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Contents

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Generalized Cayley maps and their Petrie duals*

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Abstract

Cayley maps are embeddings of Cayley graphs in orientable surfaces which possess a group of orientation preserving automorphisms acting regularly on the vertices. We generalize the concept of a Cayley map by considering embeddings of Cayley graphs in both orientable and non-orientable surfaces and by requiring a group of automorphisms acting regularly on vertices that does not have to consist entirely of orientation preserving automorphisms. This leads to new families of maps in both the orientable and non-orientable cases. Since the Petrie dual operator preserves the property of being a generalized Cayley map, throughout the paper we consider the action of this operator on our maps.

Keywords: Cayley map, regular embedding, automorphism group of a map.

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1 Introduction

A *map* is a cellular embedding of a connected graph on some surface; the map is *orientable* if the surface is. By a widely adopted definition (cf. [16]), an orientable map $\mathcal{M}$ is a Cayley map if the group of orientation-preserving automorphisms of $\mathcal{M}$ contains a subgroup acting regularly on the vertex set of $\mathcal{M}$. This way of introducing Cayley maps has been motivated by Sabidussi's well known characterization of Cayley graphs in terms of a subgroup of automorphisms of the graph acting regularly on its vertex set [18]. Apart from the history of the development of the theory of maps, however, when carrying Sabidussi's characterization over to maps there is no reason to restrict to orientation-preserving automorphisms, or even to orientable surfaces. The latter may have been first realized by Tucker [19], and in this paper we argue for the return of his way of treating Cayley maps.

In order not to clash with customary terminology we will say that a map $\mathcal{M}$ on an arbitrary surface (orientable or not) is a *generalized Cayley map* if the automorphism group of $\mathcal{M}$ contains a subgroup H that is a regular permutation group on the vertex set of $\mathcal{M}$.

In our view, Tucker's definition by means of existence of a regular action of a group of automorphisms has two major advantages: it is a generalization of the currently used definition that is simple, and in line with the original definition of a Cayley map. To the best of our knowledge there have been two other attempts at defining non-orientable Cayley maps: the one by Abas [1] via strategically placed cross-cups on the carrier surface, and another one by Kwak and Kwon [12] who developed a full theory of generalized Cayley maps (equivalent to Tucker's and our approach) but starting off from introducing generalized Cayley maps in a way that appears to be hard to use in applications.

The main difference between (original and orientable) Cayley maps and generalized Cayley maps is that the carrier surfaces of the latter may be non-orientable, and even in the orientable case the subgroup required to act regularly on vertices does not have to be orientation-preserving. Maps admitting such subgroups appear frequently in the context of regular maps with automorphism groups acting quasi-primitively on the vertices [6], and our study of such maps was the main source of motivation for the present paper.

At several places, we employ the Petrie dual operator (e.g., [22]) which proves particularly useful since it preserves the class of generalized Cayley maps. For example, we consider orientable generalized Cayley maps whose Petrie dual is again an orientable generalized Cayley map, as well as non-orientable generalized Cayley maps whose Petrie dual is orientable.

The outline of the paper is as follows. In order to emphasize parallels between the original Cayley maps on orientable surfaces and the generalized Cayley maps, the three sections following this Introduction are devoted to reviewing the background and context of highly symmetric maps, Cayley maps, and the Petrie dual operator; with Sections 5 and 6 containing the new results. We begin in Section 2 by reviewing some of the most important properties of regular maps, followed in Section 3 by reviewing algebraic machinery for dealing with orientable Cayley maps in their original setting. Section 4 discusses Petrie duals of maps in some detail. We discuss generalized Cayley maps on orientable surfaces in Section 5 and generalized Cayley maps on non-orientable surfaces in Section 6.

We can summarize the results obtained in our paper as follows. A generalized Cayley map may be both orientable and non-orientable. The orientable generalized Cayley maps come in two kinds. First, there are the original orientable Cayley maps in which the group acting regularly on the vertices of the map consists entirely of orientation preserving automorphisms. Then there are orientable generalized Cayley maps with bipartite underlying graphs in which the group acting regularly on the vertices of the map contains a subgroup of index 2 consisting of orientation preserving automorphisms while the other half of the regularly acting group consists of orientation reversing automorphisms (Theorems 5.1 and 5.2). There are infinitely many examples of orientable generalized Cayley maps which are not the original Cayley maps (Lemma 5.3 and Example 5.5), and infinitely many orientable generalized Cayley maps which are self-Petrie-dual (Remark 5.6). In Remark 5.7 we present orientable generalized Cayley maps that simultaneously admit an orientation preserving group acting regularly on their vertices and a vertex-regular group half of which consists of orientation reversing automorphisms (hence, these maps are classical Cayley maps and orientable maps of the second kind at the same time).

In Remark 6.2 we present an infinite family of non-orientable generalized Cayley maps. Taking advantage of the well-known fact that the Petrie dual of an orientable map $\mathcal{M}$ is orientable if and only if the underlying graph of $\mathcal{M}$ is bipartite (Theorem 4.2 in our paper), we argue the existence of infinitely many examples of the original orientable Cayley maps whose Petrie duals are orientable (Example 6.6) and infinitely many examples of the original orientable Cayley maps whose Petrie duals are non-orientable (any orientable Cayley map with a non-bipartite underlying graph). The Petrie dual of a non-orientable generalized Cayley map may be both orientable and non-orientable. We classify the non-orientable generalized Cayley maps whose Petrie dual is orientable (Corollary 6.5) and exhibit infinitely many examples of both non-orientable generalized Cayley maps whose Petrie dual is orientable and non-orientable generalized Cayley maps whose Petrie dual is non-orientable (Remark 6.2).

2 Regular, orientably-regular, reflexible and chiral maps

For a map $\mathcal{M}$, regions (faces) of its barycentric subdivision are the *flags* of the map. Every flag is a triangular region; informally, its three 'corners' are a vertex, the 'midpoint' of an edge incident with the vertex, and the 'centre' of a face incident with both the vertex and the edge. As long as no face of the map contains two occurrences of an edge on its boundary (no maps with such a degeneracy will be considered here), a flag can be identified with a triple (v, e, F), where v is a vertex, e an edge incident with v, and F is a face incident with both v and e. A pair of distinct flags (v, e, F) and (v', e', F') of $\mathcal{M}$ are *incident* if they share a segment of the skeleton of the barycentric subdivision; in case when no face of the map contains two occurrences of an edge on its boundary, two flags are incident if precisely two of the three equalities $v = v'$, $e = e'$, $F = F'$ hold.

Let r_0, r_1 and r_2 be involutory fixed-point-free permutations of the flag set of a map $\mathcal{M}$ formed by two-cycles consisting of incident flags, with r_0 swapping faces that share a face-center-to-edge-midpoint segment, r_1 swapping faces that share the vertex-to-face-center segment, and r_2 swapping faces that share the vertex-to-edge-midpoint segment; again, if no face of the map contains two occurrences of an edge on its boundary, the flag (v, e, F) is mapped by r_0 to (v', e, f), $v \neq v'$, by r_1 to (v, e', F), $e' \neq e$, and by r_2 to (v, e, F'), $F' \neq F$. The (flag-transitive) permutation group generated by r_0, r_1 and r_2 is the *monodromy group* $\mathrm{Mon}(\mathcal{M})$ of the map. The three generators of the monodromy group can be interpreted as 'gluing instructions' to assemble the map from its set of flags; in other words, knowledge of $\mathrm{Mon}(\mathcal{M})$ in terms of the action of the generators r_0, r_1 and r_2 on the set its flags is equivalent to knowing the map $\mathcal{M}$. Note that r_0 and r_2 commute; if the map is finite, then the orders of r_0r_1 and r_1r_2 are equal to the least common multiples of face lengths and vertex valencies, respectively.

It is well known that the carrier surface of $\mathcal{M}$ is orientable if and only if $\langle r_1r_2, r_0r_2\rangle = \langle r_1r_2, r_0r_1\rangle$ is a subgroup of index 2 in $\mathrm{Mon}(\mathcal{M})$. In such a case we speak about an orientable map; letting $\rho = r_1r_2$ and $\lambda = r_0r_2$ one may consider the 'orientable part' $\mathrm{Mon}^+(\mathcal{M}) = \langle \rho, \lambda\rangle$, the index-2-subgroup of $\mathrm{Mon}(\mathcal{M})$. Here, $\mathrm{Mon}^+(\mathcal{M})$ can be regarded as a permutation group acting on the *dart set* $\mathcal{D}$ of the map (i.e., on the set of directed edges of the map). Then ρ is a permutation that cyclically permutes, at each vertex v, the darts emanating from v in accord with a chosen orientation of the carrier surface of the map, and λ is an an involution interchanging the two darts belonging to the same edge.

An *automorphism* of a map $\mathcal{M}$ with monodromy group $\mathrm{Mon}(\mathcal{M}) = \langle r_0, r_1, r_2\rangle$ is any permutation of the flag set $\mathcal{F}$ of $\mathcal{M}$ that preserves incidence of flags. Equivalently, a permutation of $\mathcal{F}$ is an automorphism of $\mathcal{M}$ if and only if it commutes with all of r_0, r_1 and r_2; hence the *automorphism group* $\mathrm{Aut}(\mathcal{M})$ of the map is simply the centralizer of the monodromy group $\mathrm{Mon}(\mathcal{M})$ in the symmetric group $S_{\mathcal{F}}$ on the set $\mathcal{F}$. Since $\mathrm{Mon}(\mathcal{M})$ is transitive on $\mathcal{F}$, it follows that $\mathrm{Aut}(\mathcal{M})$ acts freely on the set set $\mathcal{F}$.

For an orientable analogue, let now $\mathcal{M}$ be an orientable map with dart set $\mathcal{D}$ and with $\mathrm{Mon}^+(\mathcal{M}) = \langle \rho, \lambda\rangle$ being a subgroup of $\mathrm{Mon}(\mathcal{M})$ of index 2. An *orientation-preserving* automorphism of $\mathcal{M}$ is any permutation of $\mathcal{D}$ commuting with ρ and λ. It follows that the group $\mathrm{Aut}^+(\mathcal{M})$ of all orientation-preserving automorphisms of an orientable map is the centralizer of $\mathrm{Mon}^+(\mathcal{M})$ in the symmetric group $S_{\mathcal{D}}$ on the set $\mathcal{D}$. Again, transitivity of $\mathrm{Mon}^+(\mathcal{M})$ on $\mathcal{D}$ implies that the group $\mathrm{Aut}^+(\mathcal{M})$ acts freely on the set $\mathcal{D}$. If such an orientable map $\mathcal{M}$ admits an automorphism commuting with λ but inverting ρ, then the automorphism is *orientation-reversing*.

Finally, let us discuss the highest 'level of symmetry'of maps. A map $\mathcal{M}$ with monodromy group $\mathrm{Mon}(\mathcal{M})$ is called *regular* if the automorphism group $\mathcal{M}$ acts regularly on the set $\mathcal{F}$ of flags. In this case, the groups $\mathrm{Aut}(\mathcal{M})$ and $\mathrm{Mon}(\mathcal{M})$ are abstractly isomorphic, so that in the case of a finite map the order of both groups is equal to $|\mathcal{F}|$. If $\mathcal{M}$ is a regular *orientable* map, then its automorphism group $\mathrm{Aut}(\mathcal{M})$ contains the group $\mathrm{Aut}^+(\mathcal{M})$ of orientation-preserving automorphisms as a subgroup of index 2, its other coset in $\mathrm{Aut}(\mathcal{M})$ being the collection of all orientation-reversing automorphisms of the map. The orientation-reversing automorphisms are sometimes referred to as *reflections*, giving such maps the name *reflexible*. An orientable map $\mathcal{M}$ is *orientably-regular* if the group $\mathrm{Aut}^+(\mathcal{M})$ of orientation preserving automorphisms of $\mathcal{M}$ acts regularly on its set of darts. An orientably regular map that is not reflexible is *chiral*, and in such a case $\mathrm{Aut}(\mathcal{M}) = \mathrm{Aut}^+(\mathcal{M})$.

3 Cayley maps

A *Cayley map* is an orientable map $\mathcal{M}$ that admits a group of orientation preserving automorphisms G acting regularly on its set of vertices. To make this definition more precise, one needs to consider the induced action of map automorphisms of $\mathcal{M}$ on the vertex set of the underlying graph associating the map automorphism ψ with the vertex permutation $\overline{\psi}$ mapping v to v' whenever $\psi(v, e, F) = (v', e', F')$. It is easy to see that $\overline{\psi}$ is a well defined graph automorphism of the underlying graph of $\mathcal{M}$ defined via its action on the vertices (of the map or the graph). The induced action of $\mathrm{Aut}(\mathcal{M})$ on the vertices is almost universally faithful; with the very few exceptions listed in [14, Proposition 9]. The automorphisms contained in the automorphism group of a Cayley map which acts regularly on the vertices of the map correspond in this way to automorphisms of its underlying graph forming a group acting regularly on the graph's vertices, making it into a Cayley graph [18]. A *Cayley graph* $C(G, X)$ is a graph whose vertex set can be identified with the elements of a group G generated by a set X closed under taking inverses and not containing the identity 1_G, with the pairs of adjacent vertices consisting of all pairs g, gx with $g \in G$ and $x \in X$. A graph Γ is isomorphic to a Cayley graph $C(G, X)$ if and only if $\mathrm{Aut}\Gamma$ contains a subgroup G acting regularly on the vertices of Γ [18], which justifies the remark we have made about the underlying graphs of Cayley maps being Cayley. Hence, the automorphism group of a Cayley map $\mathcal{M}$ containing a group G of orientation preserving automorphisms acting regularly on the vertices of $\mathcal{M}$ is necessarily contained in the automorphism group of its underlying Cayley graph $C(G, X)$, $G \leq \mathrm{Aut}C(G, X)$ (after applying the necessary restriction to vertices). The action of G on $C(G, X)$ is defined for each $g \in G$ via left multiplication: $A_g(h) = gh$ for all $g, h \in H$, and we shall denote the group $\{A_g \mid g \in G\}$ by G_L; it is always a subgroup of $\mathrm{Aut}C(G, X)$. The set of darts of the Cayley map $\mathcal{M}$ thus takes the form $\{(h, hx) \mid h \in G, x \in X\}$ and each $A_g \in G_L$ 'extends' to a permutation of the darts of $\mathcal{M}$ mapping the dart (h, hx) to (gh, ghx). If each of the extensions of A_g, $g \in G$, is to preserve the orientation of $\mathcal{M}$, there must exist a cyclic permutation p of X having the property that the dart $(g, gp(x))$ is always the dart lying immediately next to the dart (g, gx) in the local surface neighborhood of g (sharing the face with (g, gx)), for all $g \in G$ and $x \in X$ [16]. Thus, we can talk about the *local permutation* of the darts (g, gx), $x \in X$, emanating from g, determined by the cyclic permutation p of X. This gives rise to an equivalent definition of a Cayley map as an orientable embedding of a Cayley graph $C(G, X)$ in an orientable surface determined by the local rotation scheme with the property that the local surface ordering corresponding to the vertex $g \in G$, ρ_g, acting cyclically on the darts (g, x), $x \in X$, does not depend on g, and is determined by a fixed cyclic permutation p of X. Such Cayley map is denoted by $CM(G, X, p)$, and the equivalence of this 'local rotation definition' and the definition via the existence of an orientation preserving automorphism group acting regularly on its vertices is a well established fact in the theory of Cayley maps [16].

As argued above, the group $G_L = \{A_g \mid g \in G\}$ (isomorphic to G) is always a subgroup of the group $\mathrm{Aut}^+CM(G, X, p)$ of orientation preserving automorphisms, and $CM(G, X, p)$ is orientably regular if and only if there exists an orientation preserving automorphism $A \in \mathrm{Aut}^+CM(G, X, p)$ which fixes the identity 1_G and maps the darts emanating from 1_G in the same order as p: $A((1_G, x)) = (1_G, p(x))$, for all $x \in X$. This has been shown to be equivalent to the existence a special identity-fixing permutation φ called skew-morphism in [7], where it was first defined. Given a group G, a permutation $\varphi\colon G \to G$ of the elements of G that fixes the identity of G, $\varphi(1_G) = 1_G$, is said to be a *skew-morphism* of G with associated *power function* $\pi\colon G \to \mathbb{N}$ if the equation

$$\varphi(gh) = \varphi(g)\varphi^{\pi(g)}(h) \tag{3.1}$$

is satisfied for all $g, h \in G$. A Cayley map $CM(G, X, p)$ is orientably regular if and only if there exists a skew-morphism φ of G with the property $\varphi(x) = p(x)$, for all $x \in X$. Furthermore, a group G admits the existence of an orientably regular Cayley map $CM(G, X, p)$ if and only if G admits the existence of a skew-morphism φ with an orbit X that generates G and is closed under taking inverses (in which case, the desired orientably regular map is the map $CM(G, X, \varphi|_X)$) [7]. The classification of finite groups G that admit the existence of an orientably regular Cayley map $CM(G, X, p)$ is a hard problem and the topic of many articles. The orientation preserving automorphism group of an orientably regular Cayley map with skew-morphism φ and power function π takes the form $\mathrm{Aut}^+(CM(G,X,p)) \cong G_L \langle\varphi\rangle$ with the product multiplication defined by the rule

$$a\varphi = \varphi^{\pi(a)}\varphi(a),$$

for all $a \in G$. A Cayley map $CM(G,X,p)$ with the property $p(x^{-1}) = (p(x))^{-1}$ satisfied by all $x \in X$ is called a *balanced Cayley map*. Since a map $CM(G,X,p)$ is balanced if and only G_L is normal in $\mathrm{Aut}^+(CM(G,X,p))$, we will break with the long line of tradition, and we will call such map a *normal Cayley map*. This is in line with the name given to Cayley graphs $C(G,X)$ with G_L normal in $\mathrm{Aut}C(G,X)$.

Let us complete this section with a necessary condition for a Cayley map to be regular. Even though it can be deduced from [5] and [7], we are not aware of this condition being stated explicitly before, and so we also provide a short proof.

Theorem 3.1. *If a Cayley map $\mathcal{M} = CM(G,X,p)$ is regular, there exists a pair of skew-morphisms φ,ψ of G that preserve X and the restriction of φ to X is equal to p, while the restriction of ψ to X is equal to p^{-1}.*

Proof. If $\mathcal{M} = CM(G,X,p)$ is regular, it is also orientably regular, hence there exists a skew-morphism φ of G that preserves X and whose restriction to X is equal to p [7]. Moreover, if $\mathcal{M}$ is regular, it is isomorphic to its mirror reflection $CM(G,X,p^{-1})$, which is therefore also regular, hence orientably regular, and there exists a skew-morphism ψ of G that preserves X and whose restriction to X is equal to p^{-1}. □

It is interesting to point out that the above necessary condition is not sufficient. For example, the skew-morphism whose existence guarantees the orientable regularity of a normal Cayley map $CM(G,X,p)$ is well-known to be a group automorphism of G [21]. The inverse of a group automorphism is always a group automorphism, and hence a skew-morphism. Thus, every orientably regular normal Cayley map admits a skew-morphism whose restriction to X is p and whose inverse is a skew-morphism (and its restriction to X is p^{-1}). However, not every normal orientably regular Cayley map is regular. To mention just one famous example, all orientably regular embeddings of complete graphs have been shown to be normal Cayley maps, however, the only orientably regular embeddings of complete graphs which are also regular are the orientable embeddings of complete graphs of prime power order not exceeding 4 [8].

4 Petrie dual

The well-known duality operation switching the roles of vertices and faces of a map $\mathcal{M}$ preserves some of the most important topological characteristics of $\mathcal{M}$, such as the orientability and the genus, but generally changes the underlying graph of $\mathcal{M}$. The less well-known Petrie duality has the advantage of preserving both the embedded graph and the action of $\mathrm{Aut}(\mathcal{M})$ on it. This makes the Petrie dual operation more useful when dealing with maps with automorphism groups acting regularly on vertices of the underlying graph.

The *Petrie dual* of a map $\mathcal{M}$ is the map $P(\mathcal{M})$ with the same vertices and edges as $\mathcal{M}$ (and thus, the same underlying graph), however, the faces of $P(\mathcal{M})$ are determined by the *Petrie* (or zig-zag) *walks* of $\mathcal{M}$ which visit vertices of $\mathcal{M}$ along the edges while switching sides (i.e., faces) in the middle of every next edge situated along the boundary of the face of $\mathcal{M}$. The key point of our interest in the Petrie dual lies in the fact that the two maps $\mathcal{M}$ and $P(\mathcal{M})$ have the same vertex set and the same automorphism group. To put this claim on a more precise footing, we turn again to the monodromy groups. If $\langle r_0, r_1, r_2 \rangle$ is the monodromy group of a map $\mathcal{M}$, the monodromy group of the Petrie dual $P(\mathcal{M})$ is the group $\langle r_0 r_2, r_1, r_2 \rangle$ (which can be easily seen from the fact that the two flags associated with the opposite dart get swapped in the Petrie dual construction). Thus, $P(P(\mathcal{M})) = \mathcal{M}$, and, since $\langle r_0, r_1, r_2 \rangle = \langle r_0 r_2, r_1, r_2 \rangle$, we obtain

$$\mathrm{Aut}(\mathcal{M}) = C_{\mathbb{S}_\mathcal{F}}(\langle r_0, r_1, r_2 \rangle) = C_{\mathbb{S}_\mathcal{F}}(\langle r_0 r_2, r_1, r_2 \rangle) = \mathrm{Aut}P(\mathcal{M}).$$

Therefore, the two automorphism groups act on the same set of flags in exactly the same way, and any properties of the automorphism group of $\mathcal{M}$ with regard to the vertices of $\mathcal{M}$ are shared by the automorphism group of $P(\mathcal{M})$. In particular, if $\mathrm{Aut}(\mathcal{M})$ contains a subgroup acting regularly on the set of vertices of $\mathcal{M}$, so does the group $\mathrm{Aut}P(\mathcal{M})$. Since we make repeated use of this observation throughout our paper, we state it in the form of a theorem.

Theorem 4.1. *A map $\mathcal{M}$ is a generalized Cayley map if and only if its Petrie dual $P(\mathcal{M})$ is also a generalized Cayley map.*

Even though the Petrie dual pair shares with the original map the same underlying graph and the same automorphism group, the two maps have usually quite distinct topological properties. It is, for example, possible (even common) that $\mathcal{M}$ is orientable while $P(\mathcal{M})$ need not be. For an example, consider $\mathcal{M}$ to be the tetrahedron as an embedding of K_4 on the sphere. Then $P(\mathcal{M})$ is an embedding of K_4 in the projective plane as a map of type $\{4, 3\}$.

Since we will have to be able to distinguish between various situations with regard to the orientablity vs. non-orientability, we recall the following well-known result (e.g., [15, Remark 7]).

Theorem 4.2. *If $\mathcal{M}$ is orientable, then $P(\mathcal{M})$ is orientable if and only if the underlying graph of $\mathcal{M}$ is bipartite.*

Next, recall the concept of a *ribbon graph* which is constructed from a map $\mathcal{M}$ by cutting out an open neighborhood of each face-center and keeping small bands around the edges of $\mathcal{M}$ and small circles around the vertices. The ribbon graph of the Petrie dual of an orientable embedding of a bipartite graph is the ribbon graph of the original embedding with all bands twisted (cut off at one of the end vertices and glued back after being rotated by 180 degrees) [4].

Remark 4.3. There is a large number of examples of Cayley maps throughout the literature. For example, four of the five Platonic solids (all but the dodecahedron) are Cayley maps. All orientably regular embeddings of complete graphs are also Cayley maps [8]. Since none of the complete graphs but K_2 are bipartite, their Petrie duals are non-orientable generalized Cayley maps.

On the other hand, the underlying graph of the dodecahedron is not a Cayley graph, and hence the dodecahedron and its Petrie dual are neither Cayley maps nor generalized Cayley maps.

5 Orientable generalized Cayley maps and their Petrie duals

Let us begin by pointing out that the underlying graph of *any* generalized Cayley map (orientable or not) is a Cayley graph. This observation follows from the same line of arguments as that included in our section on Cayley maps, namely, it follows from the fact that the group of automorphisms of the underlying graph induced by the group G of map automorphisms acting regularly on the vertices of the map acts regularly on the vertices of the graph. Thus, as is well-known, the set of vertices of the underlying graph of a generalized Cayley map can be identified with the elements of G, and the action of these automorphisms on the elements of G is that of left-multiplication in G.

It follows that generalized Cayley maps $\mathcal{M}$ are precisely embeddings of Cayley graphs $C(G, X)$ into (orientable or non-orientable) surfaces satisfying the property that all left multiplications in G viewed as permutations of the elements of G 'extend' into automorphisms of the embedding, and that the order of the automorphism group of $\mathcal{M}$ acting regularly on the vertices of $\mathcal{M}$ must be equal to its number of vertices, i.e., must be equal to $|G|$.

Thus, orientable generalized Cayley maps come in two kinds. First, there are the embeddings $\mathcal{M}$ of Cayley graphs $C(G, X)$ in orientable surfaces having the property that all left multiplications in G extend into orientation preserving automorphisms of $\mathcal{M}$, which are precisely the classical Cayley maps $\mathcal{M} = CM(G, X, p)$ [16]. An orientable generalized Cayley map $\mathcal{M}$ that is not of this kind, i.e., that is not a Cayley map, must then be an embedding of a Cayley graph $C(G, X)$ in an orientable surface having the property that at least one of the left multiplications in G extends to an orientation-reversing automorphism of $\mathcal{M}$. Recall that any automorphism group of an orientable map that contains at least one orientation reversing automorphism must contain an index 2 subgroup of orientation preserving automorphisms. This means that orientable generalized Cayley maps which are not Cayley maps are embeddings of Cayley graphs $C(G, X)$ in which a subgroup H_L of G_L of index 2 extends into orientation preserving automorphisms and the other coset of this subgroup in G_L extends into orientation reversing automorphisms.

Recall that an orientable embedding of $C(G, X)$ is determined by choosing a cyclic permutation ρ_g of the elements in X for each $g \in G$. In the case of Cayley maps $CM(G, X, p)$, for every $g \in G$, the permutation ρ_g of the elements of X must be equal to p. This, of course, cannot be the case for orientable generalized Cayley maps which are not Cayley maps, i.e., there must exist elements $f, g \in G$ such that $\rho_f \neq \rho_g$. However, the distribution of local rotations in orientable generalized Cayley maps which are not Cayley maps is only slightly more complicated than that of Cayley maps. Namely, all such maps are still determined by a single cyclic permutation p of X which becomes the local rotation for the vertices contained in a subgroup H of index 2 in G and whose inverse p^{-1} becomes the local rotation for the rest of them. We shall denote these maps by $GCM(G, H, X, p)$ and note that each such map is an orientable embedding of a Cayley graph $C(G, X)$ having the properties that H is a subgroup of index 2 in G, p is a cyclic permutation of X, and the embedding of $C(G, X)$ is determined by the local rotations $\rho_h = p$, for all $h \in H$, and $\rho_k = p^{-1}$, for all $k \in G \setminus H$.

Using the notation introduced above, we classify in the following theorem orientable generalized Cayley maps in terms of their local orientations.

Theorem 5.1. *Let $\mathcal{M}$ be an orientable generalized Cayley map. Then $\mathcal{M}$ is either a Cayley map, $\mathcal{M} = CM(G, X, p)$, or a map $\mathcal{M} = GCM(G, H, X, p)$ defined above.*

Conversely, every $CM(G, H, X, p)$, i.e., every orientable embedding of a Cayley graph $C(G, X)$ having the property that $\rho_h = p$, for all $h \in H$, where H is a subgroup of index 2 in G and p is a cyclic permutation of X, and $\rho_k = p^{-1}$, for all $k \in G \setminus H$, is an orientable generalized Cayley map.

Finally, the maps $GCM(G, H, X, p)$ and $GCM(G, H, X, p^{-1})$ are isomorphic.

Proof. Let $\mathcal{M}$ be an orientable generalized Cayley map that is not a Cayley map, let $G \leq \mathrm{Aut}(\mathcal{M})$ act regularly on the vertices of M, and identify the vertices of $\mathcal{M}$ with the elements of G. Let H be the subgroup of index 2 in G of automorphisms preserving the orientation in $\mathcal{M}$ whose existence has been argued prior to the statement of this theorem.

First, let h be an element of H. This means that the associated map automorphism A_h mapping the darts (g, x), $g \in G$, $x \in X$, to the darts (hg, x) preserves the orientation in $\mathcal{M}$. It follows that $A_h\rho(g, x) = \rho A_h(g, x)$, for all $g \in G$ and $x \in X$, and thus $(hg, \rho_g(x)) = (hg, \rho_{hg}(x))$, for all $g \in G$ and $x \in X$. Since multiplication by h preserves H and its coset, the above identities yield that $\rho_h = \rho_{h'}$, for all $h, h' \in H$, and $\rho_k = \rho_{k'}$, for all $k, k' \in G \setminus H$. Taking next an element $k \in G \setminus H$, its corresponding automorphism A_k reverses the orientation in $\mathcal{M}$, and hence satisfies the identity $A_k\rho = \rho^{-1}A_k$. Thus, $(kg, \rho_g(x)) = (kg, \rho_{kg}^{-1}(x))$, for all $g \in G$ and $x \in X$. It follows that $\rho_g = \rho_{kg}^{-1}$, for all $g \in G$. If we denote the cyclic permutation of X assigned to the elements of H by p, using the fact that the multiplication by k swaps H and its coset yields that the elements in $G \setminus H$ are assigned the local permutation p^{-1}. This completes the proof of the first statement of our theorem.

The proof of the second part of the theorem is essentially the reverse of the above. If one chooses the local orientations as described in the theorem, it is easy to see that left multiplication by the elements of H preserves and left multiplication by the elements from $G \setminus H$ reverses the orientation of the obtained map.

The veracity of the final statement of our theorem can be verified by showing that the mapping φ_k defined on the set of darts of $GCM(G, H, X, p)$ by mapping the dart (g, gx) to the dart (kg, kgx) in $GCM(G, H, X, p^{-1})$, for any fixed $k \in G \setminus H$ and all $g \in G$ and $x \in X$, is a map isomorphism between the maps $GCM(G, H, X, p)$ and $GCM(G, H, X, p^{-1})$. This relatively simple task is left to the reader. $\square$

To obtain examples of orientable generalized Cayley maps that are not Cayley maps, one needs to consider embeddings of Cayley graphs of even order. To obtain an 'easy' example, one could consider the generalized Cayley map $GCM(G, H, X, p)$ for the groups $G = \mathbb{Z}_{2n}$, $n \geq 1$, $H = \langle 2 \rangle$, $X = G \setminus \{0\}$, and any cyclic permutation p of X. All of these maps are embeddings of the complete graphs K_{2n} in orientable surfaces. However, it is easy to see that the cases $n = 1$ and $n = 2$ result in Cayley maps, and even for $n \geq 3$, it is hard to show that the resulting maps are not Cayley. Thus, to construct an infinite family of orientable generalized Cayley maps which are provably not Cayley maps, one might rely on the Petrie dual operator again. Specifically, in what follows, we shall consider orientable generalized Cayley maps whose Petrie dual is orientable again. Because of Theorem 4.2, each such map must be bipartite (which is also a sufficient condition). Thus, we obtain the following:

Theorem 5.2. *Let $\mathcal{M}$ be an orientable generalized Cayley map. The Petrie dual of $\mathcal{M}$ is orientable if and only if $\mathcal{M} = CM(G, X, p)$ is a Cayley map for a group G that contains* *a subgroup H of index 2 for which $X \subseteq G \setminus H$ or if $\mathcal{M} = GCM(G, H, X, p)$ where H is a subgroup of G of index 2 for which $X \subseteq G \setminus H$.*

Furthermore, if the Petrie dual of an orientable generalized Cayley map $\mathcal{M}$ is orientable, the two maps $CM(G, X, p)$ and $GCM(G, H, X, p)$ are mutual Petrie duals.

Proof. In order for a Cayley graph $C(G, X)$ to be bipartite, it is easy to see (and well known) that it must be a Cayley graph of an even order group G that possesses a subgroup H of index 2, and X must be a subset of the non-trivial coset of H in G. The rest of the first part of the theorem follows from Theorem 5.1.

Suppose now that $\mathcal{M} = CM(G, X, p)$ with X satisfying the required condition with respect to a subgroup H of G. It is again not too hard to see that the Petrie dual to $\mathcal{M}$ reverses the order of elements from X for each element of H (or for each element of $G \setminus H$; the two maps $GCM(G, H, X, p)$ and $GCM(G, H, X, p^{-1})$ are isomorphic as argued in Theorem 5.1). □

Clearly, an orientable generalized Cayley map $\mathcal{M} = GCM(G, H, X, p)$ is not simultaneously a Cayley map if and only if the group $\mathrm{Aut}(\mathcal{M})$ contains no orientation preserving subgroup acting regularly on the vertices of $\mathcal{M}$. One way to make sure no such group exists, is to find a pair of vertices u, v of $\mathcal{M}$ for which there is no orientation preserving automorphism mapping u to v. This is the approach we take to construct an infinite family of orientable generalized Cayley maps which are not Cayley maps.

Lemma 5.3. *Let G be a finite group with a subgroup H of index 2 and a generating set $X \subseteq G \setminus H$. If $CM(G, X, p)$ is chiral, its Petrie dual $GCM(G, H, X, p)$ is a generalized Cayley map that is not a Cayley map.*

Proof. Recall that an orientably regular map is chiral if it admits no orientation reversing automorphisms. Note also, that since $CM(G, X, p)$ is bipartite and connected, any automorphism $\varphi \in \mathrm{Aut}CM(G, X, p)$ that maps a vertex $u \in H$ to a vertex $v \in G \setminus H$ maps all elements of H onto the elements of $G \setminus H$. This means that all the automorphisms in $\mathrm{Aut}GCM(G, H, X, p)$ mapping elements of H to elements in $G \setminus H$ are orientation reversing, and no orientation preserving automorphisms of $GCM(G, H, X, p)$ map the elements of H to the elements in $G \setminus H$. It follows that $GCM(G, H, X, p)$ is not a Cayley map. □

Remark 5.4. There are many examples of chiral bipartite Cayley maps. One good source of such maps is the paper [10] the authors of which construct infinitely many orientably regular embeddings of $K_{n,n}$, with $n = p^e$ where p is an odd prime and $e \geq 1$, which are Cayley maps for cyclic and dihedral groups and are chiral. Lemma 5.3 yields that all their Petrie duals are generalized Cayley maps that are not Cayley maps. A simpler family of chiral bipartite Cayley maps, brought to our attention by one of our referees, is also formed by the torus maps $\{4, 4\}_{b,c}$, for b, c both non-zero, $b + c$ even, and $b \neq c$ to guarantee chirality, and the torus maps $\{6, 3\}_{b,c}$. The chiral bipartite $\{4, 4\}_{b,c}$ maps have orders $b^2 + c^2$ and arise by identifying opposite sides of squares with corners $(0, 0)$, (b, c), $(b-c, b+c)$ and $(-c, b)$ in a unit rectangular grid.

Example 5.5. Consider the dihedral groups

$$\mathbb{D}_n = \left\langle a, b \mid a^n = b^2 = 1,\ bab^{-1} = a^{-1} \right\rangle,$$

$n \geq 3$. Taking $G = \mathbb{D}_n$, $H = \langle a \rangle$ and $X = \{b, ba, \ldots, ba^{n-1}\}$ yields a pair of mutually Petrie dual generalized Cayley maps $CM(G, X, p)$ and $GCM(G, H, X, p)$, for every cyclic permutation p of X. All such maps are orientable embeddings of the complete bipartite graph $K_{n,n}$. Taking the specific permutation $p = (b, ba, \ldots, ba^{n-1})$ yields two (possibly isomorphic) embeddings of $K_{n,n}$. First, the Cayley map

$$CM(\mathbb{D}_n, \{b, ba, \ldots, ba^{n-1}\}, (b, ba, \ldots, ba^{n-1}))$$

is an orientable embedding of $K_{n,n}$ with n faces of length $2n$ and of genus $\frac{(n-1)(n-2)}{2}$, while its Petrie dual

$$GCM(\mathbb{D}_n, \langle a \rangle, \{b, ba, \ldots, ba^{n-1}\}, (b, ba, \ldots, ba^{n-1}))$$

is an orientable embedding of $K_{n,n}$ with the same number of faces of the same length and of the same genus.

The excluded case $n = 2$ results in an embedding of $K_{2,2}$ in the sphere of the form

$$CM(\mathbb{Z}_2^2, \{(0,1), (1,1)\}, ((0,1), (1,1)))$$

which is its own Petrie dual because the inverse of the permutation $((0,1), (1,1))$ is $((0,1), (1,1))$ again.

The toroidal maps for case $n = 3$ are pictured below.

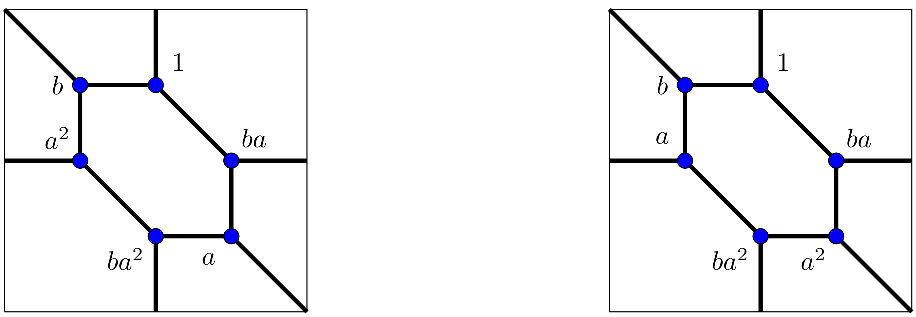

Figure 1: Maps $CM(\mathbb{D}_3, \{b, ba, ba^2\}, (b, ba, ba^2))$ and $GCM(\mathbb{D}_3, \langle a \rangle, \{b, ba, ba^2\}, (b, ba, ba^2))$.

Clearly, the two maps pictured in Figure 1 are isomorphic. Hence, the smallest maps $CM(\mathbb{D}_3, \{b, ba, ba^2\}, (b, ba, ba^2))$ and $CM(\mathbb{Z}_2^2, \{(0,1), (1,1)\}, ((0,1), (1,1))))$ are self-Petrie-dual. This might come as a surprise, as the same permutations of vertices (coming from left multiplications) cannot simultaneously extend into orientation preserving and orientation reversing automorphisms. It is, however, easy to notice that relabeling the vertices of $CM(\mathbb{D}_3, \{b, ba, ba^2\}, (b, ba, ba^2))$ changes the permutation actions of left-multiplications, and hence left multiplications with respect to one labeling may extend to orientation preserving while left multiplications with respect to another labeling may extend to orientation reversing automorphisms. Since in non-bipartite cases the Petrie dual of an orientable map is non-orientable, the case of bipartite orientable maps is the only case where one may encounter orientable self-Petrie-dual generalized Cayley maps.

We devote the rest of this section to the investigation of such possibility, i.e., to the study of orientable self-Petrie-dual generalized Cayley maps. Namely, suppose an orientable generalized Cayley map $\mathcal{M}$ is self-Petrie-dual. Then, $\mathcal{M}$ is bipartite and Theorem 5.1 implies that the automorphism group of $\mathcal{M}$ contains two subgroups acting regularly on the vertices, a subgroup of orientation preserving automorphisms as well as a subgroup of automorphisms half of which is orientation reversing. This yields a necessary condition for $\mathcal{M}$ being self-Petrie-dual, namely, $\mathcal{M}$ must be a Cayley map admitting at least one orientation reversing automorphism. Thus, chiral bipartite Cayley maps are never self-Petrie-dual (in Example 5.4, we have already encountered infinitely many such maps constructed in [10]). In fact, it is well-known that no chiral maps are self-Petrie-dual.

These observations appear to suggest that in order to construct bipartite Cayley maps that are self-Petrie-dual, one should consider maps with many orientation preserving and many orientation reversing automorphisms. In fact, searching through the literature for orientable self-Petrie-dual maps resulted only in regular orientable self-Petrie-dual maps [2, 11, 17], with the methods used in [11] or [17] relying on regularity and producing large maps of large genera. An infinite family of regular examples also arises as follows.

Remark 5.6. For every positive integer $n \geq 2$ that is relatively prime to $\varphi(n)$ (i.e., there exists a unique group of order n), there exists exactly one orientably regular embedding of $K_{n,n}$ [9]. That means that such embedding is necessarily isomorphic to its mirror reflection, hence regular, as well as self-Petrie dual. Kwak and Kwon in [13] classified the orientable regular self-Petrie embeddings of $K_{n,n}$.

Nevertheless, even in the extreme case when $\mathcal{M}$ is regular, it might happen that *any* orientation preserving subgroup $K \leq \mathrm{Aut}(\mathcal{M})$ of order half the number of vertices of $\mathcal{M}$ paired with *any* orientation reversing automorphism $\psi \in \mathrm{Aut}(\mathcal{M})$ generate a group of order larger than the number of vertices of $\mathcal{M}$. In that case, $\mathrm{Aut}(\mathcal{M})$ does not contain a subgroup of automorphisms containing an orientation reversing automorphism and acting regularly on the vertices. Hence, $\mathcal{M}$ cannot be isomorphic to any $GCM(G, H, X, p)$, and hence cannot be self-Petrie-dual. To illustrate this possibility, in the following example we present an infinite family of bipartite regular Cayley maps none of which are self-Petrie-dual. We are thankful to Gareth Jones who pointed out this example to us.

Remark 5.7. The authors of [3] considered regular embeddings of $K_{n,n}$ having the property that the orientation preserving automorphism group of the embedding that does not move the bipartite sets is not metacyclic; in which case n is necessarily a power of 2. For each such $n \geq 8$, they construct four non-isomorphic regular orientable embeddings of $K_{n,n}$, denoted $\mathcal{N}(n; k, l)$, $k, l \in \{0, 1\}$. They prove that the maps $\mathcal{N}(n; 0, 0)$ and $\mathcal{N}(n; 1, 0)$ are self-Petrie-dual, while $\mathcal{N}(n; 0, 1)$ and $\mathcal{N}(n; 1, 1)$ are Petrie duals of each other. They also show that all four maps are Cayley maps. This means, in particular, that each of the maps $\mathcal{N}(n; 0, 1)$ and $\mathcal{N}(n; 1, 1)$ is both a Cayley map and an orientable generalized Cayley map, and as such, they both admit both an orientation preserving as well as an orientation reversing automorphism group acting regularly on the vertices of the maps.

