© Strojniški vestnik 48(2002)11,591-600
ISSN 0039-2480
UDK 658.512.2:004.41
Izvirni znanstveni članek (1.01)

© Journal of Mechanical Engineering 48(2002)11,591-600
ISSN 0039-2480
UDC 658.512.2:004.41
Original scientific paper (1.01)

Optimiranje oblike konstrukcij: tristranični projektni element

Structural Shape Optimization: A Trilateral Design Element

Marko Kegl

Prispevek obravnava izpeljavo tristraničnega projektnega elementa za uporabo pri optimalnem projektiranju oblike konstrukcij. Osnova za izpeljavo novega elementa je tristranična Bézierjeva ploskev, ki je običajno parametrizirana z uporabo težiščnih koordinat. V prispevku je uporabljena drugačna parametrizacija, ki je bolj prilagojena postopkom optimizacije oblike. Na podlagi te ploskve je definiran projektni element - Bézierjevo telo, katerega mreža nadzornih točk ima v topološkem pomenu obliko tristranične prizme. Uporaba izpeljanega elementa je ponazorjena na dveh številčnih zgledih.

(Ključne besede: projektiranje konstrukcij, optimiranje oblik, elementi projektni)

This paper considers the derivation of a trilateral design element for the shape optimization of structures. The element derivation is based on a trilateral Bézier patch, usually being parametrized by barycentric coordinates. Here, another type of parametrization is used, which is more convenient for employment in optimization procedures. Based on this patch the design element is derived - a Bézier body whose control points represent a trilateral prism in the topological sense. The use of the derived element is illustrated by two numerical examples.

(Keywords: structural design, shape optimization, element design)

0 UVOD

Področje optimizacije oblike konstrukcij se je začelo intenzivneje razvijati približno pred dvema desetletjema. Kmalu se je izkazalo, da pri optimizaciji oblike naletimo na nove težave, ki jih pri običajni optimizaciji nismo poznali [1]. Te so bile več ali manj vezane na dejstvo, da se mreža končnih elementov pri optimizaciji oblike spreminja. To pripelje do kvarjenja mreže ter do nenatančnih rezultatov pri analizi odziva in občutljivosti.

Stopnja kvarjenja mreže je zelo odvisna od načina njene parametrizacije. Imam [2] je tako v svojem delu predstavil različne zamisli parametrizacije in enega od njih poimenoval tehnika projektnih elementov. Projektni element je v bistvu primerno parametriziran geometrijski objekt, ki končne elemente oskrbuje s potrebnimi geometrijskimi podatki (koordinate vozlišč itn.). Imam je za projektni element predlagal 20-vozliščni izoparametrični element, malo kasneje pa sta Braibant in Fleury [3] v svojo formulacijo 2D projektnega elementa vpeljala krivulje zlepkov B. V naslednjih letih smo na področju optimizacije oblike lahko opazili vse pogostejšo uporabo osnovnih zasnov

0 INTRODUCTION

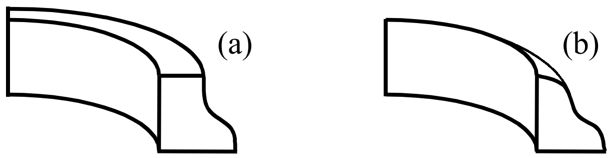
The field of structural shape optimization began to develop more rapidly about two decades ago. It soon turned out that shape optimization is accompanied by several new difficulties not known in conventional optimization [1]. These difficulties were more or less caused by the fact that during the process of shape optimization the finite-element mesh changes. This in turn leads to deterioration of the mesh and to inaccuracies in the calculation of the structural response and the sensitivity information.

The extent of the mesh deterioration depends significantly on its parametrization. Imam [2] presented in his work several parametrization concepts, one of them he termed the design-element technique. A design element is essentially a parametrized geometrical object acting as the geometrical data (nodal coordinates, etc.) provider for all the finite elements it contains. Imam proposed to employ a 20-noded isoparametric element as the design-element. Somewhat later Braibant and Fleury [3] introduced B-spline curves into the formulation of their 2D design element. In the following years, the fundamental concepts of the design element technique as well as Bézier and B-spline curves and tehnike projektnih elementov ter krivulj oziroma ploskev zlepkov B ([4] do [8]).

Pred kratkim je bil v [9] predstavljen osnutek parametrizacije oblike z uporabo tehnike projektnih elementov ter uporabe splošnega projektnega elementa. Predlagan osnutek obravnava zvezne in diskretne konstrukcije na enoten način, projektni element je definiran kot Bézierjevo telo. Topološko ima ta element obliko štiristranične prizme, zato ga bomo imenovali kar *štiristranični projektni element*. Mejna ploskev tega elementa je sestavljena iz šestih štiristraničnih Bézierjevih ploskev. Če je treba, lahko katerokoli od teh ploskev degeneriramo v tristranično ploskev s primernim pozicioniranjem nadzornih točk. To je sicer mogoče, vendar pa ni najbolj smotrno, saj pri tem uporabljamo več nadzornih točk, kakor pa jih dejansko potrebujemo. V takem primeru je zato uporaba prave tristranične ploskve mnogo bolj primerna. Glede na to je verjetno primerno definirati (razen štiristraničnega) tudi *tristranični projektni element* (sl. 1). Topološko je ta element tristranična prizma, njegova mejna ploskev pa je sestavljena iz dveh tristraničnih in treh štiristraničnih Bézierjevih ploskev.

surfaces were progressively employed in shape optimization ([4] to [8]).

