Imrich and S. Klavžar, Distinguishing Cartesian powers of graphs, *J. Graph Theory* **53** (2006), 250–260, doi:10.1002/jgt.20190.

[15] S. Klavžar, T.-L. Wong and X. Zhu, Distinguishing labellings of group action on vector spaces and graphs, *J. Algebra* **303** (2006), 626–641, doi:10.1016/j.jalgebra.2006.01.045.

[16] S. Klavžar and X. Zhu, Cartesian powers of graphs can be distinguished by two labels, *European J. Combin.* **28** (2007), 303–310, doi:10.1016/j.ejc.2005.07.001.

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Tridiagonal pairs of q-Racah type, the Bockting operator ψ, and L-operators for $U_q(L(\mathfrak{sl}_2))$

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Abstract

We describe the Bockting operator ψ for a tridiagonal pair of q-Racah type, in terms of a certain L-operator for the quantum loop algebra $U_q(L(\mathfrak{sl}_2))$.

Keywords: Bockting operator, tridiagonal pair, Leonard pair.

Math. Subj. Class.: 17B37, 15A21

1 Introduction

In the theory of quantum groups there exists the concept of an L-operator; this was introduced in [20] to obtain solutions for the Yang-Baxter equation. In linear algebra there exists the concept of a tridiagonal pair; this was introduced in [13] to describe the irreducible modules for the subconstituent algebra of a Q-polynomial distance-regular graph. Recently some authors have connected the two concepts. In [1], [4] Pascal Baseilhac and Kozo Koizumi use L-operators for the quantum loop algebra $U_q(L(\mathfrak{sl}_2))$ to construct a family of finite-dimensional modules for the q-Onsager algebra $\mathcal{O}_q$; see [2, 3, 5, **?**] for related work. A finite-dimensional irreducible $\mathcal{O}_q$-module is essentially the same thing as a tridiagonal pair of q-Racah type [**?**, Section 12], [23, Section 3]. In [22, Section 9], Kei Miki uses similar L-operators to describe how $U_q(L(\mathfrak{sl}_2))$ is related to the q-tetrahedron algebra $\boxtimes_q$. A finite-dimensional irreducible $\boxtimes_q$-module is essentially the same thing as a tridiagonal pair of q-geometric type [16, Theorem 2.7], [14, Theorems 10.3, 10.4]. Following Baseilhac, Koizumi, and Miki, in the present paper we use L-operators for $U_q(L(\mathfrak{sl}_2))$ to describe the Bockting operator ψ associated with a tridiagonal pair of q-Racah type. Before going into detail, we recall some notation and basic concepts. Throughout this paper

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$\mathbb{F}$ denotes a field. Let V denote a vector space over $\mathbb{F}$ with finite positive dimension. For an $\mathbb{F}$-linear map $A : V \to V$ and a subspace $W \subseteq V$, we say that W is an *eigenspace* of A whenever $W \neq 0$ and there exists $\theta \in \mathbb{F}$ such that $W = \{v \in V | Av = \theta v\}$; in this case θ is called the *eigenvalue* of A associated with W. We say that A is *diagonalizable* whenever V is spanned by the eigenspaces of A.

Definition 1.1. (See [13, Definition 1.1].) Let V denote a vector space over $\mathbb{F}$ with finite positive dimension. By a *tridiagonal pair* (or *TD pair*) on V we mean an ordered pair of $\mathbb{F}$-linear maps $A : V \to V$ and $A^* : V \to V$ that satisfy the following four conditions:

(i) Each of A, A^* is diagonalizable.

(ii) There exists an ordering $\{V_i\}_{i=0}^{d}$ of the eigenspaces of A such that

$$A^* V_i \subseteq V_{i-1} + V_i + V_{i+1} \qquad (0 \leq i \leq d), \tag{1.1}$$

where $V_{-1} = 0$ and $V_{d+1} = 0$.

(iii) There exists an ordering $\{V_i^*\}_{i=0}^{\delta}$ of the eigenspaces of A^* such that

$$A V_i^* \subseteq V_{i-1}^* + V_i^* + V_{i+1}^* \qquad (0 \leq i \leq \delta), \tag{1.2}$$

where $V_{-1}^* = 0$ and $V_{\delta+1}^* = 0$.

(iv) There does not exist a subspace $W \subseteq V$ such that $AW \subseteq W$, $A^*W \subseteq W$, $W \neq 0$, $W \neq V$.

We refer the reader to [12, 13, 17] for background on TD pairs, and here mention only a few essential points. Let A, A^* denote a TD pair on V, as in Definition 1.1. By [13, Lemma 4.5] the integers d and δ from (1.1) and (1.2) are equal; we call this common value the *diameter* of A, A^*. An ordering of the eigenspaces for A (resp. A^*) is called *standard* whenever it satisfies (1.1) (resp. (1.2)). Let $\{V_i\}_{i=0}^{d}$ denote a standard ordering of the eigenspaces of A. By [13, Lemma 2.4] the ordering $\{V_{d-i}\}_{i=0}^{d}$ is standard and no further ordering is standard. A similar result holds for the eigenspaces of A^*. Until the end of this section fix a standard ordering $\{V_i\}_{i=0}^{d}$ (resp. $\{V_i^*\}_{i=0}^{d}$) of the eigenspaces for A (resp. A^*). For $0 \leq i \leq d$ let θ_i (resp. θ_i^*) denote the eigenvalue of A (resp. A^*) for the eigenspace V_i (resp. V_i^*). By construction $\{\theta_i\}_{i=0}^{d}$ are mutually distinct and contained in $\mathbb{F}$. Moreover $\{\theta_i^*\}_{i=0}^{d}$ are mutually distinct and contained in $\mathbb{F}$. By [13, Theorem 11.1] the expressions

$$\frac{\theta_{i-2} - \theta_{i+1}}{\theta_{i-1} - \theta_i} \qquad \frac{\theta_{i-2}^* - \theta_{i+1}^*}{\theta_{i-1}^* - \theta_i^*}$$

are equal and independent of i for $2 \leq i \leq d-1$. For this constraint the solutions can be given in closed form [13, Theorem 11.2]. The "most general" solution is called q-Racah, and will be described shortly.

We now recall the split decomposition [13, Section 4]. For $0 \leq i \leq d$ define

$$U_i = (V_0^* + V_1^* + \cdots + V_i^*) \cap (V_0 + V_1 + \cdots + V_{d-i}).$$

For notational convenience define $U_{-1} = 0$ and $U_{d+1} = 0$. By [13, Theorem 4.6] the sum $V = \sum_{i=0}^{d} U_i$ is direct. By [13, Theorem 4.6] both

$$U_0 + U_1 + \cdots + U_i = V_0^* + V_1^* + \cdots + V_i^*,$$
$$U_i + U_{i+1} + \cdots + U_d = V_0 + V_1 + \cdots + V_{d-i}$$

for $0 \leq i \leq d$. Let $I : V \to V$ denote the identity map. By [13, Theorem 4.6] both

$$(A - \theta_{d-i} I)U_i \subseteq U_{i+1}, \qquad (A^* - \theta^*_i I)U_i \subseteq U_{i-1} \tag{1.3}$$

for $0 \leq i \leq d$.

We now describe the q-Racah case. Pick a nonzero $q \in \mathbb{F}$ such that $q^4 \neq 1$. We say that A, A^* has *q-Racah type* whenever there exist nonzero $a, b \in \mathbb{F}$ such that both

$$\theta_i = aq^{2i-d} + a^{-1}q^{d-2i}, \qquad \theta^*_i = bq^{2i-d} + b^{-1}q^{d-2i} \tag{1.4}$$

for $0 \leq i \leq d$. For the rest of this section assume that A, A^* has q-Racah type. For $1 \leq i \leq d$ we have $q^{2i} \neq 1$; otherwise $\theta_i = \theta_0$. Define an $\mathbb{F}$-linear map $K : V \to V$ such that for $0 \leq i \leq d$, U_i is an eigenspace of K with eigenvalue q^{d-2i}. Thus

$$(K - q^{d-2i} I)U_i = 0 \qquad (0 \leq i \leq d). \tag{1.5}$$

Note that K is invertible. For $0 \leq i \leq d$ the following holds on U_i:

$$aK + a^{-1}K^{-1} = \theta_{d-i} I. \tag{1.6}$$

Define an $\mathbb{F}$-linear map $R : V \to V$ such that for $0 \leq i \leq d$, R acts on U_i as $A - \theta_{d-i} I$. By (1.6),

$$A = aK + a^{-1}K^{-1} + R. \tag{1.7}$$

By the equation on the left in (1.3),

$$RU_i \subseteq U_{i+1} \qquad (0 \leq i \leq d). \tag{1.8}$$

We now recall the Bockting operator ψ. By [8, Lemma 5.7] there exists a unique $\mathbb{F}$-linear map $\psi : V \to V$ such that both

$$\psi U_i \subseteq U_{i-1} \qquad (0 \leq i \leq d), \tag{1.9}$$

$$\psi R - R\psi = (q - q^{-1})(K - K^{-1}). \tag{1.10}$$

The known properties of ψ are described in [7, 8, **?**]. Suppose we are given A, A^*, R, K in matrix form, and wish to obtain ψ in matrix form. This can be done using (1.8), (1.9), (1.10) and induction on i. The calculation can be tedious, so one desires a more explicit description of ψ. In the present paper we give an explicit description of ψ, in terms of a certain L-operator for $U_q(L(\mathfrak{sl}_2))$. According to this description, ψ is equal to $-a$ times the ratio of two components for the L-operator. Theorem 5.4 is our main result.

The paper is organized as follows. In Section 2 we review the algebra $U_q(L(\mathfrak{sl}_2))$ in its Chevalley presentation. In Section 3 we recall the equitable presentation for $U_q(L(\mathfrak{sl}_2))$. In Section 4 we discuss some L-operators for $U_q(L(\mathfrak{sl}_2))$. In Section 5 we use these L-operators to describe ψ.

2 The quantum loop algebra $U_q(L(\mathfrak{sl}_2))$

Recall the integers $\mathbb{Z} = \{0, \pm 1, \pm 2, \dots\}$ and natural numbers $\mathbb{N} = \{0, 1, 2, \dots\}$. We will be discussing algebras. An algebra is meant to be associative and have a 1. Recall the field $\mathbb{F}$. Until the end of Section 4, fix a nonzero $q \in \mathbb{F}$ such that $q^2 \neq 1$. Define

$$[n]_q = \frac{q^n - q^{-n}}{q - q^{-1}} \qquad\qquad n \in \mathbb{Z}.$$

All tensor products are meant to be over $\mathbb{F}$.

Definition 2.1. (See [10, Section 3.3].) Let $U_q(L(\mathfrak{sl}_2))$ denote the $\mathbb{F}$-algebra with generators $E_i, F_i, K_i^{\pm 1}$ $(i \in \{0, 1\})$ and relations

$$\begin{aligned}
&K_i K_i^{-1} = 1, && K_i^{-1} K_i = 1, && \\
&K_0 K_1 = 1, && K_1 K_0 = 1, && \\
&K_i E_i = q^2 E_i K_i, && K_i F_i = q^{-2} F_i K_i, && \\
&K_i E_j = q^{-2} E_j K_i, && K_i F_j = q^2 F_j K_i, && i \neq j, \\
&E_i F_j - F_j E_i = \delta_{i,j} \frac{K_i - K_i^{-1}}{q - q^{-1}}, && && \\
&E_i^3 E_j - [3]_q E_i^2 E_j E_i + [3]_q E_i E_j E_i^2 - E_j E_i^3 = 0, && && i \neq j, \\
&F_i^3 F_j - [3]_q F_i^2 F_j F_i + [3]_q F_i F_j F_i^2 - F_j F_i^3 = 0, && && i \neq j.
\end{aligned}$$

We call $E_i, F_i, K_i^{\pm 1}$ the *Chevalley generators* for $U_q(L(\mathfrak{sl}_2))$.

Lemma 2.2. (See [18, p. 35].) *We turn $U_q(L(\mathfrak{sl}_2))$ into a Hopf algebra as follows. The coproduct Δ satisfies*

$$\begin{aligned}
&\Delta(K_i) = K_i \otimes K_i, && \Delta(K_i^{-1}) = K_i^{-1} \otimes K_i^{-1}, \\
&\Delta(E_i) = E_i \otimes 1 + K_i \otimes E_i, && \Delta(F_i) = 1 \otimes F_i + F_i \otimes K_i^{-1}.
\end{aligned}$$

The counit ε satisfies

$$\varepsilon(K_i) = 1, \qquad \varepsilon(K_i^{-1}) = 1, \qquad \varepsilon(E_i) = 0, \qquad \varepsilon(F_i) = 0.$$

The antipode S satisfies

$$S(K_i) = K_i^{-1}, \qquad S(K_i^{-1}) = K_i, \qquad S(E_i) = -K_i^{-1} E_i, \qquad S(F_i) = -F_i K_i.$$

We now discuss the $U_q(L(\mathfrak{sl}_2))$-modules.

Lemma 2.3. (See [10, Section 4].) *There exists a family of $U_q(L(\mathfrak{sl}_2))$-modules*

$$\mathbf{V}(d, t) \qquad\qquad 0 \neq d \in \mathbb{N}, \qquad\qquad 0 \neq t \in \mathbb{F} \tag{2.1}$$

with this property: $\mathbf{V}(d, t)$ has a basis $\{v_i\}_{i=0}^d$ such that

$$\begin{aligned}
&K_1 v_i = q^{d-2i} v_i && (0 \leq i \leq d), && \\
&E_1 v_i = [d - i + 1]_q v_{i-1} && (1 \leq i \leq d), && E_1 v_0 = 0, \\
&F_1 v_i = [i + 1]_q v_{i+1} && (0 \leq i \leq d - 1), && F_1 v_d = 0, \\
&K_0 v_i = q^{2i-d} v_i && (0 \leq i \leq d), && \\
&E_0 v_i = t [i + 1]_q v_{i+1} && (0 \leq i \leq d - 1), && E_0 v_d = 0, \\
&F_0 v_i = t^{-1} [d - i + 1]_q v_{i-1} && (1 \leq i \leq d), && F_0 v_0 = 0.
\end{aligned}$$

The module $\mathbf{V}(d, t)$ is irreducible provided that $q^{2i} \neq 1$ for $1 \leq i \leq d$.

Definition 2.4. Referring to Lemma 2.3, we call $\mathbf{V}(d,t)$ an *evaluation module* for $U_q(L(\mathfrak{sl}_2))$. We call d the *diameter*. We call t the *evaluation parameter*.

Example 2.5. For $0 \neq t \in \mathbb{F}$ the $U_q(L(\mathfrak{sl}_2))$-module $\mathbf{V}(1,t)$ is described as follows. With respect to the basis v_0, v_1 from Lemma 2.3, the matrices representing the Chevalley generators are

$$E_1: \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad F_1: \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \qquad K_1: \begin{pmatrix} q & 0 \\ 0 & q^{-1} \end{pmatrix},$$
$$E_0: \begin{pmatrix} 0 & 0 \\ t & 0 \end{pmatrix}, \qquad F_0: \begin{pmatrix} 0 & t^{-1} \\ 0 & 0 \end{pmatrix}, \qquad K_0: \begin{pmatrix} q^{-1} & 0 \\ 0 & q \end{pmatrix}.$$

Lemma 2.6. (See [19, p. 58].) *Let U and V denote $U_q(L(\mathfrak{sl}_2))$-modules. Then $U \otimes V$ becomes a $U_q(L(\mathfrak{sl}_2))$-module as follows. For $u \in U$ and $v \in V$,*

$$\begin{aligned} K_i(u \otimes v) &= K_i(u) \otimes K_i(v), \\ K_i^{-1}(u \otimes v) &= K_i^{-1}(u) \otimes K_i^{-1}(v), \\ E_i(u \otimes v) &= E_i(u) \otimes v + K_i(u) \otimes E_i(v), \\ F_i(u \otimes v) &= u \otimes F_i(v) + F_i(u) \otimes K_i^{-1}(v). \end{aligned}$$

Definition 2.7. (See [11, p. 110].) Up to isomorphism, there exists a unique $U_q(L(\mathfrak{sl}_2))$-module of dimension 1 on which each $u \in U_q(L(\mathfrak{sl}_2))$ acts as $\varepsilon(u)I$, where ε is from Lemma 2.2. This $U_q(L(\mathfrak{sl}_2))$-module is said to be *trivial*.

Proposition 2.8. (See [22, Theorem 3.2].) *Assume that $\mathbb{F}$ is algebraically closed with characteristic zero, and q is not a root of unity. Let V denote a nontrivial finite-dimensional irreducible $U_q(L(\mathfrak{sl}_2))$-module on which each eigenvalue of K_1 is an integral power of q. Then V is isomorphic to a tensor product of evaluation $U_q(L(\mathfrak{sl}_2))$-modules.*

3 The equitable presentation for $U_q(L(\mathfrak{sl}_2))$

In this section we recall the equitable presentation for $U_q(L(\mathfrak{sl}_2))$. Let $\mathbb{Z}_4 = \mathbb{Z}/4\mathbb{Z}$ denote the cyclic group of order 4. In a moment we will discuss some objects X_{ij}. The subscripts i, j are meant to be in $\mathbb{Z}_4$.

Lemma 3.1. (See [15, Theorem 2.1], [22, Proposition 4.2].) *The algebra $U_q(L(\mathfrak{sl}_2))$ has a presentation by generators*

$$X_{01}, \quad X_{12}, \quad X_{23}, \quad X_{30}, \quad X_{13}, \quad X_{31} \tag{3.1}$$

and the following relations:

$$X_{13}X_{31} = 1, \quad X_{31}X_{13} = 1, \quad \frac{qX_{01}X_{12} - q^{-1}X_{12}X_{01}}{q - q^{-1}} = 1, \quad \frac{qX_{12}X_{23} - q^{-1}X_{23}X_{12}}{q - q^{-1}} = 1,$$
$$\frac{qX_{23}X_{30} - q^{-1}X_{30}X_{23}}{q - q^{-1}} = 1, \quad \frac{qX_{30}X_{01} - q^{-1}X_{01}X_{30}}{q - q^{-1}} = 1, \quad \frac{qX_{01}X_{13} - q^{-1}X_{13}X_{01}}{q - q^{-1}} = 1,$$
$$\frac{qX_{31}X_{12} - q^{-1}X_{12}X_{31}}{q - q^{-1}} = 1, \quad \frac{qX_{23}X_{31} - q^{-1}X_{31}X_{23}}{q - q^{-1}} = 1, \quad \frac{qX_{13}X_{30} - q^{-1}X_{30}X_{13}}{q - q^{-1}} = 1,$$
$$X_{i,i+1}^3X_{i+2,i+3} - [3]_qX_{i,i+1}^2X_{i+2,i+3}X_{i,i+1} + [3]_qX_{i,i+1}X_{i+2,i+3}X_{i,i+1}^2 - X_{i+2,i+3}X_{i,i+1}^3 = 0.$$

An isomorphism with the presentation in Definition 2.1 sends

$$\begin{aligned} &X_{01} \mapsto K_0 + q(q-q^{-1})K_0F_0, && X_{12} \mapsto K_1 - (q-q^{-1})E_1,\\ &X_{23} \mapsto K_1 + q(q-q^{-1})K_1F_1, && X_{30} \mapsto K_0 - (q-q^{-1})E_0,\\ &X_{13} \mapsto K_1, \qquad X_{31} \mapsto K_0. \end{aligned}$$

The inverse isomorphism sends

$$\begin{aligned} &E_1 \mapsto (X_{13}-X_{12})(q-q^{-1})^{-1}, && E_0 \mapsto (X_{31}-X_{30})(q-q^{-1})^{-1},\\ &F_1 \mapsto (X_{31}X_{23}-1)q^{-1}(q-q^{-1})^{-1}, && F_0 \mapsto (X_{13}X_{01}-1)q^{-1}(q-q^{-1})^{-1},\\ &K_1 \mapsto X_{13}, \qquad K_0 \mapsto X_{31}. \end{aligned}$$

Note 3.2. For notational convenience, we identify the copy of $U_q(L(\mathfrak{sl}_2))$ given in Definition 2.1 with the copy given in Lemma 3.1, via the isomorphism given in Lemma 3.1.

Definition 3.3. Referring to Lemma 3.1, we call the generators (3.1) the *equitable generators* for $U_q(L(\mathfrak{sl}_2))$.

Lemma 3.4. (See [24, Theorem 3.4].) *From the equitable point of view the Hopf algebra $U_q(L(\mathfrak{sl}_2))$ looks as follows. The coproduct Δ satisfies*

$$\begin{aligned} &\Delta(X_{13}) = X_{13}\otimes X_{13}, \qquad \Delta(X_{31}) = X_{31}\otimes X_{31},\\ &\Delta(X_{01}) = (X_{01}-X_{31})\otimes 1 + X_{31}\otimes X_{01}, && \Delta(X_{12}) = (X_{12}-X_{13})\otimes 1 + X_{13}\otimes X_{12},\\ &\Delta(X_{23}) = (X_{23}-X_{13})\otimes 1 + X_{13}\otimes X_{23}, && \Delta(X_{30}) = (X_{30}-X_{31})\otimes 1 + X_{31}\otimes X_{30}. \end{aligned}$$

The counit ε satisfies

$$\begin{aligned} &\varepsilon(X_{13}) = 1, && \varepsilon(X_{31}) = 1, && \varepsilon(X_{01}) = 1,\\ &\varepsilon(X_{12}) = 1, && \varepsilon(X_{23}) = 1, && \varepsilon(X_{30}) = 1. \end{aligned}$$

The antipode S satisfies

$$\begin{aligned} &S(X_{31}) = X_{13}, \qquad S(X_{13}) = X_{31},\\ &S(X_{01}) = 1 + X_{13} - X_{13}X_{01}, && S(X_{12}) = 1 + X_{31} - X_{31}X_{12},\\ &S(X_{23}) = 1 + X_{31} - X_{31}X_{23}, && S(X_{30}) = 1 + X_{13} - X_{13}X_{30}. \end{aligned}$$

4 Some L-operators for $U_q(L(\mathfrak{sl}_2))$

In this section we recall some L-operators for $U_q(L(\mathfrak{sl}_2))$, and describe their basic properties.

We recall some notation. Let Δ denote the coproduct for a Hopf algebra H. Then the opposite coproduct Δ^{op} is the composition

$$\Delta^{\mathrm{op}}: \quad H \xrightarrow[\Delta]{} H\otimes H \xrightarrow[r\otimes s\mapsto s\otimes r]{} H\otimes H.$$

Definition 4.1. (See [22, Section 9.1].) Let V denote a $U_q(L(\mathfrak{sl}_2))$-module and $0 \neq t \in \mathbb{F}$. Consider an $\mathbb{F}$-linear map

$$L: \quad V\otimes \mathbf{V}(1,t) \to V\otimes \mathbf{V}(1,t).$$

We call this map an *L-operator for V with parameter t* whenever the following diagram commutes for all $u \in U_q(L(\mathfrak{sl}_2))$:

$$\begin{array}{ccc} V \otimes \mathbf{V}(1,t) & \xrightarrow{\Delta(u)} & V \otimes \mathbf{V}(1,t) \\ {\scriptstyle L}\Big\downarrow & & \Big\downarrow{\scriptstyle L} \\ V \otimes \mathbf{V}(1,t) & \xrightarrow[\Delta^{\mathrm{op}}(u)]{} & V \otimes \mathbf{V}(1,t) \end{array}$$

Definition 4.2. (See [22, Section 9.1].) Let V denote a $U_q(L(\mathfrak{sl}_2))$-module and $0 \neq t \in \mathbb{F}$. Consider any $\mathbb{F}$-linear map

$$L: \quad V \otimes \mathbf{V}(1,t) \to V \otimes \mathbf{V}(1,t). \tag{4.1}$$

For $r, s \in \{0, 1\}$ define an $\mathbb{F}$-linear map $L_{rs}: V \to V$ such that for $v \in V$,

$$L(v \otimes v_0) = L_{00}(v) \otimes v_0 + L_{10}(v) \otimes v_1, \tag{4.2}$$
$$L(v \otimes v_1) = L_{01}(v) \otimes v_0 + L_{11}(v) \otimes v_1. \tag{4.3}$$

Here v_0, v_1 is the basis for $\mathbf{V}(1,t)$ from Lemma 2.3.

Lemma 4.3. *Referring to Definition 4.2, the map* (4.1) *is an L-operator for V with parameter t if and only if the following equations hold on V:*

$$K_1 L_{00} = L_{00} K_1, \qquad K_1 L_{01} = q^{-2} L_{01} K_1,$$
$$K_1 L_{10} = q^2 L_{10} K_1, \qquad K_1 L_{11} = L_{11} K_1;$$

$$L_{00} E_1 - q E_1 L_{00} = L_{10}, \qquad L_{01} E_1 - q E_1 L_{01} = L_{11} - L_{00} K_1,$$
$$L_{10} E_1 - q^{-1} E_1 L_{10} = 0, \qquad L_{11} E_1 - q^{-1} E_1 L_{11} = -L_{10} K_1;$$

$$F_1 L_{00} - q^{-1} L_{00} F_1 = L_{01}, \qquad F_1 L_{01} - q L_{01} F_1 = 0,$$
$$F_1 L_{10} - q^{-1} L_{10} F_1 = L_{11} - K_0 L_{00}, \qquad F_1 L_{11} - q L_{11} F_1 = -K_0 L_{01};$$

$$K_0 L_{00} = L_{00} K_0, \qquad K_0 L_{01} = q^2 L_{01} K_0,$$
$$K_0 L_{10} = q^{-2} L_{10} K_0, \qquad K_0 L_{11} = L_{11} K_0;$$

$$L_{00} E_0 - q^{-1} E_0 L_{00} = -t L_{01} K_0, \qquad L_{01} E_0 - q^{-1} E_0 L_{01} = 0,$$
$$L_{10} E_0 - q E_0 L_{10} = t L_{00} - t L_{11} K_0, \qquad L_{11} E_0 - q E_0 L_{11} = t L_{01};$$

$$F_0 L_{00} - q L_{00} F_0 = -t^{-1} K_1 L_{10}, \qquad F_0 L_{01} - q^{-1} L_{01} F_0 = t^{-1} L_{00} - t^{-1} K_1 L_{11},$$
$$F_0 L_{10} - q L_{10} F_0 = 0, \qquad F_0 L_{11} - q^{-1} L_{11} F_0 = t^{-1} L_{10}.$$

Proof. This is routinely checked. □

Example 4.4. (See [21, Appendix], [22, Proposition 9.2].) Referring to Definition 4.2, assume that V is an evaluation module $\mathbf{V}(d, \mu)$ such that $q^{2i} \neq 1$ for $1 \leq i \leq d$. Consider the matrices that represent the L_{rs} with respect to the basis $\{v_i\}_{i=0}^d$ for $\mathbf{V}(d, \mu)$ from Lemma 2.3. Then the following are equivalent:

(i) the map (4.1) is an L-operator for V with parameter t;

(ii) the matrix entries are given in the table below (all matrix entries not shown are zero):

operator	$(i,i-1)$-entry	(i,i)-entry	$(i-1,i)$-entry
L_{00}	0	$\frac{q^{1-i}-\mu^{-1}tq^{i-d}}{q-q^{-1}}\,\xi$	0
L_{01}	$[i]_q q^{1-i}\xi$	0	0
L_{10}	0	0	$[d-i+1]_q q^{i-d}\mu^{-1}t\xi$
L_{11}	0	$\frac{q^{i-d+1}-\mu^{-1}tq^{-i}}{q-q^{-1}}\,\xi$	0

Here $\xi \in \mathbb{F}$.

Lemma 4.5. (See [22, Proposition 9.3].) *Let U and V denote $U_q(L(\mathfrak{sl}_2))$-modules, and consider the $U_q(L(\mathfrak{sl}_2))$-module $U \otimes V$ from Lemma 2.6. Let $0 \neq t \in \mathbb{F}$. Suppose we are given L-operators for U and V with parameter t. Then there exists an L-operator for $U \otimes V$ with parameter t such that for $r, s \in \{0, 1\}$,*

$$L_{rs}(u \otimes v) = L_{r0}(u) \otimes L_{0s}(v) + L_{r1}(u) \otimes L_{1s}(v) \qquad u \in U, \quad v \in V. \tag{4.4}$$

Proof. For $r, s \in \{0, 1\}$ define an $\mathbb{F}$-linear map $L_{rs} : U \otimes V \to U \otimes V$ that satisfies (4.4). Using (4.4) and Lemma 2.6 one checks that the L_{rs} satisfy the equations in Lemma 4.3. The result follows by Lemma 4.3. □

Corollary 4.6. *Adopt the notation and assumptions of Proposition 2.8. Then for $0 \neq t \in \mathbb{F}$ there exists a nonzero L-operator for V with parameter t.*

Proof. By Proposition 2.8 along with Example 4.4 and Lemma 4.5. □

5 TD pairs and L-operators

In Section 1 we discussed a TD pair A, A^* on V. We now return to this discussion, adopting the notation and assumptions that were in force at the end of Section 1. Recall the scalars q, a, b from (1.4). Recall the map K from above (1.5).

Proposition 5.1. (See [17, p. 103].) *Assume that $\mathbb{F}$ is algebraically closed with characteristic zero, and q is not a root of unity. Then the vector space V becomes a $U_q(L(\mathfrak{sl}_2))$-module on which $K = X_{31}$, $K^{-1} = X_{13}$ and*

$$A = aX_{01} + a^{-1}X_{12}, \qquad A^* = bX_{23} + b^{-1}X_{30}.$$

Proof. This is how [17, p. 103] looks from the equitable point of view. □

Note 5.2. The $U_q(L(\mathfrak{sl}_2))$-module structure from Proposition 5.1 is not unique in general.

We now investigate the $U_q(L(\mathfrak{sl}_2))$-module structure from Proposition 5.1. Recall the map R from above (1.7).

Lemma 5.3. *Assume that the vector space V becomes a $U_q(L(\mathfrak{sl}_2))$-module on which $K = X_{31}$, $K^{-1} = X_{13}$ and*

$$A = aX_{01} + a^{-1}X_{12}, \qquad A^* = bX_{23} + b^{-1}X_{30}.$$

On this module,

(i) *R looks as follows in the equitable presentation:*

$$R = a(X_{01} - X_{31}) + a^{-1}(X_{12} - X_{13}). \tag{5.1}$$

(ii) *R looks as follows in the Chevalley presentation:*

$$R = (q - q^{-1})(aqK_0F_0 - a^{-1}E_1). \tag{5.2}$$

Proof. (i) In line (1.7) eliminate A, K, K^{-1} using the assumptions of the present lemma. (ii) Evaluate the right-hand side of (5.1) using the identifications from Lemma 3.1 and Note 3.2. □

We now present our main result. Recall the Bockting operator ψ from (1.9), (1.10).

Theorem 5.4. *Assume that the vector space V becomes a $U_q(L(\mathfrak{sl}_2))$-module on which $K = X_{31}$, $K^{-1} = X_{13}$ and*

$$A = aX_{01} + a^{-1}X_{12}, \qquad A^* = bX_{23} + b^{-1}X_{30}.$$

Consider an L-operator for V with parameter a^2. Then on V,

$$\psi = -a(L_{00})^{-1}L_{01} \tag{5.3}$$

provided that L_{00} is invertible.

Proof. Let $\widehat{\psi}$ denote the expression on the right in (5.3). We show $\psi = \widehat{\psi}$. To do this, we show that $\widehat{\psi}$ satisfies (1.9), (1.10). Concerning (1.9), by Lemma 4.3 the equation $K_0\widehat{\psi} = q^2\widehat{\psi}K_0$ holds on V. By Lemma 3.1, Note 3.2, and the construction, we obtain $K_0 = X_{31} = K$ on V. By these comments $K\widehat{\psi} = q^2\widehat{\psi}K$ on V. By this and (1.5) we obtain $\widehat{\psi}U_i \subseteq U_{i-1}$ for $0 \leq i \leq d$. So $\widehat{\psi}$ satisfies (1.9). Next we show that $\widehat{\psi}$ satisfies (1.10). Since L_{00} is invertible and $K_0K_1 = 1$ it suffices to show that on V,

$$L_{00}(\widehat{\psi}R - R\widehat{\psi}) = (q - q^{-1})L_{00}(K_0 - K_1). \tag{5.4}$$

By this and (5.2) it suffices to show that on V,

$$aqL_{00}(\widehat{\psi}K_0F_0 - K_0F_0\widehat{\psi}) - a^{-1}L_{00}(\widehat{\psi}E_1 - E_1\widehat{\psi}) + L_{00}(K_1 - K_0) = 0. \tag{5.5}$$

We examine the terms in (5.5). By Lemma 4.3 and the construction, the following hold on V:

$$\begin{aligned} L_{00}\widehat{\psi}K_0F_0 &= -aL_{01}K_0F_0 \\ &= -aq^{-2}K_0L_{01}F_0 \\ &= -aq^{-1}K_0(F_0L_{01} - a^{-2}L_{00} + a^{-2}K_1L_{11}) \end{aligned}$$

and

$$\begin{aligned} L_{00}K_0F_0\widehat{\psi} &= K_0L_{00}F_0\widehat{\psi} \\ &= q^{-1}K_0(a^{-2}K_1L_{10} + F_0L_{00})\widehat{\psi} \\ &= q^{-1}K_0(a^{-2}K_1L_{10}\widehat{\psi} - aF_0L_{01}) \end{aligned}$$

and

$$\begin{aligned} L_{00}\widehat{\psi}E_1 &= -aL_{01}E_1 \\ &= -a(qE_1L_{01} + L_{11} - L_{00}K_1) \\ &= -a(qE_1L_{01} + L_{11} - K_1L_{00}) \end{aligned}$$

and

$$\begin{aligned} L_{00}E_1\widehat{\psi} &= (L_{10} + qE_1L_{00})\widehat{\psi} \\ &= L_{10}\widehat{\psi} - qaE_1L_{01} \end{aligned}$$

and

$$L_{00}K_1 = K_1L_{00}, \qquad L_{00}K_0 = K_0L_{00}.$$

To verify (5.5), evaluate its left-hand side using the above comments and simplify the result using $K_0K_1 = 1$. The computation is routine, and omitted. We have shown that $\widehat{\psi}$ satisfies (1.10). The result follows. □

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References

[1] P. Baseilhac, An integrable structure related with tridiagonal algebras, *Nuclear Phys. B* **705** (2005), 605–619, doi:10.1016/j.nuclphysb.2004.11.014.

[2] P. Baseilhac and S. Belliard, The half-infinite XXZ chain in Onsager's approach, *Nuclear Phys. B* **873** (2013), 550–584, doi:10.1016/j.nuclphysb.2013.05.003.

[3] P. Baseilhac and K. Koizumi, A deformed analogue of Onsager's symmetry in the XXZ open spin chain, *J. Stat. Mech. Theory Exp.* **10** (2005), P10005, doi:10.1088/1742-5468/2005/10/p10005.

[4] P. Baseilhac and K. Koizumi, A new (in)finite-dimensional algebra for quantum integrable models, *Nuclear Phys. B* **720** (2005), 325–347, doi:10.1016/j.nuclphysb.2005.05.021.

[5] P. Baseilhac and K. Koizumi, Exact spectrum of the XXZ open spin chain from the q-Onsager algebra representation theory, *J. Stat. Mech. Theory Exp.* **9** (2007), P09006, doi:10.1088/1742-5468/2007/09/p09006.

[6] P. Baseilhac and K. Shigechi, A new current algebra and the reflection equation, *Lett. Math. Phys.* **92** (2010), 47–65, doi:10.1007/s11005-010-0380-x.

[7] S. Bockting-Conrad, Two commuting operators associated with a tridiagonal pair, *Linear Algebra Appl.* **437** (2012), 242–270, doi:10.1016/j.laa.2012.02.007.

[8] S. Bockting-Conrad, Tridiagonal pairs of q-Racah type, the double lowering operator ψ, and the quantum algebra $U_q(\mathfrak{sl}_2)$, *Linear Algebra Appl.* **445** (2014), 256–279, doi:10.1016/j.laa.2013.12.007.

[9] S. Bockting-Conrad and P. Terwilliger, The algebra $U_q(\mathfrak{sl}_2)$ in disguise, *Linear Algebra Appl.* **459** (2014), 548–585, doi:10.1016/j.laa.2014.07.022.

[10] V. Chari and A. Pressley, Quantum affine algebras, *Commun. Math. Phys.* **142** (1991), 261–283, `https://projecteuclid.org/euclid.cmp/1104248585`.

[11] V. Chari and A. N. Pressley, *A Guide to Quantum Groups*, Cambridge University Press, Cambridge, 1995.

[12] T. Ito, K. Nomura and P. Terwilliger, A classification of sharp tridiagonal pairs, *Linear Algebra Appl.* **435** (2011), 1857–1884, doi:10.1016/j.laa.2011.03.032.

[13] T. Ito, K. Tanabe and P. Terwilliger, Some algebra related to P- and Q-polynomial association schemes, in: A. Barg and S. Litsyn (eds.), *Codes and Association Schemes*, American Mathematical Society, Providence, Rhode Island, volume 56 of *DIMACS Series in Discrete Mathematics and Theoretical Computer Science*, pp. 167–192, 2001, papers from the DIMACS Workshop held at Rutgers University, Piscataway, NJ, November 9 – 12, 1999.

[14] T. Ito and P. Terwilliger, The q-tetrahedron algebra and its finite dimensional irreducible modules, *Comm. Algebra* **35** (2007), 3415–3439, doi:10.1080/00927870701509180.

[15] T. Ito and P. Terwilliger, Tridiagonal pairs and the quantum affine algebra $U_q(\widehat{sl}_2)$, *Ramanujan J.* **13** (2007), 39–62, doi:10.1007/s11139-006-0242-4.

[16] T. Ito and P. Terwilliger, Two non-nilpotent linear transformations that satisfy the cubic q-Serre relations, *J. Algebra Appl.* **6** (2007), 477–503, doi:10.1142/s021949880700234x.

[17] T. Ito and P. Terwilliger, The augmented tridiagonal algebra, *Kyushu J. Math.* **64** (2010), 81–144, doi:10.2206/kyushujm.64.81.

[18] J. C. Jantzen, *Lectures on Quantum Groups*, volume 6 of *Graduate Studies in Mathematics*, American Mathematical Society, Providence, Rhode Island, 1996.

[19] C. Kassel, *Quantum Groups*, volume 155 of *Graduate Texts in Mathematics*, Springer-Verlag, New York, 1995, doi:10.1007/978-1-4612-0783-2.

[20] P. P. Kulish and E. K. Sklyanin, Solutions of the Yang-Baxter equation, *J. Sov. Math.* **19** (1982), 1596–1620, doi:10.1007/bf01091463.

[21] P. P. Kulish and N. Yu. Reshetikhin, Quantum linear problem for the sine-Gordon equation and higher representations, *J. Sov. Math.* **23** (1983), 2435–2441, doi:10.1007/bf01084171.

[22] K. Miki, Finite dimensional modules for the q-tetrahedron algebra, *Osaka J. Math.* **47** (2010), 559–589, `https://projecteuclid.org/euclid.ojm/1277298918`.

[23] P. Terwilliger, Two relations that generalize the q-Serre relations and the Dolan-Grady relations, in: A. N. Kirillov, A. Tsuchiya and H. Umemura (eds.), *Physics and Combinatorics 1999*, World Scientific Publishing, River Edge, New Jersey, pp. 377–398, 2001, doi: 10.1142/9789812810199_0013, proceedings of the International Workshop held at Nagoya University, Nagoya, August 23 – 27, 1999.

[24] P. Terwilliger, The equitable presentation for the quantum group $U_q(\mathfrak{g})$ associated with a symmetrizable Kac-Moody algebra $\mathfrak{g}$, *J. Algebra* **298** (2006), 302–319, doi:10.1016/j.jalgebra.2005.11.013.

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Vertex-quasiprimitive 2-arc-transitive digraphs*

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Abstract

We study vertex-quasiprimitive 2-arc-transitive digraphs, and reduce the problem of vertex-primitive 2-arc-transitive digraphs to almost simple groups. This includes a complete classification of vertex-quasiprimitive 2-arc-transitive digraphs where the action on vertices has O'Nan-Scott type SD or CD.

Keywords: Digraphs, vertex-quasiprimitive, 2*-arc-transitive.*

Math. Subj. Class.: 05C20, 05C25

1 Introduction

A *digraph* Γ is a pair $(V, \rightarrow)$ with a set V (of vertices) and an antisymmetric irreflexive binary relation $\rightarrow$ on V. All digraphs considered in this paper will be finite. For a non-negative integer s, an *s-arc* of Γ is a sequence $v_0, v_1, \dots, v_s$ of vertices with $v_i \rightarrow v_{i+1}$ for each $i = 0, \dots, s-1$. A 1-arc is also simply called an *arc*. We say Γ is *s-arc-transitive* if the group of all automorphisms of Γ (that is, all permutations of V that preserve the relation $\rightarrow$) acts transitively on the set of s-arcs. More generally, for a group G of automorphisms of Γ, we say Γ is (G, s)*-arc-transitive* if G acts transitively on the set of s-arcs of Γ.

A transitive permutation group G on a set Ω is said to be *primitive* if G does not preserve any nontrivial partition of Ω, and is said to be *quasiprimitive* if each nontrivial normal subgroup of G is transitive. It is easy to see that primitive permutation groups are necessarily quasiprimitive, but there are quasiprimitive permutation groups that are not primitive. We say a digraph is *vertex-primitive* if its automorphism group is primitive on the vertex set. The aim of this paper is to investigate finite vertex-primitive s-arc transitive digraphs with $s \geqslant 2$. However, we will often work in the more general quasiprimitive setting.

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There are many s-arc-transitive digraphs, see for example [2, 6, 7, 8]. In particular, for every integer $k \geqslant 2$ and every integer $s \geqslant 1$ there are infinitely many k-regular (G, s)-arc-transitive digraphs with G quasiprimitive on the vertex set (see the proof of Theorem 1 of [2]). On the other hand, the first known family of vertex-primitive 2-arc-transitive digraphs besides directed cycles was only recently discovered in [3]. The digraphs in this family are not 3-arc-transitive, which prompted the following question:

Question 1.1. Is there an upper bound on s for vertex-primitive s-arc-transitive digraphs that are not directed cycles?

The O'Nan-Scott Theorem divides the finite primitive groups into eight types and there is a similar theorem for finite quasiprimitive groups, see [9, Section 5]). For four of the eight types, a quasiprimitive group of that type has a normal regular subgroup. Praeger [8, Theorem 3.1] showed that if Γ is a $(G, 2)$-arc-transitive digraph and G has a normal subgroup that acts regularly on V, then Γ is a directed cycle. Thus to investigate vertex-primitive and vertex-quasiprimitive 2-arc-transitive digraphs, we only need to consider the four remaining types. One of these types is where G is an almost simple group, that is, where G has a unique minimal normal subgroup T, and T is a nonabelian simple group. The examples of primitive 2-arc-transitive digraphs constructed in [3] are of this type. This paper examines the remaining three types, which are labelled SD, CD and PA, and reduces Question 1.1 to almost simple vertex-primitive groups (Corollary 1.6). We now define these three types and state our results.

We say that a quasiprimitive group G on a set Ω is of type SD if G has a unique minimal normal subgroup N, there exists a nonabelian simple group T and positive integer $k \geqslant 2$ such that $N \cong T^k$, and for $\omega \in \Omega$, N_ω is a full diagonal subgroup of N (that is, $N_\omega \cong T$ and projects onto T in each of the k simple direct factors of N). It is incorrectly claimed in [8, Lemma 4.1] that there is no 2-arc-transitive digraph with a vertex-primitive group of automorphisms of type SD. However, there is an error in the proof which occurs when concluding "σx also fixes $D\mathtt{t}^{-1}$". Indeed, given a nonabelian simple group T, our Construction 3.1 yields a $(G, 2)$-arc-transitive digraph $\Gamma(T)$ with G primitive of type SD. These turn out to be the only examples.

Theorem 1.2. *Let Γ be a connected $(G, 2)$-arc-transitive digraph such that G is quasiprimitive of type* SD *on the set of vertices. Then there exists a nonabelian simple group T such that $\Gamma \cong \Gamma(T)$, as obtained from* Construction 3.1. *Moreover,* $\mathrm{Aut}(\Gamma)$ *is vertex-primitive of type SD and Γ is not 3-arc-transitive.*

The remaining two quasiprimitive types, CD and PA, both arise from product actions. For any digraph Σ and positive integer m, Σ^m denotes the direct product of m copies of Σ as in Notation 2.6. The wreath product $\mathrm{Sym}(\Delta) \wr \mathrm{S}_m = \mathrm{Sym}(\Delta)^m \rtimes \mathrm{S}_m$ acts naturally on the set Δ^m with product action. Let G_1 be the stabiliser in G of the first coordinate and let H be the projection of G_1 onto $\mathrm{Sym}(\Delta)$. If G projects onto a transitive subgroup of S_m, then a result of Kovács [4, (2.2)] asserts that up to conjugacy in $\mathrm{Sym}(\Delta)^m$ we may assume that $G \leqslant H \wr \mathrm{S}_m$. A reduction for 2-arc-transitive digraphs was sought in [8, Remark 4.3] but only partial results were obtained. Our next result yields the desired reduction.

Theorem 1.3. *Let $H \leqslant \mathrm{Sym}(\Delta)$ with transitive normal subgroup N and let $G \leqslant H \wr \mathrm{S}_m$ acting on $V = \Delta^m$ with product action such that G projects to a transitive subgroup of S_m and G has component H. Moreover, assume that $N^m \leqslant G$. If Γ is a (G, s)-arc-transitive* *digraph with vertex set V such that $s \geqslant 2$, then $\Gamma \cong \Sigma^m$ for some (H, s)-arc-transitive digraph Σ with vertex set Δ.*

A quasiprimitive group of type CD on a set Ω is one that has a product action on Ω and the component is quasiprimitive of type SD, while a quasiprimitive group of type PA on a set Ω is one that acts faithfully on some partition $\mathcal{P}$ of Ω and G has a product action on $\mathcal{P}$ such that the component H is an almost simple group. When G is primitive of type PA, H is primitive and the partition $\mathcal{P}$ is the partition into singletons, that is, G has a product action on Ω. As a consequence, we have the following corollaries.

Corollary 1.4. *Suppose Γ is a connected $(G, 2)$-arc-transitive digraph such that G is vertex-quasiprimitive of type* CD. *Then there exists a nonabelian simple group T and positive integer $m \geqslant 2$ such that $\Gamma \cong \Gamma(T)^m$, where $\Gamma(T)$ is as obtained from* Construction 3.1. *Moreover, Γ is not 3-arc-transitive.*

Corollary 1.5. *Suppose Γ is a (G, s)-arc-transitive digraph such that G is vertex-primitive of type* PA. *Then $\Gamma \cong \Sigma^m$ for some (H, s)-arc-transitive digraph Σ and integer $m \geqslant 2$ for some almost simple primitive permutation group $H \leqslant \mathrm{Aut}(\Sigma)$.*

We give an example in Section 2.3 of an infinite family of $(G, 2)$-arc-transitive digraphs Γ with G vertex-quasiprimitive of PA type such that Γ is not a direct power of a digraph Σ (indeed the number of vertices of Γ is not a proper power). We leave the investigation of such digraphs open.

We note that Theorem 1.2 and Corollaries 1.4 and 1.5, reduce Question 1.1 to studying almost simple primitive groups.

Corollary 1.6. *There exists an absolute upper bound C such that every vertex-primitive s-arc-transitive digraph that is not a directed cycle satisfies $s \leqslant C$, if and only if for every (G, s)-arc-transitive digraph with G a primitive almost simple group we have $s \leqslant C$.*

Theorem 1.2 follows immediately from a more general theorem (Theorem 3.15) given at the end of Section 3. Then in Section 4, we prove Theorem 1.3 as well as Corollaries 1.4–1.5 after establishing some general results for normal subgroups of s-arc-transitive groups.

2 Preliminaries

We say that a digraph Γ is *k-regular* if both the set $\Gamma^-(v) = \{u \in V \mid u \to v\}$ of in-neighbours of v and the set $\Gamma^+(v) = \{w \in V \mid v \to w\}$ of out-neighbours of v have size k for all $v \in V$, and we say that Γ is *regular* if it is k-regular for some positive integer k. Note that any vertex-transitive digraph is regular. Moreover, if Γ is regular and (G, s)-arc-transitive with $s \geqslant 2$ then it is also $(G, s-1)$-arc-transitive.

Recall that a digraph is said to be connected if and only if its underlying graph is connected. A vertex-primitive digraph is necessarily connected, for otherwise its connected components would form a partition of the vertex set that is invariant under digraph automorphisms.

2.1 Group factorizations

All the groups we consider in this paper are assumed to be finite. An expression of a group G as the product of two subgroups H and K of G is called a *factorization* of G. The following lemma lists several equivalent conditions for a group factorization, whose proof is fairly easy and so is omitted.

Lemma 2.1. *Let H and K be subgroups of G. Then the following are equivalent.*

(a) $G = HK$.

(b) $G = KH$.

(c) $G = (x^{-1}Hx)(y^{-1}Ky)$ *for any* $x, y \in G$.

(d) $|H \cap K||G| = |H||K|$.

(e) *H acts transitively on the set of right cosets of K in G by right multiplication.*

(f) *K acts transitively on the set of right cosets of H in G by right multiplication.*

The s-arc-transitivity of digraphs can be characterized by group factorizations as follows:

Lemma 2.2. *Let Γ be a G-arc-transitive digraph, $s \geqslant 2$ be an integer, and $v_0 \to v_1 \to \cdots \to v_{s-1} \to v_s$ be an s-arc of Γ. Then Γ is (G, s)-arc-transitive if and only if $G_{v_1 \ldots v_i} = G_{v_0 v_1 \ldots v_i} G_{v_1 \ldots v_i v_{i+1}}$ for each i in $\{1, \ldots, s-1\}$.*

Proof. For any i such that $1 \leqslant i \leqslant s-1$, the group $G_{v_1 \ldots v_i}$ acts on the set $\Gamma^+(v_i)$ of out-neighbours of v_i. Since $v_{i+1} \in \Gamma^+(v_i)$ and $G_{v_1 \ldots v_i v_{i+1}}$ is the stabilizer in $G_{v_1 \ldots v_i}$ of v_{i+1}, by Frattini's argument, the subgroup $G_{v_0 v_1 \ldots v_i}$ of $G_{v_1 \ldots v_i}$ is transitive on $\Gamma_+(v_i)$ if and only if $G_{v_1 \ldots v_i} = G_{v_0 v_1 \ldots v_i} G_{v_1 \ldots v_i v_{i+1}}$. Note that Γ is (G, s)-arc-transitive if and only if Γ is $(G, s-1)$-arc-transitive and $G_{v_0 v_1 \ldots v_i}$ is transitive on $\Gamma_+(v_i)$. One then deduces inductively that Γ is (G, s)-arc-transitive if and only if $G_{v_1 \ldots v_i} = G_{v_0 v_1 \ldots v_i} G_{v_1 \ldots v_i v_{i+1}}$ for each i in $\{1, \ldots, s-1\}$. □

If Γ is a G-arc-transitive digraph and $u \to v$ is an arc of Γ, then since G is vertex-transitive we can write $v = u^g$ for some $g \in G$ and it follows that

$$v^{g^{-1}} \to v \to \cdots \to v^{g^{s-2}} \to v^{g^{s-1}} \tag{2.1}$$

is an s-arc of Γ. In this setting, Lemma 2.2 is reformulated as follows.

Lemma 2.3. *Let Γ be a G-arc-transitive digraph, $s \geqslant 2$ be an integer, v be a vertex of Γ, and $g \in G$ such that $v \to v^g$. Then Γ is (G, s)-arc-transitive if and only if*

$$\bigcap_{j=0}^{i-1} g^{-j} G_v g^j = \left(\bigcap_{j=0}^{i} g^{-(j-1)} G_v g^{j-1} \right) \left(\bigcap_{j=0}^{i} g^{-j} G_v g^j \right)$$

for each i in $\{1, \ldots, s-1\}$.

Proof. Let $v_j = v^{g^{j-1}}$ for any integer j such that $0 \leqslant j \leqslant s-1$. Then the s-arc (2.1) of Γ turns out to be $v_0 \to v_1 \to \cdots \to v_{s-1} \to v_s$, and for any i in $\{1, \ldots, s\}$ we have

$$G_{v_1 \ldots v_i} = \bigcap_{j=1}^{i} G_{v_j} = \bigcap_{j=1}^{i} g^{-(j-1)} G_v g^{j-1} = \bigcap_{j=0}^{i-1} g^{-j} G_v g^j$$

and

$$G_{v_0 v_1 \ldots v_i} = \bigcap_{j=0}^{i} G_{v_j} = \bigcap_{j=0}^{i} g^{-(j-1)} G_v g^{j-1}.$$

Hence the conclusion of the lemma follows from Lemma 2.2. □

2.2 Constructions of s-arc-transitive digraphs

Let G be a group, H be a subgroup of G, V be the set of right cosets of H in G and g be an element of $G \setminus H$ such that $g^{-1} \notin HgH$. Define a binary relation $\rightarrow$ on V by letting $Hx \rightarrow Hy$ if and only if $yx^{-1} \in HgH$ for any $x, y \in G$. Then $(V, \rightarrow)$ is a digraph, denoted by $\mathrm{Cos}(G, H, g)$. Right multiplication gives an action R_H of G on V that preserves the relation $\rightarrow$, so that $R_H(G)$ is a group of automorphisms of $\mathrm{Cos}(G, H, g)$.

Lemma 2.4. *In the above notation, the following hold.*

(a) $\mathrm{Cos}(G, H, g)$ *is* $|H{:}H \cap g^{-1}Hg|$*-regular.*

(b) $\mathrm{Cos}(G, H, g)$ *is* $R_H(G)$*-arc-transitive.*

(c) $\mathrm{Cos}(G, H, g)$ *is connected if and only if* $\langle H, g\rangle = G$*.*

(d) $\mathrm{Cos}(G, H, g)$ *is* $R_H(G)$*-vertex-primitive if and only if* H *is maximal in* G*.*

(e) Let $s \geqslant 2$ *be an integer. Then* $\mathrm{Cos}(G, H, g)$ *is* $(R_H(G), s)$*-arc-transitive if and only if for each* i *in* $\{1, \dots, s-1\}$*,*

$$\bigcap_{j=0}^{i-1} g^{-j}Hg^{j} = \left(\bigcap_{j=0}^{i} g^{-(j-1)}Hg^{j-1}\right)\left(\bigcap_{j=0}^{i} g^{-j}Hg^{j}\right).$$

Proof. Parts (a)–(d) are folklore (see for example [2]), and part (e) is derived in light of Lemma 2.3. □

Remark 2.5. Lemma 2.4 establishes a group theoretic approach to constructing s-arc-transitive digraphs. In particular, $\mathrm{Cos}(G, H, g)$ is $(R_H(G), 2)$-arc-transitive if and only if $H = (gHg^{-1} \cap H)(H \cap g^{-1}Hg)$.

Next we show how to construct s-arc-transitive digraphs from existing ones. Let Γ be a digraph with vertex set U and Σ be a digraph with vertex set V. The *direct product* of Γ and Σ, denoted $\Gamma \times \Sigma$, is the digraph (it is easy to verify that this is indeed a digraph) with vertex set $U \times V$ and $(u_1, v_1) \rightarrow (u_2, v_2)$ if and only if $u_1 \rightarrow u_2$ and $v_1 \rightarrow v_2$, where $u_i \in U$ and $v_i \in V$ for $i = 1, 2$.

Notation 2.6. For any digraph Σ and positive integer m, denote by Σ^m the direct product of m copies of Σ.

Lemma 2.7. *Let* s *be a positive integer,* Γ *be a* (G, s)*-arc-transitive digraph and* Σ *be a* (H, s)*-arc-transitive digraph. Then* $\Gamma \times \Sigma$ *is a* $(G \times H, s)$*-arc-transitive digraph, where* $G \times H$ *acts on the vertex set of* $\Gamma \times \Sigma$ *by product action.*

Proof. Let $(u_0, v_0) \rightarrow (u_1, v_1) \rightarrow \cdots \rightarrow (u_s, v_s)$ and $(u_0', v_0') \rightarrow (u_1', v_1') \rightarrow \cdots \rightarrow (u_s', v_s')$ be any two s-arcs of $\Gamma \times \Sigma$. Then $u_0 \rightarrow u_1 \rightarrow \cdots \rightarrow u_s$ and $u_0' \rightarrow u_1' \rightarrow \cdots \rightarrow u_s'$ are s-arcs of Γ while $v_0 \rightarrow v_1 \rightarrow \cdots \rightarrow v_s$ and $v_0' \rightarrow v_1' \rightarrow \cdots \rightarrow v_s'$ are s-arcs of Σ. Since Γ is (G, s)-arc-transitive, there exists $g \in G$ such that $u_i^g = u_i'$ for each i with $0 \leqslant i \leqslant s$. Similarly, there exists $h \in H$ such that $v_i^h = v_i'$ for each i with $0 \leqslant i \leqslant s$. It follows that $(u_i, v_i)^{(g,h)} = (u_i', v_i')$ for each i with $0 \leqslant i \leqslant s$. This means that $\Gamma \times \Sigma$ is a $(G \times H, s)$-arc-transitive. □

2.3 Example

In this subsection we give an example of an infinite family of $(G, 2)$-arc-transitive digraphs Γ with G vertex-quasiprimitive of PA type such that Γ is not a direct power of a digraph Σ. In fact, we prove in Lemma 2.9 that the number of vertices of Γ is not a proper power.

Let $n \geqslant 5$ be odd, $G_1 = \mathrm{Alt}(\{1, 2, \ldots, n\})$ and $G_2 = \mathrm{Alt}(\{n+1, n+2, \ldots, 2n\})$. Take permutations

$$a = (1, n+1)(2, n+2)\cdots(n, 2n), \quad b = (1, 2)(3, 4)(n+1, n+2)(n+3, n+4)$$

and

$$\begin{aligned} g = (&1, n+2, 2, n+3, 5, n+6, 7, n+8, \ldots, 2i-1, n+2i, \ldots, n-2, 2n-1, n, \\ &n+1, 3, n+4, 4, n+5, 6, n+7, \ldots, 2j, n+2j+1, \ldots, n-1, 2n). \end{aligned}$$

In fact, $g = ac$ with

$$c = (1, 3, 5, 6, 7, \ldots, n)(n+1, n+2, \ldots, 2n).$$

Let $G = (G_1 \times G_2) \rtimes \langle a \rangle$, and note that $g \in G$ as $c \in G_1 \times G_2$. Let $H = \langle a, b \rangle = \langle a \rangle \times \langle b \rangle$ and $\Gamma_n = \mathrm{Cos}(G, H, g)$.

Lemma 2.8. *For all odd $n \geqslant 5$, Γ_n is a connected $(G, 2)$-arc-transitive digraph with G quasiprimitive of PA type on the vertex set.*

Proof. As $(G_1 \times G_2) \cap H = \langle b \rangle$ we see that G is quasiprimitive of PA type on the vertex set. To show that Γ_n is connected, we shall show $\langle H, g \rangle = G$ in light of Lemma 2.4(c). Let $M = \langle H, g \rangle \cap (G_1 \times G_2)$. Then we only need to show $M = G_1 \times G_2$ since $a \in \langle H, g \rangle$.

Denote the projections of $G_1 \times G_2$ onto G_1 and G_2, respectively, by π_1 and π_2. Note that g^2 fixes both $\{1, \ldots, n\}$ and $\{n+1, \ldots, 2n\}$ setwise with

$$\pi_1(g^2) = (1, 2, 5, 7, \ldots, 2i-1, \ldots, n, 3, 4, 6, \ldots, 2j, \ldots, n-1)$$

and

$$\pi_1(g^{n+1}) = (1, 3, 2, 4, 5, \ldots, n).$$

We have $g^2 \in M$ and

$$\pi_1(g^{-(n+1)} b g^{n+1} b) = \pi_1(g^{-(n+1)} b g^{n+1}) \pi_1(b) = (3, 4)(2, 5)(1, 2)(3, 4) = (1, 2, 5),$$

which implies

$$\pi_1(M) \geqslant \pi_1(\langle g^2, b \rangle) \geqslant \pi_1(\langle g^2, g^{-(n+1)} b g^{n+1} b \rangle) = \langle \pi_1(g^2), \pi_1(g^{-(n+1)} b g^{n+1} b) \rangle = G_1$$

using the fact that the permutation group generated by a 3-cycle (α, β, γ) and an n-cycle with first 3-entries α, β, γ is A_n. It follows that

$$\pi_2(M) = \pi_2(M^a) = (\pi_2(M))^a = G_1^a = G_2,$$

and so M is either $G_1 \times G_2$ or a full diagonal subgroup of $G_1 \times G_2$. However, $c = ag \in M$ while $\pi_1(c)$ and $\pi_2(c)$ have different cycle types. We conclude that M is not a diagonal subgroup of $G_1 \times G_2$, and so $M = G_1 \times G_2$ as desired.

Now we prove that Γ_n is $(G, 2)$-arc-transitive, which is equivalent to proving that $H = (gHg^{-1} \cap H)(H \cap g^{-1}Hg)$ according to Lemma 2.4(e). In view of

$$(ab)^g = (ab)^{ac} = (ab)^c = a \tag{2.2}$$

we deduce that $a \in H \cap H^g$. Since H is not normal in $G = \langle H, g\rangle$, we have $H^g \neq H$. Consequently, $H \cap H^g = \langle a\rangle$. Then again by (2.2) we deduce that

$$H \cap H^{g^{-1}} = (H \cap H^g)^{g^{-1}} = \langle a\rangle^{g^{-1}} = \langle a^{g^{-1}}\rangle = \langle ab\rangle.$$

This yields

$$(gHg^{-1} \cap H)(H \cap g^{-1}Hg) = \langle a\rangle\langle ab\rangle = H. \tag{2.3}$$

Finally, the condition $g^{-1} \notin HgH$ holds as a consequence (see [3, Lemma 2.3]) of (2.3) and the conclusion $H^g \neq H$. This completes the proof. □

Lemma 2.9. *The number of vertices of Γ_n is not a proper power for any odd $n \geqslant 5$.*

Proof. Suppose that the number of vertices of Γ_n is m^k for some $m \geqslant 2$ and $k \geqslant 2$. Then we have

$$m^k = \frac{|G|}{|H|} = \frac{2(n!/2)^2}{4} = \frac{(n!)^2}{8} \tag{2.4}$$

If $k = 2$, then (2.4) gives $(n!)^2 = 2(2m)^2$, which is not possible. Hence $k \geqslant 3$. By Bertrand's Postulate, there exists a prime number p such that $n/2 < p < n$. Thus, the largest p-power dividing $n!$ is p, and so the largest p-power dividing the right hand side of (2.4) is p^2. However, this implies that the largest p-power dividing m^k is p^2, contradicting the conclusion $k \geqslant 3$. □

2.4 Normal subgroups

Lemma 2.10. *Let Γ be a (G, s)-arc-transitive digraph with $s \geqslant 2$, M be a vertex-transitive normal subgroup of G, and $v_1 \to \cdots \to v_s$ be an $(s-1)$-arc of Γ. Then $G = MG_{v_1\dots v_i}$ for each i in $\{1, \dots, s\}$.*

Proof. Since M is transitive on the vertex set of Γ, there exists $m \in M$ such that $v_1^m = v_2$. Denote $u_i = v_1^{m^{i-1}}$ for each i such that $0 \leqslant i \leqslant s$. Then $G_{u_0u_1\dots u_i} = mG_{u_1\dots u_iu_{i+1}}m^{-1}$ for each i such that $0 \leqslant i \leqslant s-1$, and $u_0 \to u_1 \to \cdots \to u_s$ is an s-arc of Γ since $v_1 \to v_2$ and m is an automorphism of Γ. For each i in $\{1, \dots, s-1\}$, we deduce from Lemma 2.2 that

$$G_{u_1\dots u_i} = G_{u_0u_1\dots u_i}G_{u_1\dots u_iu_{i+1}} = (mG_{u_1\dots u_iu_{i+1}}m^{-1})G_{u_1\dots u_iu_{i+1}}.$$

Let φ be the projection from G to G/M. It follows that

$$\begin{aligned}
\varphi(G_{u_1\dots u_i}) &= \varphi(m)\varphi(G_{u_1\dots u_iu_{i+1}})\varphi(m)^{-1}\varphi(G_{u_1\dots u_iu_{i+1}}) \\
&= \varphi(G_{u_1\dots u_iu_{i+1}})\varphi(G_{u_1\dots u_iu_{i+1}}) \\
&= \varphi(G_{u_1\dots u_iu_{i+1}})
\end{aligned}$$

and so $G_{u_1\dots u_i}M = G_{u_1\dots u_iu_{i+1}}M$ for each i in $\{1, \dots, s-1\}$. Again as M is transitive on the vertex set of Γ, we have $G = MG_{u_1}$. Hence

$$G = MG_{u_1} = MG_{u_1u_2} = \cdots = MG_{u_1\dots u_i} = \cdots = MG_{u_1\dots u_s}.$$

Now for each i in $\{1, \dots, s\}$, the digraph Γ is $(G, i-1)$-arc-transitive, so there exists $g \in G$ such that $(v_1^g, \dots, v_i^g) = (u_1, \dots, u_i)$. Hence

$$G = MG_{u_1 \dots u_i} = M(g^{-1} G_{v_1 \dots v_i} g) = MG_{v_1 \dots v_i}$$

by Lemma 2.1(c). □

By Frattini's argument, we have the following consequence of Lemma 2.10:

Corollary 2.11. *Let Γ be a (G, s)-arc-transitive digraph with $s \geqslant 2$, and M be a vertex-transitive normal subgroup of G. Then Γ is $(M, s-1)$-arc-transitive.*

To close this subsection, we give a short proof of the following result of Praeger [8, Theorem 3.1] using Lemma 2.10.

