ON THE RAINBOW CONNECTION OF CARTESIAN PRODUCTS AND THEIR SUBGRAPHS

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Ljubljana, June 7, 2011

On the rainbow connection of Cartesian products and their subgraphs

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Abstract

Rainbow connection number of Cartesian products and their subgraphs are considered. Previously known bounds are compared and non-existence of such bounds for subgraphs of products are discussed. It is shown that the rainbow connection number of an isometric subgraph of a hypercube is bounded above with the rainbow connection number of the hypercube. Isometric subgraphs of hypercubes with the rainbow connection number smaller as much as possible than the rainbow connection of the hypercube are constructed. The concept of c-strong rainbow coloring is introduced. In particular it is proved that the so-called Θ-coloring of an isometric subgraph of a hypercube is its unique optimal c-strong rainbow coloring.

Key words: rainbow connection; strong rainbow connection; Cartesian product of graphs; isometric subgraph; hypercube

AMS subject classification (2010): 05C15, 05C76, 05C12

1 Introduction

The concept of rainbow connection was introduced just a few years ago by Chartrand, Johns, McKeon, and Zhang [4] but it already received an amazing attention. For instance, the recent survey [11] of Li and Sun on the topic contains a list of 56 references. Let us briefly mention a sample of the related research directions.

In the seminar paper [4], the rainbow connection number $\mathrm{rc}(G)$ as well as the strong rainbow connection number $\mathrm{src}(G)$ of a connected graph G were introduced. Among other results, these two graphs invariants were determined for cycles, wheels,

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complete bipartite graphs, and complete multipartite graphs. As a consequence, the difference $\mathrm{src}(G) - \mathrm{rc}(G)$ can be arbitrarily large. In [3] it is proved that the invariants are intrinsically difficult, in fact even deciding whether $\mathrm{rc}(G) = 2$ holds for a graph G is an NP-complete problem. On a positive side, Kemnitz and Schiermeyer [8] proved that most of graphs of order n, diameter two and clique number at least $n - 3$ have rainbow connection number two, and list all the exceptions. Some more bad news: to find out whether a given edge coloring is a rainbow coloring is also an NP-complete problem [3]. Bounds on the rainbow connection number of graphs in terms of minimum degree and other graph parameters are given in [2, 9, 12].

Very recently, two groups of authors independently considered the rainbow connection of graph products [1, 5]. More precisely, the first of these papers deals with Cartesian, lexicographic and strong products, while the latter treats direct, strong and lexicographic products. So all standard graph products (see [6] for the theory of these products) have been addresses by now. Additional results on graph operations, including graph products, were reported in [10].

In this paper we are interested in the rainbow connection number of Cartesian products of graphs with the emphasize on the question what can be said about their subgraphs. In Section 3 we present and compare known upper bounds and demonstrate that there is no hope for some general bounds on the rainbow connection number of (isometric) subgraphs of Cartesian products. On the other hand, for the simplest products—hypercubes—the situation is different. We treat hypercubes in Section 4 where it is shown that the rainbow connection number of an isometric subgraph of a hypercube is bounded above with the rainbow connection number of the hypercube. Using bipartite wheels we show that there exist isometric subgraphs of hypercubes with the rainbow connection number arbitrarily smaller than the rainbow connection number of the hypercube. In the final section the concept of c-strong rainbow coloring is introduced and studied. This is in part motivated by the fact that the c-strong rainbow connection number of an isometric subgraph of an arbitrary Cartesian product graph is bounded above with the one of the product.

2   Preliminaries

In this section we collect definitions and concepts needed in the rest of the paper. All the graphs considered will be simple, finite, and connected.

An edge coloring of a connected graph G is a *rainbow coloring* if for any two vertices of G there is a path between them whose edges have pairwise different colors. Such paths are called *rainbow paths* and G is called *rainbow connected*. The least number of colors needed to make G rainbow connected is the *rainbow connection number* $\mathrm{rc}(G)$ of G. If in the above definitions the paths considered are *shortest* paths, we speak of the *strong rainbow coloring*, *strong rainbow connected graphs*, and the *strong rainbow connection number* $\mathrm{src}(G)$ of G.

