

Generalized Cayley maps and their Petrie duals*

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Abstract

Cayley maps are embeddings of Cayley graphs in orientable surfaces which possess a group of orientation preserving automorphisms acting regularly on the vertices. We generalize the concept of a Cayley map by considering embeddings of Cayley graphs in both orientable and non-orientable surfaces and by requiring a group of automorphisms acting regularly on vertices that does not have to consist entirely of orientation preserving automorphisms. This leads to new families of maps in both the orientable and non-orientable cases. Since the Petrie dual operator preserves the property of being a generalized Cayley map, throughout the paper we consider the action of this operator on our maps.

Keywords: Cayley map, regular embedding, automorphism group of a map.

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1 Introduction

A *map* is a cellular embedding of a connected graph on some surface; the map is *orientable* if the surface is. By a widely adopted definition (cf. [16]), an orientable map \mathcal{M} is a Cayley map if the group of orientation-preserving automorphisms of \mathcal{M} contains a subgroup acting regularly on the vertex set of \mathcal{M} . This way of introducing Cayley maps has been motivated by Sabidussi's well known characterization of Cayley graphs in terms of a subgroup of automorphisms of the graph acting regularly on its vertex set [18]. Apart from the history of the development of the theory of maps, however, when carrying Sabidussi's characterization over to maps there is no reason to restrict to orientation-preserving automorphisms, or even to orientable surfaces. The latter may have been first realized by Tucker [19], and in this paper we argue for the return of his way of treating Cayley maps.

In order not to clash with customary terminology we will say that a map \mathcal{M} on an arbitrary surface (orientable or not) is a *generalized Cayley map* if the automorphism group of \mathcal{M} contains a subgroup H that is a regular permutation group on the vertex set of \mathcal{M} .

In our view, Tucker's definition by means of existence of a regular action of a group of automorphisms has two major advantages: it is a generalization of the currently used definition that is simple, and in line with the original definition of a Cayley map. To the best of our knowledge there have been two other attempts at defining non-orientable Cayley maps: the one by Abas [1] via strategically placed cross-cups on the carrier surface, and another one by Kwak and Kwon [12] who developed a full theory of generalized Cayley maps (equivalent to Tucker's and our approach) but starting off from introducing generalized Cayley maps in a way that appears to be hard to use in applications.

The main difference between (original and orientable) Cayley maps and generalized Cayley maps is that the carrier surfaces of the latter may be non-orientable, and even in the orientable case the subgroup required to act regularly on vertices does not have to be orientation-preserving. Maps admitting such subgroups appear frequently in the context of regular maps with automorphism groups acting quasi-primitively on the vertices [6], and our study of such maps was the main source of motivation for the present paper.

At several places, we employ the Petrie dual operator (e.g., [22]) which proves particularly useful since it preserves the class of generalized Cayley maps. For example, we consider orientable generalized Cayley maps whose Petrie dual is again an orientable generalized Cayley map, as well as non-orientable generalized Cayley maps whose Petrie dual is orientable.

The outline of the paper is as follows. In order to emphasize parallels between the original Cayley maps on orientable surfaces and the generalized Cayley maps, the three sections following this Introduction are devoted to reviewing the background and context of highly symmetric maps, Cayley maps, and the Petrie dual operator; with Sections 5 and 6 containing the new results. We begin in Section 2 by reviewing some of the most important properties of regular maps, followed in Section 3 by reviewing algebraic machinery for dealing with orientable Cayley maps in their original setting. Section 4 discusses Petrie duals of maps in some detail. We discuss generalized Cayley maps on orientable surfaces in Section 5 and generalized Cayley maps on non-orientable surfaces in Section 6.

We can summarize the results obtained in our paper as follows. A generalized Cayley map may be both orientable and non-orientable. The orientable generalized Cayley maps come in two kinds. First, there are the original orientable Cayley maps in which the group acting regularly on the vertices of the map consists entirely of orientation preserving automorphisms. Then there are orientable generalized Cayley maps with bipartite underlying

graphs in which the group acting regularly on the vertices of the map contains a subgroup of index 2 consisting of orientation preserving automorphisms while the other half of the regularly acting group consists of orientation reversing automorphisms (Theorems 5.1 and 5.2). There are infinitely many examples of orientable generalized Cayley maps which are not the original Cayley maps (Lemma 5.3 and Example 5.5), and infinitely many orientable generalized Cayley maps which are self-Petrie-dual (Remark 5.6). In Remark 5.7 we present orientable generalized Cayley maps that simultaneously admit an orientation preserving group acting regularly on their vertices and a vertex-regular group half of which consists of orientation reversing automorphisms (hence, these maps are classical Cayley maps and orientable maps of the second kind at the same time).

In Remark 6.2 we present an infinite family of non-orientable generalized Cayley maps. Taking advantage of the well-known fact that the Petrie dual of an orientable map \mathcal{M} is orientable if and only if the underlying graph of \mathcal{M} is bipartite (Theorem 4.2 in our paper), we argue the existence of infinitely many examples of the original orientable Cayley maps whose Petrie duals are orientable (Example 6.6) and infinitely many examples of the original orientable Cayley maps whose Petrie duals are non-orientable (any orientable Cayley map with a non-bipartite underlying graph). The Petrie dual of a non-orientable generalized Cayley map may be both orientable and non-orientable. We classify the non-orientable generalized Cayley maps whose Petrie dual is orientable (Corollary 6.5) and exhibit infinitely many examples of both non-orientable generalized Cayley maps whose Petrie dual is orientable and non-orientable generalized Cayley maps whose Petrie dual is non-orientable (Remark 6.2).

2 Regular, orientably-regular, reflexible and chiral maps

For a map \mathcal{M} , regions (faces) of its barycentric subdivision are the *flags* of the map. Every flag is a triangular region; informally, its three ‘corners’ are a vertex, the ‘midpoint’ of an edge incident with the vertex, and the ‘centre’ of a face incident with both the vertex and the edge. As long as no face of the map contains two occurrences of an edge on its boundary (no maps with such a degeneracy will be considered here), a flag can be identified with a triple (v, e, F) , where v is a vertex, e an edge incident with v , and F is a face incident with both v and e . A pair of distinct flags (v, e, F) and (v', e', F') of \mathcal{M} are *incident* if they share a segment of the skeleton of the barycentric subdivision; in case when no face of the map contains two occurrences of an edge on its boundary, two flags are incident if precisely two of the three equalities $v = v', e = e', F = F'$ hold.

Let r_0, r_1 and r_2 be involutory fixed-point-free permutations of the flag set of a map \mathcal{M} formed by two-cycles consisting of incident flags, with r_0 swapping faces that share a face-center-to-edge-midpoint segment, r_1 swapping faces that share the vertex-to-face-center segment, and r_2 swapping faces that share the vertex-to-edge-midpoint segment; again, if no face of the map contains two occurrences of an edge on its boundary, the flag (v, e, F) is mapped by r_0 to (v', e, f) , $v \neq v'$, by r_1 to (v, e', F) , $e' \neq e$, and by r_2 to (v, e, F') , $F' \neq F$. The (flag-transitive) permutation group generated by r_0, r_1 and r_2 is the *monodromy group* $\text{Mon}(\mathcal{M})$ of the map. The three generators of the monodromy group can be interpreted as ‘gluing instructions’ to assemble the map from its set of flags; in other words, knowledge of $\text{Mon}(\mathcal{M})$ in terms of the action of the generators r_0, r_1 and r_2 on the set its flags is equivalent to knowing the map \mathcal{M} . Note that r_0 and r_2 commute; if the map is finite, then the orders of $r_0 r_1$ and $r_1 r_2$ are equal to the least common multiples of face

lengths and vertex valencies, respectively.