Hence, not even the existence of both types of vertex-regular groups is sufficient for making a generalized Cayley map self-Petrie. It is hard to say whether these non-metacyclic maps are in any way typical, and whether a regular bipartite Cayley map is more likely to be self-Petrie-dual or not. The following highly specialized result appears to be of some relevance toward answering this question:

Theorem 5.8 ([17]). *There are no regular, self-dual, self-Petrie-dual, normal Cayley maps with odd vertex degree.*

6 Non-orientable generalized Cayley maps and their Petrie duals

Even though constructing non-orientable generalized Cayley maps is relatively easy – it is enough to consider the Petrie dual of any non-bipartite orientable generalized Cayley map – a more explicit construction of such maps is needed. When considering only the Petrie duals of orientable generalized Cayley maps, one would never encounter a non-orientable generalized Cayley map whose Petrie dual is not orientable. To see that such maps might exist, one just needs to realize that the Petrie dual of a non-orientable generalized Cayley map whose underlying graph is bipartite (we will construct such maps) is necessarily non-orientable. The present section differs from the previous section which was concerned with maps whose groups are transitive on arcs (regular and orientably regular). In this section, we abandon the focus on such highly symmetric generalized Cayley maps and seek only to provide examples of various non-orientable generalized Cayley maps.

To specify an embedding of a graph in a non-orientable surface, one needs to specify the local orientation of outgoing darts around every vertex as well as to specify for each edge whether the two local orientations associated with its end-vertices are consistent or not. More precisely, for any given edge e incident with vertices u and v, let $\mathcal{U}$ be an open neighborhood of e which includes both u and v and all of e, but no other vertices and no other complete edges. Then $\mathcal{U}$ is homeomorphic to an open disk, and is therefore an orientable topological object. By calling the two local orientations associated with the end-vertices of e *consistent*, we mean that the local orientations around u and v within $\mathcal{U}$ are both clockwise or both anti-clockwise. We call them *inconsistent* otherwise. Referring to the ribbon graph associated with the map, the edges with inconsistent end-vertex orientations are sometimes also called *twisted*, as the strip containing the edge e and the vertices u and v is twisted before being attached to the rest of the ribbon graph through the vertices u and v. To indicate whether the end-vertex orientations of an edge are consistent or not, one usually assigns 1 or -1 to the edge, respectively (or, in a visualization of the graph embedding using local rotations, an edge with inconsistent end-vertex orientations is marked by an 'x'). Perhaps the biggest disadvantage of this description of an embedding of a graph is the fact that one does not have a guarantee that the resulting embedding is indeed non-orientable. For example, the bipartite orientable maps $GCM(G,H,X,p)$ defined in the previous section can also be described using this 'non-orientable' description by saying that $GCM(G,H,X,p)$ is the map in which the local rotation at each vertex is equal to p and all edges have inconsistent end-vertex orientations.

As explained in the previous sections, a generalized Cayley map must be an embedding of a Cayley graph $C(G,X)$ with the property that each left multiplication by the elements of G extends into a map automorphism of the embedding. One of the finer points to be dealt with in the previous section was the fact that the extensions might be either orientation preserving or orientation reversing. We do not have two kinds of map automorphisms in non-orientable maps which makes the situation a bit easier.

Let us start by describing general embeddings of Cayley graphs. Let $C(G,X)$ be a Cayley graph, let ρ_g denote the local rotation of the elements of X around g, and let $\iota_{\{g,gx\}} \in \{1,-1\}$ be the label assigned to the edge $\{g,gx\}$. The three defining involutions r_0, r_1, r_2 are then defined as follows:

$$(g, c_{\{g,gx\}}, c_F)^{r_0} = \begin{cases} (gx, c_{\{g,gx\}}, c_F), & \text{if } \iota_{\{g,gx\}} = 1, \\ (gx, c_{\{g,gx\}}, c_{F'}), & \text{if } \iota_{\{g,gx\}} = -1 \end{cases},$$

$$(g, c_{\{g,gx\}}, c_F)^{r_1} = (g, c_{\{g,g\rho_g^{\iota\{g,gx\}}(x)\}}, c_F), \text{ and } (g, c_{\{g,gx\}}, c_F)^{r_2} = (g, c_{\{g,gx\}}, c_{F'}),$$

where F' is the face adjacent to F across the edge $\{g, gx\}$.

The following characterization of generalized Cayley maps can be already found in both [19] and [12]. As stated above, it covers both the orientable and the non-orientable generalized Cayley maps.

Theorem 6.1 ([12, 19]). *A (orientable or non-orientable) map $\mathcal{M}$ is a generalized Cayley map if and only if it is an embedding of a Cayley graph $C(G, X)$ with all local cyclic permutations ρ_g, $g \in G$, equal to a fixed cyclic permutation p of X, and the twist distribution ι satisfying the property $\iota_{\{g,gx\}} = \iota_{\{g',g'x\}}$, for all $g, g' \in G$ and $x \in X$.*

The simplest interpretation of the last condition is that all edges labelled by the same generator (or its inverse) must be either all simultaneously un-twisted or all twisted, and hence, from now on, we will assume that ι acts on the set X, $\iota\colon X \to \{-1, 1\}$, mapping $x \in X$ to $\iota_{\{g,gx\}}$ (thus, in particular, $\iota(x) = \iota(x^{-1})$), for all $x \in X$. As each generalized Cayley map is determined by the four-tuple (G, X, p, ι), we will denote these maps by $GCM(G, X, p, \iota)$.

The authors of [12] generalized this simplification even further and introduced the following convenient notation.

Let $\mathcal{M} = GCM(G, X, p, \iota)$ be a generalized Cayley map. Without loss of generality, we may assume that $X = \{x_0, x_1, \ldots, x_{d-1}\}$, while $p(x_i) = x_{i+1}$, for all $i \in [d] = \{0, 1, 2, \ldots d-1\}$ (with the addition performed modulo d). If we denote the *distribution of inverses* in X by the function $\kappa\colon [d] \to [d]$, satisfying the property $(x_i)^{-1} = x_{\kappa(i)}$, and use the fact that ι assigns the same value to all edges arising from right mutiplication by x_i to define (with just a hint of abuse of the notation) $\iota\colon [d] \to \{-1, 1\}$, $\iota(i) = \iota_{\{1_g, x_i\}}$ (note that $\iota(\kappa(i)) = \iota(i)$, for all $i \in [d]$), we may associate the flags of $\mathcal{M}$ with the ordered triples from $G \times [d] \times \{-1, 1\}$, with the neighboring flags $(g, c_{(g,gx_i)}, c_F)$ and $(g, c_{(g,gx_i)}, c_{F'})$ corresponding to the triples $(g, i, 1)$ and $(g, i, -1)$. The defining involutions take then the particularly simple form:

$$(g, i, j)^{r_0} = (gx_i, \kappa(i), \iota(i)j), \ (g, i, j)^{r_1} = (g, i + j, j), \text{ and } \ (g, i, j)^{r_2} = (g, i, -j).$$

Remark 6.2. It is now easy to construct infinitely many non-orientable generalized Cayley maps. As is well-known (see e.g., [4]), an embedding of a graph determined by local rotations and a twisting function is non-orientable if and only if it contains at least one cycle with an odd number of twisted edges. Hence, for example, it is easy to construct infinitely many non-orientable embeddings of the complete bipartite graphs $K_{n,n}$. To get such an embedding, one can take a bipartite $C(G, X)$ with G containing a subgroup H of index 2, $X = G \setminus H = \{x_0, x_1, \ldots, x_{|G|/2-1}\}$, satisfying the only additional condition that $x_0^2 \neq x_2^{-1} x_1^{-1}$, choose any cyclic permutation p of X, twist the edges labeled by x_0, and observe that the 4-cycle consisting of the vertices $1_G, x_0, x_0x_1, x_0x_1x_2, x_0x_1x_2(x_0x_1x_2)^{-1} = 1_G$ obtained by successive multiplications by x_0, x_1, x_2 and $(x_0x_1x_2)^{-1}$ (which, being an odd product of elements from X, necessarily belongs to X) contains the twisted edge labeled

x_0 exactly once (if $(x_0x_1x_2)^{-1}$ were equal to x_0, we would get the identity $x_2^{-1}x_1^{-1} = x_0^2$ which we explicitely prohibited). Hence, all such generalized Cayley maps $GCM(G, X, p, \iota)$ are unorientable embeddings of complete bipartite graphs, and there are clearly infinitely many of them. Note also that all such maps, being embeddings of bipartite graphs, have the property that their Petrie dual is non-orientable.

Next, we bring in the Petrie dual operator again and fully describe its action on generalized Cayley maps. The mapping $-\iota$ used in the statement of the following theorem is the twisting function $-\iota(x) = (-1) \cdot \iota(x)$, for all $x \in X$.

Theorem 6.3. *If $\mathcal{M} = GCM(G, X, p, \iota)$, then $P(\mathcal{M}) = GCM(G, X, p, -\iota)$.*

Proof. It is easy to see that $-\iota$ is a correctly defined twisting function of a generalized Cayley map, i.e., $-\iota(x) = -\iota(x^{-1})$, for all $x \in X$. Since the two maps $GCM(G, X, p, -\iota)$ and $P(GCM(G, X, p, \iota))$ have the same underlying graphs, one way to prove our theorem is to show that the Petrie polygons of $GCM(G, X, p, \iota)$ and the faces of the map $GCM(G, X, p, -\iota)$ are identical. This can be shown by performing a careful calculation of the boundaries of the two oriented Petrie polygons of $GCM(G, X, p, \iota)$ that start at (g, i) (which might turn out to be the same polygon visiting (g, i) twice, once followed along p and once followed along p^{-1}, depending on whether we first turn 'right' or 'left') and the boundaries of the corresponding faces in $GCM(G, X, p, -\iota)$. As pointed out by one of our referees, it can also be deduced from the fact that if r_0, r_1, r_2 are the generators of the monodromy group of $GCM(G, X, p, \iota)$, then r_0r_2, r_1, and r_2 are the generators for the monodromy group of $P(GCM(G, X, p, \iota))$ (Section 4) while they can also be easily seen to be the generators of the monodromy group of $GCM(G, X, p, -\iota)$. $\square$

Based on the above theorem, obtaining the Petrie dual for a Cayley map is very simple. Namely, the Petrie dual of a Cayley map $CM(G, X, p)$ is the generalized Cayley map $GCM(G, X, p, \iota)$, where ι twists all the edges of the map, i.e., $\iota(x) = -1$, for every $x \in X$. Since all the edges are twisted, the Petrie dual of a Cayley map $\mathcal{M}$ is non-orientable if and only if $\mathcal{M}$ contains at least one odd cycle (i.e., the underlying graph is non-bipartite). Hence, Theorem 5.2 is a corollary of Theorem 6.3.

Example 6.4. To obtain a specific example of the use of Theorem 6.3, we go back to a classical example of a pair of mutually Petrie dual maps, namely the regular embedding of the tetrahedron on the sphere and its Petrie dual embedded in the projective plane. Since tetrahedron is the Cayley map $CM(\mathbb{Z}_2 \times \mathbb{Z}_2, \{(1,0), (0,1), (1,1)\}, ((1,0), (0,1), (1,1)))$ with 4 triangular faces, its Petrie dual is the generalized Cayley map

$$GCM(\mathbb{Z}_2 \times \mathbb{Z}_2, \{(1,0), (0,1), (1,1)\}, ((1,0), (0,1), (1,1)), \iota),$$

with $\iota(x) = -1$ for all $x \in \{(1,0), (0,1), (1,1)\}$. Since the tetrahedron contains odd cycles (triangles), the dual map is non-orientable, well-known to be an embedding in the projective plane with 3 faces of length 4 (which can also be easily verified via direct calculations in the generalized Cayley map).

This example can be viewed as the first member of an infinite family of examples. For every *even* $n \geq 4$, let $\mathbb{D}_n = \langle a, b \mid a^n = b^2 = 1, bab = a^{-1} \rangle$ be the dihedral group of order $2n$, $X = \{b, a^{\frac{n}{2}}, ba\}$ be a set of three involutions, and $p = (b, a^{\frac{n}{2}}, ba)$. Since n is assumed even, the Petrie dual of the trivalent Cayley map $CM(\mathbb{D}_n, X, p)$ is non-orientable (as the underlying graph of both maps contains a cycle of lenght $n+1$). For even $n \geq 4$, the Petrie dual of $CM(G, X, p)$ looks 'like' the Petrie dual of the tetrahedron: a $2n$-cycle with inside chords at every other vertex of the cycle, and an outside edge for the rest. All these Petrie duals are embeddings in the projective plane with one face of length $2n$ and all the remaining faces of length 4.

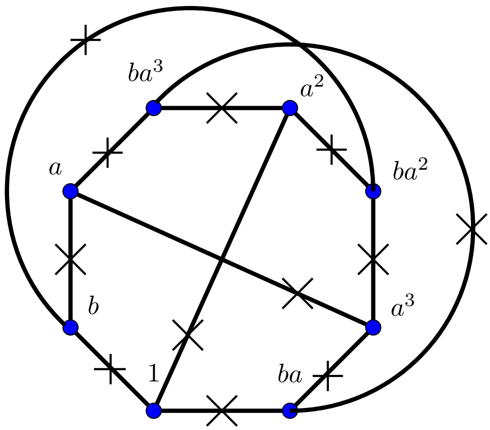

Figure 2: Representation of the Petrie dual of the map $CM(\mathbb{D}_4, \{b, a^2, ba\}, (b, a^2, ba))$ indicating local rotations and marking the edges with inconsistent end-vertex rotations of its underlying graph.

Next, we characterize non-orientable generalized Cayley maps whose Petrie dual is orientable. As should be expected, those are the 'obvious' Petrie duals of non-bipartite orientable Cayley maps, i.e., the non-bipartite non-orientable generalized Cayley maps satisfying $\iota(x) = -1$, for all $x \in X$, as well as other maps with a rather special twisting function. The Petrie duals of all other non-orientable generalized Cayley maps are non-orientable.

Corollary 6.5. *The Petrie dual of a non-orientable generalized Cayley map $GCM(G, X, p, \iota)$ is orientable if and only if the underlying Cayley graph $C(G, X)$ contains an odd cycle and every cycle of $C(G, X)$ contains an even number of untwisted edges.*

Proof. Suppose that $\mathcal{M} = GCM(G, X, p, \iota)$ is non-orientable while its Petrie dual $P(\mathcal{M}) = GCM(G, X, p, -\iota)$ is orientable. This means that $\mathcal{M}$ cannot be bipartite as that would make the orientable $P(\mathcal{M})$ bipartite and hence by Theorem 5.2 its Petrie dual $P(P(\mathcal{M})) = \mathcal{M}$ orientable; which it is not. Thus $\mathcal{M}$ contains an odd cycle. If any cycle in $\mathcal{M}$ contained an odd number of untwisted edges, that very same cycle would contain an odd number of twisted edges in $P(\mathcal{M})$; making it non-orientable. This proves one implication.

Suppose now that $\mathcal{M}$ is non-orientable, contains an odd length cycle and every cycle of $\mathcal{M}$ contains an even number of untwisted edges. Then $\mathcal{M}$ is not bipartite and its Petrie dual contains no cycles with an odd number of twisted edges. □

As stated prior to Theorem 6.3, non-orientable non-bipartite generalized Cayley maps with all edges twisted trivially satisfy the property that their Petrie dual is also non-orientable. Next, we provide an infinite family of examples that satisfy the conditions of Theorem 6.3 but contain both twisted and untwisted edges.

Example 6.6. Let Γ be a bipartite d-regular graph, d-odd, let $\mathcal{M}$ be any orientable embedding of Γ, and ρ and λ be the generators for the monodromy group G of $\mathcal{M}$. Consider the Cayley graph $C(G, \{\rho, \rho^{-1}, \lambda\})$. There is an easy way to visualize this graph. It is the underlying graph of the *truncation* $T(\mathcal{M})$ of $\mathcal{M}$, i.e., the map obtained from $\mathcal{M}$ by locally removing its vertices and replacing them by d-cycles attached to the dangling edges in the order determined by the local rotation. We claim that every cycle of $C(G, \{\rho, \rho^{-1}, \lambda\})$ contains an even number of edges labeled λ. To see this, take any cycle C of $C(G, \{\rho, \rho^{-1}, \lambda\})$. The edges labeled λ trace the 'preimage' of this cycle in Γ, which is necessarily a union of edge disjoint even cycles. Since every edge labeled λ in C corresponds to exactly one edge of the preimage, the number of edges labeled λ in C is even. Thus, any choice of p together with the twisting function $\iota(\rho) = \iota(\rho^{-1}) = -1, \iota(\lambda) = 1$ makes the non-orientable $GCM(G, X, p, \iota)$ satisfy the conditions of Theorem 6.3.

The visualization of this example is in fact quite simple. We start with a 'bipartite' map whose 'vertices' are the d-cycles formed by edges labelled ρ, all of which are twisted, with the edges labelled λ and connecting the two sets of d-cycles untwisted. This map is non-orientable, since d is required to be odd. The orientable dual untwists the d-cycles and twists the connecting edges.

To conclude this section, we consider one more classical concept from topological graph theory – the *orientable double covering* $\mathcal{M}^o$ of a non-orientable map $\mathcal{M}$ given by its underlying graph, rotation system and twisting function. The underlying graph of the double covering is the $\mathbb{Z}_2$-lift of the underlying graph of the original map with the untwisted edges given the voltage $0 \in \mathbb{Z}_2$, and the twisted edges receiving the voltage $1 \in \mathbb{Z}_2$. Informally, it is the double cover of the underlying graph of the original map with the untwisted edges each lifted into two edges connecting vertices in the same layer, and the two lifts of the twisted edges crossing from one layer to the other (for more details consult [20]). Theorem 2.4 in [20] asserts that $\mathrm{Aut}(\mathcal{M})$ lifts into the orientation preserving automorphism group $\mathrm{Aut}^+\mathcal{M}^o$, and the full group $\mathrm{Aut}(\mathcal{M})^o$ is the direct product of $\mathrm{Aut}^+\mathcal{M}^o$ with some orientation reversing involutory automorphism ψ of order 2 (that swaps the two layers of the double covering). It follows that the direct product of the lift of a subgroup of $\mathrm{Aut}(\mathcal{M})$ acting regularly on the vertices of $\mathcal{M}$ with the group $\langle\psi\rangle$ acts regularly on the vertices of $\mathcal{M}^o$, and we obtain:

Theorem 6.7. *The orientable double covering $\mathcal{M}^o$ of any non-orientable generalized Cayley map $\mathcal{M}$ is an orientable generalized Cayley map admitting a group of automorphisms containing an orientation reversing automorphism and acting regularly on its vertices.*

7 Regular generalized Cayley maps

In conclusion of our paper, we would like to direct the reader toward [12] where the authors of that paper developed a complete theory of regular generalized Cayley maps including necessary and sufficient conditions based on the existence of special permutations of the elements of the underlying group. Their language, however, is different from the one used here, and they do not consider the orientable generalized Cayley maps $GCM(G, H, X, p)$ as a special case.

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On p-gonal fields of definition*

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Abstract

Let S be a closed Riemann surface of genus $g \geq 2$ and φ be a conformal automorphism of S of prime order p such that $S/\langle\varphi\rangle$ has genus zero. Let $\mathbb{K} \leq \mathbb{C}$ be a field of definition of S. We prove the existence of a field extension $\mathbb{F}$ of $\mathbb{K}$, of degree at most $2(p-1)$, for which S is definable by a curve of the form $y^p = F(x) \in \mathbb{F}[x]$, in which case φ corresponds to $(x,y) \mapsto (x, e^{2\pi i/p}y)$. If, moreover, φ is also definable over $\mathbb{K}$, then $\mathbb{F}$ can be chosen to be at most a quadratic extension of $\mathbb{K}$. For $p = 2$, that is when S is hyperelliptic and φ is its hyperelliptic involution, this fact is due to Mestre (for even genus) and Huggins and Lercier-Ritzenthaler-Sijslingit in the case that $\mathrm{Aut}(S)/\langle\varphi\rangle$ is non-trivial.

Keywords: Riemann surfaces, p-gonal curves, automorphisms.

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1 Introduction

In [23], H. A. Schwarz proved that the group $\mathrm{Aut}(S)$ of conformal automorphisms of a closed Riemann surface S of genus $g \geq 2$ is finite. Later, in [17], A. Hurwitz obtained the upper bound $|\mathrm{Aut}(S)| \leq 84(g-1)$ (this is known as the Hurwitz's bound).

Let $p \geq 2$ be a prime integer. We say that a closed Riemann surface S is *cyclic p-gonal* if there exists some $\varphi \in \mathrm{Aut}(S)$ of order p such that the quotient orbifold $S/\langle\varphi\rangle$ has genus zero. In this case, φ is called a *p-gonal automorphism* and the cyclic group $\langle\varphi\rangle$ a *p-gonal group* of S. The case $p = 2$ corresponds to S being hyperelliptic and φ its (unique) hyperelliptic involution. The case $p = 3$ was studied by R. D. M. Accola in [1]. In [10], G. González-Diez proved that p-gonal groups are unique up to conjugation in $\mathrm{Aut}(S)$. In [13], it was observed that, if $p \geq 5n - 7$, where $n \geq 3$ is the number of fixed

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points of φ, then $\langle\varphi\rangle$ is the unique p-group in $\mathrm{Aut}(S)$. Results concerning automorphisms of p-gonal Riemann surfaces can be found, for instance, in [2, 3, 4, 5, 11, 27].

As a consequence of the Riemann-Roch theorem, a closed Riemann surface S can be described by an (either affine or projective) irreducible complex algebraic curve, i.e., after desingularization (if it is non-smooth) and filling at some punctures in the affine case, it carries a Riemann surface structure which is biholomorphic to that of S (see Remark 1.1 for the case of cyclic p-gonal surfaces). A subfield $\mathbb{K}$ of the field $\mathbb{C}$ of complex numbers is called a *field of definition* of S (or that S is *definable* over $\mathbb{K}$) if there is an irreducible algebraic curve representing S, which is defined as the common zeroes of some polynomials with coefficients in $\mathbb{K}$. The intersection of all the fields of definition of S is called the *field of moduli* of S. In general, it is not a field of definition (see Section 3).

If we are given a (finite) group $G < \mathrm{Aut}(S)$ and the geometrical structure of the quotient orbifold S/G, then it is not a simple task to find an algebraic curve for S reflecting the action of G. A family of surfaces for which algebraic models are well known is the case of cyclic p-gonal Riemann surfaces, which we proceed to recall below.

Let S be a p-gonal Riemann surface, $\varphi \in \mathrm{Aut}(S)$ be a p-gonal automorphism and $\pi\colon S \to \widehat{\mathbb{C}}$ be a regular branched cover with $\langle\varphi\rangle$ its deck group. Let $\{a_1,\dots,a_m\} \subset \widehat{\mathbb{C}} = \mathbb{C}\cup\{\infty\}$ be the set of branch values of π. If $a_j \neq \infty$, for every $j = 1,\dots,m$, then there exist integers $n_1,\dots,n_m \in \{1,\dots,p-1\}$, $n_1+\cdots+n_m \equiv 0 \bmod p$, such that S is defined by the affine, irreducible and smooth *p-gonal curve* with equation

$$E:\ y^p = F(x) = \prod_{j=1}^{m}(x-a_j)^{n_j} \in \mathbb{C}[x]. \tag{1.1}$$

If one of the branch values is equal to ∞, say $a_m = \infty$, then in (1.1) we delete the corresponding factor $(x-a_m)^{n_m}$ and assume $n_1+\cdots+n_{m-1} \not\equiv 0 \bmod p$. In this affine algebraic model, $\pi(x,y) = x$ and $\varphi(x,y) = (x,\omega_p y)$, where $\omega_p = e^{2\pi i/p}$. In the hyperelliptic case, i.e., $p=2$, in the above one has $m \in \{2g+1, 2g+2\}$ and $n_j = 1$.

Remark 1.1. The affine curve (1.1) is smooth at those points (x,y), where $y \neq 0$. At a point $(a_j,0)$, the curve is smooth exactly when $n_j = 1$ (anyway, if $n_j > 1$, it has a neighborhood homeomorphic to a disc). An irreducible projective algebraic curve defining S is obtained from the above affine one as

$$\widehat{E}:\ y^p z^{n_1+\cdots+n_m-p} = \prod_{j=1}^{m}(x-a_j z)^{n_j}. \tag{1.2}$$

As in the affine model, the projective curve $\widehat{E}$ is smooth at the points $[x:y:1]$, where $y \neq 0$. At the points $[a_j:0:1]$ it is smooth if and only if $n_j = 1$ (again, in the other cases there is a neighborhood homeomorphic to a disc). The curve is also non-smooth at the point $[0:1:0]$. After normalization of the curve, one obtains a closed Riemann surface which is biholomorphic to S. In this case, $\pi([x:y:z]) = x/z$ and $\varphi([x:y:z]) = [x:\omega_p y:z]$.

If $\mathbb{F}$ is a subfield of $\mathbb{C}$ such that in (1.1) we have $F(x) \in \mathbb{F}[x]$, then we say that $\mathbb{F}$ is a *p-gonal field of definition* of S (and that S is *cyclically p-gonally defined over* $\mathbb{F}$). Note that there are infinitely many different p-gonal fields of definition for S (for instance, if T is a Möbius transformation, then we may replace the values a_j by $T(a_j)$).

Given a field of definition of a p-gonal Riemann surface S, it is not clear at first sight if it is a p-gonal field of definition. Also, it might be that a minimal p-gonal field of definition is not a minimal field of definition (see the exceptional case $(m, p) = (4, 3)$ in Section 4.1). This paper aims to provide an argument to show that, given any field of definition $\mathbb{K}$ of S, there is a p-gonal field of definition $\mathbb{F}$ which is an extension of degree at most $2(p-1)$ over $\mathbb{K}$.

If φ is an automorphism of S, then we say that S and φ are simultaneously defined over $\mathbb{K}$ if there is an algebraic curve model of S, defined over $\mathbb{K}$, such that φ is given by a rational map on it with coefficients in $\mathbb{K}$.

Theorem 1.2. *Let S be a cyclic p-gonal Riemann surface of genus $g \geq 2$, with a p-gonal automorphism φ, and let $\mathbb{K}$ be a field of definition of S. Then*

(1) *There is p-gonal field of definition of S, this being an extension of degree at most $2(p-1)$ of $\mathbb{K}$ (which is also a field of definition of φ).*

(2) *If both S and φ are simultaneously defined over $\mathbb{K}$, then there is a p-gonal field of definition of S, this being an extension of degree at most two of $\mathbb{K}$.*

(3) *If in* Equation (1.1) *$n_1 = \cdots = n_m$, then there is a p-gonal field of definition of S, this being an extension of degree at most two of $\mathbb{K}$.*

Remark 1.3. Theorem 1.2 is still valid if we change $\mathbb{C}$ to any algebraically closed field, where in positive characteristic we need to assume that p is different from the characteristic.

Remark 1.4. For each integer $n \geq 2$, not necessarily prime, the definition of cyclic n-gonal Riemann surface S, n-gonal automorphism φ and n-gonal group $\langle \varphi \rangle$ is the same as for the prime situation. In the particular case that every fixed point of a non-trivial power φ^k is also a fixed point of φ, the definition of an n-gonal curve is the same as in (1.1), but replacing p by n and assuming each the exponent n_j to be relatively prime to n. In this case, under the assumption that S has a unique n-gonal group $\langle \varphi \rangle$ (this is the situation for generalized superelliptic Riemann surfaces [15]), then the arguments of the proof of Theorem 1.2 allows us to obtain that: if $\mathbb{K}$ is a field of definition of S, then there is an n-gonal field of definition of S, this being an extension of degree at most $2\phi(n)$ of $\mathbb{K}$, where $\phi(n)$ is the ϕ-Euler function.

2 An application to hyperelliptic Riemann surfaces

Let S be a hyperelliptic Riemann surface (i.e., $p = 2$) with hyperelliptic involution φ and let $\mathbb{K}$ be a field of definition of S. As φ is unique, one may consider the group $\mathrm{Aut}_{red}(S) := \mathrm{Aut}(S)/\langle \varphi \rangle$, called the reduced group of automorphisms of S.

For even genus, in [22], J-F. Mestre proved that S is also hyperelliptically definable over $\mathbb{K}$. If the genus is odd, then the previous fact is in general false; as can be seen from examples in [8, 9, 20, 21]. In [16], B. Huggins proved that if $\mathrm{Aut}_{red}(S)$ is neither trivial nor cyclic, then S is also hyperelliptically definable over $\mathbb{K}$. In [21], R. Lercier, C. Ritzenthaler and J. Sijslingit proved that S can be hyperelliptically defined over a quadratic extension of $\mathbb{K}$ if the reduced group is a non-trivial cyclic group. Our theorem asserts that this fact is still valid even if the reduced group is trivial.

Corollary 2.1. *If $\mathbb{K}$ is a field of definition of a hyperelliptic Riemann surface, then it is hyperelliptically definable over an extension of degree at most two of $\mathbb{K}$.*

3 An application to fields of moduli

Let S be a closed Riemann surface and let C be an irreducible algebraic curve representing it. The *field of moduli* $\mathcal{M}_S$ of S is the fixed field of the group $\Gamma_C = \{\sigma \in \mathrm{Aut}(\mathbb{C}/\mathbb{Q}) : C^\sigma \cong C\}$; this field does not depend on the choice of the algebraic model C. In [18], S. Koizumi proved that $\mathcal{M}_S$ coincides with the intersection of all fields of definition of S, but in general it might not be a field of definition [6, 7, 12, 16, 19]. If $\mathrm{Aut}(S)$ is trivial (the generic situation for $g \geq 3$), then Weil's descent theorem [25] asserts that $\mathcal{M}_S$ is a field of definition of S. In [26], J. Wolfart proved that if $S/\mathrm{Aut}(S)$ is the Riemann sphere with exactly 3 cone points (i.e., S is quasiplatonic), then $\mathcal{M}_S$ is also a field of definition of S. In a more general setting, if $S/\mathrm{Aut}(S)$ has genus zero, then it is known that S is definable over an extension of degree at most two of $\mathcal{M}_S$ (see [14] for a more general statement).

Now, let S be a p-gonal Riemann surface of genus $g \geq 2$ and let $G = \langle \varphi \rangle < \mathrm{Aut}(S)$ be a p-gonal group. As previously noted, S is either definable over $\mathcal{M}_S$ or over a suitable quadratic extension of it (but it might not be cyclically p-gonally definable over such a minimal field of definition). In the case that G is not a unique p-gonal subgroup, in [28], A. Wootton noted that S can be cyclically p-gonally defined over an extension of degree at most 2 of its field of moduli. In the case that G is the unique p-gonal subgroup, the quotient group $\mathrm{Aut}(S)/G$ is called the *reduced group* of S. In [19], A. Kontogeorgis proved that if the reduced group is neither trivial nor a cyclic group, then S can always be defined over its field of moduli. So, a direct consequence of Theorem 1.2 is the following.

Corollary 3.1. *Let S be a cyclic p-gonal Riemann surface with a p-gonal group $G = \langle \varphi \rangle$.*

(1) *If G is not a normal subgroup of* $\mathrm{Aut}(S)$*, then S is cyclically p-gonally definable over an extension of degree at most two of $\mathcal{M}_S$.*

(2) *If G is a normal subgroup of* $\mathrm{Aut}(S)$ *and* $\mathrm{Aut}(S)/G$ *is different from the trivial group or a cyclic group, then S is cyclically p-gonally definable over an extension of degree at most* $2(p-1)$ *of $\mathcal{M}_S$. Moreover, if φ also is defined over $\mathcal{M}_S$, then the extension can be chosen to be of degree at most two.*

(3) *If $G =$* $\mathrm{Aut}(S)$*, then S is cyclically p-gonally definable over an extension of degree at most* $4(p-1)$ *of its field of moduli. Moreover, if φ also is defined over $\mathcal{M}_S$, then the extension can be chosen of degree at most* 4.

As every hyperelliptic Riemann surface is definable over an extension of degree at most two of its field of moduli, Corollary 2.1 asserts the following.

Corollary 3.2. *Every hyperelliptic Riemann surface is hyperelliptically definable over an extension of degree at most* 4 *of its field of moduli. Moreover, if either* (i) *the genus is even or* (ii) *the genus is odd and the reduced group is not trivial, then the hyperelliptic Riemann surface is hyperelliptically defined over an extension of degree at most* 2 *of its field of moduli.*

Examples of hyperelliptic Riemann surfaces with a trivial reduced group that cannot be defined over their field of moduli were provided by C. J. Earle [6, 7] and G. Shimura [24]. The same type of examples, but with a non-trivial cyclic reduced group, were provided by B. Huggins [16].

4 Proof of Theorem 1.2

We assume the p-gonal Riemann surface S to be provided by an irreducible curve C, defined over a subfield $\mathbb{K}$ of $\mathbb{C}$. If $\overline{\mathbb{K}}$ is the algebraic closure of $\mathbb{K}$ inside $\mathbb{C}$, then (in this algebraic model) the p-gonal automorphism φ is given by a rational map defined over $\overline{\mathbb{K}}$. We divide the arguments depending on the uniqueness of the cyclic group $\langle\varphi\rangle$.

4.1 The case when $\langle\varphi\rangle$ is not unique

The following result, due to A. Wootton, describes those cases where the uniqueness fails.

Theorem 4.1 ([28, A. Wootton]). *Let S be a cyclic p-gonal Riemann surface of genus $g \geq 2$ and let $m = 2(g+p-1)/(p-1)$. If (m,p) is different from any the following tuples*

$$\text{(i) } (3,7), \quad \text{(ii) } (4,3), \quad \text{(iii) } (4,5), \quad \text{(iv) } (5,3), \quad \text{(v) } (p,p),\ p \geq 5, \quad \text{(vi) } (2p,p),\ p \geq 3,$$

then S has a unique p-gonal group.

In the same paper, Wootton describes the exceptional cyclic p-gonal Riemann surfaces, ie., where the p-gonal group is non-unique.

(i) Case $(m,p) = (3,7)$ corresponds to Klein's quartic (a non-hyperelliptic Riemann surface of genus 3) $x^3y+y^3z+z^3x = 0$, whose group of automorphisms is $\mathrm{PGL}_2(7)$ (of order 168). This surface is cyclically 7-gonally defined as $y^7 = x^2(x-z)z^4$.

(ii) Case $(m,p) = (4,3)$ corresponds to the genus 2 Riemann surface defined hyperelliptically by $y^2z^3 = x(x^4-z^4)$, whose group of automorphisms is $\mathrm{GL}_2(3)$ (of order 48). This surface is cyclically 3-gonally defined as $y^3z^3 = (x^2-z^2)(x^2-(15\sqrt{3}-26)z^2)^2$.

(iii) Case $(m,p) = (4,5)$ corresponds to the genus 4 non-hyperelliptic Riemann surface, called Bring's curve, which is the complete intersection of the quadric $x_1x_4+x_2x_3 = 0$ and the cubic $x_1^2x_3 + x_2^2x_1 + x_3^2x_4 + x_4^2x_2 = 0$ in the 3-dimensional complex projective space. Its group of automorphisms is $\mathfrak{S}_5$, the symmetric group in five letters $\mathfrak{S}_5$. This surface is cyclically 5-gonally defined as $y^5z^5 = (x^2-z^2)(x^2+z^2)^4$.

(iv) Case $(m,p) = (5,3)$ corresponds to the genus 3 non-hyperelliptic closed Riemann surface $x^4+y^4+z^4+2i\sqrt{3}z^2y^2 = 0$, whose group of automorphisms has order 48. The quotient of that surface by its group of automorphisms has signature $(0;2,3,12)$. This surface is cyclically 3-gonally defined as $y^3z^3 = x^2(x^4-z^4)$.

(v) Case $(m,p) = (p,p)$, where $p \geq 5$, corresponds to the Fermat curve $x^p+y^p+z^p = 0$, whose group of automorphisms is $\mathbb{Z}_p^2 \rtimes \mathfrak{S}_3$. This is already in a p-gonal form as $y^p = -z^p - x^p$.

(vi) Case $(m,p) = (2p,p)$, where $p \geq 3$. There is a 1-dimensional family with group of automorphisms $\mathbb{Z}_p^2 \rtimes \mathbb{Z}_2^2$ (the quotient by that group has signature $(0;2,2,2,p)$). Also, there is a surface with group of automorphisms $\mathbb{Z}_p^2 \rtimes D_4$ (the quotient by that group has signature $(0;2,4,2p)$. These surfaces are cyclically p-gonally defined as $y^pz^p = (x^p-a^pz^p)(x^p-z^p/a^p) = x^{2p}-(a^p+1/a^p)x^pz^p+z^{2p}$.

Note that, in all the above exceptional cases, the surface S is cyclically p-gonally defined over an extension of degree at most 2 over the field of moduli. In fact, with only the exception of case (ii), S is cyclically p-gonally defined over its field of moduli. So, we are done in this situation.

4.2 The case when $\langle\varphi\rangle$ is unique

We now assume that $\langle\varphi\rangle$ is unique. Set $\Gamma = \mathrm{Gal}(\overline{\mathbb{K}}/\mathbb{K})$. Let us consider a rational map $\pi\colon C \to \mathbb{P}^1_{\overline{\mathbb{K}}}$, defined over $\overline{\mathbb{K}}$, which is a regular branched covering with $\langle\varphi\rangle$ as its deck group and whose branch values are $a_1, ..., a_m \in \mathbb{C}$ (in fact, these values belong to $\overline{\mathbb{K}}$). Let the integers $n_1, ..., n_m \in \{1, ..., p-1\}$, $n_1 + \cdots + n_m \equiv 0 \bmod p$, be such that C is isomorphic to a p-gonal curve E with Equation (1.1).

4.2.1 Proof of Part (1)

Let us recall that φ is already defined over $\overline{\mathbb{K}}$. In the next, we note that φ is defined over an extension of $\mathbb{K}$ of degree at most $p-1$.

Claim 4.2. *The rational map φ is defined over an extension $\mathbb{K}_1$ of $\mathbb{K}$ of degree at most $p-1$.*

Proof. If $\sigma \in \Gamma$, then φ^σ is an automorphism of order p of $C^\sigma = C$. As we are assuming the uniqueness of $\langle\varphi\rangle$, we must have that $\varphi^\sigma \in \Omega := \{\varphi, \varphi^2, \ldots, \varphi^{p-1}\}$. In particular, the subgroup A of Γ consisting of those σ such that $\varphi^\sigma = \varphi$ must have index at most the cardinality of the set Ω, which is $p-1$. This asserts that φ is defined over the fixed field $\mathbb{K}_1$ of A, which is an extension of degree at most $p-1$ of $\mathbb{K}$. $\square$

Set $\Gamma_1 = \mathrm{Gal}(\overline{\mathbb{K}}/\mathbb{K}_1)$. If $\tau \in \Gamma_1$, then (as the identity $I\colon C \to C = C^\tau$ conjugates $\langle\varphi\rangle = \langle\varphi\rangle^\tau = \langle\varphi^\tau\rangle$ to itself), there is a (unique) automorphism g_τ of $\mathbb{P}^1_{\overline{\mathbb{K}}}$ such that $\pi^\tau = \pi^\tau \circ I = g_\tau \circ \pi$ (see the following diagram).

$$\begin{tikzcd} C \arrow[r, "I"] \arrow[d, "\pi"'] & C = C^\tau \arrow[d, "\pi^\tau"'] \\ \mathbb{P}^1_{\overline{\mathbb{K}}} \arrow[r, "g_\tau"] & \mathbb{P}^1_{\overline{\mathbb{K}}} \end{tikzcd}$$

As the group of automorphisms of $\mathbb{P}^1_{\overline{\mathbb{K}}}$ is given by Möbius transformations (i.e., elements of $\mathrm{PGL}_2(\overline{\mathbb{K}})$), we must have $g_\tau \in \mathrm{PGL}_2(\overline{\mathbb{K}})$.

We may apply each $\sigma \in \Gamma_1$ to the above diagram to obtain the following one

$$\begin{tikzcd} C^\sigma = C \arrow[r, "I"] \arrow[d, "\pi^\sigma"'] & C = C^{\sigma\tau} \arrow[d, "\pi^{\sigma\tau}"'] \\ \mathbb{P}^1_{\overline{\mathbb{K}}} \arrow[r, "g_\tau^\sigma"] & \mathbb{P}^1_{\overline{\mathbb{K}}} \end{tikzcd}$$

The above permits us to obtain the following diagram

$$\begin{array}{ccccc} C & \xrightarrow{I} & C = C^{\sigma} & \xrightarrow{I=I^{\sigma}} & C = C^{\sigma\tau} \\ \pi \downarrow & & \pi^{\sigma} \downarrow & & \pi^{\sigma\tau} \downarrow \\ \mathbb{P}^1_{\overline{\mathbb{K}}} & \xrightarrow{g_{\sigma}} & \mathbb{P}^1_{\overline{\mathbb{K}}} & \xrightarrow{g_{\tau}^{\sigma}} & \mathbb{P}^1_{\overline{\mathbb{K}}} \end{array}$$

As the transformation g_ρ is uniquely determined by $\rho \in \Gamma_1$, the collection $\{g_\rho\}_{\rho\in\Gamma_1}$ satisfies the co-cycle relation

$$g_{\sigma\tau} = g_\tau^\sigma \circ g_\sigma, \quad \sigma, \tau \in \Gamma_1.$$

Weil's descent theorem [25] ensures the existence of a genus zero irreducible and non-singular algebraic curve B, defined over $\mathbb{K}_1$, and an isomorphism $R\colon \mathbb{P}^1_{\overline{\mathbb{K}}} \to B$, defined over $\overline{\mathbb{K}}$, so that

$$g_\sigma \circ R^\sigma = R, \quad \sigma \in \Gamma_1.$$

Also, for $\sigma \in \Gamma_1$, we have $\{\sigma(a_1), ..., \sigma(a_m)\} = \{g_\sigma(a_1), ..., g_\sigma(a_m)\}$, so it follows that $\{R(a_1), ..., R(a_m)\}$ is Γ_1-invariant.

Let us denote by $A(n_j)$ the set of those a_k's for which $n_k = n_j$.

Claim 4.3. *Each set $R(A(n_j))$ is Γ_1-invariant.*

Proof. If $\sigma \in \Gamma_1$, then (as $\pi^\sigma = g_\sigma \circ \pi$) the set $g_\sigma(A(n_j))$ corresponds to the set of those $\sigma(a_k)$ having the same n_l (for some l), that is, $g_\sigma(A(n_j)) = \sigma(A(n_l))$. As $\varphi^\sigma = \varphi$, we must have $n_l = n_j$, that is, $g_\sigma(A(n_j)) = \sigma(A(n_j))$. This last equality implies the desired claim. □

Claim 4.4. *There is an effective $\mathbb{K}_1$-rational divisor $U \geq 0$ of degree at most two in B.*

Proof. We follow similar techniques as used by Huggins in her thesis [16] (and other authors). Let us consider any $\mathbb{K}_1$-rational meromorphic 1-form ω in B. Since B has genus zero, the canonical divisor $K = (\omega)$ is a $\mathbb{K}_1$-rational of degree -2. In this way, there is a positive integer d such that the divisor $D = R(a_1) + \cdots + R(a_m) + dK$ is $\mathbb{K}_1$-rational of degree 1 or 2. If $D \geq 0$, then we set $U := D$.

