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13 Second Quantization as Cross Product

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Abstract. In the contributions [4,5] of this proceedings the new way of the second quantization of fermions is proposed, inspired by the fact that the Clifford and Grassmann algebra by themselves offer basis in internal space, presented as creation operators on the corresponding vacuum state, which together with their Hermitian conjugated annihilation partners fulfill all the requirements for the second quantized fermions, provided that the part of the basis in the ordinary space is orthogonal. In the Hilbert space of indefinite number of fermions it is assumed that each fermion has to distinguish from all the others either in ordinary or in internal space or in both spaces. The purpose of this contribution is to generalize this last requirement for either fermions or bosons.

Povzetek. V prispevkih [4,5] tega zbornika predstavita avtorja nov način druge kvantizacije fermionov. Cliffordov in Grassmanov prostor ponudita namreč bazo v notranjem prostoru fermionov, ki jo določajo kreacijski operatorji na vakuumskem stanju, ti pa skupaj s Hermitsko andjungiranimi operatorji (annihilacijskimi operatorji) izpolnjujejo vse Diracove zahteve za fermione v drugi kvantizaciji pod pogojem, da je baza v prostoru gibalnih količin ortogonalna. V Hilbertovem prostoru nedoločenega števila fermionov mora vsakemu fermionu ustrezati drugačen notranji prostor ali drugačna gibalna količina, V drugi kvantizaciji je Hilbertov prostor direkten produkt neskončne množice Hilbertovih prostorov za izbrano vrednost gibalne količine. Namen tega prispevka je posplošiti ta drugi del zahteve tako za fermione kot za bozone.

Keywords: second quantization, bosons, fermions, cross product

13.1 Introduction

We present in this contribution the possibility to make a new step in the new way of the second quantization of fermions, presented in the contributions [4,5] of this proceedings, for indefinite number of fermions and bosons.

It is the purpose of the present discussion to seek to use such a formulation of second quantized theories to generalize them to possibly quite new types of second quantization like theories. This is inspired from the type of theory put forward by one of us as being unification theory of spin, charges and families [1–5]

* H.B. Nielsen presented the talk.

Second quantization as Cross product

It is rather trivial and welknown that a second quantized (free) theory of bosons has a second quantized Hilbert space, that can be written as a Cartesian cross product over an (infinite) set of (smaller) Hilbert spaces, each of which is attached for example to the momentum, and tells how many particles have just this momentum.

Simplest case: A scalar without internal degrees of freedom

If we think of a charged scalar - like π^+ - it may be natural to even include in our "momentum" also the sign of the energy and use that as the 'factorisation **p**arameter" **p**. We like to do it as abstract and general as possible, so we now use the letter **p** and you can think of it as "(factorization) **p**arameter" or as momentum as you like.

In the π^+ case we take the "factorization parameter" **p** *to be:*

$$\mathbf{p} = (\vec{p}, sign(E)). \tag{13.1}$$

The general form as factorized space:

The Hilbert space for the second quantized boson system can always be written like

$$\mathcal{H} = \bigotimes_{\mathbf{p}} \mathcal{H}_{\mathbf{p}}. \tag{13.2}$$

In the π^+ example, where $\mathbf{p} = (\vec{p}, sign(E))$, the Hilbert space $\mathcal{H}_{\mathbf{p}}$ is actually that of harmonic oscillator for which the number operator counts the number of π^+ particles with just the **p**-specification **p**.

(Here we stepped too fast over the Dirac sea for bosons problem, but that is not so crucial just now; just think of antiparticles instead, when formally $sign(E) < 0$.)

Dream of generalization(s)

In the formulation as the Cartesian product

$$\mathcal{H} = \bigotimes_{\mathbf{p}} \mathcal{H}_{\mathbf{p}} \tag{13.3}$$

one could dream about making a new - and perhaps interesting theory - by replacing the Hilbert spaces that are factors in the Cartesian product such as $\mathcal{H}_{\mathbf{p}}$ by some Hilbert spaces with a different structure, e.g. different dimensionality.

E.g. Could we decide that all these harmonic oscillators could only be excited up to their 7th level, after that it would not be possible to put more π^+ in with a given **p** ?