It is well known that the carrier surface of \mathcal{M} is orientable if and only if $\langle r_1 r_2, r_0 r_2 \rangle = \langle r_1 r_2, r_0 r_1 \rangle$ is a subgroup of index 2 in $\text{Mon}(\mathcal{M})$. In such a case we speak about an orientable map; letting $\rho = r_1 r_2$ and $\lambda = r_0 r_2$ one may consider the ‘orientable part’ $\text{Mon}^+(\mathcal{M}) = \langle \rho, \lambda \rangle$, the index-2-subgroup of $\text{Mon}(\mathcal{M})$. Here, $\text{Mon}^+(\mathcal{M})$ can be regarded as a permutation group acting on the *dart set* \mathcal{D} of the map (i.e., on the set of directed edges of the map). Then ρ is a permutation that cyclically permutes, at each vertex v , the darts emanating from v in accord with a chosen orientation of the carrier surface of the map, and λ is an involution interchanging the two darts belonging to the same edge.

An *automorphism* of a map \mathcal{M} with monodromy group $\text{Mon}(\mathcal{M}) = \langle r_0, r_1, r_2 \rangle$ is any permutation of the flag set \mathcal{F} of \mathcal{M} that preserves incidence of flags. Equivalently, a permutation of \mathcal{F} is an automorphism of \mathcal{M} if and only if it commutes with all of r_0, r_1 and r_2 ; hence the *automorphism group* $\text{Aut}(\mathcal{M})$ of the map is simply the centralizer of the monodromy group $\text{Mon}(\mathcal{M})$ in the symmetric group $S_{\mathcal{F}}$ on the set \mathcal{F} . Since $\text{Mon}(\mathcal{M})$ is transitive on \mathcal{F} , it follows that $\text{Aut}(\mathcal{M})$ acts freely on the set \mathcal{F} .

For an orientable analogue, let now \mathcal{M} be an orientable map with dart set \mathcal{D} and with $\text{Mon}^+(\mathcal{M}) = \langle \rho, \lambda \rangle$ being a subgroup of $\text{Mon}(\mathcal{M})$ of index 2. An *orientation-preserving* automorphism of \mathcal{M} is any permutation of \mathcal{D} commuting with ρ and λ . It follows that the group $\text{Aut}^+(\mathcal{M})$ of all orientation-preserving automorphisms of an orientable map is the centralizer of $\text{Mon}^+(\mathcal{M})$ in the symmetric group $S_{\mathcal{D}}$ on the set \mathcal{D} . Again, transitivity of $\text{Mon}^+(\mathcal{M})$ on \mathcal{D} implies that the group $\text{Aut}^+(\mathcal{M})$ acts freely on the set \mathcal{D} . If such an orientable map \mathcal{M} admits an automorphism commuting with λ but inverting ρ , then the automorphism is *orientation-reversing*.

Finally, let us discuss the highest ‘level of symmetry’ of maps. A map \mathcal{M} with monodromy group $\text{Mon}(\mathcal{M})$ is called *regular* if the automorphism group \mathcal{M} acts regularly on the set \mathcal{F} of flags. In this case, the groups $\text{Aut}(\mathcal{M})$ and $\text{Mon}(\mathcal{M})$ are abstractly isomorphic, so that in the case of a finite map the order of both groups is equal to $|\mathcal{F}|$. If \mathcal{M} is a regular *orientable* map, then its automorphism group $\text{Aut}(\mathcal{M})$ contains the group $\text{Aut}^+(\mathcal{M})$ of orientation-preserving automorphisms as a subgroup of index 2, its other coset in $\text{Aut}(\mathcal{M})$ being the collection of all orientation-reversing automorphisms of the map. The orientation-reversing automorphisms are sometimes referred to as *reflections*, giving such maps the name *reflexible*. An orientable map \mathcal{M} is *orientably-regular* if the group $\text{Aut}^+(\mathcal{M})$ of orientation preserving automorphisms of \mathcal{M} acts regularly on its set of darts. An orientably regular map that is not reflexible is *chiral*, and in such a case $\text{Aut}(\mathcal{M}) = \text{Aut}^+(\mathcal{M})$.

3 Cayley maps

A *Cayley map* is an orientable map \mathcal{M} that admits a group of orientation preserving automorphisms G acting regularly on its set of vertices. To make this definition more precise, one needs to consider the induced action of map automorphisms of \mathcal{M} on the vertex set of the underlying graph associating the map automorphism ψ with the vertex permutation $\bar{\psi}$ mapping v to v' whenever $\psi(v, e, F) = (v', e', F')$. It is easy to see that $\bar{\psi}$ is a well defined graph automorphism of the underlying graph of \mathcal{M} defined via its action on the vertices (of the map or the graph). The induced action of $\text{Aut}(\mathcal{M})$ on the vertices is almost universally faithful; with the very few exceptions listed in [14, Proposition 9]. The automorphisms contained in the automorphism group of a Cayley map which acts regularly on

the vertices of the map correspond in this way to automorphisms of its underlying graph forming a group acting regularly on the graph's vertices, making it into a Cayley graph [18]. A *Cayley graph* $C(G, X)$ is a graph whose vertex set can be identified with the elements of a group G generated by a set X closed under taking inverses and not containing the identity 1_G , with the pairs of adjacent vertices consisting of all pairs g, gx with $g \in G$ and $x \in X$. A graph Γ is isomorphic to a Cayley graph $C(G, X)$ if and only if $\text{Aut}\Gamma$ contains a subgroup G acting regularly on the vertices of Γ [18], which justifies the remark we have made about the underlying graphs of Cayley maps being Cayley. Hence, the automorphism group of a Cayley map \mathcal{M} containing a group G of orientation preserving automorphisms acting regularly on the vertices of \mathcal{M} is necessarily contained in the automorphism group of its underlying Cayley graph $C(G, X)$, $G \leq \text{Aut}C(G, X)$ (after applying the necessary restriction to vertices). The action of G on $C(G, X)$ is defined for each $g \in G$ via left multiplication: $A_g(h) = gh$ for all $g, h \in H$, and we shall denote the group $\{A_g \mid g \in G\}$ by G_L ; it is always a subgroup of $\text{Aut}C(G, X)$. The set of darts of the Cayley map \mathcal{M} thus takes the form $\{(h, hx) \mid h \in G, x \in X\}$ and each $A_g \in G_L$ 'extends' to a permutation of the darts of \mathcal{M} mapping the dart (h, hx) to (gh, ghx) . If each of the extensions of A_g , $g \in G$, is to preserve the orientation of \mathcal{M} , there must exist a cyclic permutation p of X having the property that the dart $(g, gp(x))$ is always the dart lying immediately next to the dart (g, gx) in the local surface neighborhood of g (sharing the face with (g, gx)), for all $g \in G$ and $x \in X$ [16]. Thus, we can talk about the *local permutation* of the darts (g, gx) , $x \in X$, emanating from g , determined by the cyclic permutation p of X . This gives rise to an equivalent definition of a Cayley map as an orientable embedding of a Cayley graph $C(G, X)$ in an orientable surface determined by the local rotation scheme with the property that the local surface ordering corresponding to the vertex $g \in G$, ρ_g , acting cyclically on the darts (g, x) , $x \in X$, does not depend on g , and is determined by a fixed cyclic permutation p of X . Such Cayley map is denoted by $CM(G, X, p)$, and the equivalence of this 'local rotation definition' and the definition via the existence of an orientation preserving automorphism group acting regularly on its vertices is a well established fact in the theory of Cayley maps [16].