Let us assume D is not effective. Let us consider the Riemann-Roch space $L(D)$, consisting of those non-constant rational maps $\phi\colon B \to \mathbb{P}^1_{\overline{\mathbb{K}}}$ whose divisors satisfy $(\phi) + D \geq 0$ together with the constant ones. As the divisor D is $\mathbb{K}_1$-rational, for every $\sigma \in \Gamma_1$ and every $\phi \in L(D)$, it follows that $\phi^\sigma \in L(D)$. This, in particular, permits us to observe that we can find a basis of $L(D)$ consisting of rational maps defined over $\mathbb{K}_1$. One of the elements of such a basis must be a non-zero constant map. As, by Riemann-Roch's theorem, $L(D)$ has dimension 2 (if D has degree one) or 3 (if D has degree two), we may find a non-constant $f \in L(D)$ belonging to such a basis (defined over $\mathbb{K}_1$). In this case, we may take $U = (f) + D \geq 0$. □

By Claim 4.4, there is an effective $\mathbb{K}_1$-rational divisor U of degree 1 or 2 and $U \geq 0$. We have three possibilities:

(1) $U = s$, where $s \in B$ is $\mathbb{K}_1$-rational; or

(2) $U = 2t$, where $t \in B$ is $\mathbb{K}_1$-rational; or

(3) $U = r + q$, where $r, q \in B$, $r \neq q$, and $\{r, q\}$ is Γ_1-invariant.

In cases (1) and (2) we have the existence of a $\mathbb{K}_1$-rational point in B. In this case, we set $\mathbb{K}_2 = \mathbb{K}_1$. In case (3) we have a point (say r) in B which is rational over a quadratic extension $\mathbb{K}_2$ of $\mathbb{K}_1$.

Let $b \in B$ be a $\mathbb{K}_2$-rational point (whose existence is provided above). By Riemann-Roch's theorem, the Riemann-Roch space $L(b)$ (where b is thought of as a divisor of degree one) has dimension 2. Similarly as above, we may choose a basis $\{1, L\}$ of $L(b)$, with each element defined over $\mathbb{K}_2$. In this case, $L\colon B \to \widehat{\mathbb{C}}$ turns out to be an isomorphism defined over $\mathbb{K}_2$.

We have that $Q = L \circ R \circ \pi\colon C \to \widehat{\mathbb{C}}$ is a Galois (branched) covering with deck group $\langle \varphi \rangle$ and whose branch values are $\{L(R(a_1)), ..., L(R(a_m))\}$. It follows that S is p-gonally defined by

$$y^p = F(x) = \prod_{j=1}^{m} (x - L(R(a_j)))^{n_j}.$$

As the sets $\{L(R(a_1)), ..., L(R(a_m))\}$ and $L(R(A(n_j)))$ are $\mathrm{Gal}(\overline{\mathbb{K}}/\mathbb{K}_2)$-invariant (by Claim 4.3 and the fact that $\mathbb{K}_1$ is a subfield of $\mathbb{K}_2$), it follows that $F(x) = \prod_{j=1}^{m} (x - L(R(a_j)))^{n_j} \in \mathbb{K}_2[x]$. As $\mathbb{K}_2$ is an extension of degree at most two of $\mathbb{K}_1$ and the last one is an extension of degree at most $p-1$ of $\mathbb{K}$, we are done.

4.2.2 Proof of Parts (2) and (3)

If φ is already defined over $\mathbb{K}$ then we assume $\mathbb{K}_1 = \mathbb{K}$ (i.e., we set $\Gamma_1 = \Gamma$) in the above arguments. Similarly, if in Equation (1.1) we have that $n_1 = \cdots = n_m = n$, then there will be only one set $A(n)$. In this case, in the previous arguments, we do not need to use Claim 4.3 (where it was needed for the choice of $\mathbb{K}_1$) and we may work as in the proof of Part (1) with $\mathbb{K}$ instead of $\mathbb{K}_1$.

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An online bin-packing problem with an underlying ternary structure

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Abstract

Following an orginal idea by Knödel, an online bin-packing problem is considered where the large items arrive in double-packs. The dual problem where the small items arrive in double-packs is also considered. The enumerations have a ternary random walk flavour, and for the enumeration, the kernel method is employed.

Keywords: Knödel walks, third-order recursion, kernel method, coefficient extraction, state diagram.

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1 Introduction

Walter Knödel introduced the following online bin-packing problem [3]: There are bins of size 1, and random items of size $\frac{2}{3}$ (large items) and of size $\frac{1}{3}$ (small items) appear and are put into the boxes. A typical scenario is that a number j of partially filled boxes exist, and the number j becomes $j+1$ resp. $j-1$, depending on whether a the new item is of large resp. small type. "At random" means that both types appear with the same probability $\frac{1}{2}$.

In my collection of examples [4], I showed how to deal with the Knödel problem using the kernel method. I was, however, not the only author who was intrigued by such questions; a notable paper is by Michael Drmota [1], which is of a more probabilistic type, whereas I tried to emphasize the combinatorial point of view.

The present paper has a certain 'ternary' flavour: the next section deals with the instance of large items appearing in double-packs. The handler breaks off the double-packs, and then treats the items as Knödel would have done. Typically, the number of partially filled boxes increases by 2 or decreases by 1. In order to keep the system balanced, we assume that the small items appear twice as often as the double-packs.

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The last section deals with the dual problem, where the small items appear in double-packs and the large items as single units.

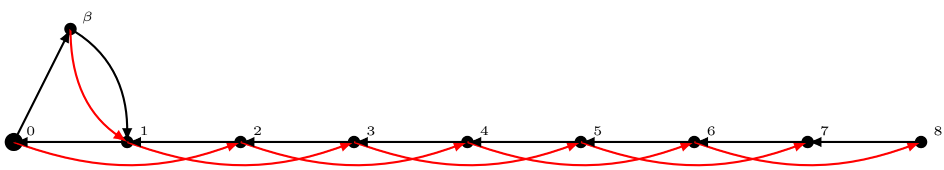
The kernel method is used to obtain all the relevant enumerations. The recent paper [5] served as an inspiration, but deals with a different issue. It must be said that, when [4] was prepared, such ternary questions would have been outside of my reach. Luckily, now, they are not.

We confine ourselves here just to enumerations, deriving explicit generating functions in one or two variables. Questions of a more probabilistic nature are not treated.

2 The first model

The following items arrive at random: a double-pack of items, each of size $\frac{2}{3}$, and an item of size $\frac{1}{3}$. We could equip the set-up with general probabilities p and $q = 1 - p$, but we restrict ourselves to the 'balanced' case where the single items are twice as likely as the double-packs, so we set $p = \frac{1}{3}$ and $q = \frac{2}{3}$.

The following state diagram (we show only a finite part of it) describes the situation. There are states representing 'i boxes filled to $\frac{2}{3}$'; a double-pack pushes the i to $i + 2$, and a single item reduces it to $i - 1$. There is an exceptional state, called β, standing for one box, filled to $\frac{1}{3}$. The red edges represent an arrival of a double-pack, and will be labelled by pz; the black edges represent an arrival of a single item, and will be labelled by qz.



From the state diagram, we set off an infinite set of generating functions in the variable z, where the coefficient of z^n is the probability that n random steps lead to state i, for $i \geq 0$ or $i = \beta$. Mostly, we just write f_i instead of $f_i(z)$. The following system of recursions can be read off immediately:

$$\begin{aligned} f_0 &= 1 + qzf_1, \quad f_\beta = qzf_0, \\ f_1 &= zf_\beta + qzf_2 = qz^2 f_0 + qzf_2, \\ f_i &= pzf_{i-2} + qzf_{i+1}, \quad i \geq 2. \end{aligned}$$

Our method to solve this system is the kernel method. For that, we introduce a bivariate generating function $F(u,z)$, but we mostly write just $F(u)$:

$$\begin{aligned}
F(u) &= \sum_{i\ge 0} u^i f_i(z)\\
&= 1 + qzf_1 + qz^2uf_0 + qzuf_2 + \sum_{i\ge 2} u^i \Big[pzf_{i-2} + qzf_{i+1}\Big]\\
&= 1 + qzf_1 + qz^2uf_0 + pzu^2F(u) + \sum_{i\ge 1} u^i qzf_{i+1}\\
&= 1 + qzf_1 + qz^2uf_0 + pzu^2F(u) + \frac{qz}{u}\Big(F(u) - f_0 - uf_1\Big)\\
&= 1 + qz^2uf_0 + pzu^2F(u) + \frac{qz}{u}\Big(F(u) - f_0\Big)
\end{aligned}$$

Note that $f_0 = F(0)$. It is beneficial to introduce the new variable $u = zU$; doing this, powers of z that appear are multiples of 3. Later, it will be convenient to set $x = z^3$. As can be seen, the numbers of steps leading to a state i belong to just one residue class modulo 3. We compute

$$F(u) = \frac{-3U - 2z^3U^2f_0 + 2f_0}{z^3U^3 - 3U + 2} = \frac{-3U - 2xU^2f_0 + 2f_0}{xU^3 - 3U + 2}.$$

As it is common using the kernel method, setting $U = 0$ leads to a void equation. However, factorizing the denominator is the method of choice. There is 'bad' factor in the denominator, which must also appear in the numerator, which allows us to compute f_0 and consequently the whole bivariate generating function. In order to deal with the ternary equation successfully, we further set $x = z^3 = \frac{27}{4}t(1-t)^2$ and we find the 3 roots

$$U_1 = \frac{2}{3(1-t)}, \quad U_2 = \frac{1}{\sigma}, \quad U_3 = \frac{1}{\tau},$$

with

$$\sigma = \frac{3}{4}(t - \sqrt{4t - 3t^2}\,), \quad \tau = \frac{3}{4}(t + \sqrt{4t - 3t^2}).$$

A motivation for this substitution is the Lagrange inversion formula and/or the enumeration of ternary trees; see also [6]. Plugging $U = \frac{2}{3(1-t)}$ into the numerator (this is the bad factor, as explained a little bit later), leads to

$$f_0 = \frac{1}{(1-t)(1-3t)}$$

and furthermore to the simplified numerator

$$\frac{-3U - 2xU^2f_0 + 2f_0}{U - \frac{2}{3(1-t)}} = \frac{1}{1-3t}\Big(-3 + \frac{27}{2}t(t-1)U\Big).$$

The variable x is given in terms of t. The inverse relation is of interest. It can be obtained by the Lagrange inversion formula or, as here, by contour integration:

$$\begin{aligned}
[x^k]t &= \frac{1}{2\pi i}\oint \frac{dx}{x^{k+1}}t = \frac{1}{2\pi i}\frac{27}{4}\Big(\frac{4}{27}\Big)^{k+1}\oint \frac{dt(1-t)(1-3t)}{t^{k+1}(1-t)^{2k+2}}t\\
&= \frac{1}{2\pi i}\Big(\frac{4}{27}\Big)^{k}\oint \frac{dt(1-3t)}{t^{k}(1-t)^{2k+1}} = \Big(\frac{4}{27}\Big)^{k}[t^{k-1}]\frac{1-3t}{(1-t)^{2k+1}}\\
&= \Big(\frac{4}{27}\Big)^{k}\bigg[\binom{3k-1}{k-1}-3\binom{3k-2}{k-2}\bigg],
\end{aligned}$$

which, after simplification, gives us

$$t=\sum_{k\ge1}\frac{1}{k}\binom{3k-2}{k-1}\frac{2^{2k}}{3^{3k}}x^k.$$

A similar computation leads to

$$\frac{1}{1-t}=\sum_{k\ge0}\frac{1}{2k+1}\binom{3k}{k}\frac{2^{2k+1}}{3^{3k+1}}x^k.$$

From this we infer that for $z\sim 0$, $U\sim\frac{2}{3}$, or $u\sim\frac{2}{3}z$, explaining why we are talking about the bad factor. We continue the computation:

$$\begin{aligned}
F(u) &= \frac{1}{1-3t}\Big(-3+\frac{27}{2}t(t-1)U\Big)\frac{1}{x(U-\frac{1}{\sigma})(U-\frac{1}{\tau})}\\
&= \frac{1}{1-3t}\Big(-3+\frac{27}{2}t(t-1)U\Big)\frac{\frac{9}{4}t(t-1)}{x(1-\sigma U)(1-\tau U)}\\
&= \frac{1}{(1-3t)(1-t)}\Big(1-\frac{9}{2}t(t-1)U\Big)\frac{1}{(1-\sigma U)(1-\tau U)}.
\end{aligned}$$

Partial fraction decomposition leads to (we use the abbreviation $W=\sqrt{4t-3t^2}$)

$$\begin{aligned}
\frac{1}{(1-\sigma U)(1-\tau U)} &= \frac{1}{2}\Big(1-\frac{t}{W}\Big)\frac{1}{1-\sigma U}+\frac{1}{2}\Big(1+\frac{t}{W}\Big)\frac{1}{1-\tau U}\\
&= \frac{1}{2}\Big[\frac{1}{1-\sigma U}+\frac{1}{1-\tau U}\Big]+\frac{t}{2W}\Big[\frac{1}{1-\tau U}-\frac{1}{1-\sigma U}\Big]\\
&= \frac{1}{2}\Big[\frac{1}{1-\sigma U}+\frac{1}{1-\tau U}\Big]+\frac{3t}{4(\tau-\sigma)}\Big[\frac{1}{1-\tau U}-\frac{1}{1-\sigma U}\Big].
\end{aligned}$$

For the further simplification we will resort to two identities going by the name of Girard-Waring formula, see e. g. [2]:

$$X^m+Y^m=\sum_{0\le k\le m/2}(-1)^k\binom{m-k}{k}\frac{m}{m-k}(XY)^k(X+Y)^{m-2k};$$

$$\frac{X^m-Y^m}{X-Y}=\sum_{0\le k\le (m-1)/2}(-1)^k\binom{m-1-k}{k}(XY)^k(X+Y)^{m-1-2k}.$$

Of course, we will apply them with $X = \tau$ and $Y = \sigma$. Then

$$\begin{aligned}
&[U^m]\frac{1}{2}\Big[\frac{1}{1-\sigma U}+\frac{1}{1-\tau U}\Big]\\
&\quad=\frac{1}{2}\sum_{0\le k\le m/2}(-1)^k\binom{m-k}{k}\frac{m}{m-k}\Big(\frac{9}{4}t(t-1)\Big)^k\Big(\frac{3}{2}t\Big)^{m-2k}\\
&\quad=\frac{1}{2}\Big(\frac{3}{2}\Big)^m\sum_{0\le k\le m/2}(-1)^k\binom{m-k}{k}\frac{m}{m-k}t^k(t-1)^kt^{m-2k}\\
&\quad=\frac{1}{2}\Big(\frac{3}{2}\Big)^m\sum_{0\le k\le m/2}(-1)^k\binom{m-k}{k}\frac{m}{m-k}(t-1)^kt^{m-k}
\end{aligned}$$

and

$$\begin{aligned}
&[U^m]\frac{3t}{4(\tau-\sigma)}\Big[\frac{1}{1-\tau U}-\frac{1}{1-\sigma U}\Big]\\
&\quad=\frac{3t}{4}\sum_{0\le k\le (m-1)/2}(-1)^k\binom{m-1-k}{k}\Big(\frac{9}{4}t(t-1)\Big)^k\Big(\frac{3}{2}t\Big)^{m-1-2k}\\
&\quad=\Big(\frac{3}{2}\Big)^{m-1}\frac{3t}{4}\sum_{0\le k\le (m-1)/2}(-1)^k\binom{m-1-k}{k}(t-1)^kt^{m-1-k}\\
&\quad=\frac{1}{2}\Big(\frac{3}{2}\Big)^m\sum_{0\le k\le (m-1)/2}(-1)^k\binom{m-1-k}{k}(t-1)^kt^{m-k}.
\end{aligned}$$

Combining the two leads to a pleasant simplification:

$$\begin{aligned}
&[U^m]\frac{1}{2}\Big[\frac{1}{1-\sigma U}+\frac{1}{1-\tau U}\Big]+[U^m]\frac{3t}{4(\tau-\sigma)}\Big[\frac{1}{1-\tau U}-\frac{1}{1-\sigma U}\Big]\\
&\quad=\Big(\frac{3}{2}\Big)^m\sum_{0\le k\le m/2}(-1)^k\binom{m-k}{k}(t-1)^kt^{m-k},
\end{aligned}$$

or simpler

$$[U^m]\frac{1}{(1-\sigma U)(1-\tau U)}=\Big(\frac{3}{2}\Big)^m\sum_{0\le k\le m/2}(-1)^k\binom{m-k}{k}(t-1)^kt^{m-k}.$$

We need a second similar term:

$$\begin{aligned}
&[U^m]\Big(-\frac{9}{2}t(t-1)U\Big)\frac{1}{(1-\sigma U)(1-\tau U)}\\
&\quad=-\frac{9}{2}t(t-1)[U^{m-1}]\frac{1}{(1-\sigma U)(1-\tau U)}\\
&\quad=-\frac{9}{2}t(t-1)\Big(\frac{3}{2}\Big)^{m-1}\sum_{0\le k\le m/2}(-1)^k\binom{m-1-k}{k}(t-1)^kt^{m-1-k}\\
&\quad=-3\Big(\frac{3}{2}\Big)^m\sum_{0\le k\le m/2}(-1)^k\binom{m-1-k}{k}(t-1)^{k+1}t^{m-k}.
\end{aligned}$$

Putting everything together we found

$$F(u) = \frac{1}{(1-3t)(1-t)} \sum_{m\ge 0} \frac{u^m}{z^m}\Big(\frac{3}{2}\Big)^m \sum_{0\le k\le m/2} (-1)^k \binom{m-k}{k}(t-1)^k t^{m-k}$$
$$- \frac{3}{(1-3t)(1-t)} \sum_{m\ge 0} \frac{u^m}{z^m}\Big(\frac{3}{2}\Big)^m \sum_{0\le k\le m/2} (-1)^k \binom{m-1-k}{k}(t-1)^{k+1} t^{m-k}.$$

Reading off the coefficients of z^j for $j \ge 1$ as well is now done with Cauchy's integral formula; the contours are always small circles (or equivalent) around the origin. The starting point is

$$[z^n u^j]F(u) = \Big(\frac{3}{2}\Big)^j [z^{n+j}] \frac{1}{(1-3t)(1-t)} \sum_{0\le k\le j/2} (-1)^k \binom{j-k}{k}(t-1)^k t^{j-k}$$
$$- \Big(\frac{3}{2}\Big)^j [z^{n+j}] \frac{3}{(1-3t)(1-t)} \sum_{0\le k\le j/2} (-1)^k \binom{j-1-k}{k}(t-1)^{k+1} t^{j-k}$$

and we will treat the two sums separately. There is only a contribution if $n+j \equiv 0 \bmod 3$. (This is also clear from the combinatorial context.) Assume this and set $N := \frac{n+j}{3}$.

Step 1:

$$\begin{aligned}
&[x^N] \frac{1}{(1-3t)(1-t)} \sum_{0\le k\le j/2} (-1)^k \binom{j-k}{k}(t-1)^k t^{j-k}\\
&= \frac{1}{2\pi i}\oint \frac{dx}{x^{N+1}} \frac{1}{(1-3t)(1-t)} \sum_{0\le k\le j/2} (-1)^k \binom{j-k}{k}(t-1)^k t^{j-k}\\
&= \frac{1}{2\pi i}\oint \frac{27}{4} \frac{dt}{\big(\frac{27}{4}t(1-t)^2\big)^{N+1}} \sum_{0\le k\le j/2} (-1)^k \binom{j-k}{k}(t-1)^k t^{j-k}\\
&= \frac{1}{2\pi i}\oint \Big(\frac{4}{27}\Big)^N dt \sum_{0\le k\le j/2} (-1)^k \binom{j-k}{k}(t-1)^{k-2N-2} t^{j-k-N-1}\\
&= \Big(\frac{4}{27}\Big)^N [t^{N-j+k}] \sum_{0\le k\le j/2} (-1)^k \binom{j-k}{k}(t-1)^{k-2N-2}\\
&= \Big(\frac{4}{27}\Big)^N (-1)^{N-j} \sum_{0\le k\le j/2} \binom{j-k}{k}\binom{k-2N-2}{N-j+k}.
\end{aligned}$$

Step 2:

$$\begin{aligned}
&[x^N] \frac{3}{(1-3t)(1-t)} \sum_{0\le k\le j/2} (-1)^k \binom{j-1-k}{k}(t-1)^{k+1} t^{j-k}\\
&= 3\Big(\frac{4}{27}\Big)^N [t^{N-j+k}] \sum_{0\le k\le j/2} (-1)^k \binom{j-1-k}{k}(t-1)^{k-2N-1}\\
&= 3\Big(\frac{4}{27}\Big)^N (-1)^{N-j} \sum_{0\le k\le j/2} \binom{j-1-k}{k}\binom{k-2N-1}{N-j+k}.
\end{aligned}$$

We put all the results of this section together in a theorem.

Theorem 2.1. *The generating function $F(u) = F(u,z)$ has the following explicit form:*

$$F(u) = \frac{1}{(1-3t)(1-t)}\Big(1-\frac{9}{2}t(t-1)U\Big)\frac{1}{(1-\sigma U)(1-\tau U)}$$

Here, $u = zU$, $z^3 = x = \frac{27}{4}t(1-t)^2$, and

$$\sigma = \frac{3}{4}(t-\sqrt{4t-3t^2}\,), \qquad \tau = \frac{3}{4}(t+\sqrt{4t-3t^2}\,).$$

Note that $(1-\sigma U)(1-\tau U) = 1-\frac{3}{2}tU+\frac{9}{4}t(t-1)U^2$. Written in the new variable U, only powers of z that are multiples of 3 appear. Further, we get the representation sorted by powers of u:

$$\begin{aligned}F(u,z) &= \frac{1}{(1-3t)(1-t)}\sum_{m\ge 0}\frac{u^m}{z^m}\Big(\frac{3}{2}\Big)^m\sum_{0\le k\le m/2}(-1)^k\binom{m-k}{k}(t-1)^k t^{m-k}\\ &-\frac{3}{(1-3t)(1-t)}\sum_{m\ge 0}\frac{u^m}{z^m}\Big(\frac{3}{2}\Big)^m\sum_{0\le k\le (m-1)/2}(-1)^k\binom{m-1-k}{k}(t-1)^{k+1}t^{m-k}.\end{aligned}$$

Reading off coefficients of $z^N u^j$, where $N = \frac{n+j}{3}$ leads to

$$\begin{aligned}[z^N u^j]F(u,z) &= \Big(\frac{4}{27}\Big)^N(-1)^{N-j}\sum_{0\le k\le j/2}\binom{j-k}{k}\binom{k-2N-2}{N-j+k}\\ &-3\Big(\frac{4}{27}\Big)^N(-1)^{N-j}\sum_{0\le k\le (j-1)/2}\binom{j-1-k}{k}\binom{k-2N-1}{N-j+k}.\end{aligned}$$

For the special state β, the following series representation holds:

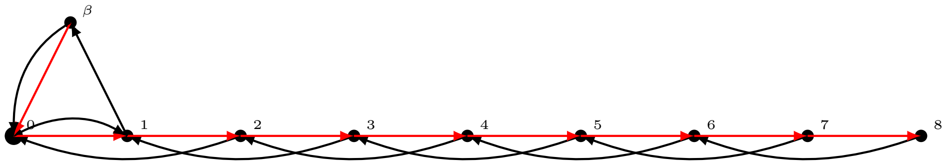
$$f_\beta(z) = \sum_{n\ge 0}\frac{2^{2n+1}}{3^{3n+1}}\binom{3n+1}{n}z^{3n+1}.$$

The computation for the special state was not shown yet:

$$\begin{aligned}[z^{3n+1}]f_\beta &= \frac{2}{3}[x^n]\frac{1}{(1-t)(1-3t)} = \frac{2}{3}\frac{1}{2\pi i}\oint\frac{dx}{x^{n+1}}\frac{1}{(1-t)(1-3t)}\\ &= \frac{2}{3}\frac{27}{4}\frac{1}{2\pi i}\oint\frac{dt}{(\frac{27}{4})^{n+1}t^{n+1}(1-t)^{2n+2}} = \frac{2}{3}\Big(\frac{4}{27}\Big)^n\frac{1}{2\pi i}\oint\frac{dt}{t^{n+1}(1-t)^{2n+2}}\\ &= \frac{2}{3}\Big(\frac{4}{27}\Big)^n[t^n]\frac{1}{(1-t)^{2n+2}} = \frac{2^{2n+1}}{3^{3n+1}}\binom{3n+1}{n}.\end{aligned}$$

3 The dual model

Now, the red edges mean the arrival of the large objects (size $\frac{2}{3}$) and the black edges mean a double-pack of the small edges (size $\frac{1}{3}$ each). To keep the system balanced, the large objects should arrive twice as often as the double-packs of small edges. Again, the generating function g_i refers to paths of length n leading eventually into state i. After n steps, only a state i can be reached with $n \equiv i \bmod 3$. The state diagram and the recursions are immediate:



We work only with $p = \frac{2}{3}$, $q = \frac{1}{3}$. Directly from the state diagram,

$$\begin{aligned} g_0 &= 1 + zg_\beta + qzg_2 = 1 + qz^2 g_1 + qzg_2, \\ g_\beta &= qzg_1, \quad g_1 = zg_0 + qzg_3, \\ g_i &= pzg_{i-1} + qzg_{i+2}, \quad i \geq 2. \end{aligned}$$

Summing the recursions,

$$\begin{aligned} G(u) &= \sum_{i\geq 0} u^i g_i(z) = g_0 + ug_1 + \sum_{i\geq 2} u^i \Big(pzg_{i-1} + qzg_{i+2} \Big) \\ &= g_0 + uzg_0 + qzug_3 + pzu \sum_{i\geq 1} u^i g_i + \frac{qz}{u^2} \sum_{i\geq 4} u^i g_i \\ &= g_0 + uzg_0 + pzuG(u) - pzug_0 + \frac{qz}{u^2} \sum_{i\geq 3} u^i g_i \\ &= g_0 + uzg_0 + pzuG(u) - pzug_0 + \frac{qz}{u^2} (G(u) - g_0 - ug_1 - u^2 g_2) \\ &= g_0 + quzg_0 + pzuG(u) + \frac{qz}{u^2} G(u) - \frac{qz}{u^2} g_0 - \frac{qz}{u} g_1 - qzg_2 \\ &= g_0 + quzg_0 + pzuG(u) + \frac{qz}{u^2} G(u) - \frac{qz}{u^2} g_0 - \frac{qz}{u} g_1 + 1 + qz^2 g_1 - g_0. \end{aligned}$$

Solving, we find with $V = uz$:

$$G(u) = \frac{-V^3 g_0 - 3V^2 - g_1 V^2 z^2 + z^3 g_0 + g_1 V z^2}{2V^3 - 3V^2 + x}.$$

Now we factorize the denominator, using the same substitutions $x = z^3$ and $x = \frac{27}{4} t(1 - t)^2$:

$$2(V - \tfrac{3}{2}(1-t))(V - \sigma)(V - \tau) = 2V^3 - 3V^2 + x.$$

This time, both, $(V - \sigma)$ and $(V - \tau)$ are bad factors. Plugging into the numerator, we find two equations, and the solutions:

$$g_0 = \frac{4}{(1-3t)(4-3t)}, \quad g_1 = \frac{27t(1-t)}{z^2(1-3t)(4-3t)}.$$

It can be noted that $\widetilde{V} := V^{-1}$, with the denominator of the previous section; thus, the three roots carry over. Dividing out the bad factors, we find

$$\frac{-V^3 g_0 - 3V^2 - g_1 V^2 z^2 + z^3 g_0 + g_1 V z^2}{(V-\sigma)(V-\tau)} = \frac{12(t-1) - 4V}{(1-3t)(4-3t)}.$$

Altogether:

$$\begin{aligned}
G(u) &= \frac{6(t-1)-2V}{(1-3t)(4-3t)}\frac{1}{V-\frac32(1-t)}\\
&= \frac{6(1-t)+2V}{(1-3t)(4-3t)\frac32(1-t)}\frac{1}{1-\frac{2}{3(1-t)}V}\\
&= 2\frac{2+\frac{2}{3(1-t)}V}{(1-3t)(4-3t)}\frac{1}{1-\frac{2}{3(1-t)}V}.
\end{aligned}$$

Furthermore

$$\begin{aligned}
[V^j]G(u) &= \frac{2}{(1-3t)(4-3t)}\Big[2\Big(\frac23\frac{1}{1-t}\Big)^j+\Big(\frac23\frac{1}{1-t}\Big)^j\Big]\\
&= \frac{6}{(1-3t)(4-3t)}\Big(\frac23\frac{1}{1-t}\Big)^j, \quad j\ge 1,
\end{aligned}$$

and

$$[u^j]G(u) = z^j\frac{6}{(1-3t)(4-3t)}\Big(\frac23\frac{1}{1-t}\Big)^j.$$

Now let us consider $j+3N$ steps to reach state j, and then

$$\begin{aligned}
[z^{j+3N}u^j]G(u) &= [x^N]\frac{6}{(1-3t)(4-3t)}\Big(\frac23\frac{1}{1-t}\Big)^j\\
&= \frac{1}{2\pi i}\oint\frac{dx}{x^{N+1}}\frac{6}{(1-3t)(4-3t)}\Big(\frac23\frac{1}{1-t}\Big)^j\\
&= \frac{27}{4}\Big(\frac{4}{27}\Big)^{N+1}\frac{1}{2\pi i}\oint\frac{dt(1-t)(1-3t)}{t^{N+1}(1-t)^{2N+2}}\frac{6}{(1-3t)(4-3t)}\Big(\frac23\frac{1}{1-t}\Big)^j\\
&= \Big(\frac{4}{27}\Big)^N\Big(\frac23\Big)^j\frac{1}{2\pi i}\oint\frac{dt}{t^{N+1}(1-t)^{2N+j+1}}\frac{6}{(4-3t)}\\
&= \Big(\frac{4}{27}\Big)^N\Big(\frac23\Big)^{j-1}[t^N]\frac{1}{(1-t)^{2N+j+1}}\frac{1}{(1-\frac34t)}\\
&= \Big(\frac{4}{27}\Big)^N\Big(\frac23\Big)^{j-1}\sum_{i=0}^{N}\Big(\frac34\Big)^{N-i}\binom{2N+j+i}{i}\\
&= \sum_{i=0}^{N}\frac{2^{2i+j-1}}{3^{2N+i+j-1}}\binom{2N+j+i}{i}.
\end{aligned}$$

The coefficients of g_0 are different:

$$[z^{3N}]g_0 = \sum_{i=0}^{N}\frac{2^{2i}}{3^{2N+i}}\binom{2N+i}{i}.$$

Furthermore,

$$[z^{3N+1}]g_\beta = \frac13[z^{3N}]g_1 = \sum_{i=0}^{N}\frac{2^{2i}}{3^{2N+i+1}}\binom{2N+1+i}{i}.$$

Here are the main results of this section:

Theorem 3.1. *The generating function $G(u) = G(u, z)$ has the following explicit form:*

$$G(u) = 2\frac{2 + \frac{2}{3(1-t)}V}{(1-3t)(4-3t)}\frac{1}{1 - \frac{2}{3(1-t)}V}.$$

Here, $u = \frac{V}{z}$, $z^3 = x = \frac{27}{4}t(1-t)^2$. Written in the new variable V, only powers of z that are multiples of 3 appear. Further, we get the representation sorted by powers of u:

$$[V^j]G(u) = \frac{6}{(1-3t)(4-3t)}\Big(\frac{2}{3}\frac{1}{1-t}\Big)^j, \quad j \geq 1,$$

and

$$[u^j]G(u) = z^j\frac{6}{(1-3t)(4-3t)}\Big(\frac{2}{3}\frac{1}{1-t}\Big)^j.$$

Reading off coefficients of $z^{j+3N}u^j$ leads to

$$[z^{j+3N}u^j]G(u,z) = \sum_{i=0}^{N}\frac{2^{2i+j-1}}{3^{2N+i+j-1}}\binom{2N+j+i}{i}.$$

For the special cases, the following series representation holds:

$$[z^{3N}]g_0 = \sum_{i=0}^{N}\frac{2^{2i}}{3^{2N+i}}\binom{2N+i}{i},$$

$$[z^{3N+1}]g_\beta = \frac{1}{3}[z^{3N}]g_1 = \sum_{i=0}^{N}\frac{2^{2i}}{3^{2N+i+1}}\binom{2N+1+i}{i}.$$

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Variants of the domination number for flower snarks

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Abstract

We consider the flower snarks, a widely studied infinite family of 3–regular graphs. For the Flower snark J_n on $4n$ vertices, it is trivial to show that the domination number of J_n is equal to n. However, results are more difficult to determine for variants of domination. The Roman domination, weakly convex domination, and convex domination numbers have been determined for flower snarks in previous works. We add to this literature by determining the independent domination, 2-domination, total domination, connected domination, upper domination, secure domination and weak Roman domination numbers for flower snarks.

Keywords: Flower, snarks, domination, variants, secure.

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1 Introduction

Consider a graph G containing the vertex set V and edge set E. A subset of the vertices $S \subset V$ is said to be a *dominating set* if every vertex in $V \setminus S$ is adjacent to at least one vertex in S. Then, the *domination problem* is to determine the size of the smallest dominating set in a given graph G, which is known as the *domination number* of G and is denoted by $\gamma(G)$.

There are obvious real-world applications for the domination problem. For example, suppose that the vertices of a graph correspond to locations in a secure site, and that each location needs to remain under observation by guards. If a guard at one location is able to simultaneously observe another, there is an edge between the corresponding vertices. Then, by placing guards at the sites corresponding to any dominating set, all locations are under observation. Clearly, it is desirable to do so with as few guards as possible.

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However, in many real-world applications, the definition of domination may be unsuitable in some way, and so variants of domination have been described to handle such cases. Continuing the example in the previous paragraph, suppose that the guards carry out their observation from the top of large towers. The vantage point from these towers enables them to observe adjacent locations, but does not permit them to observe their own location. Then, their location would need to be observed by a guard at an adjacent location. In such a situation, we are seeking not a dominating set, but a *total dominating set*, which we define formally now along with several other variants of dominating sets that we will consider in this manuscript.

Definition 1.1. Consider a dominating set $S \subseteq V$. Then:

- S is a *(weakly) convex dominating set* if S is (weakly) convex in G.
- S is an *independent dominating set* if S is an independent set in G.
- S is a *minimal dominating set* if any proper subset of S is not a dominating set.
- S is a 2*-dominating set* if any vertex $v \notin S$ is adjacent to at least two vertices in S.
- S is a *total dominating set* if every vertex in V is adjacent to at least one vertex in S.
- S is a *connected dominating set* if the subgraph of G induced by the vertices in S is connected.
- S is a *secure dominating set* if, for every vertex $v \in V \setminus S$, there exists a vertex $w \in S$ such that $vw \in E$, and $(S \setminus \{w\}) \cup \{v\}$ is a dominating set.

Analogously to the domination number, we define $\gamma_{wcon}(G)$ to be the weakly convex domination number, $\gamma_{con}(G)$ to be the convex domination number, $i(G)$ to be the independent domination number, $\gamma_2(G)$ to be the 2-domination number, $\gamma_t(G)$ to be the total domination number, $\gamma_c(G)$ to be the connected domination number, and $\gamma_s(G)$ to be the secure domination number. We also define the upper domination number, $\Gamma(G)$, as follows.

Definition 1.2. The *upper domination number*, denoted by $\Gamma(G)$, is equal to the size of the largest minimal dominating set.

In addition, we consider two more variants of domination. Suppose that for our graph G, we have a function $f\colon V \to \{0, 1, 2\}$. Then, the *weight* of f, denoted $w(f)$, is equal to $\sum_{v \in V} f(v)$.

Definition 1.3. f is a *Roman dominating function* on G if it satisfies the condition that every vertex v for which $f(v) = 0$ is adjacent to at least one vertex w for which $f(w) = 2$.

In the following definition, we say that a vertex v is *undefended* with respect to f if $f(v) = 0$ and for every vertex w adjacent to v, $f(w) = 0$.

Definition 1.4. f is a *weak Roman dominating function* on G if, for every vertex v such that $f(v) = 0$, there is at least one vertex w, adjacent to v and satisfying the following: if we define a new function $g\colon V \to \{0, 1, 2\}$ defined by $g(v) = 1$, $g(w) = f(w) - 1$, and $g(u) = f(u)$ for all $u \neq \{v, w\}$, then there are no undefended vertices with respect to g.

Then, the Roman domination number $\gamma_R(G)$ is the minimum weight over all Roman dominating functions, and equivalently for the weak Roman domination number $\gamma_r(G)$. It is worth noting that if we further demand that $f(v) \leq 1$ for all $v \in V$ then weak Roman domination is equivalent to secure domination. Hence, it is clear that $\gamma_r(G) \leq \gamma_s(G)$.

Domination numbers are known for some infinite families of graphs. Other than trivial results such as for paths or cycles, perhaps the most famous result is the sprawling effort over a 27 year period [2, 8, 14, 16, 20, 36] to provide a complete characterisation of domination numbers for grid graphs $G(n, m)$ of all possible sizes, consisting of 23 special cases before settling into a standard formula for $n, m \geq 16$. Other results for domination include generalized Petersen graphs [13, 24, 41], Cartesian products involving cycles [1, 9, 28], King graphs [40], Latin square graphs [29], hypercubes [3], Sierpiński graphs [34], Knödel graphs [12], and various graphs from chemistry [26, 32], among others.

However, far fewer results are known for variants of domination beyond trivial results such as for paths, cycles, stars, wheels, or complete graphs. We summarise the most noteworthy of these results for the variants of domination considered in this paper. For upper domination, results are known for various graphs based on chessboards [4, 11, 18, 39, 40]. For total domination, results are known for some grid graphs $G(n, m)$ (for $n \leq 6$) [15, 21], Knödel graphs [27], and various graphs from chemistry [26]. For connected domination, results are known for trees [33], circulant graphs [35], Centipede graphs [38], and various graphs from chemistry [26]. For weak Roman domination, results are known for various graphs based on chessboards [30], Cartesian products involving complete graphs [37], and Helm graphs and Web graphs [23]. For secure domination, results are known for some grid and torus graphs (for $n \leq 3$) [17], various Cartesian products of stars, cycles, paths and complete graphs [37], and middle graphs [31].

In recent works, the following results were shown for flower snarks (which will be defined in the next section).

Theorem 1.5 ([25, Maksimovic et al. (2018)], [22, Kratica et al. (2020)])**.** *Consider the flower snark J_n, for $n \geq 3$. Then we have $\gamma_R(J_n) = \gamma_{wcon}(J_n) = 2n$, and $\gamma_{con}(J_n) = 4n$.*

We now add to the above literature by proving the following additional results for flower snarks:

Theorem 1.6. *Consider the flower snark J_n, for $n \geq 3$. Then,*

$$\gamma(J_n) = i(J_n) = n, \qquad \gamma_s(J_n) = \gamma_r(J_n) = \left\lceil \frac{3n+1}{2} \right\rceil,$$

$$\gamma_2(J_n) = \begin{cases} \left\lceil \frac{5n}{3} \right\rceil, & \text{if } n \neq 1 \bmod 3, \\ \frac{5n+4}{3}, & \text{if } n = 1 \bmod 3, \end{cases} \qquad \gamma_t(J_n) = \begin{cases} \left\lceil \frac{3n}{2} \right\rceil, & \text{if } n \neq 2 \bmod 4, \\ \frac{3n}{2}+1, & \text{if } n = 2 \bmod 4, \end{cases}$$

$$\gamma_c(J_n) = \Gamma(J_n) = \begin{cases} 2n, & \text{if } n \text{ is even,} \\ 2n-1, & \text{if } n \text{ is odd.} \end{cases}$$

In most cases, the proofs will be by induction, and hence it will be necessary to first prove the results for some number of base cases. Rather than provide these proofs here, we will simply use mixed-integer linear programming formulations of each variant of domination from literature to handle the base cases.

2 Flower snarks

The *chromatic index* of a graph is the minimum required number of colours to color the edges of the graph, such that no two incident edges have the same colour. By Vizing's theorem, it is known that all 3–regular graphs have chromatic index 3 or 4. The latter case is rare, and simple, connected, bridgeless 3–regular graphs with chromatic index 4 are called *snarks*. It is common to further add the restriction that the girth should be at least 5, with such graphs known as *nontrivial snarks*. Flower snarks [19], discovered by Isaacs in 1975, were the first known infinite family of nontrivial snarks, and are denoted by J_n. They are defined as follows.

Definition 2.1 (Flower snarks). For $n \geq 3$, take the union of n copies of $K_{1,3}$. Denote the degree 3 vertex in the i-th copy as a^i, and the other three vertices in the i-th copy as b^i, c^i and d^i. Then construct an n-cycle through vertices $b^1, b^2, \ldots, b^n$, and a $2n$-cycle through vertices $c^1, c^2, \ldots, c^n, d^1, d^2, \ldots, d^n$.

In order to have the properties of a nontrivial snark, n must be odd and $n \geq 5$. However, for other values of $n \geq 3$ a 3–regular graph is nonetheless obtained by this construction. In this paper we will consider all $n \geq 3$, and for convenience we will refer to all of them as flower snarks.

For the remainder of this document, we will consider various kinds of dominating sets of flower snarks. As such, it is convenient to go over some brief terms and notation here. Recall that J_n contains n copies of $K_{1,3}$. We will refer to the i-th copy as J^i, and its four vertices as a^i, b^i, c^i and d^i. Note that this notation does not include n, as typically n will be fixed in our considerations. Also, we will say that a copy J^i has *weight* k in a dominating set S, if S contains k vertices from J^i. We will use the term *pattern* to describe a sequence of weights of consecutive copies of J_n in S. For example, if we say that S contains the pattern 121, it means there is a set of three consecutive copies J^{i-1}, J^i, J^{i+1} which have weights $1, 2$ and 1 respectively in S. Further to this, in a set of consecutive copies of J_n, we will use the term *configuration* to describe the specific allocation of its vertices to S. Finally, we will define w_i^S to be equal to the number of copies with weight i in S.

Flower snarks can be visualised in various ways. A common method is to distribute the copies of $K_{1,3}$ in a circle, using curved edges for the $2n$-cycle and straight edges elsewhere. We display one such drawing of J_n in part (a) of Figure 1, for $n = 7$. However, for our purposes it will be convenient to focus only on a small section of a flower snark at a time, and so we will use the drawing style displayed in part (b) of Figure 1, with each copy displayed vertically. As indicated in Figure 1 we will assume that in these drawings, the bottom vertex in copy J^i is b^i, followed by a^i, c^i and d^i. When useful to avoid confusion, a label will be given above each copy.

It is worth noting that, when viewing only a section of J_n, vertices b^i, c^i and d^i are all essentially equivalent, and this will be useful in simplifying many of the upcoming proofs. In a global sense this is not the case, as there is a “twist” in the final copy in which c^n links to d^1, and d^n links to c^1. However, due to the symmetry of the flower snark, when viewing any copy locally we can choose to relabel the vertices so that this twist occurs elsewhere in the graph. Hence, in the arguments that follow, whenever we are viewing only a portion of the graph we will always assume that the twist occurs elsewhere in the graph.