The *Cartesian product* $G \square H$ of graphs G and H has the vertex set $V(G) \times V(H)$. Vertices (g, h) and (g', h') of $G \square H$ are adjacent whenever $gg' \in E(G)$ and $h = h'$, or $g = g'$ and $hh' \in E(H)$. The subgraph of $G \square H$ induced on vertices $\{(g, h) \mid g \in V(G)\}$ is a is a *G-layer* (through h). *H-layers* are defined analogously. Clearly, G-layers and H-layers are isomorphic to G and H, respectively. The Cartesian product operation is commutative and associative, hence the Cartesian product of more factors is well-defined. The simplest multiple Cartesian products $\square_{i=1}^{d} K_2$ are known as hypercubes Q_d. The Cartesian product of graphs is connected if and only if all of its factors are connected.

Recall that the *k-wheel* W_k, $k \geq 3$, is the graph obtained from C_k by adding a vertex and connecting it to every vertex of C_k. For $k \geq 3$, the *bipartite wheel* BW_k is the graph obtained from the k-wheel W_k by subdividing each of the edges of the outer cycle of W_k with one vertex. In particular, BW_3 is the graph obtained from Q_3 be removing one of its vertices and $BW_4 = P_3 \square P_3$.

The graph distance considered is the standard shortest paths distance. By $\mathrm{ecc}(v) = \max\{d_G(u, v) \mid u \in V(G)\}$ we denote the *eccentricity* of a vertex $v \in V(G)$. The *diameter* of G, $\mathrm{diam}(G)$, is the length of a longest shortest path, in other words, $\mathrm{diam}(G) = \max\{\mathrm{ecc}(v) \mid v \in V(G)\}$. Note that

$$\mathrm{diam}(G) \leq \mathrm{rc}(G) \leq \mathrm{src}(G).$$

The *radius* of G is defined as $r(G) = \min\{\mathrm{ecc}(v) \mid v \in V(G)\}$.

A subgraph H of connected graph G is *isometric* if $d_H(u, v) = d_G(u, v)$ holds for all $u, v \in V(H)$. If additionally every shortest u, v-path lies completely in H, we say that H is a *convex* subgraph of G. G is a *partial cube* if for some integer d, G is an isometric subgraph of Q_d. Edges uv and $u'v'$ of G are in relation Θ if $d(u, u') + d(v, v') \neq d(u, v') + d(u', v)$. Relation Θ is reflexive, symmetric, but generally not transitive. Winkler [13] proved that a (connected) graph G is a partial cube if and only G is a bipartite graph with transitive relation Θ. Hence for partial cubes G, relation Θ partitions $E(G)$ into equivalence classes that will be referred as Θ-classes. The least number d needed for a partial cube G to isometrically embed into Q_d is the *isometric dimension* $\mathrm{idim}(G)$ of G. Finally, a graph is a *median graph* if for any triple of its vertices u, v, w there exists a unique vertex that lies on a shortest u, v-path, on a shortest u, w-path and on a shortest v, w-path. It is well-known that median graphs are partial cubes.

3 Cartesian products and their subgraphs

In this section we present two known upper bounds for the rainbow connection number of Cartesian products, compare the two bounds, and address the problem if there is a relation between the rainbow connection number of a Cartesian product and its subgraphs. We state all results for two factors since generalizations to more

factors are straightforward due to the associativity of the Cartesian product and since the distance in a product is just the sum of distances between the projections onto factors.

Let ℓ_G and ℓ_H be edge colorings of graphs G and H. Then the coloring $\ell_{G \square H}$ defined with $\ell_{G \square H}((g,h)(g',h)) = \ell_G(gg')$ if $gg' \in E(G)$ and $\ell_{G \square H}((g,h)(g,h')) = \ell_H(hh')$ if $hh' \in E(H)$ will be called a *product coloring* (of $G \square H$ with respect to ℓ_G and ℓ_H). In other words, a product coloring inherits colorings of layers from the colorings of the corresponding factors.

Given graphs G and H equipped with disjoint optimal rainbow colorings, Li and Sun [10] noticed that the product coloring of $G \square H$ gives

$$\mathrm{rc}(G \square H) \leq \mathrm{rc}(G) + \mathrm{rc}(H). \tag{1}$$

They also observed that the bound is tight, we state this fact for the later reference.