As argued above, the group $G_L = \{A_g \mid g \in G\}$ (isomorphic to G) is always a subgroup of the group $\text{Aut}^+CM(G, X, p)$ of orientation preserving automorphisms, and $CM(G, X, p)$ is orientably regular if and only if there exists an orientation preserving automorphism $A \in \text{Aut}^+CM(G, X, p)$ which fixes the identity 1_G and maps the darts emanating from 1_G in the same order as p : $A((1_G, x)) = (1_G, p(x))$, for all $x \in X$. This has been shown to be equivalent to the existence a special identity-fixing permutation φ called skew-morphism in [7], where it was first defined. Given a group G , a permutation $\varphi: G \rightarrow G$ of the elements of G that fixes the identity of G , $\varphi(1_G) = 1_G$, is said to be a *skew-morphism* of G with associated *power function* $\pi: G \rightarrow \mathbb{N}$ if the equation

$$\varphi(gh) = \varphi(g)\varphi^{\pi(g)}(h) \quad (3.1)$$

is satisfied for all $g, h \in G$. A Cayley map $CM(G, X, p)$ is orientably regular if and only if there exists a skew-morphism φ of G with the property $\varphi(x) = p(x)$, for all $x \in X$. Furthermore, a group G admits the existence of an orientably regular Cayley map $CM(G, X, p)$ if and only if G admits the existence of a skew-morphism φ with an orbit X that generates G and is closed under taking inverses (in which case, the desired orientably regular map is the map $CM(G, X, \varphi|_X)$) [7]. The classification of finite groups G that admit the existence of an orientably regular Cayley map $CM(G, X, p)$ is a hard

problem and the topic of many articles. The orientation preserving automorphism group of an orientably regular Cayley map with skew-morphism φ and power function π takes the form $\text{Aut}^+(CM(G, X, p)) \cong G_L \langle \varphi \rangle$ with the product multiplication defined by the rule

$$a\varphi = \varphi^{\pi(a)}\varphi(a),$$

for all $a \in G$. A Cayley map $CM(G, X, p)$ with the property $p(x^{-1}) = (p(x))^{-1}$ satisfied by all $x \in X$ is called a *balanced Cayley map*. Since a map $CM(G, X, p)$ is balanced if and only G_L is normal in $\text{Aut}^+(CM(G, X, p))$, we will break with the long line of tradition, and we will call such map a *normal Cayley map*. This is in line with the name given to Cayley graphs $C(G, X)$ with G_L normal in $\text{Aut}C(G, X)$.

Let us complete this section with a necessary condition for a Cayley map to be regular. Even though it can be deduced from [5] and [7], we are not aware of this condition being stated explicitly before, and so we also provide a short proof.

Theorem 3.1. *If a Cayley map $\mathcal{M} = CM(G, X, p)$ is regular, there exists a pair of skew-morphisms φ, ψ of G that preserve X and the restriction of φ to X is equal to p , while the restriction of ψ to X is equal to p^{-1} .*

Proof. If $\mathcal{M} = CM(G, X, p)$ is regular, it is also orientably regular, hence there exists a skew-morphism φ of G that preserves X and whose restriction to X is equal to p [7]. Moreover, if \mathcal{M} is regular, it is isomorphic to its mirror reflection $CM(G, X, p^{-1})$, which is therefore also regular, hence orientably regular, and there exists a skew-morphism ψ of G that preserves X and whose restriction to X is equal to p^{-1} . \square

It is interesting to point out that the above necessary condition is not sufficient. For example, the skew-morphism whose existence guarantees the orientable regularity of a normal Cayley map $CM(G, X, p)$ is well-known to be a group automorphism of G [21]. The inverse of a group automorphism is always a group automorphism, and hence a skew-morphism. Thus, every orientably regular normal Cayley map admits a skew-morphism whose restriction to X is p and whose inverse is a skew-morphism (and its restriction to X is p^{-1}). However, not every normal orientably regular Cayley map is regular. To mention just one famous example, all orientably regular embeddings of complete graphs have been shown to be normal Cayley maps, however, the only orientably regular embeddings of complete graphs which are also regular are the orientable embeddings of complete graphs of prime power order not exceeding 4 [8].

4 Petrie dual

The well-known duality operation switching the roles of vertices and faces of a map \mathcal{M} preserves some of the most important topological characteristics of \mathcal{M} , such as the orientability and the genus, but generally changes the underlying graph of \mathcal{M} . The less well-known Petrie duality has the advantage of preserving both the embedded graph and the action of $\text{Aut}(\mathcal{M})$ on it. This makes the Petrie dual operation more useful when dealing with maps with automorphism groups acting regularly on vertices of the underlying graph.

The *Petrie dual* of a map \mathcal{M} is the map $P(\mathcal{M})$ with the same vertices and edges as \mathcal{M} (and thus, the same underlying graph), however, the faces of $P(\mathcal{M})$ are determined by the *Petrie* (or zig-zag) *walks* of \mathcal{M} which visit vertices of \mathcal{M} along the edges while switching sides (i.e., faces) in the middle of every next edge situated along the boundary of the face

of \mathcal{M} . The key point of our interest in the Petrie dual lies in the fact that the two maps \mathcal{M} and $P(\mathcal{M})$ have the same vertex set and the same automorphism group. To put this claim on a more precise footing, we turn again to the monodromy groups. If $\langle r_0, r_1, r_2 \rangle$ is the monodromy group of a map \mathcal{M} , the monodromy group of the Petrie dual $P(\mathcal{M})$ is the group $\langle r_0 r_2, r_1, r_2 \rangle$ (which can be easily seen from the fact that the two flags associated with the opposite dart get swapped in the Petrie dual construction). Thus, $P(P(\mathcal{M})) = \mathcal{M}$, and, since $\langle r_0, r_1, r_2 \rangle = \langle r_0 r_2, r_1, r_2 \rangle$, we obtain

$$\text{Aut}(\mathcal{M}) = C_{\mathbb{S}_{\mathcal{F}}}(\langle r_0, r_1, r_2 \rangle) = C_{\mathbb{S}_{\mathcal{F}}}(\langle r_0 r_2, r_1, r_2 \rangle) = \text{Aut}P(\mathcal{M}).$$

Therefore, the two automorphism groups act on the same set of flags in exactly the same way, and any properties of the automorphism group of \mathcal{M} with regard to the vertices of \mathcal{M} are shared by the automorphism group of $P(\mathcal{M})$. In particular, if $\text{Aut}(\mathcal{M})$ contains a subgroup acting regularly on the set of vertices of \mathcal{M} , so does the group $\text{Aut}P(\mathcal{M})$. Since we make repeated use of this observation throughout our paper, we state it in the form of a theorem.

Theorem 4.1. *A map \mathcal{M} is a generalized Cayley map if and only if its Petrie dual $P(\mathcal{M})$ is also a generalized Cayley map.*

Even though the Petrie dual pair shares with the original map the same underlying graph and the same automorphism group, the two maps have usually quite distinct topological properties. It is, for example, possible (even common) that \mathcal{M} is orientable while $P(\mathcal{M})$ need not be. For an example, consider \mathcal{M} to be the tetrahedron as an embedding of K_4 on the sphere. Then $P(\mathcal{M})$ is an embedding of K_4 in the projective plane as a map of type $\{4, 3\}$.

Since we will have to be able to distinguish between various situations with regard to the orientability vs. non-orientability, we recall the following well-known result (e.g., [15, Remark 7]).