Various proofs in this paper will go as follows. We will begin with a set of copies of J_n for which the pattern is known, as well as possibly knowing in advance that some vertices are either in S, or not in S. From there, depending on the variant of domination being

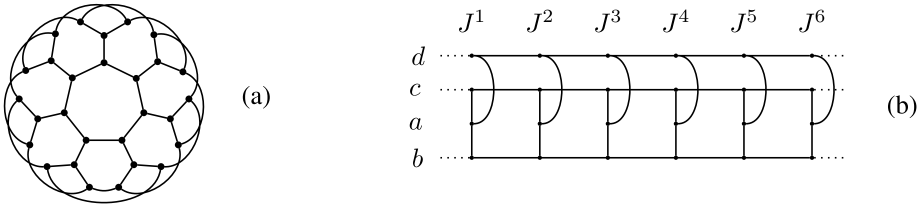


Figure 1: In part (a) a common drawing of the flower snark J_7. In part (b), a section of a larger flower snark consisting of six copies.

considered, and the structure of J_n, we will go on to prove that certain vertices either must be in S, or must not be in S. For these proofs, figures will be provided, using the following convention. The pattern known in advance will be indicated by listing the corresponding weight underneath each copy. Vertices which are known in advance to be in S will be marked with ●, and vertices which are known in advance not to be in S will be marked with ×. Then, vertices which are subsequently shown (up to equivalence) to be in S will be marked with ○, while vertices which are subsequently shown not to be in S will be marked with ⁄. We demonstrate this convention with a simple example.

Example 2.2. Suppose we have four copies J^1, J^2, J^3, J^4 which meet the pattern 1111, and that we know that $d^2 \in S$ and $c^1 \notin S$. This situation is displayed in Figure 2. Clearly, since $d^2 \in S$ and J^2 has weight 1, we know that $a^2 \notin S$, $b^2 \notin S$, and $c^2 \notin S$. These three are marked with × in Figure 2 as no argument was needed to establish they are not in S. Then, the only remaining vertex which can dominate c^2 is c^3, and hence $c^3 \in S$. Since this was argued in the proof, c^3 is marked with a ○ in Figure 2. Also, since J^3 has weight 1 the vertices a^3, b^3, and d^3 cannot be in S, and so they are marked with ⁄. Then, the only remaining vertex which can dominate b^2 is b^1, which we similarly mark in Figure 2. Finally, the only remaining vertex which can dominate b^3 is b^4, which we again mark in Figure 2. Hence, we now know exactly which vertices in J^1, J^2, J^3, J^4 are contained in S.

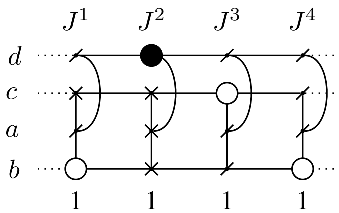


Figure 2: The situation described in Example 2.2.

To conclude this section, we note that it is trivial to determine the domination and independent domination numbers for flower snarks.

Lemma 2.3. *For $n \geq 3$, we have $\gamma(J_n) = i(J_n) = n$.*

Proof. The graph J_n contains n copies of $K_{1,3}$. For each copy J^i, the vertex a^i is adjacent only to other vertices in J^i, and so any dominating set must contain at least one vertex from each copy. Hence, $n \leq \gamma(J_n) \leq i(J_n)$. Then, it suffices to note that the set $\{a^1, a^2, \ldots, a^n\}$ is an independent dominating set, and hence $i(J_n) \leq n$, leading to the result. □

Corollary 2.4. *For any variant of dominating set considered in this paper, each copy must have weight at least* 1.

3 Upper domination

In this section, we will determine the upper domination number for flower snarks. We begin by considering what weights are possible for copies in a minimal dominating set.

Lemma 3.1. *Consider the graph J_n for $n \geq 3$. If S is a minimal dominating set for J_n then the weight of each copy is either* $1, 2$ *or* 3.

Proof. We know from Corollary 2.4 that each copy must have positive weight. Then suppose that a copy J^i has weight 4. It can be easily checked that $S \setminus \{a^i\}$ is also a dominating set, contradicting the assumption that S is minimal. □

In the following lemma, we use the term i *depends on* j to imply that $S \cap N[i] = \{j\}$. Note that since S is minimal, for every vertex in S there must be at least one other vertex which depends on it.

Lemma 3.2. *Consider the graph J_n for $n \geq 4$, and a minimal dominating set S. If a copy J^i has weight* 3 *in S, then both adjacent copies J^{i-1} and J^{i+1} have weight* 1 *in S.*

Proof. Suppose it is not the case, that is, J^i has weight 3 and at least one of J^{i-1} and J^{i+1} has weight greater than 1. Since b^i, c^i and d^i are all equivalent in this framing, there are only two cases to consider; if $a^i \in S$ and if $a^i \notin S$.

First, consider the case when $a^i \in S$. Then there are two other vertices from J^i also in S. Without loss of generality, suppose $b^i \in S$ and $c^i \in S$. This situation is shown in Figure 3 part (a). Then, since S is minimal, the removal of a^i does not result in a dominating set. This is only possible if d^i depends on a^i. Hence, $d^{i-1} \notin S$ and $d^{i+1} \notin S$.

Then, consider b^i. Since S is minimal, at least one of b^{i-1} and b^{i+1} must depend on b^i. Without loss of generality, suppose it is the former. This implies that both $b^{i-1} \notin S$ and $a^{i-1} \notin S$. Then, $c^{i-1} \in S$, or else copy J_{i-1} has weight 0 which contradicts Lemma 3.1. Then, since S is minimal, c^{i+1} must depend on c^i, which implies that $c^{i+1} \notin S$ and $a^{i+1} \notin S$. Hence, from Lemma 3.1 copy J^{i+1} also has weight 1, contradicting the initial assumption.

Second, consider the case when $a^i \notin S$. Then, S contains b^i, c^i, and d^i. Since S is minimal and a^i does not depend on b^i, at least one of b^{i-1} or b^{i+1} must depend on b^i. Without loss of generality, suppose it is the former. This implies that both $b^{i-1} \notin S$ and $a^{i-1} \notin S$. We can make analogous arguments for c^i and d^i. Clearly, the dependent vertices can not all be from the same copy, or else that copy has weight 0. Hence, two of the dependent vertices are in one copy and one is in the other; without loss of generality we will assume that b^{i-1} is dependent on b^i, c^{i-1} is dependent on c^i, and d^{i+1} is dependent on d^i. Hence, J^{i-1} has weight 1, and so by assumption, J^{i+1} has weight 2, and $b^{i+1} \in S$ and $c^{i+1} \in S$. But then, by similar arguments, it must be the case that b^{i+2} depends on b^{i+1}, and c^{i+2} depends on c^{i+1}, implying that J^{i+2} has weight 1, and $d^{i+2} \in S$. This situation is displayed in Figure 3 part (b). Finally, it can be seen from this figure that $S \setminus \{d^i\}$ is a dominating set, contradicting the initial assumption that S is minimal. □

Figure 3: The two situations described in the proof of Lemma 3.2.

Recalling that w_i^S is the number of copies to have weight i in S, Lemma 3.2 implies the following.

Corollary 3.3. *Consider the graph J_n for $n \geq 4$. If S is a minimal dominating set for J_n then $w_1^S \geq w_3^S$, with equality occurring if and only if $w_1^S = w_3^S = \frac{n}{2}$.*

We are now ready to prove the main result of this section.

Theorem 3.4. *Consider a graph J_n for $n \geq 3$. Then,*

$$\Gamma(J_n) = \begin{cases} 2n, & \text{if } n \text{ is even,} \\ 2n-1, & \text{if } n \text{ is odd.} \end{cases}$$

Proof. We use the first formulation for upper domination from [6] to confirm that $\Gamma(J_3) = 5$. Then, suppose that S is a minimal dominating set for J_n, for $n \geq 4$. From Lemma 3.1 we have $w_0^S = w_4^S = 0$. Hence, we have $w_1^S + w_2^S + w_3^S = n$, and $w_1^S + 2w_2^S + 3w_3^S = |S|$. Combining these, we obtain $|S| = 2n + w_3^S - w_1^S$. From Corollary 3.3 we know that $w_1^S \geq w_3^S$, and the inequality is strict if n is odd. Hence, $|S| \leq 2n$ if n is even, and $|S| \leq 2n-1$ if n is odd. Then we just need to obtain the corresponding lower bounds.

Suppose that n is even. It is easy to check that if we repeat the configuration displayed in Figure 4 part (a) $n/2$ times, what results is a minimal dominating set with weight $2n$. Hence, if n is even, we have $\Gamma(J_n) \geq 2n$. Then suppose that n is odd. Again, it is easy to check that if we repeat the configuration displayed in Figure 4 part (a) $(n-1)/2$ times, and then use the configuration displayed in Figure 4 part (b) for the final copy, what results is a minimal dominating set with weight $2n-1$. Hence, if n is odd, $\Gamma(J_n) \geq 2n-1$. □

Figure 4: The configuration for upper domination which gives the desired lower bound for $\Gamma(J_n)$ for $n \geq 3$. Part (a) can be repeated as many times as necessary. Then, if n is odd, use part (b) to finish. The result is a minimal dominating set with weight $2n$ if n is even, or weight $2n-1$ if n is odd.

4 Upper bounds by construction

In the upcoming sections, we will determine lower bounds for the 2-domination, total domination, connected domination, secure domination, and weak Roman domination numbers of flower snarks. To obtain equality, we will require corresponding upper bounds, which we provide here. We leave it as an exercise to the reader to verify that the configurations given here satisfy the requirements of the various kinds of domination, and that they imply the appropriate upper bounds for Theorem 1.6.

For 2-domination, the configuration shown in Figure 5 can be repeated as often as necessary, truncating the final time to get the desired size.

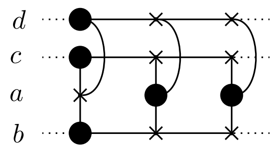


Figure 5: The configuration for 2-domination which gives the desired upper bound for $\gamma_2(J_n)$. The result is a 2-dominating set of weight $\lceil \frac{5n}{3} \rceil$ if $n \neq 1 \bmod 3$, or weight $\lceil \frac{5n}{3} \rceil + 1$ if $n = 1 \bmod 3$.

For total domination, the configuration shown in Figure 6 part (a) can be repeated as often as necessary. Then the configurations in parts (b), (c) and (d) can be used to complete the remaining copies.

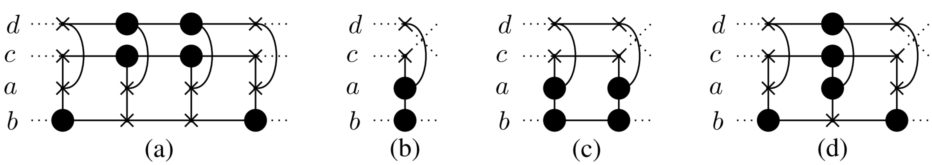


Figure 6: The configuration for total domination which gives the desired upper bound for $\gamma_t(J_n)$ for $n \geq 3$. Part (a) can be repeated as many times as necessary. If $n = 0 \bmod 4$ this is sufficient. If $n = 1 \bmod 4$, use (b) to finish. If $n = 2 \bmod 4$, use part (c) to finish. If $n = 3 \bmod 4$, use part (d) to finish. The result is a total dominating set of weight $\lceil \frac{3n}{2} \rceil$ if $n \neq 2 \bmod 4$, or weight $\frac{3n}{2} + 1$ if $n = 2 \bmod 4$.

For connected domination, the configuration shown in Figure 7 can be repeated as often as necessary, truncating the final time to get the desired size. If $n \neq 1 \bmod 4$ the result is connected dominating. If $n = 1 \bmod 4$, then removing d^1 and adding b^1 gives the desired result.

For secure domination, the configuration shown in Figure 8 part (a) can be repeated as often as necessary. Then the configurations in parts (b), (c), (d), and (e) can be used to complete the remaining copies. Since secure domination is an upper bound for weak Roman domination, these configurations also give an upper bound for weak Roman domination.

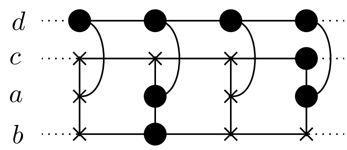


Figure 7: The configuration for connected domination which gives the desired upper bound for $\gamma_c(J_n)$ for $n \geq 3$. It may be repeated as many times as necessary, truncating the final time to get the final size. If $n = 1 \bmod 4$, then in the first copy, remove d^1 and add b^1. The result is a connected dominating set of weight $2n$ if n is even, or $2n - 1$ if n is odd.

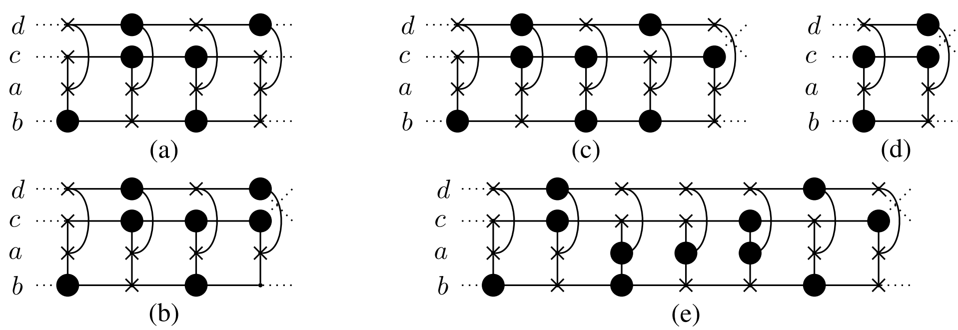


Figure 8: The configuration for secure domination which gives the desired upper bound for $\gamma_s(J_n)$ for $n \geq 4$. Part (a) can be repeated as many times as necessary. If $n = 0 \bmod 4$, use part (b) to finish. If $n = 1 \bmod 4$, use part (c) to finish. If $n = 2 \bmod 4$, use part (d) to finish. If $n = 3 \bmod 4$, use part (e) to finish. In all cases, the result is a secure dominating set of weight $\lceil \frac{3n+1}{2} \rceil$.

5 2-domination

In this section, we will determine the 2-domination numbers for flower snarks.

Lemma 5.1. *Consider the graph J_n for $n \geq 3$, and a 2-dominating set S. If the copy J^i has weight 1 in S, then $a^i \in S$.*

Proof. Recall that a^i is adjacent only to other vertices in J^i. From the definition of 2-domination, if $a^i \notin S$ then it must have at least two neighbours in S. Since J^i has weight 1, this is impossible, and so $a^i \in S$. □

Theorem 5.2. *Consider the graph J_n for $n \geq 3$, and a 2-dominating set S. Then any copy with weight 1 in S has a neighbouring copy with weight 3 or 4 in S.*

Proof. Without loss of generality, suppose that J^2 has weight 1. Then, from Lemma 5.1 we have $a^2 \in S$. Then, suppose that its neighbours J^1 and J^3 both have weight less than 3. At this stage, vertices b^2, c^2 and d^2 have only one neighbour in S, so they each need at least one more. Without loss of generality, suppose that $b^1 \in S$. Then, consider the case when $c^1 \in S$. Since J^1 has weight less than 3, this implies that $a^1 \notin S$ and $d^1 \notin S$. However, it is then impossible for d^1 to have two neighbours in S. This situation is displayed in part (a) of Figure 9.

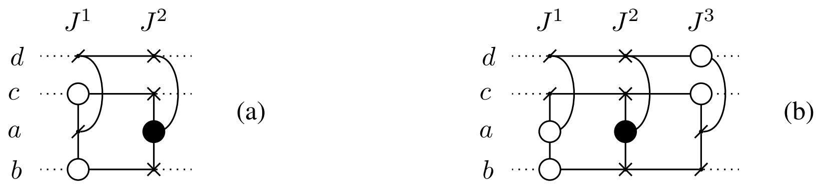


Figure 9: The two situations described in the proof of Theorem 5.2. In part (a), d^1 cannot have two neighbours in S. In part (b), copy b^3 cannot have two neighbours in S.

Hence, we must have $c^1 \notin S$. Then, c^1 must have two neighbours in S, which implies that $a^1 \in S$. Since J^1 has weight less than 3, this implies that $d^1 \notin S$. Then, in order for c^2 and d^2 to have two neighbours in S, we must have $c^3 \in S$ and $d^3 \in S$, respectively. Since J^3 has weight less than 3, this implies that $a^3 \notin S$ and $b^3 \notin S$. However, it is then impossible for b^3 to have two neighbours in S, completing the proof. This situation is displayed in part (b) of Figure 9. □

We are now ready to prove the main result of this section.

Theorem 5.3. *Consider the graph J_n for $n \geq 3$. Then,*

$$\gamma_2(J_n) = \begin{cases} \left\lceil \frac{5n}{3} \right\rceil, & \text{if } n \neq 1 \bmod 3, \\ \frac{5n+4}{3}, & \text{if } n = 1 \bmod 3. \end{cases}$$

Proof. The upper bound was established in Section 4. Then, from Corollary 2.4 we know that each copy has weight at least 1. This, combined with Theorem 5.2, implies that any three set of consecutive copies have weight at least 5. Hence we have $\gamma_2(S) \geq \lceil \frac{5n}{3} \rceil$. Hence, Theorem 5.3 is true for $n \neq 1 \bmod 3$.

Suppose that $n = 1 \bmod 3$, and we have a 2-dominating set S such that $|S| < \frac{5n+4}{3}$. Hence, we must have $|S| = \frac{5n+1}{3}$. For each copy J^i, denote by $w^3(i)$ the combined weights of J^i, J^{i+1}, and J^{i+2}, where the superscripts are taken modulo n. Recall that $w^3(i) \geq 5$. Hence there must be a single value k such that $w^3(k) = 6$ and $w^3(j) = 5$ for all $j \neq k$. Consider the copies J^k, J^{k+1}, J^{k+2}. We will consider the possible patterns they could have, and show that each is impossible. Since $w^3(k)$ has weight 6, the only possible patterns (up to symmetry) for copies J^k, J^{k+1} and J^{k+2} are $123, 132, 222$, and 312.

Suppose that J^k, J^{k+1} and J^{k+2} meet either the pattern 123 or the pattern 132. Since $w^3(k+1) = 5$, this implies that J^{k+3} has weight 0, which contradicts Corollary 2.4.

Suppose next that J^k, J^{k+1} and J^{k+2} meet the pattern 222. Since $w^3(k+1) = w^3(k+2) = 5$, this implies that J^{k+3} has weight 1, and J^{k+4} has weight 2. However, this contradicts Theorem 5.2 since J^{k+3} has no neighbour with weight 3 or 4. Hence, this is impossible.

Finally, suppose that J^k, J^{k+1} and J^{k+2} meet pattern 312. Since $w^3(k+1) = w^3(k+2) = w^3(k+3) = 5$, this implies that J^{k+3} has weight 2, J^{k+4} has weight 1, and J^{k+5} has weight 2. Again, this contradicts Theorem 5.2 since J^{k+4} has no neighbour with weight 3 or 4. Hence, all cases are impossible, and so $|S| \geq \frac{5n+4}{3}$, completing the proof. □

6 Total domination

In this section, we will determine the total domination numbers for flower snarks. We begin by identifying three patterns which cannot occur in total dominating sets of J_n.

Theorem 6.1. *Consider the graph J_n for $n \geq 3$, and a total dominating set S. Then S does not contain the patterns* $111, 1121,$ *or* 12121.

Proof. From Corollary 2.4, each copy of J_n has weight at least 1 in S. Also, if a copy J^i has weight 1, then $a^i \notin S$ because otherwise a^i itself is not dominated.

Suppose that S has the pattern 111. Without loss of generality, suppose the three copies meeting this pattern are J^1, J^2, J^3. As indicated above, $a^2 \notin S$. Regardless of which vertex from J^2 is in S, it does not dominate any of b^2, c^2, d^2. Also, any vertex from J^1 dominates at most one vertex from J^2, and likewise for any vertex from J^3. Hence there is at least one vertex in J^2 which is not dominated, contradicting the assumption that S is a total dominating set.

Then, suppose that S has the pattern 1121. Without loss of generality, suppose the four copies meeting this pattern are J^1, J^2, J^3, J^4. Using an equivalent argument to above, it must be the case that b^2, c^2, d^2 are dominated by one vertex from J^1 and two vertices from J^3. Hence, $a^3 \notin S$. This means none of the vertices from J^3 dominate any of b^3, c^3, d^3. Also, any vertex from J^2 dominates at most one vertex from J^3, and likewise for any vertex from J^4. Hence there is at least one vertex in J^3 which is not dominated, contradicting the assumption that S is a total dominating set.

Finally, suppose that S has the pattern 12121. Without loss of generality, suppose the five copies meeting this pattern are $J^1, \ldots, J^5$. Since J^1 and J^3 are both weight 1, they can collectively dominate at most two vertices from J^2. Hence, $a^2 \in S$, and by an equivalent argument, $a^4 \in S$. Note that none of the vertices from J^3 can dominate any of b^3, c^3 or d^3. Also, the vertices from J^2 dominate at most one vertex from J^3, and likewise for the vertices from J^4. Hence there is at least one vertex in J^3 which is not dominated, contradicting the assumption that S is a total dominating set. □

An immediate observation arising from Corollary 2.4 and Theorem 6.1 is that any four consecutive copies must collectively have weight at least 6 in S, leading to the following corollary.

Corollary 6.2. *For $n \geq 4$, $\gamma_t(J_n) \geq \left\lceil \frac{3n}{2} \right\rceil$.*

We are now ready to prove the main result of this section.

Theorem 6.3. *Consider the graph J_n for $n \geq 3$. Then,*

$$\gamma_t(J_n) = \begin{cases} \left\lceil \frac{3n}{2} \right\rceil, & \text{if } n \neq 2 \bmod 4, \\ \frac{3n}{2} + 1, & \text{if } n = 2 \bmod 4. \end{cases}$$

Proof. The upper bound was established in Section 4. We use the formulation for total domination from [7] to confirm that $\gamma_t(J_3) = 5$. Then, suppose there is some value $k \geq 4$ such that Theorem 6.3 is false. If $k \neq 2 \bmod 4$ then we have $\gamma_t(J_k) \leq \left\lceil \frac{3k}{2} \right\rceil - 1$, contradicting Corollary 6.2. Hence, we must have $k = 2 \bmod 4$. Hence, $\gamma_t(J_k) \leq \frac{3k}{2}$, and from Corollary 6.2 this implies that $\gamma_t(J_k) = \frac{3k}{2}$.

Suppose that we have a total dominating set S with weight $\frac{3k}{2}$. Note that this implies that *every* set of four consecutive copies of J_n has weight 6 in S; call this property 1. It is clear that property 1 implies that no copy has weight 4 in S. If a copy has weight 3 in S, then in order to satisfy property 1, the next three copies must have weight 1, which contradicts Theorem 6.1. Hence, every copy has weight 1 or 2 in S; call this property 2. It can be easily checked that there are only two ways to satisfy properties 1 and 2 without contradicting Theorem 6.1; either S contains the repeated pattern $1212\dots12$, or S contains the repeated pattern $11221122\dots1122$. In the former case, S contains the pattern 12121 which by Theorem 6.1 is impossible. Hence, we must have the latter case. Also, the latter case is impossible because $k = 2 \bmod 4$. Hence, all cases are impossible, completing the proof. □

7 Connected domination

In this section, we will determine the connected domination numbers for flower snarks. The following three remarks are obvious, but we list them here to aid the readability of two upcoming proofs.

Remark 7.1. Suppose that S is a connected dominating set for a graph G. If a new graph H is created by either (a) adding an edge between two non-adjacent vertices in G, or (b) deleting a vertex $v \notin S$ from G, then S is also a connected dominating set for H.

Remark 7.2. Suppose that S is a connected dominating set for a graph G, and that G contains a degree 1 vertex $v \in S$ whose neighbour has degree larger than 1. If a new graph H is created by deleting v from G, then $S \setminus \{v\}$ is a connected dominating set for H.

Remark 7.3. Suppose that S is a connected dominating set for a graph G, and that G contains a triangle uvw such that v is degree 2, and $v \in S$. If a new graph H is created by deleting v from G, then $S \setminus \{v\}$ is a connected dominating set for H.

In the following, we define $V^S(J^i) := S \cap (a^i, b^i, c^i, d^i)$, that is, $V^S(J^i)$ is the set of vertices from J^i that are contained in S.

Theorem 7.4. *Consider the graph J_n for $n \geq 4$, and let S be a connected dominating set for J_n. Suppose that there is a copy J^i such that either $a^i \notin S$, or J^i has weight 2 in S. Then $\overline{S} := S \setminus V^S(J^i)$ is a connected dominating set for J_{n-1}.*

Proof. Consider first the situation when $a^i \notin S$, and consider the graph J_{n-1}. One may think of J_{n-1} as being constructed by starting with J_n and then "smoothing out" copy J^i, in the following sense. First, we add edges connecting b^{i-1} to b^{i+1}, c^{i-1} to c^{i+1} and d^{i-1} to d^{i+1}, and then we delete the vertex set $V(J^i)$. Suppose that we are midway through this process, having added all three edges, and having deleted a^i. Call this intermediate graph G, and note from Remark 7.1 that S is a connected dominating set for G. Then, note that in G, b^i is a degree 2 vertex and is part of a triangle $b^{i-1}b^ib^{i+1}$. Now, suppose that we delete b^i from G, to obtain a new intermediate graph G_2. If $b^i \notin S$ then from Remark 7.1 we can see that S is a connected dominating set for G_2. If $b^i \in S$ then from Remark 7.3 we can see that $S \setminus \{b^i\}$ is a connected dominating set for G_2. Applying analogous arguments for c^i and d^i, we obtain the result. The graphs G and G_2 are displayed in Figure 10.

We next consider the situation when J^i has weight 2 in S. If $a^i \notin S$ then the previous paragraph applies. If $a^i \in S$ then there must be one more vertex from J^i also in S; without loss of generality, suppose that $b^i \in S$. Again, we construct J_{n-1} by adding edges to and deleting vertices from J_n. Suppose that we are midway through this process, having added all three edges, and having deleted c^i and d^i. Call this intermediate graph G_3. From Remark 7.1 we know that S is a connected dominating set for G_3. Note that in G_3, a^i is a degree 1 vertex. Remark 7.2 implies that we can obtain a second intermediate graph, G_4, by deleting a^i, and that $S \setminus \{a^i\}$ is a connected dominating set for G_4. Finally, note that in G_4, b^i is degree 2 vertex and is part of a triangle $b^{i-1}b^ib^{i+1}$. Hence, Remark 7.3 implies the result. The graphs G_3 and G_4 are displayed in Figure 10. □

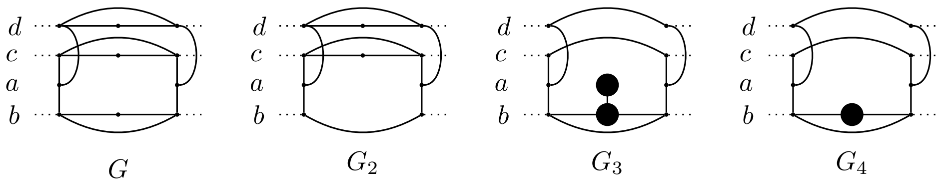


Figure 10: Sections of the four intermediate graphs G, G_2, G_3, G_4 from the proof of Theorem 7.4.

We also make use of the concept of "smoothing out" a copy in the following lemma.

Lemma 7.5. *Consider the graph J_n for $n \geq 4$, and let S be a connected dominating set for J_n. Suppose that there is a copy J^i with weight 4 in S. Then for any positive integer $k \leq n-3$, we have that $\overline{S} := S \setminus (V^S(J^{i+1}) \cup \ldots \cup V^S(J^{i+k}))$ is a connected dominating set for J_{n-k}.*

Proof. Suppose that we delete from J_n all vertices from each of the copies $J^{i+1}, \ldots, J^{i+k}$. Call this intermediate graph G. Then, we add edges connecting b^i to b^{i+k+1}, c^i to c^{i+k+1} and d^i to d^{i+k+1}, and call the resulting graph G_2. Since we can always relabel the vertices so that the twist occurs elsewhere in the graph, it can then be seen that G_2 is isomorphic to J_{n-k}. Both G and G_2 are displayed in Figure 11. Since $b^i \in S$, $c^i \in S$, and $d^i \in S$, it is clear that vertices b^{i+k+1}, c^{i+k+1} and d^{i+k+1} are dominated in G_2. Also, from Corollary 2.4 we know that J^{i+k+1} has weight at least 1, and so a^{i+k+1} is dominated. Hence, $\overline{S}$ is a dominating set for G_2. Next, we need to show that $\overline{S}$ is a connected dominating set for G_2. Recall the intermediate graph G. It is clear that the subgraph of G induced by $\overline{S}$ is either connected, or has exactly two connected components. In the former case, the subgraph of G_2 induced by $\overline{S}$ is also connected. In the latter case, it is clear that there must be an edge vw in G_2 such that $v \in J^i$, $w \in J^{i+k+1}$ and both v, w are contained in S, and hence the subgraph of G_2 induced by $\overline{S}$ is connected. Either way, we obtain the desired result. □

Lemma 7.6. *Suppose that there is an odd value $n \geq 5$ such that $\gamma_c(J_{n-1}) = 2n-2$. If S is a connected dominating set for J_n such that $|S| = 2n-1$, then w_4^S is even.*

Proof. Suppose that $w_2^S > 0$. Then there is a copy J^i with weight 2. From Theorem 7.4, we can obtain a connected dominating set for J_{n-1} of cardinality $2n-3$, contradicting the initial assumption. Hence, $w_2^S = 0$. Then, we have $w_1^S + w_3^S + w_4^S = n$, and $w_1^S + 3w_3^S + 4w_4^S = 2n-1$. Combining these two, we obtain $3w_4^S = n-1-2w_3^S$. Since n is odd, this implies that w_4^S is even. □

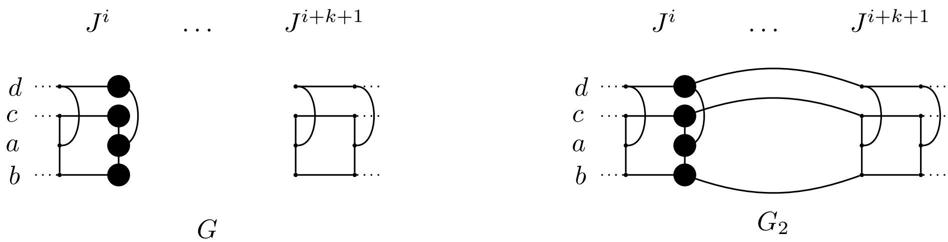


Figure 11: Sections of the two intermediate graphs G and G_2 from the proof of Lemma 7.5.

Theorem 7.7. *Consider the graph J_n for $n \geq 3$, and suppose that S is a connected dominating set. Then S does not contain the patterns* 111 *or* 1312131.

Proof. Any connected dominating set with $|S| \geq 2$ is also a total dominating set, and hence the result for pattern 111 follows immediately from Theorem 6.1.

Next, consider the case when S contains the pattern 1312131. Without loss of generality, suppose the copies meeting this pattern are $J^1, \ldots, J^7$. Suppose that $a^4 \notin S$. Then, without loss of generality, suppose that $b^4 \in S$, $c^4 \in S$, and $d^4 \notin S$. This situation is displayed in part (a) of Figure 12. Since S is dominating, it must contain at least one of d^3 and d^5. Since both J^3 and J^5 have weight 1, both options are equivalent, so without loss of generality we will assume that $d^5 \in S$. Then $a^5 \notin S$, $b^5 \notin S$, and $c^5 \notin S$. In order for S to be connected, we must have $b^3 \in S$ and $c^3 \in S$, but this is impossible since J^3 has weight 1. Hence, it must be the case that $a^4 \in S$.

Again, without loss of generality, suppose that $b^4 \in S$, $c^4 \notin S$ and $d^4 \notin S$. This situation is displayed in part (b) of Figure 12. Since S is connected, it must contain at least one b^3 or b^5. As in the previous paragraph, without loss of generality we will assume that $b^5 \in S$. Then, in order for S to be dominating, we must have $c^6 \in S$ and $d^6 \in S$. Now, suppose that $b^6 \in S$. Then, since S is connected, it must contain both c^7 and d^7, but this is impossible since J^7 has weight 1. Hence, it must be the case that $b^6 \notin S$, and hence $a^6 \in S$. Then, since S is connected, we must have $b^3 \in S$ and $b^2 \in S$. In order for S to be dominating, we must have $c^2 \in S$ and $d^2 \in S$, and since J^2 has weight 3 this implies that $a^2 \notin S$. Finally, to ensure S is connected, it must contain each of b^1, c^1 and d^1, but this is impossible since J^1 has weight 1. □

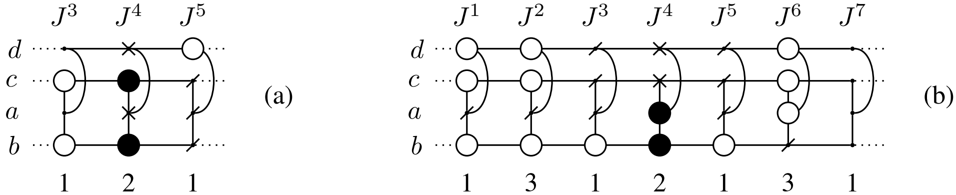


Figure 12: The two situations described in the proof of Theorem 7.7. In part (a), J^3 has too many vertices in S. In part (b), J^1 has too many vertices in S.

Theorem 7.8. *Consider the graph J_n for even $n \geq 6$, and a connected dominating set S for J_n such that $|S| = 2n-1$. If $\gamma_c(J_{n-1}) = 2n-3$, and $w_4^S = 0$, then S does not contain the pattern* 112.

Proof. Without loss of generality, suppose that one set of copies meeting the pattern 112 is J^2, J^3, J^4, and consider also J^1 and J^5. Since J^2 and J^3 have weight 1, we know that $a^2 \notin S$ and $a^3 \notin S$. Without loss of generality, suppose that $b^2 \in S$. Then, suppose that $b^3 \in S$ as well. This situation is displayed in part (a) of Figure 13. Since S is dominating, it must also contain each of c^1, d^1, c^4, and d^4. Since J^4 has weight 2, we have $a^4 \notin S$ and $b^4 \notin S$. Since S is connected, it implies that $b^1 \in S$. Since there are no copies of weight 4 in S, this implies that $a^1 \notin S$. However, by Theorem 7.4, we can then obtain a connected dominating set for J_{n-1} of cardinality $2n-4$, contradicting the assumption that $\gamma_c(J_{n-1}) = 2n-3$. Hence, the assumption that $b^3 \in S$ must be false.

We instead have $b^2 \in S$ and $b^3 \notin S$. Without loss of generality, suppose that $c^3 \in S$. This situation is displayed in part (b) of Figure 13. Since S is dominating, it must also contain d^4. Since S is connected, we have $c^4 \in S$, and since J^4 has weight 2, we have $a^4 \notin S$ and $b^4 \notin S$. Then, since S is dominating, we have $b^5 \in S$, and since S is connected we have $c^5 \in S$ and $d^5 \in S$. Since there are no copies of weight 4 in S, it implies that $a^5 \notin S$. However, by Theorem 7.4, we can the obtain a connected dominating set for J_{n-1} of cardinality $2n-4$, which again contradicts the assumption that $\gamma_c(J_{n-1}) = 2n-3$, completing the proof. $\square$

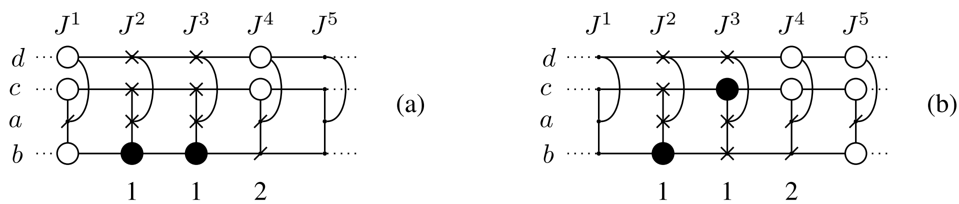


Figure 13: The two situations described in the proof of Theorem 7.8. In both parts, there is a copy with weight 3, but with the a vertex not contained in S.

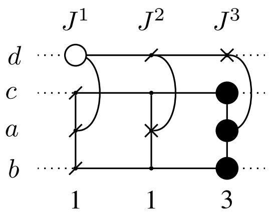
In the next theorem, we will require the following concept. Suppose that we have a connected dominating set S for J_n. Denote by $J_n(S)$ the subgraph of J_n induced by S. By definition, $J_n(S)$ is connected. Then, consider a set of consecutive copies $J^i, \ldots, J^{i+k}$, and define $\overline{S} = S \cap (V(J^i) \cup \ldots \cup V(J^{i+k}))$. If $J_n(\overline{S})$ is a disconnected graph, then we say that the copies $J^i, \ldots, J^{i+k}$ are *locally disconnected* in S. Since S itself is connected, it is clear that if S contain two sets of locally disconnected consecutive copies, they must overlap by two or more copies.

Theorem 7.9. *Consider the graph J_n for even $n \geq 6$, and a connected dominating set S for J_n such that $|S| = 2n-1$. If $\gamma_c(J_{n-1}) = 2n-3$, then any set of consecutive copies meeting the patterns* 113 *or* 3123 *are locally disconnected.*

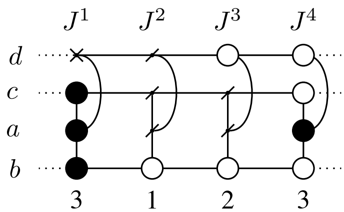
Proof. Consider first the pattern 113. Without loss of generality, suppose that one set of copies meeting this pattern is J^1, J^2, J^3, and that this set of copies is not locally disconnected. Suppose that $a^3 \notin S$. Then, from Theorem 7.4 there exists a connected dominating set for J_{n-1} with cardinality $2n-4$, which is impossible since by assumption

$\gamma_c(J_{n-1}) = 2n-3$. Hence, $a^3 \in S$. Then, without loss of generality, suppose that $b^3 \in S$, $c^3 \in S$, and $d^3 \notin S$. This situation is displayed in part (a) of Figure 14. Since J^2 has weight 1, it is clear that $a^2 \notin S$. Then, in order for S to be dominating, it must contain at least one of d^1 and d^2. However, if $d^2 \in S$, then S does not contain b^2 or c^2, and hence J^1, J^2, J^3 are locally disconnected, contradicting the initial assumption. Hence, $d^2 \notin S$, and so $d^1 \in S$. But then, because J^1 has weight 1, S does not contain a^1, b^1, or c^1, which again implies that J^1, J^2, J^3 are locally disconnected, contradicting the initial assumption. Hence, the initial assumption must be false, and J^1, J^2, J^3 are locally disconnected.

Next, consider the pattern 3123. Without loss of generality, suppose that one set of copies meeting this pattern is J^1, J^2, J^3, J^4, and that this set of copies if not locally disconnected. Using an identical argument as from the previous paragraph, we must have $a^1 \in S$ and $a^4 \in S$. Then, without loss of generality, suppose that $b^1 \in S$ and $c^1 \in S$. This situation is displayed in part (b) of Figure 14. Since J^2 has weight 1, and the set of copies is not locally disconnected, we have $a^2 \notin S$ and $d^2 \notin S$. The remaining two choices are equivalent; without loss of generality, suppose that $b^2 \in S$. Then, since S is dominating, and the set of copies is not locally disconnected, we must have $b^3 \in S$ and $d^3 \in S$. Finally, since S is dominating, and the set of copies is not locally disconnected, we must have $b^4 \in S$, $c^4 \in S$ and $d^4 \in S$. However, this is impossible since J^4 has weight 3. Hence, the initial assumption must be false, and J^1, J^2, J^3, J^4 are locally disconnected. □



(a)



(b)

Figure 14: The two situations described in the proof of Theorem 7.9. In part (a), the copies are locally disconnected. In part (b), copy J^4 has too many vertices in S.

Theorem 7.7, along with the fact that the patterns 113 and 3123 overlap by at most one copy, leads to the following corollary.

Corollary 7.10. *Consider the graph J_n for even $n \geq 6$, and a connected dominating set S for J_n such that $|S| = 2n-1$. If $\gamma_c(J_{n-1}) = 2n-3$, then S contains at most one instance the patterns* 113 *or* 3123.

We are finally ready to prove the main result of this section.

Theorem 7.11. *Consider the graph J_n for $n \geq 3$. Then,*

$$\gamma_c(J_n) = \begin{cases} 2n, & \text{if } n \text{ is even,} \\ 2n-1, & \text{if } n \text{ is odd.} \end{cases}$$

Proof. The upper bounds are provided in Section 4. We use the MTZ formulation for connected domination from [10] to confirm that $\gamma_c(J_3) = 5, \gamma_c(J_4) = 8, \gamma_c(J_5) = 9,$

and $\gamma_c(J_6) = 12$. Suppose there is some value $k \geq 7$ such that Theorem 7.11 is true for $n = 3, \ldots, k-1$, but false for $n = k$. We will first consider the case when k is odd. Then there is a connected dominating set S for J_k such that $|S| = 2k-2$. Clearly, S contains a copy, say J^i, with weight 1, and $a^i \notin S$. Then from Theorem 7.4, it implies that there is a connected dominating set for J_{k-1} with cardinality $2k-3$, which contradicts the assumption that Theorem 7.11 is true for J_{k-1}.

Hence, k must be even. Then, there is a connected dominating set S for J_k such that $|S| = 2k-1$. Suppose that there are two copies of J_k with weight 2 in S. Then, from Theorem 7.4 we can obtain a connected dominating set for J_{k-2} with cardinality $2k-5$, which contradicts the initial assumption. Hence, there must be at most one copy of J_k with weight 2 in S. That is, either $w_2^S = 0$ or $w_2^S = 1$.

Suppose that $w_2^S = 0$. Then we have $w_1^S + w_3^S + w_4^S = k$, and $w_1^S + 3w_3^S + 4w_4^S = 2k-1$. From the latter equation, it is clear that w_1^S and w_3^S must have different parity. Hence, from the former equation, w_4^S must be odd. Combining the two equations, we obtain $2w_1^S = k + 1 + w_4^S$. Note that this implies that more than half of the copies have weight 1 in S. From Theorem 7.7, S cannot contain the pattern 111. Hence, it can be seen that S contains the pattern 11 at least $w_4^S + 1$ times. From Corollary 7.10 we know there is at most one instance of the pattern 113 in S. Hence, there is at least w_4^S instances of the pattern 11 where both adjacent copies have weight 4, which is impossible.

Hence, we must have $w_2^S = 1$. Then, we have $w_1^S + w_3^S + w_4^S = k-1$, and $w_1^S + 3w_3^S + 4w_4^S = 2k-3$. From the latter equation, it is clear that w_1^S and w_3^S must have different parity. Hence, from the former equation, w_4^S must be even. Combining the two equations, we obtain $2w_1^S = k + w_4^S$. Similar to before, this implies that at least half of the copies have weight 1 in S. From Theorem 7.7, S cannot contain the pattern 111. Hence, it can be seen that S contains the pattern 11 at least w_4^S times. Now, suppose that S contains the pattern 4114. From Lemma 7.5, we can then obtain a connected dominating set for J_{k-3} with cardinality $2k-7$ which has one fewer copy of weight 4 than S, which in turn violates Lemma 7.6. Hence, every instance of the pattern 11 has a copy of weight 2 or 3 next to it, and from Corollary 7.10 this means there is at most one instance of the pattern 11. This, in turn, implies that $w_4^S \leq 1$, and since w_4^S is even, we have $w_4^S = 0$. Hence, $w_1^S = \frac{k}{2}$. If there are no instances of the pattern 11, then S must contain the pattern 1312131, violating Theorem 7.7. Hence there is exactly one instance of the pattern 11. This means that there must be one other instance in S of two consecutive copies having non-unit weight. The only options are that S contains the pattern 23, or the pattern 33.

