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The Möbius function of $\mathrm{PSU}(3, 2^{2^n})$

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Abstract

Let G be the simple group $\mathrm{PSU}(3, 2^{2^n})$, $n > 0$. For any subgroup H of G, we compute the Möbius function $\mu_L(H, G)$ of H in the subgroup lattice L of G, and the Möbius function $\mu_{\bar{L}}([H], [G])$ of $[H]$ in the poset $\bar{L}$ of conjugacy classes of subgroups of G. For any prime p, we provide the Euler characteristic of the order complex of the poset of non-trivial p-subgroups of G.

Keywords: Unitary groups, Möbius function, subgroup lattice.

Math. Subj. Class.: 20G40, 20D30, 05E15, 06A07

1 Introduction

The Möbius function $\mu(H, G)$ on the subgroups of a finite group G is defined recursively by $\mu(G, G) = 1$ and $\sum_{K \geq H} \mu(K, G) = 0$ if $H < G$. This function was used in 1936 by Hall [12] to enumerate k-tuples of elements of G which generate G, for a given k.

The combinatorial and group-theoretic properties of the Möbius function were investigated by many authors; see the paper [14] by Hawkes, Isaacs, and Özaydin. The Möbius function is defined more generally on a locally finite poset $(\mathcal{P}, \leq)$ by the recursive definition $\mu(x, x) = 1$, $\mu(x, y) = 0$ if $x \not\leq y$, and $\sum_{x \leq z \leq y} \mu(z, y) = 0$ if $x \leq y$; for instance, the poset taken into consideration may be the subgroup lattice L of a finite group G ordered by inclusion. Mann [19, 20] studied $\mu(H, G)$ in the broader context of profinite groups G and defined a probabilistic zeta function $P(G, s)$ associated to G, related to the probability of generating G with s elements when G is positively finitely generated.

The Möbius function on a poset $\mathcal{P}$ also appears in the context of topological invariants of the order simplicial complex $\Delta(\mathcal{P})$ associated to $\mathcal{P}$, see the works of Brown [2] and

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Quillen [25]; if $\mathcal{P}$ is the subgroup lattice of a finite group G, then the reduced Euler characteristic of $\Delta(\mathcal{P})$ is equal to $\mu(\{1\}, G)$. This motivates the search for $\mu(\{1\}, G)$ independently of the knowledge of $\mu(H, G)$ for other subgroups H of G, see for instance [26, 27] and the references therein; $\mu(\{1\}, G)$ is often called the *Möbius number* of G. Shareshian provided a formula in [26] for $\mu(\{1\}, \mathrm{Sym}(n))$, and computed $\mu(\{1\}, G)$ in [27] when $G \in \{\mathrm{PGL}(2, q), \mathrm{PSL}(2, q), \mathrm{PGL}(3, q), \mathrm{PSL}(3, q), \mathrm{PGU}(3, q), \mathrm{PSU}(3, q)\}$ with q odd or G is a Suzuki group $\mathrm{Sz}(2^{2h+1})$.

Consider the poset $\bar{L}$ of conjugacy classes $[H]$ of subgroups H of a finite group G, ordered as follows: $[H] \leq [K]$ if and only if H is contained in some conjugate of K in G. After Hawkes, Isaacs, and Özaydin [14], we denote by $\lambda(H, G)$ the Möbius function $\mu([H], [G])$ in $\bar{L}$, while $\mu(H, G)$ is the Möbius function in L. Some attempt was done to search relations between the Möbius functions $\mu(H, G)$ and $\lambda(H, G)$; Hawkes, Isaacs, and Özaydin [14] proved that, if G is solvable, then

$$\mu(\{1\}, G) = |G'| \cdot \lambda(\{1\}, G). \tag{1.1}$$

The property (1.1), which we call (μ, λ)-property, does not hold in general for non-solvable groups; see [1]. Pahlings [23] proved that, if G is solvable, then

$$\mu(H, G) = [N_{G'}(H) : H \cap G'] \cdot \lambda(H, G) \tag{1.2}$$

for any subgroup H of G. The analysis of the generalized (μ, λ)-property (1.2), although false in general for non-solvable groups, is of interest since it relates the Möbius functions $\mu(H, G)$ and $\lambda(H, G)$.

A lot of work was done by several authors about probabilistic functions for groups; see for instance [6, 10, 19, 20]. In particular, Mann posed in [19] a conjecture, the validity of which would imply that the sum

$$\sum_{H} \frac{\mu(H, G)}{[G : H]^s}$$

over all subgroups $H < G$ of finite index of a positively finitely generated profinite group G is absolutely convergent for s in some right complex half-plane and, for $s \in \mathbb{N}$ large enough, represents the probability of generating G with s elements. Lucchini [18] showed that this problem can be reduced so that Mann's conjecture is reformulated as follows: there exist two constants $c_1, c_2 \in \mathbb{N}$ such that, for any finite monolithic group G with non-abelian socle,

1. $|\mu(H, G)| \leq [G : H]^{c_1}$ for any $H < G$ such that $G = H\,\mathrm{soc}(G)$, and
2. the number of subgroups $H < G$ of index n in G such that $H\,\mathrm{soc}(G) = G$ and $\mu(H, G) \neq 0$ is upper bounded by n^{c_2}, for any $n \in \mathbb{N}$.

It seems natural to investigate this conjecture on finite monolithic groups starting by almost simple groups. Mann's conjecture has been shown to be satisfied by the alternating and symmetric groups [3], as well as by those families of groups G for which $\mu(H, G)$ has been computed for any subgroup H; namely, $\mathrm{PSL}(2, q)$ [8, 12], $\mathrm{PGL}(2, q)$ [8], the Suzuki groups $\mathrm{Sz}(2^{2h+1})$ [9], and the Ree groups $R(3^{2h+1})$ [24].

In this paper, we take into consideration the three dimensional projective special unitary group $G = \mathrm{PSU}(3, q)$ over the field with $q = 2^{2^n}$ elements, for any positive n (note that $\mathrm{PSU}(3, q) = \mathrm{PGU}(3, q)$ as $3 \nmid (q + 1)$). In particular, the following results are obtained.

(i) We compute $\mu(H,G)$ for any subgroup H of G, as summarized in Table 1. This shows that the groups $\mathrm{PSU}(3,2^{2^n})$ satisfy Mann's conjecture.

(ii) We compute $\lambda(H,G)$ for any subgroup H of G, as summarized in Table 1. This shows that the groups $\mathrm{PSU}(3,2^{2^n})$ satisfy the (μ,λ)-property, but do not satisfy the generalized (μ,λ)-property.

(iii) We compute the Euler characteristic $\chi(\Delta(L_p\setminus\{1\}))$ of the order complex of the poset $L_p\setminus\{1\}$ of non-trivial p-subgroups of G, for any prime p, as summarized in Table 2.

For the subgroups listed in Table 1, the isomorphism type determines a unique conjugacy class in G.

Table 1: Subgroups H of $G=\mathrm{PSU}(3,q)$, $q=2^{2^n}$, with $\mu(H)\neq 0$ or $\lambda(H)\neq 0$.

Isomorphism type of H	$\lvert H\rvert$	$N_G(H)$	$\mu(H,G)$	$\lambda(H,G)$
G	$q^3(q^3+1)(q^2-1)$	H	1	1
$(E_q\,.\,E_{q^2})\rtimes C_{q^2-1}$	$q^3(q^2-1)$	H	-1	-1
$\mathrm{PSL}(2,q)\times C_{q+1}$	$q(q^2-1)(q+1)$	H	-1	-1
$(C_{q+1}\times C_{q+1})\rtimes\mathrm{Sym}(3)$	$6(q+1)^2$	H	-1	-1
$C_{q^2-q+1}\rtimes C_3$	$3(q^2-q+1)$	H	-1	-1
$E_q\rtimes C_{q^2-1}$	$q(q^2-1)$	H	1	1
$(C_{q+1}\times C_{q+1})\rtimes C_2$	$2(q+1)^2$	H	1	1
$\mathrm{Sym}(3)$	6	$\mathrm{Sym}(3)\times C_{q+1}$	$q+1$	1
C_3	3	$C_{q^2-1}\rtimes C_2$	$\frac{2(q^2-1)}{3}$	1
C_2	2	$(E_q\,.\,E_{q^2})\rtimes C_{q+1}$	$-\frac{q^3(q+1)}{2}$	-1

Table 2: Euler characteristic of the order complex of the poset of proper p-subgroups of G.

Prime p	$p\nmid\lvert G\rvert$	$p=2$	$p\mid(q+1)$	$p\mid(q-1)$	$p\mid(q^2-q+1)$
$\chi(\Delta(L_p\setminus\{1\}))$	0	q^3+1	$-\frac{q^6-2q^5-q^4+2q^3-3q^2}{3}$	$\frac{q^6+q^3}{2}$	$-\frac{q^6+q^5-q^4-q^3}{3}$

The paper is organized as follows. Section 2 contains preliminary results on the Möbius functions $\mu(H,G)$ and $\lambda(H,G)$ and the relation between the Möbius function and the Euler characteristic of the order complex; this section contains also preliminary results on the groups $G=\mathrm{PSU}(3,2^{2^n})$, whose elements are described geometrically in their action on the Hermitian curve associated to G. Sections 3 and 4 are devoted to the determination of $\mu(H,G)$ and $\lambda(H,G)$, respectively, for any subgroup H of G. Section 5 provides the Euler characteristic of the order complex of the poset of proper p-subgroups of G, for any prime p.

2 Preliminary results

Let $(\mathcal{P}, \leq)$ be a finite poset. The Möbius function $\mu_{\mathcal{P}} \colon \mathcal{P} \times \mathcal{P} \to \mathbb{Z}$ is defined recursively as follows:

$$\mu_{\mathcal{P}}(x,y) = 0 \quad \text{if} \quad x \not\leq y; \qquad \mu_{\mathcal{P}}(x,x) = 1; \qquad \sum_{x \leq z \leq y} \mu_{\mathcal{P}}(z,y) = 0 \quad \text{if} \quad x < y.$$

If $x < y$, then $\mu_{\mathcal{P}}(x,y)$ can be equivalently defined by

$$\sum_{x \leq z \leq y} \mu_{\mathcal{P}}(x,z) = 0.$$

To the poset $\mathcal{P}$ we can associate a simplicial complex $\Delta(\mathcal{P})$ whose vertices are the elements of $\mathcal{P}$ and whose i-dimensional faces are the chains $a_0 < \cdots < a_i$ of length i in $\mathcal{P}$; $\Delta(\mathcal{P})$ is called the *order complex* of $\mathcal{P}$. Provided that $\mathcal{P}$ has a least element 0, the Euler characteristic of the order complex of $\mathcal{P} \setminus \{0\}$ is computed as follows (see [28, Proposition 3.8.6]):

$$\chi(\Delta(\mathcal{P} \setminus \{0\})) = -\sum_{x \in \mathcal{P} \setminus \{0\}} \mu_{\mathcal{P}}(0,x).$$

Given a finite group G, we will consider the following two Möbius functions associated to G.

(i) The Möbius function on the subgroup lattice L of G, ordered by inclusion. We will denote $\mu_L(H,G)$ simply by $\mu(H)$.

(ii) The Möbius function on the poset $\bar{L}$ of conjugacy classes $[H]$ of subgroups H of G, ordered as follows: $[H] \leq [K]$ if and only if H is contained in the conjugate gKg^{-1} for some $g \in G$. We will denote $\mu_{\bar{L}}([H],[G])$ simply by $\lambda(H)$.

Two facts will be used to compute $\mu(H)$. The first easy fact is that, if H and K are conjugate in G, then $\mu(H) = \mu(K)$. The second fact is due to Hall [12, Theorem 2.3], and is stated in the following lemma.

Lemma 2.1. *If $H < G$ satisfies $\mu(H) \neq 0$, then H is the intersection of maximal subgroups of G.*

For any prime p, let L_p be the subposet of L given by all p-subgroups of G, so that

$$\chi(\Delta(L_p \setminus \{1\})) = -\sum_{H \in L_p \setminus \{1\}} \mu_{L_p}(\{1\},H). \tag{2.1}$$

By a result of Brown [2], $\chi(\Delta(L_p \setminus \{1\}))$ is congruent to 1 modulo the order $|G|_p$ of a Sylow p-subgroup of G. In order to compute explicitly $\chi(\Delta(L_p \setminus \{1\}))$ we will use the following result of Hall [12, Equation (2.7)]:

Lemma 2.2. *Let H be a p-group of order p^r. If H is not elementary abelian, then $\mu_{L_p}(\{1\},H) = 0$. If H is elementary abelian, then $\mu_{L_p}(\{1\},H) = (-1)^r p^{\binom{r}{2}}$.*

We describe now the group G which will be considered in the following sections. Let n be a positive integer, $q = 2^{2^n}$, $\mathbb{F}_q$ be the finite field with q element, and $\bar{\mathbb{F}}_q$ be the algebraic closure of $\mathbb{F}_q$. Let $\mathcal{U}$ be a non-degenerate unitary polarity of the plane $\mathrm{PG}(2,q^2)$ over $\mathbb{F}_{q^2}$, and $\mathcal{H}_q \subset \mathrm{PG}(2,\bar{\mathbb{F}}_q)$ be the Hermitian curve defined by $\mathcal{U}$. The following homogeneous equations define models for $\mathcal{H}_q$ which are projectively equivalent over $\mathbb{F}_{q^2}$:

$$X^{q+1} + Y^{q+1} + Z^{q+1} = 0; \qquad (2.2)$$

$$X^q Z + X Z^q - Y^{q+1} = 0. \qquad (2.3)$$

The models (2.2) and (2.3) are called the Fermat and the Norm-Trace model of $\mathcal{H}_q$, respectively. The set of $\mathbb{F}_{q^2}$-rational points of $\mathcal{H}_q$ is denoted by $\mathcal{H}_q(\mathbb{F}_{q^2})$, and consists of the q^3+1 isotropic points of $\mathcal{U}$. The full automorphism group $\mathrm{Aut}(\mathcal{H}_q)$ of $\mathcal{H}_q$ is defined over $\mathbb{F}_{q^2}$, and coincides with the unitary subgroup $\mathrm{PGU}(3,q)$ of $\mathrm{PGL}(3,q^2)$ stabilizing $\mathcal{H}_q(\mathbb{F}_{q^2})$, of order $|\mathrm{PGU}(3,q)| = q^3(q^3+1)(q^2-1)$.