Theorem 4.2. *If \mathcal{M} is orientable, then $P(\mathcal{M})$ is orientable if and only if the underlying graph of \mathcal{M} is bipartite.*

Next, recall the concept of a *ribbon graph* which is constructed from a map \mathcal{M} by cutting out an open neighborhood of each face-center and keeping small bands around the edges of \mathcal{M} and small circles around the vertices. The ribbon graph of the Petrie dual of an orientable embedding of a bipartite graph is the ribbon graph of the original embedding with all bands twisted (cut off at one of the end vertices and glued back after being rotated by 180 degrees) [4].

Remark 4.3. There is a large number of examples of Cayley maps throughout the literature. For example, four of the five Platonic solids (all but the dodecahedron) are Cayley maps. All orientably regular embeddings of complete graphs are also Cayley maps [8]. Since none of the complete graphs but K_2 are bipartite, their Petrie duals are non-orientable generalized Cayley maps.

On the other hand, the underlying graph of the dodecahedron is not a Cayley graph, and hence the dodecahedron and its Petrie dual are neither Cayley maps nor generalized Cayley maps.

5 Orientable generalized Cayley maps and their Petrie duals

Let us begin by pointing out that the underlying graph of *any* generalized Cayley map (orientable or not) is a Cayley graph. This observation follows from the same line of arguments as that included in our section on Cayley maps, namely, it follows from the fact that the group of automorphisms of the underlying graph induced by the group G of map automorphisms acting regularly on the vertices of the map acts regularly on the vertices of the graph. Thus, as is well-known, the set of vertices of the underlying graph of a generalized Cayley map can be identified with the elements of G , and the action of these automorphisms on the elements of G is that of left-multiplication in G .

It follows that generalized Cayley maps \mathcal{M} are precisely embeddings of Cayley graphs $C(G, X)$ into (orientable or non-orientable) surfaces satisfying the property that all left multiplications in G viewed as permutations of the elements of G ‘extend’ into automorphisms of the embedding, and that the order of the automorphism group of \mathcal{M} acting regularly on the vertices of \mathcal{M} must be equal to its number of vertices, i.e., must be equal to $|G|$.

Thus, orientable generalized Cayley maps come in two kinds. First, there are the embeddings \mathcal{M} of Cayley graphs $C(G, X)$ in orientable surfaces having the property that all left multiplications in G extend into orientation preserving automorphisms of \mathcal{M} , which are precisely the classical Cayley maps $\mathcal{M} = CM(G, X, p)$ [16]. An orientable generalized Cayley map \mathcal{M} that is not of this kind, i.e., that is not a Cayley map, must then be an embedding of a Cayley graph $C(G, X)$ in an orientable surface having the property that at least one of the left multiplications in G extends to an orientation-reversing automorphism of \mathcal{M} . Recall that any automorphism group of an orientable map that contains at least one orientation reversing automorphism must contain an index 2 subgroup of orientation preserving automorphisms. This means that orientable generalized Cayley maps which are not Cayley maps are embeddings of Cayley graphs $C(G, X)$ in which a subgroup H_L of G_L of index 2 extends into orientation preserving automorphisms and the other coset of this subgroup in G_L extends into orientation reversing automorphisms.

Recall that an orientable embedding of $C(G, X)$ is determined by choosing a cyclic permutation ρ_g of the elements in X for each $g \in G$. In the case of Cayley maps $CM(G, X, p)$, for every $g \in G$, the permutation ρ_g of the elements of X must be equal to p . This, of course, cannot be the case for orientable generalized Cayley maps which are not Cayley maps, i.e., there must exist elements $f, g \in G$ such that $\rho_f \neq \rho_g$. However, the distribution of local rotations in orientable generalized Cayley maps which are not Cayley maps is only slightly more complicated than that of Cayley maps. Namely, all such maps are still determined by a single cyclic permutation p of X which becomes the local rotation for the vertices contained in a subgroup H of index 2 in G and whose inverse p^{-1} becomes the local rotation for the rest of them. We shall denote these maps by $GCM(G, H, X, p)$ and note that each such map is an orientable embedding of a Cayley graph $C(G, X)$ having the properties that H is a subgroup of index 2 in G , p is a cyclic permutation of X , and the embedding of $C(G, X)$ is determined by the local rotations $\rho_h = p$, for all $h \in H$, and $\rho_k = p^{-1}$, for all $k \in G \setminus H$.

Using the notation introduced above, we classify in the following theorem orientable generalized Cayley maps in terms of their local orientations.

Theorem 5.1. *Let \mathcal{M} be an orientable generalized Cayley map. Then \mathcal{M} is either a Cayley map, $\mathcal{M} = CM(G, X, p)$, or a map $\mathcal{M} = GCM(G, H, X, p)$ defined above.*

Conversely, every $CM(G, H, X, p)$, i.e., every orientable embedding of a Cayley graph $C(G, X)$ having the property that $\rho_h = p$, for all $h \in H$, where H is a subgroup of index 2 in G and p is a cyclic permutation of X , and $\rho_k = p^{-1}$, for all $k \in G \setminus H$, is an orientable generalized Cayley map.

Finally, the maps $GCM(G, H, X, p)$ and $GCM(G, H, X, p^{-1})$ are isomorphic.

Proof. Let \mathcal{M} be an orientable generalized Cayley map that is not a Cayley map, let $G \leq \text{Aut}(\mathcal{M})$ act regularly on the vertices of \mathcal{M} , and identify the vertices of \mathcal{M} with the elements of G . Let H be the subgroup of index 2 in G of automorphisms preserving the orientation in \mathcal{M} whose existence has been argued prior to the statement of this theorem.

First, let h be an element of H . This means that the associated map automorphism A_h mapping the darts (g, x) , $g \in G$, $x \in X$, to the darts (hg, x) preserves the orientation in \mathcal{M} . It follows that $A_h \rho(g, x) = \rho A_h(g, x)$, for all $g \in G$ and $x \in X$, and thus $(hg, \rho_g(x)) = (hg, \rho_{hg}(x))$, for all $g \in G$ and $x \in X$. Since multiplication by h preserves H and its coset, the above identities yield that $\rho_h = \rho_{h'}$, for all $h, h' \in H$, and $\rho_k = \rho_{k'}$, for all $k, k' \in G \setminus H$. Taking next an element $k \in G \setminus H$, its corresponding automorphism A_k reverses the orientation in \mathcal{M} , and hence satisfies the identity $A_k \rho = \rho^{-1} A_k$. Thus, $(kg, \rho_g(x)) = (kg, \rho_{kg}^{-1}(x))$, for all $g \in G$ and $x \in X$. It follows that $\rho_g = \rho_{kg}^{-1}$, for all $g \in G$. If we denote the cyclic permutation of X assigned to the elements of H by p , using the fact that the multiplication by k swaps H and its coset yields that the elements in $G \setminus H$ are assigned the local permutation p^{-1} . This completes the proof of the first statement of our theorem.

The proof of the second part of the theorem is essentially the reverse of the above. If one chooses the local orientations as described in the theorem, it is easy to see that left multiplication by the elements of H preserves and left multiplication by the elements from $G \setminus H$ reverses the orientation of the obtained map.