Proposition 3.1 ([10]) *Let G and H be graphs with $\mathrm{rc}(G) = \mathrm{diam}(G)$ and $\mathrm{rc}(H) = \mathrm{diam}(H)$. Then $\mathrm{rc}(G \square H) = \mathrm{rc}(G) + \mathrm{rc}(H)$.*

Proof. Using (1) and the fact that $\mathrm{diam}(G \square H) = \mathrm{diam}(G) + \mathrm{diam}(H)$ we have:

$$\begin{aligned}
\mathrm{diam}(G) + \mathrm{diam}(H) &= \mathrm{diam}(G \square H) \leq \mathrm{rc}(G \square H) \leq \mathrm{rc}(G) + \mathrm{rc}(H) \\
&= \mathrm{diam}(G) + \mathrm{diam}(H),
\end{aligned}$$

hence the assertion. $\square$

On the other hand, Basavaraju et al. [1] proved the following upper bound:

$$\mathrm{rc}(G \square H) \leq 2r(G \square H). \tag{2}$$

To prove (2) a construction more involved than the one to obtain (1) is required. The next result demonstrates tightness of (2):

Proposition 3.2 ([1]) *Let G and H be graphs with $\mathrm{diam}(G) = 2r(G)$ and $\mathrm{diam}(H) = 2r(H)$. Then $\mathrm{rc}(G \square H) = 2r(G \square H)$.*

Proof. Similarly as in the proof of Proposition 3.1, the fact $r(G \square H) = r(G) + r(H)$ and (2) yield:

$$\begin{aligned}
\mathrm{diam}(G) + \mathrm{diam}(H) &= \mathrm{diam}(G \square H) \leq \mathrm{rc}(G \square H) \leq 2r(G \square H) = 2r(G) + 2r(H) \\
&= \mathrm{diam}(G) + \mathrm{diam}(H),
\end{aligned}$$

and we are done. $\square$

Proposition 3.2 in particular implies that for any $n, m \geq 2$,

$$\mathrm{rc}(K_{1,n} \square K_{1,m}) = 4.$$

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This example shows that (2) can be arbitrarily better than (1). Indeed, inequality (1) asserts that $\mathrm{rc}(K_{1,n} \square K_{1,m}) \leq n + m$.

On the other hand, (2) can also be worse than (1). To see this, consider any graphs G and H with $\mathrm{diam}(G) = r(G) = \mathrm{rc}(G)$ and $\mathrm{diam}(H) = r(H) = \mathrm{rc}(H)$. Then Proposition 3.1 implies that (1) gives the exact result $\mathrm{rc}(G \square H) = r(G) + r(H)$, while (2) asserts $\mathrm{rc}(G \square H) \leq 2r(G) + 2r(H)$. For a simple concrete example consider products of complete graphs for which $\mathrm{rc}(K_n \square K_m) = 2$.

We turn now our attention to subgraphs of Cartesian products and pose a question whether we can bound $\mathrm{rc}(X)$ in terms of $\mathrm{rc}(G \square H)$, provided X is a subgraph of $G \square H$. Clearly, $\mathrm{rc}(X)$ can be arbitrarily smaller than $\mathrm{rc}(G \square H)$. But it can also be arbitrarily bigger, as the example of Figure 1 shows. There P_{20} is a subgraph of $P_5 \square P_4$, $\mathrm{rc}(P_{20}) = 19$, and $\mathrm{rc}(P_5 \square P_4) = \mathrm{rc}(P_5) + \mathrm{rc}(P_4) = 7$ by Proposition 3.1. Of course, the example easily generalizes to $P_n \square P_m$ for any n and m.
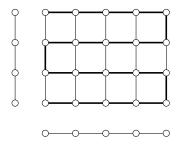

Figure 1: The product $P_5 \square P_4$

Hence not much can be said about general subgraphs of Cartesian products. But what if X is an isometric subgraph of $G \square H$? Also in this case, $\mathrm{rc}(X)$ can be arbitrarily bigger than $\mathrm{rc}(G \square H)$, we will demonstrate this in the next section. Finally, note that considering convex subgraphs gives no new information because convex subgraphs of Cartesian products are precisely subproducts that project onto convex subgraphs of the factors [7].

4   Isometric subgraphs of hypercubes

The message of the previous section was that not much can be said in general about the rainbow connection number of (isometric) subgraphs of Cartesian products. For the simplest Cartesian products—hypercubes—the situation is different:

Proposition 4.1 *Let G be an isometric subgraph of Q_n. Then $\mathrm{rc}(G) \leq \mathrm{rc}(Q_n) = n$.*

Proof. Clearly, $\mathrm{rc}(Q_1) = 1$. For $n \geq 2$ the assertion $\mathrm{rc}(Q_n) = n$ follows by an inductive use of Proposition 3.1.