Suppose S contains the pattern 23. Without loss of generality, suppose that the copies J^3, J^4 meet this pattern. Since this is the only instance of S having two consecutive copies of weight other than 1, we know that J^2 has weight 1. From Theorem 7.8 we know that J^1 cannot have weight 1, or else S would contain the pattern 112. The only remaining option is that J^1 has weight 3, and so copies J^1, J^2, J^3, J^4 meet the pattern 3123. Then, since J^3 has weight 2 and $w_2^S = 1$, there are no other copies with weight 2. Hence, the one instance of the pattern 11 which occurs elsewhere in the graph must be followed by a copy of weight 3. That is, S contains both the patterns 113 and 3123, which from Corollary 7.10 is impossible. Therefore, S must not contain the pattern 23, and instead contains the pattern 33.

Finally, without loss of generality, suppose that the copies J^1, J^2 meet the pattern 33. There is one vertex from J^1 which is not contained in S. Suppose we define S_2 which is equal to the union of S and this one vertex. Hence, S_2 is a connected dominating set for J_k

with cardinality $2k$. Then, from Lemma 7.5, we can obtain a connected dominating set for J_{k-1} with cardinality $2k-3$, which contains exactly one copy with weight 4, contradicting Lemma 7.6.

Hence, in all cases a contradiction is reached, completing the proof. □

8 Weak Roman domination and secure domination

In this section, we determine (simultaneously) the weak Roman domination numbers and the secure domination numbers for flower snarks. Recall that for the weak Roman domination number, rather than seeking to construct a set S, we instead look to define a function $f\colon V \to \{0,1,2\}$. In order to use language consistent with the rest of this paper, in this section we define the weight of a copy J^i to be equal to $f(a^i)+f(b^i)+f(c^i)+f(d^i)$. Note that this is somewhat more ambiguous than in previous sections; for instance, if a copy has weight 2, it may have two vertices with weight 1, or a single vertex with weight 2. We will deal with these ambiguities when they arise. Similarly to in previous sections, we will say that f contains a pattern if there is a set of consecutive copies whose weights meet that pattern.

Although there is a technical definition for weak Roman domination (e.g. see Definition 1.4), it is useful to provide an intuitive interpretation. Recall that for standard domination, one interpretation is that the set of vertices needs to be protected. If a guard is placed at a vertex, then it protects that vertex, and also all adjacent vertices. A configuration of guards which protects all vertices corresponds to a dominating set. A similar interpretation applies for weak Roman domination. Suppose that at each vertex we can place up to two guards. As before, a vertex is protected if there is at least one guard there, or there is at least one guard among its adjacent vertices. Then we further consider the notion of a vertex being *defended*. We say a vertex v is defended if there is at least one guard at v, or alternatively, if it possible to relocate one guard from an adjacent vertex w to v, in such a way that all vertices are still protected by the resulting configuration of guards. In the latter case, we will say that w *can defend* v. Then a weak Roman dominating function is one in which every vertex is defended.

Throughout this section, we will often use an argument which goes as follows; suppose there is exactly one guard at v, and that v defends another adjacent vertex w. In order to do so, the guard from v would move to w. However, doing so may mean another vertex u, also adjacent to v, becomes unprotected. In such a case, we will say that v *cannot defend* w *without leaving* u *unprotected.* Another alternative is as follows; suppose again that v defends w. In order to do so, the guard from v would move to w. However, doing so may mean that u will be unprotected unless there is a guard at another vertex, say x. In such a case, we will say *in order for* v *to defend* w*, there must be a guard at* x *to avoid leaving* u *unprotected.*

Although the following result is simple, we include it here as it will be used many times in this section.

Lemma 8.1. *Consider the graph* J_n*, for* $n \geq 3$*, and a weak Roman dominating function* f*. If a copy* J^i *has weight* 1 *in* f*, then none of the vertices from* J^i *can defend any vertices from* J^{i-1} *or* J^{i+1}*.*

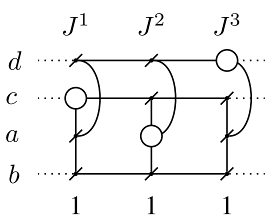
Proof. Since J^i has weight 1, there is exactly one vertex with positive weight. If $f(a^i)=1$, then the result follows immediately since a^i is not adjacent to any vertices from J^{i-1}

or J^{i+1}. Then, suppose instead that $f(a^i) = 0$ and one of the other vertices in J^i has weight 1. Then that vertex cannot defend any vertex from J^{i-1} or J^{i+1} without leaving a^i unprotected. □

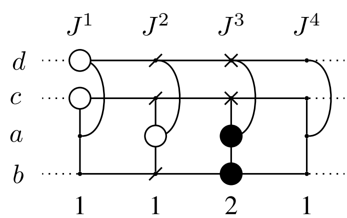
Theorem 8.2. *Consider the graph J_n, for $n \geq 3$, and a weak Roman dominating function f. Then f does not contain the patterns* 111, 1121 *or* 1212121.

Proof. Suppose that f contains the pattern 111. Without loss of generality, suppose that copies J^1, J^2, J^3 meet the pattern 111 in f. This situation is displayed in part (a) of Figure 15. Each vertex in J^2 must be defended, but since J^1 and J^3 have weight 1, from Lemma 8.1 none of their vertices can defend any vertices from J^2. Hence, each vertex in J^2 must be defended solely by vertices in J^2. It is clear that this implies that $f(a^2) = 1$. Then, consider b^2. In order for a^2 to defend b^2, there must be a guard at either c^1 or c^3 to avoid leaving c^2 unprotected. Likewise, in order for a^2 to defend b^2, there must be a guard at either d^1 or d^3 to avoid leaving d^2 unprotected. Since J^1 and J^3 have weight 1, at most one vertex from each can have a guard. Without loss of generality, suppose that $f(c^1) = 1$ and $f(d^3) = 1$. Then a^2 cannot defend c^2 without leaving b^2 unprotected. Hence, this is impossible, and f does not contain the pattern 111.

Next, suppose that f contains the pattern 1121. Without loss of generality, suppose that copies J^1, J^2, J^3, J^4 meet the pattern 1121 in f. Since J^2 and J^4 have weight 1, from Lemma 8.1 none of their vertices can defend any vertices from J^3. Hence, each vertex from J^3 must be defended solely by vertices in J^3. Clearly, this implies that $f(a^3) \geq 1$. There are two possibilities, either $f(a^3) = 2$, or $f(a^3) = 1$ and there is a guard at another vertex in J^3. Suppose first that $f(a^3) = 2$. Then there are no vertices from J^1 or J^3 that can defend any vertices from J^2. Hence, J^2 is in an equivalent situation to J^2 in the previous paragraph and so this situation is impossible. Instead, suppose that $f(a^3) = 1$ and, without loss of generality, that $f(b^3) = 1$. This situation is displayed in part (b) of Figure 15. Clearly, vertices c^2 and d^2 can only be defended by vertices from J^2, and hence we must have $f(a^2) = 1$. Then, in order for a^2 to defend c^2, there must be a guard at d^1 to avoid leaving d^2 unprotected. Similarly, in order for a^2 to defend d^2, there must be a guard at c^1 to avoid leaving c^2 unprotected. This is impossible since J^1 has weight 1, and so f does not contain the pattern 1121.



(a)



(b)

Figure 15: The first two situations described in the proof of Theorem 8.2. In part (a), a^2 cannot defend c^2 without leaving b^2 unprotected. In part (b), there are too many guards in J^1.

Finally, suppose that f contains the pattern 1212121. Without loss of generality, suppose that copies $J^1, \ldots, J^7$ meet the pattern 1212121 in f. Since J^1 and J^3 have weight 1, from Lemma 8.1 none of their vertices can defend any vertices from J^2. Hence, each vertex from J^2 must be defended solely by vertices in J^2, which implies that $f(a^2) \geq 1$.

Analogous arguments imply that $f(a^4) \geq 1$ and $f(a^6) \geq 1$. Now, consider J^2. There are two possibilities, either $f(a^2) = 2$, or $f(a^2) = 1$ and there is a guard at another vertex in J^2. Suppose first that $f(a^2) = 2$. Then there are no vertices from J^2 that can defend any vertices from J^3, and at most one vertex from J^4 can defend a vertex from J^3. Hence, J^3 is in an equivalent situation to J^2 in the previous paragraph, and so this situation is impossible.

Instead, suppose that $f(a^2) = 1$ and, without loss of generality, that $f(b^2) = 1$. This situation is displayed in Figure 16. Then, in order for a^2 to defend c^2, there must be a guard at either d^1 or d^3 to avoid leaving d^2 unprotected. Similarly, in order for a^2 to defend d^2, there must be a guard at either c^1 or c^3 to avoid leaving c^2 unprotected. Since J^1 and J^3 have weight 1, at most one vertex from each may have a guard. Without loss of generality, suppose that $f(c^1) = 1$ and $f(d^3) = 1$. Then, since f is dominating, we must have $f(c^4) = 1$ in order to protect c^3. Now, in order for a^4 to defend d^4, there must be a guard at b^5 to avoid leaving b^4 unprotected. Then, since f is dominating, we must have $f(d^6) = 1$ in order to protect d^5. Finally, consider vertex c^5. There is only one adjacent vertex with a guard that can defend it, c^4. However, c^4 cannot defend c^5 without leaving c^3 unprotected. Thus, c^5 is not defended. This is impossible, and hence f does not contain the pattern 1212121. □

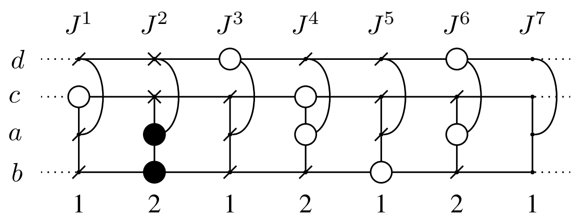


Figure 16: The third situation described in the proof of Theorem 8.2. Here, c^4 cannot defend c^5 without leaving c^3 unprotected.

Lemma 8.3. *Consider the graph J_n, for $n \geq 4$, and a weak Roman dominating function f with weight $w(f) \leq \frac{3n}{2}$. Then any set of four consecutive copies have a combined weight of 6 in f, every copy has weight either 1 or 2 in f, and $w(f) = \frac{3n}{2}$.*

Proof. Denote by $w^4(i)$ the combined weights of $J^i, J^{i+1}, J^{i+2}, J^{i+3}$ where the superscripts are taken modulo n. Now, by Corollary 2.4 we know that no copies have weight 0, and so $w^4(i) \geq 4$. Suppose $w^4(i) = 4$. This implies that $J^i, J^{i+1}, J^{i+2}, J^{i+3}$ all have weight 1. However, this means f contains the pattern 111, which by Theorem 8.2 is impossible. Then, suppose $w^4(i) = 5$. Up to symmetry, there are only two possibilities; copies $J^i, J^{i+1}, J^{i+2}, J^{i+3}$ either have the pattern 1112 or 1121. Both of these options are impossible by Theorem 8.2. Hence, we have $w^4(i) \geq 6$. Then, by assumption, we have $\sum_{i=1}^{n} w^4(i) \leq 6n$. This is only possible if all $w^4(i) = 6$, and also implies that $w(f) = \frac{3n}{2}$.

Since each copy has weight at least one, and each set of four consecutive copies has a combined weight of six, it is clear that any individual copy must have weight at most three. Suppose there is a copy J^i with weight 3 in f, then the next three copies must each have weight 1. However, by Theorem 8.2 this is impossible. Hence, each copy J^i must have weight either 1 or 2 in f. □

Lemma 8.4. *Consider the graph J_n, for $n \geq 3$, and a weak Roman dominating function f with weight $w(f)$. Suppose there is an integer $k \geq 3$ such that in f, the configuration of guards in copies $J^{i+1}, \ldots, J^{i+k}$ is identical to the configuration of guards in $J^{i+k+1}, \ldots, J^{i+2k}$, and suppose that $J^{i+1}, \ldots, J^{i+k}$ collectively contain g guards. Then there is a weak Roman dominating function for J_{n-k} with weight equal to $w(f) - g$.*

Proof. Suppose that we "smooth out" copies $J^{i+1}, \ldots, J^{i+k}$ in the way displayed in Figure 11. That is, we add edges connecting b^i to b^{i+k+1}, c^i to c^{i+k+1}, and d^i to d^{i+k+1}, and then we delete the copies $J^{i+1}, \ldots, J^{i+k}$. Call the resulting graph G_2. It is easy to check that G_2 is isomorphic to J_{n-k}. Then, define a new function f_2 such that $f_2(v) = f(v)$ if $v \in V \setminus \{V(J^{i+1}) \cup \ldots \cup V(J^{i+k})\}$, and $f_2(v)$ is undefined otherwise. Since $J^{i+1}, \ldots, J^{i+k}$ collectively contain g guards in f, it is clear that $w(f_2) = w(f) - g$. Then if we can show that f_2 is a weak Roman dominating function for G_2, the result follows immediately.

For a given function, one can check to see if a vertex v is defended by observing the configuration of guards in vertices no more than distance three away from v. Then, for any vertex v in G_2, consider the configuration of graphs (according to f_2) in the subgraph induced by the vertices at most distance 3 from v. By assumption, this is identical to the corresponding configuration of guards (according to f) in the subgraph induced by the vertices at most distance 3 from v in J_n. Hence, all vertices of G are defended in f_2, and so f_2 is a weak Roman dominating function for G, leading to the desired result. □

In the proof of the following theorem, whenever there are two guards at a vertex, we will mark that vertex with a ■.

Theorem 8.5. *Consider the graph J_n, for $n \geq 9$, and a weak Roman dominating function f. If f contains the pattern 21122112, then there exists a weak Roman dominating function for J_{n-4} with weight equal to $w(f) - 6$.*

Proof. Without loss of generality, suppose that copies $J^1, \ldots, J^8$ meet the pattern 21122112 in f. Consider copy J^4, and suppose that $f(a^4) = 2$. This situation is displayed in part (a) of Figure 17. Then, no vertices from J^4 can defend any vertices from J^3, and by Lemma 8.1 the same is also true for J^2. Hence, all four vertices in J^3 must be defended by vertices in J^3. This implies that $f(a^3) = 1$. Then, in order for a^3 to defend b^3, there must be a guard at c^2 to avoid leaving c^3 unprotected, and there must also be a guard at d^2 to avoid leaving d^3 unprotected. Since J^2 has weight 1, this is impossible, and so $f(a^4) \neq 2$.

Then, suppose that $f(a^4) = 1$. Then there is another vertex in J^4 with a guard. Without loss of generality, suppose that $f(b^4) = 1$. This situation is displayed in part (b) of Figure 17. Using a similar argument as in the previous paragraph, vertices c^3 and d^3 must be defended by vertices from J^3, and hence $f(a^3) = 1$. Then, in order for a^3 to defend c^3, there must be a guard at d^2 to avoid leaving d^3 unprotected. Likewise, in order for a^3 to defend d^3, there must be a guard at c^2 to avoid leaving c^3 unprotected. Since J^2 has weight 1, this is impossible. Hence we can conclude that $f(a^4) = 0$. It is clear that identical arguments can be made to conclude that $f(a^1) = f(a^5) = f(a^8) = 0$ as well.

Then, suppose that there is a vertex in J^4 with two guards. Without loss of generality, suppose that $f(b^4) = 2$. This situation is displayed in part (c) of Figure 17. An identical argument to that in the previous paragraph can be used to show that $f(a^3) = 1$, and this again implies that $f(c^2) = f(d^2) = 1$. Since J^2 has weight 1, this is impossible. Hence, exactly two vertices in J^4 have weight 1. Again, identical arguments can be made to reach the same conclusion for each of J^1, J^5 and J^8.

Now, without loss of generality, suppose that $f(b^4) = f(c^4) = 1$. Then, d^4 must be defended by either d^3 or d^5, and by Lemma 8.1 it cannot be d^3. Hence, we must have $f(d^5) = 1$, and there must be one more vertex in J^5 with a guard, either b^5 or c^5. Due to symmetry, either choice may be made without loss of generality; we will choose $f(b^5) = 1$. Now, consider vertex d^3. From Lemma 8.1 it cannot be defended by d^2. Hence, we must have either $f(a^3) = 1$ or $f(d^3) = 1$.

Suppose that $f(a^3) = 1$. Then, c^3 must be defended by at least one of a^3 or c^4 (from Lemma 8.1 it cannot be defended by c^2). Suppose first that a^3 defends c^3. This situation is displayed in part (d) of Figure 17. In order for a^3 to defend c^3, there must be a guard at d^2 to avoid leaving d^3 unprotected. Then, in order to defend b^2 and c^2 we must have $f(b^1) = 1$ and $f(c^1) = 1$ respectively. But then, b^1 cannot defend a^1 without leaving b^2 unprotected, and c^1 cannot defend a^1 without leaving c^2 unprotected. Hence, a^1 is not defended, which is impossible, and so we conclude that a^3 cannot defend c^3. Hence, c^4 must defend c^3. This situation is displayed in part (e) of Figure 17. In order for c^4 to defend c^3, there must be a guard at c^6 to avoid leaving c^5 unprotected. Also, the only vertex which can defend d^4 is d^5, but in order to do so, there must be a guard at d^7 to avoid leaving d^6 unprotected. Finally, consider a^5, which can only be defended by either b^5 or d^5. However, b^5 cannot defend a^5 without leaving b^6 unprotected, and d^5 cannot defend a^5 without leaving d^4 unprotected. Hence, all of these cases are impossible, and so conclude that $f(a^3) = 0$, and accordingly, $f(d^3) = 1$.

The current situation is displayed in part (f) of Figure 17. Now consider c^3. From Lemma 8.1 it is clear that c^2 cannot defend c^3, so c^4 must defend c^3. In order for c^4 to defend c^3, there must be a guard at c^6 to avoid leaving c^5 unprotected. Likewise, from Lemma 8.1, d^3 cannot defend d^4, so d^5 must defend d^4. In order for d^5 to defend d^4, there must be a guard at d^7 to avoid leaving d^6 unprotected. Furthermore, from Lemma 8.1, c^6 cannot defend c^5, so c^4 must defend c^5. In order for c^4 to defend c^5, there must be a guard at c^2 to avoid leaving c^3 unprotected. Finally, from Lemma 8.1, the only vertices which can defend b^2 and d^2 are b^1 and d^1 respectively, and so $f(b^1) = f(d^1) = 1$. Likewise, from Lemma 8.1, the only vertices which can defend b^7 and c^7 are b^8 and c^8 respectively, so $f(b^8) = f(c^8) = 1$.

At this point, it can be seen that the configuration of guards in J^1, J^2, J^3, J^4 is identical to the configuration of guards in J^5, J^6, J^7, J^8. Then, the result follows immediately from Lemma 8.4. □

We are finally ready to prove the main result of this section.

Theorem 8.6. *Consider the graph J_n, for $n \geq 3$. Then,* $\gamma_s(J_n) = \gamma_r(J_n) = \left\lceil \dfrac{3n+1}{2} \right\rceil$.

Proof. The (coincident) upper bounds for both weak Roman domination and secure domination are provided in Section 4. Note that $\gamma_r(J_n) \leq \gamma_s(J_n)$, so a corresponding lower bound for weak Roman domination will also serve as a lower bound for secure domination. We use the formulation for weak Roman domination from [7] to confirm that $\gamma_r(J_3) = 5$, $\gamma_r(J_4) = 7$, $\gamma_r(J_5) = 8$, $\gamma_r(J_6) = 10$, $\gamma_r(J_7) = 11$, and $\gamma_r(J_8) = 13$, and the formulation from [5] to confirm the corresponding results for secure domination. Then, suppose there is a value $k \geq 9$ such that Theorem 8.6 is true for $n = 3, \ldots, k-1$, but is not true for $n = k$. That is, there exists a weak Roman dominating function f for J_k with

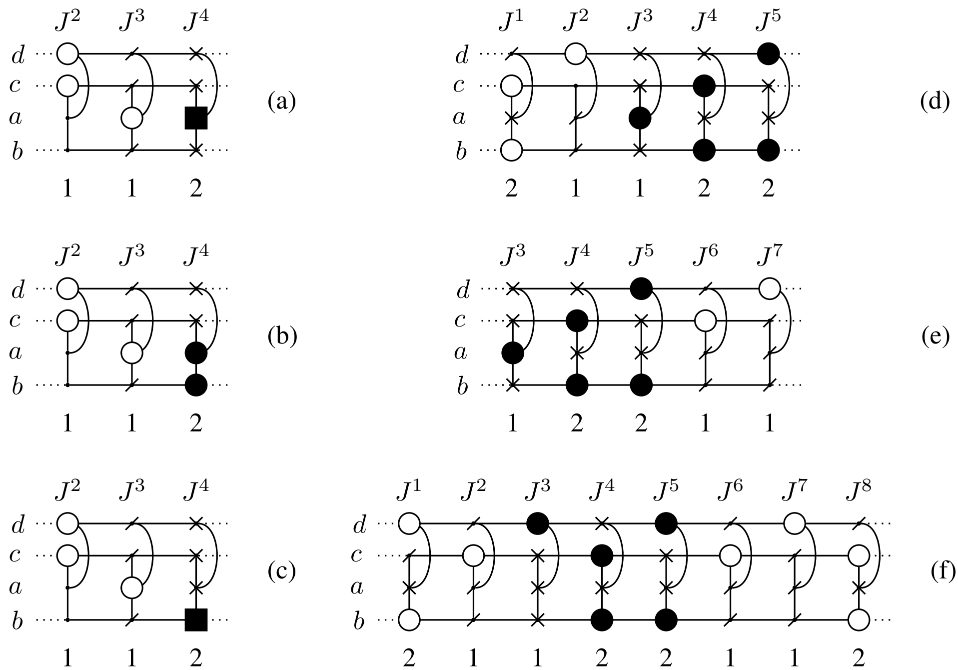


Figure 17: The six situations described in the proof of Theorem 8.5. In parts (a), (b) and (c), there are too many guards in copy J^2. In part (d), a^1 is undefended. In part (e), a^5 is undefended. In part (f), the configuration of guards in J^1, J^2, J^3, J^4 is identical to that in J^5, J^6, J^7, J^8.

weight $w(f) < \left\lceil \frac{3k+1}{2} \right\rceil$. This implies that $w(f) \leq \frac{3k}{2}$. Hence, from Lemma 8.3 we have $w(f) = \frac{3k}{2}$, each set of four consecutive copies has weight 6, and each copy has weight 1 or 2.

It is easy to check that there are only two possibilities. Either f has the repeating pattern $121212\ldots12$, or f has the repeating pattern $21122112\ldots2112$. Note that it is impossible to combine these two patterns without there being a set of four consecutive copies with weight not equal to 6. Then, since $k \geq 9$, by Theorem 8.2 the pattern $121212\ldots12$ is impossible. Hence, f must have the repeating pattern $21122112\ldots2112$. Then, by Theorem 8.5 there is weak Roman dominating function for J_{k-4} with weight equal to $\frac{3k}{2} - 6 < \left\lceil \frac{3(k-4)+1}{2} \right\rceil$. This contradicts our initial assumption, completing the proof. □

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Optimal strategies in fractional games: vertex cover and domination

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Abstract

In a hypergraph $\mathcal{H} = (V, \mathcal{E})$ with vertex set V and edge set $\mathcal{E}$, a real-valued function $f\colon V \to [0,1]$ is a fractional transversal if $\sum_{v \in E} f(v) \geq 1$ holds for every $E \in \mathcal{E}$. Its size is $|f| := \sum_{v \in V} f(v)$, and the fractional transversal number $\tau^*(\mathcal{H})$ is the smallest possible $|f|$.

We consider a game scenario where two players have opposite goals, one of them trying to minimize and the other to maximize the size of a fractional transversal constructed incrementally. We prove that both players have strategies to achieve their common optimum, and they can reach their goals using rational weights.

Keywords: Fractional vertex cover, fractional transversal game, fractional domination game.

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1 Introduction

Let $\mathcal{H} = (V, \mathcal{E})$ be a finite hypergraph, where V is the finite vertex set and $\mathcal{E}$ is the edge set, a set system over the underlying set V. We assume that every edge contains at least one vertex; that is, $\mathcal{E} \subseteq 2^V \setminus \{\emptyset\}$. A hypergraph is *$k$-uniform* if $|E| = k$ holds for all $E \in \mathcal{E}$. A set $T \subseteq V$ is a *transversal*[1] of $\mathcal{H}$ if every edge is covered by a vertex of T, which formally means that $T \cap E \neq \emptyset$ holds for all $E \in \mathcal{E}$. Its real relaxation, called *fractional transversal*, is a function $f\colon V \to [0,1]$ such that $\sum_{v\in E} f(v) \geq 1$ holds for every $E \in \mathcal{E}$. The size of f is defined as $|f| := \sum_{v\in V} f(v)$. The *transversal number* $\tau(\mathcal{H})$ and the *fractional transversal number* $\tau^*(\mathcal{H})$ of $\mathcal{H}$ are the minimum cardinality $|T|$ of a transversal and minimum value $|f|$ of a fractional transversal, respectively.

The *transversal game* is a competitive optimization version of hypergraph transversals, which was introduced in [9] and studied further in [10]. It is played on a hypergraph $\mathcal{H}$ by two players called *Edge-hitter* and *Staller*. They take turns choosing a vertex. The game is over when all edges are covered, and the length of the game is the number of vertices chosen by the players until the end of the game. Edge-hitter wants to finish the game as soon as possible, while Staller wants to delay the end. To prevent Staller from making completely useless moves, we stipulate that the chosen vertex must be contained in at least one previously uncovered edge.

Assuming that both players play optimally[2] and Edge-hitter starts, the length of the game on $\mathcal{H}$ is uniquely determined. It is called the *game transversal number* of $\mathcal{H}$ and is denoted by $\tau_g(\mathcal{H})$. Analogously, the *Staller-start game transversal number* of $\mathcal{H}$, denoted by $\tau_g'(\mathcal{H})$, is the length of the game under the same rules when Staller makes the first move. Among other results, it was proved in [9] that $|\tau_g(\mathcal{H}) - \tau_g'(\mathcal{H})| \leq 1$ always holds. We further recall that, denoting by $n(\mathcal{H})$ and $m(\mathcal{H})$ the number of vertices and edges in $\mathcal{H}$ respectively, $\frac{4}{11}(n(\mathcal{H}) + m(\mathcal{H}))$ is a (sharp) upper bound on $\tau_g(\mathcal{H})$ if $\mathcal{H}$ does not contain one-element edges and it is not isomorphic to the cycle C_4.

Below we shall refer to this game as the *integer game*, as opposed to its fractional version which we will introduce in the next section.

The important motivation of this approach are the *domination game* [7] and the *total domination game* [17], where in fact the transversal game is played on the 'closed neighborhood hypergraph' and on the 'open neighborhood hypergraph' of a graph, respectively.[3] Further variants studied so far include the disjoint domination [14], connected domination [2], and fractional domination [15] games on graphs, and the domination games on hypergraphs [13]. Some of the most recent results can be found in [3, 4, 8, 11, 12, 16, 18, 19, 20, 21, 22, 23]. For a thorough survey and list of further

[1] In various areas of discrete mathematics and computer science, a transversal is called vertex cover, or hitting set, or blocking set. It is also equivalent to the set cover in the dual hypergraph.

[2] A *strategy* of a player means that every possible state of the game is associated with a move he/she will play if that situation arises. From Edge-hitter's point of view, the value $v_E(S)$ of a strategy S is the smallest integer k such that, if Edge-hitter plays according to S, the transversal game always finishes in at most k moves (no matter which strategy is applied by Staller). We say that S is an *optimal strategy* for Edge-hitter, if $v_E(S)$ is the possible smallest value over the family of all strategies. Similarly, from Staller's point of view, a strategy S can be associated with the value $v_S(S)$ that is the largest integer k such that, if Staller follows strategy S, the length of the game is always at least k; further S is an *optimal strategy* for Staller, if $v_S(S)$ is the largest value over the family of all strategies. The reader may find more about optimal strategies and the uniqueness of the corresponding parameters in [5, Section 1.2].

[3] Recall from the literature that the closed and open neighborhood hypergraphs of a graph G are defined on the same vertex set V as G, and the closed (resp. open) neighborhood hypergraph consists of edges corresponding to the closed (resp. open) neighborhoods of vertices in G.

references see the book [5].

Our results

In Section 2 we introduce the rules of the game and prove that its value is well-defined. We present some examples, showing that it makes a difference whether Edge-hitter or Staller starts. Moreover, edges that are supersets of other edges of the hypergraph may influence the game value, in contrast to the standard non-game version of the transversal number.

In Section 3 we compare the game transversal number with other related parameters, and prove a monotonicity property, implying that changing the starting player can affect the value of the game by at most 1.

The rules of the game allow the players to split their moves into infinitely many submoves. In Section 4 and 5 we give some structural results showing that the full generality of the moves allowed by our rules is not needed. Namely, any infinite move is equivalent to a finite move, and Edge-hitter can restrict his strategy to moves in which every permutation of submoves is equally good.

In Section 6 we prove that the game can be modeled in a way that leads to an optimization problem solvable via the theory of piecewise linear continuous rational functions. From this, we derive that the game value is rational for every finite hypergraph; moreover both players can achieve their goals using rational submoves.

Consequences concerning domination games and several conjectures are given in the concluding Section 7.

2 Fractional transversal game

Let $\mathcal{H} = (V, \mathcal{E})$ be a hypergraph. In the context of the fractional transversal game, we will consider a *cover function* $t\colon V \to [0,1]$ that is updated after each move during the game. We denote by $|t|$ the sum $\sum_{v\in V} t(v)$. Given a cover function t, the corresponding *load function* is $\ell\colon \mathcal{E} \to [0,1]$ defined by the rule

$$\ell(E) = \ell(E,t) = \min\Big\{1, \sum_{v\in E} t(v)\Big\}$$

for every $E \in \mathcal{E}$. If $\ell \equiv 1$, we say that $\mathcal{H}$ is fully covered. We shall write t_i and ℓ_i for the cover and load functions after the i^{th} move.

The game begins with $t_0 \equiv 0$ and therefore with $\ell_0 \equiv 0$. It is finished when the hypergraph becomes fully covered. Edge-hitter and Staller take turns making moves under the following rules. As long as $\ell \not\equiv 1$, the next player performs a *move*, which is a sequence $(v_{i_1}, w_1), (v_{i_2}, w_2), \dots$ of arbitrary length (possibly infinite). It consists of the *submoves* (v_{i_k}, w_k), $k = 1, 2, \dots$, where $v_{i_1}, v_{i_2}, \dots$ are vertices of $\mathcal{H}$ with any number of repetitions allowed, and the *weights* $w_1, w_2, \dots$ are real numbers from $[0,1]$.

We say that a submove (v_{i_k}, w_k) is *legal* if it increases the load of some edge by w_k. In a *legal move*, a player makes a series of legal submoves such that the sum of the weights equals 1 or the move completes the game, whichever comes first. Formally, the i^{th} move $(v_{i_1}, w_1), (v_{i_2}, w_2), \dots$ is legal, if the following conditions hold:

($*$) For every $k \geq 1$ there exists an edge $E \in \mathcal{E}$ such that $v_{i_k} \in E$ and

$$\ell_{i-1}(E) + \Bigg(\sum_{\substack{v_{i_s}\in E \\ 1\leq s\leq k-1}} w_s \Bigg) + w_k \leq 1\,.$$

(∗∗) The *total weight constraint*: $\sum_{k\geq 1} w_k \leq 1$, and if the move does not end the game, then $\sum_{k\geq 1} w_k = 1$.

The cover function t_i can gradually be reached from t_{i-1} by adding the weight w_k to $t(v_{i_k})$ after each submove; this process converts also the corresponding load function from ℓ_{i-1} to ℓ_i.

Suppose that a fractional transversal game $\mathcal{G}$ finishes with the q^{th} move. The *value* $|\mathcal{G}|$ *of the game* is defined as the value $|t_q|$ of the cover function obtained at the end, that is the sum of the weights that have been spent during the game. The goal of Edge-hitter is to achieve a value $|\mathcal{G}|$ as small as possible, while Staller wants a large $|\mathcal{G}|$.

Assuming that Edge-hitter starts the fractional transversal game on $\mathcal{H}$, we consider the set of upper bounds,

$$U_{\mathcal{H}} = \{a \in \mathbb{R} : \text{Edge-hitter has a strategy that ensures } |\mathcal{G}| \leq a\}$$

and the set of lower bounds,

$$L_{\mathcal{H}} = \{b \in \mathbb{R} : \text{Staller has a strategy that ensures } |\mathcal{G}| \geq b\}.$$

Formally the *game fractional transversal number* $\tau_g^*(\mathcal{H})$ is defined as

$$\tau_g^*(\mathcal{H}) = \inf(U_{\mathcal{H}}).$$

The Staller-start game fractional transversal number $\tau_g^{*\prime}(\mathcal{H})$ is defined similarly, under the condition that the first move is made by Staller.

The following assertion shows that $\tau_g^*(\mathcal{H})$ is also equal to $\sup(L_{\mathcal{H}})$, and the situation is similar if Staller starts the game. The proof is essentially the same as the one for the game fractional domination number in [15].

Proposition 2.1. *For every hypergraph $\mathcal{H}$ we have $\inf(U_{\mathcal{H}}) = \sup(L_{\mathcal{H}})$, and the analogous equality holds for the Staller-start game, too.*

Proof. First, assume that $\inf(U_{\mathcal{H}}) < \sup(L_{\mathcal{H}})$ and consequently, there exist two reals x and y satisfying $\inf(U_{\mathcal{H}}) < x < y < \sup(L_{\mathcal{H}})$. By definition, $x \in U_{\mathcal{H}}$ and, therefore, Edge-hitter can ensure that, under every strategy of Staller, the value of the game is at most x. Similarly, $y \in L_{\mathcal{H}}$ and Staller has a strategy that ensures $|\mathcal{G}| \geq y$ whatever strategy is followed by Edge-hitter. This is a contradiction that establishes $\inf(U_{\mathcal{H}}) \geq \sup(L_{\mathcal{H}})$.

Now, we prove the reverse inequality. By definition, $z < \inf(U_{\mathcal{H}})$ implies that Edge-hitter does not have a strategy to achieve $|\mathcal{G}| \leq z$. That is, against each strategy of Edge-hitter there is a strategy of Staller which results in $|\mathcal{G}| > z$. We may infer that $z \in L_{\mathcal{H}}$ and therefore $z \leq \sup(L_{\mathcal{H}})$. Since it holds for every $z < \inf(U_{\mathcal{H}})$, we conclude $\inf(U_{\mathcal{H}}) \leq \sup(L_{\mathcal{H}})$. This completes the proof of the proposition. □

Later, in Section 6, we will show that $\inf(U_{\mathcal{H}}) = \min(U_{\mathcal{H}})$ and $\sup(L_{\mathcal{H}}) = \max(L_{\mathcal{H}})$. Therefore, Edge-hitter and Staller have optimal strategies under which, respectively, $|\mathcal{G}| \leq \tau_g^*(\mathcal{H})$ and $|\mathcal{G}| \geq \tau_g^*(\mathcal{H})$ are achieved.

2.1 Examples for the fractional transversal game

(1) Our first example is the 4-cycle $C_4 = v_1v_2v_3v_4v_1$, which can also be considered as a 2-uniform hypergraph. It is easy to check that $\tau^*(C_4) = 2$, while $\tau_g(C_4) = 3$ and $\tau'_g(C_4) = 2$ were proved for the integer games [9]. Now we prove that $\tau^*_g(C_4) = 5/2$.

- For the upper bound, the following strategy of Edge-hitter ensures that the sum of the weights spent during the game is at most $5/2$. His first move is $(v_1, \frac{1}{4}), (v_2, \frac{1}{4})$, $(v_3, \frac{1}{4})$, $(v_4, \frac{1}{4})$; it results in $\ell_1 \equiv \frac{1}{2}$ and $\sum_{E\in\mathcal{E}} \ell_1(E) = 2$. Then, for the first $\frac{1}{2}$ of the weight spent by Staller, each of her submoves necessarily increases the load of both incident edges; and each of her remaining submoves increases the load of at least one edge. Therefore, the total load increases by at least $\frac{1}{2} \times 2 + \frac{1}{2} = \frac{3}{2}$ to $\sum_{E\in\mathcal{E}} \ell_2(E) \geq \frac{7}{2}$, and Edge-hitter can achieve $\sum_{E\in\mathcal{E}} \ell_3(E) = 4$ by spending at most $\frac{1}{2}$ in the final move. This proves $\tau^*_g(C_4) \leq 5/2$.

- Let us show the reverse inequality. We note that the first move of the game has the same effect as a sequence $(v_1, w_1), (v_2, w_2), (v_3, w_3), (v_4, w_4)$ of submoves with $\sum_{i=1}^{4} w_i = 1$.[4] After this move of Edge-hitter, $\sum_{E\in\mathcal{E}} \ell_1(E) = 2$ and hence, there is an edge E with $\ell_1(E) \leq \frac{1}{2}$. By symmetry, we may assume that $\ell_1(v_1v_2) \leq \frac{1}{2}$. Let Staller play the move $(v_3, w_1 + w_4), (v_4, w_2 + w_3)$. The move is legal as $\ell_1(v_2v_3) = w_2 + w_3 = 1 - (w_1 + w_4)$ and $\ell_1(v_4v_1) = w_1 + w_4 = 1 - (w_2 + w_3)$. We then have $\ell_2(v_1v_2) = \ell_1(v_1v_2) \leq \frac{1}{2}$ and Edge-hitter needs to spend at least $\frac{1}{2}$ to finish the game. This strategy of Staller shows $\tau^*_g(C_4) \geq 5/2$.

If Staller starts the fractional transversal game on C_4 with the move $(v_1, w_1), (v_2, w_2)$, $(v_3, w_3), (v_4, w_4)$, then Edge-hitter can ensure $|\mathcal{G}| = 2$ by playing $(v_1, w_3), (v_2, w_4)$, $(v_3, w_1), (v_4, w_2)$. Indeed, ℓ_2 assigns $w_1 + w_2 + w_3 + w_4 = 1$ to every edge of the graph. Therefore, $\tau^{*\prime}_g(C_4) \leq 2$. Since $\tau^{*\prime}_g(C_4) \geq \tau^*(C_4) = 2$ also holds[5], we get $\tau^{*\prime}_g(C_4) = 2$.

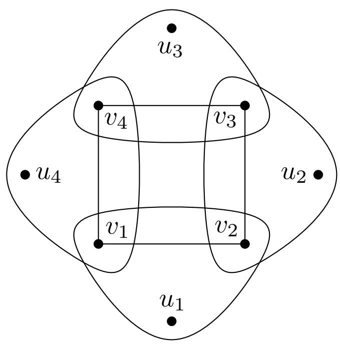

Figure 1: A hypergraph $\mathcal{H}$ with nested edges.

[4]Theorem 4.1 and Lemma 5.1 will give conditions for this replacement property in general. For the first move of the game, it is much easier to see as, for each edge E, its load $\ell_1(E)$ equals the sum of the weights assigned to the vertices of E.

[5]See the proof of Proposition 3.1(i) for a simple explanation.

(2) Now we modify the previous example C_4 by adding four new vertices $u_1, \ldots, u_4$ and four new edges $\{v_1, v_2, u_1\}, \ldots, \{v_4, v_1, u_4\}$ to get the hypergraph $\mathcal{H}$ shown in Figure 1. When the fractional (or integer) transversal number is considered, each edge that is a superset of another edge can be deleted, which implies $\tau^*(\mathcal{H}) = \tau^*(C_4) = 2$. We show that the situation is different for the fractional transversal game on $\mathcal{H}$, that is $\tau_g^*(\mathcal{H}) = 3 \neq \tau_g^*(C_4)$. Suppose that Edge-hitter starts the fractional transversal game $\mathcal{G}$ on $\mathcal{H}$.

- We first show that Edge-hitter can ensure that the value $|\mathcal{G}|$ of the game is at most 3. An optimal first move for him is $(v_1, \frac{1}{4}), (v_2, \frac{1}{4}), (v_3, \frac{1}{4}), (v_4, \frac{1}{4})$. Then, independently of Staller's reply, Edge-hitter plays $(v_1, w_1), (v_2, w_2), (v_3, w_3), (v_4, w_4)$ as his second move, where w_i equals $\frac{1}{4}$ or, if $(v_i, \frac{1}{4})$ is not a legal submove, w_i is the maximum legal weight for v_i. After this move, $\ell_3 \equiv 1$ and we may infer that $|\mathcal{G}| \leq 3$. This proves $\tau_g^*(\mathcal{H}) \leq 3$.
- Our second claim is that Staller has a strategy that always results in $|\mathcal{G}| \geq 3$. As each vertex of $\mathcal{H}$ belongs to at most two 3-element edges, after Edge-hitter's first move the sum of the loads of the 3-element edges is at most 2. Thus, Staller can play a legal move that does not assign weights to v_1, v_2, v_3, v_4. After this move, the sum of the loads of the 2-element edges remains at most 2, and Edge-hitter has to spend a weight of at least 1 to finish the game. This shows $\tau_g^*(\mathcal{H}) \geq 3$. We therefore conclude $\tau_g^*(\mathcal{H}) = 3 > \tau_g^*(C_4)$.

(3) The removal of the edges which are *subsets* of other edges in a hypergraph $\mathcal{F}$ may also change the values of the parameters. For instance, let $\mathcal{F}$ be the hypergraph obtained from C_4 by adding the 4-element edge $E = \{v_1, v_2, v_3, v_4\}$. As the load of E equals 1 after the first move in the game, it is easy to see that $\tau_g^*(\mathcal{F}) = \tau_g^*(C_4) = 5/2$, while the removal of all 2-element edges results in a one-edge hypergraph $\mathcal{F}'$ with $\tau_g^*(\mathcal{F}') = 1$.

3 Some basic facts and the Continuation Principle

In this section we first observe some simple inequalities which are analogous to the ones in other games concerning graph domination and hypergraph transversal, most notably to the fractional domination game [15].

Proposition 3.1.

