A new generalization of generalized Petersen graphs[*]

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Abstract

We discuss a new family of cubic graphs, which we call group divisible generalised Petersen graphs ($GDGP$-graphs), that bears a close resemblance to the family of generalised Petersen graphs, both in definition and properties. The focus of our paper is on determining the algebraic properties of graphs from our new family. We look for highly symmetric graphs, e.g., graphs with large automorphism groups, and vertex- or arc-transitive graphs. In particular, we present arithmetic conditions for the defining parameters that guarantee that graphs with these parameters are vertex-transitive or Cayley, and we find one arc-transitive $GDGP$-graph which is neither a CQ graph of Feng and Wang, nor a generalised Petersen graph.

Keywords: Generalised Petersen graph, arc-transitive graph, vertex-transitive graph, Cayley graph, automorphism group.

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1   Introduction

Generalised Petersen graphs $GP(n, k)$ (the name and notation coined in 1969 by Watkins [18], with the subclass with n and k relatively prime considered already by Coxeter in 1950 [6]) constitute one of the central families of algebraic graph theory. While there are many reasons for the interest in this family, with a bit of oversimplification one could say that among the most important is the simplicity of their description (requiring just two parameters n and k) combined with the richness of the family that includes the well-known Petersen and dodecahedron graphs, as well as large families of vertex-transitive graphs and seven symmetric (arc-transitive) graphs.

Our motivation for studying the new family of *group divisible generalised Petersen graphs* ($GDGP$-graphs; introduced in [11] under the name SGP-graphs) lies in the fact that they share the above characteristics with the generalised Petersen graphs. They include all vertex-transitive generalised Petersen graphs but the dodecahedron as a proper subclass, are easily defined via a sequence of integral parameters, and contain graphs of various levels of symmetry.

Historically, ours is certainly not the first attempt at generalising generalised Petersen graphs. In 1988, the family of *I-graphs* was introduced in the Foster Census [2]. This family differs from the generalised Petersen graphs in allowing the span on the outer rim to be different from 1: The *I*-graph $I(n, j, k)$ is the cubic graph with vertex set $\{u_i, v_i \,|\, i \in \mathbb{Z}_n\}$ and edge set $\{\{u_i, u_{i+j}\}, \{u_i, v_i\}, \{v_i, v_{i+k}\} \,|\, i \in \mathbb{Z}_n\}$. However, the only *I*-graphs that are vertex-transitive are the original generalised Petersen graphs which are the graphs $I(n, 1, k)$ [1, 14].

The family of *GI-graphs* introduced in [5] by Conder, Pisanski and Žitnik in 2014 is a further generalisation of *I*-graphs. For positive integers $n \geq 3$, $m \geq 1$, and a sequence K of elements in $\mathbb{Z}_n - \{0, \frac{n}{2}\}$, $K = (k_0, k_1, \ldots, k_{m-1})$, the *GI*-graph $GI(n; k_0, k_1, \ldots, k_{m-1})$ is the graph with vertex set $\mathbb{Z}_m \times \mathbb{Z}_n$ and edges of two types:

(i) an edge from (u, v) to (u', v), for all distinct $u, u' \in \mathbb{Z}_m$ and all $v \in \mathbb{Z}_n$,

(ii) edges from (u, v) to $(u, v \pm k_u)$, for all $u \in \mathbb{Z}_m$ and all $v \in \mathbb{Z}_n$.

The *GI*-graphs are $(m+1)$-regular, thus cubic when $m = 2$, which is the case that covers the *I*-graphs, with the subclass of the $GI(n; 1, k)$ graphs covering the generalised Petersen graphs.

Another generalisation is due to Lovrečič-Saražin, Pacco and Previtali, who extended the class of generalised Petersen graphs to the so-called *supergeneralised Petersen graphs* [15]. Let $n \geq 3$ and $m \geq 2$ be integers and $k_0, k_1, \ldots, k_{m-1} \in \mathbb{Z}_n - \{0\}$. The vertex-set of the supergeneralised Petersen graph $P(m, n; k_0, \ldots, k_{m-1})$ is $\mathbb{Z}_m \times \mathbb{Z}_n$ and its edges are of two types:

(i) an edge from (u, v) to $(u + 1, v)$, for all $u \in \mathbb{Z}_m$ and all $v \in \mathbb{Z}_n$,

(ii) edges from (u, v) to $(u, v \pm k_u)$, for all $u \in \mathbb{Z}_m$ and all $v \in \mathbb{Z}_n$.

Note that $GP(n, k)$ is isomorphic to $P(2, n; 1, k)$.

Finally, in 2012, Zhou and Feng [20] modified the class of generalised Petersen graphs in order to classify cubic vertex-transitive non-Cayley graphs of order $8p$, for any prime p [20]. In their definition, the subgraph induced by the outer edges is a union of two n-cycles. Let $n \geq 3$ and $k \in \mathbb{Z}_n - \{0\}$. *The double generalised Petersen graph* $DP(n, k)$ is defined to have the vertex set $\{x_i, y_i, u_i, v_i \,|\, i \in \mathbb{Z}_n\}$ and the edge set equal to the union of the outer
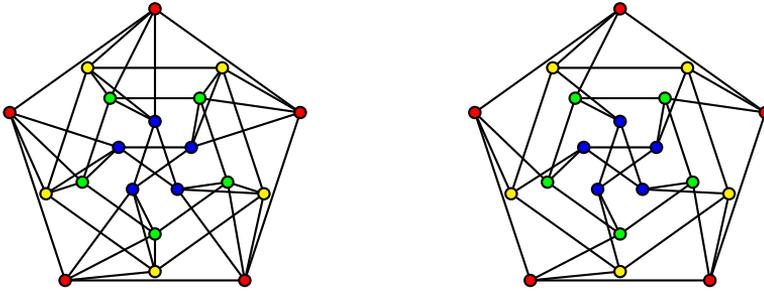
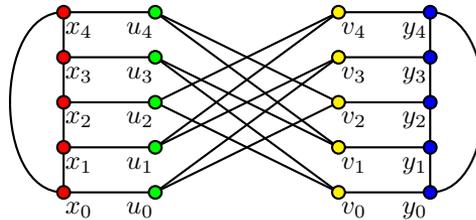Figure 1: $GI(5; 1, 1, 1, 2)$ and $P(4, 5; 1, 1, 1, 2)$.

Figure 2: $DP(5, 2)$.

edges $\{\{x_i, x_{i+1}\}, \{y_i, y_{i+1}\} \mid i \in \mathbb{Z}_n\}$, the inner edges $\{\{u_i, v_{i+k}\}, \{v_i, u_{i+k}\} \mid i \in \mathbb{Z}_n\}$, and the spokes $\{\{x_i, u_i\}, \{y_i, v_i\} \mid i \in \mathbb{Z}_n\}$.

Even though non-empty intersections exist between the above classes and the class of group divisible generalised Petersen graphs considered in our paper, none of these is significant, and we believe that ours is, in a way, the most natural generalisation of generalised Petersen graphs.

2 Generalised Petersen graphs

Let us review the basic properties of generalised Petersen graphs. A *generalised Petersen graph* $GP(n, k)$ is determined by integers n and k, $n \geq 3$ and $\frac{n}{2} > k \geq 1$. The vertex set $V(GP(n, k)) = \{u_i, v_i \mid i \in \mathbb{Z}_n\}$ is of order $2n$ and the edge set $E(GP(n, k))$ of size $3n$ consists of edges of the form

$$\{u_i, u_{i+1}\}, \quad \{u_i, v_i\}, \quad \{v_i, v_{i+k}\}, \tag{2.1}$$

where $i \in \mathbb{Z}_n$. Thus, $GP(n, k)$ is always a trivalent graph, the Petersen graph is the graph $GP(5, 2)$, the dodecahedron is $GP(10, 2)$, and the ADAM graph is $GP(24, 5)$.

