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The Geometry of Venice: Topographical Observation and Statistical Appraisal

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Abstract

Having observed that ancient Venice belfries are located in such a way that they generate many Pythagorean triangles, having a great number of vertices in common, it has been decided to test the null hypothesis of random location by statistical and probabilistic methods. A simple index, called Pythagorean Ratio, is proposed, for checking which triangles are to be considered as Pythagorean. Then, a Monte Carlo simulation is performed, generating samples of “random belfries” in the historical kernel of Venice; a Poisson model seems to fit very well the number X of Pythagorean triangles. Combining this number with the number of connections, the null hypothesis is rejected. Adding a further belfry (S.Simeon Grande) to the original group of belfries, the significance becomes even higher.

1 Introduction

Ordering one’s own environment has many advantages: it lowers the cost of protection and improves quality of life. With this aim, the most important points of observation and transmission of visual information are the tops of topographical data, such as towers and belfries. The history of Venice gives a strong evidence of it. Just from the top of S.Marco belfry, on August 25th, 1609, the Signoria (chief family) and the same Doge (chief of State) looked through the first Galileo’s telescope. The Venetian senate confirmed the utility of this tool, doubled the scientist’s salary and, moreover, imposed the “top secret” on this business (*Non detur exemplum exordii*) (Tiepolo *et al.* 1985). Right now, in Venice, towers and belfries lack of a clear historic documentation, and as watch towers, have maybe sometimes been built before the corresponding churches.

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A particular working hypothesis, based on ancient topography, economy and statistical assessment, shows us that belfries and ancient watch towers were deliberately positioned at the vertex of particular geometrical forms; this means that belfries were not built at random, but according to precise rules. In this paper a statistical method is proposed, based on simulation, for checking this working hypothesis.

In particular, Luigi Calzavara has written the archaeological and topographical considerations (Chapters 1, 2, 3), while Maurizio Brizzi has performed the statistical analysis (Chapter 4).

2 Some notes about Pythagorean triangles

The belfries of Venice seem to be sited according to an ancient triangulation based on the right-angled triangle. The master builder had a practical method for determining the right angle of a building: he had to take a ring of rope, divide it into twelve *equal* parts, with a knot to mark each section, and then pull it out with a divergent force at the third (3), seventh (3+4) and twelfth (3+4+5) knots, so creating a right-angled triangle. This was a hands-on version of the first Pythagorean Triplet (3,4,5). In fact this method will always produce a shape reflecting the well known Pythagorean equation: $3^2+4^2=5^2$. This ends the master builder's work with the first of an infinite sequence of Pythagorean triples, a system already well known in ancient Babylon and documented by Tablet 322 of the Plimpton collection at Columbia University. In this tablet, dating about 1500 B.C., written in cuneiform alphabet, 15 Pythagorean triplets are reported. These triplets allow us to build Pythagorean triangles with a minimum angle width from 31°05' and 44°46', and are useful to topographical tracing.

At this point, the topographer's experience begins. This time, however, rings of rope or wooden rods were no longer used to build a right-angle but to trace the vertex points of a specific right-angled triangle. Each time, depending on the city's particular building requirements, the most suitable ring had to be devised. The most suitable ring links the triplet of Pythagorean numbers, and hence the theoretical shape, with the most suitable measurement of the cord segment i.e. with the distance between the vertex points. In this way a bi-univocal correspondence was established between specific numerical values of the infinite number of Pythagorean triples and the infinite points on the plane. Measuring instruments are generally bivalent. They are employed in building procedure but also when verifying what already built.

In this particular work, we have been studying Pythagorean triangles during forty years, in order to determine the different phases of building development in the city of Venice, and to evaluate, even statistically, the concrete evidence of a building process.