Recently, a concept of shape parametrization was presented [9] using the design-element technique and a general-purpose design element. With the proposed concept, continuum and discrete structures are parametrized in a unified way. The employed design element is a rational Bézier body. Topologically, this convenient element represents a quadrilateral prism, so it will be termed here as the *quadrilateral design element*. The boundary surface of this element consists of six quadrilateral Bézier patches. If necessary, any of these patches can be degenerated to a trilateral patch by adequate positioning of the control points. This is possible, but not very efficient, since one needs to define and employ more control points than are actually needed. In such a case the use of a genuine trilateral patch represents a much better choice. Therefore, it seems to be advantageous to define (except for the quadrilateral) also a *trilateral design element* (Figure 1). Topologically, the trilateral element represents a trilateral prism. Its boundary surface consists of two trilateral and three quadrilateral Bézier patches.



Sl. 1. *Štiristranični (a) in tristranični (b) prizmatični projektni element*
Fig. 1. *A quadrilateral (a) and a trilateral (b) prismatic design element*

1 TEHNIKA PROJEKTNIH ELEMENTOV

Obravnavajmo konstrukcijo, ki v geometrijskem smislu pomeni telo B v stvarnem 3D prostoru. Pri tehniki projektnih elementov si celo telo B zamislimo kot sestavljeno iz preprostejših delov D_i, ki imajo znane geometrijske lastnosti in jih je preprosto parametrizirati (slika 2). Te sestavne dele imenujemo *projektne elemente*.

Naj simbol D označuje poljuben projektni element. Vzemimo tudi, da D v geometrijskem smislu pomeni Bézierjevo telo. Obliko in lego Bézierjevega telesa določajo lege njegovih nadzornih točk, nanj pa lahko gledamo tudi kot na preslikavo f enotske kocke $U = [0,1]^3$ v stvarni 3D prostor [9]. Preslikava f preslika torej vsako točko $\mathbf{s} \in U$ v točko stvarnega 3D prostora s krajevnim vektorjem $\mathbf{r} \in D$ (slika 3). Nadzorne točke Bézierjevega telesa imajo v tej preslikavi vlogo parametrov, ki vplivajo na izračun $\mathbf{r}$ oziroma na lego in obliko projektnega elementa D. Torej lahko zapišemo $\mathbf{r}=f(\mathbf{s},\mathbf{q}_{ijk})$, kjer smo s $\mathbf{q}_{ijk}$ simbolično označili krajevne vektorje nadzornih točk telesa D.

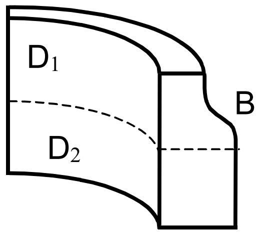
Spreminjanje oblike telesa D dosežemo preprosto tako, da spreminjamo lege njegovih

1 THE DESIGN ELEMENT TECHNIQUE

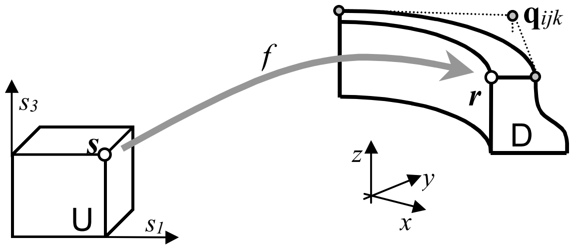
Let us consider a structure, in geometrical sense, being represented by the body B in real 3D space. By the design element technique the whole body B can be thought of as being assembled of simpler parts D_i, which exhibit known geometrical properties and can be parametrized in a simple way. These individual parts are termed the *design elements*.

Let the symbol D denote a generic design element. Let us also assume that in a geometrical sense D represents a Bézier body. The shape and the position of a Bézier body are determined by the positions of its control points. It can also be thought of as a mapping f from the unit cube $U = [0,1]^3$ into the real 3D space [9]. The mapping f maps every point $\mathbf{s} \in U$ into a point with a position vector $\mathbf{r} \in D$ in the real 3D space (Figure 3). The Bézier body control points act in this mapping as parameters that affect the calculation of $\mathbf{r}$, i.e. the position and shape of the design element D. Thus, we can write $\mathbf{r}=f(\mathbf{s},\mathbf{q}_{ijk})$, where the symbol $\mathbf{q}_{ijk}$ was used to denote symbolically the control point position vectors of the body D.

The change of the shape of D can simply be achieved by changing the positions of its control



Sl. 2. *Telo B predstavljata projektna elementa D_1 in D_2*
Fig. 2. *The body B is represented by the design elements D_1 and D_2*



Sl. 3. *Preslikava iz U v D pod vplivom parametrov* $\mathbf{q}_{ijk}$
Fig. 3. *A mapping from U into D influenced by the parameters* $\mathbf{q}_{ijk}$

nadzornih točk $\mathbf{q}_{ijk}$. Preostane le še, da na primeren način vključimo uporabo ustrezne diskretizacijske metode, na primer metode končnih elementov. Za uporabo te metode potrebujemo mrežo končnih elementov, ki mora pokrivati celotno območje D. Vendar to ni vse, saj moramo imeti v mislih tudi dejstvo, da bomo obliko telesa D spreminjali – oblika mreže končnih elementov mora tem spremembam ustrezno slediti. To najenostavneje dosežemo tako, da mrežo oziroma vozlišča posameznih končnih elementov definiramo v enotski kocki U. Krajevne vektorje vozlišč mreže v stvarnem prostoru pa izračunamo z uporabo preslikave *f* (sl. 4). Tako bo mreža končnih elementov avtomatično in povsem natančno sledila spremembam oblike telesa D.

Glede na zgoraj opisano lahko rečemo, da je projektni element podan s preslikavo *f*. Z izrazom *izpeljava projektnega elementa* mislimo torej na izpeljavo splošnega izraza za *f* ter na izpeljavo vseh potrebnih izrazov, ki so odvisni od *f* in jih potrebujemo pri izračunavanju vhodnih geometrijskih podatkov za mrežo končnih elementov. V dosedanjem delu [9] smo izpeljali in na raznih primerih uporabili štiristranični projektni element, ta prispevek pa prikazuje izpeljavo tristraničnega elementa.