We will call the u_i vertices the *outer vertices*, the v_i vertices the *inner vertices*, and the three distinct forms of edges displayed in (2.1) *outer edges, spokes*, and *inner edges*, respectively. Graphs introduced in this paper will also contain vertices and edges of these types. We will use the symbols Ω, Σ and I, respectively, to denote the three n-sets of edges. The n-circuit induced in $GP(n, k)$ by Ω will be called the *outer rim*. If d denotes the greatest common divisor of n and k, then I induces a subgraph which is the union of d pairwise-disjoint $\frac{n}{d}$-circuits, called *inner rims*. The parameter k also denotes the *span* of
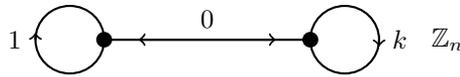
Figure 3: Voltage graph for the generalised Petersen graph $GP(n,k)$.

the inner rims (which is the distance, as measured on the outer rim, between the outer rim neighbors of two vertices adjacent on an inner rim).

The class of generalised Petersen graphs is well understood and has been studied by many authors. In 1971, Frucht, Graver and Watkins [9] determined their automorphism groups. They proved that $GP(n,k)$ *is vertex-transitive if and only if* $k^2 \equiv \pm 1$ mod n *or* $(n,k) = (10,2)$. Later, Nedela and Škoviera [16], and (independently) Lovrečič-Saražin [13] proved that a generalised Petersen graph $GP(n,k)$ is Cayley if and only if $k^2 \equiv 1$ mod n. Recall that a *Cayley graph* $Cay(G,X)$, where G is a group generated by the set X which does not contain the identity 1_G and is closed under taking inverses, is the graph whose vertices are the elements of G and edges are the pairs $\{g, xg\}$, $g \in G$, $x \in X$.

3   Polycirculants and voltage graphs

A non-identity automorphism of a graph is (m,n)-*semiregular* if its cycle decomposition consists of m cycles of length n. Graphs admitting (m,n)-semiregular automorphisms are called m-*circulants* (if one chooses to suppress the parameter m, they are sometimes called *polycirculants*). If $m = 1, 2, 3$, or 4, an m-circulant is said to be a *circulant*, a *bicirculant*, a *tricirculant*, or a *tetracirculant*, respectively. It is easy to see that generalised Petersen graphs are bicirculants; the corresponding automorphism consists of the two cycles $(u_0, u_1, u_2, \ldots, u_{n-1}), (v_0, v_1, v_2, \ldots, v_{n-1})$.

The reader is also most likely familiar with the fact that generalised Petersen graphs can be defined in a nice and compact way using the language of voltage graphs (a more detailed treatment may be found for example in [10]):

If Γ is an undirected graph, we associate each edge of Γ with a pair of opposite arcs and denote the set of all such arcs by $D(\Gamma)$. A *voltage assignment* on Γ is any mapping α from $D(\Gamma)$ into a group G that satisfies the condition $\alpha(e^{-1}) = (\alpha(e))^{-1}$ for all $e \in D(\Gamma)$ (with e^{-1} being the opposite arc of e, and $(\alpha(e))^{-1}$ being the inverse of $\alpha(e)$ in G). The *lift* (sometimes called the *derived regular cover*) of Γ with respect to a voltage assignment α on Γ is a graph denoted by Γ^α. The vertex set $V(\Gamma^\alpha)$ consists of $|V(\Gamma)| \cdot |G|$ vertices $u_g = (u, g), (u, g) \in V(\Gamma) \times G$. Two vertices u_g and v_f are adjacent in Γ^α if $e = (u, v)$ is an arc of Γ and $f = g \cdot \alpha(e)$ in G.

All generalised Petersen graphs $GP(n,k)$ are lifts of the dumbbell graph $\mathcal{D}$ which consists of two vertices joined by an edge and loops attached to them. The corresponding voltage assignment $\alpha : D(\mathcal{D}) \longrightarrow \mathbb{Z}_n$ assigns 0 to the arcs connecting the two vertices, 1 and -1 to the arcs of the loop at one of the vertices, and k and $-k$ to the arcs of the loop at the other vertex (Figure 3).

Similarly, the I-graph $I(n, j, k)$ is a derived regular cover of the dumbbell graph with 0 assigned to the 'handle', and the values $j, -j$ and $k, -k \in \mathbb{Z}_n$ assigned to the loops (Figure 4).

The GI-graph $GI(n; a, b, c, d)$ is a lift of the complete graph K_4, and so is the super-generalised Petersen graph $P(4, n; a, b, c, d)$ (see Figures 5 and 6, respectively).
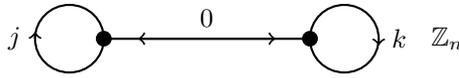
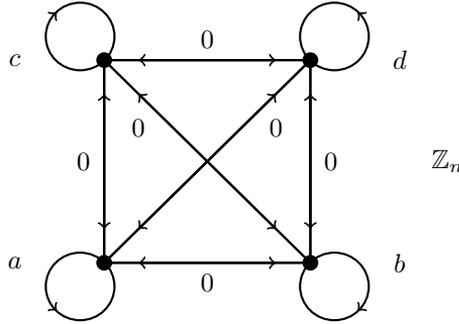Figure 4: Voltage graph for the I-graph $I(n, j, k)$.

Figure 5: Voltage graph for the GI-graph $GI(n; a, b, c, d)$.

Recently, Conder, Estélyi, and Pisanski in [4] considered more general voltage assignments, and thus generalised the double generalised Petersen graphs even further.

There is another family of polycirculants that will prove useful later in our paper, introduced by Feng and Wang in 2003 [7]. Their graphs are called CQ graphs, and were originally introduced as octacirculants (for their voltage graph description see Figure 7). The definition of $CQ(k, n)$ used in [7] makes sense for any k, n such that $\gcd(k, n) = 1$, which is equivalent to $k \in \mathbb{Z}_n^*$. Frelih and Kutnar [8] later correctly showed that each $CQ(k, n)$ is in fact a tetracirculant. However, their voltage graph depiction is not correct. One needs two different voltage graphs, depending on the parity of m. The correct voltage graphs are depicted in Figures 8 and 9.

Moreover, in the definition of $CQ(k, m)$ used by Frelih and Kutnar the inverse k^{-1} is
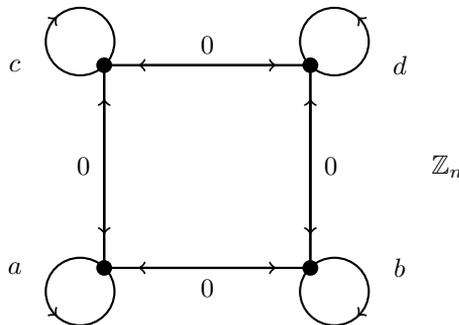

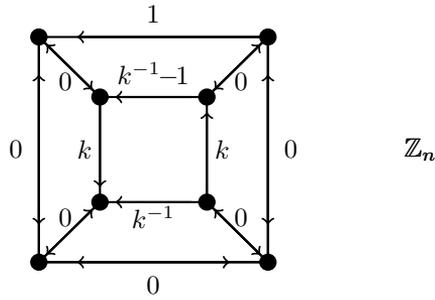Figure 6: Voltage graph for the supergeneralised Petersen graph $P(4, n; a, b, c, d)$.