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Let now G be an isometric subgraph of Q_n. Color Q_n by the product coloring, where each of its factors K_2 is colored with a private color and let G be equipped with the induced coloring. Let $u, v \in V(G)$. Then, as G is isometric in Q_n, there exists a shortest u,v-path in G, say P, that lies completely in Q_n. As P is a geodesic, the product coloring of Q_n assigns different colors to its edges. Hence G is rainbow connected and we have used at most n colors. $\qquad\square$

Corollary 4.2 *Let G be a partial cube with* $\mathrm{idim}(G) = \mathrm{diam}(G)$. *Then* $\mathrm{rc}(G) = \mathrm{idim}(G)$.

Proof. Combine the fact that $\mathrm{rc}(G) \geq \mathrm{diam}(G)$ with Proposition 4.1. $\qquad\square$

We continue with a specific class of partial cubes that will enable us to answer a question raised in Section 3.

Lemma 4.3 *For any $k \geq 4$,* $\mathrm{rc}(BW_k) = 4$.

Proof. Denote the central vertex of BW_k with x. For $i = 1, \ldots, k$ let y_i be a neighbor of x (in ordered way through the cycle), and z_i the vertex of degree 2 between y_i and y_{i+1}, where $i + 1$ is meant cyclically. It is easy to see, that $\mathrm{diam}(BW_k) = 4$ for $k \geq 4$. To complete the proof we thus need to construct a rainbow coloring of BW_k using four colors.

Suppose first k is even. Color the edges xy_i with color 3 if i is odd, and with color 4 otherwise. Edges $y_i z_i$ and $y_i z_{i-1}$ get color 1 for odd i, for even i they get color 2. Any of the non-neighbors (z_i) of x can be reached from x using either colors 4 and 2 or colors 3 and 1. Let $1 \leq i < j \leq k$. Now find a rainbow path between y_i and y_j. If i and j are of different parity, then we can take a path via x using colors 3 and 4. Otherwise, take the path $y_i, x, y_{j-1}, z_{j-1}, y_j$ (a path of colors $3, 4, 2, 1$ or $4, 3, 1, 2$). Next, take z_i and z_j. Then $z_i, y_{i+1}, x, y_{j+1}, z_j$ is a rainbow path if i and j are of different parity, otherwise we can take the path $z_i, y_{i+1}, x, y_j, z_j$. The only case left is when we take y_i and z_j, where $1 \leq i, j \leq k$. Then the path y_i, x, y_t, z_j is a rainbow path, where y_t is the neighbor of z_j with t of different parity as i.

Let now k be odd. As above, edges $y_i z_i$ and $y_i z_{i-1}$ get color 1 for odd i, for even i they get color 2. For $1 \leq i < k$, edge xy_i gets color 3 if i is odd, and color 4 otherwise. We color the remaining edge xy_k with color 2. With the same arguments as in (i) follows that we have a rainbow path between any two vertices in $BW_k - \{z_{k-1}, y_k, z_k\}$. The path $z_{k-1}, y_{k-1}, x, y_1, z_k$ is a rainbow path. Hence it remains to see, that there exists a rainbow path between any vertex from $BW_k - \{z_{k-1}, y_k, z_k\}$ and any vertex from $\{z_{k-1}, y_k, z_k\}$. Clearly, from y_k we can achieve any vertex in $BW_k - \{z_{k-1}, y_k, z_k\}$ by going to x (color 2) and then using the colors $3, 1$ or 4. From z_{k-1} (resp. z_k) we can reach y_i with colors $1, 2, 3$ or $1, 2, 4$ and we can reach z_i with colors $2, 4, 3, 1$ (resp. $1, 3, 4, 2$). $\qquad\square$

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Consider products $BW_k \square BW_k$, $k \geq 4$. Then Lemma 4.3 and Proposition 3.1 imply that $\mathrm{rc}(BW_k \square BW_k) = 8$. On the other hand, $K_{1,k}$ is an isometric subgraph of $BW_k \square BW_k$ with $\mathrm{rc}(K_{1,k}) = k$, which demonstrates that in general Cartesian products the rainbow connection number of an isometric subgraph can be arbitrarily bigger than the one of the product.