We could of course postulate such a "theory" but it would be rather strange physically. *A postulate of only up to 7 particles per* **p** *would violate locality*

In a big universe particles with the same momentum are so far from each other that one cannot from locality feel if there are more or less than 7 particles in the same momentum eigenstate.

If we use $\tilde{\mathbf{x}}$ *instead of* **p** *then locality would be automatic.*

If one thinks of a discretized (d-1)-space, i.e. really a (perhaps a bit irregular) lattice, and take the state of the universe to be described by the a state in the Hilbert space $\mathcal{H}$, then factorization of the type

$$\mathcal{H} = \bigotimes_{\vec{x}} \mathcal{H}_{\vec{x}} \tag{13.4}$$

i.e. where we as "factorization parameter" use the spatial position $\vec{x}$ - the lattice point, if discretized - this Cartesian product would be automatically suited for locality, one should just only provide it with local interaction, but could for the structure and operators acting on the single factors $\mathcal{H}_{\vec{x}}$ be very free since everything would be o.k..

Usual second quantization for the Norma's *spin-charge-family* theory

Once one has decided on the inner degrees of freedom, the statistics – fermion or boson – and of dimension of space time and thus of the dimension of the momentum vectors, one would than think that *there is only one way to second quantize.*

This way will then turn out in the boson case to indeed be of the form that the full second quantized Hilbert space $\mathcal{H}$ takes the product form, and thus *be written in the product way*.

However, if one starts by a product form and has not gotten it via the standard procedure, then we would feel a priori unsafe if this would be a physically meaningful way or not.

It probably depends strongly on the details.

A couple of trivialities on component numbers

i. A Dirac (rather Weyl) massless spinor in an even number d of space time dimensions has $2^{\frac{d}{2}-1}$ components.

ii. In Norma's *spin-charge-family* theory ([3] and the references therein) there is not only the usual Dirac spin index with $2^{\frac{d}{2}-1}$ components, but a quite analogous family index again with the $2^{\frac{d}{2}-1}$ components. So in this model the number of components could be marked by two Dirac indices, or instead using another but equivalent formalism with projection and nilpotent "operators". But in any case of these two formalisms the number of components for a full fermion particle is the square of the number for an ordinary Dirac construction. The number of components is therefore 2^{d-2}. One can learn in Ref. [4,5] in this proceedings that:

a. Only operators of an odd character can offer the second quantization fermions.

b. The operators of an odd character split into two parts, Hermitian conjugated to each other.

iii. If we ignore momentum and look at one single momentum only, then the number of different states one could produce by having for this single momentum various possible numbers with the 2^{d-2} different components filled or unfilled would be $2^{2^{d-2}}$. Let us add that the rest of possibilities belong to either the Hermitian conjugated partners or have the evenness Character and do not fulfil the anticommutation relations for fermions (and probably even not for bosons. In any case the number is much much more than the number of components.

Standard second quantization procedure in factor language

Before telling this standard procedure of quantizing fermions by the factorization into the Cartesian product of "subHilbert" spaces, we have to admit that one *cannot* do that without some essential modification, which we though postpone to discuss below in the section called "The problem of fermions".

However, we are for the moment interested in reaching to the point, where we can see the problems when one attempts to make a new way of second quantizing by postulating some algebraic structure for the operators acting on the "subHilbert" spaces $\mathcal{H}_{\mathbf{p}}$ going into the Cartesian product. For this problem presumably the statistics being fermion or boson statistics may however not matter so much, so our postponing is not so crucial for that.

iv. Let us first look for a fixed momentum **p** and calculate which states are needed to describe the possibilities for filling with the allowed number of particles (up to one for fermions, and up to infinity for bosons) all the internal states.

v. Then we construct the Hilbert space $\mathcal{H}_{\mathbf{p}}$,of which is just the number of different ways of filling particles into the different combinations of internal states.

vi. Then finally you can take the Cartesian product and get the genuine Hilbert space for the full second quantized theory.

Standard way $\dim(\mathcal{H}_{\mathbf{p}}) = 2^{2^{d-2}}$ for Norma's theory.