Figure 7: Original voltage graph for the octacirculant graph $CQ(k, n), \gcd(k, n) = 1$, which appeared in [7].
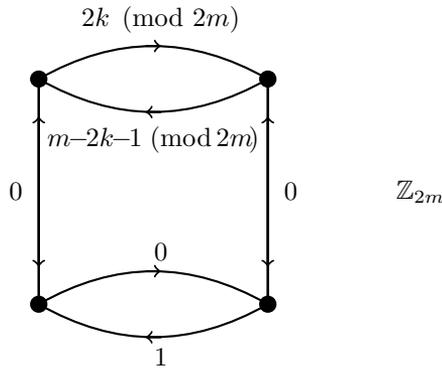

Figure 8: Corrected voltage graph for the tetracirculant graph $CQ(k, m)$, k odd, m even.
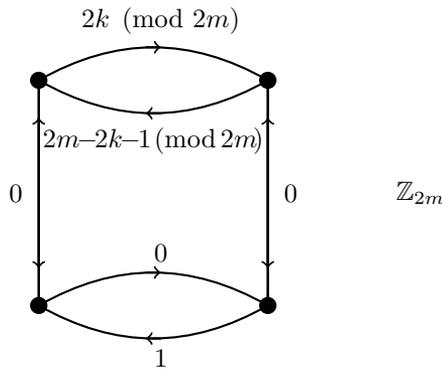

Figure 9: Corrected voltage graph for the tetracirculant graph $CQ(k, m)$, k odd, m odd.

not needed. Hence, their voltage graphs define a family of graphs that is more general than that of Feng and Wang [7]. In our paper, we shall use this more general definition.

4 $GDGP$-graphs

The graphs we shall focus on in this paper are defined as follows.

Definition 4.1. Let $n \geq 3$ and $m \geq 2$ be positive integers such that m divides n, let a be a non-zero element of $\mathbb{Z}_m$, and let $K = (k_0, k_1, \ldots, k_{m-1})$ be a sequence of elements from $\mathbb{Z}_n$ all of which are congruent to a modulo m and satisfy the requirement $k_j + k_{j-a} \not\equiv 0$ (mod n), for all $j \in \mathbb{Z}_m$.

The graph $GDGP_m(n; K)$, or alternatively $GDGP_m(n; k_0, k_1, \ldots, k_{m-1})$, has the vertex set $\{u_i, v_i \mid i \in \mathbb{Z}_n\}$ of order $2n$ and the edge set of size $3n$:

$$\{u_i, u_{i+1}\}, \ \{u_i, v_i\}, \ \{v_{mi+j}, v_{mi+j+k_j}\}, \tag{4.1}$$

where $i \in \mathbb{Z}_{\frac{n}{m}}$, $j \in \mathbb{Z}_m$, and the arithmetic operations are performed modulo n.

While the $GDGP$-graphs defined above share the outer rim edges and the spokes with the generalised Petersen graphs (and are therefore all connected), the inner edges are determined by the more complicated rule $\{\{v_{mi+j}, v_{mi+j+k_j}\} \mid i \in \mathbb{Z}_{\frac{n}{m}}, j \in \mathbb{Z}_m\}$ applied in groups of size $m \geq 2$.

The choices made in our definition guarantee that the $GDGP$-graphs are cubic. This claim is obviously true for the outer vertices u_i. To prove the claim for the inner vertices v_{mi+j}, $i \in \mathbb{Z}_{\frac{n}{m}}$, $j \in \mathbb{Z}_m$, it is enough to show that each inner vertex v_{mi+j} of $GDGP_m(n; K)$ is incident with exactly two edges of the type determined by the third rule of (4.1). Thus, assume that $\{v_{mi'+j'}, v_{mi'+j'+k_{j'}}\}$ is incident with v_{mi+j}. Then, either $v_{mi+j} = v_{mi'+j'}$ or $v_{mi+j} = v_{mi'+j'+k_{j'}}$. If $v_{mi+j} = v_{mi'+j'}$, then $i = i', j = j'$, and $k_j = k_{j'}$ is uniquely determined; there is exactly one such edge. If $v_{mi+j} = v_{mi'+j'+k_{j'}}$, then $mi + j = mi' + j' + k_{j'}$, thus $j \equiv j' + k_{j'} \equiv j' + a$ (mod m), which uniquely determines $j' = j - a \in \mathbb{Z}_m$ as well as $k_{j'}$. The equation $mi + j = mi' + j' + k_{j'}$ then uniquely determines i', and therefore there is exactly one edge $\{v_{mi'+j'}, v_{mi'+j'+k_{j'}}\}$ for which $v_{mi+j} = v_{mi'+j'+k_{j'}}$. To complete the argument, note that the two edges $\{v_{mi+j}, v_{mi+j+k_j}\}$ and $\{v_{mi+j-k_{j-a}}, v_{mi+j}\}$ are necessarily different, since we assume that $k_j + k_{j-a} \not\equiv 0$ (mod n), for all $j \in \mathbb{Z}_m$. Let us observe for future reference that the three neighbors of v_{mi+j} are the vertices u_{mi+j}, v_{mi+j+k_j} and $v_{mi+j-k_{j-a}}$.

Being 2-regular, the graph induced by the inner vertices v_i, $i \in \mathbb{Z}_n$ consists of disjoint cycles. If we denote the order of a in $\mathbb{Z}_m$ by $o_m(a)$, it is easy to see that the length of the inner cycle containing v_{mi+j} is the product of $o_m(a)$ with the order $o_n(k_j + k_{j+a} + k_{j+2a} + \ldots + k_{j+(o_m(a)-1)a})$ of the element $k_j + k_{j+a} + k_{j+2a} + \ldots + k_{j+(o_m(a)-1)a}$ in $\mathbb{Z}_n$ (with the indices calculated modulo m). Thus, if a is chosen to be a generator for $\mathbb{Z}_m$ (i.e., $o_m(a) = m$), all inner cycles in $GDGP_m(n; k_0, k_1, \ldots, k_{m-1})$ are of the same length $m \cdot o_n(k_0 + k_1 + \ldots + k_{m-1})$. In particular, if $m = 2$, a is by definition necessarily congruent to 1 (mod 2), and is therefore a generator for $\mathbb{Z}_2$, hence all inner cycles of the graphs $GDGP_2(n; k_0, k_1)$ are of length $2 \cdot o_n(k_0 + k_1)$.

Example 4.2. Consider the graph $GDGP_2(8; 1, 3)$ in Figure 10. Both 1 and 3 are congruent to 1 (mod 2), which is a generator for $\mathbb{Z}_2$. The order of the sum $1 + 3 = 4$ is 2 in $\mathbb{Z}_8$, and hence the inner edges of this graph form two disjoint 4-cycles.
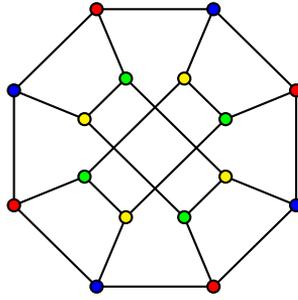
Figure 10: $GDGP_2(8; 1, 3)$.

It is easy to see that, for even n and odd k, $GP(n, k)$ is isomorphic to $GDGP_2(n; k, k)$, and for n divisible by 3 and $k \not\equiv 0 \pmod 3$, $GP(n, k)$ is isomorphic to $GDGP_3(n; k, k, k)$. We generalise this observation in the following lemma. A sequence $k_0, \ldots, k_{m-1}$ is said to be *periodic* if there exists an integer $0 < p < m$ that divides m and $k_i = k_{i+p}$, for all $i \in \mathbb{Z}_m$. The smallest p with this property is the *period* of the sequence. The proof of the following lemma is now obvious.

Lemma 4.3. *Let* $GDGP_m(n; K)$ *be a graph such that* $K = (k_0, \ldots, k_{m-1})$ *is a periodic sequence with a period* p. *If* $p = 1$, *the graph* $GDGP_m(n; K)$ *is isomorphic to the graph* $GP(n, k)$, *and if* $p > 1$, $GDGP_m(n; K)$ *is isomorphic to the graph* $GDGP_p(n; K')$, *where* $K' = (k_0, \ldots, k_{p-1})$.