2 TRISTRANIČNI PROJEKTNI ELEMENT

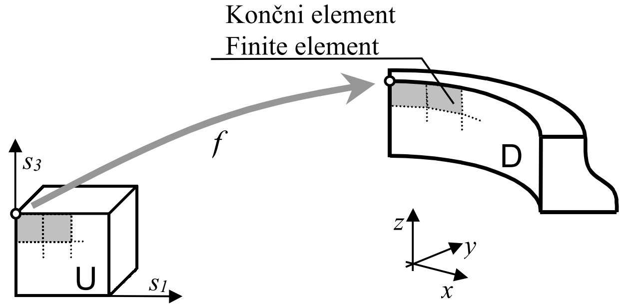
Tristranični element je zasnovan na tristranični Bézierjevi ploskvi. Za razliko od štiristranične variante

points $\mathbf{q}_{ijk}$. What still needs to be done is to introduce, in an appropriate way, the use of an adequate discretization method, for example, the finite-element method. In order to employ this method, we need some finite-element mesh, spanning the entire region of D. In doing so, we have to keep in mind that the shape of D will change – the shape of the finite-element mesh must follow those changes accordingly. The easiest way to achieve this is to define the mesh (i.e. the nodes) in the unit cube U. The nodal position vectors in the real space are then calculated by employing the mapping *f* (Figure 4). In this way the finite-element mesh will automatically and accurately follow the shape changes of the body D.

According to the discussion above one can say that the design element is defined by the mapping *f*. Thus, by the term *design element derivation* we actually mean the derivation of some generic expression for *f* as well as the derivation of all the necessary expressions that depend on *f*, and need to be employed to compute the geometrical data of the finite-element mesh. In our previous work [9] we derived and successfully employed the quadrilateral design element on several examples. This paper presents the derivation of the trilateral design element.

2 THE TRILATERAL DESIGN ELEMENT

The trilateral design element is based on a trilateral Bézier patch. In contrast to the quadrilateral



Sl. 4. *Preslikava vozlišč v stvarni prostor*
Fig. 4. *The mapping of the nodes into the real space*

je tristranična Bézierjeva ploskev parametrizirana z uporabo težiščnih koordinat $\mathbf{u} = [u,v,w]^T$. Te koordinate med sabo niso neodvisne, saj morajo zadoščati pogoju $u+v+w=1$. Krajevni vektor $\mathbf{p}$ poljubne točke na tristranični ploskvi reda n je podan z [10]:

version, a trilateral Bézier patch is conventionally parametrized by employing barycentric coordinates $\mathbf{u} = [u,v,w]^T$. These coordinates are not independent, since they must fulfill the requirement that $u+v+w=1$. The position vector $\mathbf{p}$ of a generic point on a trilateral patch of the order n is given by [10]

$$\mathbf{p} = \sum_{|\mathbf{i}|=n} B_{ijk}^{n} \mathbf{q}_{ijk} \qquad (1),$$

kjer sta $|\mathbf{i}|=i+j+k$ in $i,j,k\geq 0$. Simbol $\mathbf{q}_{ijk}$ označuje nadzorno točko, $B_{ijk}^n = \hat{B}_{ijk}^n(\mathbf{u})$ pa je bivariantni Bernsteinov polinom reda n, ki je definiran kot:

where $|\mathbf{i}|=i+j+k$ and $i,j,k\geq 0$. The symbol $\mathbf{q}_{ijk}$ denotes a control point and $B_{ijk}^n = \hat{B}_{ijk}^n(\mathbf{u})$ is the bivariate Bernstein polynomial of the order n, defined as:

$$B_{ijk}^{n} = \frac{n!}{i!\,j!\,k!} u^i v^j w^k, \; |\mathbf{i}| = n \qquad (2).$$

Vendar taka običajna parametrizacija ni najprimernejša za izpeljavo projektnega elementa. Pri tem je namreč najugodneje, da je parametriziran z neodvisnimi parametri ter da ima primerne karakteristične parametrične smeri v vsaki točki. Da bi zadostili temu pogoju, moramo težiščne koordinate nadomestiti z dvema primernima neodvisnima parametroma.

V ta namen najprej preštevilčimo nadzorne točke in pripadajoče polinome: nadzorni poligon tristranične ploskve je definiran z mrežo $N = \frac{1}{2}(n+1)(n+2)$ nadzornih točk $\mathbf{q}_{ijk}$. Vpeljimo nov dogovor ter nadzorne točke označimo s $\mathbf{q}_i$, $i=1,...,N$, kjer je $\mathbf{q}_1 \equiv \mathbf{q}_{00n}$, medtem ko so naslednje nadzorne točke tiste, pri katerih indeksi z leve naraščajo najhitreje (sl. 5a). Enak dogovor sprejmimo tudi za polinome, tako da imamo B_i^N, $i=1,...,N$, kjer je $B_1^N \equiv B_{00n}^n$.

Nova neodvisna parametra s_1 in s_2 vpeljimo tako, da dobimo primerne parametrične smeri, in sicer:

This conventional arrangement, however, is not very convenient for the derivation of a design element. A design element is most conveniently defined as a geometrical body, parametrized by three independent parameters and exhibiting three characteristic parametric directions at each point. In order to meet this requirement, the barycentric coordinates of the trilateral patch have to be replaced by two suitable and independent parameters.

For this purpose we re-numerate the control points and their corresponding polynomials: the control polygon of a trilateral patch is defined by a triangular scheme of $N = \frac{1}{2}(n+1)(n+2)$ control points $\mathbf{q}_{ijk}$. We introduce a new arrangement, denoting the control points with $\mathbf{q}_i$, $i=1,...,N$ so that $\mathbf{q}_1 \equiv \mathbf{q}_{00n}$, while the subsequent control points are the ones with the leftmost index increasing most rapidly (Figure 5a). The same arrangement is adopted for the polynomials, now denoted by B_i^N, $i=1,...,N$ where $B_1^N \equiv B_{00n}^n$.