The combinatorial properties of $\mathcal{H}_q(\mathbb{F}_{q^2})$ can be found in [16]. In particular, any line ℓ of $\mathrm{PG}(2,q^2)$ has either 1 or $q+1$ common points with $\mathcal{H}_q(\mathbb{F}_{q^2})$, that is, ℓ is either a tangent line or a chord of $\mathcal{H}_q(\mathbb{F}_{q^2})$; in the former case ℓ contains its pole with respect to $\mathcal{U}$, in the latter case ℓ doesn't. Also, $\mathrm{PGU}(3,q)$ acts 2-transitively on $\mathcal{H}_q(\mathbb{F}_{q^2})$ and transitively on $\mathrm{PG}(2,q^2)\setminus\mathcal{H}_q$; $\mathrm{PGU}(3,q)$ acts transitively also on the non-degenerate self-polar triangles $T=\{P_1,P_2,P_3\}\subset \mathrm{PG}(2,q^2)\setminus\mathcal{H}_q$ with respect to $\mathcal{U}$. Recall that, if $\sigma\in\mathrm{PGU}(3,q)$ stabilizes a point $P\in\mathrm{PG}(2,q^2)$, then σ stabilizes also the polar line of P with respect to $\mathcal{U}$, and vice versa.

The curve $\mathcal{H}_q$ is non-singular and $\mathbb{F}_{q^2}$-maximal of genus $g=\frac{q(q-1)}{2}$, that is, the size of $\mathcal{H}_q(\mathbb{F}_{q^2})$ attains the Hasse-Weil upper bound $q^2+1+2gq$. This implies that $\mathcal{H}_q$ is $\mathbb{F}_{q^4}$-minimal and $\mathbb{F}_{q^6}$-maximal, so that $\mathcal{H}_q(\mathbb{F}_{q^4})\setminus\mathcal{H}_q(\mathbb{F}_{q^2})=\emptyset$ and $|\mathcal{H}_q(\mathbb{F}_{q^6})\setminus\mathcal{H}_q(\mathbb{F}_{q^2})| = q^6+q^5-q^4-q^3$. Let Φ_{q^2} be the Frobenius map $(X,Y,Z)\mapsto(X^{q^2},Y^{q^2},Z^{q^2})$ over $\mathrm{PG}(2,\bar{\mathbb{F}}_{q^2})$; then the $\mathbb{F}_{q^6}\setminus\mathbb{F}_{q^2}$-rational points of $\mathcal{H}_q$ split into $\frac{q^6+q^5-q^4-q^3}{3}$ non-degenerate triangles $\{P,\Phi_{q^2}(P),\Phi_{q^2}^2(P)\}$. The group $\mathrm{PGU}(3,q)$ is transitive on such triangles.

Since $3\nmid(q+1)$, we have $\mathrm{PGU}(3,q)=\mathrm{PSU}(3,q)$; henceforth, we denote by G the simple group $\mathrm{PSU}(3,q)$. The following classification of subgroups of G goes back to Hartley [13]; here we use that $\log_2(q)$ has no odd divisors different from 1. The notation S_2 stands for a Sylow 2-subgroup of G, which is a non-split extension $E_q\,.\,E_{q^2}$ of its elementary abelian center of order q by an elementary abelian group of order q^2.

Theorem 2.3. *Let* $n>0$, $q=2^{2^n}$, *and* $G=\mathrm{PSU}(3,q)$. *Up the conjugation, the maximal subgroups of G are the following.*

(i) The stabilizer $M_1(P)\cong S_2\rtimes C_{q^2-1}$ *of a point* $P\in\mathcal{H}_q(\mathbb{F}_{q^2})$*, of order* $q^3(q^2-1)$.

(ii) The stabilizer $M_2(P)\cong\mathrm{PSL}(2,q)\times C_{q+1}$ *of a point* $P\in\mathrm{PG}(2,q^2)\setminus\mathcal{H}_q(\mathbb{F}_{q^2})$*, of order* $q(q^2-1)(q+1)$.

(iii) The stabilizer $M_3(T)\cong(C_{q+1}\times C_{q+1})\rtimes\mathrm{Sym}(3)$ *of a non-degenerate self-polar triangle* $T=\{P,Q,R\}\subset\mathrm{PG}(2,q^2)\setminus\mathcal{H}_q$ *with respect to* $\mathcal{U}$*, of order* $6(q+1)^2$.

(iv) The stabilizer $M_4(T)\cong C_{q^2-q+1}\rtimes C_3$ *of a triangle* $T=\{P,\Phi_{q^2}(P),\Phi_{q^2}^2(P)\}\subset\mathcal{H}_q(\mathbb{F}_{q^6})\setminus\mathcal{H}_q(\mathbb{F}_{q^2})$*, of order* $3(q^2-q+1)$.

For a detailed description of the maximal subgroups of G, both from an algebraic and a geometric point of view, we refer to [11, 21, 22].

In our investigation it is useful to know the geometry of the elements of $\mathrm{PGU}(3,q)$ on $\mathrm{PG}(2,\bar{\mathbb{F}}_q)$, and in particular on $\mathcal{H}_q(\mathbb{F}_{q^2})$. This can be obtained as a corollary of Theorem 2.3, and is stated in Lemma 2.2 with the usual terminology of collineations of projective planes; see [16]. In particular, a linear collineation σ of $\mathrm{PG}(2,\bar{\mathbb{F}}_q)$ is a (P,ℓ)-*perspectivity*, if σ preserves each line through the point P (the *center* of σ), and fixes each point on the line ℓ (the *axis* of σ). A (P,ℓ)-perspectivity is either an *elation* or a *homology* according to $P \in \ell$ or $P \notin \ell$. Lemma 2.4 was obtained in [21] in a more general form (i.e., for any prime power q).

Lemma 2.4. *For a nontrivial element $\sigma \in G = \mathrm{PSU}(3,q)$, $q = 2^{2^n}$, one of the following cases holds.*

- *(A)* $\mathrm{ord}(\sigma) \mid (q+1)$ *and σ is a homology, with center $P \in \mathrm{PG}(2,q^2) \setminus \mathcal{H}_q$ and axis ℓ_P which is a chord of $\mathcal{H}_q(\mathbb{F}_{q^2})$; (P,ℓ_P) is a pole-polar pair with respect to $\mathcal{U}$.*
- *(B)* $2 \nmid \mathrm{ord}(\sigma)$ *and σ fixes the vertices P_1,P_2,P_3 of a non-degenerate triangle $T \subset$* $\mathrm{PG}(2,q^6)$.
  - *(B1)* $\mathrm{ord}(\sigma) \mid (q+1)$, *$P_1,P_2,P_3 \in \mathrm{PG}(2,q^2) \setminus \mathcal{H}_q$, and the triangle T is self-polar with respect to $\mathcal{U}$.*
  - *(B2)* $\mathrm{ord}(\sigma) \mid (q^2-1)$ *and* $\mathrm{ord}(\sigma) \nmid (q+1)$*; $P_1 \in \mathrm{PG}(2,q^2) \setminus \mathcal{H}_q$ and $P_2,P_3 \in \mathcal{H}_q(\mathbb{F}_{q^2})$.*
  - *(B3)* $\mathrm{ord}(\sigma) \mid (q^2-q+1)$ *and $P_1,P_2,P_3 \in \mathcal{H}_q(\mathbb{F}_{q^6}) \setminus \mathcal{H}_q(\mathbb{F}_{q^2})$.*
- *(C)* $\mathrm{ord}(\sigma) = 2$*; σ is an elation with center $P \in \mathcal{H}_q(\mathbb{F}_{q^2})$ and axis ℓ_P which is tangent to $\mathcal{H}_q$ at P, such that (P,ℓ_P) is a pole-polar pair with respect to $\mathcal{U}$.*
- *(D)* $\mathrm{ord}(\sigma) = 4$*; σ fixes a point $P \in \mathcal{H}_q(\mathbb{F}_{q^2})$ and a line ℓ_P which is tangent to $\mathcal{H}_q$ at P, such that (P,ℓ_P) is a pole-polar pair with respect to $\mathcal{U}$.*
- *(E)* $\mathrm{ord}(\sigma) = 2d$ *where d is a nontrivial divisor of $q+1$; σ fixes two points $P \in \mathcal{H}_q(\mathbb{F}_{q^2})$ and $Q \in \mathrm{PG}(2,q^2) \setminus \mathcal{H}_q$, the polar line PQ of P, and the polar line of Q which passes through P.*

For a detailed description of the elements and subgroups of G, both from an algebraic and a geometric point of view, we refer to [11, 21, 22], on which our geometric arguments are based.

Throughout the paper, a nontrivial element of G is said to be of type (A), (B), (B1), (B2), (B3), (C), (D), or (E), as given in Lemma 2.4. Also, the polar line to $\mathcal{H}_q$ at $P \in \mathrm{PG}(2,q^2)$ is denoted by ℓ_P. Note that, under our assumptions, any element of order 3 in G is of type (B2). We will denote a cyclic group of order d by C_d and an elementary abelian group of order d by E_d. The center $Z(S_2)$ of S_2 is elementary abelian of order q, and any element in $S_2 \setminus Z(S_2)$ has order 4; see [11, Section 3].

3 Determination of $\mu(H)$ for any subgroup H of G

Let $n > 0$, $q = 2^{2^n}$, $G = \mathrm{PSU}(3,q)$. This section is devoted to the proof of the following theorem.

Theorem 3.1. *Let H be a proper subgroup of G. Then H is the intersection of maximal subgroups of G if and only if H is one of the following groups:*

$$\begin{array}{lll} S_2 \rtimes C_{q^2-1}, & \mathrm{PSL}(2,q) \times C_{q+1}, & C_{q^2-q+1} \rtimes C_3, \\ (C_{q+1} \times C_{q+1}) \rtimes \mathrm{Sym}(3), & E_q \rtimes C_{q^2-1}, & (C_{q+1} \times C_{q+1}) \rtimes C_2, \\ C_{q+1} \times C_{q+1}, & C_{q^2-1}, & C_{2(q+1)}, \\ C_{q+1} = Z(M_2(P)) \text{ for some } P, & E_q, & \mathrm{Sym}(3), \\ C_3, & C_2, & \{1\}. \end{array} \tag{3.1}$$

Given a type of groups in Equation (3.1)*, there is just one conjugacy class of subgroups of G of that isomorphism type.*

The normalizer $N_G(H)$ of H in G for the groups H in Equation (3.1) *are, respectively:*

$$\begin{array}{lll} H, & H, & H, \\ H, & H, & H, \\ H \rtimes \mathrm{Sym}(3), & H \rtimes C_2, & E_q \times C_{q+1}, \\ \mathrm{PSL}(2,q) \times H, & S_2 \rtimes C_{q^2-1}, & H \times C_{q+1}, \\ C_{q^2-1} \rtimes C_2, & S_2 \rtimes C_{q+1}, & G. \end{array} \tag{3.2}$$

The values $\mu(H)$ for the groups H in Equation (3.1) *are, respectively:*

$$\begin{array}{ccc} -1, & -1, & -1, \\ -1, & 1, & 1, \\ 0, & 0, & 0, \\ 0, & 0, & q+1, \\ \frac{2(q^2-1)}{3}, & -\frac{q^3(q+1)}{2}, & 0. \end{array} \tag{3.3}$$

The proof of Theorem 3.1 is divided into several propositions.

Proposition 3.2. *The group G contains exactly one conjugacy class for any group in Equation* (3.1)*.*

Proof. **Case 1:** The first four groups in Equation (3.1), i.e.,

$$S_2 \rtimes C_{q^2-1},\ \mathrm{PSL}(2,q) \times C_{q+1},\ C_{q^2-q+1} \rtimes C_3,\ \text{ and }\ (C_{q+1} \times C_{q+1}) \rtimes \mathrm{Sym}(3),$$

are the maximal subgroups of G, for which there is just one conjugacy class by Theorem 2.3.