Consequently, $GDGP_m(n; k, k, \ldots, k) \cong GP(n, k)$, for all divisors m of n and all $k \not\equiv 0 \pmod m$. To simplify our notation and arguments, *we will assume from now on that the sequence K used in the notation $GDGP_m(n; K)$ is aperiodic.*

Most importantly, not all $GDGP$-graphs are generalised Petersen graphs. Consider, for example, the graph $GDGP_2(8; 1, 3)$ constructed in Example 4.2. The subgraph induced by the inner vertices consists of two disjoint 4-cycles. While the same is true for $GP(8, 2)$, nevertheless, $GDGP_2(8; 1, 3)$ is not isomorphic to any generalised Petersen graph. To see this, note that each of the outer vertices of this graph lies on exactly one 4-cycle, while each of the inner vertices lies on two 4-cycles. Thus, no automorphism of $GDGP_2(8; 1, 3)$ interchanges the outer and inner vertices (and $GDGP_2(8; 1, 3)$ is not a vertex-transitive graph). If $GDGP_2(8; 1, 3)$ were to be isomorphic to a generalised Petersen graph, it would have to be a bicirculant and would have to admit a $(2, 8)$-semiregular automorphism. The two 8-orbits would thus necessarily consist of the outer and inner vertices. The automorphism group of the 8-cycle induced by the outer vertices is equal to the dihedral group $\mathbb{D}_8$ which contains only two 8-cycles: $(u_0, u_1, \ldots, u_7)$, and its inverse. Thus, the action of any $(2, 8)$-semiregular automorphism of $GDGP_2(8; 1, 3)$ on the outer vertices would have to be equal to one of these cycles. Since automorphisms must preserve adjacency, this would necessarily force the action of this semiregular automorphism on the inner vertices to be the cycle $(v_0, v_1, \ldots, v_7)$ or its inverse. However, neither the permutation $(u_0, u_1, \ldots, u_7)(v_0, v_1, \ldots, v_7)$ nor its inverse are graph automorphisms of $GDGP_2(8; 1, 3)$. Hence, $GDGP_2(8; 1, 3)$ is not a bicirculant, and is therefore not isomorphic to any generalised Petersen graph.
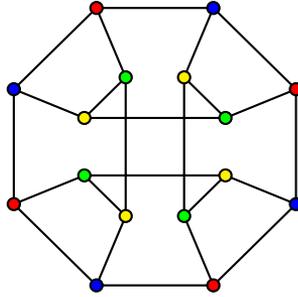
Figure 11: $GDGP_2(8; 1, 5)$.

On the other hand, it is not hard to see that the $(4, 4)$-semiregular permutation $(u_0, u_2, u_4, u_6)(u_1, u_3, u_5, u_7)(v_0, v_2, v_4, v_6)(v_1, v_3, v_5, v_7)$ is an automorphism of $GDGP_2(8; 1, 3)$, and that the following theorem holds in general.

Theorem 4.4. *For all $m \geq 2$, the permutation*

$$\alpha : V(GDGP_m(n; K)) \rightarrow V(GDGP_m(n; K)), \; u_i \mapsto u_{i+m}, \; v_i \mapsto v_{i+m}, \; i \in \mathbb{Z}_n,$$

is a $(2m, \frac{n}{m})$-semiregular automorphism of $GDGP_m(n; K)$.
Thus, every $GDGP_m(n; K)$-graph is a $2m$-circulant.

We conclude the section with an easy but useful graph-theoretical property of the $GDGP$-graphs.

Lemma 4.5. *The $GDGP_m(n; K)$ is a bipartite graph if and only if n is even and all elements in K are odd.*

5   Automorphisms of $GDGP$-graphs

Many of the ideas of this and the forthcoming sections can be demonstrated with the use of the following family of $GDGP$-graphs.

Example 5.1. For each *even $n \geq 4$*, consider the graph $GDGP_2(n; 1, n - 3)$. The graphs in this family are also known as the *crossed prism graphs* [19]. In particular, $GDGP_2(4; 1, 1) \cong GP(4, 1)$, and $GDGP_2(6; 1, 3)$ is the *Franklin graph*. The graph $GDGP_2(8; 1, 5)$ is depicted in Figure 11. All the graphs $GDGP_2(n; 1, n - 3)$ are vertex-transitive, which means, in particular, that they admit an automorphism mapping an outer vertex to an inner vertex.

Our first result follows already from our discussion of $GDGP_2(8; 1, 3)$.

Lemma 5.2. *If $\gamma \in \mathrm{Aut}(GDGP_m(n; K))$ fixes set-wise any of the sets Ω, Σ or I, then it either fixes all three sets or fixes Σ set-wise and interchanges Ω and I.*

We have observed already that the action on Ω of an Ω-preserving automorphism must belong to $\mathbb{D}_n$. Assume that an automorphism σ preserves Ω and acts on Ω as a reflection. Then one of the following occurs:

1. σ has no fixed points, in which case n is necessarily even and there exists an $s \in \mathbb{Z}_n$ such that σ swaps u_s and u_{s+1}, and, consequently, σ swaps all the pairs u_{s-i} and u_{s+1+i}, $i \in \mathbb{Z}_n$;

2. σ fixes at least one vertex, say u_s, and consequently swaps the pairs u_{s+i} and u_{s-i}, $i \in \mathbb{Z}_n$.

The same is necessarily true for the inner vertices, and in case (1) σ swaps v_{s-i} and v_{s+1+i}, $i \in \mathbb{Z}_n$, while in case (2) σ swaps the pairs v_{s+i}, v_{s-i}, $i \in \mathbb{Z}_n$.

In either case, σ is a bijection on the vertices and it preserves the outer edges and the spokes. Thus the image of an inner edge must again be an inner edge. Assume first that σ is of type (1), and consider an arbitrary inner edge $\{v_{mi+j}, v_{mi+j+k_j}\}$. Its image under σ is the pair $\{v_{2s-mi-j+1}, v_{2s-mi-j-k_j+1}\}$, which is an edge of $GDGP_m(n; K)$ if and only if $2s - mi - j - k_j + 1 \equiv 2s - mi - j + 1 + k_{j'} \pmod{n}$, i.e., $-k_j \equiv k_{j'} \pmod{n}$, where $j' \equiv 2s - mi - j + 1 \equiv 2s - j + 1 \pmod{m}$, or, $2s - mi - j + 1 \equiv 2s - mi - j - k_j + 1 + k_{j''}$ $\pmod{n}$, i.e., $k_j \equiv k_{j''} \pmod{n}$, where $j'' \equiv 2s - mi - j - k_j + 1 \equiv 2s - j - a + 1$ $\pmod{m}$. Therefore, σ is a graph automorphism of $GDGP_m(n; K)$ if and only if (at least) one the above equalities holds for each $i \in \mathbb{Z}_{\frac{n}{m}}$, $j \in \mathbb{Z}_m$. In the special case when $m = 2$, a is necessarily 1, $j'' \equiv 2s + j + 1 + 1 \equiv j \pmod{2}$, and hence:

Lemma 5.3. *For every $GDGP_2(n; K)$, and for every $s \in \mathbb{Z}_n$, the reflection σ_s swapping the pairs u_{s-i} and u_{s+1+i}, and the pairs v_{s-i} and v_{s+1+i}, for all $i \in \mathbb{Z}_n$, is a graph automorphism of $GDGP_2(n; K)$. Consequently,* $\mathrm{Aut}(GDGP_2(n; K))$ *acts transitively on the two sets of outer and inner vertices of $GDGP_2(n; K)$.*

Proof. The graph automorphism $\alpha : V(GDGP_2(n; K)) \to V(GDGP_2(n; K))$ defined in Theorem 4.4 and sending $u_i \mapsto u_{i+2}$ and $v_i \mapsto v_{i+2}$, for all $i \in \mathbb{Z}_2$, has two orbits on the outer and two orbits on the inner vertices. The reflection automorphisms σ_s mix these two orbits. $\qquad\square$

Both of our examples, $GDGP_2(8; 1, 3)$ and $GDGP_2(8; 1, 5)$, can be easily seen to be symmetric with respect to reflections about axes passing through the centers of a pair of opposing outer edges.