Proposition 2.12. *Let Γ be a $(G, 2)$-arc-transitive digraph. If G has a vertex-regular normal subgroup, then Γ is a directed cycle.*

Proof. Let N be a vertex-regular normal subgroup of G, and $u \to v$ be an arc of Γ. Then $|G|/|N| = |G_v|$, and $G = G_{uv}N$ by Lemma 2.10. Hence by Lemma 2.1(d), $|G_{uv}| \geqslant |G|/|N| = |G_v|$ and so $|G_{uv}| = |G_v|$. Consequently, Γ is 1-regular, which means that Γ is a directed cycle. □

2.5 Two technical lemmas

Lemma 2.13. *Let A be an almost simple group with socle T and L be a nonabelian simple group. Suppose $L^n \leqslant A$ and $|T| \leqslant |L^n|$ for some positive integer n. Then $n = 1$ and $L = T$.*

Proof. Note that $L^n/(L^n \cap T) \cong (L^n T)/T \leqslant A/T$, which is solvable by the Schreier conjecture. If $L^n \cap T \neq L^n$, then $L^n/(L^n \cap T) \cong L^m$ for some positive integer m, a contradiction. Hence $L^n \cap T = L^n$, which means $L^n \leqslant T$. This together with the condition that $|T| \leqslant |L^n|$ implies $L^n = T$. Hence $n = 1$ and $L = T$, as the lemma asserts. □

Lemma 2.14. *Let A be an almost simple group with socle T and S be a primitive permutation group on $|T|$ points. Then S is not isomorphic to any subgroup of A.*

Proof. Suppose for a contradiction that $S \lesssim A$. Regard S as a subgroup of A, and write $\mathrm{Soc}(S) = L^n$ for some simple group L and positive integer n. Since S is primitive on $|T|$ points, $\mathrm{Soc}(S)$ is transitive on $|T|$ points, and so $|T|$ divides $|\mathrm{Soc}(S)| = |L|^n$. Consequently, L is nonabelian for otherwise T would be solvable. Then by Lemma 2.13 we have $\mathrm{Soc}(S) = L = T$. It follows that S is an almost simple primitive permutation group with $\mathrm{Soc}(S)$ regular, contradicting [5]. □

3 Vertex-quasiprimitive of type SD

3.1 Constructing the graph $\Gamma(T)$

Construction 3.1. *Let T be a nonabelian simple group of order k with $T = \{t_1, \dots, t_k\}$. Let $D = \{(t, \dots, t) \mid t \in T\}$ be a full diagonal subgroup of T^k and let $g = (t_1, \dots, t_k)$. Define $\Gamma = \mathrm{Cos}(T^k, D, g)$ and let V be the set of right cosets of D in T^k, i.e. the vertex set of $\Gamma(T)$.*

Lemma 3.2. *$\Gamma(T)$ is a $|T|$-regular digraph.*

Proof. Suppose that $D\cap g^{-1}Dg \neq 1$. Then there exist $s, t \in T\setminus\{1\}$ such that $(s, \ldots, s) = (t_1^{-1}tt_1, \ldots, t_k^{-1}tt_k)$. Thus $s = t_i^{-1}tt_i$ for each i such that $1 \leqslant i \leqslant k$. Since $\{t_i \mid 1 \leqslant i \leqslant k\} = T$, we have $t_j = 1$ for some $1 \leqslant j \leqslant k$. It then follows from the equality $s = t_j^{-1}tt_j$ that $s = t$. Thus $t = t_i^{-1}tt_i$ for each i such that $1 \leqslant i \leqslant k$. Hence t lies in the center of T, which implies $t = 1$ as T is nonabelian simple, a contradiction. Consequently, $D \cap g^{-1}Dg = 1$, and so $\mathrm{Cos}(T^k, D, g)$ is $|T|$-regular as $|D|/|D \cap g^{-1}Dg| = |D| = |T|$.

Suppose that $g^{-1} \in DgD$. Then there exist $s, t \in T$ such that $(t_1^{-1}, \ldots, t_k^{-1}) = (st_1t, \ldots, st_kt)$. It follows that $t_i^{-1} = st_it$ for each i such that $1 \leqslant i \leqslant k$. Since $\{t_i \mid 1 \leqslant i \leqslant k\} = T$, we have $t_j = 1$ for some $1 \leqslant j \leqslant k$. Then the equality $t_j^{-1} = st_jt$ leads to $s = t^{-1}$. Thus $t_i^{-1} = t^{-1}t_it$ for each i such that $1 \leqslant i \leqslant k$. This implies that the inverse map is an automorphism of T and so T is abelian, a contradiction. Hence $g^{-1} \notin DgD$, from which we deduce that $\mathrm{Cos}(T^k, D, g)$ is a digraph, completing the proof. $\square$

Next we show that up to isomorphism, the definition of $\Gamma(T)$ does not depend on the order of $t_1, t_2, \ldots, t_k$.

Lemma 3.3. *Let $g' = (t'_1, \ldots, t'_k)$ such that $T = \{t'_1, \ldots, t'_k\}$. Then $\mathrm{Cos}(T^k, D, g) \cong \mathrm{Cos}(T^k, D, g')$.*

Proof. Since $\{t'_1, \ldots, t'_k\} = \{t_1, \ldots, t_k\}$, there exists $x \in \mathrm{S}_k$ such that $t_{i^x} = t'_i$ for each i with $1 \leqslant i \leqslant k$. Define an automorphism λ of T^k by $(g_1, \ldots, g_k)^\lambda = (g_{1^x}, \ldots, g_{k^x})$ for all $(g_1, \ldots, g_k) \in T^k$. Then λ normalizes D and $\lambda^{-1}g\lambda = g'$. Hence the map $Dh \mapsto Dh^\lambda$ gives an isomorphism from $\mathrm{Cos}(T^k, D, g)$ to $\mathrm{Cos}(T^k, D, g')$. $\square$

For any $t \in T$, let $x(t)$ and $y(t)$ be the elements of S_k such that $t_{i^{x(t)}} = tt_i$ and $t_{i^{y(t)}} = t_it^{-1}$ for any $1 \leqslant i \leqslant k$, and define permutations $\lambda(t)$ and $\rho(t)$ of V by letting

$$D(g_1, \ldots, g_k)^{\lambda(t)} = D(g_{1^{x(t)}}, \ldots, g_{k^{x(t)}})$$

and

$$D(g_1, \ldots, g_k)^{\rho(t)} = D(g_{1^{y(t)}}, \ldots, g_{k^{y(t)}})$$

for any $(g_1, \ldots, g_k) \in T^k$. For any $\varphi \in \mathrm{Aut}(T)$, let $z(\varphi) \in \mathrm{S}_k$ such that $t_{i^{z(\varphi)}} = t_i^\varphi$ for any $1 \leqslant i \leqslant k$, and define $\delta(\varphi) \in \mathrm{Sym}(V)$ by letting

$$D(g_1, \ldots, g_k)^{\delta(\varphi)} = D((g_{1^{z(\varphi^{-1})}})^\varphi, \ldots, (g_{k^{z(\varphi^{-1})}})^\varphi)$$

for any $(g_1, \ldots, g_k) \in T^k$. In particular, $\delta(\varphi)$ both permutes the coordinates and acts on each entry.

Lemma 3.4. *λ and ρ are monomorphisms from T to $\mathrm{Sym}(V)$, and δ is a monomorphism from $\mathrm{Aut}(T)$ to $\mathrm{Sym}(V)$.*

Proof. For any $s, t \in T$, noting that $x(t)x(s) = x(st)$, we have

$$\begin{aligned} D(g_1, \ldots, g_k)^{\lambda(s)\lambda(t)} &= D(g_{1^{x(s)}}, \ldots, g_{k^{x(s)}})^{\lambda(t)} \\ &= D(g_{1^{x(t)x(s)}}, \ldots, g_{k^{x(t)x(s)}}) \\ &= D(g_{1^{x(st)}}, \ldots, g_{k^{x(st)}}) \\ &= D(g_1, \ldots, g_k)^{\lambda(st)} \end{aligned}$$

for each $(g_1, \dots, g_k) \in T^k$, and so $\lambda(st) = \lambda(s)\lambda(t)$. This means that λ is a homomorphism from T to $\mathrm{Sym}(V)$. Moreover, since $\lambda(t)$ acts on V as the permutation $x(t)$ on the entries, $\lambda(t) = 1$ if and only if $x(t) = 1$, which is equivalent to $t = 1$. Hence λ is a monomorphism from T to $\mathrm{Sym}(V)$. Similarly, ρ is a monomorphism from T to $\mathrm{Sym}(V)$.

For any $\varphi, \psi \in \mathrm{Aut}(T)$, since $z(\psi^{-1})z(\varphi^{-1}) = z(\psi^{-1}\varphi^{-1}) = z((\varphi\psi)^{-1})$, we have

$$\begin{aligned} D(g_1, \dots, g_k)^{\delta(\varphi)\delta(\psi)} &= D((g_{1^{z(\varphi^{-1})}})^{\varphi}, \dots, (g_{k^{z(\varphi^{-1})}})^{\varphi})^{\delta(\psi)} \\ &= D((g_{1^{z(\psi^{-1})z(\varphi^{-1})}})^{\varphi\psi}, \dots, (g_{k^{z(\psi^{-1})z(\varphi^{-1})}})^{\varphi\psi}) \\ &= D(g_1, \dots, g_k)^{\delta(\varphi\psi)} \end{aligned}$$

for all $(g_1, \dots, g_k) \in T^k$. This means that δ is a homomorphism from $\mathrm{Aut}(T)$ to $\mathrm{Sym}(V)$. Next we prove that δ is a monomorphism. Let $\varphi \in \mathrm{Aut}(T)$ such that

$$D((g_{1^{z(\varphi^{-1})}})^{\varphi}, \dots, (g_{k^{z(\varphi^{-1})}})^{\varphi}) = D(g_1, \dots, g_k)^{\delta(\varphi)} = D(g_1, \dots, g_k) \tag{3.1}$$

for each $(g_1, \dots, g_k) \in T^k$. Take any $i \in \{1, \dots, k\}$ and $(g_1, \dots, g_k) \in T^k$ such that $g_j = 1$ for all $j \neq i$ and $g_i \neq 1$. By (3.1), there exists $t \in T$ such that $(g_{j^{z(\varphi^{-1})}})^{\varphi} = tg_j$ for each $j \in \{1, \dots, k\}$. As a consequence, we obtain $t = 1$ by taking any $j \in \{1, \dots, k\} \setminus \{i\}$ such that $j^{z(\varphi^{-1})} \neq i$. Also, for $j \in \{1, \dots, k\}$, $(g_{j^{z(\varphi^{-1})}})^{\varphi} \neq t$ if and only if $j = i$. It follows that $i^{z(\varphi^{-1})} = i$. As i is arbitrary, this implies that $z(\varphi^{-1}) = 1$, and so $\varphi = 1$. This shows that δ is a monomorphism from $\mathrm{Aut}(T)$ to $\mathrm{Sym}(V)$. $\square$

Let M be the permutation group on V induced by the right multiplication action of T^k. For any group X, the *holomorph* of X, denoted by $\mathrm{Hol}(X)$, is the normalizer of the right regular representation of X in $\mathrm{Sym}(X)$. Note that $\langle x(T), y(T), z(\mathrm{Aut}(T))\rangle = x(T) \rtimes z(\mathrm{Aut}(T)) = y(T) \rtimes z(\mathrm{Aut}(T))$ is primitive on $\{1, \dots, k\}$ and permutation isomorphic to $\mathrm{Hol}(T)$. Thus,

$$X := \langle M, \lambda(T), \rho(T), \delta(\mathrm{Aut}(T))\rangle \tag{3.2}$$

is a primitive permutation group on V of type SD with socle M, and the conjugation action of X on the set of k factors of $M \cong T^k$ is permutation isomorphic to $\mathrm{Hol}(T)$. Let $v = D \in V$, a vertex of $\Gamma(T)$. For any $t \in T$ let $\sigma(t) \in M$ be the permutation of V induced by right multiplication by $(t, \dots, t)$. Then

$$X_v/\sigma(T) = X_v/(X_v \cap M) \cong X_vM/M = X/M \cong \mathrm{Hol}(T)$$

since M acts transitively on V, and therefore

$$|X_v| = |\sigma(T)||\mathrm{Hol}(T)| = |T|^3|\mathrm{Out}(T)|. \tag{3.3}$$

Lemma 3.5. $X \leqslant \mathrm{Aut}(\Gamma(T))$.

Proof. Clearly $M \leqslant \mathrm{Aut}(\Gamma(T))$, so it remains to verify that $\lambda(T)$, $\rho(T)$ and $\delta(\mathrm{Aut}(T))$ are subgroups of $\mathrm{Aut}(\Gamma(T))$. Let $D(g_1, \dots, g_k) \in V$ and $D(g_1', \dots, g_k') \in V$. Then we have $D(g_1, \dots, g_k) \to D(g_1', \dots, g_k')$ in $\Gamma(T)$ if and only if

$$(g_1'g_1^{-1}, \dots, g_k'g_k^{-1}) \in D(t_1, \dots, t_k)D. \tag{3.4}$$

Let $t \in T$. Since (3.4) holds if and only if

$$\begin{aligned}(g'_{1^{x(t)}}g^{-1}_{1^{x(t)}},\dots,g'_{k^{x(t)}}g^{-1}_{k^{x(t)}}) &\in D(t_{1^{x(t)}},\dots,t_{k^{x(t)}})D\\ &= D(tt_1,\dots,tt_k)D\\ &= D(t_1,\dots,t_k)D,\end{aligned}$$

we conclude that $D(g_1,\dots,g_k) \to D(g'_1,\dots,g'_k)$ if and only if $D(g_1,\dots,g_k)^{\lambda(t)} \to D(g'_1,\dots,g'_k)^{\lambda(t)}$. This shows $\lambda(t) \in \mathrm{Aut}(\Gamma(T))$ for any $t \in T$. Similarly, we have $\rho(t) \in \mathrm{Aut}(\Gamma(T))$ for any $t \in T$. Let $\varphi \in \mathrm{Aut}(T)$. Then (3.4) holds if and only if

$$\begin{aligned}((g'_{1^{z(\varphi^{-1})}}g^{-1}_{1^{z(\varphi^{-1})}})^{\varphi},\dots,(g'_{k^{z(\varphi^{-1})}}g^{-1}_{k^{z(\varphi^{-1})}})^{\varphi}) &\in D((t_{1^{z(\varphi^{-1})}})^{\varphi},\dots,(t_{k^{z(\varphi^{-1})}})^{\varphi})D\\ &= D((t_1^{\varphi^{-1}})^{\varphi},\dots,(t_k^{\varphi^{-1}})^{\varphi})D\\ &= D(t_1,\dots,t_k)D.\end{aligned}$$

It follows that $D(g_1,\dots,g_k) \to D(g'_1,\dots,g'_k)$ if and only if

$$D(g_1,\dots,g_k)^{\delta(\varphi)} \to D(g'_1,\dots,g'_k)^{\delta(\varphi)},$$

and so $\delta(\varphi) \in \mathrm{Aut}(\Gamma(T))$ for any $\varphi \in \mathrm{Aut}(T)$. This completes the proof. □

Denote $H = \langle M, \lambda(T)\rangle = M \rtimes \lambda(T) \leqslant X$.

Lemma 3.6. $\Gamma(T)$ *is* $(H,2)$*-arc-transitive.*

Proof. It is readily seen that $H_v = \sigma(T) \times \lambda(T) \cong T^2$. Let $K = \{\sigma(t)\lambda(t) \mid t \in T\}$. For any $t \in T$ and any $(g_1,\dots,g_k) \in T^k$ we have

$$\begin{aligned}D(g_1,\dots,g_k)^{g^{-1}\sigma(t)\lambda(t)g} &= D(g_1t_1^{-1}t,\dots,g_kt_k^{-1}t)^{\lambda(t)g}\\ &= D(g_{1^{x(t)}}t^{-1}_{1^{x(t)}}t,\dots,g_{k^{x(t)}}t^{-1}_{k^{x(t)}}t)^g\\ &= D(g_{1^{x(t)}}(tt_1)^{-1}t,\dots,g_{k^{x(t)}}(tt_k)^{-1}t)^g\\ &= D(g_{1^{x(t)}}t_1^{-1},\dots,g_{k^{x(t)}}t_k^{-1})^g\\ &= D(g_{1^{x(t)}},\dots,g_{k^{x(t)}})\\ &= D(g_1,\dots,g_k)^{\lambda(t)}.\end{aligned}$$

Hence $g^{-1}\sigma(t)\lambda(t)g = \lambda(t)$ for all $t \in T$. Consequently, $g^{-1}Kg = \lambda(T) < H_v$ and so $K \leqslant H_v \cap gH_vg^{-1}$. Now for any elements s and t of T,

$$\sigma(s)\lambda(t) = (\sigma(s)\lambda(s))\lambda(s^{-1}t) \in K\lambda(T) = K(g^{-1}Kg).$$

It follows that

$$H_v \leqslant K(g^{-1}Kg) \leqslant (H_v \cap gH_vg^{-1})(H_v \cap g^{-1}H_vg),$$

so $H_v = (H_v \cap gH_vg^{-1})(H_v \cap g^{-1}H_vg)$. Thus by Remark 2.5, $\Gamma(T)$ is $(H,2)$-arc-transitive, as the lemma asserts. □

An immediate consequence of Lemma 3.6 is that $\Gamma(T)$ is $(X,2)$-arc-transitive. However, X is not transitive on the set of 3-arcs of $\Gamma(T)$, as we shall see in the next lemma.

Lemma 3.7. $\Gamma(T)$ *is not* $(X, 3)$*-arc-transitive.*

Proof. Suppose that $\Gamma(T)$ is $(X, 3)$-arc-transitive. Then since M is a vertex-transitive normal subgroup of X, Corollary 2.11 asserts that $\Gamma(T)$ is $(M, 2)$-arc-transitive. As a consequence, M_v is transitive on $A_2(v) := \{(v_1, v_2) \in V^2 \mid v \to v_1 \to v_2\}$, the set of 2-arcs starting from v. However, $|M_v| = |T|$ while $|A_2(v)| = |T|^2$ as $\Gamma(T)$ is $|T|$-regular. This is not possible. □

3.2 Classification

Throughout this subsection, let T be a nonabelian simple group, $k \geqslant 2$ be an interger, $D = \{(t, \ldots, t) \mid t \in T\}$ be a full diagonal subgroup of T^k, V be the set of right cosets of D in T^k, and M be the permutation group induced by the right multiplication action of T^k on V. Suppose that G is a permutation group on V with $M \leqslant G \leqslant M.(\mathrm{Out}(T) \times \mathrm{S}_k)$, and Γ is a connected $(G, 2)$-arc-transitive digraph. Let $v = D \in V$ and w be an out-neighbour of v. Then $w = D(t_1, \ldots, t_k) \in V$ for some elements $t_1, \ldots, t_k$ of T which are not all equal. Without loss of generality, we assume $t_k = 1$. Let $u = D(t_1^{-1}, \ldots, t_k^{-1}) \in V$ and $g \in M$ be the permutation of V induced by right multiplication by $(t_1, \ldots, t_k) \in T^k$. Moreover, define $\{\Omega_1, \ldots, \Omega_n\}$ to be the partition of $\{1, \ldots, k\}$ such that $t_i = t_j$ if and only if i and j are in the same part of $\{\Omega_1, \ldots, \Omega_n\}$. Note that $G_v \leqslant \mathrm{Aut}(T) \times \mathrm{S}_k$. Let α be the projection of G_v into $\mathrm{Aut}(T)$ and β be the projection of G_v into S_k. Let $A = \alpha(G_v)$ and $S = \beta(G_v)$, so that $G_v \leqslant A \times S$, where each element σ of A is induced by an automorphism of T acting on V as

$$D(g_1, \ldots, g_k)^\sigma = D(g_1^\sigma, \ldots, g_k^\sigma)$$

and each element x of S is induced by a permutation on $\{1, \ldots, k\}$ acting on V as

$$D(g_1, \ldots, g_k)^x = D(g_{1^{x^{-1}}}, \ldots, g_{k^{x^{-1}}}).$$

As $G \geqslant M$ we have $\mathrm{Inn}(T) \leqslant A \leqslant \mathrm{Aut}(T)$. Moreover, since G is 2-arc-transitive, Lemma 2.2 implies that $G_v = G_{uv}G_{vw}$. Let R be the stabilizer in S of k in the set $\{1, \ldots, k\}$.

Take any $\sigma \in A$ and $x \in S$. Then $\sigma x \in G_u$ if and only if $x^{-1}\sigma^{-1}$ fixes u, that is

$$D((t_{1^x}^{-1})^{\sigma^{-1}}, \ldots, (t_{(k-1)^x}^{-1})^{\sigma^{-1}}, (t_{k^x}^{-1})^{\sigma^{-1}}) = D(t_1^{-1}, \ldots, t_{k-1}^{-1}, 1),$$

or equivalently,

$$D(t_{k^x}t_{1^x}^{-1}, \ldots, t_{k^x}t_{(k-1)^x}^{-1}, 1) = D((t_1^{-1})^\sigma, \ldots, (t_{k-1}^{-1})^\sigma, 1). \tag{3.5}$$

Similarly, $\sigma x \in G_w$ if and only if $x^{-1}\sigma^{-1}$ fixes w, which is equivalent to

$$D(t_{k^x}^{-1}t_{1^x}, \ldots, t_{k^x}^{-1}t_{(k-1)^x}, 1) = D(t_1^\sigma, \ldots, t_{k-1}^\sigma, 1). \tag{3.6}$$

Lemma 3.8. $\langle t_1, \ldots, t_k\rangle = T$.

Proof. For all $\sigma \in \alpha(G_{uv})$, there exists $x \in S$ such that $\sigma x \in G_u$. Then (3.5) implies that $t_{k^x}t_{i^x}^{-1} = (t_i^{-1})^\sigma$ and thus $t_i^\sigma = t_{i^x}t_{k^x}^{-1}$ for all i such that $1 \leqslant i \leqslant k$. This shows that $\alpha(G_{uv})$ stabilizes $\langle t_1, \ldots, t_k\rangle$. Similarly, for all $\sigma \in \alpha(G_{vw})$, there exists

$x \in S$ such that $\sigma x \in G_w$. Then (3.6) implies that $t_i^\sigma = t_{k^x}^{-1} t_{i^x}$ for all i such that $1 \leqslant i \leqslant k$. Accordingly, $\alpha(G_{vw})$ also stabilizes $\langle t_1, \dots, t_k \rangle$. It follows that $A = \alpha(G_v) = \alpha(G_{uv}G_{vw}) = \alpha(G_{uv})\alpha(G_{vw})$ stabilizes $\langle t_1, \dots, t_k \rangle$. Hence $\langle t_1, \dots, t_k \rangle = T$ since $\mathrm{Inn}(T) \leqslant A \leqslant \mathrm{Aut}(T)$. □

Lemma 3.9. $G_{uv} \cap (A \times R) = G_{vw} \cap (A \times R)$.

Proof. Let $\sigma \in A$ and $x \in R$. Then $t_{k^x} = t_k = 1$, and thus (3.6) shows that $\sigma x \in G_w$ if and only if $t_{i^x} = t_i^\sigma$ for all i such that $1 \leqslant i \leqslant k$. Similarly, (3.5) shows that $\sigma x \in G_u$ if and only if $t_{i^x}^{-1} = (t_i^{-1})^\sigma$ for all i such that $1 \leqslant i \leqslant k$. Since this is equivalent to $t_{i^x} = t_i^\sigma$ for all i, we conclude that $\sigma x \in G_w$ if and only if $\sigma x \in G_u$. As a consequence, $G_{uv} \cap (A \times R) = G_{vw} \cap (A \times R)$. □

Lemma 3.10. $G_{uv} \cap A = G_{vw} \cap A = 1$.

Proof. In view of Lemma 3.9 we only need to prove that $G_{vw} \cap A = 1$. For any $\sigma \in G_{vw} \cap A$, (3.6) shows that $D(t_1, \dots, t_{k-1}, 1) = D(t_1^\sigma, \dots, t_{k-1}^\sigma, 1)$, and so $t_i^\sigma = t_i$ for all i such that $1 \leqslant i \leqslant k$. By Lemma 3.8, this implies that $\sigma = 1$ and so $G_{vw} \cap A = 1$, as desired. □

Lemma 3.11. *Both $\beta(G_{uv})$ and $\beta(G_{vw})$ preserve the partition $\{\Omega_1, \dots, \Omega_n\}$.*

Proof. Let $x \in \beta(G_{uv})$. Then there exists $\sigma \in A$ such that $\sigma x \in G_u$, and so (3.5) gives

$$t_{k^x} t_{i^x}^{-1} = (t_i^{-1})^\sigma \tag{3.7}$$

for all i such that $1 \leqslant i \leqslant k$. For any $i, j \in \{1, \dots, k\}$, if i and j are in the same part of $\{\Omega_1, \dots, \Omega_n\}$, then $t_i = t_j$ and so $(t_i^{-1})^\sigma = (t_j^{-1})^\sigma$, which leads to $t_{i^x} = t_{j^x}$ by (3.7). Since $t_{i^x} = t_{j^x}$ if and only if i^x and j^x are in the same part of $\{\Omega_1, \dots, \Omega_n\}$, this shows that x, hence $\beta(G_{uv})$, preserves the partition $\{\Omega_1, \dots, \Omega_n\}$. The proof for $\beta(G_{vw})$ is similar. □

Lemma 3.12. $t_1, \dots, t_k$ *are pairwise distinct.*

Proof. Let U be the subset of V consisting of the elements $D(g_1, \dots, g_k)$ with $g_i = g_j$ whenever i and j are in the same part of $\{\Omega_1, \dots, \Omega_n\}$. By Lemma 3.11, both $\beta(G_{uv})$ and $\beta(G_{vw})$ preserve the partition $\{\Omega_1, \dots, \Omega_n\}$. Then since $S = \beta(G_v) = \beta(G_{uv}G_{vw}) = \beta(G_{uv})\beta(G_{vw})$, we derive that S preserves the partition $\{\Omega_1, \dots, \Omega_n\}$. As a consequence, S stabilizes U setwise. Meanwhile, A and g stabilize U setwise. Hence $G = \langle G_v, g \rangle \leqslant \langle A \times S, g \rangle$ stabilizes U setwise, which implies $U = V$. Thus each Ω_i has size 1 and so $t_1, \dots, t_k$ are pairwise distinct. □

Lemma 3.13. $G_{uv} \cap R = G_{vw} \cap R = 1$.

Proof. In view of Lemma 3.9 we only need to prove that $G_{vw} \cap R = 1$. Let $x \in G_{vw} \cap R$. Then $t_{k^x} = t_k = 1$, and so (3.6) shows that $t_{i^x} = t_i$ for all i such that $1 \leqslant i \leqslant k$. Note that $t_1, \dots, t_k$ are pairwise distinct by Lemma 3.12. We conclude that $x = 1$ and so $G_{vw} \cap R = 1$, as desired. □

Lemma 3.14. $k = |T|$, $\{t_1, \dots, t_k\} = T$ *and* $\Gamma \cong \Gamma(T)$ *as given in* Construction 3.1. *Moreover, if G is vertex-primitive, then the induced permutation group of G on the k copies of T is a subgroup of* $\mathrm{Hol}(T)$ *containing* $\mathrm{Soc}(\mathrm{Hol}(T))$.

Proof. It follows from Lemma 3.9 that $G_{uvw} \cap (A \times R) = G_{uv} \cap (A \times R)$. Then as G is 2-arc-transitive on Γ, we have

$$\begin{aligned}\frac{|G_v|}{|G_{uv}|} = \frac{|G_{uv}|}{|G_{uvw}|} &\leqslant \frac{|G_{uv}|}{|G_{uvw} \cap (A \times R)|} \\ &= \frac{|G_{uv}|}{|G_{uv} \cap (A \times R)|} = \frac{|G_{uv}(A \times R)|}{|A \times R|} \leqslant \frac{|A \times S|}{|A \times R|} = k.\end{aligned} \tag{3.8}$$

We thus obtain $|G_v| \leqslant k|G_{uv}| = k|G_{vw}|$. From Lemma 3.10 we deduce $\beta(G_{uv}) \cong G_{uv}$ and $\beta(G_{vw}) \cong G_{vw}$. Moreover, $t_1, \ldots, t_k$ are pairwise distinct by Lemma 3.12, which implies $|T| \geqslant k$. Therefore,

$$k|S| \leqslant |T||S| \leqslant |G_v \cap A||S| = |G_v| \leqslant k|G_{uv}| = k|\beta(G_{uv})| \leqslant k|S|$$

and

$$k|S| \leqslant |T||S| \leqslant |G_v \cap A||S| = |G_v| \leqslant k|G_{vw}| = k|\beta(G_{vw})| \leqslant k|S|.$$

Hence $|G_v \cap A| = |T| = k$, $|G_v| = k|G_{uv}| = k|G_{vw}|$ and $\beta(G_{uv}) = \beta(G_{vw}) = S$. As a consequence, $T = \{t_1, \ldots, t_k\}$ by Lemma 3.12, and so $\Gamma \cong \mathrm{Cos}(T^k, D, g) \cong \Gamma(T)$. Also, (3.8) implies that $G_{uvw} = G_{uvw} \cap (A \times R)$. If $G_{uv} \cap S = 1$ or $G_{vw} \cap S = 1$, then Lemma 3.10 implies $S = \beta(G_{uv}) \cong G_{uv} \lesssim A$ or $S = \beta(G_{vw}) \cong G_{vw} \lesssim A$, contradicting Lemma 2.14. Thus $G_{uv} \cap S$ and $G_{vw} \cap S$ are both nontrivial normal subgroups of $\beta(G_{uv}) = \beta(G_{vw}) = S$.

From now on suppose that G is primitive and so S is a primitive subgroup of S_k. By Lemma 3.13, $G_{uv} \cap R = G_{vw} \cap R = 1$, so we derive that $G_{uv} \cap S$ and $G_{vw} \cap S$ are both regular normal subgroups of S. Moreover, $G_{uv} \cap S \neq G_{vw} \cap S$ for otherwise $G_{uvw} \cap S = G_{uv} \cap S$ would be a regular subgroup of S, contrary to the condition $G_{uvw} = G_{uvw} \cap (A \times R) \leqslant A \times R$. This indicates that S has at least two regular normal subgroups, and so $\mathrm{Soc}(S) = N^{2n}$ for some nonabelian simple group N and positive integer n such that $k = |N|^n$ and $S/(G_{uv} \cap S)$ has a normal subgroup isomorphic to N^n. It follows that

$$N^n \lesssim S/(G_{uv} \cap S) = \beta(G_{uv})/(G_{uv} \cap S) \cong \alpha(G_{uv})/(G_{uv} \cap A) \cong \alpha(G_{uv}) \leqslant A,$$

and then Lemma 2.13 implies that $n = 1$ and $N \cong T$. Thus, $\mathrm{Soc}(S) \cong T^2$, and so $\mathrm{Soc}(\mathrm{Hol}(T)) \leqslant S \leqslant \mathrm{Hol}(T)$. □

We are now ready to give the main theorem of this section. Recall X defined in (3.2).

Theorem 3.15. *Let T be a nonabelian simple group, $k \geqslant 2$ be an interger, and $T^k \leqslant G \leqslant T^k.(\mathrm{Out}(T) \times \mathrm{S}_k)$ with diagonal action on the set V of right cosets of $\{(t, \ldots, t) \mid t \in T\}$ in T^k. Suppose Γ is a connected $(G, 2)$-arc-transitive digraph with vertex set V. Then $k = |T|$, $\Gamma \cong \Gamma(T)$, $\mathrm{Aut}(\Gamma) = X$ is vertex-primitive of type SD with socle T^k and the conjugation action on the k copies of T permutation isomorphic to $\mathrm{Hol}(T)$, and Γ is not 3-arc-transitive.*

Proof. We have by Lemma 3.14 that $k = |T|$, $\{t_1, \ldots, t_k\} = T$ and $\Gamma \cong \Gamma(T)$. In the following, we identify Γ with $\Gamma(T)$. Let X be as in (3.2) and $Y = \mathrm{Aut}(\Gamma(T))$. Then X is vertex-primitive of type SD with socle $T^{|T|}$, and the conjugation action of X on the $|T|$ copies of T is permutation isomorphic to $\mathrm{Hol}(T)$. Also, $X \leqslant Y$ by Lemma 3.5. It follows from [1, Theorem 1.2] that Y is vertex-primitive of type SD with the same socle of X. Then again by Lemma 3.14 we have $Y_v \leqslant \mathrm{Aut}(T) \times \mathrm{Hol}(T)$. Thus by (3.3) $Y_v = X_x$. Since X is vertex-transitive, it follows that $Y = XY_v = X$, and so Γ is not 3-arc-transitive by Lemma 3.7. □

4 Product action on the vertex set

In this section, we study (G, s)-arc-transitive digraphs with vertex set Δ^m such that G acts on Δ^m by product action. We first prove Theorem 1.3.

Proof of Theorem 1.3. Let G_1 be the stabiliser in G of the first coordinate and π_1 be the projection of G_1 into $\mathrm{Sym}(\Delta)$. Then $\pi_1(G_1) = H$. Since N is normal in H and transitive on Δ, N^m is normal in G and transitive on $\Delta^m = V$. Hence Corollary 2.11 implies that Γ is $(N^m, s-1)$-arc-transitive. In particular, since $s \geqslant 2$, N^m is transitive on the set of arcs of Γ, and so Γ has arc set $A = \{u^n \to v^n \mid n \in N^m\}$ for any arc $u \to v$ of Γ.

Let $\alpha \in \Delta$, $u = (\alpha, \dots, \alpha) \in V$ and $v = (\beta_1, \dots, \beta_m)$ be an out-neighbour of u in Γ. By Lemma 2.10 we have $G = N^m G_{uv}$. Let φ be the projection of G to S_m, and we regard $\varphi(G)$ as a subgroup of $\mathrm{Sym}(V)$. Then

$$\varphi(G) \leqslant H^m G = H^m(N^m G_{uv}) = H^m G_{uv}.$$

Take any i in $\{1, \dots, m\}$. Since $\varphi(G)$ is transitive on $\{1, \dots, m\}$, there exists $x \in \varphi(G)$ such that $1^x = i$ and $x = yz$ with $y = (y_1, \dots, y_m) \in H^m$ and $z \in G_{uv}$. Note that $z \in G_{uv}$ and $x \in \mathrm{S}_m$ both fix u. We conclude that y fixes u and hence $y_j \in H_\alpha$ for each j in $\{1, \dots, m\}$. Also, $y^{-1}x = z \in G_{uv} \leqslant G_v$ implies $\beta_1^{y_1^{-1}} = \beta_i$. It follows that for each i in $\{1, \dots, m\}$ there exists $h_i \in H_\alpha$ with $\beta_i^{h_i} = \beta_1$. Let $w = (\beta_1, \dots, \beta_1) \in V$, $h = (h_1, \dots, h_m) \in (H_\alpha)^m$ and Γ^h be the digraph with vertex set V and arc set $A^h := \{u^{nh} \to v^{nh} \mid n \in N^m\}$. It is evident that $u^h = u$, $v^h = w$, and h gives an isomorphism from Γ to Γ^h. Let Σ be the digraph with vertex set Δ and arc set $I := \{\alpha^n \to \beta_1^n \mid n \in N\}$. Then $N \leqslant \mathrm{Aut}(\Sigma)$, and viewing $N^m h = hN^m$ we have

$$\begin{aligned} A^h &= \{u^{hn} \to v^{hn} \mid n \in N^m\} = \{u^n \to w^n \mid n \in N^m\} \\ &= \{(\alpha^{n_1}, \dots, \alpha^{n_m}) \to (\beta_1^{n_1}, \dots, \beta_1^{n_m}) \mid n_1, \dots, n_m \in N\}. \end{aligned}$$

This implies that $\Gamma^h = \Sigma^m$. Consequently, $\Gamma \cong \Sigma^m$.

For any $\beta \in \Delta$, denote by $\delta(\beta)$ the point in $V = \Delta^m$ with all coordinates equal to β. Then $\delta(\alpha) \to \delta(\beta_1)$ in Γ^h since $\alpha \to \beta_1$ in Σ. Let x be any element of H. Then since

$$H = h_1^{-1} H h_1 = h_1^{-1} \pi_1(G_1) h_1 = \pi_1(h)^{-1} \pi_1(G_1) \pi_1(h) = \pi_1(h^{-1} G_1 h),$$

there exists $g \in h^{-1} G_1 h$ such that $x = \pi_1(g)$. As g is an automorphism of Γ^h and $\delta(\alpha) \to \delta(\beta_1)$ in Γ^h, we have $\delta(\alpha)^g \to \delta(\beta_1)^g$ in Γ^h. Comparing first coordinates, this implies that $\alpha^{\pi_1(g)} \to \beta_1^{\pi_1(g)}$ in Σ, which turns out to be $\alpha^x \to \beta_1^x$ in Σ. In other words, $\alpha^x \to \beta_1^x$ is in I. It follows that

$$I = \{\alpha^{xn} \to \beta_1^{xn} \mid n \in N\} = \{\alpha^{nx} \to \beta_1^{nx} \mid n \in N\}$$

as $xN = Nx$. Hence x preserves I, and so $H \leqslant \mathrm{Aut}(\Sigma)$.

Let $\alpha_0 \to \alpha_1 \to \cdots \to \alpha_s$ be an s-arc of Σ. Since Γ is $(N^m, s-1)$-arc-transitive and $N^m = h^{-1} N^m h$, it follows that Γ^h is $(N^m, s-1)$-arc-transitive. Then for any $(s-1)$-arc $\alpha_1' \to \cdots \to \alpha_s'$ of Σ, since $\delta(\alpha_1) \to \cdots \to \delta(\alpha_s)$ and $\delta(\alpha_1') \to \cdots \to \delta(\alpha_s')$ are both $(s-1)$-arcs of Γ^h, there exists $(n_1, \dots, n_m) \in N^m$ such that $\delta(\alpha_i)^{(n_1, \dots, n_m)} = \delta(\alpha_i')$ for each i with $1 \leqslant i \leqslant s$. Hence $\alpha_i^{n_1} = \alpha_i'$ for each i with $1 \leqslant i \leqslant s$. Therefore, Σ is $(N, s-1)$-arc-transitive. Let $\Sigma^+(\alpha_{s-1})$ be the set of out-neighbours of α_{s-1} in

Σ. Take any $\beta \in \Sigma^+(\alpha_{s-1})$. As $\delta(\alpha_s)$ and $\delta(\beta)$ are both out-neighbours of $\delta(\alpha_{s-1})$ in Γ^h and Γ^h is $(h^{-1}Gh, s)$-arc-transitive, there exists $g \in h^{-1}Gh \leqslant H \wr \mathrm{S}_m$ such that g fixes $\delta(\alpha_0), \delta(\alpha_1), \ldots, \delta(\alpha_{s-1})$ and maps $\delta(\alpha_s)$ to $\delta(\beta)$. Write $g = (x_1, \ldots, x_m)\sigma$ with $(x_1, \ldots, x_m) \in H^m$ and $\sigma \in \mathrm{S}_m$. Then x_1 fixes $\alpha_0, \alpha_1, \ldots, \alpha_{s-1}$ and maps α_s to β. This shows that Σ is (H, s)-arc-transitive, completing the proof. $\square$

References

[1] R. W. Baddeley and C. E. Praeger, On primitive overgroups of quasiprimitive permutation groups, *J. Algebra* **263** (2003), 294–344, doi:10.1016/s0021-8693(03)00113-3.

[2] M. Conder, P. Lorimer and C. Praeger, Constructions for arc-transitive digraphs, *J. Austral. Math. Soc. Ser. A* **59** (1995), 61–80, doi:10.1017/s1446788700038477.

[3] M. Giudici, C. H. Li and B. Xia, An infinite family of vertex-primitive 2-arc-transitive digraphs, *J. Comb. Theory Ser. B* **127** (2017), 1–13, doi:10.1016/j.jctb.2017.05.002.

[4] L. G. Kovács, Primitive subgroups of wreath products in product action, *Proc. London Math. Soc.* **58** (1989), 306–322, doi:10.1112/plms/s3-58.2.306.

[5] M. W. Liebeck, C. E. Praeger and J. Saxl, On the O'Nan-Scott theorem for finite primitive permutation groups, *J. Austral. Math. Soc. Ser. A* **44** (1988), 389–396, doi:10.1017/s144678870003216x.

[6] S. P. Mansilla and O. Serra, Construction of k-arc transitive digraphs, *Discrete Math.* **231** (2001), 337–349, doi:10.1016/s0012-365x(00)00331-9.

[7] S. P. Mansilla and O. Serra, Automorphism groups of k-arc transitive covers, *Discrete Math.* **276** (2004), 273–285, doi:10.1016/s0012-365x(03)00298-x.

[8] C. E. Praeger, Highly arc transitive digraphs, *European J. Combin.* **10** (1989), 281–292, doi:10.1016/s0195-6698(89)80064-2.

[9] C. E. Praeger, Finite quasiprimitive graphs, in: R. A. Bailey (ed.), *Surveys in Combinatorics, 1997*, Cambridge University Press, Cambridge, volume 241 of *London Mathematical Society Lecture Note Series*, pp. 65–85, 1997, doi:10.1017/cbo9780511662119.005, proceedings of the British Combinatorial Conference held at the University of London, London, July 1997.

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Characterising CCA Sylow cyclic groups whose order is not divisible by four*

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Abstract

A Cayley graph on a group G has a natural edge-colouring. We say that such a graph is *CCA* if every automorphism of the graph that preserves this edge-colouring is an element of the normaliser of the regular representation of G. A group G is then said to be *CCA* if every connected Cayley graph on G is CCA.

Our main result is a characterisation of non-CCA graphs on groups that are Sylow cyclic and whose order is not divisible by four. We also provide several new constructions of non-CCA graphs.

Keywords: CCA problem, Cayley graphs, edge-colouring, Sylow cyclic groups.

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1 Introduction

All groups and all graphs in this paper are finite. Let G be a group and let S be an inverse-closed subset of G. The *Cayley graph* of G with respect to S is the edge-coloured graph $\mathrm{Cay}(G,S)$ with vertex-set G and, for every $g \in G$ and $s \in S$, an edge $\{g, sg\}$ with colour $\{s, s^{-1}\}$. Its group of colour-preserving automorphisms is denoted $\mathrm{Aut}_c(\mathrm{Cay}(G,S))$. Let $\mathrm{Aut}_{\pm 1}(G,S) = \{\alpha \in \mathrm{Aut}(G) : s^\alpha \in \{s, s^{-1}\} \text{ for all } s \in S\}$. It is easy to see that $G_R \rtimes \mathrm{Aut}_{\pm 1}(G,S) \leqslant \mathrm{Aut}_c(\mathrm{Cay}(G,S))$, where G_R is the right-regular representation of G.

Definition 1.1 ([5]). The Cayley graph $\mathrm{Cay}(G,S)$ is *CCA* (Cayley colour automorphism) if $\mathrm{Aut}_c(\mathrm{Cay}(G,S)) = G_R \rtimes \mathrm{Aut}_{\pm 1}(G,S)$. The group G is *CCA* if every connected Cayley graph on G is CCA.

In other words, a Cayley graph is CCA if and only if the colour-preserving graph automorphisms are exactly the "obvious" ones. The terminology we use for this problem largely comes from [5]. Other papers that study this problem include [2, 3, 4, 8].

Note that $\mathrm{Cay}(G,S)$ is connected if and only if S generates G. It is also easy to see that $G_R \rtimes \mathrm{Aut}_{\pm 1}(G,S)$ is precisely the normaliser of G_R in $\mathrm{Aut}_c(\mathrm{Cay}(G,S))$. In particular, $\mathrm{Cay}(G,S)$ is CCA if and only if G_R is normal in $\mathrm{Aut}_c(\mathrm{Cay}(G,S))$, c.f. [5, Remark 6.2].

In Section 2, we introduce some basic terminology and recall a few previous results on the CCA property. In Section 3, we consider wreath products of permutation groups, and produce conditions that are sufficient to determine when such a product is a non-CCA group. This generalises results from [5]. In Section 4, we give some new constructions for non-CCA graphs.

Finally, in Section 5, we obtain a characterisation of non-CCA groups whose order is not divisible by four, in which every Sylow subgroup is cyclic. This generalises the work of [3], which dealt with the case of groups of odd squarefree order.

2 Preliminaries

The identity of a group G is denoted 1_G, or simply 1 if there is no risk of confusion. We denote a dihedral group of order $2n$ by D_n, while Q_8 denotes the quaternion group of order 8 with elements $\{\pm 1, \pm i, \pm j, \pm k\}$ and multiplication defined as usual.

We now state some preliminary results and introduce some terminology related to Cayley graphs. Let Γ be a graph and let v be a vertex of Γ. The neighbourhood of v is denoted by $\Gamma(v)$. If A is a group of automorphisms of Γ, then the permutation group induced by the vertex-stabiliser A_v on the neighbourhood of v is denoted $A_v^{\Gamma(v)}$.

Lemma 2.1 ([5, Lemma 6.3]). *The vertex-stabiliser in the colour-preserving group of automorphisms of a connected Cayley graph is a 2-group.*

Definition 2.2. Let G be a group, let $\Gamma = \mathrm{Cay}(G,S)$ and let N be a normal subgroup of G. The *quotient graph* Γ/N is $\mathrm{Cay}(G/N, S/N)$, where $S/N = \{sN : s \in S\}$.

Lemma 2.3 ([3, Lemma 3.4]). *Let A be a colour-preserving group of automorphisms of $\mathrm{Cay}(G,S)$, let N be a normal subgroup of A and let K be the kernel of the action of A on the N-orbits. If $N \leqslant G$, then A/K is a colour-preserving group of automorphisms of Γ/N.*

Lemma 2.4. *Let $\Gamma = \mathrm{Cay}(G, S)$, let A be a colour-preserving group of automorphisms of Γ, let N be a normal 2-subgroup of A and let K be the kernel of the action of A on the N-orbits. If $K_v \neq 1$, then S contains an element whose order is a power of 2 that is at least 4.*

Proof. Let $v_0 = v$. Since $K_{v_0} \neq 1$, $K_{v_0}^{\Gamma(v_0)} \neq 1$ and there exists $k \in K_{v_0}$ and a neighbour u_0 of v_0 such that $u_0^k \neq u_0$. Let $u_1 = u_0^k$. Note that $K_{u_1} \neq K_{v_0}$ hence there exists $\ell \in K_{u_1}$ such that $v_0^\ell \neq v_0$. Let $v_1 = v_0^\ell$. Repeating this process, we get a monochromatic cycle $C = (u_0, v_0, u_1, v_1, ...)$ of length at least 3. By construction, $u_i \in u_0^K = u_0^N$ and $v_i \in v_0^K = v_0^N$ for all i. In particular, $|C \cap v_0^N| \in \{|C|, |C|/2\}$. Since each vertex of Γ lies in a unique monochromatic cycle of a given colour, C is a block for A. On the other hand, v_0^N is also a block for A and thus so is $C \cap v_0^N$. It follows that $|C \cap v_0^N|$ divides $|N|$ which is a power of 2. This implies that $|C|$ is also a power of 2. Since $|C| \geqslant 3$, $|C|$ is divisible by 4 and the result follows from the fact that C is monochromatic. $\square$

For a group H, let $H^2 := \langle x^2 \mid x \in H \rangle$. The following lemma is inspired by an argument contained within [5, Theorem 6.8].

Lemma 2.5. *Let $\Gamma = \mathrm{Cay}(G, S)$ be connected, let A be a colour-preserving group of automorphisms of Γ that is normalised by G and let v be a vertex of Γ. If A_v has a subgroup U such that $U \leqslant (A_v)^2$ and no other subgroup of A_v is isomorphic to U, then $U = 1$. In particular, A_v is isomorphic to neither $\mathbb{Z}_{2^n}$ for $n \geqslant 2$ nor isomorphic to D_{2^n} for $n \geqslant 3$.*

Proof. Since A is colour-preserving, $A_v^{\Gamma(v)}$ is an elementary abelian 2-group. Since $U \leqslant (A_v)^2$, it follows that U fixes all the neighbours of v. Let $s \in S$. Since A is normalised by G, we have $U^s \leqslant A$ and, by the previous observation, $U^s \leqslant A_v$. As A_v has a unique subgroup isomorphic to U, we must have $U = U^s$. Since this holds for every $s \in S$, U is normalised by G. As G is transitive and U fixes v, this implies that $U = 1$.

The second part of the lemma follows from the first. Indeed, if A_v is isomorphic to $\mathbb{Z}_{2^n}$ for $n \geqslant 2$ or to D_{2^n} for $n \geqslant 3$, then $(A_v)^2$ is non-trivial and is the unique cyclic subgroup of its order. $\square$

3 Wreath products

Proposition 3.1. *Let H be a permutation group on a set Ω, let G be a group and let $X = G \wr_\Omega H$. If*

(i) there is an inverse-closed generating set S for G and a non-identity bijection $\tau\colon G \to G$ such that τ fixes 1, and $\tau(sg) = s^{\pm 1}\tau(g)$ for every $g \in G$ and every $s \in S$, and

(ii) either H is nontrivial or $\tau \notin \mathrm{Aut}(G)$,

then X is non-CCA.

Proof. Let $m = |\Omega|$ and write $\Omega = \{1, \ldots, m\}$ such that, if H is nontrivial, then 1 is not fixed by H.

Write $X = H \ltimes (G_1 \times \cdots \times G_m)$. Note that, if $g \in G_i$ and $h \in H$, then $g^h \in G_{i^h}$. Without loss of generality, we may assume that $1_G \notin S$. Let S_i be the subset of G_i

corresponding to S, let $T = (H - \{1_H\}) \cup S_1 \cup \cdots \cup S_m$ and let $\Gamma = \mathrm{Cay}(X, T)$. Note that T generates X hence Γ is connected. We will show that Γ is non-CCA.

Define $\tau' \colon X \to X$ by $\tau' \colon hg_1g_2\cdots g_m \mapsto h\tau(g_1)g_2\cdots g_m$, where $g_i \in G_i$ and $h \in H$. Let v be a vertex of Γ and let $s \in T$. We will show that $\tau'(sv) = s^{\pm 1}\tau'(v)$ and hence τ' is a colour-preserving automorphism of Γ. Write $v = hg_1\cdots g_m$ with $h \in H$ and $g_i \in G_i$. Let $g = g_1\cdots g_m$. Note that $\tau'(v) = \tau'(hg) = h\tau'(g)$. If $s \in H$, then $\tau'(sv) = \tau'(shg) = sh\tau'(g) = s\tau'(v)$. Suppose now that $s \in S_i$ for some $i \in \Omega$. If $i^h \neq 1$, then

$$\begin{aligned}\tau'(s^h g) = \tau'(g_1\cdots s^h g_{i^h}\cdots g_m) = \tau(g_1)g_2\cdots s^h g_{i^h}\cdots g_m &= \\ = s^h\tau(g_1)g_2\cdots g_m = s^h\tau'(g).&\end{aligned}$$

If $i^h = 1$, then $s^h \in S_1 \subseteq T$ and

$$\tau'(s^h g) = \tau'(s^h g_1\cdots g_m) = \tau(s^h g_1)g_2\cdots g_m = (s^h)^{\pm 1}\tau(g_1)g_2\cdots g_m = (s^h)^{\pm 1}\tau'(g).$$

Either way, we have

$$\tau'(sv) = \tau'(hs^h g) = h\tau'(s^h g) = h(s^h)^{\pm 1}\tau'(g) = s^{\pm 1}h\tau'(g) = s^{\pm 1}\tau'(hg) = s^{\pm 1}\tau'(v).$$

This completes the proof that τ' is a colour-preserving automorphism of Γ. It remains to show that τ' is not a group automorphism of X. (Note that τ' fixes 1_X, so if $\tau' \in X_R \rtimes \mathrm{Aut}_{\pm 1}(X, T)$, then $\tau' \in \mathrm{Aut}(X)$.)

If H is nontrivial, then, since 1 is not fixed by H, there exists $h \in H$ such that $1^h \neq 1$. Let g be an element of G_1 that is not fixed by τ. We have $\tau'(gh) = \tau'(hg^h) = hg^h = gh$ but $\tau'(g)\tau'(h) = \tau(g)h$. Since $g \neq \tau(g)$, τ' is not an automorphism of X.

If H is trivial and $\tau \notin \mathrm{Aut}(G)$, then there exist $g_1, g_2 \in G$ such that $\tau(g_1g_2) \neq \tau(g_1)\tau(g_2)$. Applying τ' to the corresponding elements of G_1 shows that τ' is not an automorphism of X. This completes the proof. □

We now obtain a few corollaries of Proposition 3.1.

Corollary 3.2. *Let H be a permutation group on a set Ω and let G be a group. If G is non-CCA, then $G \wr_\Omega H$ is non-CCA.*

Proof. Since G is non-CCA, there exists a colour-preserving graph automorphism τ of a Cayley graph $\mathrm{Cay}(G, S)$ such that $\tau(1_G) = 1_G$ but τ does not normalise G_R. Since τ is colour-preserving, $\tau(sg) = s^{\pm 1}\tau(g)$ for every $g \in G$ and every $s \in S$. Finally, since τ does not normalise G_R, we have $\tau \notin \mathrm{Aut}(G)$ and the result follows from Proposition 3.1. □

Corollary 3.3. *Let H be a nontrivial permutation group on a set Ω and let G be a group. If $G = B \ltimes A$, where A is abelian of exponent greater than 2, then $G \wr_\Omega H$ is non-CCA.*

Proof. Every element of G can be written uniquely as ba with $a \in A$ and $b \in B$. Let τ be the permutation of G mapping ba to ba^{-1}. Clearly, τ fixes 1_G but, since A has exponent greater than 2, τ is not the identity. Let $S = (A \cup B) - \{1_G\}$. Note that S is an inverse-closed generating set for G. Let $g \in G$, let $s \in S$ and write $g = ba$ with $a \in A$ and $b \in B$.

If $s \in B$, then $\tau(sg) = \tau(sba) = sba^{-1} = s\tau(ba) = s\tau(g)$. Otherwise, $s \in A$, $s^b \in A$ and

$$\tau(sg) = \tau(sba) = \tau(bs^b a) = b(s^b a)^{-1} = b(s^b)^{-1}a^{-1} = s^{-1}ba^{-1} = s^{-1}\tau(g).$$

The result then follows from Proposition 3.1, since H is nontrivial. □

Corollary 3.4. *Let H be a permutation group on a set Ω and let G be a group. If*

(i) G has exponent greater than 2,

(ii) H is nontrivial when G is abelian, and

(iii) G has a generating set S with the property that $s^g = s^{\pm 1}$ for every $s \in S$ and $g \in G$,

then $G \wr_\Omega H$ is non-CCA.

Proof. We can assume without loss of generality that S is inverse-closed. Let τ be the permutation of G that maps every element to its inverse. For every $s \in S$ and $g \in G$, we have $s^g = s^{\pm 1}$ and thus $\tau(sg) = g^{-1}s^{-1} = s^{\pm 1}g^{-1} = s^{\pm 1}\tau(g)$. Since G has exponent greater than 2, τ is not the identity. If H is trivial, then G is non-abelian so that τ is not an automorphism of G. The result then follows from Proposition 3.1. □

In view of Corollary 3.4, it would be interesting to determine the groups G such that G has a generating set S with the property that $s^g = s^{\pm 1}$ for every $s \in S$ and $g \in G$. This family of groups includes abelian groups and Q_8. This family is closed under central products but it also includes examples which do not arise as central products of smaller groups in the family, for example the extraspecial group of order 32 and minus type.

4 A few constructions for non-CCA graphs

In this section, we will describe a few constructions which yield non-CCA Cayley graphs. For a group G, let K_G denote $\mathrm{Cay}(G, G - \{1\})$, the complete Cayley graph on G. We will need a result which tells us when $G_R < \mathrm{Aut}_c(\mathrm{K}_G)$. First we state some definitions.

Definition 4.1. Let A be an abelian group of exponent greater than 2, and define a map $\iota\colon A \to A$ by $\iota(a) = a^{-1}$ for every $a \in A$. The *generalised dihedral group* over A is $\mathrm{Dih}(A) = A \rtimes \langle \iota \rangle$.

Definition 4.2. Let A be an abelian group of even order and of exponent greater than 2, and let y be an element of A of order 2. The *generalised dicyclic group* over A is $\mathrm{Dic}(A, y) := \langle A, x \mid x^2 = y, a^x = a^{-1}\ \forall a \in A \rangle$. Let ι be the permutation of $\mathrm{Dic}(A, y)$ that fixes A pointwise and maps every element of the coset Ax to its inverse.

It is not hard to check that ι is an automorphism of $\mathrm{Dic}(A, y)$.

Definition 4.3. For $\alpha \in \{i, j, k\}$, let $S_\alpha = \{\pm\alpha\} \times \mathbb{Z}_2^n$ and let σ_α be the permutation of $\mathrm{Q}_8 \times \mathbb{Z}_2^n$ that inverts every element of S_α and fixes every other element.

Theorem 4.4 ([2], Classification Theorem)**.** *If G is a group, then $G_R < \mathrm{Aut}_c(\mathrm{K}_G)$ if and only if one of the following occurs:*

1. *G is abelian but is not an elementary abelian 2-group, and* $\mathrm{Aut}_c(\mathrm{K}_G) = \mathrm{Dih}(G)$,

2. *G is generalised dicyclic but not of the form* $\mathrm{Q}_8 \times \mathbb{Z}_2^n$*, and* $\mathrm{Aut}_c(\mathrm{K}_G) = G_R \rtimes \langle \iota \rangle$*, where ι is as in Definition 4.2, or*

3. $G \cong \mathrm{Q}_8 \times \mathbb{Z}_2^n$ *and* $\mathrm{Aut}_c(\mathrm{K}_G) = \langle G_R, \sigma_i, \sigma_j, \sigma_k \rangle$*, where $\sigma_i, \sigma_j, \sigma_k$ are as in Definition 4.3.*

Definition 4.5. Let B be a permutation group and let G be a regular subgroup of B. We say that (G, B) is a *complete colour pair* if G is as in the conclusion of Theorem 4.4 and $B \leqslant \mathrm{Aut}_c(\mathrm{K}_G)$.

For a graph Γ, let $\mathcal{L}(\Gamma)$ denote its line graph.

Proposition 4.6. *Let Γ be a connected bipartite G-edge-regular graph. If H is a group of automorphisms of Γ such that:*

(i) $G \leqslant H$,

(ii) the orbits of H on the vertex-set of Γ are exactly the biparts, and

(iii) for every vertex v of Γ, either

  (a) $G_v^{\Gamma(v)} = H_v^{\Gamma(v)}$*, or*

  (b) $(G_v^{\Gamma(v)}, H_v^{\Gamma(v)})$ *is a complete colour pair,*

then H is a colour-preserving group of automorphisms of $\mathcal{L}(\Gamma)$ viewed as a Cayley graph on G.

Proof. Since G acts regularly on edges of Γ, its induced action on $\mathcal{L}(\Gamma)$ is regular on vertices. Vertices of Γ induce cliques in $\mathcal{L}(\Gamma)$, which we call *special*. Clearly, H has exactly two orbits on special cliques. Moreover, special cliques partition the edges of $\mathcal{L}(\Gamma)$, and each vertex of $\mathcal{L}(\Gamma)$ is in exactly two special cliques, one from each H-orbit. Since $G \leqslant H$, the set of edge-colours appearing in special cliques from different H-orbits is disjoint.

Let v be a vertex of Γ and let C be the corresponding special clique of $\mathcal{L}(\Gamma)$. Note that $H_v^{\Gamma(v)}$ is permutation isomorphic to H_C^C, while $G_v^{\Gamma(v)} \cong G_C^C \cong G_C$. Since G is vertex-regular on $\mathcal{L}(\Gamma)$, G_C is regular on C and thus C can be viewed as a complete Cayley graph on G_C. If $(G_v^{\Gamma(v)}, H_v^{\Gamma(v)})$ is a complete colour pair, Theorem 4.4 implies that H_C^C is colour-preserving. If $G_v^{\Gamma(v)} = H_v^{\Gamma(v)}$, then since G is colour-preserving, so is H_C^C. Since G acts transitively on the special cliques within an H-orbit and G is colour-preserving, it follows that H is colour-preserving. □

Remark 4.7. In the proof of Proposition 4.6, we only use one direction of Theorem 4.4, namely that if G appears in Theorem 4.4, then $G_R < \mathrm{Aut}_c(\mathrm{K}_G)$. The converse is not used here, but it can help to identify situations where Proposition 4.6 can be used to construct non-CCA graphs.

Example 4.8. Let Γ be the Heawood graph and let H be the bipart-preserving subgroup of $\mathrm{Aut}(\Gamma)$. Note that $H \cong \mathrm{PSL}(2,7)$ and H contains an edge-regular subgroup G isomorphic to F_{21}, the Frobenius group of order 21. Moreover, for every vertex v of Γ, we have $G_v^{\Gamma(v)} \cong \mathbb{Z}_3$ while $H_v^{\Gamma(v)} \cong \mathrm{D}_3$ and $(\mathbb{Z}_3, \mathrm{D}_3)$ is a complete colour pair. By Proposition 4.6,

H is a colour-preserving group of automorphisms of $\mathcal{L}(\Gamma)$ viewed as a Cayley graph on G. Since G is not normal in H, it follows that $\mathcal{L}(\Gamma)$ is a non-CCA graph and so F_{21} is a non-CCA group.

Example 4.8 was previously studied in [3] and [5], under a slightly different guise.

Example 4.9. Let $A \cong \mathrm{Q}_8 \times \mathbb{Z}_2^m$ and $B = \mathrm{Aut}_c(K_A) = \langle A_R, \sigma_i, \sigma_j, \sigma_k \rangle$. Then A is not normal in B, and by Theorem 4.4(3), (A, B) is a complete colour pair. Let $n = |A|$ and let $\mathrm{K}_{n,n}$ be the complete bipartite graph of order $2n$. Let $G = A \times A$ and let $H = B \times B$. By Proposition 4.6, H is a colour-preserving group of automorphisms of $\mathcal{L}(\mathrm{K}_{n,n})$ viewed as a Cayley graph on G. Since A is not normal in B, G is not normal in H hence $\mathcal{L}(\mathrm{K}_{n,n})$ is a non-CCA graph and so G is a non-CCA group.

For a graph Γ, let $\mathcal{S}(\Gamma)$ denote its subdivision graph.

Corollary 4.10. *Let Γ be a connected G-arc-regular graph. If H is a group of automorphisms of Γ such that:*

(i) $G \leqslant H$, and

(ii) $(G_v^{\Gamma(v)}, H_v^{\Gamma(v)})$ is a complete colour pair for every vertex v of Γ,

then H is a colour-preserving group of automorphisms of $\mathcal{L}(\mathcal{S}(\Gamma))$ viewed as a Cayley graph on G.