3 Archaeological findings in ancient Venice

Venice was initially built as a group of pile dwellings on marshland. Church belfries had to rest on toilsome and expensive foundations, which were sometimes more expensive than the belfry itself: marshland does not allow shifting. For economic reasons therefore, belfries can be considered topographical reference points down through the centuries. Indeed it could be said that marshland induces immobility. Furthermore, every belfry has a precise position on the Cartesian grid of the official Italian cartographic system (Salzano, 1991). Consider, as a good example, S.Marco belfry, which collapsed in 1902, but was immediately rebuilt in the same location. For simplicity's sake, the survey has been limited to 50 belfries linked to churches founded by the XI century (Dorigo, 1987). Any research into the Pythagorean triangles linking Venice belfries must obviously focus on the tolerance of right angles. Choosing a broad tolerance interval would increase the number of Pythagorean triangles but would preclude statistical significance. As a result, working on a trial and error basis, we have established an optimal tolerance of $\pm$ 0.125° (1/8 of a degree). Using this value, 61 Pythagorean triangles have been identified which most importantly, link some 48 of the 50 bell-towers in the survey. The result is a geometrical mesh of triangles, each positioned as a function of the other. In this way, the bell-towers do generate a *system*, because the topographical information passes from one to the other without interruption, as on an electrical circuit. On the basis of these findings, some perfectly measurable topographical archaeology data were prepared and the results analysed statistically. The first survey considers 19 belfries connected to churches in central Venice and founded by the IX century. This system is based on four intermeshing Pythagorean triangles where several vertex points converge. For example, triangles ABC; ABD; BEC; BEF (Figure 1) have vertex B and point C in common.

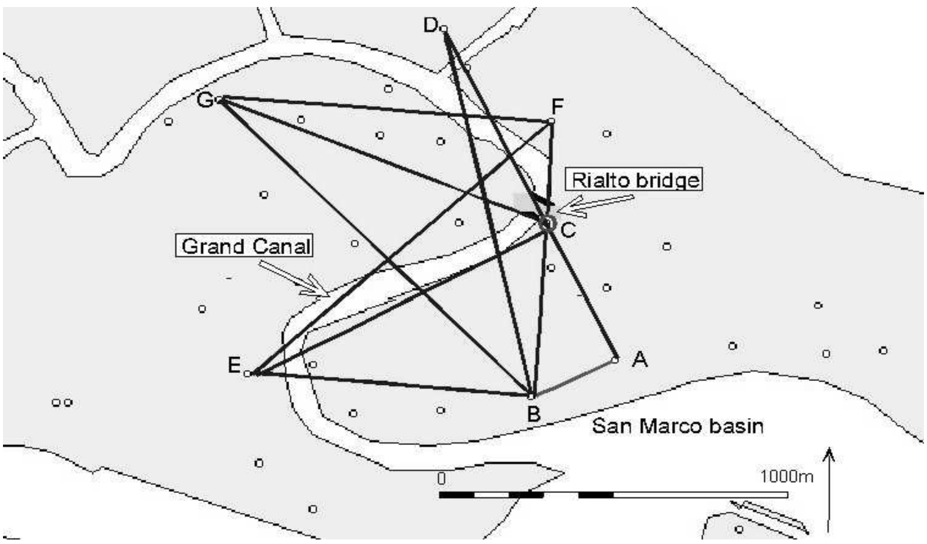


Figure 1: Map of the kernel belfries in the center of Venice.

Table 1: Some features (century, name, coordinates) of the main group of belfries.

#	century	CHURCH / BELFRY	COORDINATES (metres)	
A	9-th	S. MARCO	2 311 870.31	5 034 632.76
B	8-th	S. MOISE'	2 311 636.50	5 034 520.87
C	9-th	S. BORTOLOMIO	2 311 674.35	5 035 041.07
D	9-th	S. FOSCA	2 311 386.34	5 035 640.50
E	9-th	S. BARNABA	2 310 820.37	5 034 581.36
F	9-th	SS. APOSTOLI	2 311 697.18	5 035 348.14
G	10-th	S. SIMEON GRANDE	2 310 743.82	5 035 419.33

In Figure 1, Triangles ABC, ABD, BEC, BEF, BFG and CFG are topographical reference points in the old town. Point C corresponds to the belfry of San Bortolomio at Rialto bridge, right in the historic centre of the city. The second topographic system studied looked at the inclusion of the new belfry of San Simeon Grande – putatively dating from the *X* century – which became the vertex G of two new right-angled triangles (BFG and CFG); G is also the western boundary of the town. In this way, triangle after triangle we have been able to detect a topographical mesh linking all fifty of the belfries studied. The system appears the fruit of human design: a precise road map for Venice foundation.