In view of Proposition 4.1 we now ask, how big can the difference between $\mathrm{rc}(G)$ and $\mathrm{rc}(Q_n)$ be, where G is isometric in Q_n and $\mathrm{idim}(G) = n$. (The answer is trivial if we would not require $\mathrm{idim}(G) = n$.) We have:

Theorem 4.4 *For every $d \geq 4$ and for every $k \geq d$, there exists a median graph G with $\mathrm{diam}(G) = d = \mathrm{rc}(G)$ and $\mathrm{idim}(G) = k$.*

Proof. Lemma 4.3 gives $\mathrm{rc}(BW_{k-(d-4)}) = 4$. Connect an endvertex of a path on $d-4$ vertices with one of the vertices of degree two in $BW_{k-(d-4)}$ to construct a graph G. By this operation G is still a median graph with $\mathrm{idim}(G) = \mathrm{idim}(BW_{k-(d-4)}) + d - 4 = k$ and $\mathrm{diam}(G) = d$. Now take any 4-coloring of $BW_{k-(d-4)}$ that makes $BW_{k-(d-4)}$ rainbow connected and color the remaining edges in G each with its own (new) color. By this way G obviously gets rainbow connected where d colors are used. $\qquad\square$

The assumption $d \geq 4$ in Theorem 4.4 is unavoidable. First note that the only median graphs with $\mathrm{diam}(G) = 2$ are C_4 and $K_{1,n}$ and that $\mathrm{rc}(K_{1,n}) = \mathrm{idim}(K_{1,n}) = n - 1$. For $d = 3$, it can be shown that all median graphs G of diameter 3 can be constructed as follows. Either $G = Q_3$ or G can be obtained from one of the left three graphs from Figure 2 by attaching to the black vertices an arbitrary number of pendant vertices and by attaching to adjacent black vertices an arbitrary number of pendant squares.
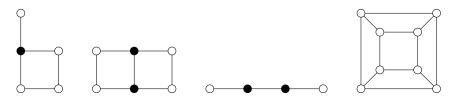

Figure 2: Generators of median graphs of diameter 3

It is straightforward to see that by attaching pendant vertices and pendant squares the rainbow connection number rises. So there is no infinite family (in the sense that we can move isometric dimension arbitrary far away from diameter) of median graphs with diameter and with rainbow connection number equal to 3.

5   Strong rainbow colorings

We now turn to strong rainbow colorings and consider the example from Figure 3. The factors C_4 and K_2 are equipped with strong rainbow colorings, however the product coloring produces an isometric subgraph of $C_4 \square K_2$ that is not (strong) rainbow colored.
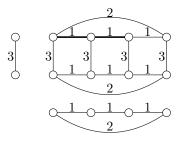

Figure 3: The product $C_4 \square K_2$

This example motivates us to introduce the following concepts. A coloring of the edges of a graph G is a *complete strong rainbow coloring, c-strong rainbow coloring* for short, if every shortest path is a rainbow path. Having a c-strong rainbow coloring of G we say that G is *c-strong rainbow connected*. The smallest number of colors needed to make G c-strong rainbow connected is the *c-strong rainbow connection number* $\overline{\mathrm{src}}(G)$ of G. Note that defining an analogous concept for the rainbow connection is not interesting, as then only the coloring where every edge has its own color would make a graph completely rainbow connected. Clearly,

$$\mathrm{diam}(G) \leq \mathrm{rc}(G) \leq \mathrm{src}(G) \leq \overline{\mathrm{src}}(G) \,.$$

The appropriateness of the c-strong rainbow colorings for Cartesian product is demonstrated with the following:

Proposition 5.1 *For any (connected) graphs G and H and any isometric subgraph X of $G \square H$,*
$$\overline{\mathrm{src}}(X) \leq \overline{\mathrm{src}}(G \square H) \leq \overline{\mathrm{src}}(G) + \overline{\mathrm{src}}(H) \,.$$

Proof. Let G and H be c-strong rainbow colored with r and s disjoint colors, respectively. Then the product coloring is a c-strong rainbow coloring of $G \square H$ using $r + s$ colors. Indeed, this follows from the fact that a shortest path in $G \square H$ projects (under the projection on G) onto a shortest path in G and projects (under the projection on H) onto a shortest path in H, cf. [7]. Moreover, the same argument also implies that $\overline{\mathrm{src}}(X) \leq \overline{\mathrm{src}}(G \square H)$ because the induced coloring of X (being

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embedded into $G \square H$) is also a c-strong rainbow coloring using at most $r + s$ colors. $\square$

Just as for rainbow connection, we can also say more about c-strong rainbow colorings for isometric subgraph of hypercubes. Before we state the result, let us introduce some related notation.