Case 2: Let $\alpha_1, \alpha_2 \in G$ have order 3, so that they are of type (B2) and α_i fixes two distinct points $P_i, Q_i \in \mathcal{H}_q(\mathbb{F}_{q^2})$. The group G is 2-transitive on $\mathcal{H}_q(\mathbb{F}_{q^2})$, and the pointwise stabilizer of $\{P_i, Q_i\}$ is cyclic of order q^2-1. Hence, $\langle\alpha_1\rangle$ and $\langle\alpha_2\rangle$ are conjugated in G.

Case 3: Let $\alpha_1, \alpha_2 \in G$ have order 2, so that they are of type (C) and α_i fixes exactly one point P_i on $\mathcal{H}_q(\mathbb{F}_{q^2})$. Up to conjugation $P_1 = P_2$, as G is transitive on $\mathcal{H}_q(\mathbb{F}_{q^2})$. The involutions fixing P_1 in G, together with the identity, form an elementary abelian group E_q, which is normalized by a cyclic group C_{q-1}; no nontrivial element of C_{q-1} commutes with any nontrivial element of E_q (see [11, Section 4]). Hence, α_1 and α_2 are conjugated under an element of C_{q-1}.

Case 4: Let $\alpha_1, \alpha_2, \beta_1, \beta_2 \in G$ satisfy $o(\alpha_i) = 3$, $o(\beta_i) = 2$, and $H_i := \langle \alpha_i, \beta_i \rangle \cong \mathrm{Sym}(3)$. As shown in the previous point, we can assume $\alpha_1 = \alpha_2$ up to conjugation. Let $P, Q \in \mathcal{H}_q(\mathbb{F}_{q^2})$ and $R \in \mathrm{PG}(2, q^2) \setminus \mathcal{H}_q$ be the fixed points of α_1. Since $\beta_i \alpha_1 \beta_i^{-1} = \alpha_1^{-1}$, we have that β_i fixes R and interchanges P and Q; β is then uniquely determined from the $\mathbb{F}_{q^2}$-rational point of PQ fixed by β (namely, the intersection between PQ and the axis of β). Since the pointwise stabilizer C_{q^2-1} of $\{P, Q\}$ acts transitively on $PQ(\mathbb{F}_{q^2}) \setminus \mathcal{H}_q$, β_1 and β_2 are conjugated, and the same holds for H_1 and H_2.

Case 5: Any two groups isomorphic to C_{q^2-1} are conjugated in G, because they are generated by elements of type (B2) and G is 2-transitive on $\mathcal{H}_q(\mathbb{F}_{q^2})$.

Case 6: Any two groups isomorphic to E_q are conjugated in G, because any such group fixes exactly one point $P \in \mathcal{H}_q(\mathbb{F}_{q^2})$, G is transitive on $\mathcal{H}_q(\mathbb{F}_{q^2})$, and the stabilizer $G_P = M_1(P)$ contains just one subgroup E_q.

Case 7: Any two groups $H_1, H_2 \cong E_q \rtimes C_{q^2-1}$ are conjugated in G. In fact, their Sylow 2-subgroups E_q coincide up to conjugation, as shown in the previous point. The normalizer $N_G(E_q)$ fixes the fixed point $P \in \mathcal{H}_q(\mathbb{F}_{q^2})$ of E_q, and hence $N_G(E_q) = M_1(P) = S_2 \rtimes C_{q^2-1}$. The complements C_{q^2-1} are conjugated by Schur-Zassenhaus Theorem; hence, H_1 and H_2 are conjugated.

Case 8: Any two groups isomorphic to $C_{2(q+1)}$ are conjugated in G, because they are generated by elements of type (E) and two elements α_1, α_2 of type (E) of the same order are conjugated in G. In fact, α_i is uniquely determined by its fixed points $P_i \in \mathcal{H}_q(\mathbb{F}_{q^2})$ and $Q_i \in \ell_{P_i}(\mathbb{F}_{q^2}) \setminus \mathcal{H}_q$; here, ℓ_{P_i} is the polar line of P_i. Up to conjugation $P_1 = P_2$, from the transitivity of G on $\mathcal{H}_q(\mathbb{F}_{q^2})$. Also, S_2 has order q^3 and acts on the q^2 points of $\ell_{P_i}(\mathbb{F}_{q^2}) \setminus \mathcal{H}_q$ with kernel E_q, hence transitively. We can then assume $Q_1 = Q_2$.

Case 9: Let Z_{P_i} be the center of $M_2(P_i)$, $i = 1, 2$. As shown in [5, Section 4], $Z_{P_i} \cong C_{q+1}$ and Z_{P_i} is made by the homologies with center P_i, together with the identity. Since G is transitive on $\mathrm{PG}(2, q^2) \setminus \mathcal{H}_q$, we have up to conjugation that $M_2(P_1) = M_2(P_2)$ and $Z_{P_1} = Z_{P_2}$.

Case 10: Any two groups $H_1, H_2 \cong C_{q+1} \times C_{q+1}$ are conjugated in G. In fact, H_i is the pointwise stabilizer of a self-polar triangle $T_i = \{P_i, Q_i, R_i\} \subset \mathrm{PG}(2, q^2) \setminus \mathcal{H}_q$ (see [5, Section 3]), and the stabilizers of T_1 and T_2 are conjugated by Theorem 2.3.

Case 11: Any two groups $H_1, H_2 \cong (C_{q+1} \times C_{q+1}) \rtimes C_2$ are conjugated in G. In fact, their subgroups $C_{q+1} \times C_{q+1}$ coincide up to conjugation as shown above, and fix pointwise a self-polar triangle $T = \{P, Q, R\} \subset \mathrm{PG}(2, q^2) \setminus \mathcal{H}_q$. Let $\beta_i \in H_i$ have order 2, $i = 1, 2$. Then β_i fixes one vertex of T and interchanges the other two vertexes. Up to conjugation in $M_3(T)$ we have $\beta_1(P) = \beta_2(P) = P$. Then $H_1 = H_2$, as they coincide with the stabilizer of P in $M_3(T)$. □

Proposition 3.3. *The normalizers $N_G(H)$ of the groups H in Equation* (3.1) *are given in Equation* (3.2).

Proof. **Case 1:** Clearly $N_G(H) = H$ for any H from the first four groups of Equation (3.1) as H is maximal in G.

Case 2: Let $H = E_q \rtimes C_{q^2-1}$. Then $H \leq M_2(P)$, where P is the unique fixed point of C_{q^2-1} in $\mathrm{PG}(2,q^2) \setminus \mathcal{H}_q$. The group H has a unique cyclic subgroup C_{q+1} of order $q+1$; namely, C_{q+1} is the center of $M_2(P)$ and is made by the homologies with center P; since q is even, H is a split extension $C_{q+1} \times (E_q \rtimes C_{q-1})$. Hence, $N_G(H) \leq N_G(C_{q+1}) = M_2(P)$. The group $H/C_{q+1} \cong E_q \rtimes C_{q-1}$ is maximal and hence self-normalizing in $M_2(P)/C_{q+1} = \mathrm{PSL}(2,q)$; thus, $N_G(E_q \rtimes C_{q-1}) = H$ and $N_G(H) = H$.

Case 3: Let $H = C_{q+1} \times C_{q+1}$. Then $N_G(H) \leq M_3(T)$, where T is the self-polar triangle fixed pointwise by H. Since H is the kernel of $M_3(T)$ in its action on T, we have $N_G(H) = M_3(T)$ and $|N_G(H)| = 6|H|$.

Case 4: Let $H = (C_{q+1} \times C_{q+1}) \rtimes C_2$. Then $C_{q+1} \times C_{q+1}$ is normal in $N_G(H)$, being the unique subgroup of index 2 in H. Hence $N_G(H) \leq M_3(T)$, where T is the self-polar triangle fixed pointwise by H. Also, $N_G(H)$ fixes the vertex P of T fixed by H, so that $N_G(H) \neq M_3(T)$. This implies $N_G(H) = H$.

Case 5: Let $H = C_{q^2-1}$. Then H is generated by an element α of type (B2) with fixed points $P, Q \in \mathcal{H}_q(\mathbb{F}_{q^2})$ and $R \in \mathrm{PG}(2,q^2) \setminus \mathcal{H}_q$. Let β be an involution satisfying $\beta(R) = R$, $\beta(P) = Q$, and $\beta(Q) = P$; then $\beta \in N_G(H)$, because H coincides with the pointwise stabilizer of $\{P,Q\}$ in G. An explicit description is the following: given $\mathcal{H}_q$ with equation (2.3), we can assume up to conjugation that $\alpha = \mathrm{diag}(a^{q+1}, a, 1)$ where a is a generator if $\mathbb{F}_{q^2}^*$ (see [11]); then take

$$\beta = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}. \tag{3.4}$$

Since $N_G(H)$ acts on $\{P,Q\}$ and $\beta \in N_G(H)$, the pointwise stabilizer H of $\{P,Q\}$ has index 2 in $N_G(H)$. This implies $N_G(H) = C_{q^2-1} \rtimes C_2$ and $|N_G(H)| = 2|H|$.

Case 6: Let $H = C_{2(q+1)}$, so that H is generated by an element α of type (E) fixing exactly two points $P \in \mathcal{H}_q(\mathbb{F}_{q^2})$ and $Q \in \ell_P(\mathbb{F}_{q^2}) \setminus \mathcal{H}_q$. Then $N_G(H)$ fixes P and Q. The subgroup E_q of $M_1(P)$ commutes with H elementwise, while any 2-element in $M_1(P) \setminus E_q$ has order 4 and does not fix Q; hence, the Sylow 2-subgroup of $N_G(H)$ is E_q. Also, $N_G(H) = E_q \rtimes C_d$, where C_d is a subgroup of C_{q^2-1} containing the subgroup C_{q+1} of H. Let C_2 be the subgroup of H of order 2; the quotient group $(C_2 \rtimes C_d)/C_{q+1} \cong C_2 \rtimes C_{\frac{d}{q+1}}$ acts faithfully as a subgroup of $\mathrm{PGL}(2,q)$ on the $q+1$ points of $\ell_Q \cap \mathcal{H}_q$. By the classification of subgroups of $\mathrm{PGL}(2,q)$ ([7]; see [17, Hauptsatz 8.27]), this implies $d = 1$; that is, $N_G(H) = E_q \rtimes C_{q+1}$ and $|N_G(H)| = \frac{q}{2}|H|$.

Case 7: Let $H = C_{q+1} = Z(M_2(P))$. Since H is the center of $M_2(P)$, $M_2(P) \leq N_G(H)$. Conversely, H is made by homologies with center P, and hence $N_G(H)$ fixes P. Thus, $N_G(H) = M_2(P)$ and $|N_G(H)| = q(q^2-1)|H|$.

Case 8: Let $H = E_q$. Since E_q has a unique fixed point P on $\mathcal{H}_q(\mathbb{F}_{q^2})$ and $E_q = Z(M_1(P))$, we have $N_G(H) \leq M_1(P)$ and $M_1(P) \leq N_G(H)$, so that $N_G(H) = M_1(P)$ and $|N_G(H)| = q^2(q^2-1)|H|$.

Case 9: Let $H = \mathrm{Sym}(3) = \langle \alpha, \beta \rangle$, with $o(\alpha) = 3$ and $o(\beta) = 2$. Let $P, Q \in \mathcal{H}_q(\mathbb{F}_{q^2})$ and $R \in \mathrm{PG}(2,q^2) \setminus \mathcal{H}_q$ be the fixed points of α; β fixes R, interchanges P and Q, and fixes another point A_β on $\ell_R \cap \mathcal{H}_q$. The group $N_G(H)$ acts on $\{P,Q\}$ and on $\{A_\beta, A_{\alpha\beta}, A_{\alpha^2\beta}\}$.

The pointwise stabilizer C_{q^2-1} has a subgroup C_{q+1} which is the center of $M_2(P)$ and fixes PQ pointwise, while any element in $C_{q^2-1} \setminus C_{q+1}$ acts semiregularly on $PQ \setminus \{P, Q\}$; hence, $C_{q^2-1} \cap N_G(H) = C_{3(q+1)}$. If an element $\gamma \in N_G(H)$ fixes $\{P, Q\}$ pointwise, then γ fixes a point in $\{A_\beta, A_{\alpha\beta}, A_{\alpha^2\beta}\}$, and hence $\gamma \in \{\beta, \alpha\beta, \alpha^2\beta\}$. Therefore, $N_G(H) = C_{3(q+1)} \rtimes C_2 = H \times C_{q+1}$ and $|N_G(H)| = (q+1)|H|$.

Case 10: Let $H = C_3$ and α be a generator of H, with fixed points $P, Q \in \mathcal{H}_q(\mathbb{F}_{q^2})$ and $R \in \mathrm{PG}(2, q^2) \setminus \mathcal{H}_q$. The normalizer $N_G(H)$ fixes R and acts on $\{P, Q\}$. There exists an involution $\beta \in G$ normalizing H and interchanging P and Q (see Equation (3.4)). Then the pointwise stabilizer of $\{P, Q\}$ has index 2 in $N_G(H)$. Also, the pointwise stabilizer of $\{P, Q\}$ in G is cyclic of order $q^2 - 1$. Then $N_G(H) = C_{q^2-1} \rtimes C_2$ and $|N_G(H)| = \frac{2(q^2-1)}{3}|H|$.