Next, let us consider σ of type (2). The image of an arbitrary edge $\{v_{mi+j}, v_{mi+j+k_j}\}$ is the pair $\{v_{2s-mi-j}, v_{2s-mi-j-k_j}\}$, which is an edge if and only if $2s - mi - j - k_j \equiv 2s - mi - j + k_{j'} \pmod{n}$, i.e., $-k_j \equiv k_{j'} \pmod{n}$, for $j' \equiv 2s - mi - j \equiv 2s - j$ $\pmod{m}$, or $2s - mi - j \equiv 2s - mi - j - k_j + k_{j''} \pmod{n}$, i.e., $k_j \equiv k_{j''} \pmod{n}$, for $j'' \equiv 2s - mi - j - k_j \equiv 2s - j - a \pmod{m}$. Comparing this result to that of Lemma 5.3, assuming $m = 2$ would require $-k_j \equiv k_{j'} \pmod{n}$, for $j' \equiv j \pmod{2}$, or $k_j \equiv k_{j''} \pmod{n}$, for $j'' \equiv j + a \pmod{2}$. The first is impossible as that would require $k_0 = k_1 = \frac{n}{2}$, which would violate our agreement that we do not consider periodic sequences, while the second possibility is explicitely prohibited in the definition of the GDGP-graphs. Hence, no $GDGP_2(n; K)$ that is not a generalised Petersen graph admits automorphisms of type (2). Specifically, it is easy to see that neither $GDGP_2(8; 1, 3)$ nor $GDGP_2(8; 1, 5)$ admits such automorphisms. There are, on the other hand, infinitely many graphs that do admit at least one such automorphism. The proof of the next lemma follows from the above discussion.

Lemma 5.4. *Let $n \geq 3$, and let K have the property that $-k_j \equiv k_{j'} \pmod{n}$, for $j' \equiv 2s - j \pmod{m}$, or let K have the property $k_j \equiv k_{j''} \pmod{n}$, for $j'' \equiv 2s - j - a$*
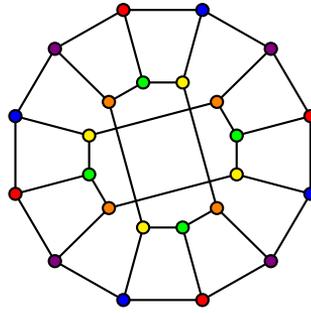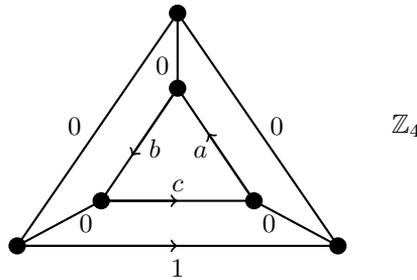
Figure 12: $GDGP_3(12; 1, 4, 1)$.

Figure 13: The voltage graph for $GDGP_3(12; 1, 4, 1)$, $a = 0, b = 0, c = 2$.

(mod m). *Then $GDGP_m(n, K)$ admits an automorphism σ that fixes u_s and v_s, and swaps the pairs u_{s+i} and u_{s-i}, v_{s+i} and v_{s-i}, $i \in \mathbb{Z}_n$.*

Example 5.5. The graphs $GDGP_3(n; a, 1, n-1)$ as well as the graphs $GDGP_3(n; 1, a, 1)$, $a \in \mathbb{Z}_n$, all admit an automorphism fixing the vertex u_0. In particular, the graph $GDGP_3(12; 1, 4, 1)$ pictured in Figure 12 is symmetric with respect to the axes passing through the vertices u_0 and u_6, and through the vertices u_3 and u_9.

All the automorphisms considered so far preserve the outer and inner rim as well as the spokes. We conclude this section by considering the graphs we started the section with, namely with the graphs $GDGP_2(n; 1, n - 3)$. As stated at the beginning of the section, all of them are vertex-transitive and admit an automorphism mapping an outer vertex to an inner vertex. Computational evidence collected in [12] suggests that the order of the full automorphism group of an $GDGP_2(n; 1, n - 3)$ is $n \cdot 2^{\frac{n}{2}}$, and it is easy to see that these graphs are neither edge- nor arc-transitive. In what follows, we present automorphisms that do not preserve the set of spokes.

Lemma 5.6. *Let $n \geq 6$. Then $GDGP_2(n; 1, n - 3)$ admits at least one automorphism which does not fix its set of spokes.*

Proof. The desired automorphism $\delta \in \mathrm{Aut}(GDGP_2(n; 1, n - 3))$ consists of just two 2-cycles: $\delta = (u_1v_0)(u_2v_3)$. Since δ moves only four vertices, to show that it is indeed
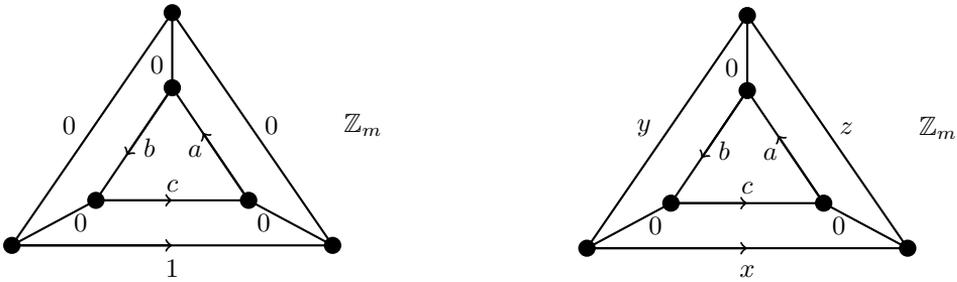
Figure 14: The voltage graphs for $GDGP_3(m; k_1, k_2, k_3)$ and $SI_3(m; l_1, l_2, l_3, k_1, k_2, k_3)$.

a graph automorphism, it suffices to show that it maps edges incident with the vertices u_1, v_0, u_2, v_3 to edges incident to the vertices u_1, v_0, u_2, v_3. This easy exercise is left to the reader. It is also easy to see that $(u_{i+1}v_i)(u_{i+2}v_{i+3})$ belongs to $\mathrm{Aut}(GDGP_2(n; 1, n-3))$ for all positive integers i divisible by 4 and smaller than $n - 3$. $\qquad\square$

Lemma 5.3 together with Lemma 5.6 yield that the graphs $GDGP_2(n; 1, n - 3)$ are indeed vertex-transitive for all $n \geq 6$.

6 Vertex-transitive and Cayley $GDGP_2$-graphs

We continue searching for vertex-transitive graphs. Lemma 5.3 asserts that $\mathrm{Aut}(GDGP_2(n; k_0, k_1))$ acts transitively on the set of the outer and the set of the inner vertices of every $GDGP_2(n; k_0, k_1)$. Thus, an $GDGP_2(n; k_0, k_1)$ graph is vertex-transitive if and only if it admits a graph automorphism mapping at least one outer vertex to an inner vertex. In this section, we present some sufficient conditions for this to happen. Note that the graphs $GDGP_2(n; k_0, k_1)$, $GDGP_2(n; k_1, k_0)$, $GDGP_2(n; -k_0, -k_1)$ and $GDGP_2(n; -k_1, -k_0)$ are all isomorphic.