Proof. Let $\Gamma' = \mathcal{S}(\Gamma)$. We show that Proposition 4.6 applies to Γ'. Clearly, Γ' is bipartite and G acts on it faithfully and edge-regularly. It is also obvious that, in its induced action on Γ', H must preserve the biparts of Γ'. Finally, let x be a vertex of Γ'. If x arose from a vertex v of Γ, then we have that $A_v^{\Gamma(v)}$ is permutation isomorphic to $A_x^{\Gamma'(x)}$ for every $A \leqslant \mathrm{Aut}(\Gamma)$. Since $(G_v^{\Gamma(v)}, H_v^{\Gamma(v)})$ is a complete colour pair, so is $(G_x^{\Gamma'(x)}, H_x^{\Gamma'(x)})$. If x arose from an edge of Γ, then x has valency 2 and, since G is arc-transitive, $G_x^{\Gamma'(x)} = H_x^{\Gamma'(x)} \cong \mathbb{Z}_2$ and $(G_x^{\Gamma'(x)}, H_x^{\Gamma'(x)})$ is a complete colour pair. □

Example 4.11. Let Γ be the Heawood graph and let $H = \mathrm{Aut}(\Gamma)$. Note that H contains an arc-regular subgroup G isomorphic to $\mathrm{AGL}(1, 7)$. Moreover, for every vertex v of Γ, we have $G_v^{\Gamma(v)} \cong \mathbb{Z}_3$ while $H_v^{\Gamma(v)} \cong \mathrm{D}_3$. By Corollary 4.10, H is a colour-preserving group of automorphisms of $\mathcal{L}(\mathcal{S}(\Gamma))$ viewed as a Cayley graph on G. Since G is not normal in H, it follows that Γ is a non-CCA graph and so $\mathrm{AGL}(1, 7)$ is a non-CCA group.

Remark 4.12. In fact, $\mathrm{AGL}(1, 7)$ is a Sylow cyclic group whose order is not divisible by four, so Example 4.11 will appear again in our characterisation of non-CCA groups of this sort, in Section 5. However, the construction we have just presented is very different from the approach we use in that section.

5 Sylow cyclic and order not divisible by four

We first introduce some notation that will be useful throughout this section. Recall that $\mathrm{PGL}(2, 7)$ has a unique conjugacy class of subgroups isomorphic to $\mathrm{AGL}(1, 7)$. The intersection of such a subgroup with the socle $\mathrm{PSL}(2, 7)$ is a Frobenius group of order 21 which we will denote F_{21}. We say that a group G is *Sylow cyclic* if, for every prime p, the Sylow p-subgroups of G are cyclic.

Our aim in this section is to characterise both the non-CCA Sylow cyclic groups whose order is not divisible by four, and the structure of the corresponding colour-preserving automorphism groups for non-CCA graphs.

Theorem 5.1. *Let G be a Sylow cyclic group whose order is not divisible by four, let $\Gamma = \mathrm{Cay}(G, S)$, let A be a colour-preserving group of automorphisms of Γ, let R be a Sylow 2-subgroup of G and let r be a generator of R. If G is not normal in A, then $G = (F \times H) \rtimes R$ and $A = (T \times J) \rtimes R$, where the following hold:*

(i) $\mathrm{PSL}(2, 7) \cong T \trianglelefteq A$,

(ii) $T \cap G = F \cong \mathrm{F}_{21}$,

(iii) $J \cap G = H \trianglelefteq J \trianglelefteq A$,

(iv) *H is self-centralising in J,*

(v) *J splits over H,*

(vi) *H is normal in A.*

Proof. To avoid ambiguity, for $g \in G$, we write $[g]$ for the vertex of Γ corresponding to g and, for $X \subseteq G$, we write $[X]$ for $\{[x] : x \in X\}$.

Let P be a Sylow 2-subgroup of A containing R. By Lemma 2.1, $A_{[1]}$ is a 2-group. Up to relabelling, we may assume that $A_{[1]} \leqslant P$. Since G is regular, we have $A = GA_{[1]}$ and $|A| = |G||A_{[1]}|$. Note that (v) and (vi) follow from the rest of the claims. Indeed, H must have odd order and, since $|A : G|$ is a power of 2, so is $|J : H|$ and thus $J = H \rtimes (P \cap J)$. As H has odd order and is normal in J, it must be characteristic in J and thus normal in A.

Since $|G|$ is not divisible by 4, it follows that G has a characteristic subgroup G_2 of odd order such that $G = G_2 \rtimes R$. By order considerations, we have $A = G_2P$.

Case 1: There is no minimal normal subgroup of A of odd order.

In this case, we have that $\mathrm{soc}(A) = T_1 \times \cdots \times T_k \times B$, where $\mathrm{soc}(A)$ is the socle of A, the T_is are non-abelian simple groups, and B is an elementary abelian 2-group. Recall that $A = G_2P$, that is, A has a 2-complement. Since this property is inherited by normal subgroups, $\mathrm{soc}(A)$ and T_i also have 2-complements for every i. This implies that, for every i, $T_i \cong \mathrm{PSL}(2, p)$ for some Mersenne prime p (see [7, Theorem 1.3] for example). Now, $|T_i|$ is divisible by 3 but the Sylow 3-subgroup of $\mathrm{soc}(A)$ is cyclic (since $|A : G|$ is a power of 2 and G is Sylow cyclic) so that $k = 1$. Let $T = T_1$. Suppose that $p > 7$ and hence $p \geqslant 31$. Note that $T_{[1]}$ has index at most 2 in some Sylow 2-subgroup of T which is isomorphic to $\mathrm{D}_{(p+1)/2}$. It follows that $T_{[1]}$ has order at least $(p + 1)/2$ and is either dihedral or cyclic. Since $p \geqslant 31$, this implies that $T_{[1]}$ contains a unique cyclic subgroup of order $(p + 1)/8$, say U, and U is contained in $(T_{[1]})^2$. By Lemma 2.5, $U = 1$, which is a contradiction. It follows that $p = 7$ and $T \cong \mathrm{PSL}(2, 7)$.

Let $\mathrm{O}_2(A)$ be the largest normal 2-subgroup of A. If $\mathrm{O}_2(A) = 1$, then $\mathrm{soc}(A) = T \cong \mathrm{PSL}(2, 7)$ and A is isomorphic to one of $\mathrm{PSL}(2, 7)$ or $\mathrm{PGL}(2, 7)$. If $A \cong \mathrm{PSL}(2, 7)$, then, as F_{21} is the only proper subgroup of $\mathrm{PSL}(2, 7)$ with index a power of 2, $G \cong \mathrm{F}_{21}$ and the theorem holds. If $A \cong \mathrm{PGL}(2, 7)$, then, for the same reason, G is isomorphic to either F_{21} or $\mathrm{AGL}(1, 7)$. If $G \cong \mathrm{F}_{21}$, then $A_{[1]}$ must be a Sylow 2-subgroup of A and thus isomorphic to D_8. In particular, $A_{[1]}$ contains a unique cyclic subgroup of order 4 and this subgroup is contained in $(A_{[1]})^2$. This contradicts Lemma 2.5. We must therefore have $G \cong \mathrm{AGL}(1, 7)$ and again the theorem holds.

We now assume that $\mathrm{O}_2(A) \neq 1$. In particular, the orbits of $\mathrm{O}_2(A)$ are of equal length, which is a power of 2 greater than 1. It follows that $|\mathrm{O}_2(A) : \mathrm{O}_2(A)_{[1]}| = |[1]^{\mathrm{O}_2(A)}| = 2$. Let K be the kernel of the action of A on the $\mathrm{O}_2(A)$-orbits. By Lemma 2.4, K is semiregular hence so is $\mathrm{O}_2(A)$. It follows that $|\mathrm{O}_2(A)| = 2$ and $\mathrm{O}_2(A)$ is central in A. This implies that $B = \mathrm{O}_2(A)$, hence $\mathrm{soc}(A) = T \times \mathrm{O}_2(A)$. Now, $A_{[1]}$ is a complement for $\mathrm{O}_2(A)$ in P, so by Gaschutz' Theorem (see for example [6, 3.3.2]), $\mathrm{O}_2(A)$ has a complement in A.

Clearly, $\mathrm{O}_2(A) \leqslant \mathrm{C}_A(T)$. We show that equality holds. Suppose, on the contrary, that $\mathrm{O}_2(A) < \mathrm{C}_A(T)$. Since $\mathrm{C}_A(T)$ is normal in A, $\mathrm{C}_A(T)/\mathrm{O}_2(A)$ must contain a minimal normal subgroup of $A/\mathrm{O}_2(A)$, say $Y/\mathrm{O}_2(A)$. Since $\mathrm{O}_2(A)$ has a complement in A, $\mathrm{O}_2(A)$ has a complement in Y, say Z. Thus $Y = \mathrm{O}_2(A) \times Z$ and Z is isomorphic to $Y/\mathrm{O}_2(A)$ which is a minimal normal subgroup of $A/\mathrm{O}_2(A)$ and therefore either an elementary abelian group of odd order, or a product of non-abelian simple groups. It follows that Z is characteristic in Y and thus normal in A. Since the action of A by conjugation on Z and on $Y/\mathrm{O}_2(A)$ are equivalent, we see that Z is a minimal normal subgroup of A. The only possibility is that $Z = T$ but, since T has trivial centre, this contradicts the fact that $Z \leqslant Y \leqslant \mathrm{C}_A(T)$. This concludes our proof that $\mathrm{C}_A(T) = \mathrm{O}_2(A) \cong \mathbb{Z}_2$.

As $\mathrm{O}_2(A)$ has a complement in A, it follows that A is isomorphic to one of $\mathrm{PSL}(2,7) \times \mathbb{Z}_2$ or $\mathrm{PGL}(2,7) \times \mathbb{Z}_2$. Suppose first that $A \cong \mathrm{PSL}(2,7) \times \mathbb{Z}_2$. Since G has even order, is not normal in A and has index a power of 2, we must have $G \cong \mathrm{F}_{21} \times \mathbb{Z}_2$ and the theorem holds with $H = J = 1$. Finally, suppose that $A \cong \mathrm{PGL}(2,7) \times \mathbb{Z}_2$. In particular, $P = Q \times \mathrm{O}_2(A)$ where $Q \cong \mathrm{D}_8$. Note that $|P : A_{[1]}| = 2$ and $A_{[1]} \cap \mathrm{O}_2(A) = 1$ hence $A_{[1]} \cong P/\mathrm{O}_2(A) \cong \mathrm{D}_8$. This contradicts Lemma 2.5.

Case 2: There exists a minimal normal subgroup of A of odd order.

Let N be a minimal normal subgroup of odd order, that is, $|N|$ is a power of some odd prime p. Let K be the kernel of the action of A on the set of N-orbits. Since the N-orbits have odd size and $K_{[1]} \leqslant A_{[1]}$ is a 2-group, $K_{[1]}$ must fix at least one point in every N-orbit. For each N-orbit B, pick $b \in G$ such that $K_{[1]} = K_{[b]}$ and $B = [b]^N$. Now the kernel of the action of K on $[1]^N$ is $K_{([1]^N)} = \bigcap_{n \in N} (K_{[1]})^n = \bigcap_{n \in N} (K_{[b]})^n = K_{([b]^N)}$. It follows that $K_{([1]^N)}$ fixes every vertex of Γ, and so K acts faithfully on $[1]^N$. Moreover, $N_{[1]} = 1$ hence $K = N \rtimes K_{[1]}$. As $|A : G|$ is a power of 2, G contains a Sylow p-subgroup of A. Since N is normal in A, it is contained in every Sylow subgroup of A, and thus $N \leqslant G$. We therefore have $GK = GNK_{[1]} = GK_{[1]}$. Since G is Sylow cyclic and N is elementary abelian, we must have $|N| = p$.

If $K_{[1]} \neq 1$, then $K_{[1]}$ must move a neighbour of $[1]$, say $[s]$ for some non-involution $s \in S$. It follows that $K_{[s]}$ must move $[1]$, necessarily to $[s^2]$ since K is colour-preserving, and thus $[s^2] \in [1]^N$. Let C be the cycle containing $[1]$ with edge-label $\{s, s^{-1}\}$. We have shown that $[1], [s^2] \in [1]^N \cap C$ and hence $|[1]^N \cap C| \geqslant 2$. Since $[1]^N$ and C are both blocks for the action of G, the former of prime order, it follows that $[1]^N \cap C = [1]^N$, that is $[1]^N \subseteq C$. Since K acts faithfully on $[1]^N$, $K_{[1]}$ acts faithfully on C and thus $|GK : G| = |K_{[1]}| \leqslant 2$. It follows that G is normal in GK, $GK = G \rtimes K_{[1]}$ and either $K = N \cong \mathbb{Z}_p$ or $K \cong \mathrm{D}_p$.

Suppose that GK/K is normal in A/K and hence GK is normal in A. We show that this implies that G is normal in A, which is a contradiction. This is trivial if $G = GK$ hence we assume that $|GK : G| = 2$ and $K \cong \mathrm{D}_p$. If G has odd order, then it is characteristic in GK and thus normal in A. We may thus assume that G has even order. Recall that G_2 is a characteristic subgroup of index 2 in G, hence G_2 is normal in GK and, since $|GK : G_2| = 4$, we have that G_2 is characteristic in GK and thus normal in A. Note that

G and $G_2 \rtimes K_{[1]}$ both have index two in GK but $G_2 \rtimes K_{[1]}$ is not semiregular, hence they are not conjugate in A. In particular, $G_2 \rtimes K_{[1]}$ and G are distinct index two subgroups of GK and thus GK/G_2 is elementary abelian of order 4. Let X be the centraliser of N in GK. Since $N \cong \mathbb{Z}_p$, $\mathrm{Aut}(N)$ is cyclic hence GK/X is cyclic and X is not contained in G_2. Since N, G_2 and GK are normal in A, so is XG_2. If $XG_2 = G$, then we are done. We thus assume that this is not the case. Note that $|XG_2 : X| = |G_2 : \mathrm{C}_{G_2}(N)|$ is odd, hence every Sylow 2-subgroup of XG_2 centralises N. Since $K_{[1]}$ has order 2 but does not centralise N, $G_2 \rtimes K_{[1]}$ is not contained in XG_2. We thus conclude that G, $G_2K_{[1]}$ and XG_2 are the three index two subgroups of GK containing G_2. One of them is normal in A, and we have seen that the other two are not conjugate in A. It follows that all three are normal in A. In particular, G is normal in A, a contradiction.

We may thus assume that GK/K is not normal in A/K. Again, we use the bar notation with respect to the mapping $A \mapsto A/K$. By Lemma 2.3, $\overline{A}$ is a colour-preserving group of automorphisms of Γ/N. By induction, we have $\overline{G} = (\overline{F} \times \overline{H}) \rtimes \overline{D}$ and $\overline{A} = (\overline{T} \times \overline{J}) \rtimes \overline{D}$, where $\overline{D}$ is a Sylow 2-subgroup of $\overline{G}$, $\mathrm{PSL}(2,7) \cong \overline{T} \trianglelefteq \overline{A}$, $\overline{T} \cap \overline{G} = \overline{F} \cong \mathrm{F}_{21}$, $\overline{J} \cap \overline{G} = \overline{H} \trianglelefteq \overline{J} \trianglelefteq \overline{A}$ and $\overline{H}$ is self-centralising in $\overline{J}$. Further, since $R \cap K = 1$ we may assume $\overline{R} = \overline{D}$. Note that $T/\mathrm{C}_T(K) \leqslant \mathrm{Aut}(K)$ is soluble since K is either cyclic or dihedral. As $T/K \cong \mathrm{PSL}(2,7)$, it follows that $T = K\mathrm{C}_T(K)$ and hence $\mathrm{C}_T(K)/\,\mathrm{Z}(K) \cong \mathrm{PSL}(2,7)$. If $K \cong \mathrm{D}_p$, then $\mathrm{Z}(K) = 1$. Set $T_0 = \mathrm{C}_T(K)$ in this case. Otherwise, $\mathrm{Z}(K) = N = K \cong \mathbb{Z}_p$ and, since the Schur multiplier of $\mathrm{PSL}(2,7)$ has order 2, we have $\mathrm{C}_T(K) = N \times T_0$ for some T_0. In both cases, $T_0 \cong \mathrm{PSL}(2,7)$ and, since both T and K are normal in A, so is T_0, which proves (i). Now, $TJ = T_0KJ = T_0J$, both T_0 and J are normal in A and $T_0 \cap J = 1$ hence $A = (T_0 \times J) \rtimes R$. Since $\overline{T_0} = \overline{T}$ there is $F_0 \leqslant T_0$ such that $\overline{F_0} = \overline{F}$. Since $F_0 \cap K = 1$ we have $F_0 \cong \overline{F_0} \cong \mathrm{F}_{21}$. Further, $F = F_0K = F_0 \times K$. Since $|GK : G| \leqslant 2$, we have that $F_0 \leqslant G$ and, since F_0 is maximal in T_0, we have $T_0 \cap G = F_0$ which is (ii).

Note that $|H|$ is not divisible by 4, hence $H = H_0 \rtimes K_{[1]}$ for some characteristic subgroup H_0 of H. In particular, $\overline{H} = \overline{H_0}$. Now $GK = FHR = F_0H_0K_{[1]}R$ so $G = F_0H_0R = (F_0 \times H_0) \rtimes R$. Since H_0 is characteristic in H, it is normal in J. Recall that $K \cap G = N \leqslant H_0$, hence $K \cap F_0R = 1$. As $\overline{J} \cap \overline{F_0R} = 1$, this implies that $J \cap F_0R = 1$. Since $H_0 \leqslant J$, we have $J \cap G = H_0(J \cap F_0R) = H_0$, which is (iii). Note that $H_0/N \cong \overline{H}$. Since $\overline{H}$ contains its centraliser in $\overline{J}$, we have $\mathrm{C}_J(H_0) \leqslant HK = H_0K_{[1]}$. As $N \leqslant H_0$ and N is self-centralising in K, we have $\mathrm{C}_J(H_0) \leqslant H_0$, which is (iv).

This concludes the proof. □

We now build on the previous result and give some information about the structure of the connection set.

Theorem 5.2. *Let G be a Sylow cyclic group whose order is not divisible by four, let $\Gamma = \mathrm{Cay}(G,S)$ be a connected non-CCA graph and let $A = \mathrm{Aut}_c(\Gamma)$. Using the notation of Theorem 5.1, write $A = (T \times J) \rtimes R$ and $G = (F \times H) \rtimes R$. Let r be the generator of R, let $Y = S \setminus (F \cup (H \rtimes R))$ and let*

$$\Gamma' = \mathrm{Cay}(F \rtimes R, (F \cap S) \cup \{r\} \cup \{s^2 : s \in Y\}).$$

Then

1. *Γ' is connected and non-CCA,*

2. *$Y \subseteq \{fz : f \in F, z \in Hr, |f| = 3, |z| = 2\}$, and*

3. *if $Y \neq \emptyset$, then $|R| = 2$, and T commutes with R.*

Proof. Since Γ is non-CCA, G is not normal in A. This yields the conclusion of Theorem 5.1. As in the proof of that theorem, for $g \in G$, we write $[g]$ for the vertex of $\mathrm{Cay}(G, S)$ corresponding to g and, for $X \subseteq G$, we write $[X]$ for $\{[x] : x \in X\}$.

Let P be a Sylow 2-subgroup of A containing R. Up to relabelling, we may assume that $A_{[1]} \leqslant P$ and thus $P = A_{[1]}R$. It follows that $[1]^{PH} = [H \rtimes R]$ is a block for A. As T is normal in A, its orbits are also blocks. One such block is $[1]^T = [F]$. As $[F] \cap [H \rtimes R] = [1]$, we find that the two block systems induced by $[F]$ and by $[H \rtimes R]$ are transverse.

The action of T on $[F]$ is equivalent to the action of $\mathrm{PSL}(2,7)$ by conjugation on its 21 Sylow 2-subgroups. In particular, if $f \in F$ and $T_{[1]} = T_{[f]}$, then $f = 1$. This observation, together with the previous paragraph, yields that the set of fixed points of $T_{[1]}$ is exactly $[H \rtimes R]$.

We first show (2) and (3). Let $s \in Y$. Note that $[Fs]$ is the orbit of T that contains s. Since $s \notin H \rtimes R$, $[s]$ is not fixed by $T_{[1]}$. Since T is colour-preserving, we have that $[s] \neq [s^{-1}] \in [Fs]$. It follows that $1 \neq s^2 \in F$. In particular, $|s^2| \in \{3, 7\}$. Since A is colour-preserving, the cycles coloured $\{s, s^{-1}\}$ form a block system for A. This means that $[\langle s^2 \rangle]$ is also a block for A, contained in the block $[F]$. Now, $\mathrm{PSL}(2,7)$ on its action on 21 points does not admit blocks of size 7, therefore $|s^2| = 3$. Since $s \notin F$ this implies $|s| = 6$. Notice also that $[\{1, s^3\}]$ is a block of A. Thus, $[s^3]$ is a fixed point of $T_{[1]}$, so $[s^3] \in [H \rtimes R]$. Since $|s^3| = 2$ but $|H|$ is odd, $s^3 \notin H$ hence $|R| = 2$ and $s^3 \in Hr$. Since H and F centralise each other and $s^3 r \in H$ and $s^2 \in F$, it follows that s^2 commutes with r. Note that $[1]^P = [R]$ is a block for A. Now, $[\langle s^2 \rangle]$ is also a block for A, being a set of vertices of even distance contained in one of the monochromatic hexagons coloured $\{s, s^{-1}\}$. It follows that $[\langle s^2, r \rangle]$ is also a block for A, of size 6 and contained in the block $[1]^{TP} = [F \rtimes R]$. Note that $\mathrm{PGL}(2,7)$ does not have blocks of size 6 in its transitive action on 42 points. It follows that $T \rtimes R \not\cong \mathrm{PGL}(2,7)$, and hence T commutes with R. Writing $f = s^4$ and $z = s^3$ concludes the proof of (2) and (3).

Let $\pi \colon G \mapsto F \rtimes R$ be the natural projection and let $s \in Y$. By the previous paragraph, we have $s^{-1} = s^2 s^3 = s^2 hr$, where $s^2 \in F$ and $h \in H$. It follows that $\pi(s^{-1}) = s^2 r$. As S is inverse-closed, we have $\pi(Y) = \{s^2 r : s \in Y\}$. Since $\langle S \rangle = G$, we have $F \rtimes R = \langle \pi(S) \rangle \leqslant \langle F \cap S, r, s^2 r : s \in Y \rangle = \langle F \cap S, r, s^2 : s \in Y \rangle$ and thus Γ' is connected. Note that $[1]^{T \rtimes R} = [F \rtimes R]$ hence $T \rtimes R \leqslant \mathrm{Aut}_c(\Gamma')$. Since $F \rtimes R$ is not normal in $T \rtimes R$, Γ' is not CCA. $\square$

The following result is, in some sense, a converse to Corollary 5.2.

Proposition 5.3. *Let G be a Sylow cyclic group whose order is not divisible by four such that $G = (F \times H) \rtimes R$ where $F \cong \mathrm{F}_{21}$, R is a Sylow 2-subgroup of G, and F and H are normal in G. Let r be the generator of R, let S be a generating set for G, let $Y = S \setminus (F \cup (H \rtimes R))$, let $S' = (F \cap S) \cup \{r\} \cup \{s^2 : s \in Y\}$, and let*

$$\Gamma' = \mathrm{Cay}(F \rtimes R, S').$$

If

1. *Γ' is connected and non-CCA,*

2. *$Y \subseteq \{fz : f \in F, z \in Hr, |f| = 3, |z| = 2\}$, and*
3. *if $Y \neq \emptyset$, then $|R| = 2$, and F commutes with R,*

then $\mathrm{Cay}(G,S)$ *is connected and non-CCA.*

Proof. Since S generates G, $\mathrm{Cay}(G,S)$ is connected. Since Γ' is a connected and non-CCA Cayley graph on $F \rtimes R$, it follows from Theorem 5.1 that there exists a group $T \rtimes R$ of colour-preserving automorphisms of Γ', with $F \leqslant T$ and $T \cong \mathrm{PSL}(2,7)$.

This yields an action of T on $F \rtimes R$. We extend this action to the vertex-set of $\mathrm{Cay}(G,S)$ in the following way: for $t \in T$ and $xh \in G$, with $x \in F \rtimes R$ and $h \in H$, let $(xh)^t = x^t h$.

Notice that if $x \in F \rtimes R$, then, since $r \in S'$ is an involution and $T \rtimes R$ is colour-preserving on Γ', for any $t \in T$ we have $(rx)^t = rx^t$.

Note that $F \leqslant T \cap G < T$. Since T is simple, it follows that $T \cap G$ is not normal in T. We claim that T is a colour-preserving group of automorphisms of $\mathrm{Cay}(G,S)$. By the previous comment, this will show that $\mathrm{Cay}(G,S)$ is non-CCA.

Let $t \in T$, let $v \in G$ and write $v = xh$ with $x \in F \rtimes R$ and $h \in H$. We will show that, for all $s \in S$, we have $(sv)^t = s^{\pm 1} v^t$.

Suppose first that $s \in S'$. (This includes the case when $s \in F$.) Since T is colour-preserving on Γ', we have $(sx)^t = s^{\pm 1} x^t$. Since $sx \in F \rtimes R$, we have $(sv)^t = (sxh)^t = (sx)^t h = s^{\pm 1} x^t h = s^{\pm 1} v^t$, as required.

Suppose next that $s \in H \rtimes R$. Write $s = h' r^i$ and $x = r^j f$, where $h' \in H$, $f \in F$ and $i, j \in \mathbb{Z}$. Let $h'' \in H$ be such that $r^{i+j} h'' = h' r^{i+j}$. Then

$$\begin{aligned}(sv)^t &= (h' r^{i+j} f h)^t = (r^{i+j} h'' f h)^t = (r^{i+j} f h'' h)^t = (r^{i+j} f)^t h'' h = r^{i+j} f^t h'' h \\ &= r^{i+j} h'' f^t h = h' r^{i+j} f^t h = s r^j f^t h = s (r^j f)^t h = s x^t h = s v^t,\end{aligned}$$

as desired.

Finally, suppose $s \in Y$. We can write $s = fz$ where $f \in F$, $z \in Hr$, $|f| = 3$ and $|z| = 2$. By (3), we have $s^3 = z$, $s^2 = f^2$ and $|s| = 6$. Since $s^3 \in H \rtimes R$, the argument of the previous paragraph shows $(s^3 v)^t = s^3 v^t$. On the other hand, since $s^2 \in S'$, we have $(s^3 v)^t = (s^2(sv))^t = s^{\pm 2}(sv)^t$. Combining these gives $(sv)^t = s^{3 \pm 2} v^t = s^{\pm 1} v^t$, as desired. □

We view Theorem 5.2 as a reduction of the CCA problem for groups of the kind appearing in its statement to the determination of non-CCA graphs on F_{21} and $\mathrm{AGL}(1,7)$. It therefore becomes of significant interest to understand the structure of such graphs.

Let x and y be elements of order 7 and 6 in $\mathrm{AGL}(1,7)$, respectively, and let $d = (y^3)^x$. Note that $\langle x, y^2 \rangle = \mathrm{F}_{21}$. Let

$$S_{21} = \{y^{\pm 2}, (xy^2)^{\pm 1}\}, \quad S_{42,1} = \{y^{\pm 2}, d\} \quad \text{and} \quad S_{42,2} = \{y^{\pm 2}, (y^{\pm 2})^d, d\}.$$

Note that $\mathrm{Cay}(\mathrm{F}_{21}, S_{21})$ is isomorphic to the line graph of the Heawood graph (see Example 4.8), while $\mathrm{Cay}(\mathrm{AGL}(1,7), S_{42,1})$ is isomorphic to the line graph of the subdivision of the Heawood graph (see Example 4.11).

Proposition 5.4.

1. *The graph* $\mathrm{Cay}(\mathrm{F}_{21}, S)$ *is connected but not CCA if and only if S is conjugate in* $\mathrm{AGL}(1,7)$ *to* S_{21}.

2. *The graph* $\mathrm{Cay}(\mathrm{AGL}(1,7), S)$ *is connected but not CCA if and only if* S *is conjugate in* $\mathrm{AGL}(1,7)$ *to one of* $S_{42,1}$ *or* $S_{42,2}$.

3. *The graph* $\mathrm{Cay}(\mathrm{F}_{21} \times \mathbb{Z}_2, S)$ *is connected but not CCA if and only if* S *is conjugate in* $\mathrm{AGL}(1,7) \times \mathbb{Z}_2$ *to some inverse-closed subset of*

$$\{y^{\pm 2}, (xy^2)^{\pm 1}, y^{\pm 2}r, (xy^2)^{\pm 1}r, r\}$$

that generates $\mathrm{F}_{21} \times \mathbb{Z}_2$*, where* $\mathbb{Z}_2 = \langle r \rangle$.

Proof. This was verified using MAGMA [1]. The proof of the first claim can also be found in [3, Proposition 2.5, Remark 2.6]. □

Remark 5.5. It can be checked that Proposition 5.4(3) yields eleven generating sets for $\mathrm{F}_{21} \times \mathbb{Z}_2$, up to conjugacy in $\mathrm{AGL}(1,7) \times \mathbb{Z}_2$.

References

[1] W. Bosma, J. Cannon and C. Playoust, The Magma algebra system. I: The user language, *J. Symbolic Comput.* **24** (1997), 235–265, doi:10.1006/jsco.1996.0125.

[2] D. P. Byrne, M. J. Donner and T. Q. Sibley, Groups of graphs of groups, *Beitr. Algebra Geom.* **54** (2013), 323–332, doi:10.1007/s13366-012-0093-7.

[3] E. Dobson, A. Hujdurović, K. Kutnar and J. Morris, On color-preserving automorphisms of Cayley graphs of odd square-free order, *J. Algebraic Combin.* **45** (2017), 407–422, doi:10.1007/s10801-016-0711-9.

[4] E. Dobson, A. Hujdurović, K. Kutnar and J. Morris, Vertex-transitive digraphs with extra automorphisms that preserve the natural arc-colouring, *Australas. J. Combin.* **67** (2017), 88–100, https://ajc.maths.uq.edu.au/pdf/67/ajc_v67_p088.pdf.

[5] A. Hujdurović, K. Kutnar, D. W. Morris and J. Morris, On colour-preserving automorphisms of Cayley graphs, *Ars Math. Contemp.* **11** (2016), 189–213, http://amc-journal.eu/index.php/amc/article/view/771.

[6] H. Kurzweil and B. Stellmacher, *The Theory of Finite Groups: An Introduction*, Universitext, Springer-Verlag, New York, 2004, doi:10.1007/b97433, translated from the 1998 German original.

[7] C. Martínez-Pérez and W. Willems, The trivial intersection problem for characters of principal indecomposable modules, *Adv. Math.* **222** (2009), 1197–1219, doi:10.1016/j.aim.2009.05.018.

[8] J. Morris, Automorphisms of circulants that respect partitions, *Contrib. Discrete Math.* **11** (2016), 1–6, http://cdm.ucalgary.ca/cdm/index.php/cdm/article/view/376.

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Combinatorial configurations, quasiline arrangements, and systems of curves on surfaces

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Abstract

It is well known that not every combinatorial configuration admits a geometric realization with points and lines. Moreover, some of them do not even admit realizations with pseudoline arrangements, i.e., they are not topological. In this paper we generalize the concept of topological configurations to a more general one (in a least possible way) such that every combinatorial configuration is realizable in this way.

In particular, we generalize the notion of a pseudoline arrangement to the notion of a quasiline arrangement by relaxing the condition that two pseudolines meet exactly once. We also generalize well-known tools from pseudoline arrangements such as sweeps and wiring diagrams. We introduce monotone quasiline arrangements as a subfamily of quasiline arrangements that can be represented with generalized wiring diagrams. We show that every incidence structure (and therefore also every combinatorial configuration) can be realized as a monotone quasiline arrangement in the real projective plane.

A quasiline arrangement with selected vertices belonging to an incidence structure can be viewed as a map on a closed surface. Such a map can be used to distinguish between two "distinct" realizations of an incidence structure as a quasiline arrangement.

Keywords: Pseudoline arrangement, quasiline arrangement, projective plane, incidence structure, combinatorial configuration, topological configuration, geometric configuration, sweep, wiring diagram, allowable sequence of permutations, maps on surfaces.

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1 Introduction

A (n_k) *configuration* consists of a set of n points and a set of n lines, such that each point is incident to the same number k of lines and each line is incident to the same number k of points. We adopt the view from Grünbaum's book [12] that configurations of points and lines come in three sorts: combinatorial, geometric and topological.

The most general are *combinatorial configurations* where sets of points and lines are abstract sets. A configuration is *geometric* if its lines are lines in Euclidean or projective plane, and its points form a subset of intersection points of these lines. Similarly, a configuration is *topological* if its lines are pseudolines in the projective plane that form a pseudoline arrangement (formal definitions will be given in later sections). An important problem is to consider whether a given combinatorial configuration can be realized as a geometric or as a topological configuration.

The smallest combinatorial (n_3) configuration, the *Fano plane*, has seven points and seven lines and is unique, up to isomorphism. Likewise, there is only one (8_3) configuration, called *Möbius-Kantor configuration*. There are three nonisomorphic (9_3) configurations, one of them is the well known Pappus configuration. While Fano plane and Möbius-Kantor configuration cannot be realized as geometric configurations (see, for example, [12, Theorem 2.1.3]), all the (9_3) configurations can. The Pappus configuration as a geometric configuration is shown in Figure 1.

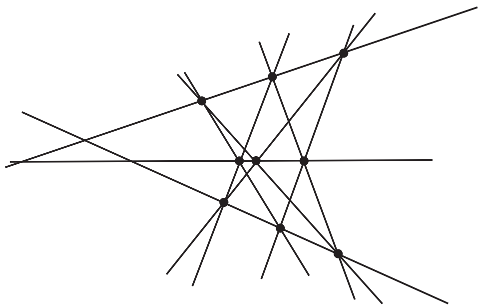


Figure 1: The Pappus configuration.

It was realized quite early by H. Schröter [17] that among the ten (10_3) configurations, exactly one, called *anti-Desargues configuration*, is not geometric. A remarkable result was proved by E. Steinitz in his thesis [18] that every combinatorial (n_3) configuration can be represented with straight lines, except for one that may be curved. He didn't notice however, that additional incidences may appear.

F. Levi [13] introduced the notion of pseudolines and showed that the combinatorial configurations (7_3) and (8_3) cannot be realized with pseudoline arrangements in the projective plane. There is a fundamental difference between the (7_3) and (8_3) configurations and the anti-Desargues configuration. Namely, the latter can be realized as a pseudoline arrangement in the projective plane, i.e. it is a topological configuration. J. Bokowski and his coworkers applied modern methods of computational synthetic geometry to study the existence or nonexistence of topological configurations [5]. B. Grünbaum, J. Bokowski, and L. Schewe [1] investigated the problem of existence of topological (n_4) configurations and showed that topological configurations exist if and only if $n \geq 17$. Recently J. Bokowski and R. Strausz [4] associated to each topological configuration a map on a surface that they call a *manifold associated to the topological configuration* and suggested a possible definition of equivalence of two topological configurations. Namely, topological configurations are distinct in a well-defined sense if and only if the associated maps are distinct.

The main purpose of this paper is to generalize the concept of topological configurations to a more general one (in the least possible way) such that every combinatorial configuration, and more generally, every combinatorial incidence structure, would be realizable in this sense. It is not hard to see that every combinatorial incidence structure can be realized by points and curves in the projective plane. However, it is not obvious that all the curves in question may, in fact, be pseudolines. It turns out that we only need to relax the condition of unique intersection of pairs of pseudolines.

In Section 4 we first generalize the notion of a pseudoline arrangement to the notion of a *quasiline arrangement* by relaxing the condition that two pseudolines meet exactly once. Then we introduce a subclass of quasiline arrangements that we call *monotone*. In Sections 5 and 6 we also generalize well-known tools from pseudoline arrangements such as sweeps, wiring diagrams, and allowable sequences of permutations. It is known that every pseudoline arrangement can be represented by a wiring diagram and conversely, every wiring diagram can be viewed as a pseudoline arrangement; see J. E. Goodman [8]. We show that every monotone quasiline arrangement can be represented by a generalized wiring diagram that is in turn also a monotone quasiline arrangement. In this respect the class of monotone quasiline arrangements is the weakest generalization of the class of pseudoline arrangements.

In Section 7 we deal with polygonal quasiline arrangements. We show that a monotone quasiline arrangement without digons is topologically equivalent to a monotone polygonal quasiline arrangement with no bends (arcs connecting two vertices of the arrangement are all straight lines).

In Section 8 we introduce a generalization of topological incidence structures that we call (*monotone*) *quasi-topological incidence structures* by allowing the set of lines to form a (monotone) quasiline arrangement instead of a pseudoline arrangement. In Section 9 we show the main result of this paper that every combinatorial incidence structure, in particular every combinatorial configuration, can be realized as a monotone quasi-topological incidence structure.

A natural parameter for a monotone quasiline arrangement is the maximal number of crossings of pairs of pseudolines. A monotone quasiline arrangement is a pseudoline arrangement if and only if this parameter is equal to 1. Similarly, combinatorial incidence structures can be stratified by the minimal number of crossings over all the representations as monotone quasi-topological incidence structures. Topological incidence structures are exactly those representations with the maximal number of crossings equal to 1. The proof of our main result is constructive and it shows that an incidence structure with v points and n lines can be realized such that the number of intersections of every two pseudolines is upper bounded by $(v+1)\,n^2$. It would be interesting to characterize those incidence structures which allow realizations with monotone quasiline arrangements such that the maximal number of intersections of two pseudolines is small and/or independent of v and n.

Finally, the concept of a map associated with a topological configuration can be extended to the quasi-topological case (Section 10).

2 Combinatorial and geometric incidence structures

In this section we review basic definitions and facts about incidence structures; see B. Grünbaum [12] or T. Pisanski and B. Servatius [15] for more background information. An *incidence structure* $\mathcal{C}$ is a triple $\mathcal{C} = (\mathcal{P}, \mathcal{L}, \mathcal{I})$, where $\mathcal{P}$ and $\mathcal{L}$ are non-empty disjoint finite sets and $\mathcal{I} \subseteq \mathcal{P} \times \mathcal{L}$. The elements of $\mathcal{P}$ are called *points* and the elements of $\mathcal{L}$ are called *lines*. The relation $\mathcal{I}$ is called *incidence relation*; if $(p, L) \in \mathcal{I}$, we say that the point p is incident to the line L or, in a geometrical language, that p lies on L. We further require that each point lies on at least two lines and each line contains at least two points. To stress the fact that these objects are of purely combinatorial nature we sometimes call them *abstract* or *combinatorial* incidence structures.

Two incidence structures $\mathcal{C} = (\mathcal{P}, \mathcal{L}, \mathcal{I})$ and $\mathcal{C}' = (\mathcal{P}', \mathcal{L}', \mathcal{I}')$ are *isomorphic*, if there exists an incidence preserving bijective mapping from $\mathcal{P} \cup \mathcal{L}$ to $\mathcal{P}' \cup \mathcal{L}'$ which maps $\mathcal{P}$ to $\mathcal{P}'$ and $\mathcal{L}$ to $\mathcal{L}'$.

Complete information about the incidence structure can be recovered also from its Levi graph with a given black and white coloring of the vertices. The *Levi graph* $G(\mathcal{C})$ of an incidence structure $\mathcal{C}$ is a bipartite graph with "black" vertices $\mathcal{P}$ and "white" vertices $\mathcal{L}$ and with an edge joining some $p \in \mathcal{P}$ and some $L \in \mathcal{L}$ if and only if p lies on L in $\mathcal{C}$.

An incidence structure is *lineal* if any two distinct points are incident with at most one common line (or equivalently, any two distinct lines are incident to at most one point). This is equivalent to saying that the Levi graph of a lineal incidence structure has girth at least 6. A (n_r, b_k)*-configuration* is a lineal incidence structure $\mathcal{C} = (\mathcal{P}, \mathcal{L}, \mathcal{I})$ with $|\mathcal{P}| = v$ and $|\mathcal{L}| = b$ such that each line is incident with the same number k of points and each point is incident with the same number r of lines. In the special case when $n = b$ (and by a simple counting argument also $r = k$) we speak of a *balanced* configuration and shorten the notation (n_k, n_k) to (n_k).

A set of lines in the real Euclidean or projective plane together with a subset of intersection points of these lines such that each line contains at least two intersection points is called a *geometric incidence structure*. From the definition it follows that each point lies on at least two lines. A geometric incidence structure together with the incidences of points and lines defines a combinatorial incidence structure, which we call the *underlying combinatorial incidence structure*. Such a combinatorial incidence structure is certainly lineal. Two geometric incidence structures are *isomorphic* if their underlying combinatorial incidence structures are isomorphic. A geometric incidence structure $\mathcal{G}$ is a *realization* of a combinatorial incidence structure $\mathcal{C}$ if the underlying combinatorial incidence structure of $\mathcal{G}$ is isomorphic to $\mathcal{C}$.

3 Pseudolines and topological incidence structures

In this section we review basic facts about pseudoline arrangements. For more background on this topic see J. E. Goodman [8] or B. Grünbaum [11]. By a projective plane we mean the real projective plane or the extended Euclidean plane. A *pseudoline* is a simple non-contractible closed curve in the projective plane. In particular, each line in the projective plane is a pseudoline.

Pseudolines and certain relationships between them inherit properties from the topological structure of the projective plane [8]. For instance:

Fact I. Any two pseudolines have at least one point in common.

Fact II. If two pseudolines meet in exactly one point they intersect transversally at that point.

A *pseudoline arrangement* $\mathcal{A}$ is a collection of at least two pseudolines in the projective plane with the property that each pair of pseudolines of $\mathcal{A}$ has exactly one point in common (at which they cross transversally). Such a point of intersection is called a *vertex* or a *crossing* of the arrangement. A crossing in which only two pseudolines meet is called *regular*. If more than two pseudolines meet in the same point, the crossing is called *singular*. Each pseudoline arrangement $\mathcal{A}$ determines an associated 2-dimensional cell complex into which the pseudolines of $\mathcal{A}$ decompose the projective plane. Its cells of dimension 0,1,2 are called *vertices*, *edges*, and *cells* (or *polygons*), respectively; see [11, p. 40]. Two pseudoline arrangements are *isomorphic* if the associated cell complexes are isomorphic; that is, if and only if there exists an incidence preserving bijective mapping between the vertices, edges, and cells of one arrangement and those of the other.

We say that a pseudoline is *polygonal*, if it is a line or it can be subdivided into a finite number of closed line segments (whereby the endpoints of the line segments occur of course twice). A pseudoline arrangement is *polygonal* if every pseudoline of the arrangement is polygonal. Note that a line arrangement is polygonal. A point on a polygonal pseudoline that is not a crossing of the arrangement is called a *bend* if two line segments meet at that point and the join of these segments is not again a line segment. A crossing v of a polygonal pseudoline arrangement is *straight*, if there exists a neighborhood $N(v)$ of v such that the intersection of $N(v)$ with every pseudoline ℓ containing v is a line segment.

It is well-known that every pseudoline arrangement is isomorphic to a a polygonal pseudoline arangement. In [11, Theorem 3.3] essentialy the following theorem is presented and a proof by induction is proposed.

Proposition 3.1. *Every pseudoline arrangement is isomorphic to a polygonal pseudoline arrangement with no bends.*

To describe pseudoline arrangements combinatorially, wiring diagrams are standard tools to use; see J. E. Goodman [8] and J. E. Goodman and R. Pollack [9].

A partial wiring diagram is a collection of x-monotone polygonal lines, called *wires*, in the Euclidean plane, each of them horizontal except for a finite number of "short" segments, where it crosses another polygonal line. *A wiring diagram* is a partial wiring diagram with the property that every two polygonal lines cross exactly once.

A wiring diagram whose wires are equally spaced can be viewed as a pseudoline arrangement: take a disk that is large enough that all the crossings are in its interior, and positioned such that the intersections of each polygonal line with the boundary of the disk are on opposite sides of the boundary of the disk. The disk can now be viewed as a disk model of the projective plane and the lines of the wiring diagram inside the disk form a polygonal pseudoline arrangement. Figure 2 (b) shows a wiring diagram that is isomorphic to the line arrangement on Figure 2 (a).

Conversely, every pseudoline arrangement can be described with a wiring diagram; see [8].

Theorem 3.2. *Every pseudoline arrangement is isomorphic to a wiring diagram.*

A pseudoline arrangement can also be viewed as an incidence structure when we define a subset of its crossings as its point set. We say that an incidence structure is *topological* if

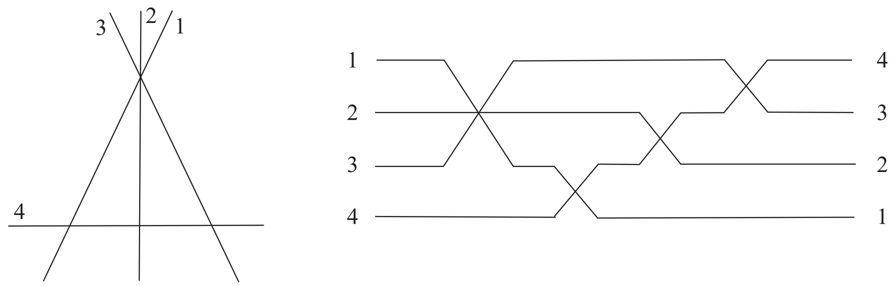


Figure 2: A pseudoline arrangement and a corresponding wiring diagram.

its points are points in the projective plane, and lines are pseudolines that form a pseudoline arrangement. A topological incidence structure $\mathcal{T}$ is a *topological realization* of a combinatorial incidence structure $\mathcal{C}$ if the underlying combinatorial incidence structure of $\mathcal{T}$ is isomorphic to $\mathcal{C}$. A topological incidence structure is *polygonal* if its lines are polygonal pseudolines. Note that any geometric incidence structure is also (polygonal) topological.

There are three distinct notions of equivalence of topological incidence structures. The weakest is combinatorial equivalence. Two topological incidence structures are *combinatorially equivalent* or *isomorphic* if they are isomorphic as combinatorial incidence structures. The strongest one is the notion of topological equivalence between pseudoline arrangements in the projective plane. Two topological incidence structures are *topologically equivalent* if there exists an isomorphism of the underlying pseudoline arrangements that induces an isomorphism of the underlying combinatorial incidence structures.

One intermediate notion is mutation equivalence; see J. Bokowski and V. Pilaud [2]. A *mutation* or a *Reidemeister move* of a pseudoline arrangement is a local transformation to another pseudoline arrangement where only one pseudoline ℓ moves across a single crossing v of the remaining arrangement. Only the position of the crossings of ℓ with the pseudolines incident to v is changed. If those crossings are not points of the incidence structure, we say that such a mutation is *admissible*. Two topological incidence structures are *mutation equivalent* if they can be modified by (possibly empty) sequences of admissible mutations to obtain topologically equivalent topological incidence structures.

As a consequence of Proposition 3.1 we have the following result.

Proposition 3.3. *Every topological incidence structure is topologically equivalent to a polygonal topological incidence structure with no bends.*

As we mentioned already, the underlying combinatorial incidence structure of a geometric incidence structure is necessary lineal. Note that the same is true for the underlying combinatorial incidence structure of a topological incidence structure. This is equivalent to saying that every incidence structure that allows a description with a wiring diagram is lineal. However, our generalization considered in the next sections covers both lineal and non-lineal incidence structures.

4 Quasiline arrangements

In this section we generalize the notion of a pseudoline arrangement in which we relax the condition on pseudoline crossings. A *quasiline arrangement* is a collection of at least two pseudolines in the real projective plane with the property that any two pseudolines have a finite number of points in common and that at each common point they cross transversally. Note that any pair of pseudolines in a quasiline arrangement meets an odd number of times. The terms such as crossings, bends, polygonal quasiline arrangements, and isomorphic quasiline arrangements are defined in the same way as for pseudoline arrangements.

To simplify the discussion we will consider only quasiline arrangements in the extended Euclidean plane with the following additional properties:

- none of the crossings of the arrangement are points at infinity and
- every pseudoline of the arrangement intersects the line at infinity exactly once.

The former condition can easily be achieved in general by a suitable projective transformation while the latter is an essential assumption. Namely, a pseudoline can intersect the line at infinity more than once.

In the sequel we define a subclass of quasiline arrangements, called monotone quasiline arrangements, that is in some sense the least generalization of the pseudoline arrangements.

It is well-known that the extended Euclidean plane is in its topological sense equivalent to the disk model of the projective plane: the projective plane is represented by a disk, with all the pairs of antipodal points on the boundary identified. The boundary of the disk is the line at infinity of the projective plane and the points on the line at infinity are points at infinity. For the rest of the section we will use the disk model of the projective plane.

Let $\mathcal{A}$ be a quasiline arrangement. We choose an orientation for the line at infinity ℓ and a point x on ℓ that is not on any of the pseudolines of the arrangement. Denote by x^+, x^- the corresponding points on the boundary of the disk representing the projective plane without identifying x^+ and x^-. These two points divide the boundary of the disk into two arcs, ℓ^+ from x^+ to x^- and ℓ^- from x^- to x^+, again by forgetting the identification of antipodal points. Starting at x^+ and moving along ℓ in the positive direction we orient a pseudoline that we meet for the first time such that it points to the interior of the circle. We call such an orientation a *monotone orientation* of the quasiline arrangement $\mathcal{A}$ and call the arrangement together with the point x^+ (or equivalently, with a monotone orientation) a *marked arrangement*. In a marked arrangement we have a natural ordering of the pseudolines as the order of intersections with ℓ starting at x^+. Moreover, the orientation of pseudolines induces the order of crossings on every pseudoline. A marked arrangement $(\mathcal{A}, x^+)$ is *proper* if the order of the intersection points of any two pseudolines is the same for both pseudolines. Observe that the property of being proper is heredity by taking subarrangements.

A quasiline arrangement $\mathcal{A}$ is a *monotone quasiline arrangement* if there exists a proper marked arrangement $(\mathcal{A}, x^+)$ for some x^+ on the boundary of the disk, representing the line at infinity. Note that it is not the same to require that there exists an orientation of the pseudolines such that the order of the intersection points on any two pseudolines is the same for both pseudolines. Figure 3 shows a quasiline arrangement in which the order of crossings is the same for any pair of pseudolines if the pseudolines are oriented alternatingly. However, it is not a monotone quasiline arrangement, since any monotone orientation of the arrangement induces different orders of crossings on the horizontal line and at least one of the other pseudolines.

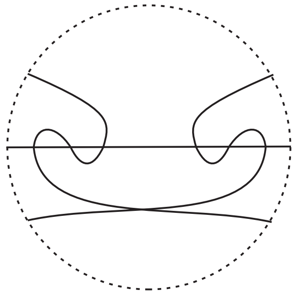
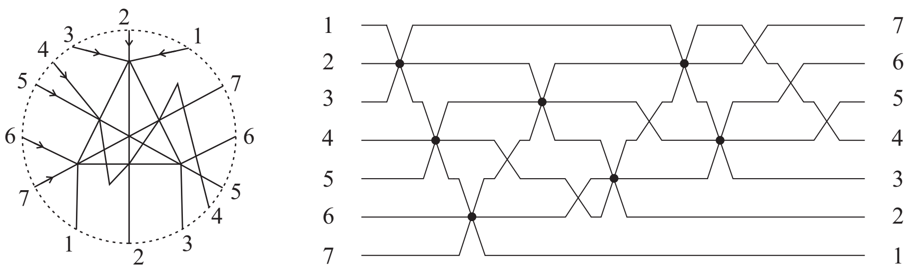


Figure 3: A quasiline arrangement that is not a monotone quasiline arrangement.

We now generalize the notion of the wiring diagram from Section 3 to be able to describe also monotone quasiline arrangements. *A generalized wiring diagram* is a partial wiring diagram with the property that every two polygonal lines cross an odd number of times. A generalized wiring diagram can be viewed as a monotone polygonal quasiline arrangement just like a wiring diagram can be viewed as a pseudoline arrangement. Figure 4 (b) shows a generalized wiring diagram which is topologically equivalent to the quasiline arrangement from Figure 4 (a). Observe that by adding seven points (corresponding to the crossings where three pseudolines cross) we arrive at the unique (7_3) configuration – the Fano plane.



(a) (b)

Figure 4: (a) A monotone quasiline arrangement and (b) its generalized wiring diagram.

One of the main results of the paper is the following generalization of Theorem 3.2.

Theorem 4.1. *Every monotone quasiline arrangement is isomorphic to a generalized wiring diagram.*

We will prove the theorem in Section 6. To do this we will need the notions of sweeping and allowable sequences of permutations, which we introduce in the next two sections.

5 Sweeping quasiline arrangements

In this section we introduce the notion of sweeping and show that every proper marked quasiline arrangement has a sweep. With a slight modification working in the disk model of the projective plane instead of the Euclidean plane, we follow S. Felsner and H. Weil [7].

Let $(\mathcal{A}, x^+)$ be a proper marked quasiline arrangement. A *sweep* of $(\mathcal{A}, x^+)$ is a sequence $c_0, c_1, \ldots, c_r$, of pseudolines such that the following conditions hold:

(1) pseudolines c_0 and c_r coincide with the line at infinity,

(2) any two pseudolines c_i and c_j intersect exactly once at x,

(3) none of the pseudolines c_i contains a vertex of the arrangement $\mathcal{A}$,

(4) each pseudoline c_i has exactly one point of intersection with each pseudoline of $\mathcal{A}$,

(5) for any two consecutive pseudolines c_i, c_{i+1} of the sequence there is exactly one vertex of the arrangement $\mathcal{A}$ between them, i.e., in the interior of the region bounded by c_i and c_{i+1} disjoint from the line at infinity.

We will show that every proper marked quasiline arrangment has a sweep. To this end we define a directed graph, or briefly a digraph, $D = D(\mathcal{A}, x^+)$ that corresponds to a marked quasiline arrangement $(\mathcal{A}, x^+)$ as follows. The vertices of D are the vertices of $\mathcal{A}$ and there is a directed edge for every pair of vertices u and v that are consecutive on an arc of some pseudoline oriented from u to v that has an empty intersection with the line at infinity. We say that such a digraph is *associated with* the marked arrangement $(\mathcal{A}, x^+)$. Note that this digraph is embedded in the plane. The undirected plane graph G underlying $D(\mathcal{A}, x^+)$ will be called the graph *associated with* the arrangement $\mathcal{A}$. Note that G depends only on the arrangement $\mathcal{A}$ and not on the orientation of the pseudolines.

In [7] S. Felsner and H. Weil prove that the digraph associated with a marked arrangement of pseudolines is acyclic. We prove the following generalization of this result.

Lemma 5.1. *The digraph associated with a proper marked quasiline arrangement is acyclic.*

Proof. Without loss of generality we may assume that the line at infinity is oriented counterclockwise. Let $(\mathcal{A}, x^+)$ be a proper marked quasiline arrangement and D its associated digraph. Note that D is a plane graph, embedded in the interior of the disk, representing the projective plane. The interior of the disk is homeomorphic to the plane, therefore Jordan curve theorem applies; see [14, p. 25].

First we observe that D contains no directed cycles of length 2, otherwise there are two lines with different orders of vertices on them and $(\mathcal{A}, x^+)$ is not a proper marked arrangement. For the same reason there are no directed cycles, all the edges of which belong to two pseudolines.

Suppose D contains a directed cycle. Let C be a directed cycle, given by the sequence of vertices and edges $v_0, e_0, v_1, \ldots, e_{t-1}, v_t = v_0$ such that no other directed cycle is contained in the area bounded by C. It is easy to see that C bounds a face of D by Jordan's theorem. Since at each vertex of the arrangement there meet at least two pseudolines, two consecutive edges of C lie on different pseudolines. Assume that C is oriented clockwise; see Figure 5. If C is oriented counterclockwise, the proof is similar.

Now consider the arrangement $\mathcal{A}'$ consisting only of the pseudolines $\ell_0, \ldots, \ell_{t-1}$ that contain the edges $e_0, \ldots, e_{t-1}$ of C consecutively. Note that also $(\mathcal{A}', x^+)$ is a proper marked arrangement. Denote by p_i the intersection of ℓ_i with the line at infinity ℓ for each i. We observe that ℓ_i and ℓ_{i+1} are distinct (the indices are taken modulo t), since two consecutive edges of C lie on different pseudolines. Without loss of generality we may assume that p_0^+ appears first after x^+.

Since p_1^+ appears after p_0^+ on the line at infinity, the arc of ℓ_1 from v_1 to p_1^+ must intersect ℓ_0 to reach ℓ^+. It can intersect ℓ_0 only between p_0^+ and v_0 (an odd number of times), otherwise we obtain a directed cycle contained in only two pseudolines. Similarly, since p_1^- appears after p_0^- on ℓ^-, the arc of ℓ_1 from v_2 to p_1^- must intersect ℓ_0 to reach ℓ^-. It can intersect ℓ_0 only between v_1 and p_0^- (an odd number of times); see Figure 5.

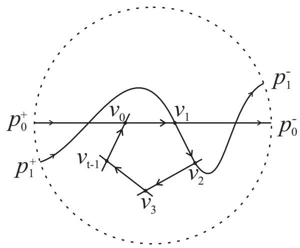


Figure 5: A directed cycle in a marked quasiline arrangement.

Now the arc of ℓ_2 between v_2 and p_2^+ cannot intersect ℓ_1 after v_2, since then we again obtain a directed cycle on two lines. Therefore it must intersect ℓ_0 after v_1 and then again before v_0 to reach ℓ^+. If $t > 2$, we see that also the arc of ℓ_2 between v_3 and p_2^- must intersect ℓ_0 to reach ℓ^-. In order to avoid self-intersection it can cross ℓ_0 only after v_1 (at least once). With the same reasoning we conclude that for each of ℓ_i, $i = 2, \ldots, t-2$, either $\ell_i = \ell_0$ or

- the arc of ℓ_i between v_i and p_i^+ first intersects ℓ_0 after v_1 at least once and then again before v_0 at least once;
- the arc of ℓ_i between v_{i+1} and p_i^- intersects ℓ_0 only after v_1 (at least once).

For the line ℓ_{t-1} there are two cases to consider.

- The pseudolines ℓ_{t-1} and ℓ_1 are distinct. As before, the arc of ℓ_{t-1} between v_{t-1} and p_{t-1}^+ first intersects ℓ_0 after v_1 at least once and then again before v_0 at least once. But then the arc of ℓ_{t-1} after v_0 cannot reach ℓ^- without intersecting ℓ_0 before v_0 thus producing a directed cycle on two pseudolines, or self-intersection. A contradiction.
- The pseudoline ℓ_{t-1} is the same as ℓ_1. Then the order of vertices $v_0 = v_t$ and v_1 is not the same for ℓ_0 and ℓ_1, a contradiction.

□

Lemma 5.2. *Every proper marked quasiline arrangement has a sweep.*

Proof. Let $(\mathcal{A}, x^+)$ be a proper marked quasiline arrangement. By Lemma 5.1, its associated digraph D is acyclic. Therefore there exist a topological sorting $v_1, \ldots, v_r$ of the vertices of D.

We will define a sweep consisting of pseudolines $c_0, c_1, \ldots, c_r$ passing through x, such that in the interior of the region bounded by c_{i-1} and c_i, $i = 1, \ldots, r$, there will be exactly one vertex of the arrangement, namely v_i.

Define c_0 and c_r to be the line at infinity.

Suppose that c_{i-1} has been defined for some $i < r-1$. Let $\ell_{i_1}, \ldots, \ell_{i_t}$ be the pseudolines of the arrangement $\mathcal{A}$ that contain v_i, in the order they intersect c_{i-1}. Take the triangle

T with sides on c_{i-1}, ℓ_{i_1} and ℓ_{i_t} and one of the vertices being v_i. This is well defined, since ℓ_{i_1} and ℓ_{i_t} intersect c_{i-1} only once by definition of c_{i-1}. Only vertices $v_1, \ldots, v_{i-1}$ of the arrangement (and all of them) are on the other side of c_{i-1} as v_i (in the interior of the disk, representing the projective plane), therefore all the lines $\ell_{i_1}, \ldots, \ell_{i_1}$ are directed towards v_i and there are no other vertices of the arrangement in the triangle T besides v_i. Define c_i to be the right boundary of an ϵ-tube around c_{i-1} and T, until it approaches x. If ϵ is small enough, only vertex v_i will be in the interior of the region bounded by c_{i-1} and c_i and c_i will intersect every line of the arrangement only once.

Clearly the pseudolines $c_0, c_1, \ldots, c_r$ obtained in this way define a sweep of the marked arrangement $(\mathcal{A}, x^+)$. □

6 Generalized allowable sequences and wiring diagrams

Wiring diagrams and allowable sequences of permutations are standard tools for describing pseudoline arrangements; see J. E. Goodman [8] and J. E. Goodman and R. Pollack [9]. We need to generalize these two notions in order to be able to describe also monotone quasiline arrangements. We introduced generalized wiring diagrams in Section 4. In this section we generalize also the notion of an allowable sequence and show how generalized wiring diagrams and generalized allowable sequences of permutations are related.

Fix $n \in \mathbb{N}$. A sequence $\Sigma = \pi_0, \ldots, \pi_r$ of permutations is called a *partial allowable sequence of permutations* if it fulfills the following properties:

(1) π_0 is the identity permutation on $\{1, \ldots, n\}$,

(2) each permutation π_i, $1 \leq i \leq r$, is obtained by the reversal of a consecutive substring M_i from the preceding permutation π_{i-1}.

For every i, $1 \leq i \leq r$, we call M_i a *move*. Move M_i represents the transition from permutation π_{i-1} to permutation π_i.

Two partial allowable sequences Σ and Σ' are *elementary equivalent* if Σ can be transformed into Σ' by interchanging two disjoint adjacent moves. Two partial allowable sequences Σ and Σ' are called *equivalent* if there exists a sequence $\Sigma = \Sigma_1, \Sigma_2, \ldots, \Sigma_m = \Sigma'$ of partial allowable sequences such that Σ_i and Σ_{i+1} are elementary equivalent for $1 \leq i < m$.

A partial allowable sequence of permutations of n elements is called an *allowable sequence of permutations* if any two elements $x, y \in \{1, \ldots n\}$ are joint members of exactly one move. The following result is easy to see.

Proposition 6.1. *Let $\Sigma = \pi_0, \ldots, \pi_r$ be an allowable sequence of permutations. Then π_r is the reverse permutation on $\{1, \ldots, n\}$.*

A partial allowable sequence of permutations $\Sigma = \pi_0, \ldots, \pi_r$ is called a *generalized allowable sequence of permutations* if any two elements $x, y \in \{1, \ldots n\}$ are joint members of an odd number of moves. We make the following observation.

Proposition 6.2. *A partial allowable sequence of permutations $\Sigma = \pi_0, \ldots, \pi_r$ is a generalized allowable sequence of permutations if and only if π_r is the reverse permutation on $\{1, \ldots, n\}$.*

To any partial allowable sequence there corresponds a partial wiring diagram:

- start drawing n horizontal lines, numbered with numbers $1, \ldots, n$ from top to bottom, at points $(0, n), \ldots, (0, 1)$,
- to each move M_i there corresponds coordinate $x_i = i$,
- at each coordinate x_i there is a crossing where the lines in the move M_i cross transversally; i.e., the order of lines from M_i is reversed.

Conversely, to any partial wiring diagram there corresponds a partial allowable sequence in the following way. We number the lines of the wiring diagram from top to bottom with numbers $1, \ldots, n$. We start with the identity permutation and after each crossing of the wiring diagram we list the lines from top to bottom.

Obviously, a partial wiring diagram, corresponding to an allowable sequence is also a wiring diagram. A partial wiring diagram, corresponding to a generalized allowable sequence is also a generalized wiring diagram.

Example 6.3. The sequence of permutations

$$1234, 3214, 3241, 3421, 4321$$

is an allowable sequence of permutations which corresponds to the wiring diagram from Figure 2.

Now we can prove Theorem 4.1.