Case 11: Let $H = C_2$ and α be a generator of H, with fixed point $P \in \mathcal{H}_q(\mathbb{F}_{q^2})$. Then $N_G(H)$ fixes P, i.e. $N_G(H) \leq M_1(P) = S_2 \rtimes C_{q^2-1}$. Since any involution of $M_1(P)$ is in the center of S_2, the Sylow 2-subgroup of $N_G(H)$ has order q^3. Let $\beta \in C_{q^2-1}$. If $o(\beta) \mid (q+1)$, then β commutes with any involution of S_2. If $o(\beta) \nmid (q+1)$, then β does not commute with any element of S_2. This implies that $N_G(H) = S_2 \rtimes C_{q+1}$, and $|N_G(H)| = \frac{q^3(q+1)}{2}|H|$. □

Lemma 3.4. *Let $\alpha \in G$ be an involution, and hence an elation, with center P and axis ℓ_P. Then there exist exactly $q^3/2$ self-polar triangles $T_{i,j} = \{P_i, Q_{i,j}, R_{i,j}\}$, $i = 1, \ldots, q^2$, $j = 1, \ldots, \frac{q}{2}$, such that α stabilizes $T_{i,j}$. Also, $P_i \in \ell_P$ and $P \in Q_{i,j}R_{i,j}$ for any i and j.*

Proof. The number of involutions in G is $(q^3+1)(q-1)$, since for any of the q^3+1 $\mathbb{F}_{q^2}$-rational points P of $\mathcal{H}_q$ the involutions fixing P form a group E_q. The number of self-polar triangles $T \subset \mathrm{PG}(2, q^2) \setminus \mathcal{H}_q$ is $[G : M_3(T)] = \frac{(q^3+1)q^3(q^2-1)}{6(q+1)^2}$. For any self-polar triangle $T = \{A_1, A_2, A_3\} \subset \mathrm{PG}(2, q^2) \setminus \mathcal{H}_q$, the number of involutions in G stabilizing T is $3(q+1)$. In fact, for any of the 3 vertexes of T there are exactly $q+1$ involutions $\alpha_1, \ldots, \alpha_{q+1}$ fixing that vertex, say A_1, and interchanging A_2 and A_3; α_i is uniquely determined by its center $A_2A_3 \cap \mathcal{H}_q$. Then, by double counting the size of

$$\{(\beta, T) \mid \beta \in G,\ o(\beta) = 2,\ T \subset \mathrm{PG}(2, q^2) \setminus \mathcal{H}_q \text{ is a self-polar triangle, } \beta \text{ stabilizes } T\},$$

α stabilizes exactly $\frac{q^3}{2}$ self-polar triangles T. For any such T, one vertex P_i of T lies on the axis of α, because α is an elation, and the other two vertexes $\{Q_{i,j}, R_{i,j}\}$ of T lie on the polar line ℓ_{P_i} of P_i. Since $M_1(P)$ is transitive on the q^2 points $P_1, \ldots, P_{q^2}$ of $\ell_P(\mathbb{F}_{q^2}) \setminus \{P\}$, any point P_i is contained in the same number $\frac{q}{2}$ of self-polar triangles $T_{i,j}$ stabilized by α. □

Lemma 3.5. *Let $\alpha \in G$ have order 3. Then there are exactly $\frac{q^2-1}{3}$ self-polar triangles*

$$T_i \subset \mathrm{PG}(2, q^2) \setminus \mathcal{H}_q, \quad i = 1, \ldots, \frac{q^2-1}{3},$$

which are stabilized by α. Also, there are exactly $\frac{2(q^2-1)}{3}$ triangles

$$\tilde{T}_j = \{P_j, \Phi_{q^2}(P_j), \Phi_{q^2}^2(P_j)\} \subset \mathcal{H}_q(\mathbb{F}_{q^6}) \setminus \mathcal{H}_q(\mathbb{F}_{q^2}), \quad j = 1, \ldots, \frac{2(q^2-1)}{3},$$

which are stabilized by α.

Proof. By Proposition 3.2, any two subgroups of G of order 3 are conjugated in G. Also, any element of order 3 is conjugated to its inverse by an involution of G. Hence, any two element of order 3 are conjugated in G.

Now the claim follows by double counting the size of

$$\{(\beta, T) \mid \beta \in G,\ o(\beta) = 3,\ T \subset \mathrm{PG}(2, q^2) \setminus \mathcal{H}_q \text{ is a self-polar triangle}, \\ \beta \text{ stabilizes } T\},$$

and

$$\{(\beta, \tilde{T}) \mid \beta \in G,\ o(\beta) = 3,\ \tilde{T} = \{P, \Phi_{q^2}(P), \Phi_{q^2}^2(P)\} \text{ with} \\ P \in \mathcal{H}_q(\mathbb{F}_{q^6}) \setminus \mathcal{H}_q(\mathbb{F}_{q^2}),\ \beta \text{ stabilizes } \tilde{T}\},$$

using the following facts. The number of elements of order 3 in G is $\binom{q^3+1}{2} \cdot 2$. The number of self-polar triangles $T \subset \mathrm{PG}(2, q^2) \setminus \mathcal{H}_q$ is $[G : M_3(T)]$. The number of elements of order 3 stabilizing a fixed self-polar triangle T is $2(q+1)^2$, because any element acting as a 3-cycle on the vertexes of T has order 3 (see [5, Section 3]). The number of triangles $\tilde{T} = \{P, \Phi_{q^2}(P), \Phi_{q^2}^2(P)\} \subset \mathcal{H}_q(\mathbb{F}_{q^6}) \setminus \mathcal{H}_q(\mathbb{F}_{q^2})$ is $[G : M_4(\tilde{T})]$. The number of elements of order 3 stabilizing a fixed triangle $\tilde{T}$ is $2(q^2 - q + 1)$, because any element in $M_4(\tilde{T}) \setminus C_{q^2-q+1}$ has order 3 (see [4, Section 4]). □

Lemma 3.6. *Let $H < G$ be isomorphic to $\mathrm{Sym}(3)$, $H = \langle\alpha\rangle \rtimes \langle\beta\rangle$. Then there are exactly $q + 1$ self-polar triangles*

$$T_i = \{P_i, Q_i, R_i\} \subset \mathrm{PG}(2, q^2) \setminus \mathcal{H}_q, \quad i = 1, \ldots, q + 1,$$

which are stabilized by H. Up to relabeling the vertexes, we have that $P_1, \ldots, P_{q+1}$ lie on the axis of the elation β, $Q_1, \ldots, Q_{q+1}$ lie on the axis of the elation $\alpha\beta$, and $R_1, \ldots, R_{q+1}$ lie on the axis of the elation $\alpha^2\beta$.

Proof. By Proposition 3.2, any two subgroups $K_1, K_2 < G$ with $K_i \cong \mathrm{Sym}(3)$ are conjugated, and $|N_G(K_i)| = 6(q + 1)$; hence, the number of subgroups of G isomorphic to $\mathrm{Sym}(3)$ is $[G : N_G(K_i)] = \frac{(q^3+1)q^3(q-1)}{6}$. The number of self-polar triangles T is $[G : M_3(T)] = \frac{(q^2-q+1)q^3(q-1)}{6}$. Then the claim on the number of self-polar triangles follows by double counting the size of

$$\{(K, T) \mid K < G,\ K \cong \mathrm{Sym}(3),\ T \subset \mathrm{PG}(2, q^2) \setminus \mathcal{H}_q \text{ is a self-polar triangle}, \\ K \text{ stabilizes } T\},$$

once we show that, for any self-polar triangle $T = \{A, B, C\}$, there are in G exactly $(q + 1)^2$ subgroups isomorphic to $\mathrm{Sym}(3)$ which stabilize T.

Let $K < M_3(T)$, $K \cong \mathrm{Sym}(3)$, $K = \langle\alpha, \beta\rangle$ with $o(\alpha) = 3$, $o(\beta) = 2$. Let P, Q, R be the fixed points of α, with $P \in \mathrm{PG}(2, q^2) \setminus \mathcal{H}_q$, $Q, R \in \mathcal{H}_q(\mathbb{F}_{q^2})$. By Proposition 3.3, $N_G(K) = K \times C_{q+1}$ where C_{q+1} is made by homologies with center P; this implies $N_G(K) \cap M_3(T) = K$. Hence, there are at least $[M_3(T) : \mathrm{Sym}(3)] = (q + 1)^2$ distinct groups $\mathrm{Sym}(3)$ stabilizing T, namely the conjugates of K through elements of $M_3(T)$. On the other side, $M_3(T)$ contains exactly $(q + 1)^2$ subgroups K of order 3, with fixed points $P \in \mathrm{PG}(2, q^2) \setminus \mathcal{H}_q$, $Q, R \in \mathcal{H}_q(\mathbb{F}_{q^2})$. Any involution β of $M_3(T)$ normalizing

K is uniquely determined by the vertex of T that β fixes, because $\beta(P) = P$, $\beta(Q) = R$, and $\beta(R) = Q$. Thus, K is contained in exactly one subgroup of $M_3(T)$ isomorphic to $\mathrm{Sym}(3)$. Therefore the number of subgroups isomorphic to $\mathrm{Sym}(3)$ which stabilize T is $(q+1)^2$.

Finally, the configuration of the vertexes of $T_1, \dots, T_{q+1}$ on the axes of the involutions of H follows from Lemma 2.4 and the fact that every involution fixes a different vertex of T_i. □

Proposition 3.7. *Any group H in Equation* (3.1) *is the intersection of maximal subgroups of G.*

Proof. **Case 1:** The first four groups of Equation (3.1) are exactly the maximal subgroups of G.

Case 2: Let $H = E_q \rtimes C_{q^2-1}$. Let $P \in \mathcal{H}_q(\mathbb{F}_{q^2})$ be the unique point of $\mathcal{H}_q$ fixed by E_q; E_q fixes ℓ_P pointwise. Also, the fixed points of C_{q^2-1} are $P, Q \in \mathcal{H}_q(\mathbb{F}_{q^2})$ and $R \in \mathrm{PG}(2, q^2) \setminus \mathcal{H}_q$, where $R \in \ell_P$ and $PQ = \ell_R$. Then $H \leq M_1(P) \cap M_2(R)$. Conversely, from $M_1(P) \cap M_2(R) \leq M_1(P)$ follows $M_1(P) \cap M_2(R) = K \rtimes C_d$ with $K \leq S_2$ and $C_d \leq C_{q^2-1}$. From $M_1(P) \cap M_2(R) \leq M_2(R)$ follows that K does not contain any element of type (D), so that $K \leq E_q$. Thus, $M_1(P) \cap M_2(R) \leq H$, and $H = M_1(P) \cap M_2(R)$.

Case 3: Let $H = (C_{q+1} \times C_{q+1}) \rtimes C_2$. Let $T = \{P, Q, R\} \subset \mathrm{PG}(2, q^2) \setminus \mathcal{H}_q$ be the self-polar triangle fixed pointwise by $C_{q+1} \times C_{q+1}$, and let P be the vertex of T fixed by C_2. Then $H \leq M_3(T) \cap M_2(P)$. Conversely, since $M_3(T) \cap M_2(P)$ fixes P and acts on $\{Q, R\}$, the pointwise stabilizer $C_{q+1} \times C_{q+1}$ of T has index at most 2 in $M_3(T) \cap M_2(P)$, so that $M_3(T) \cap M_2(P) \leq H$. Thus, $H = M_3(T) \cap M_2(P)$.

Case 4: Let $H = C_{q+1} \times C_{q+1}$. Let $T = \{P, Q, R\} \subset \mathrm{PG}(2, q^2) \setminus \mathcal{H}_q$ be the self-polar triangle fixed pointwise by $C_{q+1} \times C_{q+1}$. Since H is the whole pointwise stabilizer of T in G, we have $H = M_2(P) \cap M_2(Q) \cap M_2(R)$.

Case 5: Let $H = C_{q^2-1}$ and let α be a generator of H, with fixed points $P, Q \in \mathcal{H}_q(\mathbb{F}_{q^2})$ and $R \in \mathrm{PG}(2, q^2) \setminus \mathcal{H}_q$. The pointwise stabilizer of $\{P, Q\}$ in G is exactly H; thus, $H = M_1(P) \cap M_2(Q)$.

Case 6: Let $H = C_{2(q+1)}$ and let α be a generator of H, of type (E), with fixed points $P \in \mathcal{H}_q(\mathbb{F}_{q^2})$ and $Q \in \ell_P(\mathbb{F}_{q^2}) \setminus \mathcal{H}_q$. By Lemma 3.4 there are $\frac{q}{2}$ self-polar triangles stabilized by the involution α^{q+1} having one vertex in Q and two vertexes on ℓ_Q; let $T = \{Q, R_1, R_2\}$ be one of these triangles. Then $H \leq M_1(P) \cap M_2(Q) \cap M_3(T)$.