The veracity of the final statement of our theorem can be verified by showing that the mapping φ_k defined on the set of darts of $GCM(G, H, X, p)$ by mapping the dart (g, gx) to the dart (kg, kgx) in $GCM(G, H, X, p^{-1})$, for any fixed $k \in G \setminus H$ and all $g \in G$ and $x \in X$, is a map isomorphism between the maps $GCM(G, H, X, p)$ and $GCM(G, H, X, p^{-1})$. This relatively simple task is left to the reader. \square

To obtain examples of orientable generalized Cayley maps that are not Cayley maps, one needs to consider embeddings of Cayley graphs of even order. To obtain an ‘easy’ example, one could consider the generalized Cayley map $GCM(G, H, X, p)$ for the groups $G = \mathbb{Z}_{2n}$, $n \geq 1$, $H = \langle 2 \rangle$, $X = G \setminus \{0\}$, and any cyclic permutation p of X . All of these maps are embeddings of the complete graphs K_{2n} in orientable surfaces. However, it is easy to see that the cases $n = 1$ and $n = 2$ result in Cayley maps, and even for $n \geq 3$, it is hard to show that the resulting maps are not Cayley. Thus, to construct an infinite family of orientable generalized Cayley maps which are provably not Cayley maps, one might rely on the Petrie dual operator again. Specifically, in what follows, we shall consider orientable generalized Cayley maps whose Petrie dual is orientable again. Because of Theorem 4.2, each such map must be bipartite (which is also a sufficient condition). Thus, we obtain the following:

Theorem 5.2. *Let \mathcal{M} be an orientable generalized Cayley map. The Petrie dual of \mathcal{M} is orientable if and only if $\mathcal{M} = CM(G, X, p)$ is a Cayley map for a group G that contains*

a subgroup H of index 2 for which $X \subseteq G \setminus H$ or if $\mathcal{M} = GCM(G, H, X, p)$ where H is a subgroup of G of index 2 for which $X \subseteq G \setminus H$.

Furthermore, if the Petrie dual of an orientable generalized Cayley map \mathcal{M} is orientable, the two maps $CM(G, X, p)$ and $GCM(G, H, X, p)$ are mutual Petrie duals.

Proof. In order for a Cayley graph $C(G, X)$ to be bipartite, it is easy to see (and well known) that it must be a Cayley graph of an even order group G that possesses a subgroup H of index 2, and X must be a subset of the non-trivial coset of H in G . The rest of the first part of the theorem follows from Theorem 5.1.

Suppose now that $\mathcal{M} = CM(G, X, p)$ with X satisfying the required condition with respect to a subgroup H of G . It is again not too hard to see that the Petrie dual to \mathcal{M} reverses the order of elements from X for each element of H (or for each element of $G \setminus H$; the two maps $GCM(G, H, X, p)$ and $GCM(G, H, X, p^{-1})$ are isomorphic as argued in Theorem 5.1). \square

Clearly, an orientable generalized Cayley map $\mathcal{M} = GCM(G, H, X, p)$ is not simultaneously a Cayley map if and only if the group $\text{Aut}(\mathcal{M})$ contains no orientation preserving subgroup acting regularly on the vertices of \mathcal{M} . One way to make sure no such group exists, is to find a pair of vertices u, v of \mathcal{M} for which there is no orientation preserving automorphism mapping u to v . This is the approach we take to construct an infinite family of orientable generalized Cayley maps which are not Cayley maps.

Lemma 5.3. *Let G be a finite group with a subgroup H of index 2 and a generating set $X \subseteq G \setminus H$. If $CM(G, X, p)$ is chiral, its Petrie dual $GCM(G, H, X, p)$ is a generalized Cayley map that is not a Cayley map.*

Proof. Recall that an orientably regular map is chiral if it admits no orientation reversing automorphisms. Note also, that since $CM(G, X, p)$ is bipartite and connected, any automorphism $\varphi \in \text{Aut}CM(G, X, p)$ that maps a vertex $u \in H$ to a vertex $v \in G \setminus H$ maps all elements of H onto the elements of $G \setminus H$. This means that all the automorphisms in $\text{Aut}GCM(G, H, X, p)$ mapping elements of H to elements in $G \setminus H$ are orientation reversing, and no orientation preserving automorphisms of $GCM(G, H, X, p)$ map the elements of H to the elements in $G \setminus H$. It follows that $GCM(G, H, X, p)$ is not a Cayley map. \square

Remark 5.4. There are many examples of chiral bipartite Cayley maps. One good source of such maps is the paper [10] the authors of which construct infinitely many orientably regular embeddings of $K_{n,n}$, with $n = p^e$ where p is an odd prime and $e \geq 1$, which are Cayley maps for cyclic and dihedral groups and are chiral. Lemma 5.3 yields that all their Petrie duals are generalized Cayley maps that are not Cayley maps. A simpler family of chiral bipartite Cayley maps, brought to our attention by one of our referees, is also formed by the torus maps $\{4, 4\}_{b,c}$, for b, c both non-zero, $b + c$ even, and $b \neq c$ to guarantee chirality, and the torus maps $\{6, 3\}_{b,c}$. The chiral bipartite $\{4, 4\}_{b,c}$ maps have orders $b^2 + c^2$ and arise by identifying opposite sides of squares with corners $(0, 0)$, (b, c) , $(b - c, b + c)$ and $(-c, b)$ in a unit rectangular grid.

Example 5.5. Consider the dihedral groups

$$\mathbb{D}_n = \langle a, b \mid a^n = b^2 = 1, bab^{-1} = a^{-1} \rangle,$$

$n \geq 3$. Taking $G = \mathbb{D}_n$, $H = \langle a \rangle$ and $X = \{b, ba, \dots, ba^{n-1}\}$ yields a pair of mutually Petrie dual generalized Cayley maps $CM(G, X, p)$ and $GCM(G, H, X, p)$, for every cyclic permutation p of X . All such maps are orientable embeddings of the complete bipartite graph $K_{n,n}$. Taking the specific permutation $p = (b, ba, \dots, ba^{n-1})$ yields two (possibly isomorphic) embeddings of $K_{n,n}$. First, the Cayley map

$$CM(\mathbb{D}_n, \{b, ba, \dots, ba^{n-1}\}, (b, ba, \dots, ba^{n-1}))$$

is an orientable embedding of $K_{n,n}$ with n faces of length $2n$ and of genus $\frac{(n-1)(n-2)}{2}$, while its Petrie dual

$$GCM(\mathbb{D}_n, \langle a \rangle, \{b, ba, \dots, ba^{n-1}\}, (b, ba, \dots, ba^{n-1}))$$

is an orientable embedding of $K_{n,n}$ with the same number of faces of the same length and of the same genus.

The excluded case $n = 2$ results in an embedding of $K_{2,2}$ in the sphere of the form

$$CM(\mathbb{Z}_2^2, \{(0, 1), (1, 1)\}, ((0, 1), (1, 1)))$$

which is its own Petrie dual because the inverse of the permutation $((0, 1), (1, 1))$ is $((0, 1), (1, 1))$ again.

The toroidal maps for case $n = 3$ are pictured below.



Figure 1: Maps $CM(\mathbb{D}_3, \{b, ba, ba^2\}, (b, ba, ba^2))$ and $GCM(\mathbb{D}_3, \langle a \rangle, \{b, ba, ba^2\}, (b, ba, ba^2))$.

Clearly, the two maps pictured in Figure 1 are isomorphic. Hence, the smallest maps $CM(\mathbb{D}_3, \{b, ba, ba^2\}, (b, ba, ba^2))$ and $CM(\mathbb{Z}_2^2, \{(0, 1), (1, 1)\}, ((0, 1), (1, 1)))$ are self-Petrie-dual. This might come as a surprise, as the same permutations of vertices (coming from left multiplications) cannot simultaneously extend into orientation preserving and orientation reversing automorphisms. It is, however, easy to notice that relabeling the vertices of $CM(\mathbb{D}_3, \{b, ba, ba^2\}, (b, ba, ba^2))$ changes the permutation actions of left-multiplications, and hence left multiplications with respect to one labeling may extend to orientation preserving while left multiplications with respect to another labeling may extend to orientation reversing automorphisms. Since in non-bipartite cases the Petrie dual of an orientable map is non-orientable, the case of bipartite orientable maps is the only case where one may encounter orientable self-Petrie-dual generalized Cayley maps.