Proof of Theorem 4.1. Let $\mathcal{A}$ be a monotone quasiline arrangement with n pseudolines. Then there exists a proper marked arrangement $(\mathcal{A}, x^+)$ for some x on the line at infinity. We may label the lines of $\mathcal{A}$ by numbers $1, \ldots, n$ in the order in which they are met on the line at infinity starting at x^+. Take a sweep $c_0, c_1, \ldots, c_r$ of $(\mathcal{A}, x^+)$. It determines a sequence of permutations on the set $\{1, \ldots, n\}$: just list the lines in the order in which they are met on each c_i, $i = 0, \ldots, r$, starting at x^+. Since between any two pseudolines c_i and c_{i+1} there is exactly one vertex of the arrangement, and the order of lines that meet in that vertex is reversed when they leave the vertex, this sequence of permutations is a partial allowable sequence of permutations. Moreover, since the order of the lines of $\mathcal{A}$ is reversed at c_r, it is also a generalized allowable sequence of permutations. To a generalized allowable sequence of permutations there corresponds a generalized wiring diagram. This generalized wiring diagram is isomorphic to the original quasiline arrangement by construction. □

7 Polygonal quasiline arrangements without bends

In this section we return to the extended Euclidean plane, where the concept of straight lines and bends is well defined. Two distinct lines that are parallel in the Euclidean plane meet at the line at infinity. This is a *bend at infinity*. The notions of proper marked arrangements and monotone quasiline arrangements are also well defined in the extended Euclidean plane, by mapping homeomorphically a quasiline arrangement to the disk model of the projective plane and back, sending the line at infinity to the line at infinity.

Every pseudoline arrangement is isomorphic to a polygonal pseudoline arrangement with no bends by Proposition 3.1. With polygonal quasiline arrangements we cannot always avoid bends. The reason is that a quasiline arrangement may contain digons. A *digon* in a quasiline arrangement is cell of the associated cell complex that is adjacent to two edges and two vertices. For example, if a quasiline arrangement consists of two polygonal pseudolines, intersecting in three points, it contains three digons. We will show that digons are in fact the only reason for the need of bends in monotone polygonal quasiline arrangements. First we prove a technical lemma.

Lemma 7.1. *Let $(\mathcal{A}, x^+)$ be a proper marked quasiline arrangement that contains no digons. Let G be the undirected plane graph underlying the associated directed graph $D = D(\mathcal{A}, x^+)$ of $(\mathcal{A}, x^+)$. Then the following properties hold.*

(i) *Graph G contains no cycles of length two, i.e., G is simple.*

(ii) *Suppose $\mathcal{A}$ contains at least three crossings. Then G is 2-connected.*

Proof. (i) We will show that if G contains a cycle of length two, then it contains a face of length two which corresponds to a digon in $\mathcal{A}$. Suppose G contains a cycle C of length two with vertices v_1 and v_2. If C is the boundary of a face, we are done. Otherwise at least one pseudoline connects vertices v_1 and v_2 in the interior of C. If there are no vertices of the arrangement in the interior of C, we have at least two faces of length two in G. Otherwise there is a vertex inside C where at least two pseudolines cross. To form a crossing, at least two pseudolines that are consecutive in the cyclic order around v_1 must cross. At least one crossing w will be such that there is no other crossings between v_1 and w on two consecutive lines through v_1. They form a face of length two of G inside C with vertices v_1 and w.

(ii) Graph G is connected, since every two pseudolines of $\mathcal{A}$ cross. Let v be a vertex of G and suppose that $G - v$ is not connected. Then there exist vertices u, w which are in different components of $G - v$. If there existed two lines of $\mathcal{A}$ not incident with v, one of them incident with u and the other one incident with w (or one line not incident with v, and incident with both u and w) we would have a path connecting u and w in $G - v$. Therefore we may assume w.l.o.g. that all lines of $\mathcal{A}$ incident with u are also incident with v (there are at least two such lines, which intersect at least three times, once in u and once in v). Since $\mathcal{A}$ contains no digons, there exists a line ℓ that intersects some lines through u between u and v and is not incident with v by a similar reasoning as in (i). Such a line intersects all lines through w. If there exists a line through w not incident with v, we have a path from u to w in $G - v$, a contradiction.

If also every line incident with w is incident with v, there exists a line ℓ' that intersects some lines through w between v and w and is not incident with v, otherwise we have digons. Lines ℓ and ℓ' either intersect or ℓ is equal to ℓ' and we again have a path from u to w in $G - v$, a contradiction. □

Remark 7.2. Note that the requirement that the order of the intersection points of any two lines is the same for both lines in Lemma 7.1 is necessary, since otherwise there may be a vertex of the arrangement that is incident to all the lines of the arrangement and no two pseudolines that are consecutive around that vertex form a digon; see Figure 6. Note that the graph associated to this quasiline arrangement is not 2-connected.

Theorem 7.3. *Let $\mathcal{A}$ be a monotone quasiline arrangement. Then $\mathcal{A}$ is isomorphic to a monotone polygonal quasiline arrangement with no bends if and only if it contains no digons.*

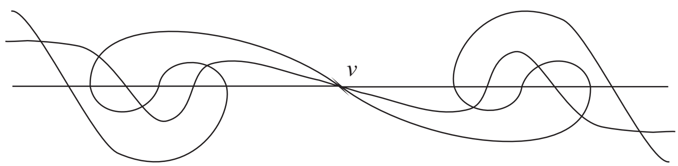


Figure 6: A quasiline arrangement such that the associated graph is not 2-connected.

Proof. If $\mathcal{A}$ contains a digon, there has to be a bend.

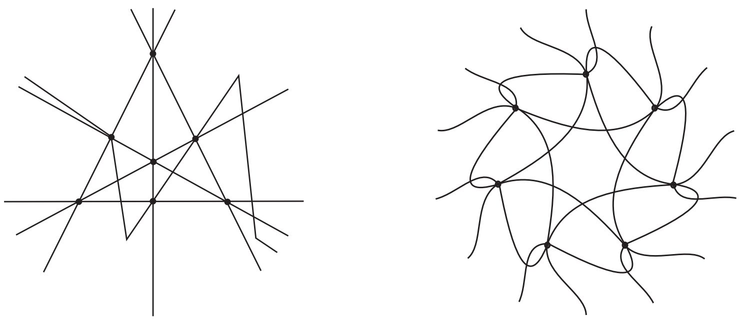
We now prove the converse. Suppose $\mathcal{A}$ contains no digons. If $\mathcal{A}$ contains only one crossing, the claim is clear. Therefore we may assume that $\mathcal{A}$ contains at least three crossings. Since $\mathcal{A}$ is a monotone quasiline arrangement, we may choose a point x^+ such that $(\mathcal{A}, x^+)$ is a proper marked arrangement. Let G be the undirected plane graph underlying its associated directed graph $D = D(\mathcal{A}, x^+)$. Graph G is simple and 2-connected by Lemma 7.1. Therefore we can draw G in the plane in such a way that the vertices on the boundary of the outer face are the vertices of a convex polygon P, all the edges of G are straight lines and the cyclic orders of edges around each vertex is preserved by [6]. To obtain a polygonal quasiline arrangement that is topologically equivalent to $\mathcal{A}$, we draw the part corresponding to G with straight lines as above. To extend it to the whole arrangement we have to add the arcs crossing the line at infinity that were omitted. These arcs connect pairs of boundary vertices of P; to each pseudoline of the arrangement there corresponds a pair of boundary vertices. It cannot happen that some line has only one common vertex with P, since in that case all the pseudolines would share a common vertex (this is not possible for monotone quasiline arrangements without digons). We connect these pairs of boundary vertices with straight lines and omit the parts of them in the interior of P, to obtain the missing arcs. In that way we assure that there are no bends at infinity. Note that any pair of pseudolines can have at most one common vertex of P, since there are no digons in $\mathcal{A}$. Therefore all the lines are distinct. The order in which the pseudolines enter P is the same as the order in which they leave P and this order is therefore reflected in the order of the straight lines that represent them. That also means that no two of these straight lines are parallel and they intersect only in the interior (these parts are omitted) or in the vertices of P since P is convex. The polygonal quasiline arrangement that we obtained therefore has no bends. Since the orders of lines around each crossing is preserved, it is homeomorphic to the original quasiline arrangement by [14, Theorem 3.3.1]. $\square$

8 Quasi-topological incidence structures

In this section the class of quasi-topological configurations based on quasiline arrangements is introduced in a way parallel to topological configurations that are based on pseudoline arrangements. An incidence structure is *quasi-topological* if its points are points in the projective plane, and lines are pseudolines that form a quasiline arrangement. A quasi-topological incidence structure is *monotone* if its underlying quasiline arrangement is monotone. A quasi-topological incidence structure $\mathcal{Q}$ is a *quasi-topological realization* of a combinatorial incidence structure $\mathcal{C}$ if the underlying combinatorial incidence structure of $\mathcal{Q}$ is isomorphic to $\mathcal{C}$.

The notions of combinatorial and topological equivalence for quasi-topological incidence structures are the same as for topological incidence structures. However, since the pseudolines are allowed to cross more than once, we extend the notion of mutation equivalence. We will also consider as admissible mutations the local transformations where one pseudoline moves across another pseudoline such that they form a digon and no other pseudolines are crossed.

Figure 7 (a) shows a monotone quasi-topological realization of the (7_3) combinatorial configuration that cannot be realized as a topological configuration; it is polygonal with three bends. However, the quasi-topological realization of the (7_3) configuration from Figure 7 (b) with a 7-fold rotational symmetry is not a monotone quasi-topological configuration, since in every monotone orientation of the pseudolines, for some of the pseudolines the crossings will be in the opposite order.



(a) (b)

Figure 7: Two different quasi-topological realizations of the (7_3) configuration.

By Theorem 4.1 all monotone quasi-topological incidence structures can be represented with wiring diagrams (where only a subset of the crossings of the wiring diagram are considered as the points of the incidence structure). Obviously, a topological incidence structure is also monotone quasi-topological. The following theorem therefore shows that the class of monotone quasi-topological incidence structures is in a sense the least generalization of the class of topological incidence structures.

Theorem 8.1. *A quasi-topological incidence structure can be represented by a generalized wiring diagram if and only if it is monotone.*

Proof. Let $\mathcal{Q}$ be a quasi-topological incidence structure. Suppose it can be represented by a generalized wiring diagram, i.e., it is topologically equivalent to the monotone quasiline arrangement that corresponds to the generalized wiring diagram. Then it must be monotone itself.

Conversely, let $\mathcal{Q}$ be monotone. Then the underlying quasiline arrangement $\mathcal{A}$ is monotone and it can be represented by a generalized wiring diagram by Theorem 4.1. □

The following result is a consequence of Theorem 7.3.

Theorem 8.2. *Let $\mathcal{Q}$ be a monotone quasi-topological incidence structure. Then $\mathcal{Q}$ is topologically equivalent to a monotone polygonal quasi-topological incidence structure with no bends if and only if the underlying quasiline arrangement contains no digons.*

9 Combinatorial incidence structures as quasiline arrangements

Not every lineal combinatorial incidence structure can be realized as a topological incidence structure. Such examples are the well-known configurations (7_3), the Fano plane, and (8_3), the Möbius-Kantor configuration. In this section we show that every combinatorial incidence structure can be realized as a monotone quasi-topological incidence structure. In view of Theorem 8.1, monotone quasi-topological incidence structures are in a sense the least generalization of topological incidence structures with the property that any combinatorial incidence structure has a realization within this class.

Theorem 9.1. *Every combinatorial incidence structure can be realized as a monotone quasi-topological incidence structure in the projective plane. Moreover, the order of pseudolines that come to each point of the quasi-topological incidence structure may be prescribed.*

Proof. Let $\mathcal{C}$ be a combinatorial incidence structure and let v be the number of points and n be the number of lines of $\mathcal{C}$. Without loss of generality we may number the lines of $\mathcal{C}$ by numbers $1, \dots, n$. To each point of $\mathcal{C}$ we assign a substring M_i of $\{1, \dots, n\}$, which corresponds to the incident lines of that point in the prescribed order; if the order of lines around points is not prescribed, it may be arbitrary. We find a quasi-topological realization of $\mathcal{C}$ in the following way:

- Let $\pi_1, \dots, \pi_v$ be permutations of $\{1, \dots, n\}$ with the property that elements from M_i appear consecutively in π_i. Let π_i' be obtained from π_i by reversing the order of elements from M_i. Let π_0' be the identity permutation and π_{v+1} the reverse permutation on $\{1, \dots, n\}$.
- Form the sequence $\pi_0', \pi_1, \pi_1', \dots, \pi_v, \pi_v', \pi_{v+1}$; if $\pi_i' = \pi_{i+1}$, take just one of them. If $\pi_i' \neq \pi_{i+1}$, insert a sequence of permutations between them to obtain a generalized allowable sequence, for $i = 0, \dots v$. This can always be done, in a bubble sort like manner by interchanging two adjacent numbers in every step.
- A generalized allowable sequence corresponds to a generalized wiring diagram which in turn corresponds to a monotone quasi-topological incidence structure. The points of the incidence structure correspond to the crossings M_i, $i = 1, \dots, v$.

□

The proof of Theorem 9.1 in fact provides us with an algorithm to construct an actual quasi-topological representation of a given combinatorial incidence structure.

We now consider the number of crossings in a quasiline arrangement. Given a quasiline arrangement $\mathcal{A}$, let p be a crossing of $\mathcal{A}$. We define the *local crossing number* of p to be $\binom{k}{2}$ if k lines cross at p. Observe that the crossing number of a regular crossing is 1. The *crossing number* of the quasiline arrangement $\mathcal{A}$ is the sum over all crossing numbers of its vertices. Given a quasi-topological incidence structure, every pair of pseudolines has at least one point in common. If such a point is not a point of the incidence structure, we call it an *unwanted crossing*.

Proposition 9.2. *Given a topological (n_k) configuration, where all the unwanted crossings are regular, the number of unwanted crossings is $\binom{n}{2} - n\binom{k}{2}$.*

The number of unwanted crossings in a quasi-topological incidence structure, obtained as in the proof of Theorem 9.1, can be high. However, if the incidence structure has v points and n lines, the number of crossings is at most $(v + 1)\, n^2$. Observe that the total number of crossings also gives an upper bound for the maximal number of crossings between two pseudolines. The following two problems are therefore natural to consider.

Problem 9.3. For a given combinatorial incidence structure determine the minimal number of crossings to realize it as a monotone quasi-topological incidence structure in the projective plane.

Problem 9.4. For a given combinatorial incidence structure determine the smallest number c such that it is realizable as a monotone quasi-topological incidence structure in the projective plane with the maximal number of crossings between every pair of pseudolines smaller than c.

A quasi-topological configuration obtained from the proof of Theorem 9.1 can have many digons. Is it possible to avoid digons? Is it any easier if we only consider lineal incidence structures?

Problem 9.5. Is every lineal combinatorial incidence structure realizable as a polygonal quasi-topological incidence structure in the projective plane with no bends?

Problem 9.6. Which combinatorial incidence structures are realizable as polygonal quasi-topological incidence structures in the projective plane with no bends?

If all the unwanted crossings are straight and there are no bends, a polygonal quasi-topological incidence structure is determined by the coordinates of the original vertices (and the directions at which the pseudolines approach infinity). Therefore also the following two problems are of interest.

Problem 9.7. Is every lineal combinatorial incidence structure realizable as a polygonal quasi-topological incidence structure in the projective plane with no bends and all the unwanted crossings straight?

Problem 9.8. Is every topological incidence structure topologically equivalent to a polygonal topological incidence structure in the projective plane with no bends and all the unwanted crossings straight?

10 Quasi-topological incidence structures as curves on surfaces

A *curve arrangement* is a collection of simple closed curves on a given surface such that every pair of curves has at most one point in common at which they cross transversely. In addition the arrangement should be cellular, i.e., the complement of the curves is a union of open discs; see J. Bokowski and T. Pisanski [3].

In this section we show that to any quasi-topological incidence structure we can associate a map M on a closed surface S that can be used to distinguish between mutation classes of quasi-topological configurations. Such a map defines an arrangement of curves on the surface S; i.e., every quasi-topological incidence structure can be viewed as a curve arrangement on some surface. This is a generalization of the work of J. Bokowski and R. Strausz [4], where only topological configurations were considered. For the background on graphs and maps we refer the reader to B. Mohar and C. Thomassen [14] or J. L. Gross and T. W. Tucker [10].

Let $\mathcal{Q}$ be a quasi-topological incidence structure. We define a graph $G = (V, E)$ corresponding to $\mathcal{Q}$ in the following way. The vertices of V are the points of $\mathcal{Q}$. Two vertices are connected by an edge if the corresponding points of $\mathcal{Q}$ are consecutive on some pseudoline (the intersection points of pseudolines that are not points of the incidence structure are ignored). Note that G is Eulerian since every pseudoline contributes two edges through a vertex. For each vertex v we choose a cyclic order π_v of edges around v. The cyclic orders of all the vertices form a *rotation system* $\pi = \{\pi_v; v \in V\}$. Now define a signature mapping $\lambda : E \to \{1, -1\}$ in the following way. For each edge $e = uv$ we check if the local orientations at u and v, determined by the cyclic order of edges around them, agree if we move from u to v along e. If they agree we set $\lambda(e) = 1$, otherwise $\lambda(e) = -1$. The pair $\Pi = (\pi, \lambda)$ is an *embedding scheme* of G. This defines a map, we denote it by $M = M(\mathcal{Q})$, on some surface S. This map is uniquely determined, up to homeomorphism; see B. Mohar and C. Thomassen [14, Theorem 3.3.1]. Note that surface S is non-orientable. This can be seen as follows. Take a cycle c of the map M that corresponds to a pseudoline. Since this is an orientation-reversing curve, starting at a vertex v and traveling along c, after returning to v the orientation at v is reversed. That means that there must be an odd number of edges on c with negative signature. Consequently the embedding of M is non-orientable by [14, Lemma 4.1.4].

A *straight-ahead walk* or a SAW in an Eulerian map is a walk that always passes from an edge to the opposite edge in the rotation at the same vertex; see T. Pisanski et al. [16]. Since the pseudolines of $\mathcal{Q}$ cross transversally at each crossing of $\mathcal{Q}$, every pseudoline corresponds to a straight-ahead walk in the map $M(\mathcal{Q})$. We have shown the following.

Theorem 10.1. *For any quasi-topological incidence structure $\mathcal{Q}$ with the set of points $\mathcal{P}$ and the set of lines $\mathcal{L}$ there exists an Eulerian map $M = M(Q)$ on a closed surface, with skeleton $G = (V, E)$ such that $\mathcal{P} = V$, and each SAW is a simple closed curve that corresponds to a line from $\mathcal{L}$.*

The maps corresponding to quasi-topological incidence structures can be used to distinguish between quasi-topological incidence structures that are not mutation equivalent.

Theorem 10.2. *If two quasi-topological incidence structures $\mathcal{Q}_1$ and $\mathcal{Q}_2$ are mutation equivalent, then $M(\mathcal{Q}_1) = M(\mathcal{Q}_2)$.*

Proof. Admissible mutations do not change the cyclic order of the pseudolines around any crossing that is a point of the incidence structure. On the other hand, the cyclic orders of the pseudolines around every vertex defines the rotation systems for maps $M(\mathcal{Q}_1)$ and $M(\mathcal{Q}_2)$. Moreover, if we choose local rotations consistently, also the embedding schemes are equal. Two maps with equal embedding schemes are considered to be the same. □

Example 10.3. Consider the quasi-topological configurations from Figure 7. The corresponding maps have 7 vertices and 21 edges. The map corresponding to the quasi-topological configuration on the left has six faces of length three and three faces of length 8, so it has Euler characteristic -5 and thus it has nonorientable genus equal to 7. The map corresponding to the quasi-topological configuration on the right has seven faces of length five and one face of lenght seven, so it has Euler characteristic -6 and thus it has nonorientable genus equal to 8. Therefore the two quasi-topological configurations are not mutation equivalent.

For a given quasi-topological incidence structure $\mathcal{Q}$ the map $M(Q)$ can be viewed as a curve arrangement on some nonorientable surface. Since every combinatorial incidence structure can be realized as a quasi-topological incidence structure, we can define the *(non)orientable genus of the incidence structure* $\mathcal{C}$ as the smallest $g = g(\mathcal{C})$ for which there exists a (non)orientable surface of genus g on which $\mathcal{C}$ can be represented with a curve arrangement.

Problem 10.4. For a given combinatorial incidence structure determine its (non) orientable genus.

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References

[1] J. Bokowski, B. Grünbaum and L. Schewe, Topological configurations (n_4) exist for all $n \geq 17$, *European J. Combin.* **30** (2009), 1778–1785, doi:10.1016/j.ejc.2008.12.008.

[2] J. Bokowski and V. Pilaud, Enumerating topological (n_k)-configurations, *Comput. Geom.* **47** (2014), 175–186, doi:10.1016/j.comgeo.2012.10.002.

[3] J. Bokowski and T. Pisanski, Oriented matroids and complete-graph embeddings on surfaces, *J. Comb. Theory Ser. A* **114** (2007), 1–19, doi:10.1016/j.jcta.2006.06.012.

[4] J. Bokowski and R. Strausz, A manifold associated to a topological (n_k) configuration, *Ars Math. Contemp.* **7** (2014), 479–485, `https://amc-journal.eu/index.php/amc/article/view/241`.

[5] J. Bokowski and B. Sturmfels, *Computational Synthetic Geometry*, volume 1355 of *Lecture Notes in Mathematics*, Springer-Verlag, Berlin, 1989, doi:10.1007/bfb0089253.

[6] N. Chiba, K. Onoguchi and T. Nishizeki, Drawing plane graphs nicely, *Acta Inform.* **22** (1985), 187–201, doi:10.1007/bf00264230.

[7] S. Felsner and H. Weil, Sweeps, arrangements and signotopes, *Discrete Appl. Math.* **109** (2001), 67–94, doi:10.1016/s0166-218x(00)00232-8.

[8] J. E. Goodman, Pseudoline arrangements, in: J. E. Goodman and J. O'Rourke (eds.), *Handbook of Discrete and Computational Geometry*, CRC Press, Boca Raton, Florida, Discrete Mathematics and its Applications, pp. 97–128, 2nd edition, 2004, doi:10.1201/9781420035315.

[9] J. E. Goodman and R. Pollack, Semispaces of configurations, cell complexes of arrangements, *J. Comb. Theory Ser. A* **37** (1984), 257–293, doi:10.1016/0097-3165(84)90050-5.

[10] J. L. Gross and T. W. Tucker, *Topological Graph Theory*, Wiley-Interscience Series in Discrete Mathematics and Optimization, John Wiley & Sons, New York, 1987.

[11] B. Grünbaum, *Arrangements and Spreads*, Regional Conference Series in Mathematics, American Mathematical Society, Providence, Rhode Island, 1972.

[12] B. Grünbaum, *Configurations of Points and Lines*, volume 103 of *Graduate Studies in Mathematics*, American Mathematical Society, Providence, Rhode Island, 2009, doi:10.1090/gsm/103.

[13] F. W. Levi, *Geometrische Konfigurationen*, S. Hirzel Verlag, Leipzig, 1929.

[14] B. Mohar and C. Thomassen, *Graphs on Surfaces*, Johns Hopkins Studies in the Mathematical Sciences, Johns Hopkins University Press, Baltimore, Maryland, 2001.

[15] T. Pisanski and B. Servatius, *Configurations from a Graphical Viewpoint*, Birkhäuser Advanced Texts, Birkhäuser, New York, 2013, doi:10.1007/978-0-8176-8364-1.

[16] T. Pisanski, T. W. Tucker and A. Žitnik, Straight-ahead walks in Eulerian graphs, *Discrete Math.* **281** (2004), 237–246, doi:10.1016/j.disc.2003.09.011.

[17] H. Schröter, Ueber die Bildungsweise und geometrische Konstruction der Konfigurationen 10_3, *Nachr. Königl. Ges. Wiss. Göttingen* (1889), 193–236.

[18] E. Steinitz, *Über die Konstruktion der Konfigurationen* n_3, Ph.D. thesis, Universität Breslau, 1894.

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Axiomatic characterization of transit functions of hierarchies

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Abstract

Transit functions provide a unifying approach to many results on intervals, convexities, and betweenness. Here we show that hierarchical structures arising in cluster analysis and phylogenetics have a natural characterization in terms of transit functions and that hierarchies are identified by multiple combinations of independent axioms.

Keywords: Transit functions, convexities, hierarchies, rooted trees, axiom systems.

Math. Subj. Class.: 05C05, 05C99, 52A01

1 Introduction

Rooted trees are commonly used in phylogeny to represent the evolutionary history of a collection of organisms. Internal vertices in these trees are interpreted as the last common ancestors for their extant descendants, which are represented by the leaves of the tree. Phylogenetic trees not only depict an evolutionary history, they also express a hierarchical classification of the organisms by identifying subtrees with taxonomic groups such as animals, plants, or insects. The tree structure of evolution thus implies a hierarchical classification of life forms. A wide variety of methods has become available in computational

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biology to infer phylogenetic trees from diverse sources of data [12], most prominently among them DNA sequences. Hierarchical clustering produces an analogous tree structure without the need to pre-suppose an evolutionary process in time that gives rise to the clusters. Instead, these intrinsic (dis)similarities of the objects under considerations are used to obtain a hierarchical classification [1]. Clustering is a topic of key importance in data mining and knowledge discovery.

Throughout this contribution, V is a finite, non-empty set. We use the notation $\subset$ to mean proper subset, and write $\subseteq$ otherwise.

Definition 1.1. A *hierarchy* on V is a set system $\mathcal{C} \in 2^V$ so that

(H1) $V \in \mathcal{C}$,

(H2) $\{x\} \in \mathcal{C}$ for all $x \in V$.

(H3) $A, B \in \mathcal{C}$ implies that either $A \cap B = \emptyset$, $A = B$, $A \subset B$, or $B \subset A$.

Hierarchies have a representation as a rooted tree T with leaf set V. Denote by $V \cup W$ the complete vertex set of T, i.e., W is the set of internal vertices of T including the root denoted by r. For simplicity we disregard the trivial case of T consisting of a single vertex. Furthermore, we require that every internal vertex has at least two children. This type of tree is sometimes called *homeomorphically irreducible*. Hence we assume from $|V| \geq 2$. There is a natural partial order $\preceq$ on $V \cup W$ so that $x \preceq y$ if and only if y lies on the path from x to the root r. The leaves are minimal elements with respect to $\preceq$, the root is the unique maximal element. For any subset $A \subseteq V \cup W$ of vertices in T, the intersection of the paths from $v \in A$ to the root is again a path. The minimal vertex in this path is the *last common ancestor* of A, denoted by $\operatorname{lca}(A)$. For each vertex $x \in V \cup W$, the subtree $T[x]$ below x consists of all vertices $y \preceq x$, i.e , all those for which the path to the root passes through x. For its leaf set we write $V[x] = V[T[x]] = \{y \in V | y \preceq x\}$. Clearly we have $\operatorname{lca}(V[x]) = x$ for all $x \in V \cup W$, and $V[x] = \{x\}$ if and only if $x \in V$. Moreover, $V[r] = V$.

There is a well-known bijection between hierarchies and homeomorphically irreducible rooted trees. Given $\mathcal{C}$, the tree $T_{\mathcal{C}}$ represents the elements of $\mathcal{C}$ as vertices so that V becomes the root, the singleton sets $\{x\}$, $x \in V$ become the leaves, and each internal vertex u of $T_{\mathcal{C}}$ corresponds to a non-trivial cluster which then can be represented as $V[T[u]] \in \mathcal{C}$. There is an edge between u and u' if and only if $V[T[u]] \neq V[T[u']]$ and there is no u'' so that $V[T[u]] \subset V[T[u'']] \subset V[T[u']]$ or $V[T[u']] \subset V[T[u'']] \subset V[T[u]]$. The tree $T_{\mathcal{C}}$ is therefore (isomorphic to) the Hasse diagram of $\mathcal{C}$ w.r.t. set inclusion. Conversely, consider an internal vertex $u \in W$. The leaf $u_1, \ldots, u_k$ of u, define a partition of $V[T[u]]$, since the leaf sets of the subtrees are disjoint, non-empty, and satisfy $V[T[u]] = \bigcup_i V[T[u_i]]$. Thus T defines a unique hierarchy $T_{\mathcal{C}}$.

With a hierarchy $\mathcal{C}$ we can naturally associate the function $R_{\mathcal{C}} : V \times V \to 2^V$, defined by

$$R_{\mathcal{C}}(x, y) = \bigcap \{A \in \mathcal{C} | x, y \in A\} \tag{1.1}$$

as the intersection of all clusters that simultaneously contain both x and y. It follows immediately from property (H3) that $R_{\mathcal{C}}(x, y)$ is itself a cluster of $\mathcal{C}$, i.e., $R_{\mathcal{C}}(x, y) \in \mathcal{C}$ for all $x, y \in V$. With a homeomorphically irreducible rooted tree T we can, in an equally natural way, associate a function $R_T : V \times V \to 2^V$ on the leaves of T by setting

$$R_T(x, y) = V[\operatorname{lca}(\{x, y\})] \tag{1.2}$$

By construction $R_T(x,y)$ consists of all the leaves of the subtree "spanned" by x and y. The bijection of trees and cluster systems, and the fact that $V[\mathrm{lca}(\{x,y\})] \in \mathcal{C}_T$ furthermore, implies that $R_T = R_{\mathcal{C}_T}$ and, conversely, $R_{\mathcal{C}} = R_{T_{\mathcal{C}}}$.

Function $R : V \times V \to 2^V$ have been studied extensively under the name *transit functions* [9]. In graph theory, they arise in a broad range of contexts involving betweenness, intervals, and convexity [2, 3, 4, 6, 8, 5, 7, 9, 10, 11]. A topic of particular interest in this field is the characterization of graph classes in terms of first order properties of their transit functions and the axiomatic characterization of a well-known transit function in terms of (transit) axioms. Therefore we ask in this contribution whether there is a set of axioms that uniquely determines the transit functions R_T of rooted trees.

Formally, a *transit function* on a non-empty set V is a function $R : V \times V \to 2^V$ satisfying the three axioms

$(t1)$ $u \in R(u,v)$ for all $u,v \in V$.

$(t2)$ $R(u,v) = R(v,u)$ for all $u,v \in V$.

$(t3)$ $R(u,u) = \{u\}$ for all $u \in V$.

Basically, a transit function describe how one can move from an element u to an element v through elements in $R(u,v)$.

A *convexity* $\mathcal{K} \subset 2^V$ is closed under intersections and both $V \in \mathcal{K}$ and $\emptyset \in \mathcal{K}$. In the context of transit function, we are primarily interested in convexities that in addition satisfy $\{x\} \in \mathcal{K}$ for all $x \in V$. A hierarchy is, therefore, a particular kind of convexity (to be precise, we have to add the empty set to the hierarchy $\mathcal{C}$ as defined by (H1), (H2), and (H3) to guarantee closure under intersections). The converse is not true, however, as exemplified by $\mathcal{K} = \{\{a,b\},\{b,c\},\{b\},\emptyset\}$.

The *canonical transit function* of a convexity $\mathcal{K}$ is given by the convex hulls spanned by x and y, i.e., it is defined by equ.(1.1) [9]. For every transit function R on V, there is an associated convexity, where a subset W of V is R-convex if $R(u,v) \subseteq W$, for every $u,v \in W$. The family of R_T-convex sets coincide with the hierarchic convexity on V.

The following non-trivial betweenness axioms were considered already by Mulder [9]:

$(b1)$ $x \in R(u,v), x \neq v \Rightarrow v \notin R(u,x)$;

$(b2)$ $x \in R(u,v) \Rightarrow R(u,x) \subseteq R(u,v)$;

(m) $x,y \in R(u,v) \Rightarrow R(x,y) \subseteq R(u,v)$.

A transit function R is called *monotone* if it satisfies axiom (m). It is satisfied by construction for the canonical transit function of a convexity, see e.g. [4, 9], and thus in particular by transit function of a hierarchy.

2 Transit functions of hierarchies

In the context of transit functions of hierarchies (or rooted trees) we encounter several additional betweenness properties:

(a) For all $u \in V$, there is $\overline{u}$ so that $R(u,\overline{u}) = V$.

(h) $R(u,v) \cap R(x,y) \neq \emptyset \Rightarrow R(u,v) \subseteq R(x,y)$ or $R(x,y) \subseteq R(u,v)$.

(h') $x \in R(u,v) \Rightarrow R(u,x) = R(u,v)$ or $R(x,v) = R(u,v)$.

(h'') $x \in R(u,v) \Rightarrow R(u,v) = R(u,x) \cup R(x,v)$.

(h''') $x \notin R(u,v) \Rightarrow R(u,x) = R(x,v)$.

(m') $R(u,v) \cap R(x,y) \neq \emptyset \Rightarrow \exists p$ and q such that $R(u,v) \cap R(x,y) = R(p,q)$

We say that a transit function is *hierarchical* if it satisfies axiom (h).

Now consider a hierarchy $\mathcal{C}$ and its transit function $R = R_{\mathcal{C}}$ defined in equ.(1.1). We easily check that (m) is satisfied also without recourse to the theory of convexities: Since $R(u,v) \in \mathcal{C}$ it follows immediately that for all $x, y \in R(u,v)$ the smallest element of $\mathcal{C}$ that contains both x and y must be a subset of $R(u,v)$. Thus $R(x,y) \subseteq R(u,v)$. Hence R satisfies the axioms (m) and thus also $(b2)$.

On the other hand, $R_{\mathcal{C}}$ does not satisfy $(b1)$, i.e., it is not a betweenness in the sense of Mulder [9]:

Example 2.1. Set $V = \{a,b,c,d\}$, $R(a,b) = R(a,c) = R(b,c) = V - \{d\}$, and $R(a,d) = R(b,d) = R(c,d) = V$. It corresponds to the hierarchy $\{\{\{a\},\{b\},\{c\}\},\{d\}\}$. Since $b \in R(a,c)$ and $c \in R(a,b)$, $(b1)$ does not hold.

Axiom (h) simply states that the transit sets form a hierarchy, which is obviously true by virtue of the defining equ.(1.1) for the transit function of any hierarchy. Axiom (m') states that the system of transit sets is closed under intersections. Property (m') is, in fact, an immediate consequence of (h) since either $R(u,v) \cap R(x,y) \neq \emptyset$ in this case implies $R(p,q) = R(u,v)$ or $R(p,q) = R(x,y)$. The canonical transit function of a convexity, and thus $R_{\mathcal{C}}$, therefore satisfies (m').

Theorem 2.2. *The collection of transit sets of a transit function R is a hierarchy, i.e., a convexity satisfying (H3), if and only if R satisfies (m), (m'), and there is a pair $p, q \in V$ so that $R(p,q) = V$.*

Proof. By construction, the canonical transit function of a convexity fulfills the three conditions. Conversely, it is well known that the transit sets are convex (w.r.t. to R) if and only if R satisfies (m). Condition (m') is necessary and sufficient to ensure that the intersection of transit sets is closed under intersection, and the existence of a pair $p, q \in V$ so that $R(p,q) = V$ is equivalent to $V \in \mathcal{K}$. $\square$

Axiom (h') is motivated by the following simple observation for rooted trees: Let u and v be two leaves of a rooted tree T. Consider an arbitrary leaf $x \in V[T[\mathrm{lca}(u,v)]]$. Then one of the three statements is true: (i) x is contained in the same subtree rooted by a child of $\mathrm{lca}(u,v)$ as v, (ii) x is contained in the same subtree rooted by a child of $\mathrm{lca}(u,v)$ as u, (iii) x is located in a subtree that is rooted at a child of $\mathrm{lca}(u,v)$ that contains neither u nor v. In the first case $\mathrm{lca}(x,v) = \mathrm{lca}(u,v)$, in the second case $\mathrm{lca}(x,u) = \mathrm{lca}(u,v)$, and in the third case $\mathrm{lca}(u,x) = \mathrm{lca}(v,x) = \mathrm{lca}(u,v)$. Thus, $V[T[\mathrm{lca}(u,v)]]$ coincides with $V[T[\mathrm{lca}(u,x)]]$ or $V[T[\mathrm{lca}(u,y)]]$.

In the following, we derive the analogous result for hierarchies without making use of the corresponding rooted tree.

Lemma 2.3. *Let $\mathcal{C}$ be a hierarchy of a finite set V. The associated transit function $R = R_{\mathcal{C}}$ defined in equ.(1.1) satisfies (h').*

Proof. Let $u, v, x \in V$ and $x \in R(u,v)$. The statement of (h') holds trivially if $x = u$ or $x = v$. If $u = v$ then $R(u,v) = \{u\}$ by $(t3)$ and thus $x = u$.

Thus, we assume that u, v and x are pairwise distinct. Define $W[u|v]$ as the largest cluster i.e., $W[u|v] \in \mathcal{C}$, so that $u \in W[u|v]$ and $v \notin W[u|v]$. By (H1) and (H3) we have $\{u\} \subseteq W[u|v] \subset R(u,v)$, hence $W[u|v]$ is well defined.

Suppose $q \in W[u|v]$. Then $W[q|v] \cap W[u|v] \neq \emptyset$ and hence $W[q|v] \subseteq W[u|v]$. By construction, $W[q|v]$ is the largest cluster in $\mathcal{C}$ that contains q but not v. Since $W[u|v]$ also has this property, we have $W[u|v] \subseteq W[q|v]$, i.e., $W[q|v] = W[u|v]$ for all $q \in W[u|v]$.

Let $p \in W[u|v]$ and $q \in W[v|u]$. Since $W[u|v] = W[p|v]$, $W[v|u] = W[q|u]$, and $R(p,q)$ are clusters of $\mathcal{C}$, $W[p|v] \cap R(p,q) \neq \emptyset$ and $W[q|u] \cap R(p,q) \neq \emptyset$, (H3) implies $W[p|v] \subset R(p,q)$ and $W[q|u] \subset R(p,q)$. Since R satisfies (m) we have $R(p,q) \subseteq R(u,v)$ since $p,q \in R(u,v)$. On the other hand, $u,v \in R(p,q)$ and $R(u,v)$ is by construction the smallest cluster that contains both u and v so that $R(u,v) \subseteq R(p,q)$. Thus $R(p,q) = R(u,v)$.

In particular, if $x \in W[u|v]$ then $R(x,v) = R(u,v)$.

Now suppose $x \in R(u,v) \setminus (W[u|v] \cup W[v|u])$. Using (H3) again we conclude $W[u|v] \cup \{x\} \subseteq R(q,x)$ for all $q \in W[u|v]$ and $R(q,x) \subseteq R(u,v)$.

By maximality of $W[u|v]$, $W[u|v] \cup \{x\}$ is not contained in any cluster of $\mathcal{C}$ that is a (proper) subset of $R(u,v)$ and does not contain v. Hence the smallest cluster containing $W[u|v] \cup \{x\}$, i.e., $R(u,x)$ also contains v and hence $R(u,v)$. By (m), $R(u,x) \subseteq R(u,v)$. Thus $R(u,x) = R(v,x) = R(u,v)$. Thus (h') holds. □

The hierarchy axiom (H1) implies immediately that there is a pair of points p,q so that $R(p,q) = V$. For any $x \in V$ we have $R(x,p) = V$ or $R(x,q) = V$, thus the transit function of a hierarchy satisfies axiom (a).

We shall see below that the transit functions of hierarchies also satisfy (h'') and (h'''). Instead of giving direct proofs, we will draw these conclusions directly from general implications between the above properties of transit functions.

3 Characterization

Theorem 3.1. *Suppose R satisfies (a), (h'), (h), and (m). Then R is the transit function of a hierarchy $\mathcal{C}$ as defined in equ.(1.1).*

Proof. It follows immediately from (h), (a), and $(t3)$ that $\mathcal{C}_R = \{R(p,q)|p,q \in V\}$ is a hierarchy. The corresponding transit function $Q := R_{\mathcal{C}_R}$ can be represented as

$$Q(u,v) = \bigcap \{R(p,q),\, p,q \in V | u,v \in R(p,q)\} \tag{3.1}$$

Thus R is the transit function of the hierarchy $\mathcal{C}$ if and only if $R = Q$. By $(t1)$ and (m) we have $Q(u,v) \subseteq R(p,q)$ for all p,q so that $u,v \in R(p,q)$. In particular, therefore $Q(u,v) \subseteq R(u,v)$. The smallest transit set $R(p,q)$ that contains both u and v (which is well-defined by virtue of the hierarchy axioms) there is contained in $R(u,v)$.

We now construct a partition $\mathbf{A}$ of $R(u,v)$ as follows: (i) $A_u := \{x \in R(u,v)| R(x,u) \neq R(u,v)\}$ and $A_v := \{x \in R(u,v)|R(x,v) \neq R(u,v)\}$. By (h') we have $R(x,v) = R(u,v)$ for $x \in A_u$ and $R(x,u) = R(u,v)$ for $x \in A_v$. (ii) The remainder $R' = R(u,v) \setminus (A_u \cup A_v)$ consists, by axiom (h') of all $x \in R(u,v)$ that satisfy $R(u,x) = R(v,x) = R(u,v)$. If R' is nonempty, choose an arbitrary $q \in R'$ and define $A_q := \{x \in R(u,q)|R(q,x) \subset R(u,x)\}$ and note that, by (h'), $R(q,u) = R(u,x) = R(u,v)$. Since $R(u,x) = R(v,x) = R(v,q)$ we $A_q = \{x \in R(v,q)|R(q,x) \subset R(v,x)\}$

as alternative representation. (iii) Now replace $R' \leftarrow R \setminus A_q$, pick a point q' in R' and construct $A_{q'} = \{x \in R(u,q') | R(q',x) \subset R(u,x)\}$. Again, we have $R(u,q') = R(u,q) = R(u,v)$, and we can replace u by v or q in the definition of $A_{q'}$. We repeat this construction until R' is empty and arrive at the $\mathbf{A} := \{A_u, A_v, A_q, \dots\}$ of $R(u,v)$.

By construction, $\mathbf{A}$ has the property that $R(p,q) \subsetneq R(u,v)$ if and only if $p,q \in A_z$ for some $A_z \in \mathbf{A}$ and $R(x,y) = R(u,v)$ if and only if x and y are contained in two different classes of $\mathbf{A}$.

Suppose $p,q \in A_z$ and there is $w \in R(p,q)$ so that $w \notin A_z$. By (m) $R(p,w) \subseteq R(p,q)$ and by construction of A_z we have $R(p,q) \subset R(u,v)$. On the other hand, $w \notin A_z$ implies, for all $p \in A_z$, that $R(p,w) = R(u,v)$, leading to the contradiction $R(p,q) \subset R(p,w)$. Thus $p,q \in A_z$ implies $R(p,q) \subseteq A_z$.

If $A_z \neq R(p,q)$, then there is $s \in A_z \setminus R(p,q)$. We have $R(p,s) \cap R(p,q) \neq \emptyset$, thus by (h) $R(p,q) \subset R(p,s)$. Repeating the argument we eventually arrive at a $\bar{p} \in A_z$ so that $R(p,\bar{p}) = A_z$, i.e., $A_z \in \mathcal{C}$.

Since u and v are contained in two different classes of $\mathbf{A}$, and $R(p,q) \subseteq A_z$ whenever $p,q \in A_z$ we can reason as follows: If $p,q \in R(u,v)$ and $u,v \in R(p,q)$ then p and q are contained in different classes of $\mathbf{A}$ and therefore $R(p,q) = R(u,v)$. The smallest transit set that contains u,v therefore is $R(u,v)$. It follows that $R(u,v) = Q(u,v)$. □

As a by-product of the proof, we have obtained, for each transit set $R(u,v)$, its partition into maximal proper subsets $R(p,q)$. This amounts to an explicit construction of the Hasse diagram of $\mathcal{C}$ directly from the transit function.

4 Implications among axioms

The characterization in Theorem 3.1 can be simplified further by observing that the conditions (a), (h'), (h), and (m) are not independent. First, find that we can simply drop (a) by virtue of $(h) \Rightarrow (a)$.

Lemma 4.1. *Let R be a transit function on V satisfying axiom (h). Then R satisfies axiom (a).*

Proof. For $u,v \in V$ consider a $v_i \in V$ so that $R(u,v) \cap R(u,v_i) \neq \emptyset$. Since (h) holds, we have $R(u,v) \subseteq R(u,v_i)$ or $R(u,v_i) \subseteq R(u,v)$. W.l.o.g. assume $R(u,v) \subseteq R(u,v_i)$. If $R(u,v_i) = V$ there is nothing to show. Otherwise, choose $v_j \in V$ such that $v_j \notin R(u,v_i)$. Again, we have $R(u,v_i) \cap R(u,v_j) \neq \emptyset$ and thus, by (h), $R(u,v_i) \subseteq R(u,v_j)$ or $R(u,v_j) \subseteq R(u,v_i)$. Since $v_j \notin R(u,v_i)$ we can rule out $R(u,v_j) \subseteq R(u,v_i)$ and conclude that $R(u,v_i) \subseteq R(u,v_j)$. Repeating this argument we obtain a chain $R(u,v) \subseteq R(u,v_i) \subseteq R(u,v_j) \subseteq \dots \subseteq R(u,v_k)$ of sets that, in each iteration increase by at least one element. Thus $R(u,v_k) = V$ for some k not larger than $|V|$, and v_k is desired element $\overline{u}$ so that $R(u,\overline{u}) = V$. □

Several, but not all pairs of the axioms (m), (m'), (h), (h'), (h''') are equivalent and characterize the transit functions of hierarchies:

Lemma 4.2. *Let R be a transit function satisfying axioms (h) and (m) on V. Then R satisfies axiom (h').*

Proof. Suppose (h) and (m) holds. For all $u,v \in V$ and all $x \in R(u,v)$ we have $R(u,x) \subseteq R(u,v)$ and $R(x,v) \subseteq R(u,v)$ because of (m). By (h) we have $R(u,x) \subseteq$

$R(x, v)$ or $R(x, v) \subseteq R(u, x)$, $R(u, x) \subseteq R(u, v)$ or $R(u, v) \subseteq R(u, x)$, and $R(x, v) \subseteq R(u, v)$ or $R(u, v) \subseteq R(x, v)$. If $R(u, x) \subseteq R(x, v)$, thus $\{x, v, u\} \in R(x, v)$ and by (m) $R(u, v) \subseteq R(x, v)$. If $R(x, v) \subseteq R(u, x)$, we have $\{x, v, u\} \subseteq R(u, x)$ and thus $R(u, v) \subseteq R(u, x)$. If (h') does not hold, there are $u, v, x \in V$ can be chosen so that neither $R(u, v) \subseteq R(u, x)$ nor $R(u, v) \subseteq R(x, v)$, a contradiction. □

Lemma 4.3. *Let R be a transit function satisfying axioms (h) and (h') on V. Then R satisfies axiom (m).*

Proof. Let $u, v \in V$ and $x, y \in R(u, v)$. From (h) we obtain that $R(u, v) \subseteq R(x, y)$ or $R(x, y) \subseteq R(u, v)$. In second case we are, done. Thus suppose $R(x, y) \nsubseteq R(u, v)$, i.e., $R(u, v) \subset R(x, y)$, and in particular $u, v \in R(x, y)$. From (h') we known that at least one of $R(u, x)$ and $R(v, x)$, and at least one of $R(u, y)$ and $R(v, y)$ equals $R(u, v)$. Since $u, v \in R(x, y)$, (h') implies $R(u, x) = R(x, y)$ or $R(u, y) = R(x, y)$ as well as $R(x, v) = R(x, y)$ or $R(v, y) = R(x, y)$. In total, this leaves 16 cases, of which all but two directly lead to the contradiction $R(u, v) = R(x, y)$. In case (i), $R(u, x) = R(v, y) = R(u, v)$ and $R(u, y) = R(v, x) = R(x, y)$. Here, $y \in R(u, x)$ and (h') implies $R(u, y) = R(u, x)$ or $R(y, x) = R(u, x)$, which further yields the contradiction $R(x, y) = R(u, y) = R(u, x) = R(u, v)$. In case (ii) $R(u, x) = R(v, y) = R(x, y)$ and $R(u, y) = R(v, x) = R(u, v)$. Here, $x \in R(u, y)$ and (h') implies $R(u, x) = R(u, v)$ or $R(x, y) = R(u, v)$, which leads to an analogous contradiction. □

Lemma 4.4. *Let R be a transit function satisfying axioms (h) and $(b2)$ on V. Then R satisfies axiom (m).*

Proof. Suppose $x, y \in R(u, v)$. By $(b2)$ $R(u, x) \subseteq R(u, v)$ and $R(u, y) \subseteq R(u, v)$. Since $u \in R(u, x) \cap R(u, y)$, (h) implies that $R(u, x) \subseteq R(u, y)$ or $R(u, y) \subseteq R(u, x)$. In the first case, $x \in R(u, y)$ and thus $R(x, y) \subseteq R(u, y) \subseteq R(u, v)$. In the second case $y \in R(u, x)$ and hence $R(x, y) \subseteq R(u, x) \subseteq R(u, y)$, i.e., (m) holds. □

Lemma 4.5. *Let R be a transit function satisfying axioms (h) and (m) on V. Then R satisfies axiom (h''').*

Proof. Suppose $z \notin R(u, v)$. By (h) $R(u, v) \subset R(u, z)$ and $R(u, v) \subset R(v, z)$, and further $R(u, z) \subseteq R(v, z)$ or $R(v, z) \subseteq R(u, z)$. In either case, $\{u, v, z\} \subseteq R(u, z)$ and $\{u, v, z\} \subseteq R(v, z)$, hence by (m) $R(u, z) \subseteq R(v, z)$ and $R(v, z) \subseteq R(u, z)$. Thus (h''') holds. □

Lemma 4.6. *Let R be a transit function satisfying axioms (h''') and (m) on V. Then R satisfies axiom (h).*

Proof. Suppose (h''') and (m) but (h) does not, i.e., there is $u, v, x, y \in V$ such that $R(x, y) \cap R(u, v) \neq \emptyset$ but neither $R(x, y) \nsubseteq R(u, v)$ nor $R(u, v) \nsubseteq R(x, y)$. Thus there is $w \in R(x, y) \cap R(u, v)$, $a \in R(u, v)$ such that $a \notin R(x, y)$ and $b \in R(x, y)$ such that $b \notin R(u, v)$. Furthermore, a, b, and w are pairwise disjoint. As a consequence of (m) we have $R(a, u) \subseteq R(u, v)$, $R(a, v) \subseteq R(u, v)$, $R(a, w) \subseteq R(u, v)$, $R(b, x) \subseteq R(x, y)$, $R(b, y) \subseteq R(x, y)$, and $R(b, w) \subseteq R(x, y)$. Since $a \notin R(x, y)$ and hence $a \notin R(b, y)$ and $a \notin R(b, w)$, (h''') implies $R(a, y) = R(a, x) = R(a, b) = R(a, w)$. Analogously, $b \notin R(u, v)$ implies $R(u, b) = R(b, v) = R(a, b) = R(b, w)$. In particular, therefore, $R(a, w) = R(b, w)$. Thus $b \in R(a, w) \subseteq R(u, v)$ and $a \in R(b, w) \subseteq R(x, y)$, contradicting the definition of a and b. Thus (h) holds. □

Lemma 4.7. *Let R be a transit function satisfying axioms (h') and (h''') on V. Then R satisfies axiom (m).*

Proof. Let $x, y \in R(u, v)$ and suppose (m) does not hold, i.e., there is $z \in R(x, y)$ such that $z \notin R(u, v)$. By (h''') we have $R(u, z) = R(v, z)$. Using (h') $x \in R(u, v)$ implies $R(u, x) = R(u, v)$ or $R(x, v) = R(u, v)$, and $y \in R(u, v)$ implies $R(u, y) = R(u, v)$ or $R(v, y) = R(u, v)$. $z \in R(x, y)$ implies $R(x, z) = R(x, y)$ or $R(z, y) = R(x, y)$. Thus z is not contained $R(u, x)$ or $R(x, v)$, and it is not contained in $R(u, y)$ or $R(v, y)$. Using (h''') again, therefore $R(u, z) = R(x, z)$ or $R(x, z) = R(z, v)$. Since $R(u, z) = R(v, z)$, we have $R(u, z) = R(v, z) = R(x, z)$. Analogously, z is not contained in $R(u, y)$ or $R(v, y)$, and thus (h''') implies $R(u, z) = R(z, y)$ or $R(v, z) = R(z, y)$, and further $R(u, z) = R(v, z) = R(y, z)$. Collecting all equalities we arrive at $R(u, z) = R(v, z) = R(x, z) = R(y, z) = R(x, y)$. Therefore $u \in R(x, y)$ and (h') implies $R(u, x) = R(x, y)$ or $R(u, y) = R(x, y)$.

In the first case $R(u, x) = R(x, y) \neq R(u, v)$ and (h') implies $R(x, v) = R(u, v)$, whence $R(x, v) \neq R(x, y)$ and finally $R(v, y) = R(x, y)$. Thus $y \in R(x, v)$, and (h') implies $R(x, y) = R(x, v)$ or $R(y, v) = R(x, v)$, i.e., $R(x, y) = R(x, v) = R(u, v)$, a contradiction.

In the second case $R(u, y) = R(x, y) \neq R(u, v)$ and (h') implies $R(v, y) = R(u, v)$, whence $R(v, y) \neq R(x, y)$ and finally $R(v, x) = R(x, y)$. Thus $x \in R(v, y)$, and (h') implies $R(x, y) = R(v, y)$ or $R(x, v) = R(v, y)$, i.e., $R(x, y) = R(v, y) = R(u, v)$, again a contradiction. Thus $R(x, y) \subseteq R(u, v)$. □

Lemma 4.8. *Let R be a transit function satisfying axioms (h') and $(b2)$ on V. Then R satisfies axioms (h'') and (m).*

Proof. Let $x \in R(u, v)$. Because of (h') we have $R(u, v) = R(u, x)$ or $R(u, v) = R(x, v)$. Condition $(b2)$ establishes that $R(u, x) \subseteq R(u, v)$ and $R(x, v) \subseteq R(u, v)$, and thus $R(u, x) \cup R(v, x) = R(u, v)$, i.e., (h'') holds. Thus, for all $y \in R(u, v)$ we have $y \in R(u, x)$ or $y \in R(x, v)$. W.l.o.g., $y \in R(x, v)$. By (h''), $R(x, v) = R(x, y) \cup R(y, v)$, hence $R(x, y) \subseteq R(u, v)$. □

Lemma 4.9. *Let R be a transit function satisfying axioms (h''') and $(b2)$ on V. Then R satisfies axiom (m).*

Proof. Let $x, y \in R(u, v)$. Suppose $R(x, y) \nsubseteq R(u, v)$ which implies that $y \notin R(u, x)$, otherwise by $(b2)$ $R(x, y) \subseteq R(u, x) \subseteq R(u, v)$, a contradiction. Also as $y \notin R(u, x)$ thus by (h''') we have $R(u, y) = R(x, y)$ which is again a contradiction as $R(u, y) \subseteq R(u, v)$ by $(b2)$. □

5 Independence of axioms

The following examples show that (m), (m'), (h), (h'), (h'') and (h''') are pairwise independent axioms.

Example 5.1. $(m) \nRightarrow (h)$.
Set $V = \{a, b, c, d, f\}$, $R(a, b) = \{a, b, c\}$, $R(d, f) = \{d, f, c\}$ and $R(x, y) = \{x, y\}$ for all other distinct pairs of points. This transit function satisfies (m) but violates (h): we $R(a, b) \cap R(d, f) \neq \emptyset$ but both $R(a, b) \setminus R(d, f) = \{a, b\}$ and $R(d, f) \setminus R(a, b) = \{d, f\}$ are non-empty.

Example 5.2. $(m) \not\Rightarrow (h')$; $(m) \not\Rightarrow (h''')$; and $(m) \not\Rightarrow (h'')$.
Let $V = \{u, v, x, y\}$, $R(u, v) = V$, and $R(p, q) = \{p, q\}$ for all other pairs of distinct points. It satisfies (m) but violates (h'). Axiom (h'') does not hold because $R(u, v) \neq R(u, x) \cup R(x, v)$. Axiom (h''') is violated by $u \notin R(x, y) = \{x, y\}$ and $R(u, x) = \{u, x\} \neq \{u, y\} = R(u, y)$.

As a consequence, the weaker conditions $(b2)$ of course also fails to imply (h), (h'), (h''), or (h''').

Example 5.3. $(h) \not\Rightarrow (m)$; $(h) \not\Rightarrow (h')$; $(m') \not\Rightarrow (m)$; $(m') \not\Rightarrow (h')$.
Let $V = \{u, v, x, y, a\}$ and consider the transit function R defined by $R(u, v) = V - \{a\}$ and $R(x, y) = V$ for all other distinct pairs $x, y \in V$. This transit function satisfies (h) and (m'). However, we have $x, y \in R(u, v)$ but $R(x, y) \nsubseteq R(u, v)$ contradicting (m). R also violates (h'), since $x \in R(u, v)$, but neither $R(u, x) = R(u, v)$, nor $R(x, v) = R(u, v)$.

Example 5.4. $(h) \not\Rightarrow (h''')$.
Set $V = \{a, b, c, d\}$, $R(a, b) = \{a, b\}$, $R(a, c) = \{a, b, c\}$, and $R(x, y) = V$ for all other distinct pairs $x, y \in V$. (h) holds, but (h''') does not hold since $c \notin R(a, b)$ but $R(a, c) \neq R(b, c)$.

Example 5.5. $(h') \not\Rightarrow (m)$.
Set $V = \{a, b, c, d\}$, $R(a, b) = R(a, d) = \{a, b, d\}$, $R(a, c) = R(b, c) = \{a, b, c\}$, and $R(b, d) = R(c, d) = \{b, c, d\}$. Here (h') holds, but not $(b2)$ and hence not (m).

Example 5.6. $(h') \not\Rightarrow (h)$; $(h') \not\Rightarrow (h''')$.
Set $V = \{a, b, c, d\}$ and $R(x, y) = \{x, y\}$ for all $x, y \in V$. Axiom (h') is trivially satisfied. Since the transit sets do not form a hierarchy, (h) does not hold. Furthermore, $c \notin R(a, b)$ and $R(a, c) = \{a, c\} \neq \{b, c\} = R(b, c)$, i.e., (h''') does not hold.

Example 5.7. $(h''') \not\Rightarrow (m)$.
Set $V = \{a, b, c, d\}$, $R(a, b) = R(a, c) = \{a, b, c\}$, and $R(x, y) = V$ for all other distinct pairs $x, y \in V$. (h''') holds but (m) does not hold since $\{b, c\} \in R(a, b) = R(a, c)$ but $R(b, c) \nsubseteq R(a, c) = R(a, b)$.

Example 5.8. $(h''') \not\Rightarrow (h)$.
Set $V = \{a, b, c, d\}$, $R(a, b) = R(b, c) = \{a, b, c\}$, $R(a, c) = \{a, c, d\}$, and $R(a, d) = R(b, d) = R(c, d) = V$. (h''') holds, but (h) is violated because $R(a, b) \nsubseteq R(a, c)$ and $R(a, c) \nsubseteq R(a, b)$. Furthermore, (h') does not hold since $d \in R(a, c)$ but $R(a, d) = \{a, d\} \neq R(a, c)$ and $R(c, d) \neq R(a, c)$.

Example 5.9. $(m) \not\Rightarrow (m')$.
Set $V = \{a, b, c, d, e, f, g\}$, $R(a, b) = \{a, b, c, d, e\}$, $R(f, g) = \{c, d, e, f, g\}$, and $R(x, y) = \{x, y\}$ for any other pair $x, y \in V$. Clearly R is monotone. Here $R(a, b) \cap R(f, g) = \{c, d, e\}$, which is convex, but there is no $p, q \in X$ such that $\{c, d, e\} = R(p, q)$, i.e., (m') does not hold.

Example 5.10. $(h') \not\Rightarrow (b2)$; $(h') \not\Rightarrow (h''')$.
Set $V = \{a, b, c, d\}$, $R(a, b) = R(b, c) = \{a, b, c\}$, $R(a, c) = R(c, d) = \{a, c, d\}$ and $R(a, d) = R(b, d) = \{a, b, d\}$. (h') holds but nor $(b2)$ and (h'''), since $c \in R(a, b)$ but $R(a, c) \nsubseteq R(a, b)$, i.e., $(b2)$ does not hold. Since $d \notin R(b, c)$ but $R(b, d) \neq R(c, d)$ implies that (h''') does not hold.

Example 5.11. $(h''') \not\Rightarrow (h)$; $(h''') \not\Rightarrow (b2)$; $(h''') \not\Rightarrow (h')$.
Set $V = \{a, b, c, d\}$, $R(a, b) = R(b, c) = \{a, b, c\}$,$R(a, c) = \{a, c, d\}$ and $R(a, d) = R(b, d) = R(c, d) = V$. (h''') holds for R. Since $R(a, b) \nsubseteq R(a, c)$ and $R(a, c) \nsubseteq R(a, b)$, thus (h) does not hold. $c \in R(a, b)$ but $R(a, c) \nsubseteq R(a, b)$ hence $(b2)$ does not hold. As $d \in R(a, c)$ but $R(a, d) \neq R(a, c)$ and $R(c, d) \neq R(a, c)$ thus (h') does not hold for R.

Hence $(b2)$, (h') and (h''') are independent axioms. While we have seen above that, some pairs of axioms are sufficient to characterize transit functions of hierarchies, this is not the case for all pairs.

Example 5.12. (h') and $(m) \not\Rightarrow (h)$.
Set $V = \{a, b, c, d\}$, $R(a, b) = \{a, b\}$, $R(a, c) = R(b, c) = \{a, b, c\}$, $R(a, d) = \{a, d\}$, $R(b, d) = \{b, d\}$ and $R(c, d) = \{c, d\}$. Clearly, both (h') and (m) are satisfied. However, $R(a, d) \cap R(b, d) \neq \emptyset$, and neither transit set is contained in the other, hence (h) does not hold.

Example 5.13. (h) and $(h''') \not\Rightarrow (m)$.
Set $V = \{a, b, c, d\}$, $R(a, b) = R(a, c) = \{a, b, c\}$ and $R(x, y) = V$ for all other distinct pair $x, y \in V$. (h''') and (h) hold, but (m) does not hold since $\{b, c\} \in R(a, b) = R(a, c)$ but $R(b, c) \nsubseteq R(a, c) = R(a, b)$.

Example 5.14. (m) and $(m') \not\Rightarrow (h'')$.
Set $V = \{a, b, c, d\}$, $R(a, b) = V$, and $R(x, y) = \{x, y\}$, for any other distinct elements $x, y \in V$. R satisfies (m) and (m') but does not satisfy (h''), since $c \in R(a, b)$ but $R(a, b) \neq R(a, c) \cup R(c, b)$.

Example 5.15. (m) and $(m') \not\Rightarrow$ there is p, q such that $R(p, q) = V$.
Set $V = \{a, b, c, d\}$, $R(a, b) = \{a, b, c\}$, $R(x, y) = \{x, y\}$, for any other distinct elements $x, y \in V$. R satisfies (m) and (m') but there does not exist any p, q such that $R(p, q) = V$.

Example 5.16. (h') and $(h'') \not\Rightarrow (h)$; (h') and $(h'') \not\Rightarrow (h''')$.
Set $V = \{a, b, c, d\}$, $R(a, c) = R(b, c) = \{a, b, c\}$, $R(x, y) = \{x, y\}$, for any other distinct elements $x, y \in V$. R satisfies (h') and (h'') but R does not satisfy (h) since $R(b, d) \cap R(b, c) \neq \emptyset$ but $R(b, d) \nsubseteq R(b, c)$ and $R(b, c) \nsubseteq R(b, d)$. Also R does not satisfy (h'''), since $d \notin R(b, c)$ but $R(b, d) \neq R(d, c)$.

Example 5.17. (m) and (m') and $(a) \not\Rightarrow (h)$.
Set $V = \{a, b, c, d\}$, $R(a, d) = R(b, d) = R(c, d) = V$, $R(a, c) = \{a, b, c\}$, $R(a, b) = \{a, b\}$, $R(b, c) = \{b, c\}$ satisfied (m), (m'), and (a) but violates (h) since $R(a, b) \cap R(b, c) = \{b\}$.

Using the results in Section 4 and the examples of non-implications among the axioms (a), (h), (h''), (h'''), $(b2)$, (m), and (m') described in Section 5, we obtain four different sets of axiomatic characterizations of the transit function of a hierarchy as a corollary of Theorem 3.1. This is stated as a Theorem below.