Since we only seek sufficient conditions, we will focus on the special case of graphs that admit automorphisms preserving the set of spokes and swapping the entire sets of outer and inner vertices. Obviously, these must be those $GDGP$-graphs in which the graphs induced by the inner vertices form a single cycle. As observed already in the discussion following the definition of the $GDGP$-graphs, this is the case if and only if the order of the element $k_0 + k_1$ in $\mathbb{Z}_n$ is equal to $\frac{n}{2}$. Thus, we shall assume from now on that $o_n(k_0 + k_1) = \frac{n}{2}$.

Suppose $\gamma \in \mathrm{Aut}(GDGP_2(n; K))$ swaps the outer and the inner rim and preserves the spokes. Because of Lemma 5.3, we may assume that γ maps u_0 to v_0. The outer rim can be mapped onto the inner rim in either the clockwise or in the counterclockwise direction. Hence, there might be two automorphisms which swap the outer and inner cycles and map u_0 to v_0.

Let γ be an automorphism which maps the outer cycle to the inner cycle in the same direction. Thus,

$$\gamma(u_{2i}) = v_{i(k_0+k_1)}, \qquad \gamma(u_{2i+1}) = v_{i(k_0+k_1)+k_0}, \qquad (6.1)$$

for all $i \in \mathbb{Z}_{\frac{n}{2}}$. Since γ is assumed to preserve the set of spokes, the image of a spoke must

be a spoke again and thus it must be the case that

$$\gamma(v_{2i}) = u_{i(k_0+k_1)}, \qquad \gamma(v_{2i+1}) = u_{i(k_0+k_1)+k_0}, \tag{6.2}$$

for all $i \in \mathbb{Z}_{\frac{n}{2}}$. On the other hand, γ must map the inner cycle to the outer cycle, and, in particular, γ must map inner edges to outer edges. For any $i \in \{0, 1, \ldots, n-1\}$, because of (6.2), the vertex v_{2i}, which is adjacent to vertices v_{2i+k_0} and v_{2i-k_1}, is mapped to the vertex $u_{i(k_0+k_1)}$, adjacent to the vertices $u_{i(k_0+k_1)+1}$ and $u_{i(k_0+k_1)-1}$. Thus, γ maps the 2-set $\{v_{2i+k_0}, v_{2i-k_1}\}$ onto the 2-set $\{u_{i(k_0+k_1)+1}, u_{i(k_0+k_1)-1}\}$. However, invoking (6.2) again,

$$\gamma(v_{2i+k_0}) = u_{(i+\frac{k_0-1}{2})(k_0+k_1)+k_0}, \text{ while } \gamma(v_{2i-k_1}) = u_{(i-\frac{k_1+1}{2})(k_0+k_1)+k_0}.$$

This gives us the congruences:

$$(i + \frac{k_0 - 1}{2})(k_0 + k_1) + k_1 \equiv i(k_0 + k_1) \pm 1 \pmod{n},$$

$$(i - \frac{k_1 + 1}{2})(k_0 + k_1) + k_1 \equiv i(k_0 + k_1) \mp 1 \pmod{n},$$

which are equivalent to the system of congruencies:

$$\begin{aligned}
\tfrac{k_0-1}{2}(k_0 + k_1) &+ k_1 \equiv \pm 1 \pmod{n}, \\
-\tfrac{k_1+1}{2}(k_0 + k_1) &+ k_1 \equiv \mp 1 \pmod{n}.
\end{aligned} \tag{6.3}$$

Moreover, applying the same ideas to the vertices v_{2i+1} yields conditions equivalent to the conditions (6.3).

Similarly, one can define an automorphism $\bar{\gamma}$ which maps the outer rim onto the inner rim in the opposite direction. In this case:

$$\begin{aligned}
\bar{\gamma}(u_{2i}) &= v_{-i(k_0+k_1)}, & \bar{\gamma}(v_{2i}) &= u_{-i(k_0+k_1)}, \\
\bar{\gamma}(u_{2i+1}) &= v_{-i(k_0+k_1)-k_1}, & \bar{\gamma}(v_{2i+1}) &= u_{-i(k_0+k_1)-k_1},
\end{aligned} \tag{6.4}$$

for all $i \in \mathbb{Z}_{\frac{n}{2}}$. Inner edges are preserved if and only if the system of congruencies

$$\begin{aligned}
\tfrac{1-k_0}{2}(k_0 + k_1) &- k_1 \equiv \pm 1 \pmod{n}, \\
\tfrac{k_1+1}{2}(k_0 + k_1) &- k_1 \equiv \mp 1 \pmod{n}
\end{aligned} \tag{6.5}$$

is satisfied.

Note that k_0, k_1 that satisfy the system (6.3) also necessarily satisfy the congruence

$$(k_0 + k_1)^2 \equiv \pm 4 \pmod{n}, \tag{6.6}$$

and parameters k_0, k_1 that satisfy (6.5) also satisfy

$$(k_0 + k_1)^2 \equiv \mp 4 \pmod{n}. \tag{6.7}$$

Recall that a necessary and sufficient condition for a vertex-transitivity of the generalised Petersen graphs $GP(n, k)$ is $k^2 \equiv \pm 1 \pmod{n}$ (except for $GP(10, 2)$)[9], which implies the congruence $(k + k)^2 \equiv \pm 4 \pmod{n}$. In this sense, the conditions (6.6) and (6.7) are generalisations of the well-known characterization of vertex-transitive generalised Petersen graphs.
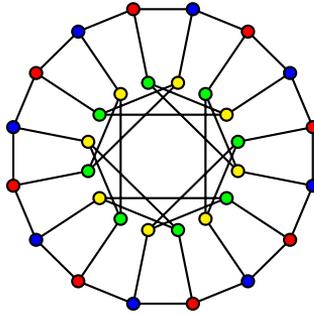
We have proved the following:

Figure 15: Graph $GDGP_2(16; 3, 11)$ which admits the automorphism γ.

Lemma 6.1. *The graph $GDGP_2(n; k_0, k_1)$ admits an automorphism that preserves the set of spokes and swaps the outer and inner vertices if and only if the order $o_n(k_0 + k_1) = \frac{n}{2}$ in $\mathbb{Z}_n$ and the parameters n, k_0, k_1 satisfy one of the systems of congruencies (6.3) or (6.5).*

Proof. We have proved in detail that at least one of the conditions is necessary. Their sufficiency follows from the fact that the vertex map of $GDGP_2(n; k_0, k_1)$ whose parameters satisfy (6.3) and which is defined by equations (6.1) and (6.2) fixes the spokes and swaps the outer and inner edges. The same holds true for the vertex map whose parameters satisfy (6.5) and which is defined via (6.4). □

It is easy to observe that the parameters of a fixed $GDGP_2(n; k_0, k_1)$ can satisfy at most one of the systems (6.3) or (6.5). Therefore, graphs $GDGP_2(n; k_0, k_1)$ admit at most one of the automorphisms γ or $\bar{\gamma}$.

The following theorem provides the sufficient condition promised at the beginning of the section. Its proof follows from Lemma 5.3 and Lemma 6.1.

Theorem 6.2. *Let $GDGP_2(n; k_0, k_1)$ be a graph whose parameters satisfy $o_n(k_0 + k_1) = \frac{n}{2}$ and one of the systems (6.3) or (6.5). Then $GDGP_2(n; k_0, k_1)$ is a vertex-transitive graph.*

Example 6.3. The parameters of the graphs $GDGP_2(n; a, n - a + 2)$ satisfy the condition $o_n(a + n - a + 2) = o_n(2) = \frac{n}{2}$ as well as the system of congruencies (6.3). Hence, they all admit the automorphism γ defined by formulas (6.1) and (6.2).

Example 6.4. The parameters of the graphs $GDGP_2(n; 1, n - 3)$ satisfy neither of the systems (6.3) or (6.5). They are nevertheless vertex-transitive and their inner edges form a single cycle. Thus, conditions (6.3) or (6.5) are sufficient but not necessary.