Let us introduce new independent parameters, s_1 and s_2, so that one gets convenient parametric lines as follows:

$$u = 1 - s_1, \; v = s_1 - s_1 s_2, \; w = s_1 s_2 \qquad (3),$$

kjer sta $s_1,s_2 \in [0,1]$. S temi zvezami postavimo odvisnost $\mathbf{u} = \hat{\mathbf{u}}(s_1,s_2)$, preprosto pa lahko tudi preverimo, da velja $u,v,w \in [0,1]$ ter $u+v+w=1$. S tem dogovorom lahko poljubno točko na ploskvi zapišemo kot:

where $s_1,s_2 \in [0,1]$. These relations establish the dependency $\mathbf{u} = \hat{\mathbf{u}}(s_1,s_2)$, and it can easily be verified that $u,v,w \in [0,1]$ and $u+v+w=1$. By adopting the above arrangement, the position vector of a generic point on the patch can be written as

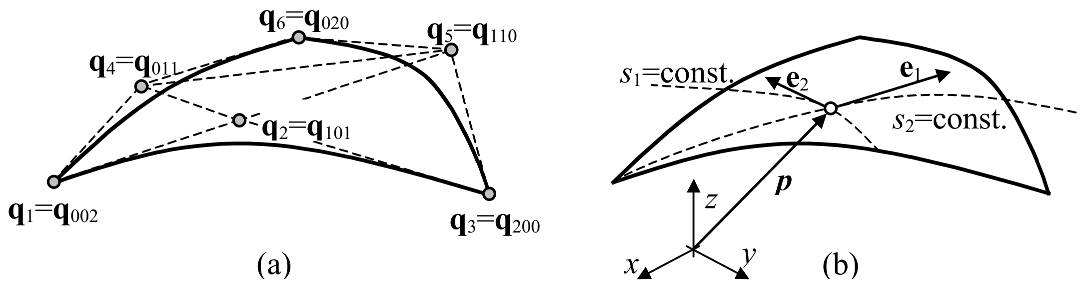
$$\mathbf{p} = \sum_{i=1}^{N} S_i^N \mathbf{q}_i \qquad (4),$$

kjer je $S_i^N = \hat{B}_i^N(\hat{\mathbf{u}}(s_1,s_2))$, tako da lahko zapišemo $S_i^N = \hat{S}_i^N(s_1,s_2)$ in posledično $\mathbf{p} = \hat{\mathbf{p}}(s_1,s_2)$. S to neodvisno parametrizacijo dobimo primerne parametrične smeri $\mathbf{e}_1 = \partial\mathbf{p}/\partial s_1$ in $\mathbf{e}_2 = \partial\mathbf{p}/\partial s_2$ (sl. 5b), ploskev pa s tem postane primerna tudi za izpeljavo projektnega elementa.

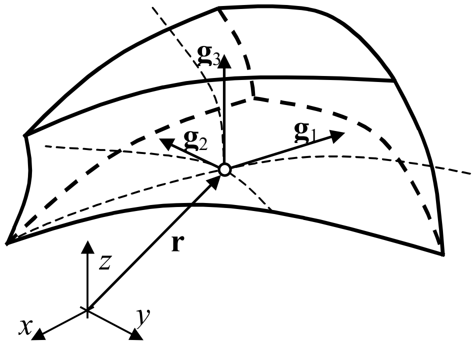
Za tristranično ploskev potrebujemo shemo N nadzornih točk. Če vzamemo M takšnih shem, dobimo tristranično prizmatično shemo $N \times M$ nadzornih točk $\mathbf{q}_{ij}$, $i=1,...,N$, $j=1,...,M$. Te nadzorne

where $S_i^N = \hat{B}_i^N(\hat{\mathbf{u}}(s_1,s_2))$ so that one can write $S_i^N = \hat{S}_i^N(s_1,s_2)$ and consequently $\mathbf{p} = \hat{\mathbf{p}}(s_1,s_2)$. This independent parametrization yields convenient parametric directions $\mathbf{e}_1 = \partial\mathbf{p}/\partial s_1$ and $\mathbf{e}_2 = \partial\mathbf{p}/\partial s_2$ (Figure 5b), which makes the patch convenient for the derivation of a design element.

For a trilateral patch one needs a triangular scheme of N control points. If one takes M such schemes, one gets a trilateral prismatic scheme of $N \times M$ control points $\mathbf{q}_{ij}$, $i=1,...,N$, $j=1,...,M$. These



Sl. 5. *Preštevilčenje nadzornih točk (a) in predlagana parametrizacija ploskve (b)*
Fig. 5. *Renumbering of control points (a) and the proposed parametrization of the patch (b)*



Sl. 6. *Tristranični projektni element z značilnimi smermi v točki* ***r***
Fig. 6. *The trilateral design element with characteristic directions at the point* ***r***

točke lahko definirajo tristranični projektni element (sl. 6), krajevni vektor njegove poljubne točke pa lahko zapišemo kot:

control points can be taken to define a trilateral prismatic design element (Figure 6) by defining the position vector of a generic point on the element by

$$\mathbf{r} = \sum_{i=1}^{N}\sum_{j=1}^{M} S_i^N U_j^M \mathbf{q}_{ij} \quad (5),$$

kjer je $U_j^M = \hat{U}_j^M(s_3)$ j-ti univariantni Bernsteinov polinom [10] reda *M*-1, izražen v odvisnosti od $s_3 \in [0,1]$. Parametrične smeri elementa so podane z $\mathbf{g}_i = \partial\mathbf{r}/\partial s_i$, *i*=1,2,3 (sl. 5).