We devote the rest of this section to the investigation of such possibility, i.e., to the study of orientable self-Petrie-dual generalized Cayley maps. Namely, suppose an orientable generalized Cayley map \mathcal{M} is self-Petrie-dual. Then, \mathcal{M} is bipartite and Theorem 5.1 implies that the automorphism group of \mathcal{M} contains two subgroups acting regularly on the vertices, a subgroup of orientation preserving automorphisms as well as a subgroup of automorphisms half of which is orientation reversing. This yields a necessary condition for \mathcal{M} being self-Petrie-dual, namely, \mathcal{M} must be a Cayley map admitting at least one orientation reversing automorphism. Thus, chiral bipartite Cayley maps are never self-Petrie-dual (in Example 5.4, we have already encountered infinitely many such maps constructed in [10]). In fact, it is well-known that no chiral maps are self-Petrie-dual.

These observations appear to suggest that in order to construct bipartite Cayley maps that are self-Petrie-dual, one should consider maps with many orientation preserving and many orientation reversing automorphisms. In fact, searching through the literature for orientable self-Petrie-dual maps resulted only in regular orientable self-Petrie-dual maps [2, 11, 17], with the methods used in [11] or [17] relying on regularity and producing large maps of large genera. An infinite family of regular examples also arises as follows.

Remark 5.6. For every positive integer $n \geq 2$ that is relatively prime to $\varphi(n)$ (i.e., there exists a unique group of order n), there exists exactly one orientably regular embedding of $K_{n,n}$ [9]. That means that such embedding is necessarily isomorphic to its mirror reflection, hence regular, as well as self-Petrie dual. Kwak and Kwon in [13] classified the orientable regular self-Petrie embeddings of $K_{n,n}$.

Nevertheless, even in the extreme case when \mathcal{M} is regular, it might happen that *any* orientation preserving subgroup $K \leq \text{Aut}(\mathcal{M})$ of order half the number of vertices of \mathcal{M} paired with *any* orientation reversing automorphism $\psi \in \text{Aut}(\mathcal{M})$ generate a group of order larger than the number of vertices of \mathcal{M} . In that case, $\text{Aut}(\mathcal{M})$ does not contain a subgroup of automorphisms containing an orientation reversing automorphism and acting regularly on the vertices. Hence, \mathcal{M} cannot be isomorphic to any $GCM(G, H, X, p)$, and hence cannot be self-Petrie-dual. To illustrate this possibility, in the following example we present an infinite family of bipartite regular Cayley maps none of which are self-Petrie-dual. We are thankful to Gareth Jones who pointed out this example to us.

Remark 5.7. The authors of [3] considered regular embeddings of $K_{n,n}$ having the property that the orientation preserving automorphism group of the embedding that does not move the bipartite sets is not metacyclic; in which case n is necessarily a power of 2. For each such $n \geq 8$, they construct four non-isomorphic regular orientable embeddings of $K_{n,n}$, denoted $\mathcal{N}(n; k, l)$, $k, l \in \{0, 1\}$. They prove that the maps $\mathcal{N}(n; 0, 0)$ and $\mathcal{N}(n; 1, 0)$ are self-Petrie-dual, while $\mathcal{N}(n; 0, 1)$ and $\mathcal{N}(n; 1, 1)$ are Petrie duals of each other. They also show that all four maps are Cayley maps. This means, in particular, that each of the maps $\mathcal{N}(n; 0, 1)$ and $\mathcal{N}(n; 1, 1)$ is both a Cayley map and an orientable generalized Cayley map, and as such, they both admit both an orientation preserving as well as an orientation reversing automorphism group acting regularly on the vertices of the maps.

Hence, not even the existence of both types of vertex-regular groups is sufficient for making a generalized Cayley map self-Petrie. It is hard to say whether these non-metacyclic maps are in any way typical, and whether a regular bipartite Cayley map is more likely to be self-Petrie-dual or not. The following highly specialized result appears to be of some relevance toward answering this question:

Theorem 5.8 ([17]). *There are no regular, self-dual, self-Petrie-dual, normal Cayley maps with odd vertex degree.*

6 Non-orientable generalized Cayley maps and their Petrie duals

Even though constructing non-orientable generalized Cayley maps is relatively easy – it is enough to consider the Petrie dual of any non-bipartite orientable generalized Cayley map – a more explicit construction of such maps is needed. When considering only the Petrie duals of orientable generalized Cayley maps, one would never encounter a non-orientable generalized Cayley map whose Petrie dual is not orientable. To see that such maps might exist, one just needs to realize that the Petrie dual of a non-orientable generalized Cayley map whose underlying graph is bipartite (we will construct such maps) is necessarily non-orientable. The present section differs from the previous section which was concerned with maps whose groups are transitive on arcs (regular and orientably regular). In this section, we abandon the focus on such highly symmetric generalized Cayley maps and seek only to provide examples of various non-orientable generalized Cayley maps.

To specify an embedding of a graph in a non-orientable surface, one needs to specify the local orientation of outgoing darts around every vertex as well as to specify for each edge whether the two local orientations associated with its end-vertices are consistent or not. More precisely, for any given edge e incident with vertices u and v , let \mathcal{U} be an open neighborhood of e which includes both u and v and all of e , but no other vertices and no other complete edges. Then \mathcal{U} is homeomorphic to an open disk, and is therefore an orientable topological object. By calling the two local orientations associated with the end-vertices of e *consistent*, we mean that the local orientations around u and v within \mathcal{U} are both clockwise or both anti-clockwise. We call them *inconsistent* otherwise. Referring to the ribbon graph associated with the map, the edges with inconsistent end-vertex orientations are sometimes also called *twisted*, as the strip containing the edge e and the vertices u and v is twisted before being attached to the rest of the ribbon graph through the vertices u and v . To indicate whether the end-vertex orientations of an edge are consistent or not, one usually assigns 1 or -1 to the edge, respectively (or, in a visualization of the graph embedding using local rotations, an edge with inconsistent end-vertex orientations is marked by an ‘x’). Perhaps the biggest disadvantage of this description of an embedding of a graph is the fact that one does not have a guarantee that the resulting embedding is indeed non-orientable. For example, the bipartite orientable maps $GCM(G, H, X, p)$ defined in the previous section can also be described using this ‘non-orientable’ description by saying that $GCM(G, H, X, p)$ is the map in which the local rotation at each vertex is equal to p and all edges have inconsistent end-vertex orientations.

As explained in the previous sections, a generalized Cayley map must be an embedding of a Cayley graph $C(G, X)$ with the property that each left multiplication by the elements of G extends into a map automorphism of the embedding. One of the finer points to be dealt with in the previous section was the fact that the extensions might be either orientation preserving or orientation reversing. We do not have two kinds of map automorphisms in non-orientable maps which makes the situation a bit easier.

Let us start by describing general embeddings of Cayley graphs. Let $C(G, X)$ be a Cayley graph, let ρ_g denote the local rotation of the elements of X around g , and let $\iota_{\{g, gx\}} \in \{1, -1\}$ be the label assigned to the edge $\{g, gx\}$. The three defining involu-

tions r_0, r_1, r_2 are then defined as follows:

$$(g, c_{\{g, gx\}}, c_F)^{r_0} = \begin{cases} (gx, c_{\{g, gx\}}, c_F), & \text{if } \iota_{\{g, gx\}} = 1, \\ (gx, c_{\{g, gx\}}, c_{F'}), & \text{if } \iota_{\{g, gx\}} = -1 \end{cases},$$

$$(g, c_{\{g, gx\}}, c_F)^{r_1} = (g, c_{\{g, g\rho_g^{\iota_{\{g, gx\}}}(x)}\}}, c_F), \text{ and } (g, c_{\{g, gx\}}, c_F)^{r_2} = (g, c_{\{g, gx\}}, c_{F'}),$$

where F' is the face adjacent to F across the edge $\{g, gx\}$.