Conversely, let $\sigma \in (M_1(P) \cap M_2(Q) \cap M_3(T)) \setminus \{1\}$. If σ fixes $\{R_1, R_2\}$ pointwise, then from $\sigma \in M_1(P)$ follows that σ is in the kernel $C_{q+1} \leq H$ of the action of $M_2(Q)$ on ℓ_Q. The quotient $(M_1(P) \cap M_2(Q) \cap M_3(T))/C_{q+1}$ acts on ℓ_Q as a subgroup of $\mathrm{PSL}(2, q)$ fixing P and interchanging R_1 and R_2. From [17, Hauptsatz 8.27] follows $(M_1(P) \cap M_2(Q) \cap M_3(T))/C_{q+1} \cong C_2$, and hence $H = M_1(P) \cap M_2(Q) \cap M_3(T)$.

Case 7: Let $H = C_{q+1} = Z(M_2(P))$. Then H is made by the homologies of G with center P, together with the identity. Thus, $H = M_1(P_1) \cap M_1(P_2) \cap M_1(P_3)$, where P_1, P_2, P_3 are distinct point in $\ell_P \cap \mathcal{H}_q$.

Case 8: Let $H = E_q$ and let P be the unique point of $\mathcal{H}_q(\mathbb{F}_{q^2})$ fixed by any element in H. Then $H = M_2(P_1) \cap M_2(P_2) \cap M_2(P_3)$, where P_1, P_2, P_3 are distinct points in $\ell_P(\mathbb{F}_{q^2}) \setminus \{P\}$.

Case 9: Let $H = C_2$, α be a generator of H with fixed point $P \in \mathcal{H}_q(\mathbb{F}_{q^2})$, and $P_1, P_2, P_3 \in \ell_P(\mathbb{F}_{q^2}) \setminus \{P\}$. Let $T = \{P_1, Q_{1,1}, R_{1,1}\}$ be a self-polar triangle stabilized by α. Then $H \leq M_2(P_1) \cap M_2(P_2) \cap M_2(P_3) \cap M_3(T)$. Since the elation α is uniquely determined by the image of one point not on its axis ℓ_P, $H \leq M_3(T)$ implies $H = M_2(P_1) \cap M_2(P_2) \cap M_2(P_3) \cap M_3(T)$.

Case 10: Let $H = C_3$. By Lemma 3.5, H stabilizes $\frac{2(q^2-1)}{3}$ triangles $\tilde{T} \subset \mathcal{H}_q(\mathbb{F}_{q^6}) \setminus \mathcal{H}_q(\mathbb{F}_{q^2})$; let $\tilde{T}_1$ and $\tilde{T}_2$ be two of them. Then $H \leq M_4(\tilde{T}_1) \cap M_4(\tilde{T}_2)$. If $H < M_4(\tilde{T}_1) \cap M_4(\tilde{T}_2)$, then there exist a nontrivial $\sigma \in G$ stabilizing pointwise both $\tilde{T}_1$ and $\tilde{T}_2$, a contradiction to Lemma 2.4. Thus, $H = M_4(\tilde{T}_1) \cap M_4(\tilde{T}_2)$.

Case 11: Let $H = \mathrm{Sym}(3)$. By Lemma 3.6, H stabilizes $q+1$ self-polar triangles $T_1, \ldots, T_{q+1}$, so that $H \leq M_3(T_1) \cap \cdots \cap M_3(T_{q+1})$. Suppose by contradiction that $H \neq M_3(T_1) \cap \cdots \cap M_3(T_{q+1})$. Then $M_3(T_1) \cap \cdots \cap M_3(T_{q+1})$ contains a nontrivial element σ fixing every triangle T_i pointwise. Since the triangles T_i's do not have vertexes in common, this is a contradiction to Lemma 2.4. Thus, $H = M_3(T_1) \cap \cdots \cap M_3(T_{q+1})$.

Case 12: Let $H = \{1\}$. Since G is simple, H is the Frattini subgroup of G. □

Proposition 3.8. *If $H < G$ is the intersection of maximal subgroups, then H is one of the groups in Equation* (3.1).

Proof. We proceed as follows: we take every subgroup $K < G$ in Equation (3.1), starting from the maximal subgroups M_i of G; we consider the intersections $H = K \cap M_i$ of K with the maximal subgroups of G; here, we assume that $K \not\leq M_i$. We show that H is again one of the groups in Equation (3.1).

Case 1: Let $K = S_2 \rtimes C_{q^2-1} = M_1(P)$ for some $P \in \mathcal{H}_q(\mathbb{F}_{q^2})$.

Let $H = K \cap M_1(Q)$, $Q \neq P$. Then H is the pointwise stabilizer of $\{P, Q\} \subset \mathcal{H}_q(\mathbb{F}_{q^2})$, which is cyclic of order q^2-1, i.e. $H = C_{q^2-1}$.

Let $H = K \cap M_2(Q)$. Suppose $Q \in \ell_P$. Then $H = E_{q^2} \rtimes C_{q^2-1}$, where E_{q^2} is made by the elations with axis PQ and C_{q^2-1} is generated by an element of type (B2) with fixed points Q, P, and another point $R \in \ell_Q$. Now suppose $Q \notin \ell_P$. Then H stabilizes ℓ_Q and hence also the point $R = \ell_P \cap \ell_Q$. Then H stabilizes QR and hence also the pole A of QR; by reciprocity, $A \in PQ$. Thus, H fixes three collinear point A, P, Q, and hence every point on AP. Then $H = C_{q+1} = Z(M_2(R))$.

Let $H = K \cap M_3(T)$, $T = \{A, B, C\}$, with P on a side of T, say $P \in AB$. Then H fixes C and acts on $\{A, B\}$. Thus, H is generated by an element of type (E) with fixed points P, C and fixed lines PC, AB; hence, $H = C_{2(q+1)}$.

Let $H = K \cap M_3(T)$, $T = \{A, B, C\}$, with P out of the sides of T. By reciprocity, no vertex of T lies on ℓ_P. This implies that no elation acts on T, so that $2 \nmid |H|$; this also implies that no homology in $M_3(T)$ fixes P, so that H has no nontrivial elements fixing T pointwise. Thus $H \leq C_3$.

Let $H = K \cap M_4(T)$. By Lagrange's theorem, $H \leq C_3$.

Case 2: Let $K = \mathrm{PSL}(2, q) \times C_{q+1} = M_2(P)$ for some $P \in \mathrm{PG}(2, q^2) \setminus \mathcal{H}_q$.

Let $H = K \cap M_2(Q)$, $Q \neq P$, and R be the pole of PQ. If $R \in PQ$, then H is the pointwise stabilizer of PQ and is made by the elations with center R; thus, $H = E_q$. If $R \notin PQ$, then H is the pointwise stabilizer of $T = \{P, Q, R\}$; thus, $H = C_{q+1} \times C_{q+1}$.

Let $H = K \cap M_3(T)$ with $T = \{A, B, C\}$. If P is a vertex of T, then $H = (C_{q+1} \times C_{q+1}) \rtimes C_2$. If P is on a side of T but is not a vertex, say $P \in AB$, then H fixes the pole $D \in AB$ of C. Then H fixes pointwise $T' = \{P, C, D\}$ and acts on $\{A, B\}$. This implies that H fixes AB pointwise and $H = C_{q+1} = Z(M_2(C))$. If P is out of the sides of T, then no nontrivial element of H fixes T pointwise; thus, $H \leq \mathrm{Sym}(3)$.

Let $H = K \cap M_4(T)$. By Lagrange's theorem, $H \leq C_3$.

Case 3: Let $K = (C_{q+1} \times C_{q+1}) \rtimes \mathrm{Sym}(3) = M_3(T)$ for some self-polar triangle $T = \{A, B, C\}$.

Let $H = K \cap M_3(T')$ with $T' = \{A', B', C'\} \neq T$. If T and T' have one vertex $A = A'$ in common, then $H = C_{2(q+1)}$ is generated by an element of type (E) fixing A and a point $D \in BC = B'C'$. If $A' \in AC \setminus \{A, C\}$, then H stabilizes $B'C'$, because $B'C'$ is the only line containing 4 points of $\{A, B, C, A', B', C'\}$. Then H fixes A', A, and C; hence also B. Since H acts on $\{B', C'\}$, H cannot be made by nontrivial homologies of center B; thus, $H = \{1\}$.

Let $H = K \cap M_4(T')$. By Lagrange's theorem, $H \leq C_3$.

Case 4: Let $K = C_{q^2-q+1} \rtimes C_3 = M_4(T)$ for some $T \subset \mathcal{H}_q(\mathbb{F}_{q^6})$. Let $H = K \cap M_4(T')$ with $T' \neq T$. Since 3 does not divide the order of the pointwise stabilized C_{q^2-q+1} of T, H contains no nontrivial elements fixing T or T' pointwise. Thus, $H \leq C_3$.

Case 5: Let $K = E_q \rtimes C_{q^2-1}$ and $P \in \mathcal{H}_q(\mathbb{F}_{q^2})$, $Q \in \ell_P \setminus \{P\}$ be the fixed points of K.

Let $H = K \cap M_1(R)$ with $R \neq P$. If $R \in \ell_Q$, then $H = C_{q^2-1}$. If $R \notin \ell_Q$, then H fixes the pole S of PR; by reciprocity $S \in PQ$, so that H fixes PQ pointwise and also $R \notin PQ$. Thus, $H = \{1\}$.

Let $H = K \cap M_2(R)$ with $R \neq Q$. If $R \in \ell_P$, then H is the pointwise stabilizer E_q of PQ. If $R \notin \ell_P$, then H fixes pointwise the self-polar triangle $\{Q, R, S\}$ where S is the pole of QR. Hence, either $H = C_{q+1} = Z(M_2(Q))$ or $H = \{1\}$ according to $P \in RS$ or $P \notin RS$, respectively.

Let $H = K \cap M_3(T)$ with $T = \{A, B, C\}$. If P is on a side of T, say $P \in BC$, then either $H = \{1\}$ or $H = C_{q+1} = Z(M_2(A))$. If P is out of the sides of T, then no nontrivial element of H can fix T pointwise; thus, $H \leq \mathrm{Sym}(3)$.

Let $H = K \cap M_4(T)$. By Lagrange's theorem, $H \leq C_3$.

Case 6: Let $K = (C_{q+1} \times C_{q+1}) \rtimes C_2 = M_3(T) \cap M_2(A)$, where $T = \{A, B, C\}$.

Let $H = K \cap M_1(P)$. If $P \in BC$, then $H = C_{2(q+1)}$ is generated by an element of type (E). If $P \notin BC$, then $H = \{1\}$.

Let $H = K \cap M_2(P)$, $P \neq A$. If $P \in \{B, C\}$, then H is the pointwise stabilizer $C_{q+1} \times C_{q+1}$ of T. If $P \in AB \setminus \{A, B\}$ or $P \in AC \setminus \{A, C\}$, then $H = C_{q+1} = Z(M_2(C))$ or $H = C_{q+1} = Z(M_2(B))$, respectively. If $P \in BC \setminus \{B, C\}$, then H fixes A, P, the pole of AP, and acts on $\{B, C\}$; thus, $H = C_{q+1} = Z(M_2(A))$. If P is not on the sides of T, then no nontrivial element of H can fix T pointwise; thus, $H \leq C_2$.

Let $H = K \cap M_3(T')$ with $T' = \{A', B', C'\} \neq T$. Since $3 \nmid |H|$, H fixes a vertex of T', say A'. If $A' = A$, then $H = C_{2(q+1)}$. If $A' \in \{B, C\}$, then H fixes T pointwise and acts on $\{B', C'\}$; thus, $H = C_{q+1} = Z(M_2(A'))$. If $A' \in (AB \cup AC) \setminus \{A, B, C\}$, then H fixes AB or AC pointwise and acts on $\{B', C'\}$; thus, $H = \{1\}$. If $A' \in BC$, then H

fixes A, A', and the pole D of AA'; as H acts on $\{B,C\}$, this implies $H=\{1\}$. If A' is not on the sides of T, then no nontrivial element of H fixes T pointwise and $H\leq C_2$.

Let $H=K\cap M_4(T')$. By Lagrange's theorem, $H\leq C_3$.

Case 7: Let $K=C_{q+1}\times C_{q+1}=M_3(T)\cap M_2(A)\cap M_2(B)\cap M_2(C)$ with $T=\{A,B,C\}$.

Let $H=K\cap M_1(P)$ or $H=K\cap M_2(P)$. If P is not on the sides of T, then $H=\{1\}$; if P is on a side of T, say $P\in BC$, then $H=C_{q+1}=Z(M_2(A))$.

Let $H=K\cap M_3(T')$ with $T'=\{A',B',C'\}$. Since K is not divisible by 2 or 3, $H\neq\{1\}$ only if H fixes T' pointwise. Up to relabeling, this implies $A'=A$, $B',C'\in BC$, and $H=C_{q+1}=Z(M_2(A))$.

Let $H=K\cap M_4(T')$. By Lagrange's theorem, $H=\{1\}$.

Case 8: Let $K=C_{q^2-1}=\langle\alpha\rangle$, with α of type (B2) fixing the points $P\in\mathrm{PG}(2,q^2)\setminus\mathcal{H}_q$ and $Q,R\in\mathcal{H}_q(\mathbb{F}_{q^2})$.