4 Statistical simulation and evaluation

4.1 The original kernel

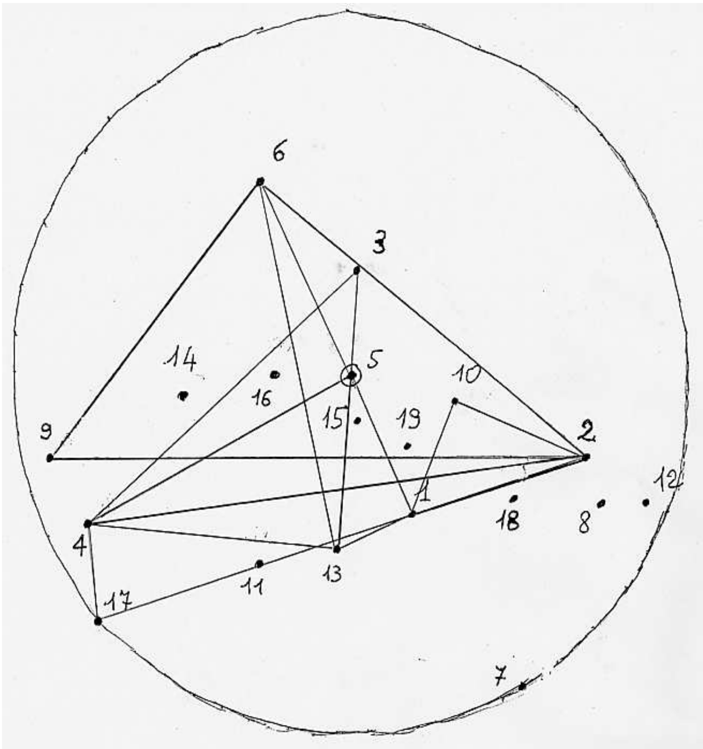
The topographical evidence described above has been submitted to a statistical appraisal, essentially based on a Monte Carlo simulation, in order to detect whether these repeated occurrence of Pythagorean triangles, and the close network of connections between them, may be attributed merely to random factors or not.

We considered an “old kernel” of nineteen belfries, connected to very ancient churches, all built before 1000 A.D., which are approximately spread over a circle, having its center in S.Bortolomio, one of the oldest religious complexes of the city. These belfries, taken three at the time, become the vertices of $C_{19,3} = \binom{19}{3} = 969$ different triangles. Using the topographic coordinates, we calculated, for each triangle, the side length, as well as two simple indices denoted with PR (Pythagorean Ratio) and MR (Maximum Ratio). Denoting with *a, b, c* the ordered side lengths of a triangle (such that $a > b > c$), we define:

$$\text{PR} = \frac{b^2 + c^2}{a^2}, \quad \text{MR} = \frac{a}{c} \tag{4.1}$$

Due to Pythagoras' theorem, PR is equal to 1 if (and only if) the main angle is right. Actually, PR may be near to 1 even when one side is very short, and this is evident if we consider that $\lim_{c \to 0} \frac{b^2 + c^2}{a^2} = 1$. Then, if MR is too large, we don't consider the resulting triangle as a possible Pythagorean one. We took into account both PR and MR values, calling hereafter "Almost Pythagorean Triangle" (APT) a triangle having $0.98 < PR < 1.02$ and $MR < 6.5$

The last inequality means, approximately, that the narrowest angle width is, at least, equal to 9° (i.e. one tenth of a right angle). With this group of belfries, we can detect seven APT's, out of 969 possible triangles (approximately 0.72%), graphically represented in Figure 2.



Belfry labels: 1 = S.Marco, 2 = S.Antonino, 3 = Apostoli, 4 = S.Barnaba, 5 = S.Bortolomio, 6 = S.Fosca, 7 = S.Giorgio Maggiore, 8 = S.Giovanni Bragora, 9 = S.Margherita, 10 = S.Maria Formosa, 11 = S.Maria Giglio, 12 = S.Martino, 13 = Moisè, 14 = S.Polo, 15 = Salvador, 16 = S.Silvestro, 17 = Trovaso, 18 = S.Zaccaria, 19 = S.Zulian.

Figure 2: Network map of 19 belfries and their Almost Pythagorean Triangles.

Let *X* be the number of APT's generated by 19 vertices (belfries). We have observed *X=7* (actually the APT's were 8, but one was dropped out, having a MR value much larger than the limit value of 6.5; we tried to check if such result was significant, defining the null and alternative hypothesis:

H_0 = "the belfries of Venice were built at random"

H_1 = "there is an underlying geometric pattern"

and making a Monte Carlo simulation. We generated, in each replication, 19 random points in a circle and counting the resulting number of APT's. We can simulate this way the distribution of *X* under the null hypothesis of random location of the points. The resulting distribution of *X* after 3,000 replications is shown in Table 2.

Table 2: Simulated distribution of X (19 belfries) and comparison with Poisson model.