The following characterization of generalized Cayley maps can be already found in both [19] and [12]. As stated above, it covers both the orientable and the non-orientable generalized Cayley maps.

Theorem 6.1 ([12, 19]). *A (orientable or non-orientable) map \mathcal{M} is a generalized Cayley map if and only if it is an embedding of a Cayley graph $C(G, X)$ with all local cyclic permutations $\rho_g, g \in G$, equal to a fixed cyclic permutation p of X , and the twist distribution ι satisfying the property $\iota_{\{g, gx\}} = \iota_{\{g', g'x\}}$, for all $g, g' \in G$ and $x \in X$.*

The simplest interpretation of the last condition is that all edges labelled by the same generator (or its inverse) must be either all simultaneously un-twisted or all twisted, and hence, from now on, we will assume that ι acts on the set $X, \iota: X \rightarrow \{-1, 1\}$, mapping $x \in X$ to $\iota_{\{g, gx\}}$ (thus, in particular, $\iota(x) = \iota(x^{-1})$), for all $x \in X$. As each generalized Cayley map is determined by the four-tuple (G, X, p, ι) , we will denote these maps by $GCM(G, X, p, \iota)$.

The authors of [12] generalized this simplification even further and introduced the following convenient notation.

Let $\mathcal{M} = GCM(G, X, p, \iota)$ be a generalized Cayley map. Without loss of generality, we may assume that $X = \{x_0, x_1, \dots, x_{d-1}\}$, while $p(x_i) = x_{i+1}$, for all $i \in [d] = \{0, 1, 2, \dots, d-1\}$ (with the addition performed modulo d). If we denote the *distribution of inverses* in X by the function $\kappa: [d] \rightarrow [d]$, satisfying the property $(x_i)^{-1} = x_{\kappa(i)}$, and use the fact that ι assigns the same value to all edges arising from right multiplication by x_i to define (with just a hint of abuse of the notation) $\iota: [d] \rightarrow \{-1, 1\}, \iota(i) = \iota_{\{1_g, x_i\}}$ (note that $\iota(\kappa(i)) = \iota(i)$, for all $i \in [d]$), we may associate the flags of \mathcal{M} with the ordered triples from $G \times [d] \times \{-1, 1\}$, with the neighboring flags $(g, c_{(g, gx_i)}, c_F)$ and $(g, c_{(g, gx_i)}, c_{F'})$ corresponding to the triples $(g, i, 1)$ and $(g, i, -1)$. The defining involutions take then the particularly simple form:

$$(g, i, j)^{r_0} = (gx_i, \kappa(i), \iota(i)j), (g, i, j)^{r_1} = (g, i + j, j), \text{ and } (g, i, j)^{r_2} = (g, i, -j).$$

Remark 6.2. It is now easy to construct infinitely many non-orientable generalized Cayley maps. As is well-known (see e.g., [4]), an embedding of a graph determined by local rotations and a twisting function is non-orientable if and only if it contains at least one cycle with an odd number of twisted edges. Hence, for example, it is easy to construct infinitely many non-orientable embeddings of the complete bipartite graphs $K_{n,n}$. To get such an embedding, one can take a bipartite $C(G, X)$ with G containing a subgroup H of index 2, $X = G \setminus H = \{x_0, x_1, \dots, x_{|G|/2-1}\}$, satisfying the only additional condition that $x_0^2 \neq x_2^{-1}x_1^{-1}$, choose any cyclic permutation p of X , twist the edges labeled by x_0 , and observe that the 4-cycle consisting of the vertices $1_G, x_0, x_0x_1, x_0x_1x_2, x_0x_1x_2(x_0x_1x_2)^{-1} = 1_G$ obtained by successive multiplications by x_0, x_1, x_2 and $(x_0x_1x_2)^{-1}$ (which, being an odd product of elements from X , necessarily belongs to X) contains the twisted edge labeled

x_0 exactly once (if $(x_0x_1x_2)^{-1}$ were equal to x_0 , we would get the identity $x_2^{-1}x_1^{-1} = x_0^2$ which we explicitly prohibited). Hence, all such generalized Cayley maps $GCM(G, X, p, \iota)$ are unorientable embeddings of complete bipartite graphs, and there are clearly infinitely many of them. Note also that all such maps, being embeddings of bipartite graphs, have the property that their Petrie dual is non-orientable.

Next, we bring in the Petrie dual operator again and fully describe its action on generalized Cayley maps. The mapping $-\iota$ used in the statement of the following theorem is the twisting function $-\iota(x) = (-1) \cdot \iota(x)$, for all $x \in X$.

Theorem 6.3. *If $\mathcal{M} = GCM(G, X, p, \iota)$, then $P(\mathcal{M}) = GCM(G, X, p, -\iota)$.*

Proof. It is easy to see that $-\iota$ is a correctly defined twisting function of a generalized Cayley map, i.e., $-\iota(x) = -\iota(x^{-1})$, for all $x \in X$. Since the two maps $GCM(G, X, p, -\iota)$ and $P(GCM(G, X, p, \iota))$ have the same underlying graphs, one way to prove our theorem is to show that the Petrie polygons of $GCM(G, X, p, \iota)$ and the faces of the map $GCM(G, X, p, -\iota)$ are identical. This can be shown by performing a careful calculation of the boundaries of the two oriented Petrie polygons of $GCM(G, X, p, \iota)$ that start at (g, i) (which might turn out to be the same polygon visiting (g, i) twice, once followed along p and once followed along p^{-1} , depending on whether we first turn ‘right’ or ‘left’) and the boundaries of the corresponding faces in $GCM(G, X, p, -\iota)$. As pointed out by one of our referees, it can also be deduced from the fact that if r_0, r_1, r_2 are the generators of the monodromy group of $GCM(G, X, p, \iota)$, then r_0r_2, r_1 , and r_2 are the generators for the monodromy group of $P(GCM(G, X, p, \iota))$ (Section 4) while they can also be easily seen to be the generators of the monodromy group of $GCM(G, X, p, -\iota)$. \square

Based on the above theorem, obtaining the Petrie dual for a Cayley map is very simple. Namely, the Petrie dual of a Cayley map $CM(G, X, p)$ is the generalized Cayley map $GCM(G, X, p, \iota)$, where ι twists all the edges of the map, i.e., $\iota(x) = -1$, for every $x \in X$. Since all the edges are twisted, the Petrie dual of a Cayley map \mathcal{M} is non-orientable if and only if \mathcal{M} contains at least one odd cycle (i.e., the underlying graph is non-bipartite). Hence, Theorem 5.2 is a corollary of Theorem 6.3.

Example 6.4. To obtain a specific example of the use of Theorem 6.3, we go back to a classical example of a pair of mutually Petrie dual maps, namely the regular embedding of the tetrahedron on the sphere and its Petrie dual embedded in the projective plane. Since tetrahedron is the Cayley map $CM(\mathbb{Z}_2 \times \mathbb{Z}_2, \{(1, 0), (0, 1), (1, 1)\}, ((1, 0), (0, 1), (1, 1)))$ with 4 triangular faces, its Petrie dual is the generalized Cayley map

$$GCM(\mathbb{Z}_2 \times \mathbb{Z}_2, \{(1, 0), (0, 1), (1, 1)\}, ((1, 0), (0, 1), (1, 1)), \iota),$$

with $\iota(x) = -1$ for all $x \in \{(1, 0), (0, 1), (1, 1)\}$. Since the tetrahedron contains odd cycles (triangles), the dual map is non-orientable, well-known to be an embedding in the projective plane with 3 faces of length 4 (which can also be easily verified via direct calculations in the generalized Cayley map).