Let $H=K\cap M_1(A)$ or $H=K\cap M_2(A)$. Since the nontrivial elements of H are either of type (B2) or of type (A) with axis QR, we have $H=\{1\}$ unless $A\in QR$; in this case, $H=C_{q+1}=Z(M_2(P))$.

Let $H=K\cap M_3(T)$ or $H=K\cap M_4(T)$. By Lagrange's theorem, $H\leq C_3$.

Case 9: Let $K=C_{2(q+1)}=\langle\alpha\rangle$ with α of type (E) fixing the points $P\in\mathcal{H}_q(\mathbb{F}_{q^2})$ and $Q\in\mathrm{PG}(2,q^2)\setminus\mathcal{H}_q$.

Let $H=K\cap M_1(R)$ or $H=K\cap M_2(R)$. If $R\in\ell_Q$, then $H=C_{q+1}=Z(M_2(Q))$. If $R\notin\ell_Q$, then $H=\{1\}$.

Let $H=K\cap M_3(T)$; recall that $H<K$. If Q is a vertex of T, then $H=C_{q+1}=Z(M_2(Q))$. If Q is not a vertex of T, then no homology in K acts on T; hence, $H\leq C_2$.

Let $H=K\cap M_4(T)$. By Lagrange's theorem, $H=\{1\}$.

Case 10: Let $K=C_{q+1}=Z(M_2(P))$ for some $P\in\mathrm{PG}(2,q^2)\setminus\mathcal{H}_q$ and $\sigma\in K\setminus\{1\}$. Then σ fixes no points out of $\{P\}\cup\ell_P$; also, the triangles fixed by σ have one vertex in P and two vertexes on ℓ_P. Thus, $K\cap M_i=\{1\}$ for any maximal subgroup M_i of G not containing K.

Case 11: Let $K=E_q$ and $\sigma\in E_q\setminus\{1\}$. Recall that K fixes one point $P\in\mathcal{H}_q(\mathbb{F}_{q^2})$ and the line ℓ_P pointwise. Also, σ fixes no points out of ℓ_P. If σ fixes a triangle $T=\{A,B,C\}$, then one vertex of T lies on $\ell_P(\mathbb{F}_{q^2})$, say A, and σ is uniquely determined by $\sigma(B)=C$. Thus, $K\cap M_1(Q)=K\cap M_2(Q)=K\cap M_4(T')=\{1\}$ and $K\cap M_3(T)\leq C_2$.

Case 12: Let $K\in\{\mathrm{Sym}(3),C_3,C_2,\{1\}\}$. Then every subgroup of K is in Equation (3.1). □

Proposition 3.9. *The values $\mu(H)$ for the groups in Equation* (3.1) *are given in Equation* (3.3).

Proof. Let H be one of the groups in Equation (3.1). By Lemma 2.1 and Proposition 3.8, $\mu(H)$ only depends on the subgroups K of G such that $H<K$ and K is in Equation (3.1).

Case 1: If H is one of the first four groups in Equation (3.1), then H is maximal in G, and hence $\mu(H)=-1$.

Case 2: Let $H=E_q\rtimes C_{q^2-1}$. Let $P\in\mathcal{H}_q(\mathbb{F}_{q^2})$ and $Q\in\mathrm{PG}(2,q^2)\setminus\mathcal{H}_q$ be the fixed points of H. Then $H=M_1(P)\cap M_2(Q)$ and H is not contained in any other maximal subgroup of G. Thus, $\mu(H) = -\{\mu(G) + \mu(M_1(P)) + \mu(M_2(Q))\} = 1$.

Case 3: Let $H = (C_{q+1} \times C_{q+1}) \rtimes C_2$. Let $T = \{P, Q, R\}$ be the self-polar triangle stabilized by H, with $H(P) = P$. No point different from P is fixed by H. Also, if a triangle $T' = \{P', Q'\} \neq T$ is fixed by H, then P is a vertex of T', say $P = P'$, and $\{Q', R'\} \subset QR$; but $C_{q+1} \times C_{q+1}$ has orbits of length $q + 1 > |\{Q', R'\}|$, so that H cannot fix T'. Then $H = M_2(P) \cap M_3(T)$ and H is not contained in any other maximal subgroup of G. Thus, $\mu(H) = 1$.

Case 4: Let $H = C_{q+1} \times C_{q+1}$ and $T = \{P, Q, R\}$ be the self-polar triangle fixed pointwise by H. The vertexes of T are the unique fixed points of the elements of type (B1) in H. Also, any triangle $T' \neq T$ fixed by an element of type (A) in H has two vertexes on a side ℓ of T; but H has orbits of length $q + 1 > 2$ on ℓ, so that H does not fix T'. Then $H = M_3(T) \cap M_2(P) \cap M_2(Q) \cap M_2(R)$ and H is not contained in any other maximal subgroup of G.

If K is one of the groups $M_3(T) \cap M_2(P)$, $M_3(T) \cap M_2(P)$, $M_3(T) \cap M_2(P)$, then K contains H properly, and $\mu(K) = 1$ as shown in the previous point. The intersection of three groups between $M_3(T)$, $M_2(P)$, $M_2(Q)$, and $M_2(R)$ is equal to H. Thus, by direct computation, $\mu(H) = 0$.

Case 5: Let $H = C_{q^2-1}$ with fixed points $P \in \mathrm{PG}(2, q^2) \setminus \mathcal{H}_q$ and $Q, R \in \mathcal{H}_q(\mathbb{F}_{q^2})$. Then $H = M_1(Q) \cap M_1(R) = M_1(Q) \cap M_1(R) \cap M_2(P)$. We already know $\mu(M_1(Q) \cap M_2(P)) = \mu(M_1(R) \cap M_2(P)) = 1$. Moreover, C_{q^2-1} has no fixed triangles, by Lagrange's theorem, and no other fixed points. Thus, by direct computation, $\mu(H) = 0$.

Case 6: Let $H = C_{2(q+1)} = \langle \alpha \rangle$; α is of type (E), fixes the points $P \in \mathcal{H}_q(\mathbb{F}_{q^2})$ and $Q \in PG(2, q^2) \setminus \mathcal{H}_q$, and fixes the lines ℓ_P and ℓ_Q. Since α^2 is a homology with center Q, the orbits on ℓ_Q of H coincide with the orbits on ℓ_Q of the elation α^{q+1}. By Lemma 3.4, the self-polar triangles T_i stabilized by H have a vertex in Q and two vertexes on ℓ_Q; there are exactly $\frac{q}{2}$ such triangles $T_1, \dots, T_{\frac{q}{2}}$. No other triangle and no other point different from P and Q is fixed by H, so that $H = M_1(P) \cap M_2(Q) \cap M_3(T_1) \cap \cdots \cap M_3(T_{\frac{q}{2}})$ and H is not contained in any other maximal subgroup of G.

If K is the intersection of $M_2(Q)$ with one of the groups $M_1(P), M_3(T_1), \dots, M_3(T_{\frac{q}{2}})$, then $K = E_q \rtimes C_{q^2-1}$ or $K = (C_{q+1} \times C_{q+1}) \rtimes C_2$; hence, K contains H properly and $\mu(K) = 1$ as shown above. The intersection of K with a third maximal subgroup of G containing H coincides with H. Finally, the intersection of any two groups in $\{M_1(P), M_3(T_1), \dots, M_3(T_{\frac{q}{2}})\}$ coincides with H. Thus, by direct computation, $\mu(H) = 0$.

Case 7: Let $H = C_{q+1} = Z(M_2(P))$. Denote $\ell_P \cap \mathcal{H}_q = \{P_1, \dots, P_{q+1}\}$ and $\ell(\mathbb{F}_{q^2}) \setminus \mathcal{H}_q = \{Q_1, \dots, Q_{q^2-q}\}$ such that, for $i = 1, \dots, \frac{q^2-q}{2}$, $T_i = \{P, Q_i, Q_{i+\frac{q^2-q}{2}}\}$ are the self-polar triangles with a vertex in P. Then

$$H = \bigcap_{i=1}^{q+1} M_1(P_i) \cap M_2(P) \cap \bigcap_{i=1}^{q^2-q} M_2(Q_i) \cap \bigcap_{i=1}^{(q^2-q)/2} M_3(T_i)$$

and H is not contained in any other maximal subgroup of G. By direct inspection, the intersections K of some (at least two) maximal subgroups of G such that $H < K < G$ are exactly the following.

(i) $K = M_1(P_i) \cap M_1(P_j)$ for some $i \neq j$; in this case, $K = C_{q^2-1}$ and $\mu(K) = 0$.

(ii) $K = M_1(P_i) \cap M_2(P)$ with $i \in \{1, \dots, q+1\}$; in this case, $K = E_q \rtimes C_{q^2-1}$ and $\mu(K) = 1$. These $q+1$ groups are pairwise distinct.

(iii) $K = M_1(P_i) \cap M_3(T_j)$ for some i, j; in this case, $K = C_{2(q+1)}$ and $\mu(K) = 0$.

(iv) $K = M_2(P) \cap M_2(Q_i)$ for some i; in this case, $K = C_{q+1} \times C_{q+1}$ and $\mu(K) = 0$.

(v) $K = M_2(P) \cap M_3(T_i)$ with $i \in \{1, \dots, \frac{q^2-q}{2}\}$; in this case, $K = (C_{q+1} \times C_{q+1}) \rtimes C_2$ and $\mu(K) = 1$. These $\frac{q^2-q}{2}$ groups are pairwise distinct.

(vi) $K = M_2(Q_i) \cap M_3(T_i)$ or $K = M_2(Q_{i+\frac{q^2-q}{2}}) \cap M_3(T_i)$, with $i \in \{1, \dots, \frac{q^2-q}{2}\}$; in this case, $K = (C_{q+1} \times C_{q+1}) \rtimes C_2$ and $\mu(K) = 0$. These $q^2 - q$ groups are pairwise distinct.

To sum up, the only subgroups K with $H < K < G$ and $\mu(K) \neq 0$ are the maximal subgroups, $q+1$ distinct groups of type $E_q \rtimes C_{q^2-1}$, and $\frac{3(q^2-q)}{2}$ distinct groups of type $(C_{q+1} \times C_{q+1}) \rtimes C_2$. Thus, $\mu(H) = 0$.

Case 8: Let $H = E_q$. Let P be the point of $\mathcal{H}_q(\mathbb{F}_{q^2})$ fixed by H; H fixes ℓ_P pointwise. We have $H = M_1(P) \cap M_2(Q_1) \cap \cdots \cap M_2(Q_{q^2})$, where $Q_1, \dots, Q_{q^2}$ are the $\mathbb{F}_{q^2}$-rational points of $\ell_P \setminus \{P\}$; H is not contained in any other maximal subgroup of G. The intersections K of at least two maximal subgroups of G such that $H < K < G$ are exactly the q^2 groups $M_1(P) \cap M_2(Q_i) = E_q \rtimes C_{q^2-1}$, with $\mu(K) = 1$. Thus, by direct computation, $\mu(H) = 0$.

Case 9: Let $H = \mathrm{Sym}(3) = \langle \alpha, \beta \rangle$ with $o(\alpha) = 3$ and $o(\beta) = 2$. Let $P \in \mathrm{PG}(2, q^2) \setminus \mathcal{H}_q$ and $Q, R \in \mathcal{H}_q$ be the fixed points of α, and $A \in QR$ be the fixed point of β on $\mathcal{H}_q$, so that β fixes $\ell_A = AP$. By Lemma 3.6 and its proof, $H = M_2(P) \cap M_3(T_1) \cap \cdots \cap M_3(T_{q+1})$, where T_i has one vertex on $\ell_A \setminus \{P, A\}$ and the other two vertexes are collinear with A; H is not contained in any other maximal subgroup of G.

For any $i, j \in \{1, \dots, q+1\}$ with $i \neq j$, no vertex of T_j is on a side of T_i; hence, no nontrivial element of $M_3(T_i) \cap M_3(T_j)$ fixes T_i pointwise. This implies $M_3(T_i) \cap M_3(T_j) = H$. Analogously, no nontrivial element in $M_3(T_i) \cap M_2(P)$ fixes T_i pointwise, and this implies $M_3(T_i) \cap M_2(P) = H$. Thus, by direct computation, $\mu(H) = q+1$.

Case 10: Let $H = C_3 = \langle \alpha \rangle$ with fixed points $P \in \mathrm{PG}(2, q^2) \setminus \mathcal{H}_q$ and $Q, R \in \mathcal{H}_q$. By Lemma 3.5,

$$H = M_1(Q) \cap M_1(R) \cap M_2(P) \cap \bigcap_{i=1}^{(q^2-1)/3} M_3(T_i) \cap \bigcap_{i=1}^{2(q^2-1)/3} M_4(\tilde{T}_i)$$

and H is not contained in any other maximal subgroup of G. By direct inspection, the intersections K of at least two maximal subgroups of G such that $H < K < G$ are exactly the following.

(i) $K = M_1(Q) \cap M_2(P)$ or $K = M_1(R) \cap M_2(P)$; in this case, $K = E_q \rtimes C_{q^2-1}$ and $\mu(K) = 1$.