Označimo sedaj s $\mathbf{s}^0 = \left[s_1^0, s_2^0, s_3^0\right]^T$ točko, ki določa lego poljubnega vozlišča nekega končnega elementa. Nadalje predpostavimo, da sta lahko $\mathbf{s}^0$ kakor tudi vse nadzorne točke odvisne od projektnih spremenljivk, zbranih v vektorju **b**. Torej velja $\mathbf{q}_{ij} = \mathbf{q}_{ij}(\mathbf{b})$ in $\mathbf{s}^0 = \mathbf{s}^0(\mathbf{b})$.

Za analizo konstrukcije potrebujemo v splošnem za vsako vozlišče naslednje geometrijske podatke: krajevni vektor $\mathbf{r}^0$ ter morda smerni vektor $\mathbf{n}^0$ (lupine, nosilci). Če predpostavimo, da lahko $\mathbf{n}^0$ izrazimo v odvisnosti od $\mathbf{g}_i^0$, potrebujemo torej za analizo odziva in občutljivosti izraze za naslednje količine: $\mathbf{r}^0$, $\mathbf{g}_i^0$, $d\mathbf{r}^0/d\mathbf{b}$ ter $d\mathbf{g}_i^0/d\mathbf{b}$.

Krajevni vektor izrazimo preprosto kot:

where $U_j^M = \hat{U}_j^M(s_3)$ is the *j*th univariate Bernstein polynomial [10] of the order *M*-1, expressed in terms of $s_3 \in [0,1]$. The parametric direction vectors of the element are given by $\mathbf{g}_i = \partial\mathbf{r}/\partial s_i$, *i*=1,2,3 (Figure 5).

Let us now denote by $\mathbf{s}^0 = \left[s_1^0, s_2^0, s_3^0\right]^T$ a point that defines the position of a generic node of some finite element. Further, it is assumed that both $\mathbf{s}^0$ as well as all the control points depend on design variables, assembled in the vector **b**. Thus, we have $\mathbf{q}_{ij} = \mathbf{q}_{ij}(\mathbf{b})$ and $\mathbf{s}^0 = \mathbf{s}^0(\mathbf{b})$.

For the structural response analysis we need, in general, for each node, the following geometrical data: its position vector $\mathbf{r}^0$ and maybe some direction vector $\mathbf{n}^0$ (shells, beams). Assuming that $\mathbf{n}^0$ can be expressed in terms of $\mathbf{g}_i^0$, we need for the response and sensitivity analysis expressions for the following quantities: $\mathbf{r}^0$, $\mathbf{g}_i^0$, $d\mathbf{r}^0/d\mathbf{b}$ and $d\mathbf{g}_i^0/d\mathbf{b}$.

The position vector is straightforwardly expressed as:

$$\mathbf{r}^0 = \left.\sum_{i=1}^{N}\sum_{j=1}^{M} E_{ij}^{NM} \mathbf{q}_{ij}\right|_{\mathbf{s}=\mathbf{s}^0} \quad (6),$$

kje je $E_{ij}^{NM} = S_i^N U_j^M$. Smerni vektorji so podani z:

where $E_{ij}^{NM} = S_i^N U_j^M$. The direction vectors are given by:

$$\mathbf{g}_k^0 = \sum_{i=1}^{N}\sum_{j=1}^{M} \frac{\partial E_{ij}^{NM}}{\partial s_k}\mathbf{q}_{ij}\Bigg|_{\mathbf{s}=\mathbf{s}^0} , \; k=1,2,3 \quad (7),$$

kjer je:

where:

$$\frac{\partial E_{ij}^{NM}}{\partial s_k} = \begin{cases} \left(\partial S_i^N / \partial s_k\right) U_j^M , \; k=1,2 \\ S_i^N \left(\partial U_j^M / \partial s_k\right), \; k=3 \end{cases} \quad (8).$$

Odvod krajevnega vektorja vozlišča po projektnih spremenljivkah je podan z:

The design derivatives of the position vector are given by:

$$\frac{d\mathbf{r}^0}{d\mathbf{b}} = \sum_{i=1}^{N}\sum_{j=1}^{M}\left[\sum_{k=1}^{3}\left(\frac{\partial E_{ij}^{NM}}{\partial s_k}\frac{ds_k^0}{d\mathbf{b}}\right)\mathbf{q}_{ij} + E_{ij}^{NM}\frac{d\mathbf{q}_{ij}}{d\mathbf{b}}\right]_{\mathbf{s}=\mathbf{s}^0} \quad (9),$$

medtem ko so odvodi smernih vektorjev:

whereas the direction vector derivatives are:

$$\frac{d\mathbf{g}_k^0}{d\mathbf{b}} = \sum_{i=1}^{N}\sum_{j=1}^{M}\left[\sum_{m=1}^{3}\left(\frac{\partial^2 E_{ij}^{NM}}{\partial s_k \partial s_m}\frac{ds_m^0}{d\mathbf{b}}\right)\mathbf{q}_{ij} + \frac{\partial E_{ij}^{NM}}{\partial s_k}\frac{d\mathbf{q}_{ij}}{d\mathbf{b}}\right]_{\mathbf{s}=\mathbf{s}^0} , \; k=1,2,3 \quad (10).$$

Izrazi (6) do (10) so vse kar potrebujemo za izračun potrebnih geometrijskih podatkov (ter njihovih odvodov po projektnih spremenljivkah) za poljuben končni element.

The expressions (6 to 10) are all we need in order to calculate the required geometric data (as well as the design derivatives) for any finite element.

3 ZGLEDA

3 EXAMPLES

Za ponazoritev teorije bomo obravnavali dva zgleda. Prvi vsebuje preprosto ravninsko konstrukcijo, njegov namen pa je ponazoriti način uporabe tristraničnega projektnega elementa. Drug zgled vsebuje preprosto prostorsko konstrukcijo, njegov namen pa je pokazati uporabo nedegeneriranega tristraničnega elementa. V obeh zgledih sta podatka o materialu enaka: elastični modul znaša E=210 GPa, Poissonov količnik je ν=0,3.