Since there are $(2^{\frac{d}{2}-2})^2$ component combinations, namely say $2^{\frac{d}{2}-2}$ genuine Dirac components, and $2^{\frac{d}{2}-2}$ family index values, there for assumed fermion-statistic $2^{2^{d-2}}$ possibilities for filling or not filling these 2^{d-2} difference internal states.

Thus the Hilbert-space for only one momentum should have the dimension

$$\dim(\mathcal{H}_{\mathbf{p}}) = 2^{2^{d-2}} . \tag{13.5}$$

(*Notice that this space $\mathcal{H}_{\mathbf{p}}$ thus has a much bigger dimension than the space of single particle internal states, which has only dimension =* 2^{d-2} .)

We ignored at first equations of motion.

We have to modify the above simplified proposal by:
vii. Notice that using the momentum energy relation

$$E^2 - \vec{p}^2 = 0 \tag{13.6}$$

we have for each (d-1)-momentum $\vec{p}$ **two** values for the energy E of the particle, so that we should let, as already mentioned, as a possibility

$$\mathbf{p} = (\vec{p}, E)\,, \tag{13.7}$$

meaning a doubling of the space of momenta to be used.

viii. Let us take into account that the (free) equation of motion (the Dirac equation, the Weyl equation indeed) for a choice of energy $E = \pm\sqrt{\vec{p}^2}$ only allow a subspace of the internal space of states for the (single) particle,

$$(\not{p})\psi = 0\,. \tag{13.8}$$

Standard second quantization as product over $(\vec{p}, sign(E))$**.**

Letting an index $_{emr}$ denote that we have restricted the single particle sates to the states obeying the equations of motion (emr = "equation of motion restricted") we write the true standard second quantized Hilbert space

$$\mathcal{H}_{emr} = \bigotimes_{(\vec{p}, sign(E))} \mathcal{H}_{(\vec{p}, sign(E)), emr}, \tag{13.9}$$

where now $\mathcal{H}_{(\vec{p}, sign(E)), emr}$ is constructed from space of single particle internal states obeying the Dirac equation and having $E = sign(E)\sqrt{\vec{p}^2}$, which because of the restriction by the equation of motion has only half the dimensionality of $2^{d/2-1}$ in the simple Dirac case or half of 2^{d-2} in the case with families. So

$$\dim(\mathcal{H}_{(\vec{p}, sign(E)), emr}) = 2^{2^{d-1}/2} = 2^{2^{d-2}}\,. \tag{13.10}$$

13.2 The problem of fermions

Yet a problem for Cartesian product form for fermions.

For just constructing the Hilbert space we could claim that this Cartesian product procedure is o.k. even for fermions, but for the *creation and annihilation operators or the field operators for fermions there is a problem more:*

If we take a true Cartesian product and let it be understood that the creation and annihilation operators for a state with $(\vec{p}, sign(E)) = \mathbf{p}$ *alone shall act on the Cartesian product factor* $\mathcal{H}_{\mathbf{p}}$*, then we cannot make such fermion creation or annihilation operators for different* $\mathbf{p}$ *anticommute!* Operators acting alone on different Cartesian product factors will namely always commute.

Suggested trick to solve the anticommutation problem:

Use operators $(-1)^{F_{\mathbf{p}}}$, where $F_{\mathbf{p}}$ is the fermion number for the fermions in the Cartesian factor $\mathcal{H}_{\mathbf{p}}$.

That is to say to construct the "true creation or annihilation operators" – $b^{\dagger}(i;\mathbf{p})$ or $b(i;\mathbf{p})$ – for the $\mathbf{p}$ Cartesian factor we modify the truly "local ones", $c^{\dagger}(i;\mathbf{p})$ and $c(i;\mathbf{p})$ defined so as to *only act on the Cartesian factor* $\mathcal{H}_{\mathbf{p}}$, not touching the other factors, by multiplying it with a lot of factors of the form $(-1)^{F_{\mathbf{p}'}}$.