X_i	n_i	f_i	F_i	$F°_i$	$F°_i$
0	81	2.70%	2.70%	2.93%	*2.93%*
1	309	10.30%	13.00%	10.34%	*13.27%*
2	574	19.13%	32.13%	18.26%	*31.53%*
3	645	21.50%	53.63%	21.48%	*53.01%*
4	535	17.84%	71.47%	18.96%	*71.97%*
5	403	13.43%	84.90%	13.39%	*85.36%*
6	254	8.47%	93.47%	7.88%	*93.24%*
7	**121**	**4.03%**	**97.50%**	**3.97%**	**97.21%**
8	46	1.53%	98.93%	1.75%	*98.96%*
9	19	0.64%	99.67%	0.69%	*99.65%*
≥10	13	0.43%	100.00%	0.35%	*100.00%*

Note – f_j and F_j are, respectively, the simple and cumulative observed frequencies; $f°_j$ and $F°_j$ are the corresponding theoretical frequencies, under a Poisson model with $\lambda=3.53$**.** The observed value and its statistical features are written in **bold**.

The empirical simulated mean value is $\bar{x} = 3.530$, and the variance is almost equal, being $V_X = 3.534$; we easily realised that Poisson random variable, with $\lambda = \bar{x} = 3.53$, fits almost perfectly the distribution of X. The last two columns show the theoretical values of p.d.f. and c.d.f. under a Poisson model; these values are all very near to "empirical" ones, and the Kolmogorov distance, i.e. the maximum distance between empirical and theoretical c.d.f. has a very low value ($D_K = 0.62\%$)**.** The very good fit of Poisson model seems to give reliability to these empirical results.

We checked the significance of $X=7$ under the null hypothesis of random location, by using simulated frequencies as probabilities for our test. Looking at Table 2, we find:

$P(X > 7 | H_0) = 0.026$ (empirical) $P(X \geq 7 | H_0) = 0.066$ (empirical)

0.028 (Poisson) 0.068 (Poisson)

Since the variable is discrete, we have two different ways of interpretation: the result is significant (at 5% level) if we exclude the value $X=7$ from the critical region, and is not if we include it. We could have used a randomisation procedure to split the probability $P(X=7)$ in two parts, "within" and "outside" the critical region; actually, we preferred to use a more "concrete" decision rule, by involving connections between triangles.

Indeed we have observed that these APT's are strictly connected, having several common vertices. We took into account the total number of connections (denoted C_T) in the whole group of seven APT's and the minimum number of connections (denoted C') of each triangle; we realised that:

a) there are several connections ($C_T = 14$; notice that the maximum is 21),
b) each triangle is connected at least with other three ($C' = 3$).

These results seemed immediately very interesting, and we confirmed this first impression by performing another simulation. Starting with 19 points (labelled from 1 to 19), we generated 7 random triangles (groups of three distinct points), computing each value of C_T and C'. We reported the joint distribution in Table 3:

Table 3: Simulated distribution of C_T (total number of connections) and C' (minimum number of connections) with 7 triangles.

C_T =	***C'=0***	***1***	***2***	***3***	***4-5***	
≤ 5	2.73	1.87	0.00	0.00	0.00	*4.60*
6 – 7	9.60	14.40	0.13	0.00	0.00	*24.13*
8 – 9	7.74	22.87	4.23	0.00	0.00	*34.84*
10 – 11	2.80	11.24	9.63	0.07	0.00	*23.74*
12 – 13	0.33	2.70	5.10	1.00	0.00	*9.13*
14 – 15	0.13	0.43	1.30	**0.87**	**0.00**	*2.73*
16 – 20	0.00	0.03	0.23	**0.40**	**0.17**	*0.83*
	23.33	*53.54*	*20.62*	*2.34*	*0.17*	*100.00*

In Table 3, we have evidenced in **bold** the frequencies of values (C_T, C') which are greater or equal to observed ones ($C_T=14$, $C'=3$). Considering the joint simulated frequencies as probabilities of a test of significance, we have:

$P(C_T \geq 14, C' \geq 3 \mid H_0) = 0.0144 = 1.44\%$

Considering now simultaneously X, C_T and C', we derive the probability of having, due to mere random effects, a resulting number of APT's which is:

1. larger than observed ($X>7$)
2. equal to observation ($X=7$), but generating a number of connections greater or equal to the observed one ($C_T \geq 14$, $C' \geq 3$).