Oba zgleda sta zapisana v obliki nelinearnega optimizacijskega problema in rešena z uporabo optimizacijskega modula AMOPT ([11] in [12]).

In order to illustrate the theory, two numerical examples will be considered. The first example involves a simple planar structure and its purpose is to illustrate how to employ the trilateral design element. The second example involves a simple space structure, and it illustrates the use of a nondegenerated trilateral element. In both examples the material data is the same: the Young's modulus is E=210 GPa and the Poisson ratio is ν=0.3.

Both examples are formulated in the form of a non-linear constrained optimization problem and solved by the optimization routine AMOPT ([11] and [12]).

Zgled 1. Optimizacija tristranične podporne plošče

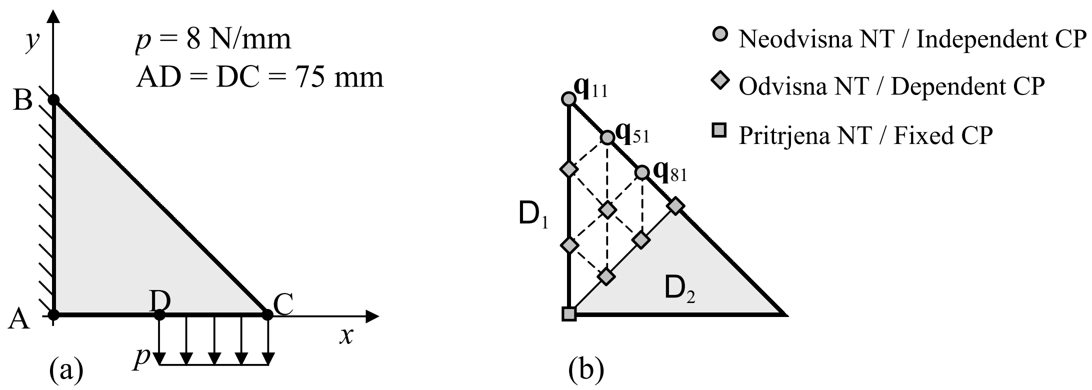
Obravnavajmo tristranično podporno ploščo, prikazano na sliki 7. Geometrijski in materialni podatki ter podatki o podprtju in obremenitvi so podani na sliki 7a. Naš namen je določiti obliko konstrukcije tako, da bo njena teža najmanjša in bodo hkrati izpolnjeni vsi omejitveni pogoji. Natančneje, želimo spreminjati višino plošče (navpično lego točke B) in obliko obrisa BC tako, da bo teža najmanjša. Postavljeni omejitveni pogoji se nanašajo na navpični pomik točke C in na napetosti vzdolž robov BC in AC.

Da bi lahko definirali optimizacijsko nalogo, konstrukcijo razdelimo na dva projektna elementa (degenerirana v ravninska lika). Zaradi zagotavljanja dovolj velike prilagodljivosti, vzamemo dva elementa, vsak z 10×1=10 nadzornimi točkami (NT). Na sliki 7b so prikazane NT prvega projektnega elementa, medtem ko so NT drugega razporejene na podoben način. Vpeljimo sedaj projektne spremenljivke **b**, s katerimi definiramo neodvisno gibljive nadzorne točke prvega elementa: $\mathbf{q}_{11}=[0,b_1]^T$, $\mathbf{q}_{51}=[b_2,b_3]^T$ in $\mathbf{q}_{81}=[b_4,b_5]^T$. Če naredimo podobno še pri drugem projektnem

Example 1. Optimization of a trilateral supporting plate

Let us consider the trilateral supporting plate shown in Figure 7. The geometrical and material data as well as the support and loading conditions are given in Figure 7a. Our objective is to determine the shape of the structure so that its weight will be minimal and, at the same time, all of the imposed constraints will be fulfilled. More precisely, we can vary the height of the plate (vertical position of point B) and the shape of the contour BC so that the weight will be minimised. The imposed constraints are related to the vertical displacement of the point C and to the stresses along the edges BC and AC.

In order to define the design problem, we start by partitioning the structure into trilateral design elements (degenerated to a flat surface). For flexibility reasons we take two elements with 10×1=10 control points (CP) each. Figure 7b illustrates the CP of the first design element, while the second one is defined in a similar way. Then we introduce design variables **b** by defining the independently movable CP of the first element as: $\mathbf{q}_{11}=[0,b_1]^T$, $\mathbf{q}_{51}=[b_2,b_3]^T$ and $\mathbf{q}_{81}=$



Sl. 7. *Tristranična podporna plošča (a) in njena predstavitev z 2 projektnima elementoma (b)*
Fig. 7. *The trilateral supporting plate (a) and its representation by 2 design elements (b)*

elementu, dobimo skupaj 9 projektnih spremenljivk. Omejitvene pogoje postavimo tako: navpični pomik točke C mora biti manjši od 0,2 mm, medtem ko morajo biti Misesove napetosti vzdolž robov BC in AC manjše od 100 MPa. Konstrukcijo modeliramo z 8 vozliščnimi ravninskimi končnimi elementi tipa Serendipity.

Obravnavali bomo dva primera: primer I, kjer vrednosti projektnih spremenljivk niso omejene in primer II, pri katerem (na primer zaradi pritrditve) postavimo naslednje dodatne omejitve: $b_1,b_3 \geq 80$ mm in $b_2,b_4 \geq 20$ mm.

Z uporabo lastnega optimizacijskega algoritma lahko oba primera rešimo v manj ko 10 iteracijah (6 iteracij za primer I in 8 za primer II). Pri obeh končnih oblikah (sl. 8) imamo hkrati delujoče tako napetostne pogoje kakor tudi pogoj za pomik. Glede na začetno konstrukcijo, se je prostornina optimalne konstrukcije zmanjšala za 48% v primeru I oziroma za 46% v primeru II.