Associate in fact to each essentially momentum $\mathbf{p}$ a subset of this kind of essential momenta $\mathbf{B}(\mathbf{p})$ and define

$$b^{\dagger}(i;\mathbf{p}) = \left[\prod_{\mathbf{p}'\in\mathbf{B}(\mathbf{p})} (-1)^{F_{\mathbf{p}'}}\right] c^{\dagger}(i;\mathbf{p}) \tag{13.11}$$

$$b(i;\mathbf{p}) = \left[\prod_{\mathbf{p}'\in\mathbf{B}(\mathbf{p})} (-1)^{F_{\mathbf{p}'}}\right] c(i;\mathbf{p}) \tag{13.12}$$

13.3 Dream of Algebra

Although we for fermions must introduce the modification from $c^{\dagger}(i;\mathbf{p})$ to $b^{\dagger}(i;\mathbf{p})$ in order to achieve the anticommutativity of the annihilation operators $b(i;\mathbf{p})$, when we build up the Hilbert space construction from a Cartesian product, we might dream of using this Cartesian product idea to make a generalization of the algebra for the operators acting on one of these Hilbert spaces $\mathcal{H}_{\mathbf{p}}$ (we could call them factor-Hilbert spaces) from which the Cartesian product is made up to a more general algebra, say $\mathbf{F}$. That is to say we imagine an algebra $\mathbf{F}$ consisting of operators acting on the Hilbert space $\mathcal{H}_{\mathbf{p}}$.

We can easily think of e.g. a couple operators/elements $f, g \in \mathbf{F}$, which e.g. anticommute $\{f, g\}_{+} = 0$. Of course we shall then have such algebra elements for every factor-Hilbert-space $\mathcal{H}_{\mathbf{p}}$, and correspondingly we should of course distinguish analogous algebra elements related to different factor-Hilbert-spaces or equivalently different $\mathbf{p}$ as we decided to enumerate these factor-Hilbert-spaces. That is to say we should write $f(\mathbf{p})$ for the operator of a given structure in $\mathbf{F}$ when it acts on $\mathcal{H}_{\mathbf{p}}$.

But now if we do not even make the modification of inserting the $(-1)^{F_{\mathbf{p}'}}$-factors when in the ordering we had to have $\mathbf{p}$ and $\mathbf{p}'$ were in a certain relative order - say $\mathbf{p}' < \mathbf{p}$ - then of course any $f(\mathbf{p})$ and any $g(\mathbf{p})$ at one $\mathbf{p}$ will commute with any $f(\mathbf{p}')$ and any $g(\mathbf{p}')$ at **another** "momentum" $\mathbf{p}' \neq \mathbf{p}$, independent of how f and g for the same $\mathbf{p}$ may happen to commute or anticommute.

In other words **we cannot prevent the commutation due to independent factor-Hilbert-spaces for the operators**, what ever we take the local algebra to be, i.e. it does not modify this commutation to let the operators say anticommute locally, it does not help even if say $\{f(\mathbf{p}), g(\mathbf{p})\}_{+} = 0$ to prevent $[f(\mathbf{p}), g(\mathbf{p}')] = 0$ for $\mathbf{p} \neq \mathbf{p}'$.

13.3.1 Even with $(-1)^{F_p}$-factors

Even if we improve our purely Cartesian product construction with the $(-1)^{F_p}$-factors as above, it will not bring us to get the commutation or anticommutation to progress from the "local" to the inter **p** commutator or anticommutator so easily. If we indeed include the type of factor (from (13.11,13.12)) being the product over the factors $(-1)^{F_{p'}}$ for all $\mathbf{p}'$ which are say "smaller" in the ordering than the **p** considered, then we will achieve that we get anticommutation all operators $g(\mathbf{p})$ say at **p** with all the ones at another place $\mathbf{p}'$ provided both operators carry a fermion number in the sense that they shift the value of the fermion number $F_{\mathbf{p}}$ for their factor Hilbert space by their action. So if e.g. two operators are fermionic in this quantum number F sense and even if they commuted when at the same site, they will anticommute when they are at different sites. If oppositely they anticommute locally they will again anticommute when at different sites(= different **p**'s).