Example 6.5. The thesis [11] contains yet another family of graphs whose parameters do not satisfy (6.3) or (6.5), but nevertheless includes vertex-transitive graphs. These are the graphs $GDGP_2(8a + 4; 1, 4a - 1)$, with a being a positive integer. The inner rim of these graphs does not form a single cycle. The smallest graph in this family is the graph $GDGP_2(12; 1, 3)$ isomorphic to the truncated octahedral graph. It is known that the truncated octahedral graph is the Cayley graph $Cay(G, X)$, where $G = S_4$ and $X = \{(1234), (1432), (12)\}$, and the group of automorphisms has order 48. Another member of the family is the graph $GDGP_2(20; 1, 7)$.
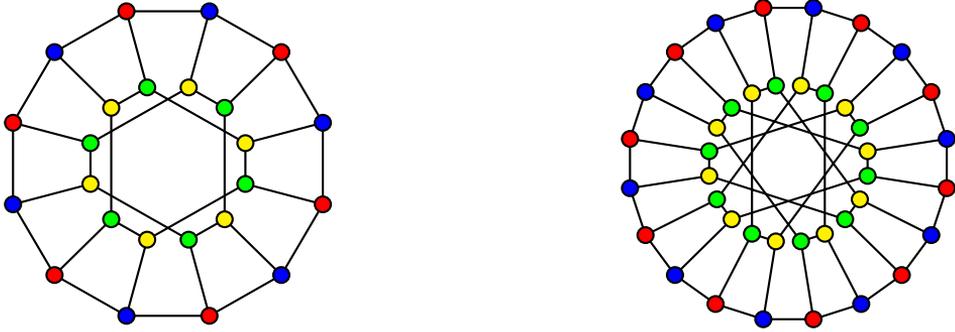
Figure 16: Graphs $GDGP_2(12; 1, 3)$ and $GDGP_2(20; 1, 7)$.

Example 6.6. Based on computational evidence, there are other families of vertex-transitive $GDGP_2$-graphs with parameters that do not satisfy (6.3) or (6.5). In the thesis [12], relying on exhaustive search of all $GDGP_2(n; k_0, k_1)$ with $n \leq 300$, six more graphs have been found whose parameters do not satisfy (6.3) or (6.5) but are vertex-transitive. These are the graphs

$$GDGP_2(96, 21, 49), GDGP_2(96, 27, 47), GDGP_2(192, 45, 97),$$

$$GDGP_2(192, 51, 95), GDGP_2(288, 69, 145), GDGP_2(288, 75, 143).$$

Since all their orders are multiples of 96, their existence suggests a possible infinite family.

Once again referring to the characterization of vertex-transitive generalised Petersen graphs, we observe $GP(n, k)$ is a Cayley graph if and only if $k^2 \equiv 1 \pmod{n}$, and thus vertex-transitive generalised Petersen graphs whose parameters satisfy the congruence relation $k^2 \equiv -1 \pmod{n}$ are not Cayley [9]. We show that this is not the case for the graphs considered in this section. Namely, we show that all graphs $GDGP_2(n; k_0, k_1)$ whose parameters satisfy (6.3) or (6.5) are Cayley graphs. We will somewhat abbreviate our arguments. Detailed proofs of the claims made in this part can be found in [11].

Let $GDGP_2(n; k_0, k_1)$ be a graph admitting one of the automorphisms γ or $\bar{\gamma}$, α be the automorphism from Theorem 4.4, and β be the automorphism of $GDGP_2(n; k_0, k_1)$ that maps u_i to u_{1-i} and v_i to v_{1-i}, $0 \leq i \leq n - 1$. The groups $G_\Sigma = \langle \alpha, \beta, \gamma \rangle$ and $\bar{G}_\Sigma = \langle \alpha, \beta, \bar{\gamma} \rangle$ are subgroups of $\mathrm{Aut}(GDGP_2(n; k_0, k_1))$ that preserve the set of spokes. It is easy to verify that if $(k_0 + k_1)^2 \equiv 4 \pmod{n}$, then

$$G_\Sigma = \langle \alpha, \beta, \gamma \,|\, \alpha^{\frac{n}{2}} = \beta^2 = \gamma^2 = 1, \ \beta\alpha\beta = \alpha^{-1}, \ \gamma\alpha\gamma = \alpha^{\frac{k_0+k_1}{2}}, \ \beta\gamma = \gamma\beta\alpha^{\frac{k_0-1}{2}} \rangle,$$

and

$$\bar{G}_\Sigma = \langle \alpha, \beta, \bar{\gamma} \,|\, \alpha^{\frac{n}{2}} = \beta^2 = \bar{\gamma}^2 = 1, \ \beta\alpha\beta = \alpha^{-1}, \ \bar{\gamma}\alpha\bar{\gamma} = \alpha^{-\frac{k_0+k_1}{2}}, \ \beta\bar{\gamma} = \bar{\gamma}\beta\alpha^{-\frac{k_1+1}{2}} \rangle.$$

On the other hand, if $(k_0 + k_1)^2 \equiv -4 \pmod{n}$, then $GDGP_2(n; k_0, k_1)$ is isomorphic to the generalised Petersen graph $GP(n, k_0)$.

Theorem 6.7. *The following statements are true for all $GDGP_2(n; k_0, k_1)$:*

1. *If γ is an automorphism of the graph $GDGP_2(n; k_0, k_1)$, then $GDGP_2(n; k_0, k_1)$ is isomorphic to the Cayley graph $Cay(G_\Sigma, \{\beta, \alpha\beta, \gamma\})$.*

2. *If $\bar\gamma$ is an automorphism of the graph $GDGP_2(n; k_0, k_1)$, then $GDGP_2(n; k_0, k_1)$ is isomorphic to the Cayley graph $Cay(\bar G_\Sigma, \{\beta, \alpha\beta, \bar\gamma\})$.*

Proof. Leaving out the technical details, we claim that the map

$$\varphi : GDGP_2(n; k_0, k_1) \longrightarrow Cay(G_\Sigma, \{\beta, \alpha\beta, \gamma\}),$$

defined on the vertices of $GDGP_2(n; k_0, k_1)$ via the formulas

$$\begin{aligned}
u_{2i} &\mapsto \alpha^i, & u_{2i+1} &\mapsto \beta\alpha^i, \\
v_{2i} &\mapsto \gamma\alpha^i, & v_{2i+1} &\mapsto \gamma\beta\alpha^i,
\end{aligned}$$

is an isomorphism between $GDGP_2(n; K)$ and $Cay(G_\Sigma, \{\beta, \alpha\beta, \gamma\})$.

Similarly, the map

$$\bar\varphi : GDGP_2(n; k_0, k_1) \longrightarrow Cay(\bar G_\Sigma, \{\beta, \alpha\beta, \bar\gamma\}),$$

defined via

$$\begin{aligned}
u_{2i} &\mapsto \alpha^i, & u_{2i+1} &\mapsto \beta\alpha^i, \\
v_{2i} &\mapsto \bar\gamma\alpha^i, & v_{2i+1} &\mapsto \bar\gamma\beta\alpha^i,
\end{aligned}$$

is an isomorphism between the graphs $GDGP_2(n; k_0, k_1)$ and $Cay(\bar G_\Sigma, \{\beta, \alpha\beta, \bar\gamma\})$. $\square$

7 Symmetric $GDGP_2$-graphs

One of the main goals of our paper is to determine which of the $GDGP$-graphs are highly symmetric. In the previous section, we have presented sufficient conditions for $GDGP_2$-graphs being vertex-transitive. For the rest of our paper, we are going to consider even a higher level of symmetry, namely, we are going to address the question which $GDGP_2$-graphs are symmetric (arc-transitive), i.e., which $GDGP_2$-graphs possess enough automorphisms to map any arc of the graph to any other arc.