This example can be viewed as the first member of an infinite family of examples. For every even $n \geq 4$, let $\mathbb{D}_n = \langle a, b \mid a^n = b^2 = 1, bab = a^{-1} \rangle$ be the dihedral group of order $2n$, $X = \{b, a^{\frac{n}{2}}, ba\}$ be a set of three involutions, and $p = (b, a^{\frac{n}{2}}, ba)$. Since n is assumed even, the Petrie dual of the trivalent Cayley map $CM(\mathbb{D}_n, X, p)$ is non-orientable (as the underlying graph of both maps contains a cycle of length $n + 1$). For even $n \geq 4$,

the Petrie dual of $CM(G, X, p)$ looks ‘like’ the Petrie dual of the tetrahedron: a $2n$ -cycle with inside chords at every other vertex of the cycle, and an outside edge for the rest. All these Petrie duals are embeddings in the projective plane with one face of length $2n$ and all the remaining faces of length 4.

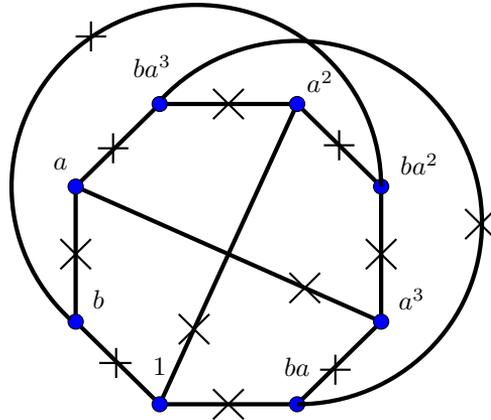


Figure 2: Representation of the Petrie dual of the map $CM(\mathbb{D}_4, \{b, a^2, ba\}, (b, a^2, ba))$ indicating local rotations and marking the edges with inconsistent end-vertex rotations of its underlying graph.

Next, we characterize non-orientable generalized Cayley maps whose Petrie dual is orientable. As should be expected, those are the ‘obvious’ Petrie duals of non-bipartite orientable Cayley maps, i.e., the non-bipartite non-orientable generalized Cayley maps satisfying $\iota(x) = -1$, for all $x \in X$, as well as other maps with a rather special twisting function. The Petrie duals of all other non-orientable generalized Cayley maps are non-orientable.

Corollary 6.5. *The Petrie dual of a non-orientable generalized Cayley map $GCM(G, X, p, \iota)$ is orientable if and only if the underlying Cayley graph $C(G, X)$ contains an odd cycle and every cycle of $C(G, X)$ contains an even number of untwisted edges.*

Proof. Suppose that $\mathcal{M} = GCM(G, X, p, \iota)$ is non-orientable while its Petrie dual $P(\mathcal{M}) = GCM(G, X, p, -\iota)$ is orientable. This means that \mathcal{M} cannot be bipartite as that would make the orientable $P(\mathcal{M})$ bipartite and hence by Theorem 5.2 its Petrie dual $P(P(\mathcal{M})) = \mathcal{M}$ orientable; which it is not. Thus \mathcal{M} contains an odd cycle. If any cycle in \mathcal{M} contained an odd number of untwisted edges, that very same cycle would contain an odd number of twisted edges in $P(\mathcal{M})$; making it non-orientable. This proves one implication.

Suppose now that \mathcal{M} is non-orientable, contains an odd length cycle and every cycle of \mathcal{M} contains an even number of untwisted edges. Then \mathcal{M} is not bipartite and its Petrie dual contains no cycles with an odd number of twisted edges. \square

As stated prior to Theorem 6.3, non-orientable non-bipartite generalized Cayley maps with all edges twisted trivially satisfy the property that their Petrie dual is also non-orientable. Next, we provide an infinite family of examples that satisfy the conditions of Theorem 6.3 but contain both twisted and untwisted edges.

Example 6.6. Let Γ be a bipartite d -regular graph, d -odd, let \mathcal{M} be any orientable embedding of Γ , and ρ and λ be the generators for the monodromy group G of \mathcal{M} . Consider the Cayley graph $C(G, \{\rho, \rho^{-1}, \lambda\})$. There is an easy way to visualize this graph. It is the underlying graph of the *truncation* $T(\mathcal{M})$ of \mathcal{M} , i.e., the map obtained from \mathcal{M} by locally removing its vertices and replacing them by d -cycles attached to the dangling edges in the order determined by the local rotation. We claim that every cycle of $C(G, \{\rho, \rho^{-1}, \lambda\})$ contains an even number of edges labeled λ . To see this, take any cycle C of $C(G, \{\rho, \rho^{-1}, \lambda\})$. The edges labeled λ trace the ‘preimage’ of this cycle in Γ , which is necessarily a union of edge disjoint even cycles. Since every edge labeled λ in C corresponds to exactly one edge of the preimage, the number of edges labeled λ in C is even. Thus, any choice of p together with the twisting function $\iota(\rho) = \iota(\rho^{-1}) = -1, \iota(\lambda) = 1$ makes the non-orientable $GCM(G, X, p, \iota)$ satisfy the conditions of Theorem 6.3.

The visualization of this example is in fact quite simple. We start with a ‘bipartite’ map whose ‘vertices’ are the d -cycles formed by edges labelled ρ , all of which are twisted, with the edges labelled λ and connecting the two sets of d -cycles untwisted. This map is non-orientable, since d is required to be odd. The orientable dual untwists the d -cycles and twists the connecting edges.

To conclude this section, we consider one more classical concept from topological graph theory – the *orientable double covering* \mathcal{M}^o of a non-orientable map \mathcal{M} given by its underlying graph, rotation system and twisting function. The underlying graph of the double covering is the \mathbb{Z}_2 -lift of the underlying graph of the original map with the untwisted edges given the voltage $0 \in \mathbb{Z}_2$, and the twisted edges receiving the voltage $1 \in \mathbb{Z}_2$. Informally, it is the double cover of the underlying graph of the original map with the untwisted edges each lifted into two edges connecting vertices in the same layer, and the two lifts of the twisted edges crossing from one layer to the other (for more details consult [20]). Theorem 2.4 in [20] asserts that $\text{Aut}(\mathcal{M})$ lifts into the orientation preserving automorphism group $\text{Aut}^+ \mathcal{M}^o$, and the full group $\text{Aut}(\mathcal{M})^o$ is the direct product of $\text{Aut}^+ \mathcal{M}^o$ with some orientation reversing involutory automorphism ψ of order 2 (that swaps the two layers of the double covering). It follows that the direct product of the lift of a subgroup of $\text{Aut}(\mathcal{M})$ acting regularly on the vertices of \mathcal{M} with the group $\langle \psi \rangle$ acts regularly on the vertices of \mathcal{M}^o , and we obtain:

Theorem 6.7. *The orientable double covering \mathcal{M}^o of any non-orientable generalized Cayley map \mathcal{M} is an orientable generalized Cayley map admitting a group of automorphisms containing an orientation reversing automorphism and acting regularly on its vertices.*

7 Regular generalized Cayley maps

In conclusion of our paper, we would like to direct the reader toward [12] where the authors of that paper developed a complete theory of regular generalized Cayley maps including necessary and sufficient conditions based on the existence of special permutations of the elements of the underlying group. Their language, however, is different from the one used here, and they do not consider the orientable generalized Cayley maps $GCM(G, H, X, p)$ as a special case.

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