Empirically simulated values:

$$P(X>7 \mid H_0) + P(X=7, C_T \geq 14, C' \geq 3 \mid H_0) = 0.0260 + 0.0403 \cdot 0.0144 =$$
$$= 0.0260 + 0.0006 = 0.0266 = 2.66\%$$

Theoretical values (Poisson model):

$$P(X>7 \mid H_0) + P(X=7, C_T \geq 14, C' \geq 3 \mid H_0) = 0.0279 + 0.0397 \cdot 0.0144 =$$
$$= 0.0279 + 0.0006 = 0.0285 = 2.85\%$$

This result, based on the number of APT's and connections, has to be considered much more significant than the mere value of *X*, and gives us sufficient elements for rejecting the null hypothesis. But this conclusion becomes clearer and stronger if we extend the group of belfries by adding a new unit: the bell tower of S.Simeon Grande.

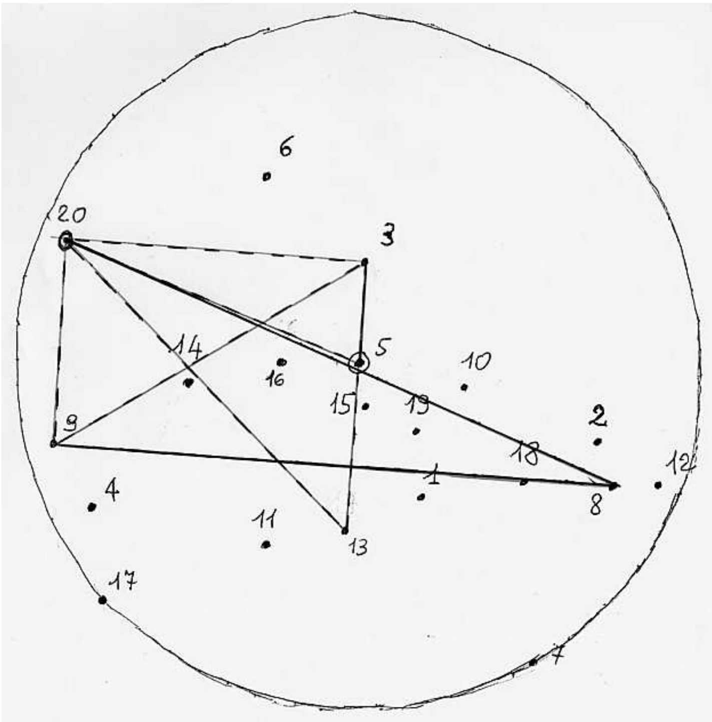


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Figure 3: New connections generated by S.Simeon Grande with the old kernel of belfries.

4.2 The 20th belfry: S. Simeon Grande

The bell tower of S. Simeon Grande was built in the first half of X century, just some year after the previous belfries, and its location, not far from the historical kernel of the city, seems to be not chosen at random. Indeed, if we add this new unit to the kernel group of 19 belfries, the number of APT's increases sensibly, becoming 12 instead of 7. This means that this "new" belfry makes five new right angles with the older ones, as shown in Figure 3.

This result seems very relevant even at a first sight, but we checked it by a further simulation procedure. Now we have $C_{20,3} = 1140$ triangles, and 12 of them (1.05%) have been classified as almost Pythagorean. If we indicate with X^* the number of APT's generated by 20 points, we have an observed value $X^*=12$ and we can repeat the same simulation described before. The main results are shown in Table 4. Considering the enlarged group of 20 belfries, the result seems much more significant than before, being $P(X^*\geq 12) = 0.0014$ (empirical) and $P(X^*\geq 12) = 0.0011$ (theoretical). Poisson model, again, fits the simulated results very well, and the Kolmogorov distance is even lower ($D_K = 0.36\%$). As told before, this almost perfect goodness-of-fit seems to give more reliability to these simulated results.

Table 4: Simulated distribution of X^* (20 belfries) and comparison with Poisson model.

X^*_i	n_i	f_i	F_i	$F°_i$	$F°_i$
0 - 1	266	8.87%	8.87%	8.52%	8.52%
2	402	13.40%	22.27%	14.00%	22.52%
3	575	19.17%	41.44%	19.09%	41.61%
4	583	19.43%	60.87%	19.52%	61.13%
5	481	16.03%	76.90%	15.97%	77.10%
6	323	10.77%	87.67%	10.88%	87.98%
7	208	6.93%	94.60%	6.36%	94.34%
8	91	3.03%	97.63%	3.25%	97.59%
9	43	1.43%	99.06%	1.48%	99.07%
10	20	0.67%	99.73%	0.60%	99.67%
11	4	0.13%	99.86%	0.22%	99.89%
12	**2**	**0.07%**	**99.93%**	**0.08%**	**99.97%**
13+	2	0.07%	100.00%	0.03%	100.00%

Note – f_j and F_j are, respectively, the simple and cumulative observed frequencies; $f°_j$ and $F°_j$ are the corresponding theoretical frequencies, under a Poisson model with $\lambda=4.09$. The observed value and its statistical features are written in **bold**.