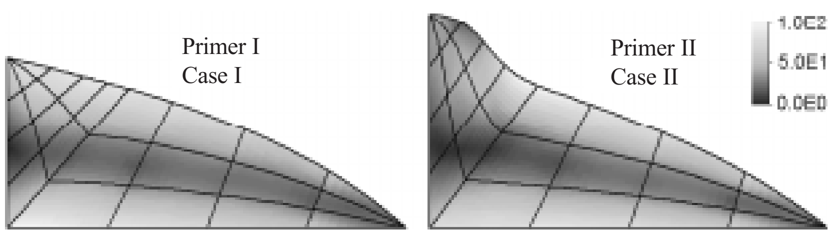
V tem zgledu smo uporabili dva projektna elementa z deset NT. Ker imata elementa štiri skupne NT, smo torej skupaj morali definirati šestnajst NT. Za primerjavo rezultatov smo enak problem rešili tudi z uporabo dveh štiristraničnih projektnih elementov, vsak z 4×4×1=16 NT. Rezultati so bili identični, vendar smo v slednjem primeru morali definirati dvaindvajset NT. V primerjavi s šestnajst NT to pomeni povečanje obsega priprave podatkov za 37,5%. To lahko pomeni bistveno razliko pri zahtevnih optimizacijskih nalogah z zapletenimi oblikami in mnogimi projektnimi spremenljivkami.

$[b_4,b_5]^T$. Adopting a similar arrangement for the second element, we finally get a total of nine design variables. The constraints are imposed as follows: the vertical displacement of point C has to be less than 0.2 mm while the Mises stresses along BC and AC have to be less than 100 MPa. The structure is modeled by 8-node plane finite elements of the Serendipity type.

Two cases will be considered: Case I where the values of thedesign variables are not limited, and Case II where (e.g. for fixing purposes) we impose the following limits: $b_1,b_3 \geq 80$ mm and $b_2,b_4 \geq 20$ mm.

By employing our own optimization algorithm, both problems could be solved successfully within ten iterations (six iterations for Case I and eight for Case II). At the final design (Figure 8), displacement and stress constraints were active in both cases. Compared to the initial volume, the optimum volumes are reduced by 48% and 46% for cases I and II, respectively.

In this example we employed two trilateral design elements with ten CP each. Since four CP are common to both design elements, we had to define a total of sixteen CP. In order to compare the results, we solved the same problem by employing two quadrilateral design elements with 4×4×1=16 CP each. The results were identical, but for this arrangement we had to define a total of twenty-two CP. Compared to sixteen, this is an increase of 37.5%. For complex design problems with sophisticated shapes and many design variables such an increase can represent a substantial difference.



Sl. 8. *Optimalne oblike z razporeditvijo napetosti*
Fig. 8. *Optimal shapes with stress distribution*

Zgled 2. Optimizacija tristranične konzole

Obravnavajmo tristranično konzolo, ki jo prikazuje slika 9. Konzola je togo podprta pri z=0 (ploskev ABG) in obremenjena na zgornji ploskvi ABCD z eno od dveh enakomerno porazdeljenih sil: (I) p_y= - 40 N/cm² in (II) p_x=30 N/cm². Obe obremenitvi sta ločeno upoštevani v okviru enega optimizacijskega problema.

Naš namen je določiti obliko robov BG in CH kakor tudi obliko ploskve BCHG tako, da bo teža konzole najmanjša. Pri katerikoli od obeh obremenitev Misesove napetosti ne smejo preseči meje 100 MPa.

Za parametrizacijo mreže končnih elementov bomo uporabili en tristranični projektni element s 15×3=45 nadzornimi točkami. Nadzorne točke, ki določajo obliko ploskve BCHG so odvisne od osemnajst projektnih spremenljivk. Konstrukcija je diskretizirana z uporabo 20-vozliščnih končnih elementov tipa Serendipity.

Tudi ta problem smo lahko rešili zelo hitro. Postopek reševanja je bil stabilen, optimalna oblika je bila dobljena po petih iteracijah. Kakor je razvidno iz preglednice, je optimizacijski postopek izpolnil postavljene pogoje (na začetku so ti bili prekoračeni) in hkrati zmanjšal prostornino konstrukcije. Optimalna oblika je prikazana na sliki 10.

Še komentar k izbiri projektnega elementa. Za uporabo tristranične variante smo potrebovali 15×3=45 nadzornih točk. Če bi uporabili štiristranično varianto, bi za enako prilagodljivost potrebovali

Example 2. Optimization of a trilateral cantilever

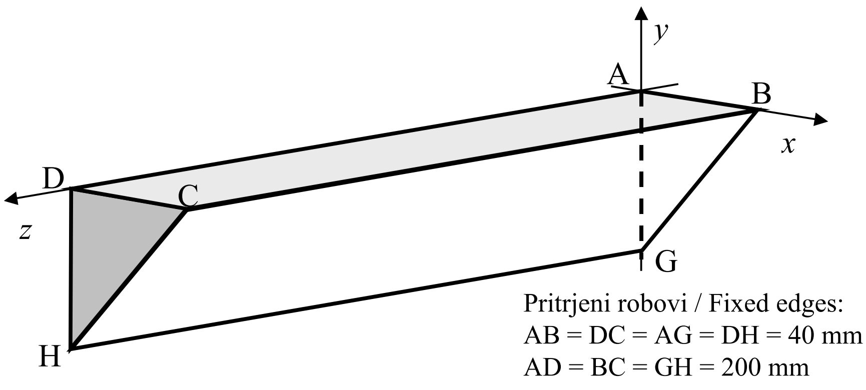
Let us consider the trilateral cantilever shown in Figure 9. The cantilever is rigidly supported at z=0 (face ABG) and loaded at the upper face ABCD by either one of two evenly distributed loads: (I) p_y= - 40 N/cm² and (II) p_x=30 N/cm². Both loads are considered simultaneously, within a single design problem.