(i) *For every hypergraph $\mathcal{H}$, it holds that*

$$\tau^*(\mathcal{H}) \leq \tau_g^*(\mathcal{H}) < 2\tau^*(\mathcal{H}) \qquad \textit{and} \qquad \tau^*(\mathcal{H}) \leq \tau_g^{*\prime}(\mathcal{H}) < 2\tau^*(\mathcal{H}) + 1.$$

(ii) *There is no universal constant C with $\tau_g(\mathcal{H}) \leq C \cdot \tau_g^*(\mathcal{H})$, and not even with $\tau(\mathcal{H}) \leq C \cdot \tau_g^*(\mathcal{H})$. The same holds true for $\tau_g^{*\prime}(\mathcal{H})$, too.*

Proof. No matter which player starts the game, at the end the cover function t_q is a fractional transversal. This implies the lower bounds $\tau_g^*(\mathcal{H}) \geq \tau^*(\mathcal{H})$ and $\tau_g^{*\prime}(\mathcal{H}) \geq \tau^*(\mathcal{H})$.

Concerning a fractional transversal game $\mathcal{G}$ on $\mathcal{H}$ and the upper bounds in (i), we can write the value of the game in the form $|\mathcal{G}| = W + W'$, where W and W' denote the total sum of weights assigned by Edge-hitter and Staller, respectively. To keep the claimed bounds, first Edge-hitter can fix an optimal fractional transversal f, i.e. one with $|f| =$

$\tau^*(\mathcal{H})$. After that, in his moves he can apply the strategy to play submoves (v_{i_j}, w_j) with the largest possible weights w_j which are not only allowed by $(*)$ but also respect the inequalities $t_{i-1}(v_{i_j}) + w_j \le f(v_{i_j})$. If such a legal submove with a positive weight does not exist anymore, then $\mathcal{H}$ is fully covered and the game is finished.

This strategy yields $W \le \tau^*(\mathcal{H})$, with strict inequality if the game is finished by Staller. We also have $W' \le W$ or $W' \le W + 1$, depending on whether the first move is made by Edge-hitter or Staller, both with strict inequalities if the game is finished by Edge-hitter. Since only one of the players can make the last move, the claimed strict upper bounds follow.

For the proof of (ii) we apply the following result of Alon [1]: For every $\epsilon > 0$ and for any sufficiently large k, there is a k-uniform hypergraph $\mathcal{H} = (V, \mathcal{E})$ such that $\tau(\mathcal{H}) \ge (1-\epsilon)\frac{\ln k}{k}(|V| + |\mathcal{E}|)$. On the other hand, a very simple fractional transversal f with $|f| = |V|/k$ may be constructed by assigning $f(v) = 1/k$ to each vertex $v \in V$. Therefore, $\tau^*(\mathcal{H}) \le \frac{|V|}{k}$ and we obtain

$$(1/2 - \epsilon)\ln k < \sup_{\mathcal{H}} \frac{\tau(\mathcal{H})}{2\tau^*(\mathcal{H})} < \sup_{\mathcal{H}} \frac{\tau(\mathcal{H})}{\tau_g^*(\mathcal{H})} \le \sup_{\mathcal{H}} \frac{\tau_g(\mathcal{H})}{\tau_g^*(\mathcal{H})}$$

due to the obvious fact $\tau \le \tau_g$ and the inequality $\tau_g^* < 2\tau^*$ from (i). For $\tau_g^{*\prime}(\mathcal{H})$ the proof is similar, by the second part of (i). □

Proposition 3.2. *The upper bounds in* Proposition 3.1(i) *are tight apart from an additive constant at most* 2.

Proof. Consider the complete bipartite graph $G = K_{k,k^2}$ on $k + k^2$ vertices as a 2-uniform hypergraph. Clearly, $\tau^*(G) = k$. In any submove of a fractional transversal game, while G is not fully covered, Staller can always select a vertex from the bigger partite class. Following this strategy, during $k-1$ moves, Staller increases the sum of the loads by at most $(k-1)k$. As G has maximum degree k^2, $k-1$ moves of Edge-hitter increase the loads by at most $(k-1)k^2$. Hence, no matter whether Edge-hitter or Staller starts the game, after $2k-2$ moves we have

$$\sum_{E \in \mathcal{E}} \ell_{2k-2}(E) \le (k-1)k + (k-1)k^2 = k^3 - k < |E|,$$

therefore the game is not over yet. This shows $\tau_g^*(G) > 2k - 2 = 2\tau^*(G) - 2$ and, similarly, $\tau_g^{*\prime}(\mathcal{H}) \ge 2\tau^*(G) - 1$ follows if Staller starts the game. □

A monotone property of the game fractional transversal number is expressed in the following idea, which provides a useful tool in simplifying several arguments. Let a hypergraph $\mathcal{H}$ with a pre-defined load function ℓ be given, which we consider as a non-zero starting configuration. We ask about the value $|\mathcal{G}|$ of the game started by Edge-hitter, where the game is finished when ℓ is completed to a load function under which $\mathcal{H}$ is fully covered. The rules are the same as they were in the case of $\ell_0 \equiv 0$, but here we have $\ell_0 = \ell$, while the value of the game is still computed by starting with the cover function $t_0 \equiv 0$. Under these conditions and assuming that Edge-hitter starts the fractional transversal game $\mathcal{G}$ on hypergraph $\mathcal{H}$ with the pre-defined ℓ, we consider the sets

$$U_{\mathcal{H}|\ell} = \{a \in \mathbb{R} : \text{Edge-hitter has a strategy that ensures } |\mathcal{G}| \le a\},$$
$$L_{\mathcal{H}|\ell} = \{b \in \mathbb{R} : \text{Staller has a strategy that ensures } |\mathcal{G}| \ge b\}.$$

Then the *game fractional transversal number with predefined load function* ℓ is defined as $\tau_g^*(\mathcal{H}|\ell) = \inf(U_{\mathcal{H}|\ell})$. It can be shown with a proof analogous to that of Proposition 2.1 that $\inf(U_{\mathcal{H}|\ell}) = \sup(L_{\mathcal{H}|\ell})$ always holds.

The corresponding value $\tau_g^{*\prime}(\mathcal{H}|\ell)$ is defined analogously for the Staller-start game on $\mathcal{H}|\ell$.

The imagination strategy is a useful technique applied in many proofs when domination and transversal games are considered (see e.g. [3, 6, 9, 15, 17, 20]). It was introduced and first used in [7]; for a detailed explanation and examples we refer the reader to [5, Chapter 2.2].

Theorem 3.3 (Continuation Principle). *If ℓ and ℓ' are load functions on the hypergraph $\mathcal{H} = (V, \mathcal{E})$ such that $\ell(E) \leq \ell'(E)$ holds for every $E \in \mathcal{E}$, then $\tau_g^*(\mathcal{H}|\ell) \geq \tau_g^*(\mathcal{H}|\ell')$, and similarly $\tau_g^{*\prime}(\mathcal{H}|\ell) \geq \tau_g^{*\prime}(\mathcal{H}|\ell')$.*

Proof. Assume for a contradiction that $\tau_g^*(\mathcal{H}|\ell) < \tau_g^*(\mathcal{H}|\ell')$, and choose two reals t_1, t_2 with $\tau^*(\mathcal{H}|\ell) < t_1 < t_2 < \tau_g^*(\mathcal{H}|\ell')$. We use the imagination strategy between the following two games:

Game 1: Edge-hitter plays on $\mathcal{H}|\ell$ applying a strategy which ensures that the value of the game is at most t_1. Staller's moves in Game 1 are defined according to her moves in Game 2.

Game 2: Staller plays on $\mathcal{H}|\ell'$ applying a strategy which ensures that the value of the game is at least t_2. Edge-hitter's moves in Game 2 are defined according to his moves in Game 1.

Assume that Edge-hitter starts the game. He chooses his first move in Game 1 according to the prescribed strategy. We copy this move into Game 2 if it is a legal move there, or choose an appropriate replacement move for Edge-hitter in Game 2. In the next turn, Staller replies with a move according to her prescribed strategy in Game 2 and we copy the same move into Game 1. (We will see that it is always legal.) Then, Edge-hitter replies in Game 1 and we copy it or make the according move for Edge-hitter in Game 2. The two parallel games continue this way until at least one of them is finished.

The moves essentially are copied (or interpreted) between Game 1 and Game 2 such that $\ell(E) \leq \ell'(E)$ remains true for all $E \in \mathcal{E}$ after every move. If this inequality is valid before Staller's move in Game 2, then her next move in $\mathcal{H}|\ell'$ is legal in $\mathcal{H}|\ell$ as well, so that we can simply copy it into Game 1. The condition $\ell(E) \leq \ell'(E)$ for all $E \in \mathcal{E}$ clearly remains valid for the new load functions.

Suppose now that the inequality $\ell(E) \leq \ell'(E)$ is true for all $E \in \mathcal{E}$ before Edge-hitter's move in Game 1. If it is legal, we simply copy it into Game 2 and the inequality remains valid. In the other case one or more submoves (v_{i_k}, w_k) made in Game 1 are not legal in Game 2. We then choose the maximum w_k' such that (v_{i_k}, w_k') is a legal submove in Game 2. The remaining weight $w_k - w_k'$ can be distributed between arbitrary vertices such that the submoves are legal. Observe, however, that if this happens, all loads on the edges incident with v_{i_k} reach 1 after the submove (v_{i_k}, w_k') in Game 2. We infer that $\ell(E) \leq \ell'(E)$ remains true for all $E \in \mathcal{E}$ after Edge-hitter's move.

It follows that the loads will never become smaller in Game 2 than the corresponding ones in Game 1. Thus, the values g_1 and g_2 of Games 1 and 2 satisfy $g_1 \geq g_2$. By the strategies of the players, it is true that $t_1 \geq g_1$ and $g_2 \geq t_2$. We therefore obtain

$$t_1 \geq g_1 \geq g_2 \geq t_2 > t_1$$

and this contradiction proves $\tau_g^*(\mathcal{H}|\ell) \geq \tau_g^*(\mathcal{H}|\ell')$.

The analogous conclusion can be reached in the Staller-start game as well, literally by the same argument, deriving a contradiction from the assumption $\tau_g^{*\prime}(\mathcal{H}|\ell) < \tau_g^{*\prime}(\mathcal{H}|\ell')$. $\square$

We obtain the following immediate consequence.

Theorem 3.4. *The game fractional transversal numbers for the Staller-start and for the Edge-hitter-start games on $\mathcal{H}$ may differ by at most* 1.

Proof. Consider the Staller-start game. Whatever Staller moves first, she assigns total weight 1, and creates a situation which is at least as favorable for Edge-hitter as the all-zero load at the beginning of the original transversal game. Then, due to Theorem 3.3, Edge-hitter can ensure that the game ends using at most $\tau_g^*(\mathcal{H})$ further weight. This proves $\tau_g^{*\prime}(\mathcal{H}) \leq \tau_g^*(\mathcal{H}) + 1$.

Similarly, if Edge-hitter starts, after his first move he is in at least as favorable position as with the all-zero load at the beginning of the Staller-start game. This proves the reverse inequality $\tau_g^*(\mathcal{H}) \leq \tau_g^{*\prime}(\mathcal{H}) + 1$. $\square$

4 Infinite moves are not necessary

The definition of a legal move in the transversal game admits the option that a player splits the value 1 into an infinite number of pieces; e.g., $w_k = 2^{-k}$. It turns out, however, that each legal move on $\mathcal{H} = (V, \mathcal{E})$ is equivalent to a move which consists of at most $|V|$ submoves.

Theorem 4.1. *Every legal move in a fractional transversal game can be replaced with a legal move such that each vertex occurs in at most one submove of it and the two moves result in the same load function.*

Proof. First, consider a vertex v which occurs in two different submoves (v_{i_j}, w_j) and (v_{i_k}, w_k) of a move. That is, $v = v_{i_j} = v_{i_k}$ and we may assume $j < k$. By the condition $(*)$, there exists an edge $E \in \mathcal{E}$ such that $v \in E$ and the second submove (v_{i_k}, w_k) increases the load of E by exactly w_k. If the submove (v_{i_j}, w_j) is deleted from the sequence and the weight w_k is replaced by $w_j + w_k$ in the k^{th} submove, the submove and the whole move remain legal and result in the same load function as before. Performing this modification repeatedly we can achieve that every vertex occurs in either zero or exactly one or infinitely many submoves of the move in question. This already proves the statement if the move contains only a finite number of submoves.

Now, assume that the move is infinite. Then, the sequence of submoves can be split into two, such that the first part is finite, and in the second infinite part every vertex (which is present there) is repeated infinitely many times. Consider this infinite subsequence $S = (v_{i_s}, w_s), \dots$. By renaming the vertices of $\mathcal{H}$ if necessary, we may assume that $\{v_1, \dots, v_\ell\}$ is the set of the vertices which are present in S. We prove that the finite sequence $S' = (v_1, \sum_{j\colon i_j=1} w_j), \dots, (v_\ell, \sum_{j\colon i_j=\ell} w_j)$ is equivalent to S. Clearly, S and S' yield the same load function after the move. So, it is enough to prove that S' is legal. Assume for a contradiction that $(v_k, \sum_{j\colon i_j=k} w_j)$ is not a legal submove in S', and let k be the smallest such index. Then, after the $(k-1)^{\text{st}}$ submove of S', every edge E which contains v_k has a load $\ell(E) > 1 - \sum_{j\colon i_j=k} w_j$ and, moreover, there is a positive constant

ϵ such that $\min_{v_k \in E} \ell(E) + \sum_{j:\ i_j=k} w_j = 1+\epsilon$. Now, consider S again. There is an index $p = p(\epsilon)$ such that $\sum_{j \geq p} w_j < \epsilon$ and hence, before the p^{th} submove of S, each edge containing v_k is fully covered. As v_k occurs infinitely often in S, and also the occurrences after the p^{th} submove are legal, this is a contradiction. □

5 Edge-hitter's moves are transposable

We say that a finite move $(v_{i_1}, w_1), \dots, (v_{i_k}, w_k)$ is *transposable* if for any permutation $\beta(1), \dots, \beta(k)$ of $1, \dots, k$, the move $(v_{i_{\beta(1)}}, w_{\beta(1)}), \dots, (v_{i_{\beta(k)}}, w_{\beta(k)})$ is legal. We will show that from the point of view of Edge-hitter, we can restrict our attention to transposable moves. Note that every transposable move is legal, but not conversely.

We first give a characterization of transposable moves:

Lemma 5.1. *A move $(v_{i_1}, w_1), \dots, (v_{i_k}, w_k)$, where $w_j > 0$ for all $1 \leq j \leq k$ and no vertices are repeated, is transposable if and only if the total weight constraint $(**)$ is satisfied, and after performing the entire move, for every vertex v_{i_j},*

$$\min_{E \ni v_{i_j}} \sum_{v_i \in E} t(v_i) \leq 1. \tag{5.1}$$

Proof. A legal move must satisfy the two constraints $(*)$ and $(**)$.

The total weight constraint $(**)$ for a legal move is explicitly required in the lemma, and it is insensitive to permuting the submoves.

Let us turn to $(*)$. If $\sum_{v_i \in E} t(v_i) \leq 1$ for an edge E containing v_{i_j}, then omitting the submove (v_{i_j}, w_j) from the move we obtain $\sum_{v_i \in E \setminus \{v_{i_j}\}} t(v_i) \leq 1 - w_j$. Hence (v_{i_j}, w_j) is a legal submove no matter when it is performed during the move. This means that the move is transposable whenever the condition (5.1) is satisfied for all j.

In the other direction, assume that for some v_{i_j}, the left-hand side of (5.1) is bigger than 1. Consider a permutation in which (v_{i_j}, w_j) is the last submove. Then (v_{i_j}, w_j) is not legal because $(*)$ is violated. Consequently the move is not transposable. □

Theorem 5.2. *If a finite legal move $m = (v_{i_1}, w_1), \dots, (v_{i_k}, w_k)$ is not transposable in the fractional transversal game, then it can be replaced by a transposable (and legal) move after which no edge gets smaller load than after m.*

Proof. First, consider the legal move $m = (v_{i_1}, w_1), \dots, (v_{i_k}, w_k)$ and the move $m' = (v_{i_2}, w_2), \dots, (v_{i_k}, w_k), (v_{i_1}, w_1)$ that is obtained by the cyclic permutation $\beta = 2, \dots, k, 1$. It is clear that condition $(*)$ in the definition remains true for the first $k-1$ submoves of m' and consequently, these submoves are legal. For the last (and not necessarily legal) submove, determine w_1^* as the maximum weight which results in a legal submove with v_{i_1}. If $w_1^* \geq w_1$, then m' is legal and gives exactly the same load function as m. If $w_1^* < w_1$, then the same load function is obtained after the submove (v_{i_1}, w_1^*) as after m, because in both cases every edge incident with v_{i_1} is fully covered and the loads of the other edges are unchanged.

In the latter case, the sum of the weights is decreased by $w_1 - w_1^*$. After this change, the submove (v_{i_1}, w_1^*) will be legal in any permutation of $(v_{i_1}, w_1^*), (v_{i_2}, w_2), \dots, (v_{i_k}, w_k)$. That is, if a permutation is not legal after this replacement, this is due to another vertex. The same is true if some weights w_s are replaced by smaller weights.

We repeat this step for the modified sequence with permutation $\beta = 3, \ldots, k, 1, 2$, then with $\beta = 4, \ldots, k, 1, 2, 3$, and so on, finally with $\beta = k, 1, 2, \ldots, k-1$, keeping all modifications incrementally. Decreasing the weight of the last submove in each step if necessary, at the end a legal transposable move m^* is obtained, which yields the same load function as m and satisfies $\sum_{j=1}^{k} w_j^* \leq \sum_{j=1}^{k} w_j$. If $\Delta = \sum_{j=1}^{k} w_j - \sum_{j=1}^{k} w_j^*$ is positive, we use the weight Δ to increase the loads of some non-fully covered edges in an arbitrary way. The total absolute change of weights is 2Δ, or possibly less if the game is over. Finally, we normalize the move by eliminating any multiple occurrences of vertices, using Theorem 4.1.

The redistribution of Δ and the normalization may lead to a move that is not transposable. If so, we repeat the modification described above.

If the process terminates after a finite number of iterations, then the last version of the move is transposable by construction, and the proof is complete. Otherwise we obtain an infinite sequence $\Delta^{(1)}, \Delta^{(2)}, \ldots$ of re-distributions from the total unit weight of the move. The total load of edges increases by at least $\sum_{i \geq 1} \Delta^{(i)}$; hence this sum converges because altogether the total load is at most $|\mathcal{E}|$.

On the other hand, the weight of a vertex changes by at most $\Delta^{(i)}$ in the i^{th} iteration, hence the local changes in weight (at least one of which is negative in each iteration) in absolute value sum up to at most $2\sum_{i \geq 1} \Delta^{(i)}$, and therefore their sum also converges.

Let $m^{(i)}$ denote the move constructed in the i^{th} iteration, *before* the re-distribution of the weight $\Delta^{(i)}$. We know that this move is transposable. Let $w_v^{(i)}$ denote the weight used for vertex v in the corresponding submove of $m^{(i)}$, and set $w_v^{(i)} = 0$ if v does not appear in this move.

We have shown that the limit of these weights $w_v^* = \lim_{i \to \infty} w_v^{(i)} = 0$ exists. Let p denote the number of vertices with a positive weight in w^*. Relabeling these vertices in an arbitrary order, we define a move $m^* = (v_1, w_1^*), \ldots, (v_p, w_p^*)$ with p submoves. By continuity, the loads achieved by the moves $m^{(i)}$ converge to the corresponding loads after the move m^*.

We claim that m^* is transposable. This will complete the proof because the loads never decrease, hence under w^* no edge gets smaller load than by move m.

For showing that m^* is transposable, we use the characterization of Lemma 5.1. The inequality (5.1) follows from the fact that its left-hand side is a continuous function of the weights.

Let us finally check that the total weight constraint ($**$) is satisfied: Since $\sum_v w_v^{(i)} \leq 1$ for all i, this inequality is satisfied in the limit. If in some iteration, the current move terminates the game (i.e., each edge gets load at least 1 while the total weight in the move is at most 1), this will hold in all successive moves, since the loads never decrease, and hence it holds also for m^*, by continuity. Therefore, we need to show $\sum_v w_v^* = 1$ only when none of the moves terminate the game. In this case, the difference $1 - \sum_v w_v^{(i)}$ is bounded by Δ_i, which goes to 0, and hence $\sum_v w_v^* = 1$ in the limit. □

Remark 5.3. Based on Theorem 5.2, Edge-hitter may restrict his strategy to transposable moves. On the other hand, the result suggests that Staller is advised to perform moves, if possible, which are ‘very non-transposable’ in a sense.

6 Algorithm for computing the value of the game

We consider an equivalent version of the game, the *structured game*, which is easier to analyze.

Each *move* consists of $n + 1$ *rounds*. Each round consists of n *submoves*, which are dedicated to the vertices $v_1, \dots, v_n$ in succession. In each submove, the player whose turn it is can decide the amount w, the weight spent in the submove, by which the cover value of $t(v_i)$ is increased, subject to the usual rules: The increase must be useful, i.e. each submove must satisfy the condition ($*$), and it must be within the *budget constraint* of total weight 1 to be spent per move. It is possible to skip a submove by simply choosing w to be zero.

The first n rounds are identical, but the last round is special: In each submove, the weight is greedily chosen as the largest possible legal weight, hence not allowing any freedom in choosing w for the player in those submoves. This ensures that the whole move spends a total weight of 1 unless a cover is obtained.

There are n moves, alternating between the two players. This is enough to ensure that a cover is constructed when the game terminates. Every move consists of $n^2 + n$ submoves, and in total, the game consists of $N = n^3 + n^2$ submoves. We illustrate this for a small example with $n = 4$ vertices, where the Edge-hitter starts. The sequence of submoves is

$$\begin{array}{lllll}
H_1,H_2,H_3,H_4; & H_1,H_2,H_3,H_4; & H_1,H_2,H_3,H_4; & H_1,H_2,H_3,H_4; & G_1,G_2,G_3,G_4; \\
S_1,S_2,S_3,S_4; & S_1,S_2,S_3,S_4; & S_1,S_2,S_3,S_4; & S_1,S_2,S_3,S_4; & G_1,G_2,G_3,G_4; \\
H_1,H_2,H_3,H_4; & H_1,H_2,H_3,H_4; & H_1,H_2,H_3,H_4; & H_1,H_2,H_3,H_4; & G_1,G_2,G_3,G_4; \\
S_1,S_2,S_3,S_4; & S_1,S_2,S_3,S_4; & S_1,S_2,S_3,S_4; & S_1,S_2,S_3,S_4; & G_1,G_2,G_3,G_4.
\end{array}$$

Here H_i denotes a move of Edge-hitter for vertex i, and S_i denotes a move of Staller for vertex i. The greedy moves are denoted by G_i.

We do not stipulate as part of the rules that the whole budget of 1 unit must be spent during a move. This capacity is only an upper bound. It is still true that the whole budget is spent in each move if the game is played from the beginning. However, this arises as a *consequence* of the new setup, due to the greedy moves.

As soon as a cover is found, the rules imply that no more weight can be spent, and thus the game is effectively over.

Lemma 6.1. *The structured game has the same value as the original game.*

Proof. According to Theorem 4.1, we can assume that every vertex occurs at most once in a move. We can realize this in the structured game by selecting one vertex per round and leaving the weight at 0 otherwise. Thus, the structured game does not restrict the players' strategies, when compared to the original game. On the other hand, the structured game does not give the players more power: The greedy moves ensure that the total weight of 1 is used as long as it is possible. □

Example for the structured game. Consider an Edge-hitter-start game on C_4 with strategies of the players as described in Section 2.1. A corresponding structured game, where we follow the idea in the proof of Lemma 6.1, can be given as follows.

$$\begin{array}{lllll}
\text{Edge-hitter's first move:} & (v_1,\frac{1}{4}), & (v_2,0), & (v_3,0), & (v_4,0); \\
 & (v_1,0), & (v_2,\frac{1}{4}), & (v_3,0), & (v_4,0); \\
 & (v_1,0), & (v_2,0), & (v_3,\frac{1}{4}), & (v_4,0); \\
 & (v_1,0), & (v_2,0), & (v_3,0), & (v_4,\frac{1}{4}); \\
 & (v_1,0), & (v_2,0), & (v_3,0), & (v_4,0). \\
 & & & & \\
\text{Staller's first move:} & (v_1,0), & (v_2,0), & (v_3,\frac{1}{2}), & (v_4,0); \\
 & (v_1,0), & (v_2,0), & (v_3,0), & (v_4,\frac{1}{2}); \\
 & (v_1,0), & (v_2,0), & (v_3,0), & (v_4,0); \\
 & (v_1,0), & (v_2,0), & (v_3,0), & (v_4,0); \\
 & (v_1,0), & (v_2,0), & (v_3,0), & (v_4,0). \\
 & & & & \\
\text{Edge-hitter's second move:} & (v_1,0), & (v_2,\frac{1}{6}), & (v_3,0), & (v_4,0); \\
 & (v_1,0), & (v_2,0), & (v_3,0), & (v_4,0); \\
 & (v_1,0), & (v_2,0), & (v_3,0), & (v_4,0); \\
 & (v_1,0), & (v_2,0), & (v_3,0), & (v_4,0); \\
 & (v_1,\frac{1}{3}), & (v_2,0), & (v_3,0), & (v_4,0).
\end{array}$$

For demonstration, the last move makes use of a greedy submove.

By Theorem 5.2, it is not a disadvantage for Edge-hitter restricting himself to transposable moves. Consequently, his submoves can always be performed in the order $(v_1, w_1), \ldots, (v_k, w_k)$, and a single round $H_1, H_2, \ldots, H_n$ followed by n greedy submoves would be a sufficient model for Edge-hitter's moves. For simplicity, we have however chosen to treat the two players uniformly. Note that the greedy submoves are necessary also in case of Edge-hitter. Otherwise, for example, he might pass on the first move and transform the game to the Staller-start version, which sometimes admits a smaller game value (as in the example of C_4).

Consider the situation after the j^{th} submove, $0 \le j \le N$. Let $\vec{x} \in [0,1]^{\mathcal{E}}$ be an arbitrary load vector, and let $r \in [0,1]$ be the budget for the current move that is still available. If j is written in the form $j = k(n^2+n)+i$, i.e. $k = \lfloor \frac{j}{n^2+n} \rfloor$ and $0 \le i < n^2+n$, then $1-r$ is the total weight spent in the last i submoves.

We define

$$T_j(\vec{x}, r)$$

as the sum of weights spent during the remaining part of the game, if both players play optimally, starting from the current situation. If j is large and the entries of $\vec{x}$ are small, it may happen that a complete fractional cover is not reached, because the game necessarily ends after the n^{th} round. Nevertheless, we have chosen our definition because it makes T_j well-defined for arbitrary $\vec{x}$ and r. (The definition of $T_j(\vec{x}, r)$ is related to the game fractional transversal number with *predefined load function* used in the proof of the Continuation Principle (Theorem 3.3)[6].)

The value of the original game is $T_0(\vec{0}, 1)$.

We will derive a backward recursion for the functions T_j, and thus show that they are piecewise linear and continuous.

[6]In particular, if the j^{th} submove is the last submove in a move of Staller and ℓ is the load function corresponding to $\vec{x}$, then we would expect $T_j(\vec{x}, 1)$ to be $\tau_g^*(\mathcal{H}|\ell)$. However this does not hold in general because, as just discussed, the structured game may terminate too early.

Given j, we know the type of the j^{th} submove (H, S, or G) and the vertex v_i to which it applies. We denote the maximum permitted weight by

$$w_i^{\max}(\vec{x}, r) = \min\{r, \max\{\, 1 - x_E \mid E \ni v_i \,\}\}\,, \tag{6.1}$$

where x_E denotes the entry of $\vec{x}$ that corresponds to the edge $E \in \mathcal{E}$. For the result of increasing the cover value of v_i by w we write

$$\textit{update}_i(\vec{x}, w) = \vec{x}\,' \text{ with } x'_E = \begin{cases} x_E, & \text{if } v_i \notin E, \\ \min\{1, x_E + w\}, & \text{if } v_i \in E. \end{cases} \tag{6.2}$$

With these definitions, the recursion for a submove H_i for Edge-hitter can be written easily:

$$T_{j-1}(\vec{x}, r) = \min\{\, w + T_j(\textit{update}_i(\vec{x}, w), r - w) \mid 0 \le w \le w_i^{\max}(\vec{x}, r) \,\} \tag{6.3}$$

If the submove is for Staller (S_i), the recursion is the same as (6.3), except that $\min$ is replaced by $\max$. In the greedy submoves G_i, we always choose $w = w_i^{\max}(\vec{x}, r)$:

$$T_{j-1}(\vec{x}, r) = w_i^{\max}(\vec{x}, r) + T_j(\textit{update}_i(\vec{x}, w_i^{\max}(\vec{x}, r)), r - w_i^{\max}(\vec{x}, r)) \tag{6.4}$$

The last greedy submove G_n of each move is an exception: Since a different move is about to start, the budget r is reset to 1. Thus, when j is a multiple of $n^2 + n$, then

$$T_{j-1}(\vec{x}, r) = w_n^{\max}(\vec{x}, r) + T_j(\textit{update}_n(\vec{x}, w_n^{\max}(\vec{x}, r)), 1). \tag{6.5}$$

As the recursion anchor, we use the value after the final move, which is simply

$$T_N(\vec{x}, r) = 0. \tag{6.6}$$

Theorem 6.2. *Each function $T_j(\vec{x}, r)$ for $0 \le j \le N$ is a piecewise linear continuous function with finitely many linear pieces defined on $[0,1]^{\mathcal{E}} \times [0,1]$. Moreover, all T_j are rational in the sense that each linear piece has rational coefficients and rational constant part. As a consequence, the boundaries between regions of the domain with different linear functions can be described by linear equations with rational coefficients.*

Proof. We will call a function with all the desired properties — piecewise linear, continuous, and rational, with finitely many linear pieces — a PLCR function.

The proof proceeds by backward recursion from T_N down to T_0. The function T_N from (6.6) is obviously PLCR.

The sum, difference, maximum, or minimum of two PLCR functions is again PLCR, and the same holds true when substituting one PLCR function into another. It follows directly that the functions $w_i^{\max}$ and $\textit{update}_i$ are PLCR functions on the domain $[0,1]^{\mathcal{E}} \times [0,1]$. This allows us to perform the induction step in the recursions (6.4)–(6.5) for G_i.

In the recursion (6.3) we additionally have a minimization (or, in the analogous recursion for Staller, a maximization) over some range of values w. It has the form

$$\min\{\, F(\vec{x}, r, w) \mid 0 \le w \le w_i^{\max}(\vec{x}, r) \,\}$$

with the PLCR function

$$F(\vec{x}, r, w) := w + T_j(\textit{update}_i(\vec{x}, w), r - w)$$

To get rid of the varying upper bound on w, we rewrite the recursion in terms of another PLCR function

$$\hat{F}(\vec{x}, r, w) = F(\vec{x}, r, \min\{w, w_i^{\max}(\vec{x}, r)\}$$

as

$$T_{j-1}(\vec{x}, r) = \min\{\, \hat{F}(\vec{x}, r, w) \mid 0 \le w \le 1 \,\}.$$

Lemma 6.3 below establishes that T_{j-1} is a PLCR function.

The same argument applies to the recursion for Staller (S_i), where $\min$ is replaced by $\max$. □

Lemma 6.3. *Suppose that $\hat{F}(y, w)\colon [0,1]^m \times [0,1] \to \mathbb{R}$ is a PLCR function. Then the function $T(y)\colon [0,1]^m \to \mathbb{R}$ defined by minimizing over w:*

$$T(y) := \min\{\, \hat{F}(y, w) \mid 0 \le w \le 1 \,\} \tag{6.7}$$

is also a PLCR function.

Proof. We first show that T is continuous. Since $\hat{F}$ is PLCR, it is Lipschitz-continuous. Let L be its Lipschitz constant with respect to the ∞-norm. (We can compute L as the maximum L_1-norm of all coefficient vectors of the linear pieces of $\hat{F}$.) It follows that the function T in (6.7) is also Lipschitz-continuous with Lipschitz-constant L. To see this, let $\|y_0 - y_1\| \le \varepsilon$, and let $T(y_0) = \hat{F}(y_0, w_0)$ for some w_0. Then $T(y_1) \le \hat{F}(y_1, w_0) \le \hat{F}(y_0, w_0) + L\varepsilon = T(y_0) + L\varepsilon$. The converse bound $T(y_0) \le T(y_1) + L\varepsilon$ follows in the same way.

We still need to show that T is piecewise linear. For an intuitive way to see this, one can interpret the minimization over w geometrically. The graph of $\hat{F}\colon [0,1]^m \times [0,1] \to \mathbb{R}$ is a subset of $\mathbb{R}^{m+2}$. Taking the minimum over all w amounts to projecting away the coordinate corresponding to w and taking the lower envelope (with respect to the last coordinate) in the projection in $\mathbb{R}^{m+1}$. Figure 2 shows a two-dimensional illustration. This picture can also be interpreted as a three-dimensional view of the graph of a bivariate function $\hat{F}(y, w)$ when the viewing direction is parallel to the w-axis. (In this hypothetical example, the resulting minimum is discontinuous; this cannot happen when $\hat{F}$ is continuous and its domain is the box $[0,1]^m \times [0,1]$.)

Formally, we conduct the proof as follows. We know that the domain $[0,1]^{m+1}$ of $\hat{F}$ splits into finitely many rational convex $(m+1)$-dimensional polytopes P on which $\hat{F}$ is linear:

$$\hat{F}(y, w) = a_P y + b_P w + c_P, \qquad \text{for } (y, w) \in P$$

for some rational coefficient vector a_P and rational coefficients b_P and c_P. We can thus write $T(y)$ as the minimum of finitely many functions $T_P(y)$ of the form

$$T_P(y) := \min\{\, a_P y + b_P w + c_P \mid 0 \le w \le 1, (y, w) \in P \,\}, \tag{6.8}$$

where the minimum of an empty set is taken as ∞.

For fixed y, the minimum in (6.8) depends on the sign of b_P. If $b_P > 0$, the minimum is achieved on a boundary point that lies on some facet P' of P whose outer normal has negative w-coordinate. On such a facet, w can be expressed as a linear function of y, and thus, T_P can be written as a linear function

$$T_{P'}(y) = a_{P'} y + c_{P'}, \qquad \text{for } y \in \bar{P}', \tag{6.9}$$

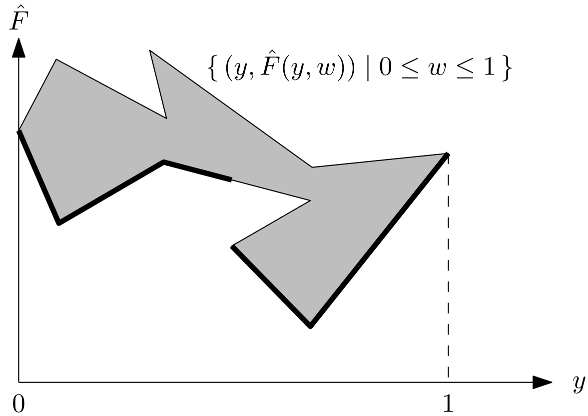

Figure 2: The lower envelope of a polyhedral set in 2 dimensions ($m = 1$).

where $\bar{P}'$ is the projection of the facet P' to $[0,1]^m$. Thus, $T_P(y)$ is the minimum of finitely many functions $T_{P'}(y)$, with the understanding that $T_{P'}(y)$ is taken as ∞ when y is outside its domain $\bar{P}'$.

The situation is similar for $b_P < 0$. When $b_P = 0$, then $\hat{F}$ does not depend on w and we can simply write

$$T_P(y) = a_P y + c_P, \quad \text{for } y \in \bar{P}, \tag{6.10}$$

where $\bar{P}$ is the projection of P.

In summary, the function $T(y)$ can be written as the minimum of finitely many pieces $T_P(y)$, each of which can in turn be written as the minimum of finitely many *linear* pieces (6.9) or (6.10). All these pieces have rational coefficients and rational domain boundaries, and since continuity of T has already been established, the PLCR property of T follows. □

The proof of Theorem 6.2 is constructive and, in principle, it provides an algorithm for computing the value $T_0(\vec{0}, 1)$ of the game. From this, we obtain the following important corollary.

Theorem 6.4. *For every finite hypergraph $\mathcal{H} = (V, \mathcal{E})$, the game fractional transversal number $\tau_g^*(\mathcal{H})$ and its Staller-start version $\tau_g^{*\prime}(\mathcal{H})$ are rational. Moreover, each player in every step has an optimal move with only rational weights, provided that the weights in all previous submoves were rational.*

Remark 6.5. It is not true in general that every optimal strategy uses only rational weights. A simple counterexample is the graph C_4 (Example 1 from Section 2.1), where Staller can start by placing x and $1 - x$ on two vertices with any $x \in [0, 1]$, no matter if x is rational or irrational.

7 Concluding remarks and open problems

Putting the fractional domination game [15] into a more general context, in this paper we introduced the fractional transversal game on hypergraphs. Among other results, we proved that the game value is rational, and both players have optimal strategies using rational weights and with a finite number of submoves. Since a dominating set of a graph is a transversal of the closed neighborhood hypergraph, and a total dominating set is a transversal of the open neighborhood hypergraph, the following consequence is immediate.

Theorem 7.1. *The fractional versions of both the domination game and the total domination game have rational game values (game fractional domination number and game fractional total domination number) on every graph.*

We conclude this paper with some conjectures and open questions.

Conjecture 7.2. *If each of the first $2k-1$ $(k \geq 1)$ moves was an integer move in the fractional transversal game, i.e. of the form $(v_{i_1}, 1)$, then Staller has an integer move in the $(2k)^{\mathrm{th}}$ turn, which is optimal in the fractional transversal game.*

This means that fractional moves would be advantageous for Edge-hitter only. If true then this conjecture implies the following weaker one.

Conjecture 7.3. *For every hypergraph H, $\tau_g^*(H) \leq \tau_g(H)$.*

Perhaps the following stronger version of Conjecture 7.2 is also true.

Conjecture 7.4. *Starting from any cover function, there is an optimal strategy for Staller where, in every submove, she always spends the largest legal weight.*

These conjectures could be approached by implementing the algorithm that is implicit in the proof of Theorem 6.2 by computer. We have not derived an estimate for the complexity (number of pieces) of the piecewise linear continuous functions $T_j(\vec{x}, r)$ that are involved in the construction. If the growth of the complexity is not too steep, there is hope to solve some examples of moderate size, beyond the range of small examples that we considered in Section 2.1, and this could shed some light on the conjectures.

One would naturally expect that $T_0(\vec{x}, r)$ is monotonically decreasing in $\vec{x}$, for fixed r, and moreover, that it is Lipschitz-continuous with Lipschitz constant 1. In other words, in every linear piece, the coefficient of each variable x_i is between 0 and -1. We have not explored these properties.

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The Sierpiński domination number*

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Abstract

Let G and H be graphs and let $f\colon V(G) \to V(H)$ be a function. The Sierpiński product of G and H with respect to f, denoted by $G \otimes_f H$, is defined as the graph on the vertex set $V(G) \times V(H)$, consisting of $|V(G)|$ copies of H; for every edge gg' of G there is an edge between copies gH and $g'H$ of H associated with the vertices g and g' of G, respectively, of the form $(g, f(g'))(g', f(g))$. In this paper, we define the Sierpiński domination number as the minimum of $\gamma(G \otimes_f H)$ over all functions $f\colon V(G) \to V(H)$. The upper Sierpiński domination number is defined analogously as the corresponding maximum.

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After establishing general upper and lower bounds, we determine the upper Sierpiński domination number of the Sierpiński product of two cycles, and determine the lower Sierpiński domination number of the Sierpiński product of two cycles in half of the cases and in the other half cases restrict it to two values.

Keywords: Sierpiński graph, Sierpiński product, domination number, Sierpiński domination number.

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1 Introduction

Sierpiński graphs represent a very interesting and widely studied family of graphs. They were introduced in 1997 in the paper [15], where the primary motivation for their introduction was the intrinsic link to the Tower of Hanoi problem, for the latter problem see the book [11]. Intensive research of Sierpiński graphs led to a review article [12] in which state of the art up to 2017 is summarized and unified approach to Sierpiński-type graph families is also proposed. Later research on Sierpiński graphs includes [2, 3, 6, 19, 23].

Sierpiński graphs have a fractal structure, the basic graphs of which are complete graphs. In 2011, Gravier, Kovše, and Parreau [7] introduced a generalization in such a way that any graph can act as a fundamental graph, and called the resulting graphs generalized Sierpiński graphs. We refer to the papers [1, 4, 5, 13, 14, 16, 17, 20, 21, 22, 24] for investigations of generalized Sierpiński graphs in the last few years.

An interesting generalization of Sierpiński graphs in the other direction has recently been proposed by Kovič, Pisanski, Zemljič, and Žitnik in [18]. Namely, in the spirit of classical graph products, where the vertex set of a product graph is the Cartesian product of the vertex sets of the factors, they introduced the Sierpiński product of graphs as follows. Let G and H be graphs and let $f\colon V(G) \to V(H)$ be an arbitrary function. The *Sierpiński product of graphs G and H with respect to f*, denoted by $G \otimes_f H$, is defined as the graph on the vertex set $V(G) \times V(H)$ with edges of two types:

- *type*-1 *edge*: $(g, h)(g, h')$ is an edge of $G \otimes_f H$ for every vertex $g \in V(G)$ and every edge $hh' \in E(H)$,
- *type*-2 *edge*: $(g, f(g'))(g', f(g))$ is an edge of $G \otimes_f H$ for every edge $gg' \in E(G)$.

We observe that the edges of type-1 induce $n(G) = |V(G)|$ copies of the graph H in the Sierpiński product $G \otimes_f H$. For each vertex $g \in V(G)$, we let gH be the copy of H corresponding to the vertex g. A type-2 edge joins vertices from different copies of H in $G \otimes_f H$, and is called a *connecting edges* of $G \otimes_f H$. A vertex incident with a connecting edge is called a *connecting vertex*. We observe that two different copies of H in $G \otimes_f H$ are joined by at most one edge. A copy of the graph H corresponding to a vertex of the graph G in the Sierpiński product $G \otimes_f H$ is called an *H-layer*.

Let G and H be graphs and H^G be the family of functions from $V(G)$ to $V(H)$. We introduce new types of domination, the *Sierpiński domination number*, denoted by $\gamma_S(G, H)$, as the minimum over all functions f from H^G of the domination number of the Sierpiński product with respect to f, and *upper Sierpiński domination number*, denoted

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by $\Gamma_S(G,H)$, as the maximum over all functions $f \in H^G$ of domination number of the Sierpiński product with respect to f. That is,

$$\gamma_{\rm S}(G,H) \coloneqq \min_{f\in H^G}\{\gamma(G\otimes_f H)\}$$

and

$$\Gamma_{\rm S}(G,H) \coloneqq \max_{f\in H^G}\{\gamma(G\otimes_f H)\}\,.$$

In this paper, we initiate the study of Sierpiński domination in graphs. In Section 1.1 we present the graph theory notation and terminology we follow. In Section 2 we discuss general lower and upper bounds on the (upper) Sierpiński domination number. Our main contribution in this introductory paper is to determine the upper Sierpiński domination number of the Sierpiński product of two cycles, and to determine the lower Sierpiński domination number of the Sierpiński product of two cycles in half of the cases and in the other half cases restrict it to two values.