Moreover, this set of 12 APT's have a impressively relevant network of connections; again, we indicated with a star (*) the results when extended to 20 belfries:

- the total number is $C_T^* = 35$ (the maximum is 66);

- the minimum number is $C'^{*} = 4$ (each triangle of the group is connected at least with four other triangles).

We have then repeated the simulation procedure described above, generating now twelve triangles and counting C_T^{*} and C'^{*}. We derived the joint frequencies reported in Table 5.

Table 5: Simulated joint distribution of C_T^{*} and C'^{*} with twelve triangles (% values).

C_T^{*} =	C'^{*}=0	1	2	3	4	5 +	
< 20	0.16	1.16	0.60	0.00	0.00	0.00	*1.92*
20 - 24	1.44	10.52	15.48	1.64	0.00	0.00	*29.08*
25 - 29	1.76	11.16	22.36	9.80	0.28	0.00	*45.36*
30 - 34	0.44	3.48	6.92	6.80	1.60	0.00	*19.24*
35 - 39	0.12	0.40	0.92	1.68	**0.84**	**0.00**	*3.96*
≥ 40	0.04	0.08	0.08	0.08	**0.08**	**0.08**	*0.44*
	3.96	*26.80*	*46.36*	*20.00*	*2.80*	*0.08*	*100.00*

In Table 5 we indicated, like in Table 3, in **bold** the frequencies of simulated values which are not less than the observed ($C_T^{*} = 35$, $C'^{*} = 4$). Considering the simulated frequencies as probabilities for our test, we can write:

$$P(C_T^{*} \geq 35, C'^{*} \geq 4 \mid H_0) = 0.0100 = 1.00\%$$

Considering, as done before, the variables X (number of APT's), C_T (total number of connections) and C' (minimum number of connections) simultaneously, we derive the probability of having, due to mere random effects, a global result greater or equal to the observed one:

Empirically simulated values:

$$P(X^{*} > 12 \mid H_0) + P(X^{*}=12, C_T^{*} \geq 35, C'^{*} \geq 4 \mid H_0) =$$
$$= 0.0007 + 0.0007 \cdot 0.0100 = 0.0007 + 0.000007 = 0.000707 \cong 0.071\% .$$

Theoretical values (Poisson model):

$$P(X^{*} > 12 \mid H_0) + P(X^{*}=12, C_T^{*} \geq 35, C'^{*} \geq 4 \mid H_0) =$$
$$= 0.0008 + 0.0003 \cdot 0.0100 = 0.0008 + 0.000003 = 0.000803 \cong 0.080\% .$$

This result induces us to reject, at a significance level smaller than 1/1000, and therefore with a dramatically reduced probability of Type-1 error, the null hypothesis of random location of belfries in Venice historical kernel.

5 Concluding remarks

The reported results, especially considering the group of 20 belfries nearest to very ancient Rialto bridge, show a frequency above expectation of Pythagorean triangles and a remarkable network of connections between them. This induces us to reject the null hypothesis of random location versus the hypothesis of existence of a well determined systematic geometric pattern. Actually, as written before, 48 out of 50 belfries, adjacent to ancient churches built before the end of XI century, are connected by a huge network of 61 Pythagorean triangles. Evidently, it would be worthwhile also to understand the motivation of this pattern, thus opening a new line of research.

It is likely that such a geometric pattern, linked to belfries location, gives a better urbanistic efficiency to the city, helping to order and rationalize the topographic network and to improve the channels of oral and visual communication. Evidently, a sound network of acoustic and visual guidelines could give a better sense of security to Serenissima Republic citizens.

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References

[1] Dorigo, W. (1987): *Venezia Origini.* Venezia: Electa.

[2] Salzano, E. (1991): *Atlante di Venezia*. Comune di Venezia e Marsilio.

[3] Tiepolo, M.F. *et al.* (1985): *Ambiente scientifico veneziano tra Cinquecento e Seicento.* Archivio di stato di Venezia.