Theorem 5.18. *A transit function R on a non-empty set is the transit function of a hierarchy if and only if R satisfies one of the following combinations of two axioms:*

1. (h) *and* $(b2)$,

2. (h''') *and* $(b2)$,
3. (h) *and* (h'),
4. (h') *and* (h''').

Since each of these four pairs of axioms implies (m), and (m) implies $(b2)$, we can also replace $(b2)$ by (m) in Thm. 5.18. The implications between axioms and combinations of axioms discussed in this contribution are summarized in Figure 1.

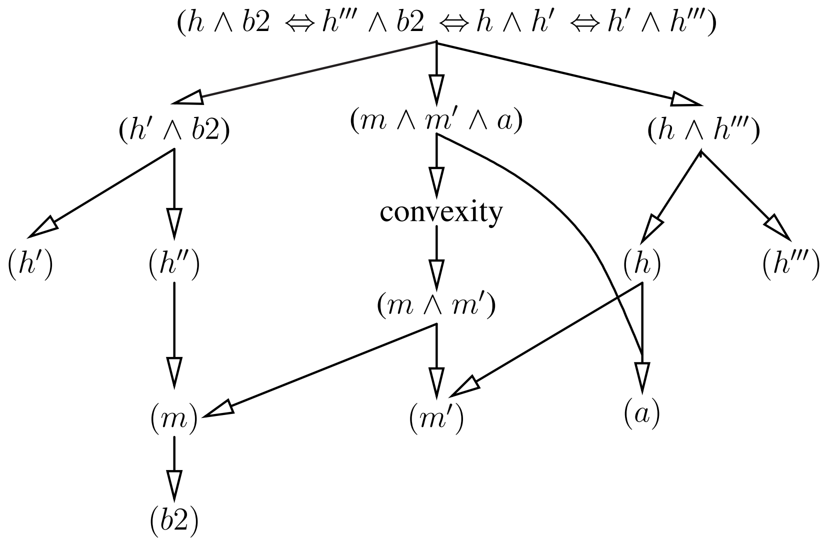

Figure 1: Summary of implications between axioms. None of the implication arrows is reversible, and only the implications implied by the transitive closure of this diagram hold.

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References

[1] C. C. Aggarwal and C. K. Reddy (eds.), *Data Clustering: Algorithms and Applications*, Chapman & Hall/CRC Data Mining and Knowledge Discovery Series, CRC Press, 2013.

[2] J. R. Calder, Some elementary properties of interval convexities, *J. London Math. Soc.* **3** (1971), 422–428, doi:10.1112/jlms/s2-3.3.422.

[3] M. Changat, S. Klavžar and H. M. Mulder, The all-paths transit function of a graph, *Czechoslovak Math. J.* **51** (2001), 439–448, doi:10.1023/a:1013715518448.

[4] M. Changat and J. Mathew, Induced path transit function, monotone and Peano axioms, *Discrete Math.* **286** (2004), 185–194, doi:10.1016/j.disc.2004.02.017.

[5] M. Changat, J. Mathew and H. M. Mulder, The induced path function, monotonicity and betweenness, *Discrete Appl. Math.* **158** (2010), 426–433, doi:10.1016/j.dam.2009.10.004.

[6] M. Changat, H. M. Mulder and G. Sierksma, Convexities related to path properties on graphs, *Discrete Math.* **290** (2005), 117–131, doi:10.1016/j.disc.2003.07.014.

[7] M. Changat, P. G. Narasimha-Shenoi and I. M. Pelayo, The longest path transit function of a graph and betweenness, *Util. Math.* **82** (2010), 111–127.

[8] M. Changat, G. N. Prasanth and J. Mathews, Triangle path transit functions, betweenness and pseudo-modular graphs, *Discrete Math.* **309** (2009), 1575–1583, doi:10.1016/j.disc.2008.02.043.

[9] H. M. Mulder, Transit functions on graphs (and posets), in: M. Changat, S. Klavžar, H. M. Mulder and A. Vijayakumar (eds.), *Convexity in Discrete Structures*, Ramanujan Mathematical Society, Mysore, volume 5 of *Ramanujan Mathematical Society Lecture Notes Series*, pp. 117–130, 2008, joint proceedings of the International Instructional Workshop held in Thiruvananthapuram, March 22 – April 2, 2006 and the International Workshop on Metric and Convex Graph Theory held in Barcelona, June 12 – 16, 2006.

[10] H. M. Mulder and L. Nebeský, Axiomatic characterization of the interval function of a graph, *European J. Combin.* **30** (2009), 1172–1185, doi:10.1016/j.ejc.2008.09.007.

[11] L. Nebeský, The induced paths in a connected graph and a ternary relation determined by them, *Math. Bohem.* **127** (2002), 397–408, http://mb.math.cas.cz/mb127-3/5.html.

[12] C. Semple and M. Steel, *Phylogenetics*, volume 24 of *Oxford Lecture Series in Mathematics and its Applications*, Oxford University Press, Oxford, 2003.

[13] P. F. Stadler and C. R. Stephens, Landscapes and effective fitness, *Comments Theor. Biol.* **8** (2003), 389–431, doi:10.1080/08948550302439.

[14] P. F. Stadler and G. P. Wagner, Algebraic theory of recombination spaces, *Evol. Comput.* **5** (1997), 241–275, doi:10.1162/evco.1997.5.3.241.

[15] M. L. J. van de Vel, *Theory of Convex Structures*, volume 50 of *North-Holland Mathematical Library*, North-Holland Publishing Company, Amsterdam, 1993.

[16] G. P. Wagner and P. F. Stadler, Complex adaptations and the structure of recombination spaces, in: C. L. Nehaniv and M. Ito (eds.), *Algebraic Engineering*, World Scientific, pp. 96–115, 1999, doi:10.1142/9789814527958, joint proceedings of the Kyoto Workshop on Formal Languages and Computer Systems held in Kyoto, Japan, March 18 – 21, 1997 and the First International Conference on Semigroups and Algebraic Engineering held in Aizu, Japan, March 24 – 28, 1997.

[17] G. P. Wagner and P. F. Stadler, Quasi-independence, homology and the unity of type: a topological theory of characters, *J. Theoret. Biol.* **220** (2003), 505–527, doi:10.1006/jtbi.2003.3150.

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On the distribution of subtree orders of a tree*

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Abstract

We investigate the distribution of the number of vertices of a randomly chosen subtree of a tree. Specifically, it is proven that this distribution is close to a Gaussian distribution in an explicitly quantifiable way if the tree has sufficiently many leaves and no long branchless paths. We also show that the conditions are satisfied asymptotically almost surely for random trees. If the conditions are violated, however, we exhibit by means of explicit counterexamples that many other (non-Gaussian) distributions can occur in the limit. These examples also show that our conditions are essentially best possible.

Keywords: Subtrees, normal distribution, homeomorphically irreducible trees, random trees.

Math. Subj. Class.: 05C05

1 Introduction

By a subtree of a tree, we mean any nonempty connected subgraph; obviously, such a subgraph is again a tree. The distribution of the number of vertices of a randomly chosen subtree of a tree was first studied by Jamison in two papers [5, 6], in which he investigates the average subtree order of a tree, i.e. the mean number of vertices of a subtree. Among his main results is the fact that the average order of subtrees of an n-vertex tree is at least $(n+2)/3$, with equality only for the path. The problems that Jamison proposed in his papers received considerable attention recently [4, 14, 16], as did other aspects of subtrees in trees, specifically extremal problems, whose study was initiated by Székely and Wang [12, 13]. Jamison's question whether the average order is always at least $n/2$ for homeomorphically irreducible trees, i.e. trees without vertices of degree 2, was only answered (affirmatively) very recently by Vince and Wang [14], who also showed that the average subtree order of such a tree is less than $3n/4$.

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Many other of Jamison's questions remain open to date. A question of his that was also discussed in the 2011 edition of the Combinatorics REGS [7] reads as follows:

Question 1.1. Given a tree T of order n, let $s_k(T)$ denote the number of subtrees of order k. When is it true that the numbers $s_2(T), \dots, s_n(T)$ form a unimodal list (weakly increasing at first, then weakly decreasing)? In particular, is it unimodal when T has no vertices of degree 2?

It should be noted here that $s_1(T) = n$ and $s_2(T) = n - 1$ for every tree T of order n, so $s_1(T)$ cannot be included if a unimodal list is to be obtained. The question seems to be fairly hard, and we will not actually answer it in this paper. However, we provide a related result: if a tree has sufficiently many leaves and no long branchless paths (this will be made more precise later), then the distribution of the subtree orders is close to a Gaussian distribution in an explicitly quantifiable way. In particular, this is the case for trees without vertices of degree 2. Moreover, the conditions we impose are usually satisfied: for random trees, they are valid asymptotically almost surely.

Asymptotic normality of the distribution does of course not imply unimodality, nor the other way around, but the two are clearly connected, so our result provides evidence that the answer to Question 1.1 might be affirmative. It should also be pointed out that our main result parallels a classic theorem of Godsil [3] on matchings: if $G_1, G_2, \dots$ is a sequence of graphs, then the distribution of the size of matchings in G_n (suitably renormalised) converges to a Gaussian distribution, provided that the variance tends to infinity. See [8] for a recent extension.

Godsil's theorem is based on properties of the matching polynomial, in particular the fact that all its zeros are real. Indeed, it is well known that a polynomial with positive coefficients and only real zeros has log-concave (and thus unimodal) coefficients, so Question 1.1 could be answered affirmatively if all zeros of the polynomial

$$\sum_{k=2}^{n} s_k(T)u^k$$

were real for every T. This "subtree polynomial" was already considered by Jamison himself in [5]. More recently, Yan and Yeh [18] studied a weighted version, and Martin et al. [9] considered a bivariate generalisation involving the number of leaves.

Unfortunately, the subtree polynomial does not have real roots only as the matching polynomial does, so the situation for subtrees of trees turns out to be more intricate than for matchings of graphs. As a simple concrete counterexample, consider the star S_4 with four vertices: we have

$$\sum_{k=1}^{4} s_k(S_4)u^k = 4u + 3u^2 + 3u^3 + u^4,$$

a polynomial with two non-real roots. Even if the first coefficient is removed, we get

$$\sum_{k=2}^{4} s_k(S_4)u^k = 3u^2 + 3u^3 + u^4,$$

which still has two non-real roots.

However, we obtain a central limit theorem for the distribution of subtree orders analogous to Godsil's theorem under some technical conditions. Our approach is of a rather different nature, and we hope that it might also prove useful to deal with other problems, such as a conjecture of Alavi, Malde, Schwenk and Erdős [1] concerning the independence polynomial of trees that parallels Question 1.1. Our main theorem can be stated as follows:

Theorem 1.2. *Let $T_1, T_2, \ldots$ be a sequence of trees such that $|T_n|$, i.e. the number of vertices of T_n, goes to infinity, the proportion of leaves among all vertices is bounded below by a positive constant, and the length of the longest branchless path in T_n is at most $|T_n|^{1/2-\epsilon}$ for some fixed ϵ (and sufficiently large n). Then the order distribution of the subtrees of T_n (suitably renormalised) converges weakly to a Gaussian distribution.*

It is easy to find both examples and counterexamples for the normal distribution: for instance, if T_n is an n-vertex star, then the distribution of the subtree orders is essentially a binomial distribution, which converges to a Gaussian law. On the other hand, if one considers the sequence of n-vertex paths, then the limit distribution is quite different. This and other examples and counterexamples will be discussed in Section 2, where we also show that the technical conditions of Theorem 1.2 are indeed important and also essentially best possible.

The main part of the paper is organised as follows: we first obtain some auxiliary results and prove two versions of our main theorem (a central and a local limit theorem, see Theorem 4.8 and Theorem 4.10 respectively) for rooted trees in Section 4 before passing on to unrooted trees in Section 5. Rooted trees are more accessible because one can use a recursive approach, and we will see that an appropriate root can always be chosen in such a way that most subtrees contain the root. In Section 6, we will see that in the "generic" case of random trees, the conditions of our main theorem are satisfied, so that the Gaussian distribution is indeed the typical limit distribution of subtree orders.

Notation. Throughout this paper, we make frequent use of the Vinogradov symbol $\ll$ interchangeably with the $\mathcal{O}$-notation: $f(T) \ll g(T)$ or $f(T) = \mathcal{O}(g(T))$ means that $f(T) \leq Kg(T)$ for a suitable positive constant K and all (sufficiently large) trees T. If further variables are included in an $\mathcal{O}$-term, the constant K is always independent of those, unless mentioned otherwise.

2 Examples and counterexamples

For a tree T, we let $\mathcal{S}(T)$ denote the set of all subtrees of T, i.e. all connected induced subgraphs of T. The polynomial associated with this set, which we call the *subtree polynomial* of T, is denoted by $S(T, u)$:

$$S(T, u) = \sum_{\tau \in \mathcal{S}(T)} u^{|\tau|}.$$

The total number of subtrees is clearly $S(T, 1)$, for which we will simply write $S(T)$. Our goal will be to prove central and local limit theorems for the coefficients of this polynomial. Note also that $S_u(T, 1) = \frac{\partial}{\partial u} S(T, u)\big|_{u=1}$ is the total number of vertices in T's subtrees, so $S_u(T, 1)/S(T)$ is the mean subtree order. Likewise, the variance is given by

$$\frac{S_{uu}(T, 1) + S_u(T, 1)}{S(T)} - \left(\frac{S_u(T, 1)}{S(T)}\right)^2. \tag{2.1}$$

Before we get to the proof of the main theorem, let us briefly discuss some examples and counterexamples to illustrate its statement.

2.1 The star

If $T = S_n$ is a star of order n, then every subtree either consists of the centre and an arbitrary set of leaves, or it is a single leaf. Thus we have

$$S(T, u) = nu + \sum_{k=2}^{n} \binom{n-1}{k-1} u^k$$

and in particular $S(T) = 2^{n-1} + n - 1$. We see that the distribution of subtree orders is essentially a binomial distribution, which gives us a Gaussian distribution in the limit.

2.2 The path

The distribution of subtree orders of a path P_n turns out to be quite different: every subtree is again a path and uniquely characterised by its endpoints. We obtain

$$S(P_n, u) = \sum_{k=1}^{n} (n - k + 1) u^k.$$

If we divide the subtree orders by n and take the limit, we obtain a distribution whose density is given by $f(t) = 2(1 - t)$ on the interval $[0, 1]$.

The examples that we consider in the following are all constructed by suitably combining paths and stars. Depending on how this is done, a variety of different limit distributions can be obtained. Of course, there does not even have to be a limit distribution at all: this is not the case, for example, if we consider a sequence of trees of increasing orders, alternating between paths and stars.

2.3 The broom

The simplest possible combination of a star and a path is the broom, consisting of a path of k vertices and ℓ leaves attached to one of its ends (the "centre" of the broom, denoted v in Figure 1). Here, the limit as $k + \ell \to \infty$ depends very much on the relative sizes of k and ℓ. If k is fixed, then there is very little difference to a star, and we obtain a Gaussian limit distribution. On the other hand, if ℓ is fixed, then we have essentially the same order distribution as for a path (and exactly the same in the limit). As soon as ℓ grows faster than $\log_2 k$, almost all subtrees contain the broom's centre v (i.e., the proportion of such subtrees tends to 1). This is because there are $k2^\ell$ subtrees containing it, as opposed to $\mathcal{O}(k^2 + \ell)$ not containing it.

Subtrees containing the centre v have a distribution that is a convolution of a binomial distribution (stemming from the leaves attached to v) and a discrete uniform distribution (stemming from the path). In the limit, the distribution with greater variance dominates. Since the variances are of order k^2 and ℓ respectively, we have three phases:

(i) $k^2/\ell \to 0$: the leaves dominate, and a suitably renormalised version of the order distribution converges to a normal distribution.

(ii) $k^2/\ell \to a > 0$: the limit distribution is a convolution of a (continuous) uniform distribution and a Gaussian distribution.

(iii) $k^2/\ell \to \infty$ (but $k/2^\ell \to 0$): the long path dominates, and the renormalised order distribution converges to a uniform distribution.

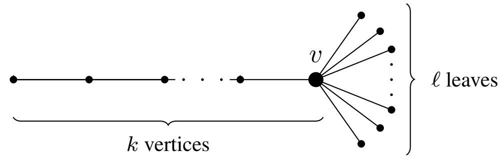

Figure 1: The broom.

2.4 The extended star

Figure 2 shows an extended star, obtained by attaching d (≥ 3) paths of length k to a common vertex v. For fixed d, we obtain (by the same argument as in the previous example) a convolution of d uniform distributions in the limit as $k \to \infty$. As soon as d also tends to infinity, however, the limit is Gaussian again (showing that the conditions of Theorem 1.2 are important, but not strictly necessary).

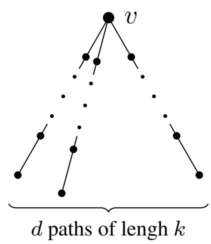

Figure 2: The extended star.

2.5 A discontinuous limit distribution

By suitably choosing the parameters of a double-star (see Figure 3), we can even obtain a discontinuous limit distribution. Such a tree consists of a path of length k and leaves attached to the two endpoints v_1 and v_2 (ℓ and r leaves, respectively). We set $\ell = 3n$, $r = n + c$ for some constant c, and $k = 2^n$. The same argument that we used for the broom shows that almost all subtrees contain v_1 in this case. The probability that v_2 is contained as well is easily found to be $2^c/(1 + 2^c)$ in the limit. In this case, the subtree order is $2^n + \mathcal{O}(n)$. Otherwise, it essentially follows a discrete uniform distribution on the interval $[1, 2^n]$ (the leaves attached to v_1 only playing a minor role). So if we divide the subtree orders by 2^n, we obtain a limit distribution that is a mix of the uniform distribution on $[0, 1]$ and a point measure at 1, which means that its distribution function has a discontinuity at 1.

We remark that another choice of parameters is interesting as well: if we set $\ell = r = 3n$ and $k = 2^n$, then almost all subtrees contain both v_1 and v_2 (and the probability that this is not the case is as low as $\mathcal{O}(4^{-n})$). Thus the subtree order distribution is essentially a convolution of two binomial distributions, and the variance is $\mathcal{O}(n)$. This shows that

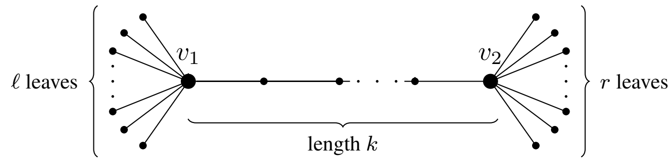

Figure 3: The double-star.

the variance of the subtree order distribution can be as low (in order of magnitude) as the logarithm of the order of the underlying tree, and we conjecture that it cannot be less, i.e. that (2.1) is bounded below by $K \log |T|$ for some positive constant K. On the other hand, the order of magnitude of the variance can be as high as $|T|^2$, as the example of the path shows.

2.6 Short branchless paths are insufficient

The two conditions of Theorem 1.2 (short branchless paths, many leaves) ensure that the trees T_n are not too "path-like". However, as we exhibit now, neither of the two conditions suffices on its own to ensure a Gaussian limit distribution. The broom is a simple example showing that even a proportion of leaves tending to 1 may not be enough: if we choose k and ℓ such that $\ell = ak^2$ for some fixed constant a, then we obtain a convolution Gaussian-Uniform in the limit rather than a pure normal distribution. This example also explains why $\sqrt{|T_n|}$ is the threshold for the length of branchless paths.

Finding a counterexample that satisfies the condition on paths, but does not have sufficiently many leaves, is a little bit more complicated. It can be constructed as follows (see Figure 4): fix positive constants α, β, γ such that $\beta < \alpha < \frac{1}{2}$, $\alpha + \gamma = 1$ and $2\alpha > \beta + \gamma$. Start with a central vertex v, which is connected to $\ell + 1 = \lfloor n^\gamma \rfloor$ vertices $w_0, w_1, w_2, \dots, w_\ell$ by paths of length $\lfloor n^\alpha \rfloor$. To each of these vertices except w_0, we attach $\lfloor n^\beta \rfloor$ leaves. Note that the order of the resulting tree T_n is $|T_n| \sim n^\alpha \cdot n^\gamma = n$, so that there are no branchless paths of length $|T_n|^{1/2-\epsilon}$ if $\epsilon < \frac{1}{2} - \alpha$ and n is sufficiently large. On the other hand, the number of leaves is $L(T_n) \sim n^\beta \cdot n^\gamma = o(n)$ (note, however, that the exponent $\beta + \gamma$ can be made arbitrarily close to 1 with an appropriate choice of α, β, γ).

The limit distribution is not Gaussian in this case: the same argument that we used in previous examples shows that $v, w_1, w_2, \dots, w_\ell$ and thus also the paths connecting them are part of almost all subtrees. The remaining "random" part is the same as for a broom consisting of a path of length (approximately) n^α and (approximately) $n^{\beta+\gamma}$ leaves. Since $2\alpha > \beta + \gamma$ by our choice, we are in the situation where the limit distribution as $n \to \infty$ is uniform.

3 Preliminary results

Before we start with the actual proof of our main result, let us fix some notation and prove some auxiliary inequalities.

3.1 Definitions and notation

Most of the time, we will be working with rooted trees, since they allow for a recursive approach. Thus we first define an analogue of the polynomial $S(T, u)$ for rooted trees.

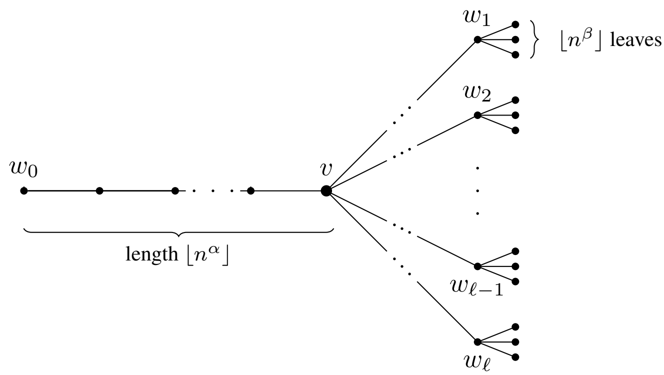

Figure 4: The final counterexample.

Consider a tree T with root v_0, and let $\mathcal{S}^\bullet(T)$ be the set of all subtrees of T containing v_0. The generating polynomial for subtrees containing the root is denoted by $S^\bullet(T, u)$:

$$S^\bullet(T, u) = \sum_{\tau \in \mathcal{S}^\bullet(T)} u^{|\tau|}.$$

The main reason for considering this polynomial is the fact that it can be computed recursively from the root branches. For a vertex v of T, we let $T(v)$ be the branch of T rooted at v (consisting of v and all its descendants). Suppose that $v_1, v_2, \ldots, v_d$ are the root's children. It is not hard to see that $S^\bullet(T, u)$ satisfies the following recursive formula:

$$S^\bullet(T, u) = S^\bullet(T(v_0), u) = u \prod_{j=1}^{d} \left(1 + S^\bullet(T(v_j), u)\right). \tag{3.1}$$

This follows from the fact that a subtree of T that contains the root v_0 induces either the empty tree or a subtree that contains v_j in the branch $T(v_j)$ for each v_j.

Notation. For the convenience of the reader we list some further notation that is used throughout this paper:

(i) $\mathcal{L}(T)$ and $L(T)$ are the set and the number of leaves, respectively.

(ii) $\mathcal{I}(T)$ and $I(T)$ are the set and the number of interior vertices, respectively.

(iii) By a branchless path or 2-path, we mean a path in which all vertices, except for the endpoints, have degree 2. We let $P(T)$ denote the maximum length of a 2-path of T.

Moreover, we use $c_0, c_1, c_2, \ldots$ to denote absolute constants (that do not depend on the specific tree or any of its parameters).

3.2 Two inequalities

We begin with the following simple but useful lemma, which provides two inequalities that will be used repeatedly in the following section.

Lemma 3.1. *If T is a rooted tree with $|T| \geq 2$, then*

$$S^\bullet(T) \geq 2^{L(T)} \text{ and } L(T) \geq \frac{|T|}{2P(T)}.$$

Proof. Every subset A of $\mathcal{L}(T)$ gives rise to a subtree obtained as the union of all paths connecting the leaves in A to the root. If A is empty, we take the subtree consisting only of the root as the corresponding subtree. This proves the first inequality.

For the proof of the second inequality, we let $V_2(T)$ be the number of non-root vertices of degree 2 and $V_{\geq 3}(T)$ the number of non-root vertices of degree at least 3. Consider all maximal 2-paths (not containing the root as an inner vertex in case that the root has degree 2). To each such path, we can uniquely associate its endpoint that is further away from the root, which is either a leaf or a (non-root) vertex of degree at least 3. Thus there are $L(T) + V_{\geq 3}(T)$ such paths. Since the total number of edges, which is $|T| - 1$, is at most $P(T)$ times the number of maximal 2-paths, we obtain

$$(L(T) + V_{\geq 3}(T))P(T) \geq |T| - 1.$$

On the other hand, the handshake lemma gives us

$$2(L(T) + V_2(T) + V_{\geq 3}(T)) = 2(|T| - 1) \geq L(T) + 2V_2(T) + 3V_{\geq 3}(T) + 1,$$

the final 1 being the trivial lower bound for the root degree. Thus $L(T) \geq V_{\geq 3}(T) + 1$, and consequently

$$2L(T)P(T) \geq (L(T) + V_{\geq 3}(T) + 1)P(T) \geq (L(T) + V_{\geq 3}(T))P(T) + 1 \geq |T|,$$

which is equivalent to the second inequality in the statement of the lemma. □

4 Rooted trees

4.1 The moment generating function

In order to prove the central limit theorem for the order distribution of subtrees, we study the associated moment generating function, first only for rooted trees. Note that

$$\frac{S^\bullet(T,u)}{S^\bullet(T,1)} = \frac{S^\bullet(T,u)}{S^\bullet(T)} = \frac{1}{S^\bullet(T)} \sum_{\tau \in \mathcal{S}^\bullet} u^{|\tau|}$$

is the probability generating function for the order of a randomly chosen subtree of T that contains the root. Likewise,

$$\frac{S^\bullet(T,e^t)}{S^\bullet(T)}$$

is the moment generating function. For our purposes, it turns out to be useful to consider an auxiliary function, denoted $F(T,t)$. We define it recursively by $F(T,t) = \log(1+e^t)$ if $|T| = 1$ and

$$F(T,t) = \sum_j F(T(v_j),t) + f(T,t), \tag{4.1}$$

where

$$f(T,t) = t + \log\left(1 + \frac{1}{S^\bullet(T,e^t)}\right). \tag{4.2}$$

Here and in the following, log will always denote the principal branch of the logarithm. In view of (3.1), we have

$$1 + S^\bullet(T, e^t) = e^{F(T,t)}, \tag{4.3}$$

as can be seen by a simple induction. As a first step, we show that $S^\bullet(T, e^t)$ is bounded away from 0 if t is sufficiently small, so that we can actually take the logarithm in (4.2).

Lemma 4.1. *There exist absolute constants $\delta > 0$ and $c_0 > 0$ with the following property: if T is a tree such that the lengths of the 2-paths of T are all bounded above by some positive integer P (which can be a function of T), then we have*

$$|1 + S^\bullet(T, e^t)| \geq e^{c_0 L(T)} \tag{4.4}$$

whenever $|t| \leq \frac{\delta}{P}$. Moreover, the function $f(T, t)$ as defined in (4.2) *is analytic in the disk defined by the inequality $|t| \leq \frac{\delta}{P}$.*

Remark 4.2. It is important to bear in mind that t is complex in this context. If we were to consider only real values of t, it would e.g. be trivial that $|1 + S^\bullet(T, e^t)| = 1 + S^\bullet(T, e^t) > 1$.

Proof. We will show that the statements of the lemma hold for the following explicit constants:

$$\delta = 0.001 \quad \text{and} \quad c_0 = 0.012.$$

So we assume throughout this proof that δ is as defined above. We show first that the inequality (4.4) implies analyticity of the function $f(T, t)$ in (4.2). Note that

$$1 + \frac{1}{S^\bullet(T, e^t)} = \left(1 - \frac{1}{1 + S^\bullet(T, e^t)}\right)^{-1}.$$

If $|1 + S^\bullet(T, e^t)| \geq e^{c_0 L(T)} \geq e^{c_0}$, then $1 - \frac{1}{1+S^\bullet(T,e^t)}$ lies inside the disk with centre 1 and radius e^{-c_0}. Thus the reciprocal $\left(1 - \frac{1}{1+S^\bullet(T,e^t)}\right)^{-1}$ lies inside the disk with centre $\frac{1}{1-e^{-2c_0}}$ and radius $\frac{e^{-c_0}}{1-e^{-2c_0}}$. The principal branch of the logarithm is an analytic function inside this disk, so $f(T, t)$ is analytic.

Now we prove (4.4). Let P be an arbitrary positive integer, and let T be a tree such that no 2-path of T has length greater than P. The lemma is satisfied for $|T| = 1$: in this case, we have $S^\bullet(T, e^t) = e^t$, and it is easily verified that

$$|1 + S^\bullet(T, e^t)| = |1 + e^t| \geq 1 + e^{-\delta} > e^{c_0}$$

holds for $|t| \leq \delta$.

If $2 \leq |T| \leq 12P$, then for every subtree τ of T we have $|t||\tau| \leq 12\delta$. It follows that $\operatorname{Re}(e^{t|\tau|}) \geq e^{-12\delta}\cos(12\delta)$. Therefore,

$$\begin{aligned}\Big|1 + \sum_{\tau \in \mathcal{S}^\bullet(T)} e^{t|\tau|}\Big| &\geq 1 + \sum_{\tau \in \mathcal{S}^\bullet(T)} \operatorname{Re}\left(e^{t|\tau|}\right) \\ &\geq e^{-12\delta}\cos(12\delta)\big(1 + S^\bullet(T)\big). \end{aligned} \tag{4.5}$$

Applying Lemma 3.1 to estimate the right side of (4.5), we have

$$\begin{aligned}|1+S^\bullet(T,e^t)| &= \Big|1+\sum_{\tau\in\mathcal{S}^\bullet(T)} e^{t|\tau|}\Big| \geq e^{-12\delta}\cos(12\delta)2^{L(T)} \\ &\geq e^{c_0}2^{L(T)-1} \geq e^{c_0}e^{c_0(L(T)-1)} = e^{c_0L(T)}.\end{aligned}$$

So the proof is complete in this case, and we assume from now on that $|T| > 12P$.

For each vertex v of T, we define

$$m(v,t) = |1+S^\bullet(T(v),e^t)|.$$

Let $v_1, v_2, \dots$ be v's children. Using (3.1), we find that for $|t| \leq \frac{\delta}{P}$,

$$m(v,t) \geq e^{-\delta/P}\prod_j m(v_j,t) - 1. \tag{4.6}$$

Let A be the set of vertices in "small branches", defined formally as the set of all vertices w of T for which $|T(w)| \leq 12P$. Thus for every $w \in A$, the bound in (4.5) applies to the branch $T(w)$, and we have

$$m(w,t) = \Big|1+\sum_{\tau\in\mathcal{S}^\bullet(T(w))} e^{t|\tau|}\Big| \geq e^{-12\delta}\cos(12\delta)\big(1+S^\bullet(T(w))\big). \tag{4.7}$$

We define $m_0 = 2e^{-12\delta}\cos(12\delta) \approx 1.976$, so we can deduce from (4.7) that for every $w \in A$

$$m(w,t) \geq m_0. \tag{4.8}$$

The rest of the proof is divided into two parts: in the first part, we prove that $m(v,t)$ cannot be too small when v is outside of A. In the second part, we use recursion (4.6) to complete the proof of (4.4).

Part 1: We claim that $m(v,t) \geq 3P$ for all $v \in T \setminus A$.

Assume that the claim is not true, and let $w \in T \setminus A$ be a counterexample (i.e., $m(w,t) < 3P$) with maximum distance from the root. In addition, let $w_0 = w, w_1, \dots, w_r$ be the longest sequence of vertices (possibly, $r = 0$) such that none of these vertices lies in A, w_{j+1} is w_j's only child for $0 \leq j < r$, and w_r has either more than one child or a single child that lies in A. Now consider two different cases:

(i) Suppose that all of w_r's children, which we denote by $x_1, x_2, \dots, x_d$, lie in A. Since $w_r \notin A$, we have $|T(w_r)| > 12P$, so at least one of these children is the root of a branch of order at least $12P/d$. Without loss of generality, $|T(x_1)| \geq 12P/d$. We have

$$m(x_1,t) \geq \frac{m_0}{2}\big(1+S^\bullet(T(x_1))\big) \geq \frac{m_0}{2}(1+|T(x_1)|) \geq \frac{m_0}{2}\cdot\frac{12P}{d}$$

by (4.5); the inequality $S^\bullet(T(x_1)) \geq |T(x_1)|$ simply follows from the fact that we can associate the path from the root, which is also a subtree, to each vertex. Moreover, we know that $m(x_2,t), \dots, m(x_d,t) \geq m_0$ by (4.8). Now (4.6) gives us

$$m(w_r,t) \geq e^{-\delta/P}\cdot\frac{m_0}{2}\cdot\frac{12P}{d}\cdot m_0^{d-1} - 1 = 6e^{-\delta/P}\cdot\frac{m_0^d}{d}\cdot P - 1.$$

Using the numerical values of δ and m_0, one easily verifies that $m_0^d \geq d$ for all $d \geq 1$ and $6e^{-\delta/P} \geq 6e^{-\delta} \geq \frac{11}{2}$. Hence,

$$m(w_r, t) \geq \frac{11P}{2} - 1 \geq \frac{9P}{2}.$$

(ii) Otherwise, w_r has at least 2 children $x_1, x_2, \ldots, x_d$, at least one of which (without loss of generality, x_1) does not lie in A. By our choice of w as a counterexample to our claim with maximum distance from the root, we have $m(x_1, t) \geq 3P$. Moreover, $m(x_j, t) \geq \min(3P, m_0) = m_0$ for all other children (the lower bound $3P$ applies if $x_j \notin A$, the lower bound m_0 otherwise). It follows that

$$m(w_r, t) \geq e^{-\delta/P} \cdot 3P \cdot m_0^{d-1} - 1 \geq 3m_0 e^{-\delta/P} \cdot P - 1 \geq \frac{11}{2}P - 1.$$

Again, we obtain

$$m(w_r, t) \geq \frac{9P}{2}.$$

Now note that $w_0, w_1, \ldots, w_r$ is a branchless path, so that $r \leq P$ by definition. We apply (4.6) repeatedly to $w_{r-1}, w_{r-2}, \ldots, w_0 = w$ to obtain

$$m(w_0, t) \geq \left(e^{-\delta/P}\right)^r m(w_r, t) - \sum_{k=0}^{r-1} e^{-\delta k/P} \geq e^{-\delta} m(w_r, t) - P \geq \left(\frac{9e^{-\delta}}{2} - 1\right) P.$$

The last expression is greater than $3P$ by our choice of δ, and we reach a contradiction. So the claim is proven.

Part 2: Now we complete the proof of (4.4). Taking the logarithm of inequality (4.6), we obtain

$$\log m(v, t) + \log\left(1 + \frac{1}{m(v, t)}\right) \geq \sum_j \log m(v_j, t) - \frac{\delta}{P}. \qquad (4.9)$$

Note that the set A can be written as a disjoint union of the vertex sets of certain trees $T(y_1), T(y_2), \ldots$ rooted at $y_1, y_2, \ldots$. Iterating (4.9) from the root v_0 to the vertices $y_1, y_2, \ldots$ and applying (4.7) and Lemma 3.1 yields

$$\begin{aligned}
\log m(v_0, t) &\geq \sum_j \log m(y_j, t) - \sum_{v \in T \setminus A} \log\left(1 + \frac{1}{m(v, t)}\right) - \frac{\delta |T \setminus A|}{P} \\
&\geq \sum_j \log\left(\frac{m_0}{2} S^\bullet(T(y_j))\right) - \sum_{v \in T \setminus A} \log\left(1 + \frac{1}{m(v, t)}\right) - \frac{\delta |T \setminus A|}{P} \\
&\geq \sum_j \log\left(\frac{m_0}{2} 2^{L(T(y_j))}\right) - \sum_{v \in T \setminus A} \log\left(1 + \frac{1}{m(v, t)}\right) - \frac{\delta |T \setminus A|}{P}.
\end{aligned}$$

Furthermore, since $m_0 < 2$ and the trees $T(y_1), T(y_2), \ldots$ contain all leaves of T, we have

$$\begin{aligned}
\sum_j \log\left(\frac{m_0}{2} 2^{L(T(y_j))}\right) &\geq \sum_j \log\left(m_0^{L(T(y_j))}\right) \\
&= \log(m_0) \sum_j L(T(y_j)) = \log(m_0) L(T).
\end{aligned}$$

Now recall that $m(v,t) \geq 3P$ for all $v \notin A$, which gives us

$$\sum_{v \in T \setminus A} \log\left(1 + \frac{1}{m(v,t)}\right) + \frac{\delta |T \setminus A|}{P} \leq \left(\log\left(1 + \frac{1}{3P}\right) + \frac{\delta}{P}\right) |T \setminus A|$$
$$\leq \left(\frac{1}{3} + \delta\right) \frac{|T|}{P}.$$

Putting these bounds together, we obtain

$$\log m(v_0, t) \geq \log(m_0) L(T) - \left(\frac{1}{3} + \delta\right) \frac{|T|}{P}.$$

From Lemma 3.1, we know that

$$L(T) \geq \frac{|T|}{2P}.$$

Hence, we finally have

$$\log |1 + S^\bullet(T,t)| = \log m(v_0, t) \geq \left(\log(m_0) - \frac{2}{3} - 2\delta\right) L(T).$$

The proof of (4.4) is completed by applying the exponential function on both sides of the latter inequality and by noting that the constant $\log(m_0) - \frac{2}{3} - 2\delta$ is greater than $c_0 = 0.012$ (defined at the beginning of the proof). □

We have shown that $f(T,t)$ and consequently $F(T,t)$ can be regarded as complex analytic functions in a disk around zero, so $F(T,t)$ admits a Taylor expansion near zero, which we are now going to investigate further. By (4.3), we have

$$\mu(T) = \frac{d}{dt} F(T,t)\Big|_{t=0} = \frac{S_u^\bullet(T)}{1 + S^\bullet(T)}$$

and

$$\sigma^2(T) = \frac{d^2}{dt^2} F(T,t)\Big|_{t=0} = \frac{S_{uu}^\bullet(T)}{1 + S^\bullet(T)} + \mu(T) - \mu^2(T),$$

where we use $S_u^\bullet(T)$ as a shorthand for $S_u^\bullet(T,1) = \frac{d}{du} S^\bullet(T,u)\big|_{u=1}$ in the same way as $S^\bullet(T)$, and $S_{uu}^\bullet(T)$ is defined analogously for the second derivative. The intuition behind the notation $\mu(T)$ and $\sigma^2(T)$ is that these two quantities are essentially the average order of subtrees in $\mathcal{S}^\bullet(T)$ and the variance respectively, if we include an additional dummy subtree of order 0 in the count (compare also the considerations at the beginning of Section 2). This is asymptotically irrelevant and simplifies the following calculations.

For the rest of this section, we let γ be a fixed positive real number, let P be a positive integer that represents an upper bound on the length of all 2-paths in T, and set

$$\Delta = \frac{\delta}{2P^{1+\gamma}},$$

where $\delta = 0.001$ is as defined in the proof of Lemma 4.1. It also follows from Lemma 4.1 that for every vertex v in T, the function $F(T(v), t)$ is analytic in the disk centred at zero with radius 2Δ. So we can define the quantity

$$r(T) = \sup_{0 < |t| \leq \Delta} \left| \frac{F(T,t) - F(T,0) - \mu(T)t - \sigma^2(T)\frac{t^2}{2}}{t^3} \right|,$$

which represents the error in the second-order Taylor approximation of $F(T,t)$. Then by definition, for $|t| \le \Delta$ we have

$$F(T,t) = F(T,0) + \mu(T)t + \sigma^2(T)\frac{t^2}{2} + \mathcal{O}\Big(r(T)|t|^3\Big). \tag{4.10}$$

Next, we estimate the quantities $\sigma^2(T)$ and $r(T)$. Note first that $\sigma^2(T)$ satisfies the following additive relation that one can easily deduce from its definition and (4.1):

$$\sigma^2(T) = \frac{S^\bullet(T)}{1+S^\bullet(T)} \sum_j \sigma^2(T(v_j)) + \frac{\mu(T)^2}{S^\bullet(T)}. \tag{4.11}$$

Moreover, the recursion (4.1) also yields

$$\begin{aligned} F(T,t) - F(T,0) - \mu(T)t - \sigma^2(T)\frac{t^2}{2} &= \\ = \sum_j \bigg(F(T(v_j),t) - F(T(v_j),0) &- \mu(T(v_j))t - \sigma^2(T(v_j))\frac{t^2}{2}\bigg) \\ &+ f(T,t) - f(T,0) - f'(T,t)t - f''(T,t)\frac{t^2}{2}, \end{aligned}$$

so by the triangle inequality

$$r(T) \le \sum_j r(T(v_j)) + \sup_{0<|t|\le\Delta} \left| \frac{f(T,t) - f(T,0) - f'(T,0)t - f''(T,0)\frac{t^2}{2}}{t^3} \right|,$$

and since

$$f(T,t) - f(T,0) - f'(T,0)t - f''(T,0)\tfrac{t^2}{2} = \int_0^t \int_0^u \int_0^v f'''(T,w)\,dw\,dv\,du,$$

we have

$$r(T) \le \sum_j r(T(v_j)) + \frac{1}{6} \sup_{|t|\le\Delta} |f'''(T,t)|. \tag{4.12}$$

As in the proof of Lemma 4.1, we will now iterate (4.11) and (4.12) along the tree to obtain a lower estimate for $\sigma^2(T)$ and an upper estimate for $r(T)$. To this end, we introduce a (now slightly different) notion of "small branches" again: we let B be the set of all vertices w for which $|T(w)| \le P^{1+\gamma}$. Our first lemma gives an upper estimate for $r(T)$.

Lemma 4.3. *We have*

$$r(T) \ll |T| + \sum_{v\in\mathcal{I}(T)\cap B} \frac{|T(v)|^3}{S^\bullet(T(v))}. \tag{4.13}$$

Proof. Iterating (4.12), we have

$$r(T) \ll L(T) + \sum_{v\in\mathcal{I}(T)\setminus B} \sup_{|t|\le\Delta} |f'''(T(v),t)| + \sum_{v\in\mathcal{I}(T)\cap B} \sup_{|t|\le\Delta} |f'''(T(v),t)|.$$

The term $L(T)$ on the right side bounds the contribution from the leaves. We now consider two cases each estimating one of the sums above:

(i) We first look at the case that $v \notin B$. Cauchy's integral formula yields, for $|t| \leq \Delta$,

$$f'''(T(v),t) = \frac{3!}{2\pi i} \oint_{\mathcal{C}(t,\Delta)} \frac{f(T(v),z) - z}{(z-t)^4} dz,$$

where $\mathcal{C}(t,\Delta)$ is the circle centred at t with radius Δ. The integral representation of $f'''(T(v),t)$ gives us the bound

$$\begin{aligned} |f'''(T(v),t)| &\leq 6\Delta^{-3} \sup_{z \in \mathcal{C}(t,\Delta)} |f(T(v),z) - z| \\ &= 6\Delta^{-3} \sup_{z \in \mathcal{C}(t,\Delta)} \left| \log\left(1 + \frac{1}{S^\bullet(T(v),e^z)}\right)\right|. \end{aligned}$$

Hence,

$$\begin{aligned} \sup_{|t|\leq\Delta} |f'''(T(v),t)| &\leq 6\Delta^{-3} \sup_{|z|\leq 2\Delta} \left| \log\left(1 + \frac{1}{S^\bullet(T(v),e^z)}\right)\right| \\ &\leq 6\Delta^{-3} \sup_{|z|\leq \frac{\delta}{P}} \left| \log\left(1 + \frac{1}{S^\bullet(T(v),e^z)}\right)\right|. \end{aligned}$$

Now we can apply Lemma 4.1 to estimate $|S^\bullet(T(v),e^z)|$ for $|z| \leq \frac{\delta}{P}$ (recall that $|S^\bullet(T(v),e^z)|$ is bounded below by a constant greater than 1 in this case). We obtain

$$\sup_{|t|\leq\Delta} |f'''(T(v),t)| \ll \Delta^{-3} e^{-c_0 L(T(v))}.$$

The assumption $v \notin B$ implies $|T(v)| > P^{1+\gamma}$. In addition, we know that the lengths of all branchless paths in $T(v)$ are bounded above by P since $T(v)$ is a branch of T, so by Lemma 3.1 we have

$$L(T(v)) \geq \frac{|T(v)|}{2P} > \frac{1}{2}P^\gamma.$$

Therefore,

$$\sup_{|t|\leq\Delta} |f'''(T(v),t)| \ll \Delta^{-3} e^{-\frac{c_0}{2}P^\gamma} \ll P^{3(1+\gamma)} e^{-\frac{c_0}{2}P^\gamma},$$

which is bounded above by a constant (that depends on our choice of γ, but is independent of P). Thus

$$\sum_{v \in \mathcal{I}(T)\setminus B} \sup_{|t|\leq\Delta} |f'''(T(v),t)| \ll |T|. \tag{4.14}$$

(ii) If $v \in \mathcal{I}(T) \cap B$, then the function $f(T(v),z)$ is analytic in the closed disk centred at zero with a slightly larger radius $\frac{\delta}{|T(v)|}$ (this is greater than Δ since $v \in B$, i.e. $|T(v)| \leq P^{1+\gamma}$ by definition). To see this, we can use the same argument that gave us (4.5): for $|z| \leq \frac{\delta}{|T(v)|}$, we have

$$|S^\bullet(T(v),e^z)| \geq e^{-\delta}\cos(\delta) S^\bullet(T(v)), \tag{4.15}$$

which in turn is strictly greater than 1 by the choice we have made for δ and by the fact that $S^\bullet(T(v)) \geq 2$ since $v \in \mathcal{I}(T)$. Now for any t such that $|t| \leq \Delta$, let $\mathcal{C}(t,R)$ be the circle centred at t with radius $R = \frac{\delta}{2|T(v)|}$. Note that $\mathcal{C}(t,R)$ lies in the region of analyticity of the function $f(T(v),z)$, since if $z \in \mathcal{C}(t,R)$, we have

$$|z| \leq |t| + |z-t| \leq \frac{\delta}{2P^{1+\gamma}} + \frac{\delta}{2|T(v)|} \leq \frac{\delta}{|T(v)|}.$$

Thus, by Cauchy's integral formula, we have

$$f'''(T(v),t) = \frac{3!}{2\pi i}\oint_{\mathcal{C}(t,R)} \frac{f(T(v),z)-z}{(z-t)^4}dz,$$

from which we deduce the bound

$$|f'''(T(v),t)| \leq 48\,\delta^{-3}|T(v)|^3 \sup_{z\in\mathcal{C}(t,R)} |f(T(v),z)-z|.$$

The right side can be estimated using (4.15):

$$\sup_{z\in\mathcal{C}(t,R)} |f(T(v),z)-z| = \sup_{z\in\mathcal{C}(t,R)} \left|\log\left(1+\frac{1}{S^\bullet(T(v),e^z)}\right)\right| \ll \frac{1}{S^\bullet(T(v))}$$

uniformly for $|t| \leq \Delta$. Therefore, we obtain

$$\sup_{|t|\leq\Delta} |f'''(T(v),t)| \ll \frac{|T(v)|^3}{S^\bullet(T(v))} \tag{4.16}$$

for $v \in \mathcal{I}(T) \cap B$.

The lemma follows by combining (4.14) and (4.16). □

Let $\mathcal{P}(v)$ denote the set of all vertices on the path in T from v to the root v_0 (excluding v, but including v_0). We define

$$\eta(v) = \begin{cases} 1 & \text{if } v = v_0, \\ \prod_{w\in\mathcal{P}(v)} \frac{S^\bullet(T(w))}{1+S^\bullet(T(w))} & \text{otherwise.} \end{cases}$$

Lemma 4.4. *Suppose that $L(T) \geq \lambda|T|$ for some fixed constant $\lambda > 0$. We have*

$$\sigma^2(T) \gg |T| + \sum_{v\in\mathcal{I}(T)\cap B} \eta(v)\frac{|T(v)|^2}{S^\bullet(T(v))}. \tag{4.17}$$

The implied constant only depends on λ.

Proof. Iterating (4.11) (and noting that $\sigma^2(T) = \frac{1}{4} > 0$ if $|T| = 1$), we obtain

$$\begin{aligned}\sigma^2(T) &\gg \sum_{v\in\mathcal{L}(T)} \eta(v) + \sum_{v\in\mathcal{I}(T)} \eta(v)\frac{\mu(T(v))^2}{S^\bullet(T(v))} \\ &\geq \sum_{v\in\mathcal{L}(T)} \eta(v) + \sum_{v\in\mathcal{I}(T)\cap B} \eta(v)\frac{\mu(T(v))^2}{S^\bullet(T(v))}.\end{aligned}$$

It was shown by Jamison in [5] that the average cardinality of a subtree containing the root of a rooted tree of order n is at least $(n+1)/2$, so

$$\frac{S_u^\bullet(T(v))}{S^\bullet(T(v))} \geq \frac{|T(v)|+1}{2},$$

which implies that

$$\mu(T(v)) = \frac{S_u^\bullet(T(v))}{S^\bullet(T(v))} \cdot \frac{1}{1+S^\bullet(T(v))^{-1}} \geq \frac{|T(v)|+1}{2} \cdot \frac{1}{1+|T(v)|^{-1}} = \frac{|T(v)|}{2}.$$

So it remains to show that

$$\sum_{v \in \mathcal{L}(T)} \eta(v) \gg |T|. \tag{4.18}$$

To this end, we define a set of "exceptional branches" in such a way that $\eta(v)$ is bounded below by an explicit constant unless v lies in one of these branches. Choose two constants $\beta \in (0,1)$ and $K > (\lambda/2)^{-1/\beta}$, and let $z_1, z_2, \ldots, z_M$ be the vertices that satisfy

$$L(T(z_j)) \leq |T(z_j)|^{1-\beta} \text{ and } |T(z_j)| \geq K$$

and are closest to the root with this property (in the sense that no vertex on the path from the root to z_j satisfies both inequalities). We set $E_j = T(z_j)$ and let E be the union of all E_j. Now take any leaf v that does not lie in E, and let v' be its ancestor closest to the root that satisfies $|T(v')| < K$ (possibly, $v' = v$). Now we split the product that defines $\eta(v)$ as follows:

$$\begin{aligned}
\eta(v) &= \prod_{w \in \mathcal{P}(v)} \frac{1}{1+S^\bullet(T(w))^{-1}} \\
&= \prod_{w \in \mathcal{P}(v) \setminus \mathcal{P}(v')} \frac{1}{1+S^\bullet(T(w))^{-1}} \prod_{w \in \mathcal{P}(v')} \frac{1}{1+S^\bullet(T(w))^{-1}}.
\end{aligned}$$

There are at most K vertices in $\mathcal{P}(v) \setminus \mathcal{P}(v')$ since the set $\mathcal{P}(v) \setminus \mathcal{P}(v')$ lies entirely in $T(v')$. In addition, for every w we have the trivial bound $1+S^\bullet(T(w))^{-1} \leq 2$. Therefore,

$$\prod_{w \in \mathcal{P}(v) \setminus \mathcal{P}(v')} \frac{1}{1+S^\bullet(T(w))^{-1}} \geq 2^{-K}.$$

Furthermore, for every vertex w on the path from the root to v', we must have $|T(w)| \geq K$ by the choice of v', and $L(T(w)) > |T(w)|^{1-\beta}$ since v does not lie in E. Recall from Lemma 3.1 that $S^\bullet(T(w)) \geq 2^{L(T(w))}$. Hence we have

$$\begin{aligned}
\eta(v) &\geq 2^{-K} \prod_{w \in \mathcal{P}(v')} \left(1+2^{-L(T(w))}\right)^{-1} \\
&\geq 2^{-K} \prod_{w \in \mathcal{P}(v')} \left(1+2^{-|T(w)|^{1-\beta}}\right)^{-1} \\
&\geq 2^{-K} \prod_{j \geq K} \left(1+2^{-j^{1-\beta}}\right)^{-1}.
\end{aligned}$$

Note that the infinite product converges. So we can deduce that $\eta(v)$ is bounded below by a constant that only depends on β and K unless $v \in E$. Consequently,

$$\sum_{v \in \mathcal{L}(T)} \eta(v) \gg |\mathcal{L}(T) \setminus E|. \tag{4.19}$$

We will see that E cannot contain more than half of the leaves. We may assume that E is non-empty, for otherwise this statement is trivial. So let us assume that

$$\sum_{j=1}^{M} L(E_j) > \frac{L(T)}{2} \geq \frac{\lambda}{2}|T|.$$

By the definition of the branches $E_1, E_2, \ldots, E_M$, this gives us

$$\sum_{j=1}^{M} |E_j|^{1-\beta} \geq \frac{\lambda}{2}|T|.$$

On the other hand, since $E_1, E_2, \ldots, E_M$ are pairwise disjoint, we also have

$$\sum_{j=1}^{M} |E_j| \leq |T|.$$

Since we are assuming that E is non-empty, we have $M \neq 0$. Hence, by Jensen's inequality,

$$\frac{\lambda}{2}|T| \leq \sum_{j=1}^{M} |E_j|^{1-\beta} \leq M\left(\frac{\sum_{j=1}^{M}|E_j|}{M}\right)^{1-\beta} \leq M\left(\frac{|T|}{M}\right)^{1-\beta}.$$

It follows that

$$M \geq (\lambda/2)^{1/\beta}|T|.$$

On the other hand, each E_j contains at least K vertices, so we have

$$|T| \geq \sum_{j=1}^{M} |E_j| \geq MK.$$

Combining the last two inequalities, we obtain

$$K \leq (\lambda/2)^{-1/\beta}, \tag{4.20}$$

which contradicts the choice of K. This means that $|E| \leq L(T)/2$, so (4.19) finally yields

$$\sum_{v \in \mathcal{L}(T)} \eta(v) \gg L(T) \gg |T|,$$

which completes the proof. Note that the implied constant does indeed only depend on λ (and our choice of β and K, which was arbitrary). $\square$

To make use of the previous lemma, we also need to bound $\eta(v)$ from below for $v \in \mathcal{I}(T) \cap B$, which is achieved by the following lemma:

Lemma 4.5. *For every vertex $v \in T$ and every vertex $v' \in \mathcal{P}(v)$, we have*

$$\eta(v) \geq \eta(v')\frac{|T(v)|}{2|T(v')|}.$$

Proof. The statement is void if v is the root v_0, so we assume from now on that v is not the root. Let $v' = w_0, w_1, w_2, \ldots, w_k = v$ be the vertices of the path connecting v' and v (which form part of the path connecting v_0 and v). By definition, we have

$$\frac{\eta(v)}{\eta(v')} = \prod_{j=0}^{k-1} \frac{S^\bullet(T(w_j))}{1 + S^\bullet(T(w_j))}.$$

Clearly, $S^\bullet(T(w_j)) \geq 1 + S^\bullet(T(w_{j+1}))$ for $j = 0, 1, \ldots, k-1$; iterating further, we obtain

$$S^\bullet(T(w_j)) \geq k - j + S^\bullet(T(v)).$$

So we have, for $j = 0, 1, \ldots, k-1$,

$$\frac{S^\bullet(T(w_j))}{1 + S^\bullet(T(w_j))} \geq \frac{k - j + S^\bullet(T(v))}{1 + k - j + S^\bullet(T(v))},$$

and it follows that

$$\frac{\eta(v)}{\eta(v')} \geq \prod_{j=0}^{k-1} \frac{k - j + S^\bullet(T(v))}{1 + k - j + S^\bullet(T(v))} = \frac{1 + S^\bullet(T(v))}{1 + k + S^\bullet(T(v))} \geq \frac{S^\bullet(T(v))}{k + S^\bullet(T(v))}.$$

Now we consider two cases:

(i) First, if $S^\bullet(T(v)) \geq k$ then

$$\frac{\eta(v)}{\eta(v')} \geq \frac{S^\bullet(T(v))}{k + S^\bullet(T(v))} \geq \frac{1}{2} \geq \frac{|T(v)|}{2|T(v')|}.$$

(ii) Otherwise, if $S^\bullet(T(v)) < k$, then

$$\frac{\eta(v)}{\eta(v')} \geq \frac{S^\bullet(T(v))}{2k} \geq \frac{|T(v)|}{2|T(v')|}.$$

The last inequality holds because $S^\bullet(T(v)) \geq |T(v)|$ and $|T(v')| > k$ (the latter since $T(v')$ contains the $k+1$ vertices $w_0, w_1, \ldots, w_k$).

Hence, the lemma follows. □

The bound on $\eta(v)$ is now used to bound $r(T)$ in terms of $\sigma^2(T)$.

Lemma 4.6. *Suppose that $L(T) \geq \lambda|T|$ for a fixed constant $\lambda > 0$. We have*

$$r(T) \ll P^{1+\gamma}\sigma^2(T).$$

The constant implied in this estimate depends on λ and γ, but nothing else, in particular not on P.

Proof. Recall that B consists of all vertices w for which $T(w) \leq P^{1+\gamma}$. We write B as the disjoint union of branches $T(y_1), T(y_2), \ldots$. If v lies on the path connecting the root v_0 and one of the y_j, then by definition we have

$$|T(v)| > P^{1+\gamma}.$$

By Lemma 3.1, this implies

$$L(T(v)) \geq \frac{|T(v)|}{2P} \geq \frac{|T(v)|}{2|T(v)|^{1/(1+\gamma)}} = \frac{1}{2}|T(v)|^{\gamma/(1+\gamma)}.$$

Using this inequality, we can argue as in the proof of (4.19) that $\eta(y_j)$ is bounded below by an absolute constant for every j. Applying Lemma 4.5, we deduce that for $v \in T(y_j)$,

$$\eta(v) \gg \frac{|T(v)|}{|T(y_j)|} \geq \frac{|T(v)|}{P^{1+\gamma}}.$$

Therefore,

$$\begin{aligned}\sum_{v\in\mathcal{I}(T)\cap B} \eta(v)\frac{|T(v)|^2}{S^\bullet(T(v))} &= \sum_j \sum_{v\in\mathcal{I}(T(y_j))} \eta(v)\frac{|T(v)|^2}{S^\bullet(T(v))} \\ &\gg P^{-1-\gamma} \sum_{v\in\mathcal{I}(T)\cap B} \frac{|T(v)|^3}{S^\bullet(T(v))}.\end{aligned}$$

The desired statement now follows from Lemma 4.3 and Lemma 4.4. □

As a consequence of Lemma 4.6, we now obtain the required information on the Taylor expansion of $F(T,t)$.

Proposition 4.7. *Let $\delta = 0.001$ be as previously defined, and let $\lambda, \gamma > 0$ be fixed constants. If $L(T) \geq \lambda|T|$, then we have*

$$F(T,t) = F(T,0) + \mu(T)t + \sigma^2(T)\frac{t^2}{2} + \mathcal{O}\left(P(T)^{1+\gamma}\sigma^2(T)|t|^3\right) \tag{4.21}$$

for $|t| \leq \frac{\delta}{2P(T)^{1+\gamma}}$, where the constant implied in the $\mathcal{O}$-term only depends on λ and γ.

Proof. This statement follows directly from Lemma 4.6 and (4.10). □

4.2 Central limit theorem

We are now ready to prove the central limit theorem for the order distribution of subtrees.

Theorem 4.8. *Let $T_1, T_2, \ldots$ be a sequence of rooted trees such that $|T_n| \to \infty$ as $n \to \infty$ and the following two conditions are satisfied for all sufficiently large n:*

(i) $P(T_n) \leq |T_n|^{\frac{1}{2}-\epsilon}$ *for some constant $\epsilon > 0$,*

(ii) $L(T_n) \geq \lambda|T_n|$ *for some constant $\lambda > 0$.*

Then the distribution of the random variable $X_n^\bullet$, defined as the order of a randomly chosen subtree of T_n containing the root, is asymptotically Gaussian. More precisely, if $\Phi_n^\bullet(x)$ denotes the distribution function of the renormalised random variable

$$Y_n^\bullet = \frac{X_n^\bullet - \mu(T_n)}{\sigma(T_n)},$$

then we have the following estimate for the speed of convergence:

$$\sup_{x\in\mathbb{R}} \left| \Phi_n^\bullet(x) - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-t^2/2} dt \right| = \mathcal{O}\left(|T_n|^{-\alpha}\right) \tag{4.22}$$

for every positive constant $\alpha < \epsilon/3$. The constant implied in the $\mathcal{O}$-term only depends on α and λ.

Proof. For ease of notation, we drop the dependence on n. Recall that the moment generating function of $X^\bullet = X_n^\bullet$ is

$$\mathbb{E}\left(e^{tX^\bullet}\right) = \frac{S^\bullet(T, e^t)}{S^\bullet(T)}.$$

Instead of working directly with $X^\bullet$, we use the modified random variable $X^* = X_n^*$ that also includes an empty dummy subtree. The moment generating function of this random variable is given by

$$\mathbb{E}\left(e^{tX^*}\right) = \frac{1 + S^\bullet(T, e^t)}{1 + S^\bullet(T)},$$

and if $Y^* = Y_n^* = (X_n^* - \mu(T_n))/\sigma(T_n)$ is the associated renormalised random variable, it is easy to see that the distribution functions $\Phi^\bullet$ of $Y^\bullet$ and Φ^* of Y^* differ only by very little:

$$|\Phi^\bullet(x) - \Phi^*(x)| \le \frac{1}{1 + S^\bullet(T)} \tag{4.23}$$

for all $x \in \mathbb{R}$, so it is sufficient to prove the estimate for Φ^* instead of $\Phi^\bullet$. The condition $L(T) \ge \lambda|T|$ implies

$$\sigma^2(T) \gg |T|$$

by Lemma 4.4, in particular $\sigma(T) \to \infty$ as $|T| \to \infty$. The moment generating function of the renormalised random variable Y^* is

$$\begin{aligned}\mathbb{E}\left(e^{tY^*}\right) &= e^{-\mu(T)t/\sigma(T)} \mathbb{E}\left(e^{tX^*/\sigma(T)}\right) \\ &= \exp\left(-\frac{\mu(T)t}{\sigma(T)} + F\left(T, \frac{t}{\sigma(T)}\right) - F(T, 0)\right).\end{aligned}$$

The expansion in Proposition 4.7 gives us

$$F\left(T, \frac{t}{\sigma(T)}\right) = F(T, 0) + \frac{\mu(T)}{\sigma(T)} t + \frac{t^2}{2} + \mathcal{O}\left(\frac{P(T)^{1+\gamma}}{\sqrt{|T|}} |t|^3\right)$$

and thus

$$\mathbb{E}\left(e^{tY^*}\right) = \exp\left(\frac{t^2}{2} + \mathcal{O}\left(\frac{P(T)^{1+\gamma}}{\sqrt{|T|}} |t|^3\right)\right) \tag{4.24}$$

if $|t| \leq \frac{\delta\sigma(T)}{2P(T)^{1+\gamma}}$. Note that we can choose γ freely here (the choice affects the $\mathcal{O}$-constant, though). The condition $P(T) \leq |T|^{\frac{1}{2}-\epsilon}$ allows us to choose γ in such a way that

$$\frac{P(T)^{1+\gamma}}{\sqrt{|T|}} \to 0.$$

Therefore,

$$\mathbb{E}\left(e^{tY^*}\right) \longrightarrow e^{t^2/2}$$

for any fixed t as $n \to \infty$, which would already prove a central limit theorem. For the precise error estimate, we use the following Berry-Esseen type inequality [10, Theorem 5.1]:

$$\sup_{x\in\mathbb{R}}\left|\Phi^*(x) - \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-t^2/2}dt\right| \leq c_1\int_{-M}^{M}\left|\frac{\varphi_T(t) - e^{-t^2/2}}{t}\right|dt + \frac{c_2}{M}$$

for certain absolute constants c_1, c_2, where

$$\varphi_T(t) = \int_{-\infty}^{\infty} e^{ity}d\Phi^*(y) = \mathbb{E}\left(e^{itY^*}\right).$$

In view of (4.24), we have

$$\left|\varphi_T(t) - e^{-t^2/2}\right| \ll |t|^3 e^{-t^2/2}\frac{P(T)^{1+\gamma}}{\sqrt{|T|}}$$

if $|t|^3 = \mathcal{O}\Big(\sqrt{|T|}/P(T)^{1+\gamma}\Big)$. Therefore,

$$\sup_{x\in\mathbb{R}}\left|\Phi^*(x) - \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-t^2/2}dt\right| = \mathcal{O}\left(\frac{P(T)^{1+\gamma}}{\sqrt{|T|}} + \frac{1}{M}\right)$$

for any M satisfying $M^3 = \mathcal{O}\Big(\sqrt{|T|}/P(T)^{1+\gamma}\Big)$. We choose

$$M = \left(\frac{\sqrt{|T|}}{P(T)^{1+\gamma}}\right)^{1/3}$$

and γ in such a way that

$$\frac{1}{M} = \left(\frac{P(T)^{1+\gamma}}{\sqrt{|T|}}\right)^{1/3} \leq \left(\frac{|T|^{(1+\gamma)(1/2-\epsilon)}}{\sqrt{|T|}}\right)^{1/3} = |T|^{\gamma(1-2\epsilon)/6-\epsilon/3} \leq |T|^{-\alpha}.$$

Note finally that the difference between $\Phi^\bullet(x)$ and $\Phi^*(x)$ is uniformly bounded above by $S^\bullet(T)^{-1}$ in view of (4.23). Since $S^\bullet(T) \geq |T| \geq |T|^\alpha$, this completes the proof. $\square$

4.3 Local limit theorem

Now that we have established a central limit theorem, it is natural to ask whether a local limit theorem for single coefficients of $S^\bullet(T,u)$ also holds. To be precise, given a sequence of rooted trees $T_1, T_2, \dots$ satisfying both properties of Theorem 4.8, can we give an estimate for the number of subtrees of order k, for values of k around the mean $\mu(T_n)$? In this section, we show that it is indeed possible to obtain such a result. Before we come to the proof, an estimate for $|S^\bullet(T,u)|$ when u lies on the unit circle is required. This is precisely what we state in the next lemma.