For an edge ab of a partial cube G we will denote with $\Theta(ab)$ the Θ-class of G containing ab. Removing the edges $\Theta(ab)$ from G, two connected components W_{ab} (containing the vertices that are closer to a than to b) and W_{ba} (containing the vertices that are closer to b than to a) are obtained. The subgraphs of W_{ab} and W_{ba} containing the vertices that have a neighbor in W_{ba} and W_{ab} are denoted with U_{ab} and U_{ba}, respectively. Given a partial cube G, the Θ-*coloring* is the coloring of the edges of G with Θ-classes.

Theorem 5.2 *Let G be a partial cube. Then $\overline{\mathrm{src}}(G) = \mathrm{idim}(G)$. Moreover, the Θ-coloring is a unique optimal c-strong rainbow coloring of G.*

Proof. Since no two edges on a shortest path belong to the same Θ-class of G, the Θ-coloring yields $\overline{\mathrm{src}}(G) \le \mathrm{idim}(G)$.

Let c be an arbitrary c-strong rainbow coloring of G using $\overline{\mathrm{src}}(G)$ colors. Fix $ab \in E(G)$. Let xy be an arbitrary edge in W_{ab}. The latter graph is a convex subgraph in G, hence every shortest x, a-path (y, a-path, resp.) lies completely in W_{ab} (it may happen that, for instance, $x = a$). The equality $d(x, a) = d(y, a)$ is not possible, as then we would have an odd cycle in G which cannot happen in a partial cube. Thus, without loss of generality, we may assume $d(x, a) < d(y, a)$. Denote with P a shortest x, a-path. Then y, x, P, b is a shortest y, b-path since $U_{ab} \cong U_{ba}$, U_{ab} (U_{ba}, resp.) is a convex subgraph in W_{ab} (W_{ba}, resp.) and the edges between U_{ab} and U_{ba} form a perfect matching representing exactly the edges of $\Theta(ab)$. It follows that $c(xy) \ne c(ab)$. The case when xy is an edge in W_{ba} is analogous. Therefore, no two edges of different Θ-classes can have the same color in any c-strong rainbow coloring. Hence, $\overline{\mathrm{src}}(G) \ge \mathrm{idim}(G)$.

For the uniqueness just observe that if we would have used in some Θ-class more than one color, then we would need for the whole graph strictly more than $\mathrm{idim}(G)$ colors. $\square$

As already mentioned, the difference between $\mathrm{diam}(G)$ and $\mathrm{rc}(G)$ can be arbitrarily large. We now show that the same is true for $\mathrm{src}(G)$ and $\overline{\mathrm{src}}(G)$.

Proposition 5.3 *For any $n \ge 4$, $\overline{\mathrm{src}}(W_n) = \lceil n/2 \rceil$.*

Proof. Denote the central vertex of W_n with x and the vertices of the outer cycle consecutively with $y_1, y_2, \ldots, y_n$. Any two nonadjacent vertices y_i and y_j are at distance 2 (which is the diameter of W_n), hence in this case the edges xy_i and xy_j

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must have different colors. In other words, the only pairs of edges incident to x that can have the same color are of the form xy_i and xy_j, where y_i is adjacent to y_j. Thus, $\overline{\mathrm{src}}(W_n) \geq \lceil n/2 \rceil$. Now we need to find a coloring that attains this bound.

The edge $y_i y_{i+1}$ gets color 1 for odd $i < n$, for even $i < n$ it gets color 2 and the remaining edge $y_n y_1$ gets color 3. For all i, the edge xy_i gets color $\lceil i/2 \rceil$. Checking that this is a c-strong rainbow coloring is straightforward. $\square$

Recall from [4] that $\mathrm{src}(W_n) = \lceil n/3 \rceil$. Hence the difference $\overline{\mathrm{src}}(W_n) - \mathrm{src}(W_n)$ can be arbitrarily large. In fact, the same is true also in the class of partial cubes. To see this, consider again the bipartite wheels. Then

$$\overline{\mathrm{src}}(BW_n) = \mathrm{idim}(BW_n) = n \quad \text{and} \quad \lceil n/2 \rceil \leq \mathrm{src}(BW_n) \leq \lceil n/2 \rceil + 2 \,.$$

The first inequality follows from the proof of Proposition 5.3. For the second one consider the following coloring. Using the notations from Lemma 4.3, color xy_i with $\lceil i/2 \rceil$ and color the edges around the cycle alternatively with the remaining two colors. Checking the rainbow connectedness is easy.

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