(ii) $K = M_1(Q) \cap M_1(R)$; in this case, $K = C_{q^2-1}$ and $\mu(K) = 0$.

(iii) There are exactly $\frac{q-1}{3}$ groups K containing H with $K \cong \mathrm{Sym}(3)$, and hence $\mu(K) = q+1$. In fact, any involution $\beta \in G$ satisfying $\langle H, \beta\rangle \cong \mathrm{Sym}(3)$ interchanges Q and R and fixes a point of $(QR \cap \mathcal{H}_q) \setminus \{P, Q\}$; conversely, any of the $q-1$ points $A_1, \ldots, A_{q-1}$ of $(QR \cap \mathcal{H}_q) \setminus \{P, Q\}$ determines uniquely the involution $\beta_i \in G$ such that $\beta(A_i)$, $\beta_i(Q) = R$, $\beta_i(R) = Q$, and hence $\langle H, \beta_i\rangle \cong \mathrm{Sym}(3)$. The involutions β_i, $\alpha\beta_i$, and $\alpha^2\beta_i$, together with H, generate the same group; thus, there are exactly $\frac{q-1}{3}$ groups $\mathrm{Sym}(3)$ containing H.

Thus, by direct computation, $\mu(H) = \frac{2(q^2-1)}{3}$.

Case 11: Let $H = C_2 = \langle\alpha\rangle$, where α has center P. Let $\ell_P(\mathbb{F}_{q^2}) \setminus \{P\} = \{P_1, \ldots, P_{q^2}\}$. By Lemma 3.4,

$$H = M_1(P) \cap \bigcap_{i=1}^{q^2} M_2(P_i) \cap \bigcap_{i=1}^{q^2} \bigcap_{j=1}^{q/2} M_3(T_{i,j}),$$

where the triangles $T_{i,j}$ are described in Lemma 3.4; H is not contained in any other maximal subgroup of G. By direct inspection, the intersections K of at least two maximal subgroups of G such that $H < K < G$ are exactly the following.

(i) $K = M_1(P) \cap M_2(P_i)$ for $i = 1, \ldots, q^2$; in this case, $K = E_q \rtimes C_{q^2-1}$ and $\mu(K) = 0$.

(ii) $K = M_2(P_i) \cap M_2(P_j)$ with $i \neq j$; in this case, $K = E_q$ and $\mu(K) = 0$.

(iii) $K = M_1(P) \cap M_3(T_{i,j})$; in this case, $K = E_q \rtimes C_{2(q+1)}$ and $\mu(K) = 0$.

(iv) $K = M_2(Q_i) \cap M_3(T_{i,j})$ with $i \in \{1, \ldots, q^2\}$ and $j \in \{1, \ldots, \frac{q}{2}\}$; these $\frac{q^3}{2}$ distinct groups are of type $(C_{q+1} \times C_{q+1}) \rtimes C_2$, so that $\mu(K) = 1$.

(v) There are exactly $N = \frac{q^3}{2}$ groups K containing H such that $K \cong \mathrm{Sym}(3)$, and hence $\mu(K) = q+1$. This follows by double counting the size of

$$I = \{(H, K) \mid H, K < G,\ H \cong C_2,\ K \cong \mathrm{Sym}(3),\ H < K\}.$$

Arguing as in the proof of Lemma 3.4, $|I| = (q^3+1)(q-1)N$; arguing as in the proof of Lemma 3.6, $|I| = \frac{q^3(q^3+1)(q-1)}{6} \cdot 3$. Hence, $N = \frac{q^3}{2}$.

Thus, by direct computation, $\mu(H) = -\frac{q^3(q+1)}{2}$.

Case 12: Let $H = \{1\}$. Then $\mu(H) = -\sum_{\{1\} < K \leq G} \mu(K, G)$. By the values $\mu(K)$ computed in the previous cases, Propositions 3.2, and Proposition 3.3, only the following groups K have to be considered:

(i) 1 group G;

(ii) q^3+1 groups $S_2 \rtimes C_{q^2-1}$;

(iii) $q^2(q^2-q+1)$ groups $\mathrm{PSL}(2,q) \times C_{q+1}$;

(iv) $\frac{q^3(q-1)(q^2-q+1)}{6}$ groups $(C_{q+1} \times C_{q+1}) \rtimes \mathrm{Sym}(3)$;

(v) $\frac{q^3(q+1)^2(q-1)}{3}$ groups $C_{q^2-q+1} \rtimes C_3$;

(vi) $(q^3+1)q^2$ groups $E_q \rtimes C_{q^2-1}$;

(vii) $\frac{q^3(q-1)(q^2-q+1)}{2}$ groups $(C_{q+1} \times C_{q+1}) \rtimes C_2$;

(viii) $\frac{q^3(q^3+1)(q-1)}{6}$ groups $\mathrm{Sym}(3)$;

(ix) $\frac{q^3(q^3+1)}{2}$ groups C_3;

(x) $(q^3+1)(q-1)$ groups C_2.

Thus, by direct computation, $\mu(H) = 0$. □

4 Determination of $\lambda(H)$ for any subgroup H of G

Let $n > 0$, $q = 2^{2^n}$, $G = \mathrm{PSU}(3, q)$. This section is devoted to the proof of the following theorem.

Theorem 4.1. *Let H be a proper subgroup of G. Then $\lambda(H) \neq 0$ if and only H is one of the following groups:*

$$\begin{array}{lll} E_q \rtimes C_{q^2-1}, & (C_{q+1} \times C_{q+1}) \rtimes C_2, & \mathrm{Sym}(3), \\ C_3, & S_2 \rtimes C_{q^2-1}, & \mathrm{PSL}(2, q) \times C_{q+1}, \\ (C_{q+1} \times C_{q+1}) \rtimes \mathrm{Sym}(3), & C_{q^2-q+1} \rtimes C_3, & C_2. \end{array} \tag{4.1}$$

For any isomorphism type in Equation (4.1) *there is just one conjugacy class of subgroups of G.*

If H is in the first row of Equation (4.1)*, then $\lambda(H) = -1$; if H is in the second row of Equation* (4.1)*, then $\lambda(H) = 1$.*

Proof. By Proposition 3.2, for any isomorphism type in Equation (4.1) there is just one conjugacy class of subgroups of G of that type. Hence, we can use the notation $[M_1]$, $[M_2]$, $[M_3]$ and $[M_4]$ for the conjugacy classes of $M_1(P)$, $M_2(P)$, $M_3(T)$ and $M_4(T)$, respectively. If $H = G$, then $\lambda(H) = 1$; if H is one of the groups in the second row of Equation (4.1) and $H \neq C_2$, then $\lambda(H) = -1$ as H is maximal in G.

Case 1: Firstly, we assume that H is not a subgroup of $\mathrm{Sym}(3)$, and that H is not a group of homologies, i.e. $H \not\leq C_{q+1} = Z(M_2(Q))$ for any point Q.

(i) Let $H < M_4(T)$ for some T. From $H \neq C_3$ follows that some nontrivial element in H fixes T pointwise; hence, H is not contained in any maximal subgroup of G other than $M_4(T)$. Thus, inductively, $\lambda(H) = -\{\lambda(G) + \lambda(M_4(T))\} = 0$.

(ii) Let $H < M_1(P)$ for some P; we assume in addition that $\gcd(|H|, q-1) > 1$. Here, the assumption $H \not\leq \mathrm{Sym}(3)$ reads $H \notin \{\{1\}, C_2, C_3\}$. If H contains an element of order 4, then H is not contained in any maximal subgroup of G other than $M_1(P)$. Thus, inductively, $\lambda(H) = 0$.

We can then assume that the 2-elements of H are involutions, so that $H = E_{2^r} \rtimes C_d$ with $0 \leq r \leq 2^n$ and $d \mid (q^2-1)$ (see [15, Theorem 11.49]). This implies that $H \leq M_1(P) \cap M_2(Q)$ for some $Q \in \ell_P$; the eventual nontrivial elements in H whose order divides $q+1$ are homologies with center Q. Then we have $[H] \leq [M_1]$, $[H] \leq [M_2]$; by Lagrange's theorem, $[H] \not\leq [M_4]$. From the assumptions $\gcd(|H|, q-1) > 1$ and $H \not\leq \mathrm{Sym}(3)$ follows $[H] \not\leq [M_3]$.

If $H = E_q \rtimes C_{q^2-1}$, then no proper subgroup of $M_1(P)$ or $M_2(Q)$ contains H properly; thus, $\lambda(H) = 1$. If $H \neq E_q \rtimes C_{q^2-1}$, then $H < E_q \rtimes C_{q^2-1} = M_1(P) \cap$

$M_2(Q)$ up to conjugation. Thus, inductively, the only classes $[K]$ with $[H] \leq [K]$ and $\lambda(K) \neq 0$ are $[K] \in \{[G], [M_1], [M_2], [E_q \rtimes C_{q^2-1}]\}$. This implies $\lambda(H) = 0$.

(iii) Let $H < M_2(Q)$ for some Q, and assume also $H \not\leq M_1(P)$ for any P. As $H \not\leq C_3$, we have $[H] \not\leq [M_4]$. The group $\bar{H} := H/(H \cap Z(M_2(Q)))$ acts as a subgroup of $\mathrm{PSL}(2,q)$ on $\ell_Q \cap \mathcal{H}_q$; we assume in this point that H is one of the following groups (see [17, Hauptsatz 8.27]): $\mathrm{PSL}(2, 2^{2^h})$ with $0 < h \leq n$; a dihedral group of order $2d$ where d is a divisor of $q-1$ greater than 3; $\mathrm{Alt}(5)$. Then, by Lagrange's theorem, $[H] \not\leq [M_3]$. Thus, inductively, G and $M_2(Q)$ are the only groups K with $H < K$ and $\lambda(K) \neq 0$, so that $\lambda(H) = 0$.

Note that, since we are under the assumptions $H \not\leq M_1(P)$ for any P, $H \not\leq \mathrm{Sym}(3)$, and $H \not\leq C_{q+1} = Z(M_2(Q))$, we have that the only subgroups $\bar{H}$ of $\mathrm{PSL}(2,q)$ for which $\lambda(H)$ still has not been computed are the cyclic or dihedral groups of order d or $2d$ (respectively), where d is a nontrivial divisor of $q+1$.

(iv) Let $H < M_3(T)$ for some T, and assume also $H \not\leq M_1(P)$ for any P. As $H \not\leq C_3$, we have $[H] \not\leq [M_4]$. Here, the assumption $H \not\leq \mathrm{Sym}(3)$ means that some nontrivial element of H fixes T pointwise. Hence, the assumption $H \not\leq C_{q+1} = Z(M_2(Q))$ for any vertex Q of T, together with $H \not\leq M_1(P)$, implies that H contains some element of type (B1). Write $H = L \rtimes K$, with $K \leq \mathrm{Sym}(3)$ and $L < C_{q+1} \times C_{q+1}$.

If $K = C_3$ or $K = \mathrm{Sym}(3)$, then $[H] \not\leq [M_2]$; thus, inductively, G and $M_3(T)$ are the only groups K with $H < K$ and $\lambda(K) \neq 0$, so that $\lambda(H) = 0$.

If $K = C_2$ and $L = C_{q+1} \times C_{q+1}$, then $H \leq M_2(Q)$ for some vertex Q of T. Since $\bar{H} := H/(H \cap Z(M_2(Q)))$ is dihedral of order $2(q+1)$, [17, Haptsatz 8.27] implies the non-existence of groups K with $H < K < M_2(Q)$ (except for $q = 4$ and $\bar{K} = \mathrm{Alt}(5)$; in this case, $\lambda(K) = 0$ by the previous point). Thus, $\lambda(H) = -\{\lambda(G) + \lambda(M_2(Q)) + \lambda(M_3(T))\} = 1$.

If $K = C_2$ and $L < C_{q+1} \times C_{q+1}$, then again $H \leq M_2(Q)$ with Q vertex of T. The group $\bar{H}$ is dihedral of order $2d$, where $d \mid (q+1)$; $d > 1$ because L contains elements of type (B1). By the previous point and [17, Hauptsatz 8.27], the only groups K with $H < K < M_2(Q)$ are such that $\bar{K}$ is dihedral of order dividing $q+1$. Thus, inductively, $\lambda(H) = 0$.

If $K = \{1\}$, then $H \in M_2(Q)$ for any vertex Q of T. The group $\bar{H} < \mathrm{PSL}(2,q)$ on the line $\ell_Q \cap \mathcal{H}_q$ is cyclic of order $d \mid (q+1)$; $d > 1$ because H has elements of type (B1). By [17, Hauptsatz 8.27], the groups K with $H < K < M_2(Q)$ are such that either $\bar{K}$ is cyclic of order dividing $q+1$, or we have already proved that $\lambda(K) = 0$. Thus, inductively, $\lambda(K) = 0$.

(v) Let $H < M_2(Q)$ for some Q. Let $\bar{H} \neq \{1\}$ be the induced subgroup of $\mathrm{PSL}(2,q)$ acting on $\ell_Q \cap \mathcal{H}_q$. If $\bar{H}$ is cyclic or dihedral of order d or $2d$ (respectively) with $d \mid (q+1)$, then $H \leq M_3(T)$ for some T. Hence, $\lambda(H) = 0$, as already computed in the previous point in the case $K = \{1\}$ if $\bar{H}$ is cyclic, or in the case $K = C_2$ if H is dihedral.