Lemma 4.9. *Let $\lambda, \gamma > 0$ be fixed constants, and suppose that $L(T) \geq \lambda|T|$. There exist constants δ_1, c_3, c_4 depending on λ, γ such that, with*

$$\Delta_1 = \frac{\delta_1}{2P(T)^{1+\gamma}},$$

we have

$$\frac{|1+S^\bullet(T,e^{it})|}{1+S^\bullet(T)} \leq \begin{cases} e^{-c_3 t^2 \sigma(T)^2} & \text{if } t \in [-\Delta_1, \Delta_1], \\ e^{-c_4 t^2 |T|} & \text{for all } t \in [-\pi, \pi]. \end{cases}$$

Proof. The bound corresponding to $|t| \leq \Delta_1$ follows easily from Proposition 4.7 for sufficiently small $\delta_1 \leq \delta \,(= 0.001)$. Thus it suffices to prove the second bound.

Recall that we have

$$S^\bullet(T,e^{it}) = \prod_{j=1}^{d} (1 + S^\bullet(T(v_j), e^{it}))$$

if $v_1, v_2, \dots, v_d$ are the root's children, and consequently

$$\left|1+S^\bullet(T,e^{it})\right| \leq 1 + \prod_{j=1}^{d} \left|1+S^\bullet(T(v_j),e^{it})\right|. \tag{4.25}$$

This motivates the definition of a polynomial $R(T,x)$ (for positive real x) that is similar to $S^\bullet(T,u)$: it is given by $R(T,x) = x$ for $|T| = 1$ and the recursion

$$R(T(v),x) = 1 + \prod_{j=1}^{d} R(T(v_j),x). \tag{4.26}$$

In view of (4.25), we have

$$\left|1+S^\bullet(T,e^{it})\right| \leq R(T, |1+e^{it}|) \tag{4.27}$$

and $1 + S^\bullet(T,1) = 1 + S^\bullet(T) = R(T,2)$. Note that $R(T,x)$ is a polynomial of degree $L(T)$ with positive coefficients. Therefore, it is a strictly increasing function of x, and it admits the trivial lower bound

$$R(T,x) \geq x^{L(T)} \tag{4.28}$$

for all positive x. We also define the function $G(T,x) = \log(R(T,x))$, which satisfies the recurrence

$$G(T,x) = \sum_{j=1}^{d} G(T(v_j),x) - \log\left(1 - \frac{1}{R(T,x)}\right), \tag{4.29}$$

where $G(T,x) = \log x$ (and thus $G'(T,x) = x^{-1}$) if T only has one vertex. In order to estimate $S^\bullet(T, e^{it})$ by means of (4.27), we establish a bound for the difference $G(T,2) - G(T,x)$ for x in the interval $[\sqrt{2}, 2]$. By the mean value theorem, there exists some $y \in [x,2]$ such that

$$G(T,2) - G(T,x) = (2-x)G'(T,y).$$

It is not hard to see from (4.26) that the derivative $G'(T,y)$ satisfies

$$G'(T,y) = \frac{R(T,y)-1}{R(T,y)} \sum_{j=1}^{d} G'(T(v_j),y). \tag{4.30}$$

We essentially use the same argument as in the proof of Lemma 4.4 to bound $G'(T,y)$ from below. Iterating (4.30) starting from the root of T down to the leaves, we obtain, with

$$\xi(v,y) = \begin{cases} 1 & \text{if } v \text{ is the root of } T, \\ \prod_{w \in \mathcal{P}(v)} \frac{R(T(w),y)-1}{R(T(w),y)} & \text{otherwise,} \end{cases}$$

that

$$G'(T,y) = y^{-1} \sum_{v \in \mathcal{L}(T)} \xi(v,y).$$

Recall that we are assuming $x \in [\sqrt{2}, 2]$ and thus also $y \in [\sqrt{2}, 2]$. Since $R(T(v),y) \geq y^{L(T(v))} \geq 2^{L(T(v))/2}$, the same argument that gave us (4.18) now yields

$$G'(T,y) \geq \frac{1}{2} \sum_{v \in \mathcal{L}(T)} \xi(v,y) \gg \sum_{v \in \mathcal{L}(T)} \xi(v,y) \gg |T|.$$

This implies that there exists a positive constant c_5 such that

$$G(T,x) - G(T,2) \leq c_5(x-2)|T|.$$

Equivalently, if $\sqrt{2} \leq x \leq 2$, then

$$\frac{R(T,x)}{R(T,2)} \leq e^{c_5(x-2)|T|}. \tag{4.31}$$

To complete the proof, recall that (by (4.27)) $|1 + S^\bullet(T, e^{it})|$ is bounded above by $R(T, |1+e^{it}|)$ while $R(T,2) = 1 + S^\bullet(T)$. For $|t| \leq \pi/2$, we have $|1 + e^{it}| \geq \sqrt{2}$ and

$$|1+e^{it}| - 2 = 2(\cos \tfrac{t}{2} - 1) \leq -\frac{2}{\pi^2} t^2,$$

thus

$$\frac{|1 + S^\bullet(T, e^{it})|}{1 + S^\bullet(T)} \leq \frac{R(T, |1+e^{it}|)}{R(T,2)} \leq e^{-(2c_5/\pi^2)t^2|T|} \leq e^{-c_4 t^2 |T|}$$

if we choose $c_4 \leq 2c_5/\pi^2$. For the case that $|t| \geq \pi/2$, we simply note that $R(T,x)$ is an increasing function of x, so that

$$\frac{|1 + S^\bullet(T, e^{it})|}{1 + S^\bullet(T)} \leq \frac{R(T, |1+e^{it}|)}{R(T,2)} \leq \frac{R(T, \sqrt{2})}{R(T,2)} \leq e^{-c_5(2-\sqrt{2})|T|} \leq e^{-c_4 t^2 |T|}$$

if we choose $c_4 \leq (2-\sqrt{2})c_5/\pi^2$. This completes the proof. $\square$

Now we have all required ingredients for a local limit theorem. In the following, we let $s_k^\bullet(T)$ denote the number of subtrees of order k in T that contain the root, so that

$$S^\bullet(T,u) = \sum_{k=1}^{|T|} s_k^\bullet(T)u^k.$$

Theorem 4.10. *Suppose that the sequence $T_1, T_2, \ldots$ of rooted trees satisfies the conditions of Theorem 4.8. If $k = \mu(T_n) + x\sigma(T_n)$, then we have*

$$\frac{s_k^\bullet(T_n)}{S^\bullet(T_n)} \sim \frac{e^{-x^2/2}}{\sqrt{2\pi}\sigma(T_n)},$$

uniformly for x in any fixed compact interval as $n \to \infty$.

Proof. Once again, we drop the index n for convenience. By Cauchy's integral formula, the number $s_k^\bullet(T)$ can be expressed as

$$s_k^\bullet(T) = \frac{1}{2\pi i}\oint_{C(0,1)} \Big(1 + S^\bullet(T,z)\Big)\frac{dz}{z^{k+1}},$$

where $C(0,1)$ is the unit circle. If we set $z = e^{it}$, then we obtain

$$s_k^\bullet(T) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \Big(1 + S^\bullet(T,e^{it})\Big)e^{-ikt}\,dt.$$

Choose $\gamma > 0$ and $\kappa > 0$ in such a way that $\gamma/2 + 3\kappa < \epsilon$, and set $M = |T|^\kappa/\sigma(T)$. We split the integral into two parts: the central part

$$\frac{1}{2\pi}\int_{-M}^{M} \Big(1 + S^\bullet(T,e^{it})\Big)e^{-ikt}dt,$$

and the rest. Recall that we are assuming $P(T) \leq |T|^{1/2-\epsilon}$ and that we have already established $\sigma(T)^2 \gg |T|$. Since

$$\frac{\Delta_1}{M} = \frac{\delta_1\sigma(T)}{2P(T)^{1+\gamma}|T|^\kappa} \gg |T|^{1/2-\kappa-(1/2-\epsilon)(1+\gamma)} \gg |T|^{\epsilon-\gamma/2-\kappa}$$

is greater than 1 for sufficiently large $|T|$, we have $M \leq \Delta_1 = \frac{\delta_1}{2P(T)^{1+\gamma}}$, so we can apply Proposition 4.7, which gives us, for $|t| \leq M$,

$$\begin{aligned} 1 + S^\bullet(T,e^{it}) &= \exp(F(T,it)) \\ &= \exp\Big(F(T,0) + i\mu(T)t - \sigma^2(T)\frac{t^2}{2} + \mathcal{O}\big(|T|^{3\kappa+(1/2-\epsilon)(1+\gamma)-1/2}\big)\Big) \\ &= \exp\Big(F(T,0) + i\mu(T)t - \sigma^2(T)\frac{t^2}{2}\Big)\Big(1 + \mathcal{O}\big(|T|^{-(\epsilon-\gamma/2-3\kappa)}\big)\Big). \end{aligned}$$

We plug in $k = \mu(T) + x\sigma(T)$ and obtain

$$\begin{aligned} \frac{1}{2\pi}\int_{-M}^{M} \Big(1 + S^\bullet(T,e^{it})\Big)e^{-ikt}dt &= \frac{1}{2\pi}\int_{-M}^{M} e^{F(T,0)-ix\sigma(T)t-\sigma^2(T)t^2/2}\,dt \\ &\quad + \mathcal{O}\left(|T|^{-(\epsilon-\gamma/2-3\kappa)}\int_{-M}^{M} e^{F(T,0)-\sigma^2(T)t^2/2}\,dt\right). \end{aligned}$$

Since we have

$$\begin{aligned}\int_{-M}^{M} e^{F(T,0)-ix\sigma(T)t-\sigma^2(T)t^2/2}\,dt &= \tfrac{\sqrt{2\pi}}{\sigma(T)} e^{F(T,0)-x^2/2} + \mathcal{O}\Big(\int_{M}^{\infty} e^{F(T,0)-\sigma^2(T)t^2/2}\,dt\Big) \\ &= \tfrac{\sqrt{2\pi}}{\sigma(T)} e^{F(T,0)-x^2/2} + \mathcal{O}\Big(e^{F(T,0)-\sigma^2(T)M^2/2}\Big) \\ &= \tfrac{\sqrt{2\pi}}{\sigma(T)} e^{F(T,0)-x^2/2} + \mathcal{O}\Big(e^{F(T,0)-|T|^{2\kappa}/2}\Big),\end{aligned}$$

and $e^{F(T,0)} = 1 + S^\bullet(T)$, we end up with

$$\frac{1}{2\pi}\int_{-M}^{M}\Big(1+S^\bullet(T,e^{it})\Big)e^{-ikt}dt = S^\bullet(T)\frac{e^{-x^2/2}}{\sqrt{2\pi}\sigma(T)}\Big(1+\mathcal{O}\big(|T|^{-(\epsilon-\gamma/2-3\kappa)}\big)\Big).$$

For the remaining integrals, where $|t| \geq M$, we use the estimates from Lemma 4.9. For $|t| \leq \Delta_1 = \frac{\delta_1}{2P(T)^{1+\gamma}}$, they give us

$$\frac{|1+S^\bullet(T,e^{it})|}{1+S^\bullet(T)} \leq e^{-c_3M^2\sigma(T)^2} = e^{-c_3|T|^{2\kappa}},$$

and for $|t| \geq \Delta_1$, we get

$$\frac{|1+S^\bullet(T,e^{it})|}{1+S^\bullet(T)} \leq e^{-c_4\Delta_1^2|T|} \leq e^{-\delta_1^2c_4|T|^{1-(1-2\epsilon)(1+\gamma)}/4} \leq e^{-\delta_1^2c_4|T|^{2\epsilon-\gamma}/4}.$$

Since these decay faster than any power of T, the parts of the integral for which $|t| \geq M$ will only contribute to the error term. In summary, we have

$$\frac{s_k^\bullet(T)}{S^\bullet(T)} = \frac{e^{-x^2/2}}{\sqrt{2\pi}\sigma(T_n)}\Big(1+\mathcal{O}\big(|T|^{-(\epsilon-\gamma/2-3\kappa)}\big)\Big),$$

which completes the proof. □

Remark 4.11. Theorem 4.10 provides a positive answer to Question 1.1 in an asymptotic sense for large rooted trees (and as we will see in the following section, also unrooted trees) without vertices of degree 2, since both technical conditions are trivially satisfied in this case.

5 Unrooted trees

Now that we have established both a central and a local limit theorem for the number of subtrees containing the root of a rooted tree, we would like to carry the results over to unrooted trees as well. This is achieved by means of the following lemma, which guarantees the existence of a vertex that is contained in most subtrees:

Lemma 5.1. *For every tree T, there exists a vertex v of T such that the proportion of subtrees of T that do not contain v is at most $|T|2^{-L(T)/2}$.*

Proof. Let v be a vertex that minimises the sum of the distances to all leaves, i.e. the expression $\sum_{w\in\mathcal{L}(T)} d(v,w)$ attains its minimum (this is called a "leaf centroid" in [17], in analogy to the centroid). Let $T_1, T_2, \ldots, T_k$ be the branches of T, rooted at v, and $v_1, v_2, \ldots, v_k$ the corresponding neighbours of v. The important observation about v is that none of the branches can contain more than half of the leaves: if T_j contains more than $L(T)/2$ leaves, then we have

$$\sum_{w \in \mathcal{L}(T)} d(v,w) > \sum_{w \in \mathcal{L}(T)} d(v_j,w),$$

since $d(v_j,w) = d(v,w) - 1$ if w is in T_j, and $d(v_j,w) = d(v,w) + 1$ otherwise. This would contradict the choice of v.

Let τ be a subtree of T that does not contain v. It must then be completely contained in some branch T_j. It has a unique vertex closest to v, which we denote by w. We can associate $2^{|\mathcal{L}(T) \cap (T \setminus T_j)|} \geq 2^{L(T)/2}$ subtrees to τ that contain v, obtained by adding the path from w to v as well as all non-leaves not contained in T_j and any subset of the $|\mathcal{L}(T) \cap (T \setminus T_j)|$ leaves that do not lie in T_j. Finally, we root the resulting subtrees at w.

Let the total number of subtrees of T be denoted by $S(T)$ and the number of those subtrees not containing v by $S^\circ(T)$. The construction above yields at least $2^{L(T)/2}$ rooted subtrees of T associated with every subtree τ that does not contain v. The original tree τ can be recovered uniquely from such a tree σ: it consists of the root w of σ and all vertices for which the unique path from v passes through w. Thus our construction is an injection to the set of rooted subtrees of T (whose cardinality is clearly at most $|T|S(T)$), and we obtain the inequality

$$S^\circ(T) \cdot 2^{L(T)/2} \leq |T| \cdot S(T),$$

from which the statement of the lemma follows. □

Our main theorem now follows immediately both in the central and local version:

Theorem 5.2. *Let $T_1, T_2, \ldots$ be a sequence of trees such that $|T_n| \to \infty$ as $n \to \infty$ and the following two conditions are satisfied:*

(i) $P(T_n) \leq |T_n|^{\frac{1}{2}-\epsilon}$ for some constant $\epsilon > 0$,

(ii) $L(T_n) \geq \lambda |T_n|$ for some constant $\lambda > 0$.

Then the distribution of the random variable X_n, defined as the order of a randomly chosen subtree of T_n, is asymptotically Gaussian. More precisely, if $\Phi_n(x)$ denotes the distribution function of the renormalised random variable

$$Y_n = \frac{X_n - \mathbb{E}(X_n)}{\sqrt{\mathbb{V}(X_n)}},$$

then we have the following estimate for the speed of convergence:

$$\sup_{x \in \mathbb{R}} \left| \Phi_n(x) - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-t^2/2} dt \right| = \mathcal{O}\left(|T_n|^{-\alpha}\right), \tag{5.1}$$

for any positive constant $\alpha < \epsilon/3$. The constant implied in the $\mathcal{O}$-term only depends on α and λ. Moreover, if $k = \mathbb{E}(X_n) + x\sqrt{\mathbb{V}(X_n)} \in \mathbb{N}$, then we have the local limit theorem

$$\mathbb{P}(X_n = k) \sim \frac{e^{-x^2/2}}{\sqrt{2\pi \mathbb{V}(X_n)}},$$

uniformly for x in any fixed compact interval.

Proof. As in the proofs of Theorem 4.8 and Theorem 4.10, we suppress the dependence on n for ease of notation. Choose v as in Lemma 5.1, and let $X^{(v)}$ be the random variable defined as the order of a randomly selected subtree of T containing v. By Lemma 5.1, the total variation distance between the two random variables $X = X_n$ and $X^{(v)}$, which is defined as

$$\sup_A \left|\mathbb{P}(X^{(v)} \in A) - \mathbb{P}(X \in A)\right|,$$

is $\mathcal{O}(|T|/2^{L(T)/2})$. In view of our assumption on the number of leaves, this goes to 0 even at an exponential rate. Letting $\mu(T)$ and $\sigma^2(T)$ be defined as before for the tree T rooted at v, it is also easy to see by the same argument that $\mathbb{E}(X) = \mu(T) + \mathcal{O}(1)$ and $\mathbb{V}(X) = \sigma^2(T) + \mathcal{O}(1)$ (in fact, both error terms can be made exponentially small). The two statements now follow directly from Theorem 4.8 and Theorem 4.10. □

6 Random trees

The technical conditions of Theorems 4.8, 4.10 and 5.2 are not satisfied for all possible sequences of trees, but they do hold for "generic" (randomly chosen) trees. In fact, it was shown in [11] that the length of the longest branchless path of a random labelled tree of order n is concentrated around $\log n$ for large n (with a limit distribution of double exponential type), and the number of leaves of a random labelled tree of order n is concentrated around n/e (with a Gaussian limit distribution, see e.g. [2, Section 3.2.1]). Analogous statements (with different constants) hold for other families of random trees (e.g. random plane trees, random binary trees).

If T_n denotes a random labelled tree of order n for $n = 1, 2, \ldots$, then a simple application of the Borel-Cantelli Lemma shows that the conditions of Theorem 5.2 with arbitrary $\epsilon < \frac{1}{2}$ and $\lambda < \frac{1}{e}$ are satisfied for all but finitely many T_j almost surely (for both conditions, it is not difficult to obtain bounds for the probability that they are not satisfied that go to 0 faster than any power of n). Thus we obtain the following theorem:

Theorem 6.1. *Let $T_1, T_2, \ldots$ be a sequence of uniformly random labelled trees, where the order of T_n is n, let X_n denote the order of a randomly chosen subtree of T_n, and let Φ_n be the distribution function of the renormalised random variable*

$$\frac{X_n - \mathbb{E}(X_n)}{\sqrt{\mathbb{V}(X_n)}}.$$

We have

$$\sup_{x\in\mathbb{R}} \left|\Phi_n(x) - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-t^2/2} dt\right| \to 0$$

as $n \to \infty$ almost surely.

Informally, this means that the distribution of subtree orders is close to a Gaussian distribution for almost all trees. We remark that the average subtree order of a random labelled tree T_n of order n was shown to follow a Gaussian limit distribution itself (see [15] for details).

References

[1] Y. Alavi, P. J. Malde, A. J. Schwenk and P. Erdős, The vertex independence sequence of a graph is not constrained, *Congr. Numerantium* **58** (1987), 15–23, `http://www.mta.renyi.hu/~p_erdos/1987-33.pdf`.

[2] M. Drmota, *Random Trees*, Springer, Vienna, 2009, doi:10.1007/978-3-211-75357-6.

[3] C. D. Godsil, Matching behaviour is asymptotically normal, *Combinatorica* **1** (1981), 369–376, doi:10.1007/bf02579458.

[4] J. Haslegrave, Extremal results on average subtree density of series-reduced trees, *J. Combin. Theory Ser. B* **107** (2014), 26–41, doi:10.1016/j.jctb.2014.02.003.

[5] R. E. Jamison, On the average number of nodes in a subtree of a tree, *J. Combin. Theory Ser. B* **35** (1983), 207–223, doi:10.1016/0095-8956(83)90049-7.

[6] R. E. Jamison, Monotonicity of the mean order of subtrees, *J. Combin. Theory Ser. B* **37** (1984), 70–78, doi:10.1016/0095-8956(84)90046-7.

[7] R. E. Jamison, Mean size of subtrees of a tree, 2011, REGS in Combinatorics (University of Illinois), `http://www.math.uiuc.edu/~west/regs/meantree.html`.

[8] J. Kahn, A normal law for matchings, *Combinatorica* **20** (2000), 339–391, doi:10.1007/pl00009835.

[9] J. L. Martin, M. Morin and J. D. Wagner, On distinguishing trees by their chromatic symmetric functions, *J. Combin. Theory Ser. A* **115** (2008), 237–253, doi:10.1016/j.jcta.2007.05.008.

[10] V. V. Petrov, *Limit Theorems of Probability Theory*, volume 4 of *Oxford Studies in Probability*, The Clarendon Press, New York, 1995.

[11] H. Prodinger and S. Wagner, Bootstrapping and double-exponential limit laws, *Discrete Math. Theor. Comput. Sci.* **17** (2015), 123–144, `https://www.dmtcs.org/dmtcs-ojs/index.php/dmtcs/article/view/2781.1.html`.

[12] L. A. Székely and H. Wang, On subtrees of trees, *Adv. in Appl. Math.* **34** (2005), 138–155, doi:10.1016/j.aam.2004.07.002.

[13] L. A. Székely and H. Wang, Binary trees with the largest number of subtrees, *Discrete Appl. Math.* **155** (2007), 374–385, doi:10.1016/j.dam.2006.05.008.

[14] A. Vince and H. Wang, The average order of a subtree of a tree, *J. Combin. Theory Ser. B* **100** (2010), 161–170, doi:10.1016/j.jctb.2009.05.006.

[15] S. Wagner, Additive tree functionals with small toll functions and subtrees of random trees, in: N. Broutin and L. Devroye (eds.), *23rd International Meeting on Probabilistic, Combinatorial, and Asymptotic Methods for the Analysis of Algorithms (AofA'12)*, volume AQ of *DMTCS Proc.*, pp. 67–80, 2012, `https://hal.inria.fr/hal-01197234/`.

[16] S. Wagner and H. Wang, On the local and global means of subtree orders, *J. Graph Theory* **81** (2016), 154–166, doi:10.1002/jgt.21869.

[17] H. Wang, Centroid, leaf-centroid, and internal-centroid, *Graphs Combin.* **31** (2015), 783–793, doi:10.1007/s00373-013-1401-1.

[18] W. Yan and Y.-N. Yeh, Enumeration of subtrees of trees, *Theor. Comput. Sci.* **369** (2006), 256–268, doi:10.1016/j.tcs.2006.09.002.

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Abstract

We present a technique to produce arrangements of lines with nice properties. As an application, we construct (22_4) and (26_4) configurations of lines. Thus concerning the existence of geometric (n_4) configurations, only the case $n = 23$ remains open.

Keywords: Arrangement of lines, configuration of lines.

Math. Subj. Class.: 52C30

1 Enumerating arrangements

There are several ways to enumerate arrangements of lines in the real plane. For instance, one can enumerate all wiring diagrams and thus oriented matroids. However, without a very strong local condition on the cell structure, such an enumeration is feasible only for a small number of lines. In any case, most types of interesting arrangements of more than say 20 lines can probably not be enumerated completely (nowadays by a computer).

A much more promising method is (as already noted by many authors) to exploit symmetry. In fact, most relevant examples in the literature have a non-trivial symmetry group. Symmetry reduces the degrees of freedom considerably and allows us to compute examples with many more lines. The following (very simple) algorithm is a useful tool to produce "interesting" examples of arrangements with non-trivial symmetry group:

*I would like to thank J. Bokowski and V. Pilaud for calling my attention to the subject of (n_k) configurations.
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Algorithm 1.1. Look for matroids with property P which are realizable over $\mathbb{C}$.

Enumerate arrangements(q, P):

Input: a prime power q, a property P
Output: matroids of arrangements of lines in $\mathbb{CP}^2$ with P

1. Depending on P, choose a small set of lines $\mathcal{A}_0 \subseteq \mathbb{F}_q\mathbb{P}^2$ and an $n \in \mathbb{N}$.

2. For every group $H \leq \mathrm{PGL}_3(\mathbb{F}_q)$ with $|H| = n$, compute the orbit $\mathcal{A} := H\mathcal{A}_0$.

3. If $\mathcal{A}$ has property P, then compute its matroid M. Print M if it is realizable over $\mathbb{C}$.

Remark 1.2.

1. If q is not too big, then it is indeed possible to compute all the subgroups H with $|H| = n$. However, if q is too small, then only very few matroids M will be realizable in characteristic zero.

2. If we are looking for arrangements with $m = nk$ lines, then it is good to choose $\mathcal{A}_0$ with approximately k lines.

3. This algorithm mostly produces matroids that are not orientable. Thus it is a priori not the best method if one is searching for arrangements in the real projective plane. On the other hand, most "interesting" arrangements will define a matroid that is realizable over many finite fields, such that these matroids will certainly appear in the enumeration.

4. Realizing rank three matroids with a small number of lines, depending on the matroid maybe up to 70 lines, is not easy but works in most cases (see for example [4]).

2 (n_k) configurations of lines

A configuration of lines and points is an (n_k) *configuration* if it consists of n lines and n points, each of which is incident to exactly k of the other type. It is called *geometric* if these are points and lines in the real projective plane.

There are many results concerning geometric (n_4) configurations:

1. There exist geometric (n_4) configurations of lines if and only if $n \geq 18$ except possibly for $n \in \{19, 22, 23, 26, 37, 43\}$ [3, 5].

2. There is no geometric (19_4) configuration [1].

3. There exist geometric (37_4) and (43_4) configurations [2].

Thus for the existence of geometric (n_4) configurations, only the cases $n \in \{22, 23, 26\}$ were open. Using the above algorithm we can produce examples when n is 22 and 26.

We will denote both projective lines and points with coordinates $(a : b : c)$ since points and lines are dual to each other in the plane.

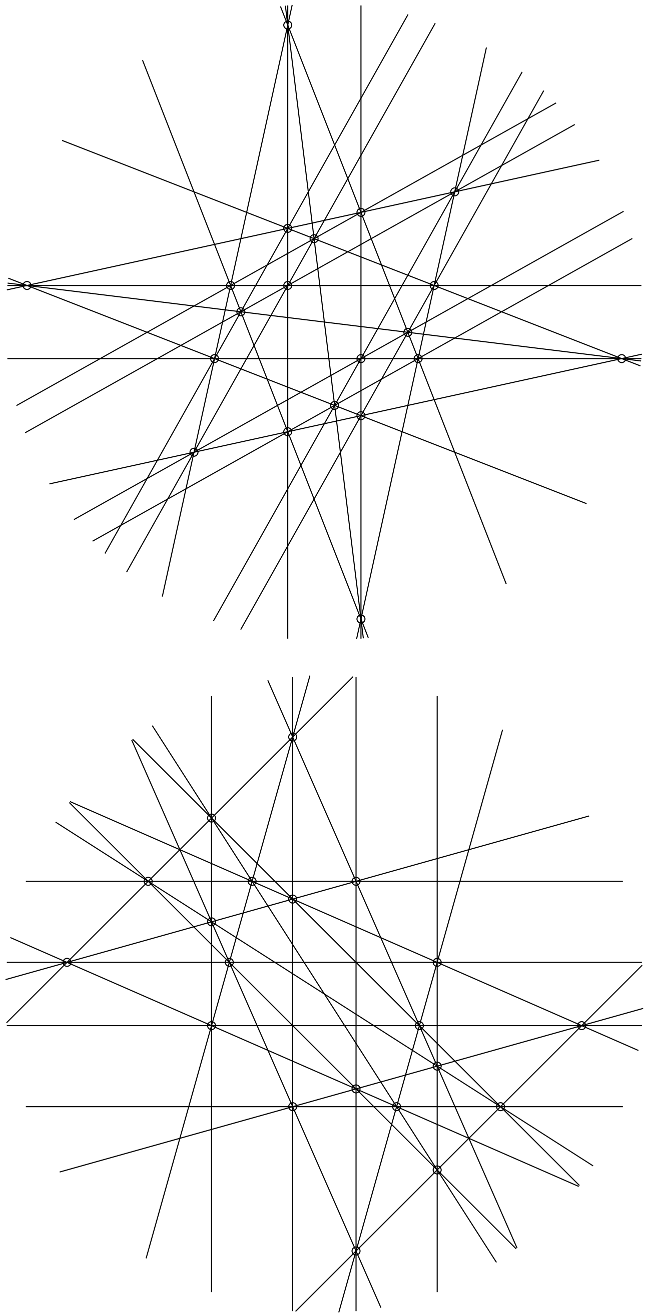

Figure 1: Two dual (22_4) configurations of lines $\left(w = \frac{-7+3\sqrt{17}}{2}\right)$.

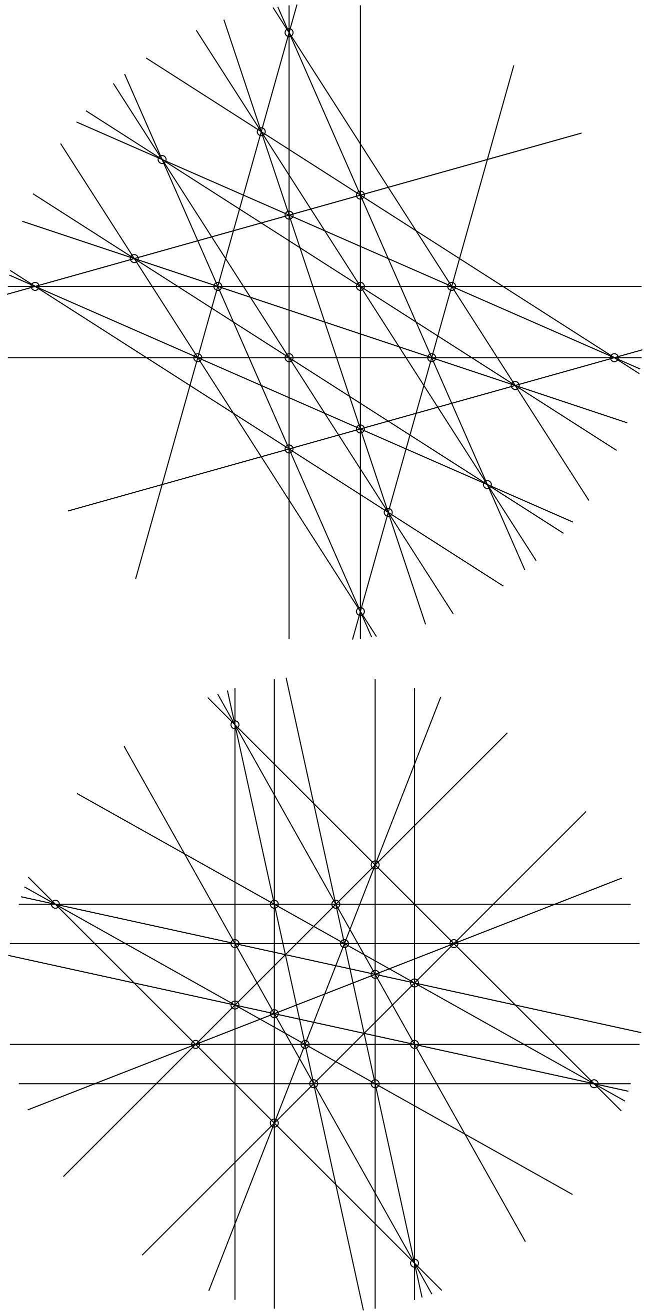

Figure 2: Two dual (22_4) configurations of lines $\left(w = \frac{-7-3\sqrt{17}}{2}\right)$.

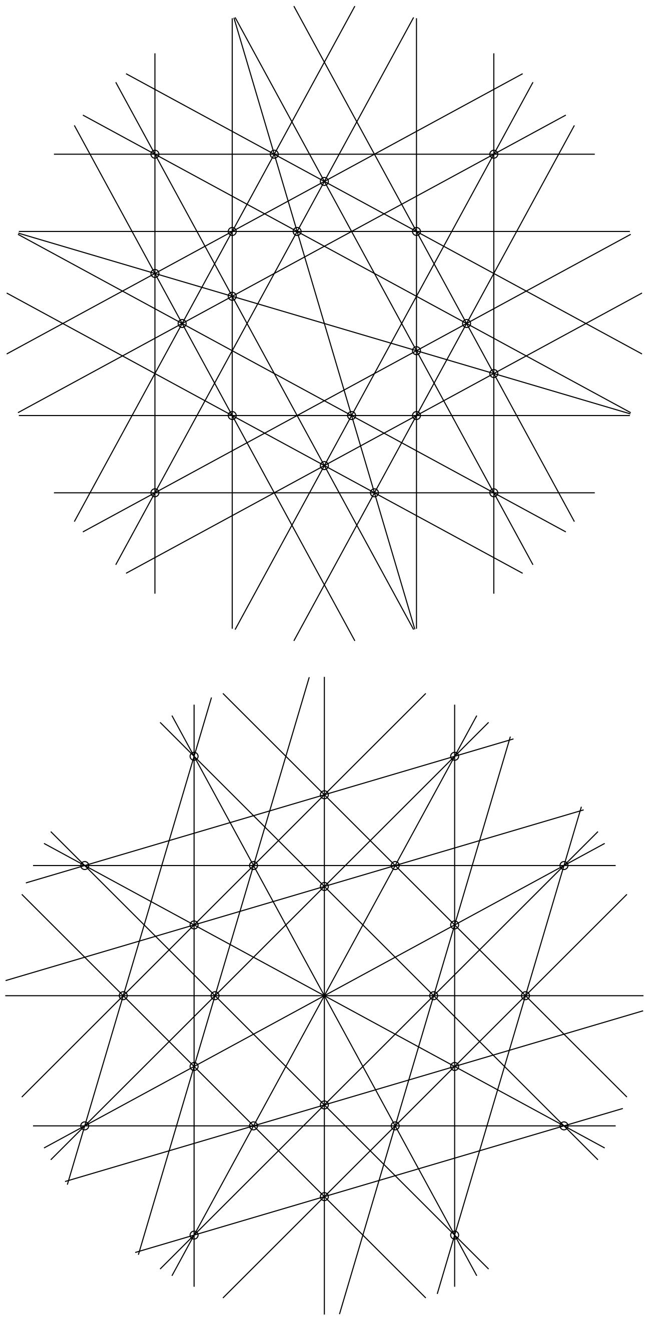

Figure 3: Two dual (26_4) configurations of lines.

2.1 (22_4) configurations

The key idea to obtain (n_4) configurations with the above algorithm is to choose an arrangement $\mathcal{A}_0$ which already has some points of multiplicity 4. This way, the orbit $\mathcal{A}$ is likely to have a large number of quadruple points as well. Indeed, starting with an arrangement in $\mathbb{F}_{19}\mathbb{P}^2$ with two quadruple points and a group H of order 4, we find the following arrangement of lines (see Figures 1 and 2):

$$\begin{aligned}\mathcal{A}_{22_4} \quad = \quad & \{(1:0:0),(0:1:0),(0:0:1),(1:1:1),(24:-5w-13:0),\\ & (24:5w+13:24w),(1:0:w),(2:0:w),\\ & (24:-5w-13:-4w+52),(24:5w+13:28w-52),\\ & (6:-w+13:-w+13),(24:-5w-13:16w+104),\\ & (48:w+65:24w),(24:5w+13:-32w+104),\\ & (18:-w+13:4w+26),(12:-w+13:0),\\ & (96:w+65:56w-104),(48:w+65:-8w+104),\\ & (48:w+65:20w+52),(39:-w+52:-w+52),\\ & (4:w+13:4w),(24:w+26:12w)\}\end{aligned}$$

where w is a root of $x^2 + 7x - 26$. Each of the 22 lines has 13 intersection points, 4 quadruple and 9 double points. The dual configuration (in which the 22 quadruple points are the lines) has 12 lines with 4 quadruple, one triple, and 7 double points, and 10 lines with 4 quadruple and 9 double points (see Figures 1 and 2).

Remark 2.1.

1. Since there are two roots w of $x^2 + 7x - 26$, we obtain two arrangements $\mathcal{A}_{22_4}$ up to projectivities. The corresponding matroids are isomorphic, but the CW complexes are different. This is why we find four arrangements including the duals.

2. The corresponding matroid has a group of symmetries isomorphic to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. This rather small group is probably the reason why this example did not appear in an earlier publication.

3. The above search finds these examples within a few seconds. The difficulty in finding such a configuration with the above algorithm is thus not about optimizing code.

2.2 (26_4) configurations

The same technique yields the following (26_4) configuration (and its dual), see Figure 3:

$$\begin{aligned}\mathcal{A}_{26_4} \quad = \quad & \{(1:0:0),(0:1:0),(0:0:1),(1:1:1),(1:-z^2-2z:z),\\ & (1:-z-2:1),(1:z:z),(2:-2z^2-4z:-2z^2-z+7),\\ & (2:-z^2-6z-7:z^2+z),(2:-z^2+7:-2z^2-z+7),\\ & (2:2z^2+4z:z^2+z),(4:0:-z^2-2z+7),\\ & (2:-z^2-6z-7:-z^2-3z),(0:4:-z^2-2z+3),\\ & (1:z^2+2z:z^2+2z),(4:-4z:-z^2-2z+7),(2:2z^2+4z:-z^2+7),\\ & (0:4:z^2-1),(2:-2z:-z^2+7),(4:2z^2-4z-14:-z^2-2z+7),\end{aligned}$$

$$(2 : z^2 - 7 : 2z^2 + 3z - 7), (2 : -2z^2 - 4z : -z^2 + 7),$$
$$(2 : 4z^2 + 4z - 14 : 3z^2 + 2z - 7), (11 : -2z^2 - 10z - 7 : -5z^2 - 3z + 21),$$
$$(2 : -2z : 3z^2 + 2z - 7), (2 : 4z^2 - 14 : 3z^2 + 2z - 7)\}$$

where z is the real root of $x^3 + 3x^2 - x - 7$.

Remark 2.2. All the matroids presented in this note have realizations which are unique up to projectivities and Galois automorphisms. For $\mathcal{A}_{26_4}$ there is a complex realization which may not be transformed into a real arrangement by a projectivity, namely when z is a complex root of $x^3 + 3x^2 - x - 7$.

2.3 (23_4) configurations

The arrangement of lines

$$\begin{aligned}\mathcal{A}_{23_4} = \{&(0 : 0 : 1), (0 : 1 : 0), (1 : 0 : 0), (2 : 0 : 1), (1 : 0 : 1),\\ &(1 : -1 : 1), (1 : 1 : 1), (2 : 2 : i + 1), (1 : 1 : i), (1 : -i : 0),\\ &(2 : -2i : i + 1), (1 : -i : i + 1), (1 : -i + 2 : i),\\ &(5 : -3i + 4 : i + 2), (2 : -i + 1 : i + 1), (5 : -2i + 1 : i + 2),\\ &(5 : -i - 2 : i + 2), (5 : -i + 2 : -i + 2), (5 : -i + 2 : i + 3),\\ &(5 : -i + 2 : 3i + 4), (1 : i : 0), (1 : i : -i), (1 : i : i)\}\end{aligned}$$

where $i = \sqrt{-1}$ has 25 intersection points of multiplicity 4. The right choice of 23 points yields a (23_4) configuration in the complex projective plane.

Remark 2.3. Notice that the above algorithm produces many more non isomorphic examples over finite fields and even (at least) three more examples over the complex numbers. Thus these results give no hint concerning the existence of geometric (23_4) configurations.

References

[1] J. Bokowski and V. Pilaud, On topological and geometric (19_4) configurations, *European J. Combin.* **50** (2015), 4–17, doi:10.1016/j.ejc.2015.03.008.

[2] J. Bokowski and V. Pilaud, Quasi-configurations: building blocks for point-line configurations, *Ars Math. Contemp.* **10** (2016), 99–112, `http://amc-journal.eu/index.php/amc/article/view/642`.

[3] J. Bokowski and L. Schewe, On the finite set of missing geometric configurations (n_4), *Comput. Geom.* **46** (2013), 532–540, doi:10.1016/j.comgeo.2011.11.001.

[4] M. Cuntz, Minimal fields of definition for simplicial arrangements in the real projective plane, *Innov. Incidence Geom.* **12** (2011), 49–60, `http://www.iig.ugent.be/online/12/volume-12-article-5-online.pdf`.

[5] B. Grünbaum, Connected (n_4) configurations exist for almost all n—second update, *Geombinatorics* **16** (2006), 254–261, `http://geombina.uccs.edu/?page_id=609\#issue2`.

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The size of algebraic integers with many real conjugates

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Abstract

In this paper we show that the relative normalised size with respect to a number field $\mathbb{K}$ of an algebraic integer $\alpha \neq -1, 0, 1$ is greater than 1 provided that the number of real embeddings s of $\mathbb{K}$ satisfies $s \geq 0.828n$, where $n = [\mathbb{K} : \mathbb{Q}]$. This can be compared with the previous much more restrictive estimate $s \geq n - 0.192\sqrt{n/\log n}$ and shows that the minimum $m(\mathbb{K})$ over the relative normalised size of nonzero algebraic integers α in such a field $\mathbb{K}$ is equal to 1 which is attained at $\alpha = \pm 1$. Stronger than previous but apparently not optimal bound for $m(\mathbb{K})$ is also obtained for the fields $\mathbb{K}$ satisfying $0.639 \leq s/n < 0.827469\ldots$. In the proof we use a lower bound for the Mahler measure of an algebraic number with many real conjugates.

Keywords: Algebraic number field, relative size, relative normalised size, Mahler measure, Schur–Siegel–Smyth trace problem.

Math. Subj. Class.: 11R04, 11R06

1 Introduction

Let $\mathbb{K}$ be a number field with signature $(s(\mathbb{K}), t(\mathbb{K})) = (s, t)$ having s real embeddings $\sigma_i : \mathbb{K} \to \mathbb{R}$, $i = 1, \ldots, s$, and t conjugate pairs of complex embeddings $\sigma_{i+j}, \overline{\sigma_{i+j}} : \mathbb{K} \to \mathbb{C}$, $j = 1, \ldots, t$. Clearly,

$$n = n(\mathbb{K}) := [\mathbb{K} : \mathbb{Q}] = s + 2t.$$

For any $\alpha \in \mathbb{K}$ we define

$$\|\alpha\|_{\mathbb{K}} := \Big(\sum_{i=1}^{s} \sigma_i(\alpha)^2 + \sum_{j=1}^{t} |\sigma_{s+j}(\alpha)|^2 \Big)^{1/2}, \tag{1.1}$$

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and

$$m_{\mathbb{K}}(\alpha) := \frac{\|\alpha\|_{\mathbb{K}}^2}{s+t}. \tag{1.2}$$

Also, put

$$m(\mathbb{K}) := \min_{\alpha \in \mathcal{O}_{\mathbb{K}} \setminus \{0\}} m_{\mathbb{K}}(\alpha), \tag{1.3}$$

where $\mathcal{O}_{\mathbb{K}}$ is the ring of integers of $\mathbb{K}$.

For any number field $\mathbb{K}$ we have $\pm 1 \in \mathcal{O}_{\mathbb{K}}$ and $\| \pm 1\|_{\mathbb{K}} = s(\mathbb{K}) + t(\mathbb{K})$, so that $m_{\mathbb{K}}(\pm 1) = 1$ (see (1.1) and (1.2)). By (1.3), this yields $m(\mathbb{K}) \leq 1$. The lower bound

$$m(\mathbb{K}) \geq \frac{1}{1 + s/n},$$

where $n = [\mathbb{K} : \mathbb{Q}]$ and $s = s(\mathbb{K})$, follows from [19, Lemma 1.1(ii)]. The stronger bound

$$m(\mathbb{K}) \geq \frac{2^{s/n}}{1 + s/n} \tag{1.4}$$

is given in [16, Theorem 5.11]. In particular, the inequality (1.4) implies $m(\mathbb{K}) = 1$ if $\mathbb{K}$ is a totally complex field ($s(\mathbb{K}) = 0$) or a totally real field ($t(\mathbb{K}) = 0$).

A motivation for introducing and studying the quantities $\|\alpha\|_{\mathbb{K}}$, $m_{\mathbb{K}}(\alpha)$ and $m(\mathbb{K})$ is given in [7]; see also a subsequent paper [6]. There, we call $\|\alpha\|_{\mathbb{K}}$ the *relative size* of α with respect to the number field $\mathbb{K}$ and $\sqrt{m_{\mathbb{K}}(\alpha)}$ the *relative normalised size* of α (with respect to $\mathbb{K}$ again). Briefly speaking, it is related to some earlier work on certain lattices defined by number fields, when in the ring of integers $\mathcal{O}_{\mathbb{K}}$ of a number field $\mathbb{K}$ with signature (s, t), one considers the vectors

$$(\sigma_1(\alpha), \dots, \sigma_s(\alpha), \Re(\sigma_{s+1}(\alpha)), \Im(\sigma_{s+1}(\alpha)), \dots, \Re(\sigma_{s+t}(\alpha)), \Im(\sigma_{s+t}(\alpha)))$$

in $\mathbb{R}^n$ defined for $\alpha \in \mathcal{O}_{\mathbb{K}}$ (see [2, Chapter 8, Section 7]). This can be applied to show that the class number of a number field is finite [11]. The norm defined in (1.1) has been considered by Pethő and Schmitt [13]; see also a subsequent paper [5]. A different but at the same time quite similar to (1.1) norm related to certain number field codes has been also considered in [9]; see also [3].

Since the minimum of the function $h(x) := 2^x/(1 + x)$ in the interval $x \in [0, 1]$ is attained at $x_0 := 1/\log 2 - 1 \notin \mathbb{Q}$ and equals $h(x_0) = (e \log 2)/2$, the inequality

$$\frac{2^{s/n}}{1 + s/n} > \frac{e \log 2}{2} = 0.942084\dots$$

holds for any integers $s \leq n$, where $s \geq 0$ and $n \geq 1$. Hence, by (1.4),

$$\frac{e \log 2}{2} < m(\mathbb{K}) \leq 1.$$

In particular, for any number field $\mathbb{K}$ we have either $m(\mathbb{K}) = 1$ or $m(\mathbb{K}) < 1$.

A large class of fields for which $m(\mathbb{K}) = 1$ was described in [7, Theorem 3.3], where we showed that for a number field $\mathbb{K}$ with signature (s, t) we have $m(\mathbb{K}) = 1$ if

$$t \leq 0.096\sqrt{s/\log s}. \tag{1.5}$$

The following theorem relaxes the bound (1.5) on t to the bound $t \leq 0.086n$ with the same conclusion and so strengthens the above result considerably. (Note that in view of $n = s + 2t$ the bound (1.5) is essentially equivalent to $t \leq 0.096\sqrt{n/\log n}$.)

Theorem 1.1. *For each number field $\mathbb{K}$ of degree n and signature (s, t) satisfying $t \leq 0.086n$ we have $m(\mathbb{K}) = 1$.*

Observe that $t \leq 0.1038s$ implies $t \leq 0.086n$ which is also equivalent to $s \geq 0.828n$. On the other hand, by [7, Theorem 3.5], for each integer $s \geq 2$ there exist infinitely many number fields $\mathbb{K}$ with signature (s, s) (so that $t = s = n/3$) for which $m(\mathbb{K}) < 1$. This shows that Theorem 1.1 is best possible up to the constant. Moreover, the constant 0.086 cannot be replaced by the constant $1/3$.

Below, the bound (1.4) will also be improved for fields $\mathbb{K}$ of degree n with signature (s, t) satisfying $0.639 \leq s/n < 0.828$ (see Corollary 2.4 in Section 2). Here, the constants 0.639 and 0.828 are just three decimal digit approximations from above of some presumably transcendental constants (see Proposition 2.1 below for the definition of $\lambda_0 = 0.827469\ldots$).

In the next section we state Theorem 2.2 which is the main result of this paper. Section 3 contains some auxiliary results. The proofs of Proposition 2.1, Theorem 2.2 and Theorem 2.5 will be given in Sections 4, 5 and 6, respectively.

2 Main results

Throughout, we shall use the following notation for fixed $\lambda > 0$:

$$g(\lambda) := \left(2^{-1/\lambda} + \sqrt{1 + 2^{-2/\lambda}}\right)^{\lambda}, \tag{2.1}$$

$$F(\lambda, x) := x g(\lambda)^{1/x} + 2(1 - x) g(\lambda)^{-1/(1-x)} - 2 + x, \tag{2.2}$$

where $0 < x \leq 1$ and, by definition, $F(\lambda, 1) = g(\lambda) - 1$. Finally, the function $\varphi(\lambda)$ is defined for positive λ as follows

$$\varphi(\lambda) := \min_{0 < x \leq 1} F(\lambda, x). \tag{2.3}$$

Here, the minimum in (2.3) is attained, since $F(\lambda, x) \to +\infty$ as $x \to 0+$ in view of $g(\lambda) > 1$.

With this notation, we will show that

Proposition 2.1. *The function $\varphi(\lambda)$ is increasing for $\lambda \geq 0.581$ and positive for $\lambda > \lambda_0 := 0.827469\ldots$. Here, $\varphi(\lambda_0) = 0$, $\varphi(0.828) = 0.000389\ldots$ and $\varphi(1) = 0.176732\ldots$.*

More values of the function $\varphi(\lambda)$ are given in Table 1. Here, for each $\lambda \in [0.83, 1]$ the constant $x_0(\lambda)$ is the point of absolute minimum of $F(\lambda, x)$ in the interval $0 < x \leq 1$, so that $\varphi(\lambda) = F(\lambda, x_0(\lambda))$.

Now, we can state the main result of this paper.

Theorem 2.2. *Let $\mathbb{K}$ be a number field with signature $(s(\mathbb{K}), t(\mathbb{K}))$ and degree $n = s(\mathbb{K}) + 2t(\mathbb{K})$ over $\mathbb{Q}$ satisfying $s(\mathbb{K}) \geq 0.581n$, and let $\alpha \neq -1, 0, 1$ be an algebraic integer in $\mathbb{K}$. Then,*

$$m_{\mathbb{K}}(\alpha) \geq 1 + \frac{\varphi(\lambda)}{1 + \lambda}, \tag{2.4}$$

where $\lambda := s(\mathbb{K})/n$ and the function $\varphi(\lambda)$ is defined in (2.1)-(2.3). In particular, the inequality $\varphi(\lambda) > 0$ holds for each $\lambda \in (\lambda_0, 1]$, that is, for $s(\mathbb{K}) > \lambda_0 n$, where $\lambda_0 = 0.827469\ldots$.

Table 1: Values of $\varphi(\lambda)$ in the range $0.83 \leq \lambda \leq 1$.

λ	$g(\lambda)$	$x_0(\lambda)$	$\varphi(\lambda)$	$\frac{\varphi(\lambda)}{1+\lambda}$
0.83	1.418557	0.529769	0.001865	0.001019
0.84	1.429308	0.532299	0.009447	0.005134
0.85	1.440180	0.534841	0.017362	0.009385
0.88	1.473522	0.542547	0.043126	0.022939
0.90	1.496362	0.547749	0.061991	0.032627
0.93	1.531545	0.555648	0.092837	0.048102
0.95	1.555624	0.560980	0.115104	0.059027
0.98	1.592687	0.569077	0.151060	0.076292
0.99	1.605296	0.571802	0.163726	0.082274
1.00	1.618033	0.574542	0.176732	0.088366

In fact, the inequalities (5.7) and (5.8) which will be proved in Section 5 can be stronger than (2.4) under some additional assumptions.

In particular, selecting (in Theorem 2.2) $\mathbb{K} := \mathbb{Q}(\alpha)$, we obtain the following:

Corollary 2.3. *Let $\alpha \neq -1, 0, 1$ be an algebraic integer of degree $d = s + 2t$ with s real conjugates α_i, $i = 1, \ldots, s$, and t pairs of complex conjugates $\alpha_{s+j}, \overline{\alpha_{s+j}}$, $j = 1, \ldots, t$. Then, for $\lambda = s/d > \lambda_0$ we have*

$$m_{\mathbb{Q}(\alpha)}(\alpha) = \frac{|\alpha_1|^2 + \cdots + |\alpha_{s+t}|^2}{s+t} \geq 1 + \frac{\varphi(\lambda)}{1+\lambda},$$

where $\varphi(\lambda) > 0$ is defined in (2.1)-(2.3).

By (1.3), Theorem 2.2 immediately implies Theorem 1.1 stated in Section 1. In the range $0.581 \leq \lambda = s/n \leq \lambda_0$ Theorem 2.2 implies the following:

Corollary 2.4. *For a number field $\mathbb{K}$ of degree n and signature (s, t) satisfying $0.581 \leq \lambda = s/n \leq \lambda_0$ we have*

$$m(\mathbb{K}) \geq 1 + \frac{\varphi(\lambda)}{1+\lambda}.$$

Note that for each λ satisfying $0.639 \leq \lambda \leq \lambda_0$ the inequality of Corollary 2.4 strengthens the bound (1.4). In particular, for $\lambda = 0.639$ we have

$$m(\mathbb{K}) \geq 1 + \frac{\varphi(\lambda)}{1+\lambda} = 0.950175\ldots,$$

whereas the bound (1.4) yields the weaker inequality

$$m(\mathbb{K}) \geq \frac{2^\lambda}{1+\lambda} = 0.950121\ldots.$$

For further comparison of the functions $1 + \varphi(\lambda)/(1+\lambda)$ and $2^\lambda/(1+\lambda)$ see Table 2.

If the number of complex conjugates $2t$ of an algebraic integer is very small compared to its degree d (which is large) then the constant 1.088366 corresponding to the case $\lambda = 1$

Table 2: Values of $1 + \frac{\varphi(\lambda)}{1+\lambda}$ vs $\frac{2^\lambda}{1+\lambda}$ for $0.64 \le \lambda \le 0.82$.

λ	$g(\lambda)$	$\frac{2^\lambda}{1+\lambda}$	$1 + \frac{\varphi(\lambda)}{1+\lambda}$
0.64	1.237064	0.950200	0.950299
0.65	1.245534	0.951011	0.951621
0.70	1.289701	0.955591	0.960509
0.75	1.336871	0.961024	0.973167
0.80	1.387027	0.967278	0.989505
0.82	1.407927	0.970003	0.997042

in Corollary 2.3 (see Table 1) can be improved, by using the results on the so-called Schur–Siegel–Smyth trace problem. The problem is named after the authors of the first three estimates of the trace of a totally positive algebraic integer [15], [17], [18]. The method of auxiliary functions introduced by Smyth in [18] was used in all subsequent papers on this subject. Specifically, we shall use the result of Liang and Wu [12] (see Lemma 3.5 below). See also some recent related papers [4] and [14].

Theorem 2.5. *There exist two absolute positive constants D and δ such that if $d \ge D$ and $t < \delta d/\log d$ then for each algebraic integer α of degree $d = s + 2t$ with s real conjugates α_i, $i = 1, \dots, s$, and t pairs of complex conjugates $\alpha_{s+j}, \overline{\alpha_{s+j}}$, $j = 1, \dots, t$, the inequality*

$$m_{\mathbb{Q}(\alpha)}(\alpha) = \frac{|\alpha_1|^2 + \cdots + |\alpha_{s+t}|^2}{s+t} > 1.79192 \tag{2.5}$$

holds.

For large d Theorem 2.5 not only gives a better bound, but also the condition $t < \delta d/\log d$ is less restrictive than the corresponding condition $t \le 0.096\sqrt{d/\log d}$ of [7, Theorem 3.3].

3 Auxiliary results

Lemma 3.1. *Let $\alpha \ne -1, 0, 1$ be an algebraic number of degree d over $\mathbb{Q}$ with signature (s, t), where $\lambda = s/d > 0$. Then,*

$$M(\alpha) \ge \left(2^{-1/\lambda} + \sqrt{1 + 2^{-2/\lambda}}\right)^{s/2}. \tag{3.1}$$

In particular, for $s \ge 0.581d$ we have

$$M(\alpha) > 1.090691^d. \tag{3.2}$$

Proof. The inequality (3.1) was proved by Garza (it is the main result in [8]). In [10], Höhn gave an alternative proof of this result.

By (2.1), (3.1), and $s = \lambda d$, we deduce that

$$M(\alpha) \ge \left(2^{-1/\lambda} + \sqrt{1 + 2^{-2/\lambda}}\right)^{\lambda d/2} = g(\lambda)^{d/2}. \tag{3.3}$$

Evidently, the function $g(\lambda)$ is increasing in $\lambda > 0$, so its smallest value in the interval $[0.581, 1]$ is attained at $\lambda = 0.581$. Thus, (3.3) implies (3.2) in view of $g(0.581)^{1/2} = 1.090691\dots$. □

We will also need the following inequality.

Lemma 3.2. *For any number fields* $\mathbb{L} \subseteq \mathbb{K}$ *with signatures* $(s(\mathbb{L}), t(\mathbb{L}))$ *and* $(s(\mathbb{K}), t(\mathbb{K}))$*, respectively, we have* $s(\mathbb{K})t(\mathbb{L}) \leq s(\mathbb{L})t(\mathbb{K})$*.*

Proof. By the primitive element theorem, write $\mathbb{L} = \mathbb{Q}(\alpha)$ and $\mathbb{K} = \mathbb{Q}(\beta)$. Then, $\alpha = P(\beta)$ with some $P \in \mathbb{Q}[x]$. Without restriction of generality we may assume that $\beta_1, \dots, \beta_s$ are the real conjugates of β and $\beta_{s+1}, \overline{\beta_{s+1}}, \dots, \beta_{s+t}, \overline{\beta_{s+t}}$ are the complex conjugates of β. Here, $s = s(\mathbb{K})$ and $t = t(\mathbb{K})$. Note that in the list $\sigma(P(\beta))$, where σ runs through all $s + 2t$ automorphisms of the field $\mathbb{K}$, each conjugate of α appears $[\mathbb{K} : \mathbb{L}]$ times. In particular, each of the numbers $P(\beta_i)$, where $1 \leq i \leq s$, is real, so the number of real conjugates of α is at least $s/[\mathbb{K} : \mathbb{L}]$. This yields

$$s(\mathbb{L}) \geq \frac{s}{[\mathbb{K} : \mathbb{L}]} = \frac{s[\mathbb{L} : \mathbb{Q}]}{[\mathbb{K} : \mathbb{L}][\mathbb{L} : \mathbb{Q}]} = \frac{s[\mathbb{L} : \mathbb{Q}]}{[\mathbb{K} : \mathbb{Q}]} = \frac{s(s(\mathbb{L}) + 2t(\mathbb{L}))}{s + 2t}.$$

Multiplying both sides by $s + 2t$ we obtain the required inequality. □

Lemma 3.3. *Let* $k \leq d$ *be two positive integers and let* $S \geq 1$*,* ρ *and* $y_1 \geq \dots \geq y_k \geq 1 \geq y_{k+1} \geq \dots \geq y_d$ *be real numbers such that*

$$y_1 + \dots + y_k + S(y_{k+1} + \dots + y_d) \geq S(d - k) + k + \rho.$$

Then, for any positive numbers $w_1, \dots, w_d$ *satisfying*

$$\max_{1 \leq i \leq d} w_i \leq S \min_{1 \leq i \leq d} w_i$$

and $w_1 + \dots + w_d = 1$ *we have*

$$w_1 y_1 + \dots + w_d y_d \geq 1 + \rho \min_{1 \leq i \leq d} w_i.$$

Proof. Put $z_i := y_i - 1$ for each $i = 1, \dots, d$. Then,

$$z_1 \geq \dots z_k \geq 0 \geq z_{k+1} \geq \dots \geq z_d \tag{3.4}$$

and

$$z_1 + \dots + z_k + S(z_{k+1} + \dots + z_d) \geq \rho. \tag{3.5}$$

Now, by (3.4), the bound $0 < \max_{1 \leq i \leq d} w_i \leq S \min_{1 \leq i \leq d} w_i$ and (3.5), it follows that

$$\begin{aligned}
\sum_{i=1}^{d} w_i z_i = \sum_{i=1}^{k} w_i z_i + \sum_{i=k+1}^{d} w_i z_i &\geq \min_{1 \leq i \leq k} w_i \sum_{i=1}^{k} z_i + \max_{k+1 \leq i \leq d} w_i \sum_{i=k+1}^{d} z_i \\
&\geq \min_{1 \leq i \leq d} w_i \sum_{i=1}^{k} z_i + \max_{1 \leq i \leq d} w_i \sum_{i=k+1}^{d} z_i \\
&\geq (z_1 + \dots + z_k + S(z_{k+1} + \dots + z_d)) \min_{1 \leq i \leq d} w_i \\
&\geq \rho \min_{1 \leq i \leq d} w_i.
\end{aligned}$$

Combined with $z_i = y_i - 1$ and $\sum_{i=1}^{d} w_i = 1$ this implies the required estimate. □

Lemma 3.4. *Let $k \le d$ be two integers, where $k \ge 0$, $d \ge 2$, and let α be an algebraic integer of degree d with signature (s,t) satisfying $s \ge 0.581d$ whose conjugates $\alpha_1, \dots, \alpha_d$ are labeled so that*

$$|\alpha_1| \ge \cdots \ge |\alpha_k| \ge 1 \ge |\alpha_{k+1}| \ge \cdots \ge |\alpha_d|.$$

Then,

$$|\alpha_1|^2 + \cdots + |\alpha_k|^2 + 2(|\alpha_{k+1}|^2 + \cdots + |\alpha_d|^2) \ge 2d - k + d\varphi(\lambda), \tag{3.6}$$

where $\lambda = s/d$ and $\varphi(\lambda)$ defined in (2.1)-(2.3).

Proof. Note that $k \ge 1$. Indeed $k = 0$ can only happen if all α_i, $i = 1, \dots, d$, are of modulus 1. So, by Kronecker's theorem, α must be a root of unity which is not the case. If $k = d$ then, by the arithmetic and geometric mean inequality (referred to as AM-GM below) and (3.3), the left side of (3.6) is at least

$$d|\mathrm{Norm}(\alpha)|^{2/d} = dM(\alpha)^{2/d} \ge dg(\lambda),$$

where $g(\lambda)$ is defined in (2.1). Since $g(\lambda)$ is increasing in $\lambda > 0$ and $g(0.581) = 1.189607\dots$, we find that the left side of (3.6) is at least $1.189d$. This is greater than its right side, since

$$2d - k + d\varphi(\lambda) = d + d\varphi(\lambda) \le d + d\varphi(1) < d + 0.18d = 1.18d$$

(see Proposition 2.1 and Table 1). In all what follows we thus assume that $0 < k < d$.