1.1 Notation and terminology

We generally follow the graph theory notation and terminology in the books [8, 9, 10] on domination in graphs. Specifically, let G be a graph with vertex set $V(G)$ and edge set $E(G)$, and of order $n(G) = |V(G)|$ and size $m(G) = |E(G)|$. For a subset S of vertices of a graph G, we denote by $G - S$ the graph obtained from G by deleting the vertices in S and all edges incident with vertices in S. If $S = \{v\}$, then we simply write $G - v$ rather than $G - \{v\}$. The subgraph induced by the set S is denoted by $G[S]$. We denote the path, cycle and complete graph on n vertices by P_n, C_n, and K_n, respectively. For $k \ge 1$ an integer, we use the notation $[k] = \{1,\dots,k\}$ and $[k]_0 = \{0,1,\dots,k\}$. We generally label vertices of the considered graphs by elements of $[n]$. In this case, the mod function over the set $[n]$ is to be understood in a natural way, more formally, we apply the following operation for $t \ge 1$: $t \bmod^* n = (t-1) \bmod n + 1$.

A vertex *dominates* itself and its neighbors, where two vertices are neighbors in a graph if they are adjacent. A *dominating set* of a graph G is a set S of vertices of G such that every vertex in G is dominated by a vertex in S. The *domination number*, $\gamma(G)$, of G is the minimum cardinality of a dominating set of G. A dominating set of cardinality $\gamma(G)$ is called a *γ-set of G*. A thorough treatise on dominating sets can be found in [8, 9].

If S is a set of vertices in a graph G, then we will use the notation $G|S$ to denote that the vertices in the set S are assumed to be dominated and hence $\gamma(G|S)$ is the minimum number of vertices in the graph G needed to dominate $V(G) \setminus S$. We note that it could be that a vertex in S is still a member of a such a minimum dominating set no matter that we do not need to dominate the vertices in S themselves. If $S = \{x\}$, then we simply denote $G|S$ by $G|x$ rather than $G|\{x\}$.

2 General lower and upper bounds

We present in this section general lower and upper bounds on the (upper) Sierpiński domination number.

Theorem 2.1. *If G and H are graphs, then*

$$n(G)\gamma(H) - m(G) \le \gamma_{\rm S}(G,H) \le \Gamma_{\rm S}(G,H) \le n(G)\gamma(H)\,.$$

Proof. Let $G \otimes_f H$ be an arbitrary Sierpiński product of graphs G and H and let X be a γ-set of $G \otimes_f H$. Assuming for a moment that all the connecting edges are removed from $G \otimes_f H$, we obtain $n(G)$ disjoint copies of H for which we clearly need $n(G)\gamma(H)$ vertices in a minimum dominating set. Consider now an arbitrary connecting edge $e = (g, f(g'))(g', f(g))$ of $G \otimes_f H$. If no end-vertex of e lies in X, then clearly $\gamma(G \otimes_f H - e) = \gamma(G \otimes_f H)$. Similarly, if both end-vertices of e lie in X, then $\gamma(G \otimes_f H - e) = \gamma(G \otimes_f H)$. Hence the only situation in which e has an effect on $\gamma(G \otimes_f H)$ is when $(g, f(g')) \in X$ and $(g', f(g)) \notin X$ (or the other way around). But in this case, the effect of the presence of the edge e is that because $(g, f(g'))$ dominates one vertex of $g'H$, the edge e might reduce the domination number by 1. That is, each connecting edge can drop the domination number of $G \otimes_f H$ by at most 1, which proves the left inequality. The other two inequalities are clear. □

To show that the lower bound of Theorem 2.1 is achieved, we show later in Theorem 3.10 that for $n \geq 3$ and $k \geq 1$, if we take $G = C_n$ and $H = C_{3k+1}$ where $n \equiv 0 \pmod 4$, then $\gamma_{\rm S}(G, H) = kn = n(G)\gamma(H) - m(G)$. The upper bound of Theorem 2.1 is obtained, for example, for the Sierpiński product of two complete graphs. More generally, to achieve equality in the upper bound of Theorem 2.1 we require the graph H to have the following property.

Theorem 2.2. *The equality in $\Gamma_{\rm S}(G, H) \leq n(G)\gamma(H)$ is achieved if and only if there exists a vertex $x \in V(H)$ such that $\gamma(H|x) = \gamma(H)$.*

Proof. Suppose that H has a vertex x that satisfies $\gamma(H|x) = \gamma(H)$. In this case, we consider the Sierpiński product $G \otimes_f H$ with the function $f\colon V(G) \to V(H)$ defined by $f(v) = x$ for every vertex $v \in V(G)$. Consequently each connecting edge in the product is of the form $(g, x)(g', x)$. Thus, if X is a γ-set of $G \otimes_f H$, then $|X \cap V(gH)| = \gamma(H)$ because the only vertex of gH that can be dominated from outside gH is (g, x), but we have assume that $\gamma(H|x) = \gamma(H)$. Therefore, $\Gamma_S(G, H) = n(G)\gamma(H)$.

For the other implication suppose that $\gamma(H|x) < \gamma(H)$ for every vertex $x \in V(H)$. For an arbitrary edge $g_1g_2 \in E(G)$, if D corresponds to a γ-set of the product $G \otimes_f H$, then $|D \cap V((G \otimes_f H)[V(g_1H) \cup V(g_2H)])| \leq 2\gamma(H) - 1$. Consequently, $\Gamma_S(G, H) < n(G)\gamma(H)$. □

To conclude this section we describe large classes of graphs for which the second and the third inequality of Theorem 2.1 are both equality.

Proposition 2.3. *If G and H are graphs such that $\Delta(G) < n(H)$ and $\gamma(H) = 1$, then $\Gamma_{\rm S}(G, H) = \gamma_{\rm S}(G, H) = n(G)$.*

Proof. Let G and H be graphs such that $\Delta(G) < n(H)$ and $\gamma(H) = 1$. Thus, by Theorem 2.1, $\gamma_{\rm S}(G, H) \leq \Gamma_{\rm S}(G, H) \leq n(G)$ and it is straightforward that the Sierpiński product of graphs G and H can be dominated by taking one dominating vertex from each H-layer to the dominating set. It remains to show that the inequality $\gamma_{\rm S}(G, H) \geq n(G)$ also holds. Suppose that $\gamma_{\rm S}(G, H) \leq n(G) - 1$. Let D be a dominating set of $G \otimes_f H$, where f is such that it minimizes the domination number. Therefore there is an H-layer, denote it by H', of $G \otimes_f H$ such that $D \cap V(H') = \emptyset$. Since there are at most $\Delta(G)$ connecting edges incident with vertices from each H-layer and $\Delta(G) < n(H)$,then all the vertices from an H-layer cannot be dominated by the vertices from the neighboring layers. Therefore we have $D \cap V(H'') \neq \emptyset$ for each H''-layer of $G \otimes_f H$. Thus $\gamma_{\rm S}(G, H) \geq n(G)$ and the result follows. □

3 The Sierpiński domination number of cycles

Let us recall firstly the domination number of a path and a cycle.

Fact 3.1. *For $n \geq 3$, $\gamma(P_n) = \gamma(C_n) = \lceil \frac{n}{3} \rceil$.*

In this section, we shall prove the following results.

Theorem 3.2. *For $n \geq 3$, $k \geq 1$, and $p \in [2]_0$,*

$$\gamma_{\mathrm{S}}(C_n, C_{3k+p}) \in \begin{cases} \{kn\}; & p = 0, \\ \{kn, kn+1\}; & p = 1, \\ \{kn + \left\lfloor \frac{n}{2} \right\rfloor, kn + \left\lfloor \frac{n}{2} \right\rfloor + 1\}; & p = 2. \end{cases}$$

Moreover, if $n \equiv 0 \mod 4$, then $\gamma_{\mathrm{S}}(C_n, C_{3k+1}) = kn$ and $\gamma_{\mathrm{S}}(C_n, C_{3k+2}) = kn + \left\lfloor \frac{n}{2} \right\rfloor$.

Theorem 3.3. *For $n \geq 3$, $k \geq 1$, and $p \in [2]_0$,*

$$\Gamma_{\mathrm{S}}(C_n, C_{3k+p}) = \begin{cases} kn; & p = 0, \\ kn + \left\lceil \frac{n}{3} \right\rceil; & p = 1, \\ (k+1)n; & p = 2. \end{cases}$$

In order to prove Theorems 3.2 and 3.3, we consider three cases, depending on the value of p.

3.1 The cycle C_n and cycles C_{3k+1}

To determine $\Gamma_S(C_n, C_{3k+1})$, we prove a slightly more general result. For this purpose, we define a class of graphs $\mathcal{H}_k$ as follows.

Definition 3.4. For $k \geq 1$, let $\mathcal{H}_k$ be the class of all graphs H that have the following properties.

(a) $\gamma(H) = k + 1$ and $\gamma(H - v) = k$ for every vertex $v \in V(H)$.

(b) If $x, y \in V(H)$, then there exists a γ-set of H that contains x and y, where $x = y$ is allowed.

We show, for example, that for every $k \geq 1$, the cycle C_{3k+1} belongs to the class $\mathcal{H}_k$.

Proposition 3.5. *For $k \geq 1$, the class $\mathcal{H}_k$ of graphs contains the cycle C_{3k+1}.*

Proof. For $k \geq 1$, let $H \cong C_{3k+1}$. Since $\gamma(C_n) = \gamma(P_n) = \lceil n/3 \rceil$, property (a) in Definition 3.4 holds. To prove that property (b) in Definition 3.4 holds, let $x, y \in V(H)$. Since H is vertex-transitive, every specified vertex belongs to some γ-set of H. In particular, if $x = y$, then property (b) is immediate. Hence, we may assume that $x \neq y$. Let H be the cycle $v_1 v_2 \ldots v_{3k+1} v_1$, where renaming vertices if necessary, we may assume that $x = v_1$. Let $y = v_i$, and so $i \in [3k+1] \setminus \{1\}$.

Let $H' = H - N[\{x, y\}]$, that is, H' is obtained from H by removing x and y, and removing all neighbors of x and y. If H' is connected, then H' is a path $P_{3(k-2)+j}$ for some j where $j \in [3]$. In this case, $\gamma(H') = k - 1$. If H' is disconnected, then H' is the disjoint union of two paths P_{k_1} and P_{k_2}, where $k_1 + k_2 = 3(k-2)+1$. Thus renaming k_1 and k_2 if necessary, we may assume that either $k_1 = 3j_1$ and $k_2 = 3j_2 + 1$ where $j_1 \geq 1$, $j_2 \geq 0$, and $j_1 + j_2 = k - 2$ or $k_1 = 3j_1 + 2$ and $k_2 = 3j_2 + 2$ where $j_1, j_2 \geq 0$ and $j_1 + j_2 = k - 3$. In both cases, $\gamma(H') = \lceil k_1/3 \rceil + \lceil k_2/3 \rceil = k - 1$. Letting D' be a γ-set of H', the set $D = D' \cup \{x, y\}$ is a dominating set of H of cardinality $k + 1 = \gamma(H)$, implying that D is a γ-set of H that contains both x and y. Hence, property (b) holds. □

For $n \geq 3$ an integer, a *circulant graph* $C_n\langle L\rangle$ with a given list $L \subseteq \{1, \ldots, \lfloor \frac{1}{2}n \rfloor\}$ is a graph on n vertices in which the ith vertex is adjacent to the $(i+j)$th and $(i-j)$th vertices for each j in the list L and where addition is taken modulo n. For example, for $n = 3k+1$ where $k \geq 1$ and $L = \{1\}$, the circulant graph $C_n\langle L\rangle$ is the cycle C_{3k+1}, which, by Proposition 3.5, belongs to the class $\mathcal{H}_k$. More generally, for $n = k(2p+1)+1$ where $k \geq 1$, $p \geq 1$, and $L = [p]$, the circulant graph $C_n\langle L\rangle$ belongs to the class $\mathcal{H}_k$. We omit the relatively straightforward proof. These examples of circulant graphs serve to illustrate that for each $k \geq 1$, one can construct infinitely many graphs in the class $\mathcal{H}_k$. We determine next the upper Sierpiński domination number $\Gamma_S(C_n, H)$ of a cycle C_n and a graph H in the family $\mathcal{H}_k$.

Theorem 3.6. *For $n \geq 3$ and $k \geq 1$, if $H \in \mathcal{H}_k$, then*

$$\Gamma_{\mathrm{S}}(C_n, H) = kn + \left\lceil \frac{n}{3} \right\rceil.$$

Proof. For $n \geq 3$ and $k \geq 1$, let $G \cong C_n$ and let $H \in \mathcal{H}_k$. Let G be the cycle given by $g_1 g_2 \ldots g_n g_1$. In what follows, we adopt the following notation. For each $i \in [n]$, we denote the copy $g_i H$ of H corresponding to the vertex g_i simply by H_i. We proceed further with two claims. The first claim establishes a lower bound on $\Gamma_{\mathrm{S}}(C_n, H)$, and the second claim establishes an upper bound on $\Gamma_{\mathrm{S}}(C_n, H)$.

Claim 3.7. $\Gamma_{\mathrm{S}}(C_n, H) \geq kn + \left\lceil \frac{n}{3} \right\rceil$.

Proof. Let $f\colon V(G) \to V(H)$ be a constant function, that is, we select $h \in V(H)$ and for every vertex $g \in V(G)$, we set $f(g) = h$. Let D_G be a γ-set of G. Thus, $|D_G| = \gamma(C_n) = \lceil n/3 \rceil$. By property (a) in Definition 3.4, for every vertex $g \in V(G)$, there exists a γ-set of gH that contains the vertex $(g, f(g)) = (g, h)$. If $g \in D_G$, let D_g be a γ-set of gH that contains the vertex $(g, f(g)) = (g, h)$, and so $|D_g| = \gamma(H) = k + 1$. If $g \in V(G) \setminus D_G$, let D_g be a γ-set of $gH - (g, f(g)) = gH - (g, h)$, and so in this case $|D_g| = \gamma(H - h) = \gamma(H) - 1 = k$. Let

$$D = \bigcup_{g \in V(G)} D_g.$$

The set D is a dominating set of $G \otimes_f H$, and so

$$\gamma(G \otimes_f H) \leq |D| = \gamma(G)(k+1) + (n - \gamma(G))k = kn + \gamma(G) = kn + \left\lceil \frac{n}{3} \right\rceil. \quad (3.1)$$

For the fixed vertex h chosen earlier, we note that the set of vertices (g, h) for all $g \in V(G)$ induces a subgraph of $G \otimes_f H$ that is isomorphic to $G \cong C_n$. We denote this copy of G by Gh. Among all γ-sets of $G \otimes_f H$, let D^* be chosen to contain as many vertices of Gh as possible. Let $D^*_g = D^* \cap V(gH)$ for every $g \in V(G)$. Further let $D^*_G = \{(g,h) \in D^* \colon g \in V(G)\}$, that is, D^*_G is the restriction of D^* to the copy of G. If a vertex $(g,h) \notin D^*_G$ and (g,h) is not dominated by D^*_G, then D^*_g is a γ-set of gH by the minimality of the set D^*. However in this case, we could replace the set D^*_g be a γ-set of gH that contains the vertex (g,h) to produce a new γ-set of $G \otimes_f H$ that contains more vertices from the copy of G than does D^*, a contradiction. Hence, the set D^*_G is a dominating set in the copy of G, and so $|D^*_g(G)| \geq \gamma(G)$. By the minimality of the set D^* and by property (a) in Definition 3.4, for each vertex $g \in V(G)$, we have $|D^*_g| = \gamma(H) = k+1$ if the vertex $(g,h) \in D^*_G$ and $|D^*_g| = \gamma(H-h) = k$ if the vertex $(g,h) \notin D^*_G$. Therefore,

$$\begin{aligned}\gamma(G \otimes_f H) &= |D^*| = |D^*_G|(k+1) + (n - |D^*_G|)k = kn + |D^*_G| \\ &\geq kn + \gamma(G) = kn + \left\lceil \frac{n}{3} \right\rceil .\end{aligned} \tag{3.2}$$

By inequalities (3.1) and (3.2), we have

$$\gamma(G \otimes_f H) = kn + \left\lceil \frac{n}{3} \right\rceil . \tag{3.3}$$

By equation (3.3), we have $\Gamma_{\rm S}(C_n, H) \geq \gamma(G \otimes_f H) = kn + \lceil n/3 \rceil$. This completes the proof of Claim 3.7. (□)

Claim 3.8. $\Gamma_{\rm S}(C_n, H) \leq kn + \left\lceil \dfrac{n}{3} \right\rceil$.

Proof. Let $f \colon V(G) \to V(H)$ be an arbitrary function. Let H_i be the ith copy of H corresponding to the vertex g_i of G for all $i \in [n]$. Let D be the dominating set of $G \otimes_f H$ constructed as follows. Let $x_i y_{i+1}$ be the connecting edge from H_i to H_{i+1} for all $i \in [n]$, where addition is taken modulo n. Thus, the vertex $x_i \in V(H_i)$ is adjacent to the vertex $y_{i+1} \in V(H_{i+1})$ in the graph $G \otimes_f H$, that is, $x_i = (g_i, f(g_{i+1}))$ and $y_{i+1} = (g_{i+1}, f(g_i))$. We note that possibly $x_i = y_i$. By property (b) in Definition 3.4, there exists a γ-set of H_i that contains both x_i and y_i. For $i \in [n]$, we define the sets $D_{i,1}$, $D_{i,2}$, and $D_{i,3}$ as follows. Let $D_{i,1}$ be a γ-set of $H_i - x_i$. Let $D_{i,2}$ be a γ-set of H_i that contains both x_i and y_i. Let $D_{i,3}$ be a γ-set of $H_i - y_i$. We note that $|D_{i,1}| = |D_{i,3}| = k$ and $|D_{i,2}| = k+1$. For $i \in [n]$, we define the set D_i as follows.

$$D_i = \begin{cases} D_{i,1}; & i \equiv 1 \pmod 3 \text{ and } i \neq n, \\ D_{i,2}; & i \equiv 2 \pmod 3 \text{ or } i \equiv 1 \pmod 3 \text{ and } i = n, \\ D_{i,3}; & i \equiv 0 \pmod 3). \end{cases}$$

For example, the set D_1 dominates all vertices of $H_1 - x_1$. The set D_2 contains the vertex y_2, which is adjacent to the vertex x_1 of H_1, and contains the vertex x_2, which is adjacent to the vertex y_3 of H_3, implying that D_2 dominates the vertex x_1 of H_1, all vertices of H_2, and the vertex y_3 of H_3. The set D_3 dominates all vertices of $H_3 - y_3$. Thus, $D_1 \cup D_2 \cup D_3$ dominates all vertices in $V(H_1) \cup V(H_2) \cup V(H_3)$ in the Sierpiński product $G \otimes_f H$. Moreover, $|D_1| + |D_2| + |D_3| = k + (k+1) + k = 3k+1$. More generally, the set $D_{3j-2} \cup D_{3j-1} \cup D_{3j}$ dominates all vertices in $V(H_{3j-2}) \cup V(H_{3j-1}) \cup V(H_{3j})$ in the Sierpiński product $G \otimes_f H$ for all $j \in \{1, \dots, \lfloor n/3 \rfloor\}$. Moreover, $|D_{3j-2}| + |D_{3j-1}| + |D_{3j}| = k + (k+1) + k = 3k+1$. If $n \equiv 1 \pmod 3$, then the set D_n is a γ-set of H_n, and in this case $|D_n| = k+1$. If $n \equiv 2 \pmod 3$, then the set $D_{n-1} \cup D_n$ dominates all vertices in $V(H_{n-1}) \cup V(H_n)$, and in this case $|D_{n-1}| + |D_n| = k + (k+1) = 2k+1$. The set

$$D = \bigcup_{i=1}^{n} D_i$$

is therefore a dominating set of $G \otimes_f H$, implying that

$$\gamma(G \otimes_f H) \le |D| = \sum_{i=1}^{n} |D_i| = kn + \left\lceil \frac{n}{3} \right\rceil.$$

This completes the proof of Claim 3.8. (□)

The proof of Theorem 3.6 follows as an immediate consequence of Claims 3.7 and 3.8. □

As a consequence of Proposition 3.5, we have the following special case of Theorem 3.6.

Corollary 3.9. *For $n \ge 3$ and $k \ge 1$,*

$$\Gamma_{\rm S}(C_n, C_{3k+1}) = kn + \left\lceil \frac{n}{3} \right\rceil.$$

We consider next the Sierpiński domination number of C_n and C_{3k+1}, and show that $\gamma_{\rm S}(C_n, C_{3k+1}) = kn$ if $n \equiv 0 \pmod 4$ and $\gamma_{\rm S}(C_n, C_{3k+1}) \in \{kn, kn+1\}$, otherwise.

Theorem 3.10. *For $n \ge 3$ and $k \ge 1$,*

$$\gamma_{\rm S}(C_n, C_{3k+1}) \in \{kn, kn+1\}.$$

Moreover, if $n \equiv 0 \mod 4$, then $\gamma_{\rm S}(C_n, C_{3k+1}) = kn$.

Proof. For $n \ge 3$ and $k \ge 1$, let $G = C_n$ and let $H = C_{3k+1}$. Let G be the cycle given by $g_1 g_2 \dots g_n g_1$. We adopt our notation employed in our earlier proofs. For notational convenience, we let $V(H) = \{1, 2, \dots, 3k+1\}$ where vertices i and $i+1$ are consecutive on the cycle H for all $i \in [3k+1]$ (and where addition is taken modulo $3k+1$, and so vertex 1 and vertex $3k+1$ are adjacent).

As before, we denote the copy $g_i H$ of H corresponding to the vertex g_i simply by H_i for each $i \in [n]$. Thus, $H_i = C_{3k+1}$ is the cycle $(g_i, 1), (g_i, 2), \dots, (g_i, 3k+1), (g_i, 1)$ for all $i \in [n]$. Recall that we denote the connecting edge from H_i to H_{i+1} by $x_i y_{i+1}$ for all $i \in [n]$, where $x_i \in V(H_i)$, $y_{i+1} \in V(H_{i+1})$, and addition is taken modulo n. Thus, $y_i = (g_i, f(g_{i-1}))$ and $x_i = (g_i, f(g_{i+1}))$ for all $i \in [n]$.

By Proposition 3.5, the graph H belongs to the class $\mathcal{H}_k$. Thus, $\gamma(H) = k+1$ and $\gamma(H - v) = k$ for every vertex $v \in V(H)$. Furthermore, if $x, y \in V(H)$ where $x = y$ is allowed, then there exists a γ-set of H that contains x and y.

By the elementary lower bound on the Sierpiński domination number given in Theorem 2.1, $\gamma_{\rm S}(G, H) \ge n(G)\gamma(H) - m(G) = kn$, noting that here $n(G) = m(G) = n$ and $\gamma(H) = k+1$. It follows that $\gamma_{\rm S}(C_n, H) \ge kn$.

To complete the proof we are going to prove that

$$\gamma_{\mathrm{S}}(C_n, H) \le kn + \left\lceil \frac{n}{4} \right\rceil - \left\lfloor \frac{n}{4} \right\rfloor .$$

Let $f\colon V(G) \to V(H)$ be the function defined by

$$f(g_i) = \begin{cases} 1; & i \bmod 4 \in \{1,2\}, \\ 3; & \text{otherwise.} \end{cases}$$

for all $i \in [n]$ where addition is taken modulo n. Adopting our earlier notation, recall that $y_i = (g_i, f(g_{i-1}))$ and $x_i = (g_i, f(g_{i+1}))$ for all $i \in [n]$. Let $n = 4\ell + j$ where $j \in [3]_0 = \{0,1,2,3\}$. We note that $f(g_{4i-3}) = f(g_{4i-2}) = 1$ and $f(g_{4i-1}) = f(g_{4i}) = 3$ for all $i \in [\ell]$. Let D_i be the unique γ-set of $H_i - y_i \cong P_{3k}$ which consists of all vertices at distance 2 modulo 3 from y_i in the graph H_i for all $i \in [n]$, and let

$$D = \bigcup_{i=1}^{n} D_i .$$

We note that $|D_i| = k$ for all $i \in [n]$, and so $|D| = kn$. For all $i \in \{2, 3, \ldots, \ell - 1\}$, the following four properties hold.

(P1) $y_{4i-3} = (g_{4i-3}, 3)$ and $x_{4i-3} = (g_{4i-3}, 1)$.

(P2) $y_{4i-2} = (g_{4i-2}, 1)$ and $x_{4i-2} = (g_{4i-2}, 3)$.

(P3) $y_{4i-1} = (g_{4i-1}, 1)$ and $x_{4i-1} = (g_{4i-1}, 3)$.

(P4) $y_{4i} = (g_{4i}, 3)$ and $x_{4i} = (g_{4i}, 1)$.

Hence for all $i \in \{2, 3, \ldots, \ell - 1\}$, the vertices x_i and y_i are at distance 2 in H_i, implying that $x_i \in D_i$. We consider four cases to determine which properties hold for the boundary conditions (that is for $i \in \{1, \ell\}$) and finally to set the upper bound on the domination number in each case.

Case 1. $n \equiv 0 \pmod 4$*, that is* $n = 4\ell$.
In this case, properties (P1) and (P4) also hold for $i = 1$ and $i = \ell$, respectively. Thus, $y_1 = (g_1, 3)$ and $x_1 = (g_1, 1)$, and $y_{4\ell} = (g_{4\ell}, 3)$ and $x_{4\ell} = (g_{4\ell}, 1)$, implying that $x_1, x_{4\ell} \in D$. The set D is therefore a dominating set of $G \otimes_f H$, and so $\gamma(G \otimes_f H) \le |D| = kn = kn + \lceil n/4 \rceil - \lfloor n/4 \rfloor$.

Case 2. $n \equiv 1 \pmod 4$*, that is* $n = 4\ell + 1$.
In this case, $y_1 = x_1 = (g_1, 1)$, and $y_{4\ell+1} = (g_{4\ell+1}, 3)$ and $x_{4\ell+1} = (g_{4\ell+1}, 1)$. In particular, property (P4) also holds for $i = \ell$, and so $x_{4\ell+1} \in D$. The set $D \cup \{x_1\}$ is therefore a dominating set of $G \otimes_f H$, and so $\gamma(G \otimes_f H) \le |D| + 1 = kn + 1 = kn + \lceil n/4 \rceil - \lfloor n/4 \rfloor$.

Case 3. $n \equiv 2 \pmod 4$*, that is* $n = 4\ell + 2$.
In this case, $y_1 = x_1 = (g_1, 1)$, and $y_{4\ell+2} = x_{4\ell+2} = (g_{4\ell+2}, 1)$. We note that neither x_1 nor $x_{4\ell+2}$ belong to the set D. The set $D \cup \{x_1\}$ is a dominating set of $G \otimes_f H$, and so $\gamma(G \otimes_f H) \le |D| + 1 = kn + 1 = kn + \lceil n/4 \rceil - \lfloor n/4 \rfloor$.

Case 4. $n \equiv 3 \pmod 4$*, that is* $n = 4\ell + 3$.

In this case, $y_1 = (g_1, 3)$ and $x_1 = (g_1, 1)$, and $y_{4\ell+3} = x_{4\ell+3} = (g_{4\ell+3}, 1)$. In particular, property (P1) also holds for $i = 1$, and so $x_1 \in D$. However, $x_{4\ell+3} \notin D$. The set $D \cup \{x_{4\ell+3}\}$ is therefore a dominating set of $G \otimes_f H$, and so $\gamma(G \otimes_f H) \leq |D| + 1 = kn + 1 = kn + \lceil n/4 \rceil - \lfloor n/4 \rfloor$.

In all four cases, $\gamma(G \otimes_f H) \leq kn + \lceil n/4 \rceil - \lfloor n/4 \rfloor$. □

3.2 The cycle C_n and cycles C_{3k+2}

In this section, we determine the Sierpiński domination number $\gamma_S(C_n, C_{3k+2})$ and the upper Sierpiński domination number $\Gamma_S(C_n, C_{3k+2})$.

Theorem 3.11. *For $n \geq 3$ and $k \geq 1$, we have $\Gamma_{\mathrm{S}}(C_n, C_{3k+2}) = (k+1)n$.*

Proof. For $n \geq 3$ and $k \geq 1$, let $G \cong C_n$ and let $H \cong C_{3k+2}$. Let $f\colon V(G) \to V(H)$ be a constant function, that is, we select $h \in V(H)$ and for every vertex $g \in V(G)$, we set $f(g) = h$. For each vertex $g \in V(G)$, let H_g denote the copy of H associated with the vertex g. Let D be a dominating set of $G \otimes_f H$, and let $D_g = D \cap V(H_g)$, and so D_g is the restriction of D to the copy H_g of H. If the vertex (g, h) does not belong to D_g, then D_g dominates all vertices on the path $H_g - (g, h) \cong P_{3k+1}$, and so $|D_g| \geq \gamma(P_{3k+1}) = k + 1$. If the vertex (g, h) does belong to D_g, then D_g dominates all vertices on the cycle $H_g \cong C_{3k+2}$, and so $|D_g| \geq \gamma(C_{3k+2}) = k + 1$. In both cases, $|D_g| \geq k + 1$. Therefore,

$$\gamma(G \otimes_f H) = |D| = \sum_{g \in V(G)} |D_g| \geq (k+1)n,$$

implying that $\Gamma_{\mathrm{S}}(C_n, C_{3k+2}) \geq (k+1)n$. By the upper bound in Theorem 2.1, we have $\Gamma_{\mathrm{S}}(G, H) \leq n(G)\gamma(H) = (k+1)n$, noting that in this case $\gamma(H) = \gamma(C_{3k+2}) = k + 1$. Consequently, $\Gamma_{\mathrm{S}}(C_n, C_{3k+2}) = (k+1)n$. □

Theorem 3.12. *For $n \geq 3$ and $k \geq 1$,*

$$\gamma_{\mathrm{S}}(C_n, C_{3k+2}) \in \{kn + \left\lfloor \frac{n}{2} \right\rfloor, kn + \left\lfloor \frac{n}{2} \right\rfloor + 1\}.$$

Moreover, if $n \equiv 0 \mod 4$, then $\gamma_{\mathrm{S}}(C_n, C_{3k+2}) = kn + \left\lfloor \frac{n}{2} \right\rfloor$.

Proof. For $n \geq 3$ and $k \geq 1$, let $G \cong C_n$ and let $H \cong C_{3k+2}$. We adopt our notation employed in our earlier proofs. Thus, the cycle G is given by $g_1 g_2 \ldots g_n g_1$, and $V(H) = \{1, 2, \ldots, 3k+2\}$ where vertices i and $i+1$ are consecutive on the cycle H for all $i \in [3k+2]$ (and where addition is taken modulo $3k+2$, and so vertex 1 and vertex $3k+2$ are adjacent). As before, we denote the copy $g_i H$ of H corresponding to the vertex g_i simply by H_i for each $i \in [n]$. Thus, $H_i = C_{3k+2}$ is the cycle $(g_i, 1), (g_i, 2), \ldots, (g_i, 3k+2), (g_i, 1)$ for all $i \in [n]$.

We adopt our notation from the proof of Theorem 3.6. Thus, we denote the connecting edge from H_i to H_{i+1} by $x_i y_{i+1}$ for all $i \in [n]$, where $x_i \in V(H_i)$, $y_{i+1} \in V(H_{i+1})$, and addition is taken modulo n. Thus, $y_i = (g_i, f(g_{i-1}))$ and $x_i = (g_i, f(g_{i+1}))$ for all $i \in [n]$.

We proceed further with two claims. The first claim establishes a lower bound on $\gamma_{\mathrm{S}}(G, H)$, and the second claim an upper bound on $\gamma_{\mathrm{S}}(G, H)$. Combining these two bounds yields the desired result in the statement of the theorem.

Claim 3.13. $\gamma_S(C_n, H) \geq kn + \left\lfloor \frac{n}{2} \right\rfloor$.

Proof. Let $f\colon V(G) \to V(H)$ be an arbitrary function. We show that

$$\gamma(G \otimes_f H) \geq kn + \left\lfloor \frac{n}{2} \right\rfloor. \tag{3.4}$$

Let D be a γ-set of $G \otimes_f H$ constructed, and let $D_i = D \cap V(H_i)$ for $i \in [n]$. If the vertex x_i is not dominated by D_i, then either $x_i \neq y_i$, in which case x_i is dominated by the vertex $y_{i+1} \in D$, or $x_i = y_i$, in which case x_i is dominated by the vertex $x_{i-1} \in D$ or the vertex $y_{i+1} \in D$. Analogously, if the vertex y_i is not dominated by D_i, then either $x_i \neq y_i$, in which case y_i is dominated by the vertex $x_{i-1} \in D$, or $x_i = y_i$, in which case y_i is dominated by the vertex $x_{i-1} \in D$ or the vertex $y_{i+1} \in D$. If a vertex is not dominated by D_i, then such a vertex is x_i or y_i, and we say that such a vertex is dominated from outside H_i.

Similarly as before, we proceed with a claim that delivers properties of sets D_i leading to the desired lower bound on the Sierpiński domination number.

Claim 3.14. *The following properties hold in the graph H_i.*

(a) *If $d(x_i, y_i) \equiv 1 \pmod 3$, then $|D_i| = k$. Further, both x_i and y_i are dominated from outside H_i.*

(b) *If $d(x_i, y_i) \not\equiv 1 \pmod 3$, then $|D_i| = k + 1$.*

Proof. Suppose that D_i contains a vertex w_i that dominates x_i. Possibly, $w_i = x_i$. In order to dominate the $3(k-1)+2$ vertices in H_i not dominated by w_i, at least k additional vertices are needed even if the vertex y_i is dominated outside the cycle H_i. Thus in this case, $|D_i| \geq k+1$, implying by the minimality of the set D that $|D_i| = k+1$. Analogously, if D_i contains a vertex that dominates y_i, then $|D_i| = k + 1$. Hence, if x_i or y_i (or both x_i and y_i) are dominated by D_i, then $|D_i| = k + 1$.

Suppose that neither x_i nor y_i is dominated by D_i, implying that both x_i and y_i are dominated from outside the cycle H_i. Thus, D_i is a dominating set of $H_i' = H_i - x_i - y_i$. If $x_i = y_i$, then $H_i' = P_{3k+1}$, and by the minimality of D we have $|D_i| = \gamma(P_{3k+1}) = k+1$. Hence, we may assume that $x_i \neq y_i$. If x_i and y_i are adjacent, then $H_i' = P_{3k}$, and by the minimality of D we have $|D_i| = \gamma(P_{3k}) = k$. Suppose that x_i and y_i are not adjacent, and so H' is the disjoint union of two paths P_{k_1} and P_{k_2}, where $k_1 + k_2 = 3k$. If $k_1 = 3j_1 + 1$ and $k_2 = 3j_2 + 2$ (or if $k_1 = 3j_1 + 2$ and $k_2 = 3j_2 + 1$) for some integers j_i and j_2 where $j_1 + j_2 = k - 1$, then $|D_i| = \lceil k_1/3 \rceil + \lceil k_2/3 \rceil = (j_1 + 1) + (j_2 + 1) = k + 1$. If $k_1 = 3j_1$ and $k_2 = 3j_2$ where $j_1 + j_2 = k$, then $|D_i| = \lceil k_1/3 \rceil + \lceil k_2/3 \rceil = j_1 + j_2 = k$. Hence if neither x_i nor y_i is dominated by D_i, then either $d(x_i, y_i) \equiv 1 \pmod 3$, in which case $|D_i| = k$, or $d(x_i, y_i) \not\equiv 1 \pmod 3$, in which case $|D_i| = k+1$. This proves properties (a) and (b) of the claim. (□)

By Claim 3.14, if $|D_i| = k$ for some $i \in [n]$, then $|D_{i-1}| = |D_{i+1}| = k + 1$ where addition is taken modulo n. Furthermore in this case when $|D_i| = k$, the vertices x_i and y_i are distinct and are both dominated from outside H_i, implying that $y_{i+1} \in D_{i+1}$ and $x_{i-1} \in D_{i-1}$. This implies that if n is even, then $|D| \geq kn + n/2$, and if n is odd, then $|D| \geq kn + (n + 1)/2$. This proves inequality (3.4).

Claim 3.15. $\gamma_S(C_n, H) \leq kn + \left\lfloor \frac{n}{2} \right\rfloor + \left\lceil \frac{n}{4} \right\rceil - \left\lfloor \frac{n}{4} \right\rfloor$.

Proof. Let $f\colon V(G) \to V(H)$ be the function defined by

$$f(g_i) = \begin{cases} 1; & i \equiv 1 \pmod 4, \\ 2; & i \equiv 2 \pmod 4, \\ 3; & \text{otherwise.} \end{cases}$$

for all $i \in [n]$ where addition is taken modulo n. Adopting our earlier notation, recall that $y_i = (g_i, f(g_{i-1}))$ and $x_i = (g_i, f(g_{i+1}))$ for all $i \in [n]$. Let $n = 4\ell + j$ where $j \in [3]_0 = \{0, 1, 2, 3\}$. We note that $f(g_{4i-3}) = 1$, $f(g_{4i-2}) = 2$, and $f(g_{4i-1}) = f(g_{4i}) = 3$ for all $i \in [\ell]$.

Case 1. $n \equiv 0 \pmod 4$.
Thus, $n = 4\ell$. We note that $y_{4i-3} = (g_{4i-3}, 3)$ and $x_{4i-3} = (g_{4i-3}, 2)$ for all $i \in [n]$, and so in the graph H_{4i-3} the vertices x_{4i-3} and y_{4i-3} are at distance 1. Moreover, $y_{4i-1} = (g_{4i-1}, 2)$ and $x_{4i-1} = (g_{4i-1}, 3)$ for all $i \in [n]$, and so in the graph H_{4i-1} the vertices x_{4i-1} and y_{4i-1} are at distance 1. This implies that $H_{4i-j} - \{x_{4i-j}, y_{4i-j}\} \cong C_{3k}$ for $j \in \{1, 3\}$. Let D_{4i-j} be a γ-set of $H_{4i-j} - \{x_{4i-j}, y_{4i-j}\}$ for $j \in \{1, 3\}$, and so $|D_{4i-j}| = k$.

We also note that $y_{4i-2} = (g_{4i-2}, 1)$ and $x_{4i-2} = (g_{4i-2}, 3)$ for all $i \in [n]$, and so in the graph H_{4i-2} the vertices x_{4i-2} and y_{4i-2} are at distance 2. Moreover, $y_{4i} = (g_{4i}, 3)$ and $x_{4i} = (g_{4i}, 1)$ for all $i \in [n]$, and so in the graph H_{4i} the vertices x_{4i} and y_{4i} are at distance 2. This implies that $H_{4i-j} - N[\{x_{4i-j}, y_{4i-j}\}] \cong C_{3(k-1)}$ for $j \in \{0, 2\}$. Let D_{4i-j} be a γ-set of H_{4i-j} that contains both vertices x_{4i-j} and y_{4i-j} for $j \in \{0, 2\}$, and so $|D_{4i-j}| = k + 1$. The set

$$D = \bigcup_{i=1}^{4\ell} D_i$$

is a dominating set of $G \otimes_f H$, and so $\gamma(G \otimes_f H) \le |D| = 4k\ell + 2\ell = kn + n/2$.

Case 2. $n \equiv 2 \pmod 4$.
Thus, $n = 4\ell + 2$ and in this case, $f(g_{4\ell+1}) = 1$ and $f(g_{4\ell+2}) = 2$. We note that in the graph $H_{4\ell+1}$, the vertices $x_{4\ell+1}$ and $y_{4\ell+1}$ are at distance 1 and in the graph $H_{4\ell+2}$ we have $x_{4\ell+2} = y_{4\ell+2}$. For $i \in [4\ell]$, we define the set D_i exactly as in the previous case. Further, let $D_{4\ell+1}$ be a γ-set of $H_{4\ell+1} - \{x_{4\ell+1}, y_{4\ell+1}\} \cong C_{3k}$, and let $D_{4\ell+2}$ be a γ-set of $H_{4\ell+2}$ containing $x_{4\ell+2}$. We note that $|D_{4\ell+1}| = k$ and $|D_{4\ell+2}| = k + 1$. The set

$$D = \bigcup_{i=1}^{4\ell+2} D_i$$

is a dominating set of $G \otimes_f H$, and so $\gamma(G \otimes_f H) \le |D| = 4k\ell + 2k + 2\ell + 1 = kn + n/2$.

Case 3. $n \equiv 1 \pmod 4$.
Thus, $n = 4\ell + 1$, and in this case, $f(g_{4\ell+1}) = 1$. Thus, $y_{4\ell+1} = (g_{4\ell+1}, 3)$ and $x_{4\ell+1} = (g_{4\ell+1}, 1)$, and so in the graph $H_{4\ell+1}$, the vertices $x_{4\ell+1}$ and $y_{4\ell+1}$ are at distance 2. For $i \in [4\ell]$, we define the set D_i exactly as in the previous cases. Further, let $D_{4\ell+1}$ be a γ-set of $H_{4\ell+1}$ that contains the vertex $x_{4\ell+1}$. We note that $|D_{4\ell+1}| = k + 1$. The set

$$D = \bigcup_{i=1}^{4\ell+1} D_i$$

is a dominating set of $G \otimes_f H$, and so $\gamma(G \otimes_f H) \leq |D| = 4k\ell + k + 2\ell + 1 = kn + (n+1)/2$.

Case 4. $n \equiv 3 \pmod 4$.
Thus, $n = 4\ell + 3$, and in this case, $f(g_{4\ell+1}) = 1$, $f(g_{4\ell+2}) = 2$, and $f(g_{4\ell+3}) = 3$. In particular, $y_{4\ell+3} = (g_{4\ell+3}, 2)$ and $x_{4\ell+3} = (g_{4\ell+3}, 1)$, and so in the graph $H_{4\ell+3}$, the vertices $x_{4\ell+3}$ and $y_{4\ell+3}$ are at distance 1. For $i \in [4\ell + 2]$, we define the set D_i exactly as in Case 2. Further, let $D_{4\ell+3}$ be a γ-set of $H_{4\ell+3}$ containing the vertex $x_{4\ell+3}$. We note that $|D_{4\ell+3}| = k + 1$. The set

$$D = \bigcup_{i=1}^{4\ell+3} D_i$$

is a dominating set of $G \otimes_f H$, and so $\gamma(G \otimes_f H) \leq |D| = 4k\ell + 3k + 2\ell + 2 = kn + (n + 1)/2$. The desired result of the claim now follows from the four cases above. $(\square)$

The proof of Theorem 3.12 follows as an immediate consequence of Claim 3.13 and Claim 3.15. $\square$

3.3 The cycle C_n and cycles C_{3k}

In this section, we determine the Sierpiński domination number $\gamma_S(C_n, C_{3k})$ and the upper Sierpiński domination number $\Gamma_S(C_n, C_{3k})$.