Our objective is to determine the shape of the edges BG and CH as well as the shape of the face BCHG so that the weight of the cantilever will be minimised. The Mises stresses are allowed to be, at most, 100 MPa when either the first or the second load is applied.

In order to parametrize the mesh, one trilateral design element with 15×3=45 control points is employed. The control points that determine the shape of the surface BCHG are design dependent, which results in a total of eighteen design variables. The structure is modeled by 20-node finite elements of the Serendipity type.

This example was also solved very quickly. The solution process was stable and a near optimum design was obtained after five iterations. As can be seen in the table, the optimization process fulfilled the constraints (initially being violated), and lowered the volume of the structure. The optimal design is depicted in Figure 10.

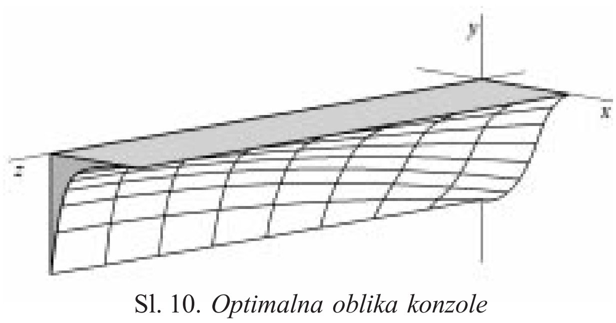
A comment on the choice of the design element. For the trilateral design element we needed 15×3=45 control points. For the same order of flexibility we would need a quadrilateral design



Sl. 9. *Tristranična konzola*
Fig. 9. *Trilateral cantilever*

Preglednica 1. *Primerjava začetnega in optimalnega projekta*
Table 1. *Comparison of the initial and optimised designs*

	Začetni Initial	Optimalni Optimum
Prostornina v cm³ Volume [cm³]	160,0	93,5
Najv. prekoračitev pogojev v % Max. constraint violation [%]	27,9	0,0



Sl. 10. *Optimalna oblika konzole*
Fig. 10. *Optimal shape of the cantilever*

element s 5×5×3=75 nadzornimi točkami. To pa je precej več kakor 45, torej je v danem primeru uporaba tristraničnega elementa precej bolj smotrna.

element with 5×5×3=75 control points. This is substantially more than 45, making, in this example, the trilateral element a far better choice.

4 SKLEP

V prispevku je bil predstavljen tristranični projektni element. Čeprav lahko namesto tristraničnega elementa vedno uporabimo splošnejšega štiristraničnega, tristranična varianta mnogokrat pomeni bolj naravni izbor. Posledica je manjše število potrebnih nadzornih točk, kar pomeni manj priprave vhodnih podatkov in manj geometrijskih pogojev. Tristranične projektne elemente lahko poljubno kombiniramo s štiristraničnimi.

4 CONCLUSION

The trilateral prismatic design element was presented in the paper. Although the quadrilateral prismatic element may be used for the same purpose, the trilateral version often represents a more natural choice. The consequence is a smaller number of required control points, meaning that less input data has to be prepared and that fewer geometrical constraints have to be imposed. The trilateral elements can be used arbitrarily in combination with the quadrilateral elements.

5 LITERATURA
5 REFERENCES

[1] Haftka, R.T., R.V.Grandhi (1986) Structural shape optimization - a survey. *Computer Methods in Applied Mechanics and Engineering,* 57, 91 106.

[2] Imam, M.H. (1982) Three dimensional shape optimization. *International Journal for Numerical Methods in Engineering*; 18, 661 673.

[3] Braibant, V., C. Fleury (1984) Shape optimal design using B splines. *Computer Methods in Applied Mechanics and Engineering;* 44, 247 267.

[4] Shyy, Y.K., C. Fleury, K. Izadpanah (1988) Shape optimal design using high order elements. *Computer Methods in Applied Mechanics and Engineering*; 71, 99 116.

[5] Maute, K., E. Ramm (1995) Adaptive topology optimization. *Structural optimization*; 10, 100 112.

[6] Ohsaki, M, T. Nakamura, Y. Isshiki (1998) Shape size optimization of plane trusses with designer's preference. *Journal of structural engineering*; 124, 1323 1330.

[7] Kegl, M., H. Antes (1998) Shape optimal design of elastic space frames with non-linear response. *International Journal for Numerical Methods in Engineering*; 43, 93 110.

[8] Wang, X., J. Zhou, Y. Hu (1999) A physics based parametrization method for shape optimization. *Computer Methods in Applied Mechanics and Engineering*; 175, 41 51.

[9] Kegl, M. (2000) Shape optimal design of structures: an efficient shape representation concept. *International Journal for Numerical Methods in Engineering*; 49, 1571 1588.

[10] Farin, G. (1993) Curves and surfaces for computer aided geometric design (2nd edn). *Academic Press,* New York.

[11] Kegl, M., M.M. Oblak (1997) Optimization of mechanical systems: on non linear first order approximation with an additive convex term. *Communications in Numerical Methods in Engineering*; 13, 13 20.

[12] Kegl, M., B.J. Butinar, B. Kegl (2002) An efficient gradient based optimization algorithm for mechanical systems. *Communications in Numerical Methods in Engineering*; 18, 363 371.

Avtorjev naslov: doc. dr. Marko Kegl
Fakulteta za strojništvo
Univerza v Mariboru
Smetanova 17
2000 Maribor
marko.kegl@uni-mb.si

Authors' Address: Doc. Dr. Marko Kegl
Faculty of mechanical eng.
University of Maribor
Smetanova 17
2000 Maribor, Slovenia
marko.kegl@uni-mb.si

Prejeto:
Received: 23.12.2002

Sprejeto:
Accepted: 31.1.2003