By AM-GM, estimating

$$|\alpha_1|^2 + \cdots + |\alpha_k|^2 \ge kM(\alpha)^{2/k}$$

and

$$|\alpha_{k+1}|^2 + \cdots + |\alpha_d|^2 \ge (d-k)\Big(\frac{|\mathrm{Norm}(\alpha)|}{M(\alpha)}\Big)^{2/(d-k)} \ge (d-k)M(\alpha)^{-2/(d-k)}$$

we find that the left side of (3.6) is at least

$$kM(\alpha)^{2/k} + 2(d-k)M(\alpha)^{-2/(d-k)}.$$

Hence, it suffices to show that

$$\frac{kM(\alpha)^{2/k} + 2(d-k)M(\alpha)^{-2/(d-k)}}{d} - 2 + \frac{k}{d} \ge \varphi(\lambda). \tag{3.7}$$

Note that the function $ky^{2/k} + 2(d-k)y^{-2/(d-k)}$ is increasing in y in the interval $[2^{d/8}, \infty)$, since its derivative $2y^{2/k-1} - 4y^{-2/(d-k)-1}$ is positive for $y > 2^{k(d-k)/(2d)}$ and the maximum of $k(d-k)$ is attained at $k = d/2$. Also, by (3.2) and $2^{1/8} = 1.090507\dots$, the inequality $M(\alpha) > 1.090691^d > 2^{d/8}$ holds. Thus, replacing $M(\alpha)$ in (3.7) by its estimate from below as in (3.3) and setting $x := k/d$, we see that it suffices to prove the inequality

$$xg(\lambda)^{1/x} + 2(1-x)g(\lambda)^{-1/(1-x)} - 2 + x \ge \varphi(\lambda) \tag{3.8}$$

for $0 < x < 1$. However, (3.8) clearly holds, by the definition of the function $\varphi(\lambda)$ in Theorem 2.2 as the minimum of the left side of (3.8) in the interval $(0, 1]$. $\square$

The next result is given [12].

Lemma 3.5. *There exist m (explicitly given) polynomials with integer coefficients $Q_1, \dots, Q_m$ and m (explicitly given) positive numbers $e_1, \dots, e_m$ such that the inequality*

$$y - \sum_{i=1}^{m} e_i \log |Q_i(y)| > 1.79193$$

holds for each $y > 0$ which is not a root of $Q_1 \dots Q_m$.

We remark that each improvement of the constant of this lemma leads to the corresponding improvement in Theorem 2.5. However, although the conjectural lower bound for the trace of a totally positive algebraic integer α is $(2 - \varepsilon)d$, where ε is an arbitrary positive number and the degree d of α is at least $d(\varepsilon)$, Serre has shown that the method of auxiliary functions as in the above lemma cannot give a constant greater than 1.8983021 (see the appendix in [1]).

4 Proof of Proposition 2.1

Note that $y = g(\lambda) > 1$ for $\lambda > 0$. Consider the function

$$f(y) := xy^{1/x} + 2(1-x)y^{-1/(1-x)}$$

in the interval $1 < y < \infty$ (here, $0 < x < 1$). Its derivative

$$f'(y) = y^{1/x-1} - 2y^{-1/(1-x)}$$

is positive if $y^{1/x+1/(1-x)} > 2$, that is, $y > 2^{x(1-x)}$. In particular, since $x(1-x) \le 1/4$, the function $f(y)$ is increasing in the interval $2^{1/4} < y < \infty$.

Thus, by (2.3) and (2.1) (which implies that $g(\lambda)$ is increasing in λ), for every fixed x in the range $0 < x < 1$ the function

$$xg(\lambda)^{1/x} + 2(1-x)g(\lambda)^{-1/(1-x)} - 2 + x$$

in increasing (in λ) for λ satisfying $g(\lambda) > 2^{1/4}$. In particular, $\varphi(\lambda)$ is increasing in λ for λ satisfying $g(\lambda) > 2^{1/4}$. Therefore, using the fact that $g(\lambda)$ is increasing in λ for $\lambda > 0$ and the actual expression (2.1), we find that

$$g(0.581) = 1.189607 \dots > 1.189207 \dots = 2^{1/4}.$$

Consequently, the function $\varphi(\lambda)$ is increasing for $\lambda \ge 0.581$. Evaluating $\varphi(\lambda)$ at $\lambda = 0.828$ gives the positive value $\varphi(0.828) = 0.000389\dots$, so $\varphi(\lambda) > 0$ for $\lambda \ge 0.828$. This, combined with evaluation of $\varphi(1) = 0.176732\dots$ and λ_0 satisfying $\varphi(\lambda_0) = 0$ completes the proof of the proposition.

5 Proof of Theorem 2.2

Let $\alpha \in \mathbb{K}$ and $\mathbb{L} = \mathbb{Q}(\alpha)$. Assume that the signature of α is (s, t) and the signature of $\mathbb{K}$ is $(s(\mathbb{K}), t(\mathbb{K}))$. Here, $\lambda = s(\mathbb{K})/n$, where $n = s(\mathbb{K}) + 2t(\mathbb{K}) = [\mathbb{K} : \mathbb{Q}]$. Put also $\lambda_1 := s(\mathbb{L})/d = s/d$, where $d = s + 2t = [\mathbb{L} : \mathbb{Q}]$. We will show that

$$\lambda_1 \ge \lambda. \tag{5.1}$$

Observe first that $t(\mathbb{L}) = 0$ implies that $s(\mathbb{L}) = d$, so that $\lambda_1 = 1$, which yields (5.1). Also, $t(\mathbb{K}) = 0$ implies $t(\mathbb{L}) = 0$, which leads to the situation we have just considered. So assume that $t(\mathbb{K}) \neq 0$ and $t = t(\mathbb{L}) \neq 0$. Then, in view of Lemma 3.2 we have $s(\mathbb{K})/t(\mathbb{K}) \leq s/t$. Adding 2 to both sides we deduce

$$\frac{n}{t(\mathbb{K})} = \frac{s(\mathbb{K}) + 2t(\mathbb{K})}{t(\mathbb{K})} = 2 + \frac{s(\mathbb{K})}{t(\mathbb{K})} \leq 2 + \frac{s}{t} = \frac{s+2t}{t} = \frac{d}{t}.$$

Therefore, $t/d \leq t(\mathbb{K})/n$. This implies (5.1), since $t/d = (1-\lambda_1)/2$ and $t(\mathbb{K})/n = (1-\lambda)/2$.

Let $\alpha_1, \dots, \alpha_s$ be the real conjugates of α. Put

$$\mathcal{C}(\alpha) := \sum_{j=1}^{t} |\alpha_{s+j}|^2 = \frac{1}{2} \sum_{i=s+1}^{d} |\alpha_i|^2.$$

Assume that for each real α_i, $1 \leq i \leq s$, it appears u_i times under the $s(\mathbb{K})$ real embeddings of $\mathbb{K}$ and $2v_i$ times under the $2t(\mathbb{K})$ complex embeddings of $\mathbb{K}$. Here, we have $u_i + 2v_i = [\mathbb{K} : \mathbb{L}]$ for each i. Also,

$$s(\mathbb{K}) = u_1 + \ldots + u_s \quad \text{and} \quad t(\mathbb{K}) = [\mathbb{K} : \mathbb{L}]t + v_1 + \ldots + v_s. \tag{5.2}$$

So, in view of (1.2) we can write

$$(s(\mathbb{K}) + t(\mathbb{K}))m_{\mathbb{K}}(\alpha) = \sum_{i=1}^{s} (u_i + v_i)\alpha_i^2 + [\mathbb{K} : \mathbb{L}]\mathcal{C}(\alpha). \tag{5.3}$$

Here, $\mathcal{C}(\alpha) = \frac{1}{2} \sum_{i=s+1}^{d} |\alpha_i|^2$. Setting

$$w_i := \frac{u_i + v_i}{s(\mathbb{K}) + t(\mathbb{K})}$$

for $i = 1, \dots, s$ and

$$w_i := \frac{[\mathbb{K} : \mathbb{L}]}{2s(\mathbb{K}) + 2t(\mathbb{K})} \tag{5.4}$$

for $i = s+1, \dots, d$, in view of (5.2) and (5.3), we derive that

$$m_{\mathbb{K}}(\alpha) = \sum_{i=1}^{d} w_i |\alpha_i|^2,$$

where $\sum_{i=1}^{d} w_i = 1$ and

$$\frac{[\mathbb{K} : \mathbb{L}]}{2s(\mathbb{K}) + 2t(\mathbb{K})} \leq w_i \leq \frac{[\mathbb{K} : \mathbb{L}]}{s(\mathbb{K}) + t(\mathbb{K})}$$

for each $i = 1, \dots, d$. Hence, by Lemma 3.3 with $S = 2$, $\rho = d\varphi(\lambda_1)$, $y_i = |\alpha_i|^2$ for $i = 1, \dots, d$, and Lemma 3.4 (with $\lambda_1 = s/d$ instead of λ), it follows that

$$m_{\mathbb{K}}(\alpha) = \sum_{i=1}^{d} w_i |\alpha_i|^2 \geq 1 + d\varphi(\lambda_1) \min_{1 \leq i \leq d} w_i.$$

Now, in case $s = d$ we have $\lambda_1 = 1$, so $\varphi(\lambda_1)$ is positive and using

$$\min_{1\le i\le d} w_i \ge \frac{[\mathbb{K}:\mathbb{L}]}{2s(\mathbb{K})+2t(\mathbb{K})} \tag{5.5}$$

we derive that

$$m_{\mathbb{K}}(\alpha) \ge 1 + \frac{d[\mathbb{K}:\mathbb{L}]\varphi(\lambda_1)}{2s(\mathbb{K})+2t(\mathbb{K})}. \tag{5.6}$$

Otherwise, when $s < n$, in view of (5.4) we have equality in (5.5). Thus, (5.6) also holds (even if $\varphi(\lambda_1)$ is negative).

Now, since $d[\mathbb{K}:\mathbb{L}] = [\mathbb{L}:\mathbb{Q}][\mathbb{K}:\mathbb{L}] = [\mathbb{K}:\mathbb{Q}] = n$ and

$$\frac{n}{2s(\mathbb{K})+2t(\mathbb{K})} = \frac{n}{n+s(\mathbb{K})} = \frac{1}{1+\lambda},$$

from (5.6) we further deduce that

$$m_{\mathbb{K}}(\alpha) \ge 1 + \frac{\varphi(\lambda_1)}{1+\lambda}. \tag{5.7}$$

Here, we have $\varphi(\lambda_1) \ge \varphi(\lambda)$, by Proposition 2.1 and the inequality (5.1). Also, by the same inequality, $1/(1+\lambda) \ge 1/(1+\lambda_1)$. So, in particular, (5.7) yields

$$m_{\mathbb{K}}(\alpha) \ge 1 + \max\left\{\frac{\varphi(\lambda)}{1+\lambda}, \frac{\varphi(\lambda_1)}{1+\lambda_1}\right\} \tag{5.8}$$

which implies the required bound.

6 Proof of Theorem 2.5

Let α be an algebraic integer with degree d greater than

$$E := 2\max_{1\le i\le m} \deg Q_i,$$

where $Q_i \in \mathbb{Z}[x]$ are given in Lemma 3.5. Applying this lemma to $y := \alpha_j^2$, where $j = 1,\dots,s$, and summing up over j we find that

$$\sum_{j=1}^{s} \alpha_j^2 > 1.79193s + \sum_{j=1}^{s}\sum_{i=1}^{m} e_i \log|Q_i(\alpha_j^2)|. \tag{6.1}$$

Note that there is nothing to prove if at least one conjugate of α is greater than $\sqrt{2d}$, because then the right side of (2.5) is greater than $2d/(s+t) \ge 2$ which is better than required. So, in all what follows without restriction of generality we may assume that $|\alpha_{s+j}| \le \sqrt{2d}$ for $j = 1,\dots,t$.

Clearly,

$$|Q_i(\alpha_{s+j}^2)| \le (D_i+1)H_i(2d)^{D_i},$$

where D_i and H_i are the degree and the height of the polynomial Q_i, respectively. Similarly, $|Q_i(\overline{\alpha_{s+j}}^2)| \le (D_i+1)H_i(2d)^{D_i}$. Note that the degree of α^2 is either d or $d/2$, so it is greater than any $D_i = \deg Q_i$ provided that $d \geq D > E$. Hence, $Q_i(\alpha_j^2) \neq 0$ for each $i = 1, \dots, m$ and each $j = 1, \dots, d$. Consequently,

$$\begin{aligned}1 \leq \prod_{j=1}^{d} |Q_i(\alpha_j^2)| &= \prod_{j=1}^{s} |Q_i(\alpha_j^2)| \prod_{j=1}^{t} |Q_i(\alpha_{s+j}^2)||Q_i(\overline{\alpha_{s+j}}^2)| \\ &\leq (2d)^{2tD_i} U_i^{2t} \prod_{j=1}^{s} |Q_i(\alpha_j^2)|,\end{aligned}$$

where $U_i := (D_i + 1)H_i$, which yields

$$\sum_{j=1}^{s} \log |Q_i(\alpha_j^2)| \geq -2tD_i \log(2d) - 2t \log U_i.$$

Summing these inequalities with weights e_i over $i = 1, \dots, m$ we derive that

$$\begin{aligned}\sum_{j=1}^{s} \sum_{i=1}^{m} e_i |\log Q_i(\alpha_j^2)| &= \sum_{i=1}^{m} \sum_{j=1}^{s} e_i |\log Q_i(\alpha_j^2)| \\ &\geq -\sum_{i=1}^{m} (2tD_i e_i \log(2d) + 2te_i \log U_i) \\ &\geq -At \log(Bd),\end{aligned}$$

where the constants $A, B > 2$ depend on the constants $e_1, \dots, e_m$ and the polynomials $Q_1, \dots, Q_m$ only. Combining this inequality with (6.1) we get

$$\sum_{j=1}^{s} \alpha_j^2 > 1.79193s - At \log(Bd).$$

To complete the proof of the theorem it suffices to show that $1.79193s - At \log(Bd) > 1.79192(s + t)$, which is equivalent to $10^{-5}s > At \log(Bd) + 1.79192t$. Multiplying both sides of this inequality by 10^5 and adding $2t$ we obtain the following equivalent inequality:

$$d = s + 2t > 10^5 At \log(Bd) + 179192t + 2t = 10^5 At \log(Bd) + 179194t.$$

We will show that the stronger inequality

$$d > 10^5(A + 2)t \log(Bd) \tag{6.2}$$

holds with the constants

$$\delta := \frac{1}{10^5(2A + 4)} \quad \text{and} \quad D := \max\{B, E + 1\}$$

depending on $e_1, \dots, e_m$ and $Q_1, \dots, Q_m$ only.

Indeed, in view of the upper bound on t, namely, $t < \delta d / \log d$, the first lower bound on d, namely, $d \geq D \geq B$, and the choice of δ the right side of (6.2) is less than

$$10^5(A + 2)\frac{\delta d}{\log d} \log(Bd) \leq 10^5(A + 2)\frac{\delta d}{\log d} \log(d^2) = \delta 10^5(2A + 4)d = d.$$

This completes the proof of (6.2) and the proof the theorem.

References

[1] J. Aguirre and J. C. Peral, The trace problem for totally positive algebraic integers, in: J. McKee and C. Smyth (eds.), *Number Theory and Polynomials*, Cambridge University Press, Cambridge, volume 352 of *London Mathematical Society Lecture Note Series*, pp. 1–19, 2008, doi:10.1017/cbo9780511721274.003, proceedings of the workshop held at Bristol University, Bristol, April 3 – 7, 2006.

[2] J. H. Conway and N. J. A. Sloane, *Sphere Packings, Lattices and Groups*, volume 290 of *Fundamental Principles of Mathematical Sciences*, Springer-Verlag, New York, 3rd edition, 1999, doi:10.1007/978-1-4757-6568-7.

[3] N. Coxon, List decoding of number field codes, *Des. Codes Cryptogr.* **72** (2014), 687–711, doi:10.1007/s10623-013-9803-x.

[4] X. Dong and Q. Wu, The absolute trace of totally positive reciprocal algebraic integers, *J. Number Theory* **170** (2017), 66–74, doi:10.1016/j.jnt.2016.06.016.

[5] J. R. Doyle and D. Krumm, Computing algebraic numbers of bounded height, *Math. Comp.* **84** (2015), 2867–2891, doi:10.1090/mcom/2954.

[6] A. Dubickas, Algebraic integers with small absolute size, *Quaest. Math.* **40** (2017), 627–644, doi:10.2989/16073606.2017.1305464.

[7] A. Dubickas, M. Sha and I. E. Shparlinski, On distances in lattices from algebraic number fields, *Moscow Math. J.* **17** (2017), 239–268, `http://www.mathjournals.org/mmj/2017-017-002/2017-017-002-004.html`.

[8] J. Garza, On the height of algebraic numbers with real conjugates, *Acta Arith.* **128** (2007), 385–389, doi:10.4064/aa128-4-6.

[9] V. Guruswami, Constructions of codes from number fields, *IEEE Trans. Inform. Theory* **49** (2003), 594–603, doi:10.1109/tit.2002.808131.

[10] G. Höhn, On a theorem of Garza regarding algebraic numbers with real conjugates, *Int. J. Number Theory* **7** (2011), 943–945, doi:10.1142/s1793042111004320.

[11] S. Lang, *Algebraic Number Theory*, volume 110 of *Graduate Texts in Mathematics*, Springer-Verlag, New York, 2nd edition, 1994, doi:10.1007/978-1-4612-0853-2.

[12] Y. Liang and Q. Wu, The trace problem for totally positive algebraic integers, *J. Aust. Math. Soc.* **90** (2011), 341–354, doi:10.1017/s1446788711001030.

[13] A. Pethő and S. Schmitt, Elements with bounded height in number fields, *Period. Math. Hungar.* **43** (2001), 31–41, doi:10.1023/a:1015225430108.

[14] K. Pratt, G. Shakan and A. Zaharescu, A generalization of the Schur-Siegel-Smyth trace problem, *J. Math. Anal. Appl.* **436** (2016), 489–500, doi:10.1016/j.jmaa.2015.12.003.

[15] I. Schur, Über die Verteilung der Wurzeln bei gewissen algebraischen Gleichungen mit ganzzahligen Koeffizienten, *Math. Z.* **1** (1918), 377–402, doi:10.1007/bf01465096.

[16] I. E. Shparlinski, *Finite Fields: Theory and Computation*, volume 477 of *Mathematics and its Applications*, Kluwer Academic Publishers, Dordrecht, 1999, doi:10.1007/978-94-015-9239-0.

[17] C. L. Siegel, The trace of totally positive and real algebraic integers, *Ann. of Math.* **46** (1945), 302–312, doi:10.2307/1969025.

[18] C. J. Smyth, Totally positive algebraic integers of small trace, *Ann. Inst. Fourier (Grenoble)* **34** (1984), 1–28, doi:10.5802/aif.975.

[19] M. A. Tsfasman, Global fields, codes and sphere packings, *Astérisque* **198–200** (1991), 373–396.

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Every finite group has a normal bi-Cayley graph*

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Abstract

A graph Γ with a group H of automorphisms acting semiregularly on the vertices with two orbits is called a *bi-Cayley graph* over H. When H is a normal subgroup of $\mathrm{Aut}(\Gamma)$, we say that Γ is *normal* with respect to H. In this paper, we show that every finite group has a connected normal bi-Cayley graph. This improves a theorem by Arezoomand and Taeri and provides a positive answer to a question reported in the literature.

Keywords: Normal, bi-Cayley, Cartesian product.

Math. Subj. Class.: 05C25, 20B25

1 Introduction

Throughout this paper, groups are assumed to be finite, and graphs are assumed to be finite, connected, simple and undirected. For the group-theoretic and graph-theoretic terminology not defined here we refer the reader to [5, 23].

Let G be a permutation group on a set Ω and $\alpha \in \Omega$. Denote by G_α the stabilizer of α in G, that is, the subgroup of G fixing the point α. We say that G is *semiregular* on Ω if $G_\alpha = 1$ for every $\alpha \in \Omega$ and *regular* if G is transitive and semiregular. It is well-known that a graph Γ is a *Cayley graph* if it has an automorphism group acting regularly on its vertex set (see [4, Lemma 16.3]). If we, instead, require that the graph Γ admits a group of automorphisms acting semiregularly on its vertex set with two orbits, then we obtain the so-called *bi-Cayley graph*.

Cayley graph is usually defined in the following way. Given a finite group G and an inverse closed subset $S \subseteq G \setminus \{1\}$, the *Cayley graph* $\mathrm{Cay}(G, S)$ on G with respect to S is a

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graph with vertex set G and edge set $\{\{g, sg\} \mid g \in G, s \in S\}$. For any $g \in G$, $R(g)$ is the permutation of G defined by $R(g)\colon x \mapsto xg$ for $x \in G$. Set $R(G) := \{R(g) \mid g \in G\}$. It is well-known that $R(G)$ is a subgroup of $\mathrm{Aut}(\mathrm{Cay}(G, S))$. In 1981, Godsil [10] proved that the normalizer of $R(G)$ in $\mathrm{Aut}(\mathrm{Cay}(G, S))$ is $R(G) \rtimes \mathrm{Aut}(G, S)$, where $\mathrm{Aut}(G, S)$ is the group of automorphisms of G fixing the set S set-wise. This result has been successfully used in characterizing various families of GRRs, namely, Cayley graphs $\mathrm{Cay}(G, S)$ such that $R(G) = \mathrm{Aut}(\mathrm{Cay}(G, S))$ (see, for example, [10, 11]). A Cayley graph $\mathrm{Cay}(G, S)$ is said to be *normal* if $R(G)$ is normal in $\mathrm{Aut}(\mathrm{Cay}(G, S))$. This concept was introduced by Xu in [24], and for more results about normal Cayley graphs, we refer the reader to [8].

Similarly, every bi-Cayley graph admits the following concrete realization. Given a group H, let R, L and S be subsets of H such that $R^{-1} = R$, $L^{-1} = L$ and $R \cup L$ does not contain the identity element of H. The *bi-Cayley graph* over H relative to the triple (R, L, S), denoted by $\mathrm{BiCay}(H,\ R,\ L,\ S)$, is the graph having vertex set the union of the right part $H_0 = \{h_0 \mid h \in H\}$ and the left part $H_1 = \{h_1 \mid h \in H\}$, and edge set the union of the right edges $\{\{h_0,\ g_0\} \mid gh^{-1} \in R\}$, the left edges $\{\{h_1,\ g_1\} \mid gh^{-1} \in L\}$ and the spokes $\{\{h_0,\ g_1\} \mid gh^{-1} \in S\}$. Let $\Gamma = \mathrm{BiCay}(H, R, L, S)$. For $g \in H$, define a permutation $BR(g)$ on the vertices of Γ by the rule

$$h_i^{BR(g)} = (hg)_i, \forall i \in \mathbb{Z}_2, h \in H.$$

Then $BR(H) = \{BR(g) \mid g \in H\}$ is a semiregular subgroup of $\mathrm{Aut}(\Gamma)$ which is isomorphic to H and has H_0 and H_1 as its two orbits. When $BR(H)$ is normal in $\mathrm{Aut}(\Gamma)$, the bi-Cayley graph $\Gamma = \mathrm{BiCay}(H, R, L, S)$ will be called a *normal bi-Cayley graph* over H (see [3] or [27]).

Wang et al. in [22] determined the groups having a connected normal Cayley graph.

Proposition 1.1. *Every finite group G has a normal Cayley graph unless $G \cong C_4 \times C_2$ or $G \cong \mathrm{Q}_8 \times C_2^r (r \geq 0)$.*

Following up this result, Arezoomand and Taeri in [3] asked: Which finite groups have normal bi-Cayley graphs? They also gave a partial answer to this question by proving that every finite group $G \ncong \mathrm{Q}_8 \times C_2^r (r \geq 0)$ has at least one normal bi-Cayley graph. At the end of [3], the authors asked the following question:

Question 1.2 ([3, Question])**.** Is there any normal bi-Cayley graph over $G \cong \mathrm{Q}_8 \times C_2^r$ for each $r \geq 0$?

We remark that for every finite group $G \ncong \mathrm{Q}_8 \times C_2^r (r \geq 0)$, the normal bi-Cayley graph over G constructed in the proof of [3, Theorem 5] is not of regular valency, and so is not vertex-transitive. So it is natural to ask the following question.

Question 1.3. Is there any vertex-transitive normal bi-Cayley graph over a finite group G?

In this paper, Questions 1.2 and 1.3 are answered in positive. The following is the main result of this paper.

Theorem 1.4. *Every finite group has a vertex-transitive normal bi-Cayley graph.*

To end this section we give some notation which is used in this paper. For a positive integer n, denote by C_n the cyclic group of order n, by $\mathbb{Z}_n$ the ring of integers modulo n, by D_{2n} the dihedral group of order $2n$, and by $\mathrm{Alt}(n)$ and $\mathrm{Sym}(n)$ the alternating group and symmetric group of degree n, respectively. Denote by Q_8 the quaternion group. For two groups M and N, $N \rtimes M$ denotes a semidirect product of N by M. The identity element of a finite group G is denoted by $\mathbf{1}$.

For a finite, simple and undirected graph Γ, we use $V(\Gamma)$, $E(\Gamma)$ and $\mathrm{Aut}(\Gamma)$ to denote its vertex set, edge set and full automorphism group, respectively, and for any $u, v \in V(\Gamma)$, $u \sim v$ means that u and v are adjacent. A graph Γ is said to be *vertex-transitive* if its full automorphism group $\mathrm{Aut}(\Gamma)$ acts transitively on its vertex set. For any subset B of $V(\Gamma)$, the subgraph of Γ induced by B will be denoted by $\Gamma[B]$.

2 Cartesian products

The *Cartesian product* $X \,\square\, Y$ of graphs X and Y is a graph with vertex set $V(X) \times V(Y)$, and with vertices (u, x) and (v, y) being adjacent if and only if $u = v$ and $x \sim y$ in Y, or $x = y$ and $u \sim v$ in X.

A non-trivial graph X is *prime* if it is not isomorphic to a Cartesian product of two smaller graphs. The following proposition shows the uniqueness of the prime factor decomposition of connected graphs with respect to the Cartesian product.

Proposition 2.1 ([12, Theorem 6.6]). *Every connected finite graph can be decomposed as a Cartesian product of prime graphs, uniquely up to isomorphism and the order of the factors.*

Two non-trivial graphs are *relatively prime* (w.r.t. Cartesian product) if they have no non-trivial common factor. Now we consider the automorphisms of Cartesian product of graphs.

Proposition 2.2 ([12, Theorem 6.10]). *Suppose ϕ is an automorphism of a connected graph Γ with prime factor decomposition $\Gamma = \Gamma_1 \,\square\, \Gamma_2 \,\square \cdots \square\, \Gamma_k$. Then there is a permutation π of $\{1, 2, \ldots, k\}$ and isomorphisms $\phi_i \colon \Gamma_{\pi(i)} \to \Gamma_i$ for which*

$$\phi(x_1, x_2, \ldots, x_k) = (\phi_1(x_{\pi(1)}), \phi_2(x_{\pi(2)}), \ldots, \phi_k(x_{\pi(k)})).$$

Corollary 2.3 ([12, Corollary 6.12]). *Let Γ be a connected graph with prime factor decomposition $\Gamma = \Gamma_1 \,\square\, \Gamma_2 \,\square \cdots \square\, \Gamma_k$. If $\Gamma_1, \Gamma_2, \ldots, \Gamma_k$ are relatively prime, then* $\mathrm{Aut}(\Gamma) = \mathrm{Aut}(\Gamma_1) \times \mathrm{Aut}(\Gamma_2) \times \cdots \times \mathrm{Aut}(\Gamma_k)$.

The following theorem provides a method of constructing normal bi-Cayley graphs.

Theorem 2.4. *Let X be a connected normal bi-Cayley graph over a group H, and let Y be a connected normal Cayley graph over a group K. If X and Y are relatively prime, then $X \,\square\, Y$ is also a normal bi-Cayley graph over the group $H \times K$.*

Proof. Assume that X and Y are relatively prime. By Corollary 2.3, $\mathrm{Aut}(X \,\square\, Y) = \mathrm{Aut}(X) \times \mathrm{Aut}(Y)$. Since X is a connected normal bi-Cayley graph over H, one has $BR(H) \trianglelefteq \mathrm{Aut}(X)$, and since Y is a connected normal Cayley graph over a group K, one has $R(K) \trianglelefteq \mathrm{Aut}(Y)$. Then $BR(H) \times R(K)$ is a normal subgroup of $\mathrm{Aut}(X \,\square\, Y) = \mathrm{Aut}(X) \times \mathrm{Aut}(Y)$. Note that $BR(H)$ acts semiregularly on $V(X)$ with two orbits, and $R(K)$ acts regularly on $V(Y)$. It follows that $BR(H) \times R(K)$ acts semiregularly on $V(X) \times V(Y)$ with two orbits, and thereby $X \,\square\, Y$ is also a normal bi-Cayley graph over the group $H \times K$. $\square$

3 Normal bi-Cayley graphs over $\mathrm{Q}_8 \times C_2^r (r \geq 0)$

In this section, we shall answer Question 1.2 in positive.

3.1 The n-dimensional hypercube

For $n \geq 1$, the *n-dimensional hypercube*, denoted by Q_n, is the graph whose vertices are all the n-tuples of 0's and 1's with two n-tuples being adjacent if and only if they differ in exactly one place.

Let $N = C_2^n$ be an elementary abelian 2-group of order 2^n with a minimum generating set $S = \{s_1, s_2, s_3, \dots, s_n\}$. By the definition of Q_n, we have $\mathrm{Cay}(N, S) \cong Q_n$. For convenience of the statement, we assume that $Q_n = \mathrm{Cay}(N, S)$. If $n = 1$, then $Q_1 = \mathbf{K}_2$ and so $\mathrm{Aut}(Q_1) = N$. In what follows, assume that $n \geq 2$. It is easy to observe that for any distinct s_i, s_j there is a unique 4-cycle in Q_n passing through $\mathbf{1}, s_i, s_j$, where $\mathbf{1}$ is the identity element of N. So if a subgroup of $\mathrm{Aut}(Q_n)$ fixes S pointwise, then it also fixes every vertex at distance 2 from $\mathbf{1}$. By the connectedness and vertex-transitivity of Q_n, we have $\mathrm{Aut}(Q_n)_{\mathbf{1}}$ acts faithfully on S. It follows that $\mathrm{Aut}(Q_n)_{\mathbf{1}} \lesssim \mathrm{Sym}(n)$. On the other hand, it is easy to see that each permutation on S induces an automorphism of N, and so $\mathrm{Aut}(N, S) \cong \mathrm{Sym}(n)$. Since $\mathrm{Aut}(N, S) \leq \mathrm{Aut}(Q_n)_{\mathbf{1}}$, one has $\mathrm{Aut}(Q_n)_{\mathbf{1}} = \mathrm{Aut}(N, S) \cong \mathrm{Sym}(n)$. Consequently, we have $\mathrm{Aut}(Q_n) = R(N) \rtimes \mathrm{Aut}(N, S) \cong N \rtimes \mathrm{Sym}(n)$ (see also [25, Lemma 1.1]).

Note that Q_n is bipartite. Let $\mathrm{Aut}(Q_n)^*$ be the kernel of $\mathrm{Aut}(Q_n)$ acting on the two partition sets of Q_n. Let $E = R(N) \cap \mathrm{Aut}(Q_n)^*$. Then $E \trianglelefteq \mathrm{Aut}(Q_n)^*$ and $E \trianglelefteq R(N)$. It follows that $E \trianglelefteq \mathrm{Aut}(Q_n)^* R(N) = \mathrm{Aut}(Q_n)$. Clearly, E acts semiregularly on $V(Q_n)$ with two orbits. Thus, we have the following lemma.

Lemma 3.1. *Use the same notation as in the above three paragraphs. For any $n \geq 1$, Q_n is a normal Cayley graph over N, and Q_n is also a normal bi-Cayley graph over E.*

3.2 The Möbius-Kantor graph

The Möbius-Kantor graph $\mathrm{GP}(8, 3)$ is a graph with vertex set $V = \{i, i' \mid i \in \mathbb{Z}_8\}$ and edge set the union of the *outer edges* $\{\{i, i{+}1\} \mid i \in \mathbb{Z}_8\}$, the *inner edges* $\{\{i', (i{+}3)'\} \mid i \in \mathbb{Z}_8\}$, and the *spokes* $\{\{i, i'\} \mid i \in \mathbb{Z}_8\}$ (see Figure 1). Note that $\mathrm{GP}(8, 3)$ is a bipartite graph with bipartition sets $B_1 = \{1, 3, 5, 7, 0', 2', 4', 6'\}$ and $B_2 = \{0, 2, 4, 6, 1', 3', 5', 7'\}$.

In [26], the edge-transitive groups of automorphisms of $\mathrm{Aut}(\mathrm{GP}(8, 3))$ were determined. We first introduce the following automorphisms of $\mathrm{GP}(8, 3)$, represented as permutations on the vertex set V:

$$\begin{aligned}
\alpha &= (1\ 3\ 5\ 7)(0\ 2\ 4\ 6)(1'\ 3'\ 5'\ 7')(0'\ 2'\ 4'\ 6'),\\
\beta &= (0\ 1'\ 2)(0'\ 6'\ 3)(4\ 5'\ 6)(7\ 4'\ 2'),\\
\gamma &= (1\ 1')(2\ 6')(3\ 3')(4\ 0')(5\ 5')(6\ 2')(7\ 7')(0\ 4'),\\
\delta &= (1\ 1')(2\ 4')(3\ 7')(4\ 2')(5\ 5')(6\ 0')(7\ 3')(0\ 6').
\end{aligned}$$

By [26, Lemma 3.1], we have $\langle \alpha, \beta \rangle = \langle \alpha, \alpha^\beta \rangle \rtimes \langle \beta \rangle \cong \mathrm{Q}_8 \rtimes \mathbb{Z}_3$, where Q_8 is the quaternion group, and moreover, $\langle \alpha, \beta \rangle \trianglelefteq \mathrm{Aut}(\mathrm{GP}(8, 3))$. Clearly, $\langle \alpha, \alpha^\beta \rangle \cong \mathrm{Q}_8$ is the Sylow 2-subgroup of $\langle \alpha, \beta \rangle$, so $\langle \alpha, \alpha^\beta \rangle$ is characteristic in $\langle \alpha, \beta \rangle$, and then it is normal in $\mathrm{Aut}(\mathrm{GP}(8, 3))$ because $\langle \alpha, \beta \rangle \trianglelefteq \mathrm{Aut}(\mathrm{GP}(8, 3))$. For convenience of the statement, we let $\mathrm{Q}_8 = \langle \alpha, \alpha^\beta \rangle$. It is easy to see that Q_8 acts semiregularly on V with two orbits B_1 and B_2. Thus we have the following lemma.

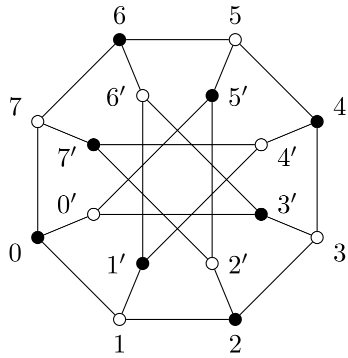

Figure 1: The Möbius-Kantor graph $\mathrm{GP}(8,3)$.

Lemma 3.2. $\mathrm{GP}(8,3)$ *is a normal bi-Cayley graph over* Q_8.

3.3 An answer to Question 1.2

Noting that $\mathrm{GP}(8,3)$ is of girth 6, $\mathrm{GP}(8,3)$ is prime. For each $r \geq 1$, it is easy to see that $Q_r = \underbrace{\mathbf{K}_2 \,\square\, \mathbf{K}_2 \,\square \cdots \square\, \mathbf{K}_2}_{n \text{ times}}$. So, Q_n and $\mathrm{GP}(8,3)$ are relatively prime. Now combining together Lemmas 3.1 and 3.2 and Theorem 2.4, we can obtain the following theorem.

Theorem 3.3. *For each* $r \geq 1$, $\mathrm{GP}(8,3) \times Q_r$ *is a vertex-transitive normal bi-Cayley graph over* $\mathrm{Q}_8 \times N$*, where* $N \cong C_2^r$.

4 Proof of Theorem 1.4

The proof of Theorem 1.4 will be completed by the following lemmas. Let G be a group. A Cayley graph $\Gamma = \mathrm{Cay}(G, S)$ on G is said to be a *graphical regular representation* (or *GRR* for short) of G if $\mathrm{Aut}(\Gamma) = R(G)$.

Lemma 4.1. *Let G be a group admitting a GRR* Γ*. Then* $\Gamma \,\square\, \mathbf{K}_2$ *is a normal bi-Cayley graph over the group G.*

Proof. If $\mathbf{K}_2$ and Γ are relatively prime, then by Corollary 2.3, we have $\mathrm{Aut}(\Gamma \,\square\, \mathbf{K}_2) = \mathrm{Aut}(\Gamma) \times \mathrm{Aut}(\mathbf{K}_2)$. Clearly, $R(G) \times \{\mathbf{1}\}$ acts semiregularly on $V(\Gamma \,\square\, \mathbf{K}_2)$ with two orbits, and $R(G) \times \{\mathbf{1}\} \trianglelefteq \mathrm{Aut}(\Gamma \,\square\, \mathbf{K}_2)$, where $\mathbf{1}$ is the identity of $\mathrm{Aut}(\mathbf{K}_2)$. It follows that $\Gamma \,\square\, \mathbf{K}_2$ is a normal bi-Cayley graph over the group G.

Suppose that $\mathbf{K}_2$ is also a prime factor of Γ. Let $\Gamma = \Gamma_1 \,\square\, \underbrace{\mathbf{K}_2 \,\square \cdots \square\, \mathbf{K}_2}_{m \text{ times}}$ be such that Γ_1 is coprime to $\mathbf{K}_2$. From Corollary 2.3 it follows that $G = \mathrm{Aut}(\Gamma) = \mathrm{Aut}(\Gamma_1) \times \mathrm{Aut}(\mathbf{K}_2 \,\square \cdots \square\, \mathbf{K}_2)$. Since Γ is a GRR of G, one has $m = 1$, and therefore $\Gamma \,\square\, \mathbf{K}_2 = \Gamma_1 \,\square\, \mathbf{K}_2 \,\square\, \mathbf{K}_2$. Then $G = \mathrm{Aut}(\Gamma_1) \times \mathrm{Aut}(\mathbf{K}_2 \,\square\, \mathbf{K}_2)$, and Γ_1 is a GRR of $\mathrm{Aut}(\Gamma_1)$. By Lemma 3.1, $\mathbf{K}_2 \,\square\, \mathbf{K}_2$ is a normal bi-Cayley graph over C_2, and by Theorem 2.4, $\Gamma \,\square\, \mathbf{K}_2$ is a normal bi-Cayley graph over $\mathrm{Aut}(\Gamma_1) \times C_2 \cong G$. $\square$

A group G is called *generalized dicyclic group* if it is non-abelian and has an abelian subgroup L of index 2 and an element $b \in G \setminus L$ of order 4 such that $b^{-1}xb = x^{-1}$ for every $x \in L$.

The following theorem gives a list of groups having no GRR (see [9]).

Theorem 4.2. *A finite group G admits a GRR unless G belongs to one of the following classes of groups:*

(I) *Class C: abelian groups of exponent greater than two;*

(II) *Class D: the generalized dicyclic groups;*

(III) *Class E: the following thirteen "exceptional groups":*

(1) $\mathbb{Z}_2^2, \mathbb{Z}_2^3, \mathbb{Z}_2^4$;
(2) D_6, D_8, D_{10};
(3) A_4;
(4) $\langle a, b, c \mid a^2 = b^2 = c^2 = 1, abc = bca = cab\rangle$;
(5) $\langle a, b \mid a^8 = b^2 = 1, bab = a^5\rangle$;
(6) $\langle a, b, c \mid a^3 = b^3 = c^2 = 1, ab = ba, (ac)^2 = (cb)^2 = 1\rangle$;
(7) $\langle a, b, c \mid a^3 = b^3 = c^3 = 1, ac = ca, bc = cb, c = a^{-1}b^{-1}cb\rangle$;
(8) $\mathrm{Q}_8 \times \mathbb{Z}_3, \mathrm{Q}_8 \times \mathbb{Z}_4$.

Lemma 4.3. *Let G be a group in Class D of Theorem 4.2. Then G has a normal bi-Cayley graph.*

Proof. If $G \cong \mathrm{Q}_8 \times C_2^r$ for some $r \geq 0$, then by Theorem 3.3 and Lemma 3.2, G has a normal bi-Cayley graph. In what follows, we assume that $G \ncong \mathrm{Q}_8 \times C_2^r$ for any $r \geq 0$. By Proposition 1.1, G has a normal Cayley graph, say Γ. If Γ is coprime to $\mathbf{K}_2$, then by Corollary 2.3, $\Gamma \,\square\, \mathbf{K}_2$ is a normal bi-Cayley graph over G.

Now suppose that $\mathbf{K}_2$ is a prime factor of Γ. Let $\Gamma = \Gamma_1 \,\square\, Q_m$, where $Q_m = \underbrace{\mathbf{K}_2 \,\square \cdots \square\, \mathbf{K}_2}_{m \text{ times}}$ and Γ_1 is coprime to $\mathbf{K}_2$. Again by Corollary 2.3, we have $\mathrm{Aut}(\Gamma) = \mathrm{Aut}(\Gamma_1) \times \mathrm{Aut}(Q_m)$. For any $x \in V(Q_m)$, set $V_x = \{(u, x) \mid u \in V(\Gamma_1)\}$, and for any $y \in V(\Gamma_1)$, set $U_y = \{(y, v) \mid v \in V(Q_m)\}$. Then $\Gamma[V_x] \cong \Gamma_1$ and $\Gamma[V_y] \cong Q_m$. Let G_{V_x} and G_{U_y} be the subgroups of G fixing V_x and U_y setwise, respectively. We shall prove the following claim.

Claim. $G = G_{V_x} \times G_{U_y}$, Γ_1 is a normal Cayley graph over a group which is isomorphic to G_{V_x}, and $G_{U_y} \cong C_2^m$.

Since Γ is vertex-transitive, by Proposition 2.2, V_x is an orbit of $\mathrm{Aut}(\Gamma_1) \times \{\mathbf{1}\}$ and $\mathrm{Aut}(\Gamma_1) \times \{\mathbf{1}\} = \mathrm{Aut}(\Gamma[V_x])$. As $\mathrm{Aut}(\Gamma_1) \times \{\mathbf{1}\} \trianglelefteq \mathrm{Aut}(\Gamma)$, each V_x is a block of imprimitivity of $\mathrm{Aut}(\Gamma)$ (namely, either $V_x^g = V_x$ or $V_x^g \cap V_x = \emptyset$ for any $g \in \mathrm{Aut}(\Gamma)$). Consider the quotient graph Γ' with vertex set $\{V_x \mid x \in V(Q_m)\}$, and V_x is adjacent to $V_{x'}$ if and only if x is adjacent to x' in Q_m. Then $\Gamma' \cong Q_m$, and $\mathrm{Aut}(\Gamma_1) \times \{\mathbf{1}\}$ is just the kernel of $\mathrm{Aut}(\Gamma)$ acting on $V(\Gamma')$. This implies that the subgroup $\mathrm{Aut}(\Gamma)_{V_x}$ of $\mathrm{Aut}(\Gamma)$ fixing V_x set-wise is just $\mathrm{Aut}(\Gamma_1) \times \mathrm{Aut}(Q_m)_x$. Since G is regular on $V(\Gamma)$, G_{V_x} is also regular on V_x, and so $\Gamma_1 \cong \Gamma[V_x]$ may be viewed as a Cayley graph on G_{V_x}. Since $G \trianglelefteq \mathrm{Aut}(\Gamma)$, one has $G_{V_x} = G \cap \mathrm{Aut}(\Gamma)_{V_x} \trianglelefteq \mathrm{Aut}(\Gamma)_{V_x}$. Note that $\{\mathbf{1}\} \times \mathrm{Aut}(Q_m)_x$ fixes every vertex in V_x. It follows that $G_{V_x} \cap (\{\mathbf{1}\} \times \mathrm{Aut}(Q_m)_x)$ is trivial, and so G_{V_x} can be viewed as a normal regular subgroup of $\mathrm{Aut}(\Gamma_1) \times \{\mathbf{1}\}$. Therefore, Γ_1 is a normal Cayley graph over some group, say $H \cong G_{V_x}$.

With a similar argument as above, we can show that Q_m is also a normal Cayley graph over some group, say $K \cong G_{U_y}$. From the argument in Section 3.1, we have $\mathrm{Aut}(Q_m) = N \rtimes \mathrm{Sym}(m)$ with $N \cong C_2^m$. We claim that $K = N$. If this is not true, then we would have $1 \neq KN/N \trianglelefteq \mathrm{Aut}(Q_m)/N \cong \mathrm{Sym}(m)$, and since K is a 2-group, the only possibility is $m = 4$. However, by Magma [6], $\mathrm{Aut}(Q_4)$ has only one normal regular subgroup which is isomorphic to C_2^4, a contradiction. Thus, $K = N \cong C_2^m$, and hence $G_{U_y} \cong C_{2^m}$.

For any $g \in G_{V_x} \cap G_{U_y}$, we have g fixes (y, x) and so $g = \mathbf{1}$ because G is regular on $V(\Gamma)$. Thus, $G_{V_x} \cap G_{U_y} = \{\mathbf{1}\}$. Then $|G_{V_x} G_{U_y}| = |G_{V_x}||G_{U_y}| = |V_x||U_y| = |V(\Gamma)| = |G|$. It follows that $G = G_{V_x} G_{U_y}$. To show that $G = G_{V_x} \times G_{U_y}$, it suffices to show that both G_{V_x} and G_{U_y} are normal in G. As G is a generalized dicyclic group, it is non-abelian and has an abelian subgroup L of index 2 and an element $b \in G \setminus L$ of order 4 such that $b^{-1}ab = a^{-1}$ for every $a \in L$.

Suppose that $G_{U_y} \nleq L$. Then there exists $g \in G_{U_y}$ such that $g = ab^i$ for some $a \in L$ and $i = 1$ or -1. Since $G_{U_y} \cong C_2^m$, g is also an involution, and so $G = L \rtimes \langle g \rangle$. Clearly, for any $a \in L$, we have $g^{-1}ag = a^{-1}$, and so $(ga)^2 = 1$. This would force that every element of G outside L is an involution, a contradiction. Thus, $G_{U_y} \leq L$, and hence $G_{U_y} \trianglelefteq G$.

Since $G = G_{V_x} G_{U_y}$, $G_{U_y} \leq L$ implies that $G_{V_x} \nleq L$. Then $|G_{V_x} : G_{V_x} \cap L| = 2$ since $|G : L| = 2$. It then follows that $G_{V_x} \cap L \trianglelefteq G$, and hence

$$G/G_{V_x} \cap L = (G_{U_y}(G_{V_x} \cap L)/G_{V_x} \cap L) \rtimes (G_{V_x}/G_{V_x} \cap L).$$

Again as G is a generalized dicyclic group and since $G_{U_y} \leq L$, the non-trivial element of $G_{V_x}/G_{V_x} \cap L$ maps every element of $G_{U_y}(G_{V_x} \cap L)/G_{V_x} \cap L$ to its inverse. Since $G_{U_y} \cong C_{2^m}$, one has $G/G_{V_x} \cap L$ is abelian, and so $G_{V_x} \trianglelefteq G$, completing the proof of the Claim.

By Lemma 3.1, we may let $Q_{m+1} = \underbrace{\mathbf{K}_2 \,\square \cdots \square\, \mathbf{K}_2}_{m+1 \text{ times}}$ be a connected normal bi-Cayley graph over $G_{U_y} \cong C_2^m$. By Claim, we may view Γ_1 as a normal Cayley graph over G_{V_x}. Since Γ_1 is coprime to $\mathbf{K}_2$, by Theorem 2.4, $\Gamma_1 \,\square\, Q_{m+1}$ is a connected normal bi-Cayley graph over $G_{V_x} \times G_{U_y} = G$. $\square$

Lemma 4.4. *Let G be a group in Class E of Theorem 4.2. Then G has a normal bi-Cayley graph.*

Proof. By Lemma 3.1, each of the groups in Class E (1) has a connected normal bi-Cayley graph.

Let $G = D_{2n} = \langle a, b \mid a^n = b^2 = 1, b^{-1}ab = a^{-1} \rangle$ with $n \geq 3$. Let $\Gamma = \mathrm{Cay}(G, \{ab, b\})$. Then Γ is a cycle of length $2n$, and so Γ is coprime to $\mathbf{K}_2$. By Theorem 2.4, $\Gamma \,\square\, \mathbf{K}_2$ is a connected normal bi-Cayley graph over G. Thus, each of the groups in Class E (2) has a connected normal bi-Cayley graph.

Let $G = \mathrm{Alt}(4)$ and let $\Gamma = \mathrm{Cay}(G, \{(1\,2\,3), (1\,3\,2), (1\,2\,4), (1\,4\,2)\})$. By Magma [6], we have $\Gamma \,\square\, \mathbf{K}_2$ is a connected normal bi-Cayley graph over $\mathrm{Alt}(4)$.

Let $G = \langle a, b, c \mid a^2 = b^2 = c^2 = 1, abc = bca = cab \rangle$ be the group in Class E (4). Let $\Gamma = \mathrm{Cay}(G, \{a, b, c\})$. By Magma [6], Γ is a connected trivalent normal Cayley graph over G and Γ has girth 6. Hence, Γ is coprime to $\mathbf{K}_2$. By Theorem 2.4, $\Gamma \,\square\, \mathbf{K}_2$ is a connected normal bi-Cayley graph over G.

Let $G = \langle a, b \mid a^8 = b^2 = 1, bab = a^5\rangle$ be the group in Class E (5). Let $\Gamma = \mathrm{Cay}(G, \{a, a^{-1}, b, a^4, a^4b\})$. By [22, Lemma 6], Γ is a connected normal Cayley graph over G, and by Magma, $\mathrm{Aut}(\Gamma \,\square\, \mathbf{K}_2) = \mathrm{Aut}(\Gamma) \times \mathbb{Z}_2$. Thus, $\Gamma \,\square\, \mathbf{K}_2$ is a normal bi-Cayley graph over G.

Let $G = \langle a, b, c \mid a^3 = b^3 = c^2 = 1, ac = ca, (ab)^2 = (cb)^2 = 1\rangle$ be the group in Class E (6). Let $\Gamma = \mathrm{Cay}(G, \{c, ca, cb\})$. By Magma [6], Γ is a connected trivalent normal Cayley graph over G and Γ has girth 6. Hence, Γ is coprime to $\mathbf{K}_2$. By Lemma 2.4, $\Gamma \,\square\, \mathbf{K}_2$ is a connected normal bi-Cayley graph over G.

Let $G = \langle a, b, c \mid a^3 = b^3 = c^3 = 1, ac = ca, bc = cb, c = a^{-1}b^{-1}cb\rangle$ be the group in Class E (7). Let $\Gamma = \mathrm{Cay}(G, \{a, b, a^{-1}, b^{-1}\})$. By Magma [6], Γ is a connected trivalent normal Cayley graph over G. Since G has order 27, Γ is coprime to $\mathbf{K}_2$. By Theorem 2.4, $\Gamma \,\square\, \mathbf{K}_2$ is a connected normal bi-Cayley graph over G.

Finally, we consider the groups in Class E (8). By Lemma 3.2, $\mathrm{GP}(8, 3)$ is a normal bi-Cayley graph over Q_8. For $n \geq 3$, let $C_n = \langle a\rangle$ and let $\Gamma = \mathrm{Cay}(C_n, \{a, a^{-1}\})$. Clearly, Γ is a normal Cayley graph over C_n. Since $\mathrm{GP}(8, 3)$ is of girth 6, $\mathrm{GP}(8, 3)$ is coprime to Γ. By Theorem 2.4, $\mathrm{GP}(8, 3) \,\square\, \Gamma$ is a connected normal bi-Cayley graph over $\mathrm{Q}_8 \times C_n$. Thus each of the groups in Class E (8) has a connected normal bi-Cayley graph. □

Lemma 4.5. *Let G be a group in Class C of Theorem 4.2. Then G has a normal bi-Cayley graph.*

Proof. Since G is abelian, G has an automorphism α such that α maps every element of G to its inverse. Set $H = G \rtimes \langle\alpha\rangle$. If H has a GRR Γ, then Γ is also a normal bi-Cayley graph over G. Suppose that H has no GRR. Then by Theorem 4.2 we have H is one of the groups in Class E (2) and (6). By Lemma 4.4, G has a normal bi-Cayley graph □

Proof of Theorem 1.4. Let G be a finite group. If G has a GRR, then by Lemma 4.1, G has a connected normal bi-Cayley graph. If G does not have a GRR, then the theorem follows from Lemmas 4.3, 4.4, 4.5 and 3.2. □

5 Final remarks

This paper would not be complete without mentioning some related work, namely on some special families of bi-Cayley graphs such as bi-circulants, bi-abelians etc. Numerous papers on the topic have been published (see, for instance, [1, 2, 7, 13, 14, 15, 16, 17, 18, 19, 20, 21]). In view of these, the following problem arises naturally.

Problem 5.1. For a given finite group H, classify or characterize bi-Cayley graphs with specific symmetry properties over H.

Let H be a finite group. We say that a bi-Cayley graph Γ of regular valency over H is a *bi-graphical regular representation* (or *bi-GRR* for short) if $\mathrm{Aut}(\Gamma) = BR(H)$. Motivated by the classification of finite groups having no GRR (see Theorem 4.2), we would like to pose the following problem.

Problem 5.2. Determine finite groups which have no bi-GRR.

References

[1] I. Antončič, A. Hujdurović and K. Kutnar, A classification of pentavalent arc-transitive bicirculants, *J. Algebraic Combin.* **41** (2015), 643–668, doi:10.1007/s10801-014-0548-z.

[2] A. Araluze, I. Kovács, K. Kutnar, L. Martínez and D. Marušič, Partial sum quadruples and bi-Abelian digraphs, *J. Combin. Theory Ser. A* **119** (2012), 1811–1831, doi:10.1016/j.jcta.2012.06.004.

[3] M. Arezoomand and B. Taeri, Normality of 2-Cayley digraphs, *Discrete Math.* **338** (2015), 41–47, doi:10.1016/j.disc.2014.10.019.

[4] N. Biggs, *Algebraic Graph Theory*, Cambridge Mathematical Library, Cambridge University Press, Cambridge, 2nd edition, 1993, doi:10.1017/cbo9780511608704.

[5] J. A. Bondy and U. S. R. Murty, *Graph Theory with Applications*, Elsevier, New York, 1976.

[6] W. Bosma, J. Cannon and C. Playoust, The `Magma` algebra system. I: The user language, *J. Symbolic Comput.* **24** (1997), 235–265, doi:10.1006/jsco.1996.0125.

[7] E. Dobson, I. Kovács and Š. Miklavič, The automorphism groups of non-edge-transitive rose window graphs, *Ars Math. Contemp.* **9** (2015), 63–75, `http://amc-journal.eu/index.php/amc/article/view/362`.

[8] Y.-Q. Feng, Z.-P. Lu and M.-Y. Xu, Automorphism groups of cayley digraphs, in: J. Koolen, J. H. Kwak and M.-Y. Xu (eds.), *Applications of Group Theory to Combinatorics*, CRC Press, Boca Raton, Florida, pp. 13–25, 2008, doi:10.1201/9780203885765.ch2.

[9] C. D. Godsil, GRR's for non-solvable groups, in: L. Lovász and V. T. Sós (eds.), *Algebraic Methods in Graph Theory. Volume 1*, number 25 in Coll. Math. Soc. J. Bolyai, pp. 221–239, 1981.

[10] C. D. Godsil, On the full automorphism group of a graph, *Combinatorica* **1** (1981), 243–256, doi:10.1007/bf02579330.

[11] C. D. Godsil, The automorphism groups of some cubic Cayley graphs, *European J. Combin.* **4** (1983), 25–32, doi:10.1016/s0195-6698(83)80005-5.

[12] R. Hammack, W. Imrich and S. Klavžar, *Handbook of Product Graphs*, Discrete Mathematics and its Applications, CRC Press, Boca Raton, Florida, 2nd edition, 2011, doi:10.1201/b10959.

[13] M. Hladnik, D. Marušič and T. Pisanski, Cyclic Haar graphs, *Discrete Math.* **244** (2002), 137–152, doi:10.1016/s0012-365x(01)00064-4.

[14] H. Koike and I. Kovács, Isomorphic tetravalent cyclic Haar graphs, *Ars Math. Contemp.* **7** (2014), 215–235, `http://amc-journal.eu/index.php/amc/article/view/302`.

[15] I. Kovács, B. Kuzman, A. Malnič and S. Wilson, Characterization of edge-transitive 4-valent bicirculants, *J. Graph Theory* **69** (2012), 441–463, doi:10.1002/jgt.20594.

[16] I. Kovács, A. Malnič, D. Marušič and Š. Miklavič, One-matching bi-Cayley graphs over abelian groups, *European J. Combin.* **30** (2009), 602–616, doi:10.1016/j.ejc.2008.06.001.

[17] A. Malnič, D. Marušič and P. Šparl, On strongly regular bicirculants, *European J. Combin.* **28** (2007), 891–900, doi:10.1016/j.ejc.2005.10.010.

[18] L. Martínez Fernández, Strongly regular m-Cayley circulant graphs and digraphs, *Ars Math. Contemp.* **8** (2015), 195–213, `http://amc-journal.eu/index.php/amc/article/view/540`.

[19] D. Marušič, Strongly regular bicirculants and tricirculants, *Ars Combin.* **25** (1988), 11–15.

[20] D. Marušič and T. Pisanski, Symmetries of hexagonal molecular graphs on the torus, *Croat. Chem. Acta* **73** (2000), 969–981, `http://hrcak.srce.hr/131972`.

[21] T. Pisanski, A classification of cubic bicirculants, *Discrete Math.* **307** (2007), 567–578, doi:10.1016/j.disc.2005.09.053.

[22] C. Wang, D. Wang and M.-Y. Xu, Normal Cayley graphs of finite groups, *Sci. China Ser. A* **41** (1998), 242–251, doi:10.1007/bf02879042.

[23] H. Wielandt, *Finite Permutation Groups*, Academic Press, New York, 1964, doi:10.1016/c2013-0-11702-3.

[24] M.-Y. Xu, Automorphism groups and isomorphisms of Cayley digraphs, *Discrete Math.* **182** (1998), 309–319, doi:10.1016/s0012-365x(97)00152-0.

[25] C. Zhou and Y.-Q. Feng, A note on Weichsel's conjecture, *Adv. Math. (China)* **36** (2007), 61–66, doi:10.11845/sxjz.2007.36.01.0061.

[26] J.-X. Zhou and Y.-Q. Feng, Edge-transitive cyclic regular covers of the Möbius-Kantor graph, *European J. Combin.* **33** (2012), 139–147, doi:10.1016/j.ejc.2011.09.034.

[27] J.-X. Zhou and Y.-Q. Feng, The automorphisms of bi-Cayley graphs, *J. Combin. Theory Ser. B* **116** (2016), 504–532, doi:10.1016/j.jctb.2015.10.004.

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Flag-transitive automorphism groups of 2-designs with $\lambda \geq (r, \lambda)^2$ and an application to symmetric designs*

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Abstract

Let $\mathcal{D}$ be a 2-(v, k, λ) design with $\lambda \geq (r, \lambda)^2$. If $G \leq \mathrm{Aut}(\mathcal{D})$ is flag-transitive, then G cannot be of simple diagonal or twisted wreath product type, and if G is product type then the socle of G has exactly two components and G has rank 3. Furthermore, we prove that if $\mathcal{D}$ is symmetric, then G must be an affine or almost simple group.

Keywords: 2*-design, automorphism group, primitivity, flag-transitivity.*

Math. Subj. Class.: 05B05, 05B25, 20B25

1 Introduction

A 2-(v, k, λ) design is an incidence structure $\mathcal{D} = (\mathcal{P}, \mathcal{B})$ where $\mathcal{P}$ is a set of v points and $\mathcal{B}$ is a set of b blocks with incidence relation such that every block is incident with exactly k points, and every 2-element subset of $\mathcal{P}$ is incident with exactly λ blocks. Let r be the number of blocks incident with a given point. The numbers v, b, r, k, and λ are the parameters of $\mathcal{D}$. A design $\mathcal{D}$ is called simple if it has no repeated blocks, and is called symmetric if $v = b$, and nontrivial if $2 < k < v - 1$. Here we always assume that $\mathcal{D}$ is simple and nontrivial. An automorphism of $\mathcal{D}$ is a permutation of the points which also permutes the blocks and preserves the incidence relation. The set of all automorphisms of $\mathcal{D}$ with the composition of maps is a group, denoted by $\mathrm{Aut}(\mathcal{D})$. Let $G \leq \mathrm{Aut}(\mathcal{D})$. If G is a primitive permutation group on the point set $\mathcal{P}$, then G is called point-primitive,

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otherwise point-imprimitive. A flag in a design is an incident point-block pair, and G is called flag-transitive if G is transitive on the set of flags.

There are many research works on flag-transitive 2-designs. It was shown in [1] that if a linear space admits a flag-transitive automorphism group G, then G is either of affine or almost simple type. Then the classification of flag-transitive linear spaces was announced in [2], and the complete proof was given by Liebeck [11] for affine type and Saxl [15] for almost simple type. In 1988, Zieschang [19] proved that if G is a flag-transitive automorphism group of a 2-design with $(r, \lambda) = 1$, then G is an affine or almost simple group. This paper study flag-transitive 2-(v, k, λ) designs under the condition that $\lambda \geq (r, \lambda)^2$. This condition has significance in design theory. On the one hand, the condition $\lambda \geq (r, \lambda)^2$ and the flag-transitivity of G implies that G is primitive [5, (2.3.7)] (also see Lemma 2.3 below), so we can use the O'Nan-Scott Theorem to analyze this type of designs. On the other hand, there exists many flag-transitive 2-designs satisfying the conditions $\lambda \geq (r, \lambda)^2$ and $(r, \lambda) > 1$. Before stating our main results, we give an example in the following.

Example 1.1. Let $\mathcal{P} = \{1, 2, 3, 4, 5, 6\}$, $G = \langle (3546), (162)(345) \rangle \cong S_5$ be a primitive group of degree 6 acting on $\mathcal{P}$. Let $B = \{1, 2, 4\}$. It is easily known that

$$\begin{aligned} B^G = \big\{ &\{1,2,4\}, \{1,3,5\}, \{4,5,6\}, \{1,3,4\}, \{1,2,6\}, \{2,3,6\}, \{1,2,5\}, \{2,4,6\}, \\ &\{1,4,6\}, \{3,5,6\}, \{2,3,4\}, \{1,2,3\}, \{1,5,6\}, \{3,4,5\}, \{1,4,5\}, \{1,3,6\}, \\ &\{2,5,6\}, \{2,4,5\}, \{2,3,5\}, \{3,4,6\} \big\}, \end{aligned}$$

and $G_B = \langle (124)(356), (12)(56) \rangle \cong D_6$ is transitive on B. Let $\mathcal{B} = B^G$. Then $\mathcal{D} = (\mathcal{P}, \mathcal{B})$ is a 2-$(6, 3, 4)$ design, and G acts flag-transitively on it.

More examples of flag-transitive 2-designs with $\lambda \geq (r, \lambda)^2$ and $(r, \lambda) > 1$ can be found in [18]. Our main theorem is the following partial improvement of Zieschang's result.