The conclusion from the remarks just above should be:

Using the starting point of the Cartesian product and only modifying by the extra factor of the type from equations (13.11,13.12) the commutation versus anticommutation of operators associated with **different p**-values depend alone on:

a. the fermion number of the operators,

b. from whether one introduce transformation (13.11, 13.12) above at all or not. But it does not depend on on how the algebra elements considered may commute or not in the "local algebra" i.e. for the **same p**-value.

13.3.2 More generally:

The above proposed method for making fermion-fields on the basis of a Cartesian product by means of an ordering of all the **p**-values is really not very attractive. In fact such an ordering does not match well with the topological structure of a momentum space or a position space except for the spatial dimension being $d_{spatial} = 1$. In higher dimensions you rather have to use the axiom of choice to even see that there exists such an ordering. We also need such a construction if we would like to make fermionization, and then this only by axiom of choice found ordering would not seem attractive at all either.

So attempting to generalize this method of constructing fermion fields from a Cartesian product is highly called for.

Now if there is in the theory some sort of gauge freedom one might not require quite as strict the properties of the extra factors introduced to convert the a priori commuting fields appearing from operators acting on different factor-Hilbert-spaces from (13.11.13.12). If one allows more freedom in the construction then one might optimistically hope to construct such factors to convert the boson-commuting operators into fermion ones to have some continuity and thus compatibility with the topology of a higher dimensional space(than just dimension =1).

We here at first write down the type of transformation to be made to construct fermions from commuting fields in a general way. Then one may investigate how much one needs to require about the multiplying factors $U(\mathbf{p},\mathbf{p}')$ converting the bosons to fermions so to speak.

Unfortunately we have not come far in developing these conditions, but just the thought of looking at it more generally might turn out useful:

$$b^\dagger(i;\mathbf{p}) = \left[\prod_{\mathbf{p}'\in\mathbf{B}(\mathbf{p})} U(\mathbf{p},\mathbf{p}')^\dagger\right] c^\dagger(i;\mathbf{p})$$
$$b(i;\mathbf{p}) = \left[\prod_{\mathbf{p}'\in\mathbf{B}(\mathbf{p})} U(\mathbf{p},\mathbf{p}')\right] c(i;\mathbf{p})$$

Not even crudely local $b^\dagger(\vec{x})$ unless the modification by $U(\vec{x},\vec{x}')$ inessential.

So there should preferably be a "gauge" transformation which could be the effect of the modification $U(\vec{x},\vec{x}')$ or "jump over correction"-replacement.

Natural that the $U(\vec{x},\vec{x}')$ depends on the direction from $\vec{x}$ to $\vec{x}'$, and thus is a function of a point on thee sphere S^{d-2}.

Also the 'gauge"like modifications must lie in a group G. So need map $S^{d-2} \to G$.

13.3.3 Anyons

To exercise constructing other statistics than bosons from the Cartesian product one would of course like to exercise with two spatial dimensions because this is the first case after the one spatial dimension case in which there are essentially no problem and fermionization is already well done. But now just 2 spatial dimensions is the interesting case in which also Leinaas Myhrheim or anyon statistics is possible[6].

With the suspicion of the gauge symmetries being important in allowing a more developed choice of the conversion factors $U(\mathbf{p},\mathbf{p}')$ a first exercise might be to even construct a system of anyons or first just a pair by electromagnetic ingredients.

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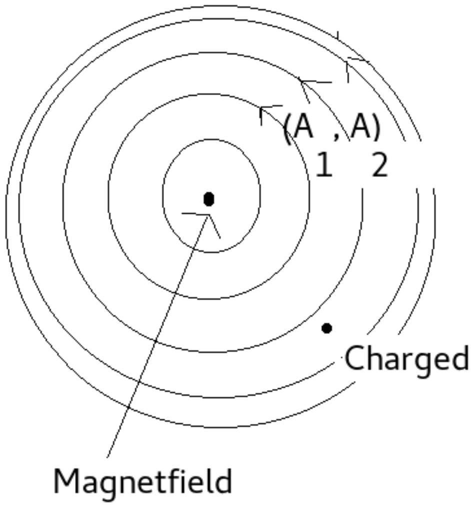

Fig. 13.1. Anyons as electric magnetic made.

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