(vi) Under the assumptions that $H \not\leq \mathrm{Sym}(3)$ and H is not a group of homologies, the only remaining case is $H < M_1(P)$ for some P with $\gcd(|H|, q-1) = 1$. In this case $H = E_{2^r} \times C_d$, where C_d is cyclic of order $d \mid (q+1)$ and made by homologies, whose axis passes through P and whose center Q lies on ℓ_P. We have $r > 0$, because $H \not\leq Z(M_2(Q))$.

If $r = 1$, then H is cyclic of order $2d$ generated by an element of type (E). By Lemma 3.4, $H \leq M_3(T)$, where T has a vertex in Q and two vertexes on ℓ_Q. Hence, $[H] \leq [M_1]$, $[H] \leq [M_2]$, $[H] \leq [M_3]$, and $[H] \not\leq [M_4]$. Let K be such that $H < K \leq G$ and K is not of the same type of H, i.e. K is not cyclic of order $2d'$ with $d' \mid (q+1)$. As shown in the previous points, $\lambda(K) \neq 0$ if and only if $[K] \in \{[G], [M_1], [M_2], [M_3], [E_q \rtimes C_{q^2-1}], [(C_{q+1} \times C_{q+1}) \rtimes C_2]\}$. Thus, inductively, $\lambda(H) = 0$.

Case 2: Let $H \leq C_{q+1} = Z(M_2(Q))$ for some Q and K be a subgroup of G properly containing H. As shown above, $\lambda(K) \neq 0$ if and only if

$$[K] \in \{[G], [M_1], [M_2], [M_3], [E_q \rtimes C_{q^2-1}], [(C_{q+1} \times C_{q+1}) \rtimes C_2]\}.$$

Thus $\lambda(Z(M_2(Q))) = 0$ and, inductively, $\lambda(H) = 0$.

Case 3: Let $H = \mathrm{Sym}(3) = \langle \alpha \rangle \rtimes \langle \beta \rangle$ with $o(\alpha) = 3$ and $o(\beta) = 2$. Let $P \in \mathrm{PG}(2, q^2) \setminus \mathcal{H}_q$ and $Q, R \in \mathcal{H}_q(\mathbb{F}_{q^2})$ be the fixed point of α, so that β fixes P and interchanges Q and R. This implies $[H] \leq [M_2]$. By Lemma 3.6, $[H] \leq [M_3]$. From the computations above and Lagrange's theorem, no class $[K]$ with $K \leq G$ other than $[G]$, $[M_2]$ and $[M_3]$ satisfies $[H] \leq [K]$ and $\lambda(H) \neq 0$. Thus, $\lambda(H) = 1$.

Case 4: Let $H = C_3$. By Lagrange's theorem and Proposition 3.2, $H < K \leq G$ and $\lambda(K) \neq 0$ if and only if

$$[K] \in \{[G], [M_1], [M_2], [M_3], [M_4], [E_q \rtimes C_{q^2-1}], [\mathrm{Sym}(3)]\}.$$

Thus, $\lambda(H) = 1$.

Case 5: Let $H = C_2$. By Lagrange's theorem and Proposition 3.2, $H < K \leq G$ and $\lambda(K) \neq 0$ if and only if

$$[K] \in \{[G], [M_1], [M_2], [M_3], [E_q \rtimes C_{q^2-1}], [(C_{q+1} \times C_{q+1}) \rtimes C_2], [\mathrm{Sym}(3)]\}.$$

Thus, $\lambda(H) = -1$.

Case 6: Let $H = \{1\}$. Collecting all the classes $[K]$ with $\lambda(K) \neq 0$, we have by direct computation $\lambda(H) = 0$. □

5 Determination of $\chi(\Delta(L_p \setminus \{1\}))$ for any prime p

Let $n > 0$, $q = 2^{2^n}$, $G = \mathrm{PSU}(3, q)$. If p is a prime number, we denote by L_p the poset of p-subgroups of G ordered by inclusion, by $L_p \setminus \{1\}$ its subposet of proper p-subgroups of G, and by $\Delta(L_p \setminus \{1\})$ the order complex of $L_p \setminus \{1\}$. In this section we determine the Euler characteristic $\chi(\Delta(L_p \setminus \{1\}))$ of $\Delta(L_p \setminus \{1\})$ for any prime p, using Equation (2.1) and Lemma 2.2. The results are stated in Theorem 5.1 and in Table 2.

Theorem 5.1. *For any prime number p one of the following cases holds:*

(i) $p \nmid |G|$ *and* $\chi(\Delta(L_p \setminus \{1\})) = 0$*;*

(ii) $p = 2$ *and* $\chi(\Delta(L_2 \setminus \{1\})) = q^3 + 1$*;*

(iii) $p \mid (q+1)$ *and* $\chi(\Delta(L_p \setminus \{1\})) = -\frac{q^6 - 2q^5 - q^4 + 2q^3 - 3q^2}{3}$*;*

(iv) $p \mid (q-1)$ *and* $\chi(\Delta(L_p \setminus \{1\})) = -\frac{q^6+q^3}{2}$*;*

(v) $p \mid (q^2-q+1)$ *and* $\chi(\Delta(L_p \setminus \{1\})) = -\frac{q^6+q^5-q^4-q^3}{3}$.

Proof. Since $|G| = q^3(q+1)^2(q-1)(q^2-q+1)$, q is even, and $3 \mid (q-1)$, the cases $p \nmid |G|$, $p=2$, $p \mid (q+1)$, $p \mid (q-1)$, and $p \mid (q^2-q+1)$ are exhaustive and pairwise incompatible. We denote by S_p a Sylow p-subgroup of G.

Case 1: Let $p \nmid |G|$. Then $\Delta(L_p \setminus \{1\}) = \emptyset$, and hence $\chi(\Delta(L_p \setminus \{1\})) = \chi(\emptyset) = 0$.

Case 2: Let $p = 2$. The group G has q^3+1 Sylow 2-subgroups, and any two of them intersect trivially; see [15, Theorem 11.133]. Any nontrivial element σ of S_2 fixes exactly one point P on $\mathcal{H}_q(\mathbb{F}_{q^2})$ which is the same for any $\sigma \in S_2$; S_2 is uniquely determined among the Sylow 2-subgroups of G by P. Hence, Equation (2.1) reads

$$\chi(\Delta(L_2 \setminus \{1\})) = -(q^3+1) \sum_{H \in L_2 \setminus \{1\},\ H(P)=P} \mu_{L_2}(\{1\}, H),$$

where P is a given point of $\mathcal{H}_q(\mathbb{F}_{q^2})$. By Lemma 2.2, we only consider those 2-groups in $M_1(P)$ which are elementary abelian. Then we consider all nontrivial subgroups H of an elementary abelian 2-group E_q of order q. For any such group $H = E_{2^r}$ of order 2^r, with $1 \le r \le 2^n$, we have $\mu_{L_2}(\{1\}, H) = (-1)^r \cdot 2^{\binom{r}{2}}$ by Lemma 2.2. Thus,

$$\chi(\Delta(L_2 \setminus \{1\})) = -(q^3+1) \sum_{r=1}^{2^n} (-1)^r\, 2^{\binom{r}{2}} \binom{2^n}{r}_2$$

where the Gaussian coefficient $\binom{2^n}{r}_2$ counts the subgroups of E_q of order 2^r. Using the property

$$\binom{2^n}{r}_2 = \binom{2^n-1}{r-1}_2 + 2^r \binom{2^n-1}{r}_2$$

we obtain

$$\begin{aligned}
&\sum_{r=1}^{2^n} (-1)^r\, 2^{\binom{r}{2}} \binom{2^n}{r}_2 \\
&\qquad = \sum_{r=1}^{2^n} (-1)^r\, 2^{\binom{r}{2}} \binom{2^n-1}{r-1}_2 + \sum_{r=1}^{2^n} (-1)^r\, 2^{\binom{r}{2}+r} \binom{2^n-1}{r}_2 \\
&\qquad = \sum_{r=0}^{2^n-1} (-1)^{r+1}\, 2^{\binom{r+1}{2}} \binom{2^n-1}{r}_2 + \sum_{r=1}^{2^n} (-1)^r\, 2^{\binom{r+1}{2}} \binom{2^n-1}{r}_2 \\
&\qquad = (-1)^0\, 2^{\binom{1}{2}} \binom{2^n-1}{0}_2 + (-1)^{2^n}\, 2^{\binom{2^n+1}{2}} \binom{2^n-1}{2^n}_2 = -1.
\end{aligned}$$

Thus, $\chi(\Delta(L_2 \setminus \{1\})) = q^3+1$.

Case 3: Let $p \mid (q+1)$. Then $S_p \le C_{q+1} \times C_{q+1}$, and hence $S_p \cong C_{p^s} \times C_{p^s}$, where $p^s \mid (q+1)$ and $p^{s+1} \nmid (q+1)$. Let H be a subgroup of S_p. By Lemma 2.2, $\mu_{L_p}(\{1\}, H) \neq 0$ only if H is elementary abelian of order p or p^2; in this cases, $\mu_{L_p}(\{1\}, C_p) = -1$ and $\mu_{L_p}(\{1\}, C_p \times C_p) = r$. Now we count the number of elementary abelian subgroups of order p or p^2 in G.

(i) A subgroup E_{p^2} of G of type $C_p \times C_p$ is uniquely determined by the maximal subgroup $M_3(T)$ such that E_{p^2} is the Sylow p-subgroup of $M_3(T)$. Hence, G contains exactly $[G : N_G(M_3(T))] = \frac{q^3(q^2-q+1)(q-1)}{6}$ elementary abelian subgroups of order p^2.

(ii) A subgroup C_p made by homologies is uniquely determined by its center $P \in \mathrm{PG}(2, q^2) \setminus \mathcal{H}_q$ of homology, because the group of homologies with center P is cyclic. Hence, G contains exactly $|\,\mathrm{PG}(2, q^2) \setminus \mathcal{H}_q| = q^2(q^2 - q + 1)$ cyclic subgroups of order p made by homologies.

(iii) A subgroup C_p which is not made by homologies is made by elements of type (B1), and fixes pointwise a unique self-polar triangle T. The Sylow p-subgroup $C_p \times C_p$ of $M_3(T)$ contains exactly 3 subgroups C_p made by homologies, namely the groups of homologies with center one of the vertexes of T. Since $C_p \times C_p$ contains $p + 1$ subgroups C_p altogether, $C_p \times C_p$ contains exactly $p - 2$ subgroups C_p not made by homologies. Thus, the number of subgroups C_p of G not made by homologies is $(p - 2) \cdot [G : N_G(M_3(T))] = \frac{q^3(q^2-q+1)(q-1)(p-2)}{6}$.

Thus, by direct computation,

$$\begin{aligned}\chi(\Delta(L_p \setminus \{1\}))\\ &= -\left\{\frac{q^3(q^2 - q + 1)(q - 1)(p - 2)}{6} \cdot r \right.\\ &\qquad \left. + \left[q^2(q^2 - q + 1) + \frac{q^3(q^2 - q + 1)(q - 1)(p - 2)}{6}\right] \cdot (-1)\right\}\\ &= -\frac{q^6 - 2q^5 - q^4 + 2q^3 - 3q^2}{3}.\end{aligned}$$

Case 4: Let $p \mid (q - 1)$. By Lemma 2.4, S_p is a subgroup of the cyclic group C_{q^2-1} fixing two points P, Q on $\mathcal{H}_q(\mathbb{F}_{q^2})$; then a proper p-subgroup H of G satisfies $\mu_{L_p}(\{1\}) \neq 0$ if and only if H has order p; in this case, $\mu_{L_p}(\{1\}, H) = -1$. Also, by Lemma 2.4, any two Sylow p-subgroups of G have trivial intersection. Then the number of subgroups C_p of G is equal to the number $\binom{q^3+1}{1}$ of couples of points in $\mathcal{H}_q(\mathbb{F}_{q^2})$; equivalently, this number is equal to $[G : N_G(C_{q^2})]$, where $|N_G(C_{q^2-1})| = 2(q^2 - 1)$ by Proposition 3.3. Thus, $\chi(\Delta(L_p \setminus \{1\})) = -\frac{q^6+q^3}{2}$.

Case 5: Let $p \mid (q^2 - q + 1)$. Then $S_p \le C_{q^2-q+1}$, and hence a proper p-subgroup H of G satisfies $\mu_{L_p}(\{1\}, H) \neq 0$ if and only if H has order p; in this case, $\mu_{L_p}(\{1\}, H) = -1$. The number of subgroups C_p of G is equal to the number of subgroups C_{q^2-q+1}, and hence to the number $[G : N_G(M_4(\tilde{T}))] = \frac{q^3(q+1)^2(q-1)}{3}$ of maximal subgroups of type $M_4(\tilde{T})$ in G. Thus, $\chi(\Delta(L_p \setminus \{1\})) = -\frac{q^3(q+1)^2(q-1)}{3} = -\frac{q^6+q^5-q^4-q^3}{3}$. □

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