Theorem 1.2. *Let $\mathcal{D}$ be a 2-(v, k, λ) design with $\lambda \geq (r, \lambda)^2$. If G is a flag-transitive automorphism group of $\mathcal{D}$, then G is of affine, almost simple type, or product type with $\mathrm{Soc}(G) \cong T \times T$, where T is a nonabelian simple group and G has rank 3.*

Flag-transitive symmetric designs with λ small have been investigated by many researchers, including Kantor [10] for finite projective planes, Regueiro [13] for $\lambda \leq 3$, Fang et al. [8] and Regueiro [14] for $\lambda = 4$, Tian and Zhou [16] for $\lambda \leq 100$. In all these cases, it was proved that if a 2-(v, k, λ) symmetric design $\mathcal{D}$ admits a flag-transitive, point-primitive automorphism group G, then G must be of affine or almost simple type. As an application of Theorem 1.2, we get the following theorem on symmetric designs.

Theorem 1.3. *Let $\mathcal{D}$ be a 2-(v, k, λ) symmetric design with $\lambda \geq (r, \lambda)^2$, which admits a flag-transitive automorphism group G. Then G is an affine or almost simple group.*

The structure of the paper is organized as follows. Section 2 gives some preliminary lemmas on flag-transitive designs and permutation groups that will apply directly to our situation. In Section 3, we prove Theorem 1.2. Our strategy is based on the O'Nan-Scott Theorem [12] on finite primitive permutation groups, so we deal with the simple diagonal type, the twisted wreath product type, and the product type in Subsections 3.1, 3.2 and 3.3, respectively. In Section 4, we give a proof of Theorem 1.3.

2 Preliminaries

The following lemma is well known.

Lemma 2.1. *The parameters v, b, k, r, λ of a 2-(v, k, λ) design satisfy the following conditions:*

(i) $vr = bk$.

(ii) $\lambda(v-1) = r(k-1)$.

(iii) $b \geq v$ *and* $k \leq r$.

Lemma 2.2. *Let $\mathcal{D}$ be a 2-(v, k, λ) design, and G be a flag-transitive automorphism group of $\mathcal{D}$. Then*

(i) $v \leq \lambda v < r^2$.

(ii) $r \mid \lambda(v-1, |G_\alpha|)$, *where G_α is the stabilizer of a point α.*

(iii) $r \mid \lambda d$ *for all nontrivial subdegrees d of G, i.e., the lengths of the G_α-orbits.*

Proof. (i) By Lemma 2.1(ii), we have $\lambda v = r(k-1) + \lambda = rk - (r - \lambda)$, and the result follows by combining it with $k \leq r$ and $1 \leq \lambda < r$.

(ii) Since G is flag-transitive and $\lambda(v-1) = r(k-1)$, we have $r \mid |G_\alpha|$ and $r \mid \lambda(v-1)$. It follows that r divides $(\lambda(v-1), |G_\alpha|)$, and hence $r \mid \lambda(v-1, |G_\alpha|)$.

For (iii), $r \mid \lambda d$ was proved in [4] and [3, p. 91]. □

The following lemma first appears in [5, (2.3.7)].

Lemma 2.3. *Let $\mathcal{D}$ be a 2-(v, k, λ) design with $\lambda \geq (r, \lambda)^2$. If $G \leq \mathrm{Aut}(\mathcal{D})$ acts flag-transitively on $\mathcal{D}$, then G is point-primitive.*

Proof. Suppose that $G \leq \mathrm{Aut}(\mathcal{D})$ is flag-transitive and $\{C_1, C_2, ..., C_t\}$ is a system of t sets of imprimitivity each of size s. Then $v = st$. The set of imprimitivity containing a point α is a union of G_α-orbits, one of which is $\{\alpha\}$, hence by Lemma 2.2(iii) we have $s \equiv 1 \pmod{\frac{r}{(r,\lambda)}}$. Then $v = st \equiv t \pmod{\frac{r}{(r,\lambda)}}$, which implies $t \equiv \frac{r(k-1)}{\lambda} + 1 \equiv 1 \pmod{\frac{r}{(r,\lambda)}}$. Now let $s = \sigma\frac{r}{(r,\lambda)} + 1$ and $t = \tau\frac{r}{(r,\lambda)} + 1$. Then

$$v = \frac{r(k-1)}{\lambda} + 1 = st = (\sigma\frac{r}{(r,\lambda)} + 1)(\tau\frac{r}{(r,\lambda)} + 1)$$

and thus

$$\sigma\tau\frac{r\lambda}{(r,\lambda)^2} + (\sigma+\tau)\frac{\lambda}{(r,\lambda)} = k - 1. \tag{2.1}$$

Since G is flag-transitive and imprimitive, we must have a solution of (2.1) with $\sigma\tau \neq 0$. Hence if $\lambda \geq (r,\lambda)^2$, then (2.1) implies $r \leq \sigma\tau r < k - 1 < k$, a contradiction. □

Lemma 2.4 ([6, Lemma 2.5])**.** *Let $\mathcal{D}$ be a symmetric design and assume that $G \leq \mathrm{Aut}(\mathcal{D})$ is a primitive rank 3 permutation group on points and blocks. If $N = \mathrm{Soc}(G)$ is non-abelian, then N is simple.*

3 Proof of Theorem 1.2

In this section, we will assume that $\mathcal{D}$ is a 2-(v,k,λ) design with $\lambda \geq (r,\lambda)^2$ and $G \leq \mathrm{Aut}(\mathcal{D})$ is flag-transitive. By Lemma 2.3, G is point-primitive. The O'Nan-Scott Theorem classifies primitive groups into five types: (i) Affine type; (ii) Almost simple type; (iii) Simple diagonal type; (iv) Product type; (v) Twisted wreath product type, see [12] for details. We will rely on the O'Nan-Scott Theorem to prove Theorem 1.2 by dealing with the cases of simple diagonal action, twisted wreath product action and product action separately.

3.1 Simple diagonal action

Proposition 3.1. *Let $\mathcal{D}$ be a 2-(v,k,λ) design with $\lambda \geq (r,\lambda)^2$. If G is a flag-transitive automorphism group of $\mathcal{D}$, then G is not of simple diagonal type.*

Proof. Suppose that G is of simple diagonal type. Then

$$G \leq W = \{(a_1,\ldots,a_m)\pi \mid a_i \in \mathrm{Aut}(T), \pi \in S_m, a_i \equiv a_j \bmod \mathrm{Inn}(T) \text{ for all } i,j\},$$

and there is $\alpha \in \mathcal{P}$ such that

$$G_\alpha \leq \{(a,\ldots,a)\pi \mid a \in \mathrm{Aut}(T), \pi \in S_m\} \cong \mathrm{Aut}(T) \times S_m,$$

and

$$M_\alpha = D = \{(a,\ldots,a) \mid a \in \mathrm{Inn}(T)\}$$

is a diagonal subgroup of $M = T_1 \times \cdots \times T_m \cong T^m$. Put $\Sigma = \{T_1,\ldots,T_m\}$, where T_i is identified with the group $\{(1,1,\ldots,t,\ldots,1) \mid t \in T\}$ where t is in the i-th position. Then G acts on Σ [12]. Moreover the set $\mathcal{P}$ of points can be identified with the set M/D of right cosets of D in M so that $\alpha = D(1,\ldots,1)$, $v = |\mathcal{P}| = |T|^{m-1}$, and for $\beta = D(t_1,\ldots,t_m)$, $s = (s_1,\ldots,s_m) \in M$, $\sigma \in \mathrm{Aut}(T)$, $\pi \in S_m$, we have the actions

$$\beta^s = D(t_1 s_1,\ldots,t_m s_m),\ \beta^\sigma = D(t_1^\sigma,\ldots,t_m^\sigma),\ \beta^\pi = D(t_{1\pi^{-1}},\ldots,t_{m\pi^{-1}}).$$

Since $M \trianglelefteq G$ and G is primitive on $\mathcal{P}$, M is transitive on $\mathcal{P}$. Since $T_1 \trianglelefteq M$ it follows that T_1 acts $\frac{1}{2}$-transitively on $\mathcal{P}$ ([17, Theorem 10.3]), and so all its orbits have equal length $c > 1$. Let Γ_1 be the orbit of T_1 containing the point α. For any $t_1 = (t,1,\ldots,1) \in T_1$, we have $\alpha^{t_1} = D(t,1,\ldots,1)$, so that

$$\Gamma_1 = \alpha^{T_1} = \{D(t,1,\ldots,1) \mid t \in T\}$$

and $|\Gamma_1| = |T| = c$. Similarly, define $\Gamma_i = \alpha^{T_i}$ for $1 < i \leq m$. Clearly $\Gamma_i \cap \Gamma_j = \{\alpha\}$ for $i \neq j$ provided that $m \geq 2$.

Choose an orbit Δ of G_α in $\mathcal{P} - \{\alpha\}$ such that $|\Delta \cap \Gamma_1| = d \neq 0$. Let $m_1 = |G_\alpha : N_{G_\alpha}(T_1)|$. Since $G_\alpha \lesssim \mathrm{Aut}(T) \times S_m$ and G^Σ is transitive on Σ, it follows that $m_1 \leq m$, and thus

$$|\Delta| = m_1 d \leq m|T|.$$

Lemma 2.2(iii) implies $r \mid \lambda m_1 d$, so $r \leq (r,\lambda) m_1 d \leq (r,\lambda) m|T|$. From $\lambda v < r^2$ and $\lambda \geq (r,\lambda)^2$ we have

$$\lambda |T|^{m-1} < r^2 \leq ((r,\lambda)m|T|)^2 \leq \lambda m^2 |T|^2.$$

As T is a nonabelian simple group, we have

$$60^{m-3} \leq |T|^{m-3} < m^2,$$

from which it follows that $m \leq 3$. Since $T \cong M_\alpha \trianglelefteq G_\alpha \lesssim \mathrm{Aut}(T) \times S_m$ and $r \mid |G_\alpha|$, r also divides $|T||\mathrm{Out}(T)|m!$. Let $a = (r, \lambda)$, so that $\frac{r}{a}(k-1) = \frac{\lambda}{a}(|T|^{m-1} - 1)$. It follows that $\frac{r}{a}$ divides $|T|^{m-1} - 1$, and so $(\frac{r}{a}, |T|) = 1$, which implies $\frac{r}{a} \mid |\mathrm{Out}(T)|m!$. Therefore,

$$|T|^{m-1} = v \leq \frac{\lambda v}{a^2} < \frac{r^2}{a^2} \leq (|\mathrm{Out}(T)|m!)^2.$$

It follows that $|T| < 4|\mathrm{Out}(T)|^2$ when $m = 2$, and $|T| < 6|\mathrm{Out}(T)|$ when $m = 3$. By [16, Lemma 2.3], T is isomorphic to one of following groups:

$$L_2(q) \text{ for } q = 5, 7, 8, 9, 11, 13, 16, 27, \text{ or } L_3(4).$$

However, from the facts $|\mathrm{Out}(L_3(4))| = 12$, $|\mathrm{Out}(L_2(q))| = 2$ for $q \in \{5, 7, 8, 11, 13, 16, 27\}$ and $|\mathrm{Out}(L_2(9))| = 4$ that $|T| > 4|\mathrm{Out}(T)|^2 > 6|\mathrm{Out}(T)|$, a contradiction. □

3.2 Twisted wreath product action

Proposition 3.2. *Let $\mathcal{D}$ be a 2-(v, k, λ) design with $\lambda \geq (r, \lambda)^2$. If G is a flag-transitive automorphism group of $\mathcal{D}$, then G is not of twisted wreath product type.*

Proof. By Lemma 2.3, G is primitive on $\mathcal{P}$. Suppose G has a twisted wreath product action. Then

$$G = T\,\mathrm{twr}_Q\ P = {}_QB \rtimes P$$

where P is a transitive permutation group on $\{1, \ldots, m\}$ with $m \geq 6$ (see [7, Theorem 4.7B(iv)]), $Q = P_1$ and $M = \mathrm{Soc}(G) = {}_QB = T_1 \times \cdots \times T_m \cong T^m$. Put $\Sigma = \{T_1, \ldots, T_m\}$, where T_i is identified with the group $\{(1, 1, \ldots, t, \ldots, 1; 1) \mid t \in T\} \cong T$ where t is in the i-th position. Then G acts on Σ (see [12]). Moreover, the set $\mathcal{P}$ of points can be identified with $G \backslash P$, the set of right cosets of P in G, so that G acts transitively on $\mathcal{P}$. Define $\alpha = P$, so that $G_\alpha = P$ and $v = |\mathcal{P}| = |T|^m$.

Similarly to the case of simple diagonal action, let $\Gamma_1 = \alpha^{T_1} = \{P(t, 1, \ldots, 1; 1) \mid t \in T\}$ so that $|\Gamma_1| = |T|$, and define $\Gamma_i = \alpha^{T_i}$ for $1 < i \leq m$. Clearly $\Gamma_i \cap \Gamma_j = \{\alpha\}$ for $i \neq j$.

Choose an orbit Δ of G_α in $\mathcal{P} - \{\alpha\}$ such that $|\Delta \cap \Gamma_1| = d \neq 0$. Let $m_1 = |G_\alpha : N_{G_\alpha}(T_1)|$. Since $G_\alpha = P$ and G^Σ is transitive on Σ, it follows that $m_1 \leq m$, and thus $|\Delta| = m_1 d \leq m|T|$. Lemma 2.2(iii) implies $\frac{r}{(r,\lambda)} \mid m_1 d$, and then $r \leq (r, \lambda) m_1 d \leq (r, \lambda) m |T|$. On the other hand, by $\lambda v < r^2$ and $\lambda \geq (r, \lambda)^2$, we have $\lambda |T|^m < r^2 \leq ((r, \lambda) m |T|)^2$. It follows that

$$60^{m-2} \leq |T|^{m-2} < m^2.$$

Thus, $m \leq 2$. However, this contradicts the fact that $m \geq 6$. □

3.3 Product action

Proposition 3.3. *Let $\mathcal{D}$ be a 2-(v, k, λ) design with $\lambda \geq (r, \lambda)^2$ admitting a flag-transitive automorphism group G and G is of product type. Then $\mathrm{Soc}(G) = T_1 \times T_2$ (where $T_i \cong T$ is a nonabelian simple group) and G has rank 3.*

Suppose that G has a product action on $\mathcal{P}$. Then there is a group K with a primitive action (of almost simple or diagonal type) on a set Γ of size $v_0 \geq 5$, where

$$\mathcal{P} = \Gamma^m, G \leq K^m \rtimes S_m = K \wr S_m \text{ and } m \geq 2.$$

The proof of Proposition 3.3 follows from the next two lemmas.

Lemma 3.4. *If G acts flag-transitively on a* 2-(v, k, λ) *design with* $\lambda \geq (r, \lambda)^2$ *and G is of product type, then* $m = 2$.

Proof. Let $H = K \wr S_m$, and let S_m act on $M = \{1, 2, \dots, m\}$. As G is flag-transitive, by Lemma 2.2(iii) we get $[G_\alpha : G_{\alpha\beta}] \geq \frac{r}{(r,\lambda)}$ for any two distinct points α, β. Since $H \geq G$, it follows that

$$[H_\alpha : H_{\alpha\beta}] \geq [G_\alpha : G_{\alpha\beta}] \geq \frac{r}{(r, \lambda)} = \frac{\lambda(v-1)}{(r, \lambda)(k-1)}. \tag{3.1}$$

Let $\alpha = (\gamma, \gamma, \dots, \gamma)$, $\gamma \in \Gamma$, $\beta = (\delta, \gamma, \dots, \gamma)$, $\gamma \neq \delta \in \Gamma$ and let $B \cong K^m$ be the base group of H. Then $B_\alpha = K_\gamma \times \cdots \times K_\gamma$, $B_{\alpha\beta} = K_{\gamma\delta} \times K_\gamma \times \cdots \times K_\gamma$. Now $H_\alpha = K_\gamma \wr S_m$, and $H_{\alpha\beta} \geq K_{\gamma\delta} \times (K_\gamma \wr S_{m-1})$. Suppose K has rank s on Γ with $s \geq 2$. We can choose a δ satisfying $[K_\gamma : K_{\gamma\delta}] \leq \frac{v_0-1}{s-1}$, so that

$$[H_\alpha : H_{\alpha\beta}] = \frac{|H_\alpha|}{|H_{\alpha\beta}|} \leq \frac{|K_\gamma|^m \cdot m!}{|K_{\gamma\delta}||K_\gamma|^{m-1} \cdot (m-1)!} \leq m\frac{v_0 - 1}{s - 1}$$

and hence by Equation (3.1),

$$\frac{\lambda(v-1)}{(r, \lambda)(k-1)} \leq [G_\alpha : G_{\alpha\beta}] \leq m\frac{v_0 - 1}{s - 1}. \tag{3.2}$$

So

$$\frac{v_0^m - 1}{v_0 - 1} \leq m\frac{(r, \lambda)(k-1)}{\lambda(s-1)}. \tag{3.3}$$

Now $(k-1)^2 \leq (r-1)(k-1) < r(k-1) = \lambda v$. Thus

$$v_0^{m-1} < m v_0^{\frac{m}{2}} \frac{(r, \lambda)}{\lambda^{\frac{1}{2}}} \leq m v_0^{\frac{m}{2}}.$$

Hence $m \leq 2$, or $m = 3$ and $v_0 < 9$.

If $m = 3$, from Equation (3.3) we have $v_0^2 + v_0 + 1 < \frac{3v_0^{\frac{3}{2}}}{s-1}$, so that $v_0 = 5$ or 6 and $s = 2$. Now, from $(k-1)^2 \leq \lambda v$ we have $\frac{(k-1)^2}{v_0^3} \leq \lambda$. On the other hand, Equation (3.3) and $\lambda \geq (r, \lambda)^2$ imply $v_0^2 + v_0 + 1 \leq \frac{3(k-1)}{\lambda^{\frac{1}{2}}}$, so that $\lambda \leq \frac{9(k-1)^2}{(v_0^2+v_0+1)^2}$. It follows that

$$\frac{(k-1)^2}{v_0^3} \leq \lambda \leq \frac{9(k-1)^2}{(v_0^2 + v_0 + 1)^2},$$

where $v_0 = 5$ or 6. Now $G \leq K \wr S_3 \leq S_{v_0} \wr S_3$ implies that G is a $\{2, 3, 5\}$-group, so by flag-transitivity, k divides $|G_B|$, and hence the only primes dividing k are $2, 3$ or 5. The only integers v_0, k, λ satisfying these conditions are $v_0 = 5$, $k = 32$, $\lambda = 8$ or 9, by using the software package GAP [9]. Then $r = 32$ or 36 which contradicts the condition $\lambda \geq (r, \lambda)^2$. Hence $m = 2$. □

Lemma 3.5. *If G acts flag-transitively on 2-(v,k,λ) design with $\lambda \geq (r,\lambda)^2$ and G is of product type, then G is a point-primitive rank 3 group and v is an odd number.*

Proof. Since G is of product type, then $m = 2$ by Lemma 3.4. From Equation (3.3), we have

$$v_0 + 1 \leq \frac{2(r,\lambda)(k-1)}{\lambda(s-1)} < \frac{2(r,\lambda)v_0}{\lambda^{\frac{1}{2}}(s-1)} \leq \frac{2v_0}{s-1}.$$

This implies $s = 2$. It follows that K acts 2-transitively on Γ, and $H = K \wr S_2$ has rank 3 with subdegrees 1, $2(v_0-1)$, $(v_0-1)^2$.

Now $G \leq H$, so each subdegree of H is the sum of some subdegrees of G and so $\frac{r}{(r,\lambda)} \mid 2(v_0-1)$. If $\frac{r}{(r,\lambda)} \neq 2(v_0-1)$, then $\frac{r}{(r,\lambda)} \leq v_0 - 1$, so that

$$r^2 \leq (r,\lambda)^2(v_0-1)^2 < (r,\lambda)^2 v_0^2 \leq \lambda v$$

which is a contradiction. Thus $\frac{r}{(r,\lambda)} = 2(v_0-1)$, by Equation (3.2), we obtain that G must have a subdegree $2(v_0-1)$ and it follows that G induces a 2-transitive group $\overline{G} \leq K$ on Γ. We conclude that G itself has rank 3 on $\mathcal{P}$ with subdegrees: 1, $2(v_0-1)$, $(v_0-1)^2$. Therefore, $\frac{r}{(r,\lambda)} \mid (v_0-1)^2$, i.e., $2 \mid v_0 - 1$. So $v = v_0^2$ is an odd number. □

Proof of Theorem 1.2. Follows immediately from Propositions 3.1, 3.2 and 3.3. □

4 Proof of Theorem 1.3

In this section, we will apply Theorem 1.2 to symmetric designs. For this purpose, we first give some basic facts on rank 3 permutation groups and symmetric designs. Lemma 4.1 first appears in [16, Lemma 1.5] with a sketch of proof. Since it is an important result for symmetric designs, we provide its proof here for completeness.

Lemma 4.1. *Let G be a finite imprimitive rank 3 permutation group on a set P. Let $P = \{\alpha\} \cup X \cup Y$ be the decomposition into G_α-orbits. Assume $|X| \leq |Y|$. Then $Q = \{\alpha\} \cup X$ is a block of the action of G. Set $\Omega = \{Q^g \mid g \in G\}$, then G acts 2-transitively on Ω.*

Proof. Let M be a maximal subgroup of G containing G_α. Then $G_\alpha < M$ and M is not transitive on P (otherwise, we have $G = MG_\alpha = M$ a contradiction). From $G_\alpha < M$, we have $M_\alpha = G_\alpha \cap M = G_\alpha$ and

$$1 + |X| + |Y| = |G : G_\alpha| = |G : M_\alpha| = |G : M||M : M_\alpha| \geq 2|M : M_\alpha|. \tag{4.1}$$

Let $R = \{\alpha^x \mid x \in M\}$. Since both X and Y are orbits of G_α, and $G_\alpha < M$, there exists $m \in M \backslash G_\alpha$ such that $\alpha^m \in X$ or Y. If $\alpha^m \in X$, then $X = \alpha^{mG_\alpha} \subseteq \alpha^M = R$, so that $Q \subseteq R$. We argue that $Q = R$. For if $Q \subsetneq R$, then there exists $m' \in M \backslash G_\alpha$ such that $y = \alpha^{m'} \in R \backslash Q \subseteq Y$, from which it follows that $Y \subsetneq R$. Hence $\{\alpha\} \cup X \cup Y = \mathcal{P} \subseteq R$, and thus $P = R$ which contradicts the fact that M is intransitive. Therefore, $Q = R = \{\alpha\} \cup X$. Similarly, if $\alpha^m \in Y$, we have $R = \{\alpha\} \cup Y$. Next, we prove that $R \neq \{\alpha\} \cup Y$, and $R = Q$ is a block.

If $R = \{\alpha\} \cup Y$, since M is transitive on R, we have $|R| = |M : M_\alpha| = 1 + |Y|$. Equation (4.1) implies that $1 + |X| + |Y| \geq 2(1 + |Y|)$, so that $|X| \geq 1 + |Y| > |Y|$ which contradicts the assumption. Thus we must have $Q = \{\alpha\} \cup X = R$. Since $Q = R =$

$\{\alpha^x \mid x \in M\}$, if $Q^g \cap Q \neq \emptyset$ for some $g \in G$, then there exist $x, y \in M$ such that $\alpha^{xg} = \alpha^y$, so that $xgy^{-1} \in G_\alpha < M$ and $g \in M$. Hence $Q^g = Q$ and Q is a block.

Since $Q = \{\alpha\} \cup X$ is a block and $\alpha \in Q$, then $G_\alpha \leq G_Q$, where G_Q is the stabilizer of the block Q. Let $\Omega = \{Q^g \mid g \in G\}$ be a block system of imprimitivity of G. As G_α is transitive on $Y = P - Q$, it follows that G_Q acts transitively on $\Omega \backslash \{Q\}$, and thus G acts 2-transitively on Ω. □

Lemma 4.2 ([16, Lemma 1.6]). *Let $\mathcal{D} = (\mathcal{P}, \mathcal{B})$ be a symmetric 2-(v, k, λ) design and $G \leq \mathrm{Aut}(\mathcal{D})$ be a point-primitive rank 3 group. Then G is also a block-primitive rank 3 group if one of the following holds for $(G, \mathcal{P})$:*

(a) The permutation group is of product or affine type.

(b) The group G is almost simple and G has no 2-transitive representation of degree d, such that d properly divides v.

Now we begin the proof of Theorem 1.3.

Proof of Theorem 1.3. Assume $\mathcal{D}$ is a 2-(v, k, λ) symmetric design with $\lambda \geq (r, \lambda)^2$, which admits a flag-transitive automorphism group G. By Theorem 1.2, G is one of the following: (i) affine type, (ii) almost simple type, or (iii) product type with $\mathrm{Soc}(G) \cong T \times T$ where T is a nonabelian simple group and G is a primitive rank 3 group. So we only need to prove that case (iii) cannot occur.

Suppose for the contrary that G has a product action on the set of points. Here G is a point-primitive rank 3 group, so we know from Lemma 4.2 that G is also a block-primitive rank 3 group. By Lemma 2.4, we have $m = 1$. This contradicts the fact that $m = 2$ (see Lemma 3.4). Hence G is of affine or almost simple type. □

References

[1] F. Buekenhout, A. Delandtsheer and J. Doyen, Finite linear spaces with flag-transitive groups, *J. Combin. Theory Ser. A* **49** (1988), 268–293, doi:10.1016/0097-3165(88)90056-8.

[2] F. Buekenhout, A. Delandtsheer, J. Doyen, P. B. Kleidman, M. W. Liebeck and J. Saxl, Linear spaces with flag-transitive automorphism groups, *Geom. Dedicata* **36** (1990), 89–94, doi:10.1007/bf00181466.

[3] H. Davies, *Automorphisms of Designs*, Ph.D. thesis, University of East Anglia, 1987.

[4] H. Davies, Flag-transitivity and primitivity, *Discrete Math.* **63** (1987), 91–93, doi:10.1016/0012-365x(87)90154-3.

[5] P. Dembowski, *Finite Geometries*, volume 44 of *Ergebnisse der Mathematik und ihrer Grenzgebiete*, Springer-Verlag, New York, 1968, doi:10.1007/978-3-642-62012-6.

[6] U. Dempwolff, Primitive rank 3 groups on symmetric designs, *Des. Codes Cryptogr.* **22** (2001), 191–207, doi:10.1023/a:1008373207617.

[7] J. D. Dixon and B. Mortimer, *Permutation Groups*, volume 163 of *Graduate Texts in Mathematics*, Springer-Verlag, New York, 1996, doi:10.1007/978-1-4612-0731-3.

[8] W. Fang, H. Dong and S. Zhou, Flag-transitive 2-$(v, k, 4)$ symmetric designs, *Ars Combin.* **95** (2010), 333–342.

[9] The GAP Group, *GAP – Groups, Algorithms, and Programming, Version 4.8.4*, 2017, `http://www.gap-system.org`.

[10] W. M. Kantor, Primitive permutation groups of odd degree, and an application to finite projective planes, *J. Algebra* **106** (1987), 15–45, doi:10.1016/0021-8693(87)90019-6.

[11] M. W. Liebeck, The classification of finite linear spaces with flag-transitive automorphism groups of affine type, *J. Combin. Theory Ser. A* **84** (1998), 196–235, doi:10.1006/jcta.1998.2897.

[12] M. W. Liebeck, C. E. Praeger and J. Saxl, On the O'Nan-Scott theorem for finite primitive permutation groups, *J. Austral. Math. Soc. Ser. A* **44** (1988), 389–396, doi:10.1017/s144678870003216x.

[13] E. O'Reilly-Regueiro, On primitivity and reduction for flag-transitive symmetric designs, *J. Combin. Theory Ser. A* **109** (2005), 135–148, doi:10.1016/j.jcta.2004.08.002.

[14] E. O'Reilly-Regueiro, Reduction for primitive flag-transitive $(v, k, 4)$-symmetric designs, *Des. Codes Cryptogr.* **56** (2010), 61–63, doi:10.1007/s10623-009-9341-8.

[15] J. Saxl, On finite linear spaces with almost simple flag-transitive automorphism groups, *J. Combin. Theory Ser. A* **100** (2002), 322–348, doi:10.1006/jcta.2002.3305.

[16] D. Tian and S. Zhou, Flag-transitive point-primitive symmetric (v, k, λ) designs with λ at most 100, *J. Combin. Des.* **21** (2013), 127–141, doi:10.1002/jcd.21337.

[17] H. Wielandt, *Finite Permutation Groups*, Academic Press, New York, 1964, doi:10.1016/c2013-0-11702-3.

[18] X. Zhan and S. Zhou, A classification of flag-transitive 2-designs with $\lambda \geq (r, \lambda)^2$ and sporadic socle, *Discrete Math.* **340** (2017), 630–636, doi:10.1016/j.disc.2016.11.025.

[19] P.-H. Zieschang, Flag transitive automorphism groups of 2-designs with $(r, \lambda) = 1$, *J. Algebra* **118** (1988), 369–375, doi:10.1016/0021-8693(88)90027-0.

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Trees with small spectral gap

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Abstract

Continuing the previous research, we consider trees with given number of vertices and minimal spectral gap. Using the computer search, we conjecture that this spectral invariant is minimized for double comet trees. The conjecture is confirmed for trees with at most 20 vertices; simultaneously no counterexamples are encountered. We provide theoretical results concerning double comets and putative trees that minimize the spectral gap. We also compare the spectral gap of regular graphs and paths. Finally, a sequence of inequalities that involve the same invariant is obtained.

Keywords: Graph eigenvalues, double comet, extremal values, numerical computation.

Math. Subj. Class.: 05C50, 65F15

1 Introduction

We use standard graph-theoretic terminology and notation. For example, for a graph G, $n = n(G)$ and $m = m(G)$ denote its order and size, while $A = A(G)$ stands for its adjacency matrix. The characteristic polynomial of A is the characteristic polynomial of G, denoted P_G. Its roots are the eigenvalues of G, denoted $\lambda_1(G) \geq \lambda_2(G) \geq \cdots \geq \lambda_n(G)$. Non-isomorphic graphs that share the same spectrum are said to be *cospectral*.

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The difference between the first two eigenvalues $\delta(G) = \lambda_1(G) - \lambda_2(G)$ is called the *spectral gap* of G. In the previous investigation (see [13]), graphs with small spectral gap are considered. The purpose of this paper is to continue this investigation in the case of trees (this part can be considered as counterpart to [13], as well), consider regular graphs with small spectral gap, and give certain upper and lower bounds for this spectral invariant.

As announced in the Abstract, we conjecture that a minimal spectral gap of trees with given order is attained for a double comet tree (the conjecture is formulated in the very beginning of the next section). We also give certain theoretical and computational results supporting this conjecture, and consider cospectrality of double comets. The conjecture is open, but nevertheless we give additional (structural and spectral) properties of a tree with minimal spectral gap.

Next, we compare the spectral gap of regular graphs and paths by showing that, under some restrictions, the spectral gap of a regular graph is bounded from below by the spectral gap of the corresponding path. Finally, we give some new bounds on δ expressed in terms of order, size, clique number, and minimal or maximal vertex degree.

We conclude this section by certain terminology and notation. The main results are given in Sections 2–4.

We use P_n, C_n, K_n, and K_{n_1,n_2} $(n_1 + n_2 = n)$ to denote the path, cycle, complete graph (or clique), and complete bipartite graph on n vertices, respectively. The disjoint union of graphs G and H is denoted by $G \cup H$, while the disjoint union of k copies of G is denoted by kG.

The *generalized double comet* $C^*(k_1, k_2, l)$, where $k_1, k_2 \geq 0$ and $l \geq 2$, is a tree obtained by attaching k_1 pendant vertices at one end of the path P_l and k_2 pendant vertices at the other end of the same path. The path P_l is referred to as the *internal path*. The *double comet* $C(k, l)$ is obtained from $C^*(k_1, k_2, l)$ for $k_1 = k_2 = k$.

2 Spectral gap of trees

We start with the following conjecture.

Conjecture 2.1. *Among all trees of order n, the spectral gap is minimized for some double comet.*

The conjecture is confirmed by computer search for trees with at most 20 vertices. In all situations, there is a unique tree attaining the minimal spectral gap. For $n \leq 8$, this is the path P_n (which is a special case of double comet), for $9 \leq n \leq 11$ this is $C(2, n-4)$, for $12 \leq n \leq 15$ this is $C(3, n-6)$, and for $16 \leq n \leq 20$ this is $C(4, n-8)$.

If Δ and Δ' $(< \Delta)$ are the largest and second largest degree in a tree T and the distance between the vertices having these degrees is at least three, then using the Interlacing Theorem (cf. [14, Theorem 1.6]), we get $\lambda_2(T) \geq \lambda_1(K_{1,\Delta'}) = \sqrt{\Delta'}$. In other words, $\sqrt{\Delta'}$ makes a lower bound on $\lambda_2(T)$. Since an increase in Δ' increases this bound, it is natural to assume that the spectral gap is minimized for a tree with two vertices of maximal degree. We provide more theoretical results supporting Conjecture 2.1.

Theorem 2.2. *In the set of trees of order n $(n \geq 2)$ and two vertices of maximal degree Δ, the largest eigenvalue λ_1 is minimized for the double comet $C(\Delta - 1, n - 2(\Delta - 1))$.*

Proof. Clearly, the statement holds for $\Delta \leq 2$. Assume next that u, v are the two vertices of degree Δ (≥ 3) in a tree T and let $N(u), N(v)$ be their open neighbourhoods. Then, removing a pendant vertex which does not belong to $N(u) \cup N(v)$ (if such a vertex does not exist, then T is the required double comet) impose a strict decrease in λ_1 (as a consequence of the Perron-Frobenius Theorem, cf. [2, Theorem 1.3.6]). In addition, by the well-known result of Hoffman and Smith [4], inserting a vertex in the unique path between u and v is again followed by a strict decrease in λ_1. Therefore, rearranging a described pendant vertex into the corresponding path gives a tree with smaller largest eigenvalue. Repeating this procedure, after finite number of steps, we arrive at $C(\Delta - 1, n - 2(\Delta - 1))$. $\square$

In the following theorem we use another result of Hoffman [3]: Let H be the graph obtained from a graph G by attaching a hanging path of an arbitrary length at any vertex v (of G) of the degree at least two. When the length of the path attached tends to infinity then the largest eigenvalue of H increases and tends to the largest root of the equation

$$f(x) = \left(\frac{x + \sqrt{x^2 - 4}}{2}\right) P_G(x) - P_{G-v}(x) = 0,$$

whenever the largest eigenvalue of H is greater than 2.

Now we have the following theorem comparing spectral gaps of generalized double comets with equal internal paths.

Theorem 2.3. *Given a generalized double comet $C^*(k_1, k_2, l)$. If $l \geq 2$ and $k_1 \leq k_2 - 2$, then $\delta(C^*(k_1, k_2, l)) > \delta(C^*(k_1 + 1, k_2 - 1, l))$.*

Proof. Let, for short, T (resp. S) stand for $C^*(k_1, k_2, l)$ (resp. $C^*(k_1 + 1, k_2 - 1, l)$).

We have $\lambda_1(T) > \lambda_1(S)$, as proved in [12], and so it remains to prove that $\lambda_2(T) \leq \lambda_2(S)$. First, we consider the three singular cases.

For $l = 2$, Using the Schwenk formula [2, Theorem 2.4.3], we compute

$$P_T(x) = x^{k_1+k_2-2} Q(x) \quad \text{and} \quad P_S(x) = x^{k_1+k_2-2} R(x),$$

where

$$\begin{aligned} Q(x) &= \left(x^4 - (k_1 + k_2 + 1)x^2 + k_1 k_2\right) \quad \text{and} \\ R(x) &= \left(x^4 - (k_1 + k_2 + 1)x^2 + k_1 k_2 - k_1 + k_2 - 1\right). \end{aligned}$$

The second largest eigenvalues of both trees coincide with second largest roots of the polynomials Q and R. Since $R(x) - Q(x) = -k_1 + k_2 - 1 > 0$, we get $\lambda_2(T) < \lambda_2(S)$.

For $l = 3$, by removing the vertex of maximal degree in T and using the interlacing argument, we get $\lambda_2(T) \leq \sqrt{k_1 + 1}$. Similarly, by removing the middle vertex of the internal path in S, we get $\lambda_2(S) \geq \sqrt{k_1 + 1}$, and we are done.

For $l = 4$, following the case $l = 2$, we get that the characteristic polynomials of trees T and S are $P_T(x) = x^{k_1+k_2-2} Q_1(x)$ and $P_S(x) = x^{k_1+k_2-2} R_1(x)$, where the polynomials Q_1 and R_1 are computed in a similar way, while their difference is $R_1(x) - Q_1(x) = (k_1 - k_2 + 1)(1 - x^2)$. Computing this value for $x = \lambda_2(T)$, we get the assertion.

Let now $l \geq 5$. For $k_1 = 2$ we directly get $\lambda_2(T) \leq 2 \leq \lambda_2(S)$, and similarly for $k_1 = 1$. Let in further T' stand for the non-trivial component of the graph obtained by removing the vertex of degree $k_2 + 1$ in T. By interlacing we have $\lambda_2(T) \leq \lambda_1(T')$. Since $k_1 \geq 3$, we may apply the result given before Theorem 2.3 to get that $\lambda_1(T')$ is less than the largest root of

$$\frac{1}{2}\left(x + \sqrt{x^2 - 4}\right) P_{K_{1,k_1}}(x) - P_{k_1 K_1}(x),$$

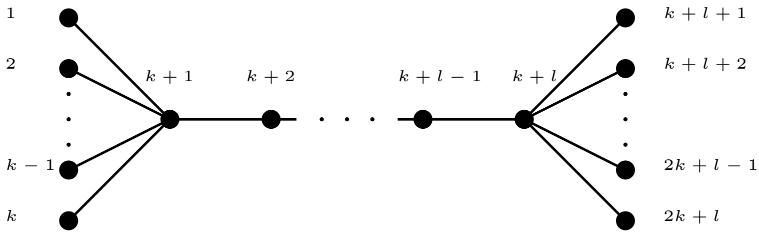

Figure 1: Double comet $C(k, l)$.

that is the largest root of

$$\frac{1}{2}\left(x+\sqrt{x^2-4}\right)(x^2-k_2)-x.$$

Setting $x=\sqrt{k_1+2}$, the last expression becomes equal to $\sqrt{k_1-2}$, i.e. it is positive in this point, which implies $\lambda_1(T')<\sqrt{k_1+2}$.

On the other hand, using interlacing once again, we get $\lambda_2(S)\geq\sqrt{k_1+2}$, which completes the proof. □

In other words, in the set of generalized double comets with fixed internal path P_l, the minimal spectral gap is attained when the cocliques on k_1 and k_2 vertices are as equal as possible.

In order to compute the spectral gap of an arbitrarily large double comet, we derive formulas for computing its largest and second largest eigenvalue. The following two results may be viewed as a counterpart to [13, Proposition 2.4].

Theorem 2.4. *If $\lambda_1(C(k,l))>2$, then $\lambda_1(C(k,l))$ is equal to $2\cosh(2t)$, where t is the unique positive root of*

$$\left(e^{2t(l-1)}+1\right)k-e^{-4t}\left(e^{4t}+1\right)\left(e^{2t(l+1)}+1\right)=0. \tag{2.1}$$

Proof. Let the vertices of $C(k,l)$ be labeled $1,2,\ldots,2k+l$ in natural order (see Figure 1), and let an eigenvector associated with $\lambda_1=\lambda_1(C(k,l))$ be denoted $x=(x_1,x_2,\ldots,x_n)^T$. Then we have

$$\lambda_1 x_i=\sum_{j\sim i}x_j \quad (1\leq i\leq 2k+l). \tag{2.2}$$

Using (2.2), we derive (2.1) in the following way. First, using the symmetry of $C(k,l)$, we may assume that $x_1=x_2=\cdots=x_k=x_{k+l+1}=x_{k+l+2}=\cdots=x_{2k+l}$ (cf. [3]). Since the coordinates of the eigenvector x are of the same sign and the eigenvector itself is determined up to a multiplicative constant, we may take $x_1=\lambda_1$. Then, we have

$$x_{k+1}=x_{k+l}=\lambda_1^2. \tag{2.3}$$

For $0\leq i\leq l-1$, using (2.2), we arrive at the following system of recurrence equations

$$x_{k+i}-\lambda_1 x_{k+i+1}+x_{k+i+2}=0 \quad (1\leq i\leq l-1). \tag{2.4}$$

The former equality, that is (2.3), may be regarded as a boundary condition for (2.4). Solving the related characteristic equation $t^2 - \lambda_1 t + 1 = 0$, we get $t_1 = \frac{1}{t_2} = \frac{\lambda_1 + \sqrt{\lambda_1^2 - 4}}{2}$, which yields

$$x_{k+i+1} = c_1 t_1^{k+i+1} + c_2 t_1^{-(k+i+1)}. \tag{2.5}$$

Setting $i = k+2$ and $i = k+l-1$ in (2.2), we deduce that

$$x_{k+2} = x_{k+l-1} = \lambda_1(\lambda_1^2 - k).$$

Next, using the first, and then the second equality in the last chain, we get $c_2 = c_1 t_1^{2k+l+1}$ and $c_1 = \frac{\lambda_1(\lambda_1^2 - k)}{t_1^{k+2} + t_1^{k+l-1}}$, respectively. Substituting these expressions in (2.5) and using our boundary condition, we get

$$\lambda_1 \left(t_1^{k+2} + t_1^{k+l-1}\right) = (\lambda_1^2 - k)\left(t_1^{k+1} + t_1^{k+l}\right).$$

Finally, using the obtained expression for t_1 and setting $\lambda_1 = 2\cosh(2t)$, after a straightforward computation we arrive at (2.1).

A simple analysis shows that the equation (2.1) has a positive real root. Moreover, it must be unique, because otherwise we would have two non-collinear eigenvectors corresponding to a simple eigenvalue λ_1. This observation completes the proof. □

Our next result is obtained in the same way. We omit the proof and refer the reader to the previous theorem and [13].

Theorem 2.5. *If $\lambda_2(C(k,l)) > 2$, then $\lambda_2(C(k,l))$ is equal to $2\cosh(2t)$, where t is the unique positive root of*

$$\left(e^{4t} - e^{2t(l+1)}\right)(k-1) + e^{2t(l+3)} - 1 = 0. \tag{2.6}$$

The last two theorems enable us to compute the spectral gap of an arbitrary double comet. All what we need is to find the positive roots of (2.1) and (2.6) by means of numerical computation. In this way we get the trees with extremely small spectral gaps. For example, we have the following results for double comets determined by parameters (k,l):

(k,l)	$(10,10)$	$(50,10)$	$(100,10)$	$(4,10)$	$(4,50)$	$(4,100)$
$\delta \approx$	0.0001	$1.7 \cdot 10^{-7}$	$1.0 \cdot 10^{-8}$	0.0073	$2.1 \cdot 10^{-12}$	$2.5 \cdot 10^{-24}$

So, Conjecture 2.1 remains open. If, for some n there exists a tree which is not a double comet but has the minimal spectral gap then, by the previous computation, its spectral gap is very close to zero. Moreover, its second largest eigenvalue must be simple, as proved in our next statement.

Theorem 2.6. *If T is a graph with minimal spectral gap in the set of trees of order n, then $\lambda_2(T) > \lambda_3(T)$.*

Proof. Assume to the contrary. Since T is a tree, it contains a vertex with two hanging paths attached. If these paths are P_{n_1} and P_{n_2} where, say $n_1 \geq n_2 \geq 2$, then let T' be the tree obtained by the relocation of the endvertex of P_{n_2} to the end of P_{n_1}. It is known from numerous literature, see for example [14, Lemma 1.29(i)], that $\lambda_1(T) > \lambda_1(T')$. Next, since $\lambda_2(T) = \lambda_3(T)$, by interlacing, we get $\lambda_2(T) \leq \lambda_2(T')$. Altogether, $\delta(T) > \delta(T')$, a contradiction. □

Recall that, according to the computer search, for $n \leq 20$, there is a unique tree that minimizes the spectral gap. So, there is another question: is such a tree unique for all n? A contribution to this question may be given by considering cospectrality of double comet graphs. It is not difficult to see that no every double comet has the unique spectrum (an easy exercise for the reader: show that $C(2, l)$ is cospectral with $C_4 \cup P_l$). However, connected graphs that are cospectral with double comets are more interesting in our context. Here we provide the following result.

Theorem 2.7. *The double comet $C(k, 2)$ is cospectral with a connected graph if and only if $k = t^2 - t + 1$, for $t \in \mathbb{N}$. The spectrum of a double comet $C(k, l)$, $3 \leq l \leq 5$, is unique in the set of connected graphs.*

Sketch proof. Observe that cospectral graphs have equal orders and sizes, and therefore if a connected graph is cospectral with a double comet, then such a graph must be a tree. Next, recall from [6] that the multiplicity of zero (also known as nullity) in the spectrum of a tree is $n - 4$ if and only if that tree is a generalized double comet $C^*(k_1, k_2, l)$, for $2 \leq l \leq 3$. Similarly, the trees of nullity $n - 6$ are those illustrated in Figure 2. Clearly, double comets specified in this theorem are covered by these sets of trees, and so to consider their cospectrality we need to compare their spectra with spectra of trees with equal number of zero eigenvalues.

For $l = 2$, the possible candidates are the mentioned two generalized double comets with nullity equal to $n - 4$. Recall (from the proof of Theorem 2.3) that the characteristic polynomial of $C^*(k_1, k_2, 2)$ is:

$$P_{C^*(k_1,k_2,2)}(x) = x^{k_1+k_2-2} \left(x^4 - (k_1 + k_2 + 1)x^2 + k_1 k_2\right).$$

Also, using the same method, we compute

$$P_{C^*(k_1,k_2,3)}(x) = x^{k_1+k_2-1} \left(x^4 - (k_1 + k_2 + 2)x^2 + k_1 k_2 + k_1 + k_2\right).$$

By setting $k_1 = k_2 = k$ in the first polynomial, we get the characteristic polynomial of $C(k, 2)$. Next, comparing the characteristic polynomials of $C(k, 2)$ and $C^*(k_1, k_2, 2)$, we get $k_1 + k_2 = k$ and $k_1 k_2 = k^2$, which implies $k_1 = k_2$, i.e. the only solution is obtained when the corresponding trees are isomorphic.

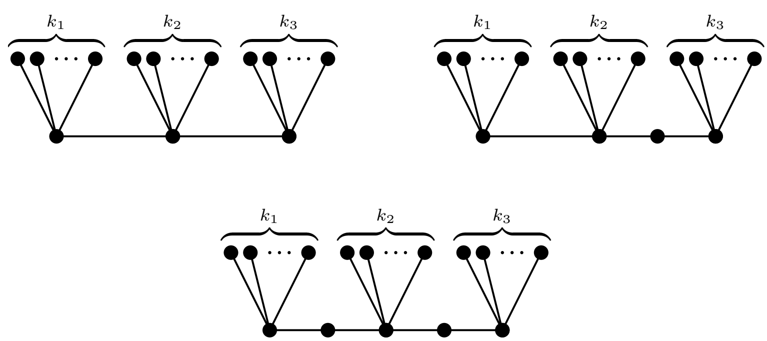

Figure 2: Trees of nullity $n - 6$.

Taking into account $C^*(k_1, k_2, 3)$, we get $k_1 + k_2 = 2k - 1$ and $k_1 k_2 = (k - 1)^2$, which means that k_1 and k_2 are the solutions of $s^2 - (2k - 1)s + (k - 1)^2 = 0$, i.e. they are of the form $\frac{2k-1\pm\sqrt{4k-3}}{2}$. Since we are interested in integral solutions, $4k-3$ must be a perfect square (clearly, an odd integer). Setting $4k-3=(2t-1)^2$, $t\in\mathbb{N}$, we get $k=t^2-t+1$. In addition, each k of this form gives two distinct positive values for k_1 and k_2, and we are done.

The cases $l\in\{3,4,5\}$ are considered in exactly the same way, i.e. by comparing the corresponding characteristic polynomials and, in some situations, by reducing the procedure by using theoretic reasoning based on more sophisticated results like eigenvalue interlacing. In all cases we get that the double comet under consideration is not cospectral with any of possible candidates, and the proof follows. $\square$

So, in general case, a double comet may be cospectral with another graph. Even more, it may occur that this graph is connected (and then, it is also a tree). On the other hand, in our computational results double comets that are cospectral with other trees do not appear in the role of those with minimal spectral gap.

3 Comparing the spectral gap of paths and regular graphs

It is proved in [13] that the spectral gap of the path P_n is always less than the spectral gap of the cycle C_n. In other words, the spectral gap of any connected 2-regular graph is greater than $\delta(P_n)$. Perhaps surprisingly, but a similar conclusion cannot be deduced for all regular graphs. A counterexample is the regular graph illustrated in Figure 3. Its spectral gap is close to 0.105, while we simultaneously have $\delta(P_{14})\approx 0.129$ (both values are rounded to three decimal places).

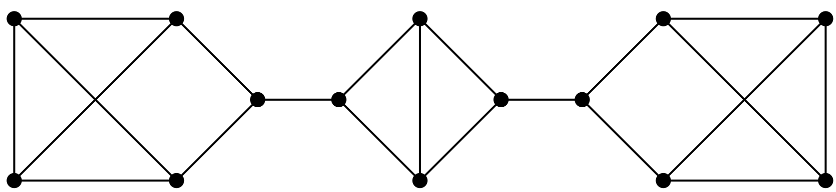

Figure 3: A regular graph whose spectral gap is smaller than the spectral gap of the corresponding path.

If the vertex degree is sufficiently large, then we have the following result.

Theorem 3.1. *Let G be a connected r-regular graph of order n $(n\geq 3)$. If $r\geq 22$, then*

$$\delta(G)>\delta(P_n).$$

Proof. If $n\leq 2r+1$, then

$$\lambda_2(G)=-\lambda_n(\overline{G})\leq\lambda_1(\overline{G})=n-r-2,$$

and so $\delta(G)=\lambda_1(G)-\lambda_2(G)\geq r-(n-r-2)\geq 1$. On the other hand, $\delta(P_n)<1$ holds for any $n\geq 5$, while the cases $n=3$ and $n=4$ are resolved easily.

Let now $n \geq 2r+2$. Considering $\delta(P_n)$ and using the Taylor series, we get

$$\begin{aligned}
\delta(P_n) &= 2\left(\cos\frac{\pi}{n+1} - \cos\frac{2\pi}{n+1}\right) \\
&< 2\left(1 - \frac{1}{2}\left(\frac{\pi}{n+1}\right)^2 + \frac{1}{4!}\left(\frac{\pi}{n+1}\right)^4 - 1\right. \\
&\qquad \left. + \frac{1}{2}\left(\frac{2\pi}{n+1}\right)^2 - \frac{1}{4!}\left(\frac{2\pi}{n+1}\right)^4 + \frac{1}{6!}\left(\frac{2\pi}{n+1}\right)^6\right) \\
&= 2\left(\frac{3}{2}\left(\frac{\pi}{n+1}\right)^2 - \frac{15}{4!}\left(\frac{\pi}{n+1}\right)^4 + \frac{1}{6!}\left(\frac{2\pi}{n+1}\right)^6\right) \\
&= 3\left(\frac{\pi}{n+1}\right)^2\left(1 - \frac{5}{12}\left(\frac{\pi}{n+1}\right)^2 + \frac{2^3}{5\cdot 3^3}\left(\frac{\pi}{n+1}\right)^4\right) \\
&< 3\left(\frac{\pi}{n+1}\right)^2.
\end{aligned}$$

We next use the inequalities for regular graphs, $\delta \geq \frac{4}{nD}$ [7] and $D \leq 3\lfloor\frac{n}{r+1}\rfloor - 1$ [1], where D stands for the diameter. We compute

$$\begin{aligned}
\delta(G) \geq \frac{4}{nD} &\geq \frac{4}{n\left(3\lfloor\frac{n}{r+1}\rfloor - 1\right)} \geq \frac{4}{n\left(3(\frac{n}{r+1}) - 1\right)} \\
&= \frac{4(r+1)}{n(3n-r-1)} > \frac{4(r+1)}{3n^2}.
\end{aligned}$$

Since, for $r \geq 22$ it holds $3\left(\frac{\pi}{n+1}\right)^2 < \frac{4(r+1)}{3n^2}$, we get the assertion. □

4 Bounds for spectral gap

In this section we give some bounds on $\delta(G)$. We start with the following lemma.

Lemma 4.1. *Given a connected graph G, let K denote its proper subgraph isomorphic to either a complete graph K_p or a complete bipartite graph $K_{p,q}$. If H is a graph obtained from G by deleting all edges belonging to K, then*

$$\lambda_2(G) \leq \lambda_1(H).$$

Proof. Assume that G has n vertices, and K has k vertices. Since K is a proper subgraph, $k < n$ must hold. If A is the adjacency matrix of $K \cup (n-k)K_1$ and B is the adjacency matrix of H then, by applying the Courant-Weyl inequality $\lambda_2(A+B) \leq \lambda_2(A) + \lambda_1(B)$ (cf. [2, Theorem 1.3.5]), we get

$$\lambda_2(G) \leq \lambda_2(K \cup (n-k)K_1) + \lambda_1(H) = 0 + \lambda_1(H). \qquad \square$$

In what follows, we use the following upper bound for λ_1^k (k being a positive integer) in terms of the clique number ω and the number of k-walks w_k in G [9],

$$\lambda_1^k \leq \frac{\omega - 1}{\omega} w_k. \tag{4.1}$$

Theorem 4.2. *For a connected graph G with n vertices, m edges, and the clique number ω ($\omega \geq 3$),*

$$\delta \geq \omega - 1 - \sqrt{(\omega-2)\left(\frac{2m}{\omega-1} - \omega\right)}. \tag{4.2}$$

Proof. Setting $k = 2$ in (4.1), we get $\lambda_1^2 \leq 2\frac{\omega-1}{\omega}m$. Deleting the edges belonging to a largest clique of G and using Lemma 4.1, we get

$$\lambda_2(G)^2 \leq 2\frac{\omega-2}{\omega-1}\left(m - \binom{\omega}{2}\right),$$

or

$$\lambda_2(G) \leq \sqrt{(\omega-2)\left(\frac{2m}{\omega-1} - \omega\right)}.$$

On the other hand, $\lambda_1(G) \geq \omega - 1$, and we get the assertion. □

Analyzing the lower bound (4.2), we get $\omega - 1 - \sqrt{(\omega-2)\left(\frac{2m}{\omega-1} - \omega\right)} \geq 0$ whenever

$$m \leq \frac{1}{2} + \omega(\omega-1) + \frac{1}{2\omega-4}.$$

In other words, this lower bound gives a non-trivial result only if the above inequality (on m) holds. Moreover, although the previous theorem is easily proved, the obtained inequality can give a good estimate in some cases. If we consider the graph G obtained from the complete graph K_p by attaching k pendant edges at one of its vertices then (4.2) gives $\delta \leq p - 1 - \sqrt{k\frac{p-2}{p-1}}$. Setting $p = 10$ and $k = 1$ we get $\delta \approx 8.572$. The deviation form the exact value is close to 0.390.

In the following theorem we provide more lower bounds on δ. The first of them is based on the following inequality for a graph with n vertices, m edges and minimal vertex degree $d_{\min}$ [5, 8],

$$\lambda_1 \leq \frac{d_{\min} - 1 + \sqrt{(d_{\min}+1)^2 + 4(2m - d_{\min}n)}}{2}. \tag{4.3}$$

Theorem 4.3. *Let G be a connected r-regular graph with n vertices and let H be obtained form G by deleting*

(a) all edges belonging to a clique K_p ($p < n$),

(b) all edges belonging to a proper induced subgraph $K_{p,q}$ ($p \leq q$),

then

(a) $\delta(G) \geq \frac{r+p-\sqrt{4n(p-1)+(r+2)^2-3p^2-2rp}}{2}$,

(b) $\delta(G) \geq \frac{r+q+1-\sqrt{4nq+(r-q+1)^2-8pq}}{2}$ *if H is connected, and*

$\delta(G) \geq \frac{r+p+1-\sqrt{4np+(r-p+1)^2-8pq-4p(r-p)}}{2}$ *if H is disconnected.*

Proof. Consider (a). By Lemma 4.1, $\lambda_2(G) \leq \lambda_1(H)$. The graph H has n vertices, $\frac{rn}{2} - \binom{p}{2}$ edges, and the minimal vertex degree in H is $r - p + 1$. Using the inequality (4.3), we get

$$\lambda_2(G) \leq \lambda_1(H) \leq \frac{r - p + \sqrt{4n(p-1) + (r+2)^2 - 3p^2 - 2rp}}{2},$$

and since $\lambda_1(G) = r$, we get the assertion.

Consider now (b). If H is connected then it has n vertices, $\frac{rn}{2} - pq$ edges, and the minimal vertex degree in H is $r - q$. Using (4.3), we get the assertion. If H is disconnected then $p < q = r$, and it contains p isolated vertices. Excluding these vertices, we get the graph with $n - p$ vertices and minimal vertex degree $r - p$. The desired inequality is again obtained by using the same argument. □

Here is an upper bound for a special class of regular graphs.

Theorem 4.4. *Let G be an r-regular graph of order n. If $\overline{G}$ is triangle free, then*

$$\delta(G) \leq n\left(2 - \frac{n}{r+1}\right). \tag{4.4}$$

Equality holds if G is isomorphic to $K_{\frac{n}{2},\frac{n}{2}}$.

Proof. We have $\lambda_n(\overline{G}) \leq -\frac{(n-r-1)^2}{r+1}$ (cf. [10]), and thus $\lambda_2(G) = -1 - \lambda_n(\overline{G}) \geq -1 + \frac{(n-r-1)^2}{r+1}$. Now, $\delta(G) = r - \lambda_2(G)$ would be less than or equal to $r + 1 - \frac{(n-r-1)^2}{r+1} = n\left(2 - \frac{n}{r+1}\right)$ whenever this number is non-negative, i.e. whenever $r \geq \frac{n-2}{2}$. On the other hand, if $\overline{G}$ is triangle-free, then $\lambda_1(\overline{G}) \leq \sqrt{m(\overline{G})}$ (see [11]), i.e. $n - r - 1 \leq \sqrt{\frac{n(n-r-1)}{2}}$, which again gives $r \geq \frac{n-2}{2}$. Hence, inequality (4.4) holds for any graph specified in the theorem.

The equality is verified by direct computation. □

References

[1] L. Caccetta and W. F. Smyth, Graphs of maximum diameter, *Discrete Math.* **102** (1992), 121–141, doi:10.1016/0012-365x(92)90047-j.

[2] D. Cvetković, P. Rowlinson and S. Simić, *An Introduction to the Theory of Graph Spectra*, volume 75 of *London Mathematical Society Student Texts*, Cambridge University Press, Cambridge, 2010, doi:10.1017/cbo9780511801518.

[3] A. J. Hoffman, On limit points of spectral radii of non-negative symmetric integral matrices, in: Y. Alavi, D. R. Lick and A. T. White (eds.), *Graph Theory and Applications*, Springer, Berlin, volume 303 of *Lecture Notes in Mathematics*, pp. 165–172, 1972, proceedings of the Conference at Western Michigan University, Kalamazoo, Michigan, May 10 – 13, 1972.

[4] A. J. Hoffman and J. H. Smith, On the spectral radii of topologically equivalent graphs, in: M. Fiedler (ed.), *Recent Advances in Graph Theory*, Academia, Prague, pp. 273–281, 1975, proceedings of the Second Czechoslovak Symposium held in Prague, June 1974.

[5] Y. Hong, J.-L. Shu and K. Fang, A sharp upper bound of the spectral radius of graphs, *J. Comb. Theory Ser. B* **81** (2001), 177–183, doi:10.1006/jctb.2000.1997.

[6] S. Li, On the nullity of graphs with pendent vertices, *Linear Algebra Appl.* **429** (2008), 1619–1628, doi:10.1016/j.laa.2008.04.037.

[7] B. Mohar, The Laplacian spectrum of graphs, in: Y. Alavi, G. Chartrand, O. R. Oellermann and A. J. Schwenk (eds.), *Graph Theory, Combinatorics, and Applications, Volume 2*, John Wiley & Sons, New York, A Wiley-Interscience Publication, pp. 871–898, 1991, proceedings of the Sixth Quadrennial International Conference on the Theory and Applications of Graphs held at Western Michigan University, Kalamazoo, Michigan, May 30 – June 3, 1988.

[8] V. Nikiforov, Some inequalities for the largest eigenvalue of a graph, *Combin. Probab. Comput.* **11** (2002), 179–189, doi:10.1017/s0963548301004928.

[9] V. Nikiforov, Walks and the spectral radius of graphs, *Linear Algebra Appl.* **418** (2006), 257–268, doi:10.1016/j.laa.2006.02.003.

[10] V. Nikiforov, More spectral bounds on the clique and independence numbers, *J. Comb. Theory Ser. B* **99** (2009), 819–826, doi:10.1016/j.jctb.2009.01.003.

[11] E. Nosal, *Eigenvalues of Graphs*, Master's thesis, Univesity of Calgary, Calgary, 1970.

[12] S. K. Simić, E. M. Li Marzi and F. Belardo, On the index of caterpillars, *Discrete Math.* **308** (2008), 324–330, doi:10.1016/j.disc.2006.11.046.

[13] Z. Stanić, Graphs with small spectral gap, *Electron. J. Linear Algebra* **26** (2013), 417–432, doi:10.13001/1081-3810.1662.

[14] Z. Stanić, *Inequalities for Graph Eigenvalues*, volume 423 of *London Mathematical Society Lecture Note Series*, Cambridge University Press, Cambridge, 2015, doi:10.1017/cbo9781316341308.

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8ECM and EWM

Koper, Slovenia, Friday, 27 October 2017. Elena Resmerita of Alpen-Adria University Klagenfurt (Austria), who is Deputy-Convenor of the European Women in Mathematics network (EWM), visited the University of Primorska (Slovenia) and delivered a lecture titled *Stable reconstruction of solutions of ill-posed problems – selected topics*, at the Mathematics Research Seminar. She also visited nearby Portorož, which will be the site of the 8th European Congress in Mathematics (8EMC) in 2020. During her visit she met with the Chair and some members of the Organising committee (OC) for the 8ECM, which was joined also via the internet by Carola-Bibiane Schönlieb (University of Cambridge (UK), Convenor of the EWM), Alessandra Celletti (University of Rome Tor Vergata (Italy), Chair of the EMS Committee for Women in Mathematics), and Marie-Françoise Roy (Université de Rennes 1 (France), Chair of the IMU Committee for Women in Mathematics). The aim of the meeting was to explore ways in which questions involving gender balance and other equal opportunity and non-discrimination issues in European mathematics should be presented at the 8ECM. The OC for the 8ECM recognised the importance of these issues by forming an Equal Opportunity Sub-Committee, chaired by Karin Cvetko Vah (University of Ljubljana (Slovenia)). Several novel strategies have been put on the table, and the OC for the 8ECM has been invited to present its work on the upcoming congress at the next general meeting of the EWM, which will take place in Graz (Austria) in September 2018. It is expected that the format of the engagement of the EWM in the 8ECM will be finalised at that time.

Tomaž Pisanski

Non-commutative structures 2018: A workshop in honor of Jonathan Leech

Portorož, Slovenia, May 23 – 27, 2018

https://conferences.famnit.upr.si/e/ncs2018

Recently, non-commutative generalizations of lattices and related structures have seen an upsurge in interest, with new ideas and applications emerging, from quasilattices to skew Heyting algebras. Much of this activity derives in some way from the initiation, thirty years ago, by Jonathan Leech, of a research program into structures based on Pascual Jordan's notion of a non-commutative lattice. The first workshop on non-commutative structures aims to present the breadth of contemporary research in the area, with contributions from international and Slovenian mathematicians. We are delighted that Jonathan Leech has accepted our invitation to deliver the keynote address in this meeting, which will hopefully be the first of many.

Venue: The conference will take place at UP FTŠ – Turistica in Portorož, Slovenia.

Keynote Speaker:

- Jonathan Leech, Westmont College (CA, USA)

Confirmed Invited Speakers:

- Robert J. Bignall, Sunway University (Malaysia)
- Des FitzGerald, University of Tasmania (Tasmania, Australia)
- Mai Gehrke, CNRS & University of Côte d'Azur (France)
- Sam van Gool, University of Amsterdam (Netherlands)
- Michael Kinyon, University of Denver (CO, USA)
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Meeting of COST Action QSPACE Working Group 5 (Gender and Outreach) and Roundtable Discussion: "Gender Issues and European Women in Mathematics"

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