We have already established in Theorem 4.4 that all $GDGP_2(n; k_0, k_1)$ are tetracirculants. Since all cubic symmetric tetracirculants have been classified by Frelih and Kutnar in [8], in order to classify the symmetric $GDGP_2$-graphs (which are cubic tetracirculants), it is enough to determine which of the cubic symmetric tetracirculants listed in [8] are $GDGP_2$-graphs. Since the symmetric graphs in [8] are described in the form of the lifts, we shall achieve this goal by viewing the $GDGP_2$-graphs as lifts as well.

Recall that generalised Petersen graphs are the lifts of the dumbbell graphs from Figure 3. Note also that the dumbbell graph may be viewed as mono-gonal prism (which makes the generalised Petersen graphs bicirculant). Since the $GDGP_2$-graphs are tetracirculants, in order to view them as lifts, we need to consider base graphs of order 4. Consider the voltage graph in Figure 17 which is a di-gonal prism. In both Figures 3 and 17, the voltages along one basis cycle add up to 1. Since the voltages on the other edges must be integers, both k_0 and k_1 in Figure 17 must be odd. If we recall that the definition of the $GDGP_2(n; k_0, k_1)$-graphs also requires that the parameters k_0 and k_1 be odd, it is not hard to see that every $GDGP_2(n; k_0, k_1)$ is isomorphic to the lift described in Figure 17.

Furthermore, the voltage graph in Figure 17 is a rather special case of the more general voltage graph of Figure 7. As is well-known, the voltages along any spanning tree of the
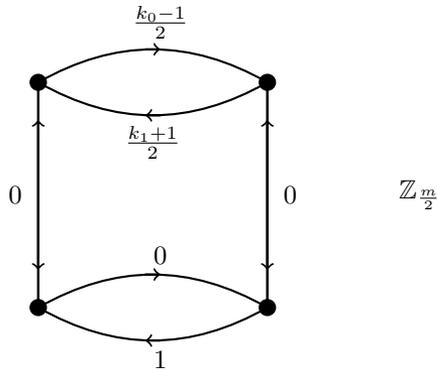
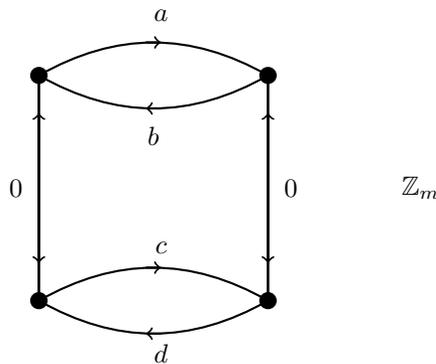Figure 17: Voltage graph for $GDGP_2(m; k_0, k_1)$.

Figure 18: Voltage graph for the tetracirculant graph $C_1(m; a, b, c, d)$.

base graph may be chosen to be equal to 0, and still, after appropriate changes of the other voltages, produce the same graph. Hence, c may always be chosen to be 0. Such graphs have been studied before, for instance in [1], under the name of C-*graphs*. However, the spanning tree used in [1] differs from our choice here. Nevertheless, each $GDGP_2$ graph is a C-graph, while the converse is not true.

Let us now continue with the task of classifying symmetric $GDGP_2$-graphs. As mentioned above, the paper [8] contains a complete classification of cubic symmetric (i.e. arc-transitive) tetracirculants.

Theorem 7.1 ([8, Theorem 1.1])**.** *A connected cubic symmetric graph is a tetracirculant if and only if it is isomorphic to one of the following graphs:*

  (i) $F008A, F020A, F020B, F024A, F028A, F032A, F040A$,

 (ii) $F016A, F048A, F056C, F060A, F080A, F096A, F112B, F120B, F224C,$
      $F240C$,

(iii) $CQ(t, m)$ *for* $2 \leq t \leq m - 3$ *satisfying* $m | (t^2 + t + 1)$,
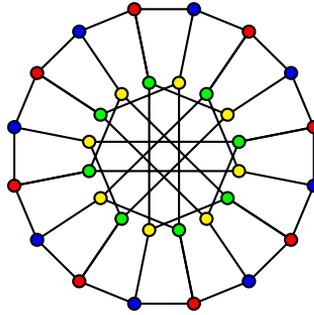
Figure 19: The Dyck graph $GDGP_2(16; 3, 7)$.

(iv) $CQ(2t - 1, 2m)$ *for* $2 \le t \le m - 1$ *satisfying* $m|(4t^2 - 2t + 1)$.

The notation FnA, FnB, etc., refers to the corresponding graphs in the Foster census [2], [3]. The graphs $CQ(t, m)$ are the lifts from Figures 8 and 9 introduced in [7].

Since all $GDGP_2$-graphs are bipartite (Lemma 4.5), we can easily rule out the tetracirculants that are not bipartite, such as $GP(5, 2)$ or $GP(10, 2)$. Graphs $F008A$, $F016A$, $F020B$, $F024A$, and $F048A$ are the generalised Petersen graphs $GP(4, 1)$, $GP(8, 3)$, $GP(10, 3)$, $GP(12, 5)$, and $GP(24, 5)$, respectively. Therefore, these graphs are not isomorphic to a non-periodic $GDGP_2(n; k_0, k_1)$. Furthermore, we checked all the sporadic cases in (i) and (ii) by our program in *SAGE-math*. That showed that the two graphs $F040A$ and $F080A$ are also not isomorphic to any non-periodic $GDGP_2(n; k_0, k_1)$.

Finally, the graph $F032A$ is isomorphic to $GDGP_2(16; 3, 7)$. It is also known under the name of the *Dyck graph*; Figure 19.

Summing up the above observations yields the following.

Theorem 7.2. *The only symmetric* $GDGP_2(n; k_0, k_1)$ *graph not isomorphic to a generalised Petersen graph* $GP(n, k)$ *or one of the graphs* $CQ(t, m)$, $2 \le t \le m - 3$, $m|(t^2 + t + 1)$ *or* $CQ(2t - 1, 2m)$, $2 \le t \le m - 1$, $m|(4t^2 - 2t + 1)$, *is the Dyck graph* $GDGP_2(16; 3, 7) = F032A$.

Note that our computer program indicates that the only arc-transitive C-graph that is not a $GDGP_2$ graph is the graph $F040A$. It can alternately be described as an SI_2-graph, i.e., a further generalisation in which one allows for spans other than 1 in both rims; see Figure 20.

Our computer experiments also indicate the following:

Conjecture 7.3.

1. *The girth of* $CQ(t, m)$, *m odd*, $\gcd(t, m) = 1$, *is equal to 6. For m even, the girth may be* 6, 8 *or* 10.

2. *Every* $CQ(t, m)$-*graph is a* $GDGP_2$ *graph. Every graph* $CQ(t, m)$ *with* $\gcd(t, m) = 1$ *is vertex transitive.*

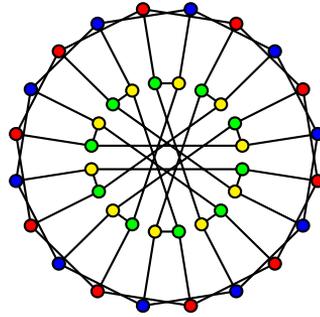We close our paper with two **open questions**:

Figure 20: F040A as $SI_2(10; 2, 2, 1, 11)$.

1. *Which of the graphs $CQ(t, m)$, $2 \leq t \leq m-3$, $m|(t^2+t+1)$, and $CQ(2t-1, 2m)$, $2 \leq t \leq m-1$, $m|(4t^2 - 2t + 1)$, are isomorphic to a $GDGP_2(n; k_0, k_1)$?*

2. *Which of the graphs $GDGP_m(n; K)$, $m > 2$, are symmetric?*

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