Theorem 3.16. *For $n \geq 3$ and $k \geq 1$,*

$$\gamma_{\mathrm{S}}(C_n, C_{3k}) = \Gamma_{\mathrm{S}}(C_n, C_{3k}) = kn.$$

Proof. We adopt our notation from the earlier sections. Let $G \cong C_n$ be the cycle $g_1 g_2 \dots g_n g_1$, and let H_i be the ith copy of C_{3k} corresponding to the vertices g_i of G for $i \in [n]$. As before, we denote the connecting edge from H_i to H_{i+1} by $x_i y_{i+1}$ for all $i \in [n]$.

Let $f\colon V(G) \to V(H)$ be an arbitrary function. Let D be a γ-set of $G \otimes_f H$, and let $D_i = D \cap V(H_i)$ for $i \in [n]$. We show that $|D_i| = k$ for all $i \in [n]$. If both vertices x_i and y_i are dominated by D_i, then D_i is a γ-set of $H_i \cong C_{3k}$, and so $|D_i| = k$. If exactly one of x_i and y_i is dominated by D_i, say x_i, then by the minimality of the set D, the set D_i is a γ-set of $H_i - y_i \cong P_{3k-1}$, and so $|D_i| = k$. Hence, we may assume that neither x_i nor y_i is dominated by D_i, for otherwise, $|D_i| = k$ and the desired bound follows.

With our assumption that neither x_i nor y_i is dominated by D_i, the set D_i is a γ-set of $H_i' = H_i - \{x_i, y_i\}$. If $x_i = y_i$, then $H_i' = P_{3k-1}$, and by the minimality of D we have $|D_i| = \gamma(P_{3k-1}) = k$. Hence, we may assume that $x_i \neq y_i$. If x_i and y_i are adjacent, then $H_i' = P_{3k-2}$, and by the minimality of D we have $|D_i| = \gamma(P_{3k-2}) = k$. Suppose that x_i and y_i are not adjacent, and so H' is the disjoint union of two paths P_{k_1} and P_{k_2}, where $k_1 + k_2 = 3k - 2$. If $k_1 = 3j_1 + 1$ and $k_2 = 3j_2$ for some integers j_i and j_2 where $j_1 + j_2 = k - 1$, then $|D_i| = \lceil k_1/3 \rceil + \lceil k_2/3 \rceil = (j_1 + 1) + j_2 = k$. Analogously, if $k_1 = 3j_1$ and $k_2 = 3j_2 + 1$, then $|D_i| = k$. If $k_1 = 3j_1 + 2$ and $k_2 = 3j_2 + 2$ for some integers j_i and j_2 where $j_1 + j_2 = k - 2$, then $|D_i| = \lceil k_1/3 \rceil + \lceil k_2/3 \rceil = (j_1 + 1) + (j_2 + 1) = k$. In all cases, $|D_i| = k$, implying that

$$\gamma(G \otimes_f H) = |D| = \sum_{i=1}^{n} |D_i| = kn.$$

Since $f\colon V(G) \to V(H)$ was chosen as an arbitrary function, and D as an arbitrary γ-set of $G \otimes_f H$, we deduce that $\gamma_{\mathrm{S}}(C_n, C_{3k}) = \Gamma_{\mathrm{S}}(C_n, C_{3k}) = \gamma(G \otimes_f H) = kn$. $\square$

4 Concluding remarks

It seems to us that in the vast majority of cases where the lower Sierpiński domination number of the Sierpiński product of two cycles is specified to two values exactly, the larger of the two is the correct value. However, the following example, which surprised us, demonstrates that there are also cases where the exact value is the smaller of the two possible values.

Let $G \cong C_{18}$ with $V(G) = [18]$ and let $H \cong C_7$ with $V(H) = [7]$ and let the function $f\colon V(G) \to V(H)$ be defined as follows:

$$\begin{aligned} f(1) &= f(4) = f(5) = f(18) = 4,\\ f(2) &= f(3) = f(6) = f(7) = 2,\\ f(8) &= f(9) = 7,\\ f(10) &= f(11) = 5,\\ f(12) &= f(13) = 3,\\ f(14) &= f(15) = 1,\\ f(16) &= f(17) = 6. \end{aligned}$$

Then Theorem 3.2 asserts that $\gamma(G \otimes_f H) \in \{36, 37\}$ and it is straightforward to check that the exact value is $\gamma(G \otimes_f H) = 36$.

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On the minisymposium problem*

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Abstract

The generalized Oberwolfach problem asks for a factorization of the complete graph K_v into prescribed 2-factors and at most a 1-factor. When all 2-factors are pairwise isomorphic and v is odd, we have the classic Oberwolfach problem, which was originally stated as a seating problem: given v attendees at a conference with t circular tables such that the ith table seats a_i people and $\sum_{i=1}^{t} a_i = v$, find a seating arrangement over the $\frac{v-1}{2}$ days of the conference, so that every person sits next to each other person exactly once.

In this paper we introduce the related *minisymposium problem*, which requires a solution to the generalized Oberwolfach problem on v vertices that contains a subsystem on m vertices. That is, the decomposition restricted to the required m vertices is a solution to the generalized Oberwolfach problem on m vertices. In the seating context above, the larger conference contains a minisymposium of m participants, and we also require that pairs of these m participants be seated next to each other for $\left\lfloor \frac{m-1}{2} \right\rfloor$ of the days.

When the cycles are as long as possible, i.e. v, m and $v-m$, a flexible method of Hilton and Johnson provides a solution. We use this result to provide further solutions when $v \equiv m \equiv 2 \pmod 4$ and all cycle lengths are even. In addition, we provide extensive

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results in the case where all cycle lengths are equal to k, solving all cases when $m \mid v$, except possibly when k is odd and v is even.

Keywords: Minisymposium problem, (generalized) Oberwolfach problem, 2-factorizations, subsystems.

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1 Introduction

We assume that the reader is familiar with the fundamentals of graph theory and of design theory and refer them to [41] and [14], respectively. In particular, a factor is a spanning subgraph and an r-factor is a factor which is r-regular, so in a 1-factor every vertex has degree one and a 2-factor is a disjoint union of cycles. Given a collection of factors, $\mathcal{F}$, an $\mathcal{F}$-factorization of a graph G is a decomposition of the edges of G into subgraphs, each of which is isomorphic to some $F \in \mathcal{F}$. If $\mathcal{F} = \{F\}$ we speak of an F-factorization.

We use K_n to denote the complete graph on n vertices and K_n^* to denote the graph K_n when n is odd and $K_n - I$, where I is a 1-factor, when n is even. Similarly, $K_{m,n}$ denotes the complete bipartite graph with parts of sizes m and n. If the parts are X and Y, respectively, we may also speak of $K_{X,Y}$. A 2-factor is called *uniform* if all of its constituent cycles are of the same length; it is called *Hamiltonian** if its cycles have longest possible lengths given the requirements of the factorization. If a 2-factor, F, consists entirely of cycles of a particular length, k say, we refer to an F-factor and an F-factorization as a C_k-factor and C_k-factorization, respectively. Given a graph G, we denote by $G[n]$ the *lexicographic product* of G with the empty graph on n vertices. Specifically, the vertex set of $G[n]$ is $V(G) \times \mathbb{Z}_n$ (where $\mathbb{Z}_n$ denotes the cyclic group of order n) and $(x, i)(y, j) \in E(G[n])$ if and only if $xy \in E(G)$, $i, j \in \mathbb{Z}_n$.

The well known Oberwolfach problem asks for a 2-factorization of K_n^* into 2-factors all of which are isomorphic to a given 2-factor F. A summary of results up until 2006 can be found in [14, Section VI.12], in particular the case of uniform factors has been solved [1, 2, 26].

Theorem 1.1 ([1, 2, 26]). *Given integers $v, k \geq 3$, there is a C_k-factorization of K_v^* if and only if $k \mid v$, except that there is no C_3-factorization of K_6^* or K_{12}^*.*

The case when all cycles in F have even length has been completely solved in [6]. The case with exactly two cycles is solved in [38]. The case of the complete graph $K_\aleph$, where $\aleph$ is any infinite cardinal has been completely solved in [15]. In the related Hamilton-Waterloo problem two 2-factors F_1 and F_2 are specified and we are asked for a factorization of K_n^* into a given number of each of the factors. There has been much recent progress in this problem, see [3, 5, 7, 10, 11, 12, 17, 27, 28, 29, 32, 39, 40]. More generally, in the generalized Oberwolfach problem we are given a set of 2-factors $F_1, F_2, \ldots, F_t$ of K_n and positive integers $\alpha_1, \alpha_2, \ldots, \alpha_t$, where $\sum \alpha_i = \lfloor \frac{n-1}{2} \rfloor$, and are asked for a factorization of K_n^* which contains α_i copies of the 2-factor F_i, see [6, 13, 21]. A major recent development gives a non-constructive asymptotic existence result for the generalized Oberwolfach problem [23].

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Other graphs have also been considered. In particular, Liu has shown the following for the complete multipartite graph.

Theorem 1.2 ([30, 31])**.** *Let k, t and u be positive integers with $k \geq 3$. There exists a C_k-factorization of $K_t[u]$ if and only if $k \mid tu$, $(t-1)u$ is even, further k is even if $t = 2$, and $(k, t, u) \notin \{(3, 3, 2), (3, 6, 2), (3, 3, 6), (6, 2, 6)\}$.*

Originally the Oberwolfach problem was stated as a seating problem:

> Given an odd number v of attendees at a conference with t circular tables such that the ith table seats a_i people and $\sum_{i=1}^{t} a_i = v$, find a seating arrangement over the $\frac{v-1}{2}$ days of the conference, so that every person sits next to each other person exactly once.

In this paper we introduce the related *minisymposium problem*. In this case we require a solution to the generalized Oberwolfach problem on v vertices such that its restriction to a subset of m vertices constitutes a solution to the generalized Oberwolfach problem on m vertices. Another way of considering the problem asks for a solution to the generalized Oberwolfach problem on v vertices which contains a subsystem on m vertices. In the seating context above, the larger conference contains a minisymposium of m participants, and we also require that pairs of these m participants be seated next to each other for $\left\lfloor \frac{m-1}{2} \right\rfloor$ of the days. A similar problem has been considered, for example, in [9] for whist tournaments.

Section 2 gives the formal definition of a minisymposium factorization and some necessary conditions for its existence, as well as introduces some special cases. In Section 3 we show how to use a flexible theorem by Hilton and Johnson [25] to solve the case of Hamiltonian* 2-factors (where the cycles are as long as possible). The same section considers the case where all cycles are of even length and $v \equiv m \equiv 2 \pmod 4$. Section 4 considers the uniform case, where all cycles have the same length. We completely solve the case where all cycles are of length m when $(v-1)m$ is even. In Section 5 we discuss and give some preliminary results on factorizations that contain more than one subsystem. We provide some concluding remarks in the final section.

2 Preliminaries

We begin by giving a formal definition of a minisymposium factorization. The minisymposium problem is equivalent to the original Oberwolfach problem when $v = m$. Hence we will generally assume that $v > m$.

Definition 2.1. Given positive integers v and m with $v \geq m$, let

$$\mathcal{F} = \left\{ F_i : \ i = 1, \ldots, \left\lfloor \frac{v-1}{2} \right\rfloor - \left\lfloor \frac{m-1}{2} \right\rfloor \right\},$$

be a collection of 2-factors on v vertices and let

$$\mathcal{G} = \left\{ (T_i, U_i) : i = 1, \ldots, \left\lfloor \frac{m-1}{2} \right\rfloor \right\},$$

where the T_i are 2-factors on m vertices and the U_i are 2-factors on $v - m$ vertices. We define a *minisymposium factorization* MSF$(\mathcal{F}, \mathcal{G})$ as a factorization of K_v^* into 2-factors $F \in \mathcal{F}$ and $G_i = T_i \cup U_i$, where $(T_i, U_i) \in \mathcal{G}$, such that $\mathcal{T} = \{T_i : \ i = 1, \ldots, \left\lfloor \frac{m-1}{2} \right\rfloor\}$ is a factorization of a subgraph of K_v^* isomorphic to K_m^*.

Note that with the notation MSF($\mathcal{F}, \mathcal{G}$), we assume that the parameters v, m, $\mathcal{T}$ and $\mathcal{U} = \{U_i \ : \ i = 1, \dots, \lfloor \frac{m-1}{2} \rfloor\}$ are defined implicitly. We may also use the notation MSF($\mathcal{F}, (\mathcal{T}, \mathcal{U})$), if we wish to explicitly refer to the factorizations $\mathcal{T}$ and $\mathcal{U}$.

An MSF($\mathcal{F}, \mathcal{G}$) can be thought of as a 2-factorization of K_v^* with a subsystem of size m. When $m = 1$ or 2, this is just a factorization of K_v^* into 2-factors in $\mathcal{F}$, which is equivalent to a solution of the generalized Oberwolfach problem, and so we will assume $m \geq 3$. Similarly, when $m = v$, this is a factorization into the T_i, so we assume that $m < v$.

Removing the subsystem, we can talk about a 2-factorization of K_v^* with a "hole" of size m. However, care must be taken when either v or m is even as the placement of the various 1-factors must be considered, as noted below.

We note that the size of $\mathcal{F}$ is

$$\left\lfloor \frac{v-1}{2} \right\rfloor - \left\lfloor \frac{m-1}{2} \right\rfloor = \begin{cases} \frac{v-m}{2} & v \equiv m \pmod 2 \\ \frac{v-m+1}{2} & v+1 \equiv m \equiv 0 \pmod 2 \\ \frac{v-m-1}{2} & v \equiv m+1 \equiv 0 \pmod 2. \end{cases}$$

- In the case when both v and m are even, we are considering a factorization of $K_v^* = K_v - I$, where I is a 1-factor, containing a factorization of a subgraph $G - J$ of K_v^* where $G \cong K_m$ and J is a 1-factor of G contained in I.
- When v is even and m is odd, we are considering a factorization of $K_v - I$, where I is a 1-factor, containing a factorization of a subgraph $G \cong K_m$. Note that none of the edges of I are contained in G.
- When v is odd and m is even, we are considering a factorization of $K_v^* = K_v$ containing a factorization of $G - J$, where $G \cong K_m$ and J is a 1-factor of G. We note that the edges of J are not covered by factors in $\mathcal{T}$, and hence must be covered by factors in $\mathcal{F}$.
- When both v and m are odd there is no 1-factor to consider.

Since in all cases $G \cong K_m$, we will henceforth refer to it as *the* K_m. We note that none of the edges of the K_m are covered by $\mathcal{F}$, except in the case of m even and v odd; in this case, the edges of the 1-factor J of the K_m are covered by $\mathcal{F}$. We use this observation in the proofs of the lemmas below, where for a given 2-factor F, we define $a_k(F)$ to be the number of cycles of length k in F.

Lemma 2.2. *For a given v and m, if there is a minisymposium factorization, MSF($\mathcal{F}, \mathcal{G}$), then for each factor $F \in \mathcal{F}$ using c_F edges in the K_m,*

$$v - m \geq -c_F + \sum_{i=3}^{v} a_i(F) \left\lceil \frac{i}{2} \right\rceil, \tag{2.1}$$

$$m \leq c_F + \sum_{i=3}^{v} a_i(F) \left\lfloor \frac{i}{2} \right\rfloor. \tag{2.2}$$

In particular, if v is even, or m is odd, all of the $c_F = 0$ and therefore,

$$\frac{\sum_{i=3}^{v} a_i(F) \left\lceil \frac{i}{2} \right\rceil}{\sum_{i=3}^{v} a_i(F) \left\lfloor \frac{i}{2} \right\rfloor} \leq \frac{v-m}{m}. \tag{2.3}$$

Proof. We first deal with the case when v is even, or m is odd. In this case none of the edges of the K_m appear in any $F \in \mathcal{F}$. Therefore, for each $F \in \mathcal{F}$, at most $\left\lfloor \frac{i}{2} \right\rfloor$ vertices of any cycle of length i in F are inside the K_m, hence

$$m \leq \sum_{i=3}^{v} a_i(F) \left\lfloor \frac{i}{2} \right\rfloor . \tag{2.4}$$

Similarly, for any cycle of length i in F, at least $\lceil \frac{i}{2} \rceil$ vertices of the cycle are not in the K_m, thus

$$\sum_{i=3}^{v} a_i(F) \left\lceil \frac{i}{2} \right\rceil \leq v - m. \tag{2.5}$$

Thus Inequality (2.3) follows.

Now, if v is odd and m is even the $\frac{m}{2}$ edges in the 1-factor J of the K_m must be used in factors from $\mathcal{F}$. Suppose that $F \in \mathcal{F}$ uses c_F of these edges. Each edge of the K_m used can increase the right hand side of Inequality (2.4) by no more than one and decrease the left hand side of Inequality (2.5) by no more than one.

□

Theorem 2.3. *For a given v and m, if there is a minisymposium factorization, MSF$(\mathcal{F}, \mathcal{G})$, then $v \geq 2m$, unless v is odd and m is even, in which case $v \geq 2m - 1$.*

Proof. When v is even or m is odd, the left hand side of Inequality (2.3) is at least 1 and therefore $v \geq 2m$.

When v is odd and m is even, $\sum_{F \in \mathcal{F}} c_F = \frac{m}{2}$. Since v is odd, each $F \in \mathcal{F}$ must contain at least one odd cycle, therefore $\sum_{i=3}^{v} a_i(F) \lceil \frac{i}{2} \rceil \geq \lceil \frac{v}{2} \rceil = \frac{v+1}{2}$. Also note that the number of factors $F \in \mathcal{F}$ is $\frac{v-m+1}{2}$. Summing Inequality (2.1) over all of the $F \in \mathcal{F}$ twice gives

$$\begin{aligned}
(v - m + 1)(v - m) &\geq 2 \sum_{F \in \mathcal{F}} \left(-c_F + \sum_{i=3}^{v} a_i(F) \left\lceil \frac{i}{2} \right\rceil \right) \\
&= -m + 2 \sum_{F \in \mathcal{F}} \sum_{i=3}^{v} a_i(F) \left\lceil \frac{i}{2} \right\rceil \\
&\geq -m + 2 \sum_{F \in \mathcal{F}} \frac{v+1}{2} \\
&= (v - m + 1)\frac{v+1}{2} - m
\end{aligned}$$

Hence,

$$\begin{aligned}
0 &\leq (v - m + 1)(v - m) - (v - m + 1)\frac{v+1}{2} + m \\
&= \frac{(v - m + 1)(v - 2m - 1) + 2m}{2} \\
&= \frac{(v - (2m - 1))(v - (m + 1))}{2}.
\end{aligned}$$

When $v = m + 1$, the factors in $\mathcal{U}$ are required to be 2-regular graphs on a single vertex, which is not possible, so $v - (m + 1) > 0$. Thus $v - (2m - 1) \geq 0$ and the result follows. $\square$

In the case where v is odd and m is even and $v = 2m - 1$, we have that $\mathcal{U}$ is a factorization of K_{v-m}. So we can interchange both the roles of m and $v - m$, as well as those of $\mathcal{T}$ and $\mathcal{U}$. Thus, without loss of generality, we may assume that $v \geq 2m$ in all cases.

There are two cases of initial special interest. Firstly, the case of *uniform* cycle lengths (when all cycles in a factor are of the same length), which we consider in detail in Section 4. Secondly, the case where the cycles are as long as possible, which in correspondence with the definition of K_n^* and the Hamiltonian-like nature of such factorizations we will call *Hamiltonian** factorizations. We formally define *Hamiltonian** factorizations in Section 3. There we show that a method of Hilton and Johnson completely settles their existence.

An MSF($\mathcal{F}, \mathcal{G}$) in which all of the factors in $\mathcal{F}$, $\mathcal{T}$ and $\mathcal{U}$ are uniform with the same cycle length k is called *uniform* and we refer to it as a UMSF(v, m, k). In this case we have the following necessary conditions.

Theorem 2.4. *If $v > m$ and a UMSF(v, m, k) exists, then $k \geq 3$, $k \mid m$ and $k \mid v$. Furthermore,*

- *if k is even, then $v \geq 2m$;*
- *if k is odd, then $v \geq \frac{2mk}{k-1}$.*

Proof. Since we are forming 2-factors with cycles of length k, we require $k \geq 3$. The divisibility conditions follow directly from the requirement for a factorization of K_v^* and K_m^* into k-cycles. If k is even, then v is even, since it is a multiple of k, and Theorem 2.3 gives $v \geq 2m$.

If k is odd, we note that for a C_k-factor $F \in \mathcal{F}$, we have $a_i(F) = \frac{v}{k}$ when $i = k$ and 0 otherwise, $\lfloor \frac{k}{2} \rfloor = \frac{k-1}{2}$ and $\lceil \frac{k}{2} \rceil = \frac{k+1}{2}$. Thus, when v is even or m is odd, Inequality (2.3) implies that $v \geq \frac{2mk}{k-1}$ and the result follows.

This leaves the case when v and k are odd and m is even. We sum Inequality (2.1) over all the $F \in \mathcal{F}$ to obtain

$$\begin{aligned}
\textstyle\sum_{F \in \mathcal{F}} (v - m) &\geq \textstyle\sum_{F \in \mathcal{F}} \left(-c_F + \sum_{i=3}^{v} a_i(F) \left\lceil \frac{i}{2} \right\rceil\right) \\
\tfrac{1}{2}(v - m + 1)(v - m) &\geq -m/2 + \textstyle\sum_{F \in \mathcal{F}} \frac{v}{k} \frac{k+1}{2} \\
2k(v - m + 1)(v - m) &\geq v(v - m + 1)(k + 1) - 2mk.
\end{aligned}$$

Rearranging and expanding in v gives

$$(k - 1)v^2 + (-3km + m + k - 1)v + 2km^2 \geq 0. \tag{2.6}$$

Let

$$f(v) = (k - 1)v^2 + (-3km + m + k - 1)v + 2km^2.$$

By Theorem 2.3, $v \geq 2m - 1$, but

$$f\,(2m - 1) = m(1 + k - 2m) < 0,$$

so v is at least as large as the larger of the two roots of f. Now

$$f\left(\frac{2mk}{k-1}-2\right)=-2(m+1-k)<0, \text{ and } f\left(\frac{2mk}{k-1}-1\right)=(k-1)m>0.$$

Thus f has its larger root between $\frac{2mk}{k-1}-2$ and $\frac{2mk}{k-1}-1$.

It is left to check that $v \neq \lfloor\frac{2mk}{k-1}-1\rfloor, \lceil\frac{2mk}{k-1}-1\rceil$. Recalling that $k \mid v$, if $v=\lfloor\frac{2mk}{k-1}-1\rfloor$ or $v=\lceil\frac{2mk}{k-1}-1\rceil$, then there exist a rational number $0 \leq \epsilon < 1$ and a positive integer a such that

$$v=\frac{2mk}{k-1}-1\pm\epsilon=ak.$$

Multiplying both sides by $\frac{k-1}{2k}$ and rearranging we have that

$$m-a\frac{k-1}{2}=(1\pm\epsilon)\frac{k-1}{2k}.$$

Since k is odd, the left side is an integer. However, $0<(1\pm\epsilon)\frac{k-1}{2k}<1$, a contradiction. We conclude that $v \geq \frac{2mk}{k-1}$. □

One interesting case in light of these necessary conditions is when $m=k$, i.e. a UMSF(v,k,k). For these parameters, since $k<v$ and $k \mid v$, we must have $v \geq 2k$, so the necessary conditions in Theorem 2.3 are satisfied. We consider these types of factorizations in Section 4.

3 Hamiltonian* and bipartite factors

Considering non-uniform factors, an obvious case to consider is a *Hamiltonian* minisymposium factorization*, which is one in which the cycles have the longest possible lengths. Specifically, the factors in $\mathcal{F}$ are all v-cycles, factors in $\mathcal{T}$ are all m-cycles and the factors in $\mathcal{U}$ are all $(v-m)$-cycles. Such an MSF$(\mathcal{F},\mathcal{G})$ is denoted by HMSF(v,m). We sometimes refer to the cycles in $\mathcal{T}$ and $\mathcal{U}$ as 'short' cycles. Because of the lengths of these cycles there are no further necessary conditions beyond those of Theorem 2.3.

In a paper on the Oberwolfach problem, Hilton and Johnson prove the following theorem on a flexible construction technique.

Theorem 3.1 ([25]). *Let m and n be integers, $1 \leq m < n$. Let $(s_1,\ldots,s_t)$, $s_i \in 1,2$, $1 \leq i \leq t$, be a composition of $n-1$. Let K_m be edge coloured with t colours $c_1,\ldots,c_t$. Let f_i be the number of edges coloured c_i and $K_m(c_i)$ be the ith colour class. This colouring can be extended to an edge-colouring of K_n in which the colour class $K_n(c_i)$ is an s_i-factor, $1 \leq i \leq t$, and when $s_i=2$, $K_n(c_i)$ contains exactly one more cycle than $K_m(c_i)$ if and only if for all $i=1,2,\ldots,t$:*

$$f_i \geq s_i(m-n/2),$$
$$s_i n \text{ is even},$$
$$\Delta(K_m(c_i)) \leq s_i.$$

This theorem is sufficient to provide a solution to the Hamiltonian* minisymposium factorization.

Corollary 3.2. *An HMSF(v, m) exists if and only if $m \geq 3$, $v \geq 2m-1$ in case m is even, and $v \geq 2m$ in case m is odd.*

Proof. The given conditions are necessary by Theorem 2.3. To prove sufficiency, we will define an edge colouring of the K_m from a decomposition of the K_m into Hamiltonian cycles and possibly a single 1-factor using Theorem 1.1. If m is odd, this defines edge colours c_i, $1 \leq i \leq (m-1)/2$. If v is also odd, extend this to a $(v-1)/2$-edge colouring of the K_m by including $(v-m)/2$ empty colour classes c_i, $(m+1)/2 \leq i \leq (v-1)/2$. Let $s_i = 2$ for all $1 \leq i \leq (v-1)/2$. If v is even, extend the colouring to a $v/2$-edge colouring of the K_m by adding empty colour classes. Let $s_i = 2$ for all $1 \leq i < v/2$ and $s_{v/2} = 1$. In both cases, it can be verified that Theorem 3.1 now gives an HMSF(v, m) as desired.

If m is even, define $m/2$ edge colour classes of the K_m from a decomposition into Hamiltonian cycles and one 1-factor. Let c_1 be the colour class of the 1-factor. If v is also even, extend this to a $v/2$-edge colouring by adding empty colour classes. Let $s_1 = 1$ and $s_i = 2$ for $2 \leq i \leq v/2$. If v is odd, extend this to a $(v-1)/2$-edge colouring by adding empty colour classes. Let $s_i = 2$ for $1 \leq i \leq v/2$. In both cases, it can be verified that Theorem 3.1 now gives an HMSF(v, m) as desired. □

Theorem 3.1 is more than just an existence result; a recursive procedure can be extracted from the proof to algorithmically build the edge decompositions. We have a more direct construction of all HMSF(v, m) which uses difference methods and decompositions of Cayley graphs [16].

Theorem 3.1 can be used much more generally to build minisymposium factorizations. Essentially it shows that it is possible to extend any 2-factorization of the K_m to one of K_v, provided that the necessary conditions hold, where the additional 2-factors are Hamiltonian, with an additional 1-factor when v is even.

When all of the cycles of the factors in $\mathcal{F}, \mathcal{U}$ and $\mathcal{T}$ are bipartite (i.e. contain only even cycles), we apply the Theorem of Häggkvist [24] (given below) to HMSF(v, m) to give us a solution to the minisymposium problem when $v \equiv m \equiv 2 \pmod 4$.

Theorem 3.3 ([24])**.** *If F is a bipartite 2-regular graph of order $2n$, then there is a factorization of $C_n[2]$ into 2 isomorphic copies of F.*

We note that in the case where the factors are bipartite, so all cycle lengths are even, v and m are both even and Theorem 2.3 gives $v \geq 2m$.

Theorem 3.4. *If $v \equiv m \equiv 2 \pmod 4$, and $\mathcal{F} = \{F_i : 1 \leq i \leq (v-m)/2\}$, $\mathcal{U} = \{U_j : 1 \leq j \leq (m-2)/2\}$ and $\mathcal{T} = \{T_j : 1 \leq j \leq (m-2)/2\}$ are sets of bipartite factors with $F_i = F_{i+1}$, $U_j = U_{j+1}$ and $T_j = T_{j+1}$ for every odd i and j, then an MSF$(\mathcal{F}, (\mathcal{T}, \mathcal{U}))$ exists if and only $v \geq 2m$.*

Proof. We note that if $v \equiv m \equiv 2 \pmod 4$, the number of the F_i is $(v-m)/2$ and the number of the U_j and the T_j is $(m-2)/2$, so the number of the F_i, U_j and T_j are all even. We take an HMSF$(v/2, m/2)$, which exists by Corollary 3.2, with factors F_i' of order $v/2$, U_i' of order $(v-m)/2$ and T_i' of order $m/2$. We blow up each vertex by 2 and apply Theorem 3.3 to factor each $F_i'[2]$ into 2 copies of F_i, each $U_j'[2]$ into 2 copies of U_j and each $T_j'[2]$ into 2 copies of T_j. □

If $v \equiv 0 \pmod 4$ or $m \equiv 0 \pmod 4$, then $v/2$ or $m/2$ would be even and the HMSF$(v/2, m/2)$ would contain 1-factors either in $K_{v/2}$ or the $K_{m/2}$. When a 1-factor is blown up as done in Theorem 3.3, it results in a C_4-factor, which prevents constructing the desired MSF unless $\mathcal{F}$, $\mathcal{T}$, and $\mathcal{U}$ already contain this kind of factor.

An immediate consequence of Theorem 3.4 is the following relating to uniform factors.

Corollary 3.5. *If $k > 3$, $k \equiv 2 \pmod 4$, $v \equiv m \equiv k \pmod{2k}$, then a UMSF$(v, m, k)$ exists if and only if $v \geq 2m$.*

4 Uniform factors

In this section we consider the case of uniform factors, i.e. when all cycles are of the same length, k. We recall from Theorem 2.4 that in order for a UMSF(v, m, k) to exist, we require that $k \geq 3$, which we will assume throughout this section. We also require $k \mid m$ and $k \mid v$. Additionally, if k is even, then $v \geq 2m$ and if k is odd, then $v \geq \frac{2mk}{k-1}$.

Corollary 3.5 gives a powerful result in the case when $k \equiv 2 \pmod 4$ and $v \equiv m \equiv k \pmod{2k}$. The case where $k = 3$ has been considered in [34, 35, 36] when v and m are both odd, and [18, 19, 20, 22, 37] when they are both even. However, the case when m and v have opposite parities appears to be completely open. We summarize these results in the following theorem.

Theorem 4.1 ([20, 35])**.** *If $v \equiv m \pmod 2$, there exists a UMSF$(v, m, 3)$ if and only if $v \geq 3m$, $v \equiv m \equiv 0 \pmod 3$, and if v, m are even, then $v, m > 12$.*

We will find the following results useful. A corollary of a result in [4] yields the following.

Theorem 4.2 ([4])**.** *If G is a Hamiltonian decomposable graph, then $G[n]$ is also Hamiltonian decomposable. In particular, $C_m[n]$ has a C_{mn}-factorization for every $m \geq 3$.*

Piotrowski [33] has shown the following result for $m \geq 4$. The case $m = 3$ is covered by Theorem 1.2.

Theorem 4.3 ([33])**.** *There exists a C_m-factorization of $C_m[n]$, except if $n = 2$ and m is odd, or when $(m, n) = (3, 6)$.*

Piotrowski [33] has also shown the following result.

Theorem 4.4 ([33])**.** *Let F be a bipartite 2-regular graph of order $2n$. The complete bipartite graph $K_2[n]$ has an F-factorization if and only if n is even, except when $n = 6$ and F consists of two 6-cycles.*

We now give some recursive constructions for uniform minisymposium factorizations.

Theorem 4.5. *Let $m \geq k \geq 3$ and $t \geq 2$ be integers. If $(t-1)m$ is even and $k \mid m$, then there is a UMSF(mt, m, k), except that there is no UMSF$(6t, 6, 3)$ UMSF$(12t, 12, 3)$, UMSF$(12, 6, 6)$, or UMSF$(2m, m, k)$ when k is odd.*

Proof. The non-existence of a UMSF$(6t, 6, 3)$ and UMSF$(12t, 12, 3)$ are covered by Theorem 4.1. Since a UMSF$(2m, m, k)$ is equivalent to a C_k-factorization of the complete bipartite graph $K_2[m]$, it clearly does not exist when the cycle length k is odd, or when $k = m = 6$ by Theorem 4.4. In all remaining cases, the following conditions simultaneously hold:

1. $(m,k) \notin \{(6,3),(12,3)\}$,
2. $(t,m,k) \neq (2,6,6)$,
3. if k is odd, then $t > 2$.

The assumptions of Theorems 1.1 and 1.2 are then satisfied. Hence there is a C_k-factorization of $K_t[m]$ and a C_k-factorization of K_m^*, which we use to fill in the parts of size m in $K_t[m]$. This completes the proof. □

Considering the necessary conditions in Theorem 2.4, we get the following corollaries.

Corollary 4.6. *Suppose that either k is even or v is odd, and $m \mid v$. Then there exists a UMSF(v,m,k) if and only if $k \mid m$, $v \geq 2m$ when k is even, and $v \geq 3m$ when k is odd, except that UMSF$(v,6,3)$, UMSF$(v,12,3)$ and UMSF$(12,6,6)$ do not exist.*

Proof. Taking $v = mt$, Theorem 2.4 gives the necessary conditions $k \mid m$ and $v \geq 2m$ when k is even. When k is odd, the necessary condition from Theorem 2.4 is $v \geq \frac{2mk}{k-1}$, but since $m \mid v$, this implies $k \geq 3m$. Given the conditions of k and v, the sufficiency comes from Theorem 4.5. □

We note that if $m \mid v$ this corollary completely solves all cases except when k is odd and v is even. One case of particular interest is when $k = m$, in this case $m \mid v$ is necessary.

Corollary 4.7. *Let $m(t-1)$ be even. Then a UMSF(tm,m,m) exists if and only if $t \geq 2$ when m is even, $t \geq 3$ when m is odd, and $(t,m) \neq (2,6)$.*

The previous results all require $m \mid v$, however the next theorem allows us to recursively construct solutions to cases where m does not divide v.

Theorem 4.8. *Assume there is a UMSF(v,m,k) and let $t \geq 1$. Then there exists a UMSF(vtk,mtk,ℓ), with $\ell \in \{k,kt\}$, in each of the following cases:*

(1) *v and m have the same parity;*

(2) *v and t are even, $\ell = tk$, and m and k are both odd, except possibly when $(k,t) = (3,2)$.*

Proof. Letting $w \in \{m,v\}$, we factorize K_{wtk}^* into $\Gamma_w = K_w^*[tk]$ and $\overline{\Gamma}_w = K_{wtk}^* - \Gamma_w$. Note that $\overline{\Gamma}_w$ is the vertex disjoint union of

(1) w copies of K_{tk}^* when w is odd, or

(2) $w/2$ copies of K_{2tk}^* when w is even.

Without loss of generality, we can assume that $\Gamma_m \subseteq \Gamma_v$ and $\overline{\Gamma}_m \subseteq \overline{\Gamma}_v$, except when v is odd and m is even. In this case the components of $\overline{\Gamma}_m$ are copies of K_{2tk}^*, while those of $\overline{\Gamma}_v$ are isomorphic to K_{tk}^*, therefore $\overline{\Gamma}_m \subseteq \overline{\Gamma}_v$ cannot hold. We proceed by constructing

(a) a C_ℓ-factorization of Γ_v containing a C_ℓ-factorization of Γ_m, and

(b) a C_ℓ-factorization of $\overline{\Gamma}_v$ containing a C_ℓ-factorization of $\overline{\Gamma}_m$, which together will provide the desired $UMSF(vtk, mtk, \ell)$.

We blow up each vertex of the UMSF(v, m, k) by tk, to obtain a $C_k[tk]$-factorization of Γ_v containing a $C_k[tk]$-factorization of Γ_m. To construct (a) it is therefore enough to factorize $C_k[tk]$ into C_ℓ-factors, $\ell \in \{k, tk\}$. By Theorem 4.3 there is a C_k-factorization of $C_k[tk]$, except when $(t, k) = (2, 3)$. In this case, the desired UMSF$(6v, 6m, 3)$ exists by Theorem 4.1. Considering that $C_k[tk] = C_k[t][k]$, by Theorem 4.2 there exists a C_{kt}-factorization of $C_k[t]$ which we blow up by k to obtain $C_{kt}[k]$-factorization of $C_k[tk]$. By Theorem 4.3, each $C_{kt}[k]$-factor can be further decomposed into C_{kt}-factors yielding a C_{kt}-factorization of $C_k[tk]$.

It is left to construct (b). If m and v have the same parity, the components of $\overline{\Gamma}_m$ and $\overline{\Gamma}_v$ are pairwise isomorphic: they are copies of K^*_{tk} or K^*_{2tk}. It is then enough to build a C_ℓ-factorization of K^*_{tk} and K^*_{2tk} for $\ell \in \{k, kt\}$. They exist by Theorem 1.1 except when $\ell = k = 3$ and one of the following two conditions hold,

1. mv is odd and $t \in \{2, 4\}$, or

2. m and v are even, and $t \in \{1, 2\}$.

In each of these cases, the existence of the desired $UMSF(3vt, 3mt, 3)$ is guaranteed by Theorem 4.1.

If v and t are even, $\ell = tk$, and both m and k are odd, the components of $\overline{\Gamma}_m$ are isomorphic to K^*_{tk}, while those of $\overline{\Gamma}_v$ are isomorphic to K^*_{2tk}. Since we can factorize K^*_{2tk} into $K_2[tk]$ and two copies of K^*_{tk}, it is enough to decompose both $K_2[tk]$ and K^*_{tk} into C_{tk}-factors. These factorizations exist by Theorem 4.4 and Theorem 1.1, respectively, except possibly when $(k, t) = (3, 2)$. □

We may now use the result on triples (Theorem 4.1) to obtain the following.

Corollary 4.9. *Let $v \equiv m \equiv 0, 3 \pmod 6$, with $v \geq 3m$ and $m \notin \{0, 6, 12\}$. Then there exists a UMSF$(3tv, 3tm, 3t)$ for all $t > 0$.*

Additionally, we may use Theorem 3.5 to obtain the following result.

Corollary 4.10. *Let $3 < k$, $k \equiv 2 \pmod 4$, $v \equiv m \equiv k \pmod{2k}$ and $v \geq 2m$. Then there exists a UMSF(vtk, mtk, k) and a UMSF(vtk, mtk, tk) for all $t > 0$.*

We note that the above result can be used to obtain UMSF's with cycle length, subsystem size or number of vertices congruent to $0 \pmod 4$ by taking t even. However, in all cases, the number of vertices and subsystem size will be divisible by k^2.

5 Multiple subsystems

A natural question to ask is if a system can have multiple subsystems. In general, it seems likely to be hard to navigate through the lattice of subsystems and all the possible ways the subsystems can be distributed across the main system. However, when the subsystems are disjoint, have small common intersections or are nested, the problem is more tractable. We give some preliminary results in the next three subsections.

5.1 Disjoint subsystems

In the uniform case, the flexibility of Theorem 1.2 allows us to create a large number of disjoint subsystems. We refer to a factorization of K_v^* into k-cycles with subsystems on disjoint vertex sets of sizes m_j for $1 \leq j \leq n$ as a UMSF$(v, \{m_j\}, k)$.

Lemma 5.1. *Let $k \mid m_j$ for $1 \leq j \leq n$. Let m be an integer such that there is a UMSF(m, m_j, k) for each $1 \leq j \leq n$. Then there exists a UMSF$(ms, \{m_j\}, k)$ for all $s \geq \max\{2, n\}$ if k is even, and for all $s \geq \max\{3, n\}$ such that $(s-1)m$ is even if k is odd, except when $(k, s, m) = (6, 2, 6)$.*

Proof. Theorem 1.2 guarantees the existence of a C_k-factorization of $K_s[m]$. For each $1 \leq j \leq n$, place a copy of the ingredient UMSF(m, m_j, k) on the jth part of size m of $K_s[m]$, and any C_k-factorization of K_m^* on each of the remaining parts. The definite exception $(k, s, m) = (6, 2, 6)$ follows from the non-existence of a UMSF$(12, 6, 6)$ (see Theorem 4.5). □

As with the uniform factorizations containing a single subsystem in this paper, the easiest case is when $m_j \mid m$ for all $1 \leq j \leq n$ and either k is even or m is odd.

Corollary 5.2. *Let $m = \operatorname{lcm}\{m_j : j = 1, \ldots, n\}$, and assume the following conditions are all satisfied:*

(1) $k \mid m_j$ *for all* $j \in \{1, \ldots, n\}$ *and* $m \mid v$;

(2) $k(m-1)$ *is even;*

(3) *if* $k = 3$, *then* $m_j \notin \{6, 12\}$ *for all* j;

(4) *if* $(k, m) = (6, 12)$, *then* $m_j \neq 6$ *for all* j;

(5) $(v, m, k) \neq (12, 6, 6)$;

(6) $v/m \geq \max\{2, n\}$ *if* k *is even;*

(7) $v/m \geq \max\{3, n\}$ *and* v *is odd if* k *is odd.*

Then there exists a UMSF$(v, \{m_j\}, k)$.

Proof. Since each m_j is a divisor of m and conditions (1)–(4) hold, we can apply either Theorem 1.1 or Corollary 4.6, as needed, to ensure the existence of a UMSF(m, m_j, k) for every $j \in \{1, \ldots, n\}$. These are the ingredient designs needed in Lemma 5.1, which can be applied in view of conditions (5)–(7). □

We note that for any fixed multiset of m_j, this corollary constructs UMSF$(v, \{m_j\}, k)$ for all but a finite number of v permitted by the necessary conditions when $m_j \mid v$ and either k is even or m is odd.

5.2 Scattered subsystems

The proof of Lemma 5.1 builds systems whose factors intersect either all of the subsystems or none of them. A balancing of the sizes of these intersections could be an interesting property. For instance, we could ask for systems whose factors do not intersect more than one subsystem. In other words, we ask for a C_k-factorization $\mathcal{F}$ of K_v^* that contains n subsystems of sizes $m_1, m_2, \dots, m_n$, such that no two factors of any of the subsystems are contained in the same factor of $\mathcal{F}$. We denote such a factorization by UMSF$(v, [m_j], k)$ and say that the subsystems are *scattered*.

Partial results in this direction can be easily obtained by making use of cycle frames. We recall that a *k-cycle frame* (k-CF) of $K_s[m]$ is a decomposition of $K_s[m]$ into holey C_k-factors; a *holey C_k-factor* is a vertex-disjoint union of k-cycles covering all vertices $K_s[m]$ except those belonging to one part. The following result, proven in [8], provides necessary and sufficient conditions for the existence of k-cycle frames.

Theorem 5.3 ([8])**.** *Let $m \geq 2$ and $k, s \geq 3$. There exists a k-cycle frame of $K_s[m]$ if and only if m is even, $m(s-1) \equiv 0 \pmod{k}$, k is even when $s = 3$, and $(k, m, s) \neq (6, 6, 3)$.*

By making use of Theorem 5.3, we obtain the following.

Lemma 5.4. *Let $u \in \{1, 2\}$. If there exists a UMSF$(2m+u, m_j, k)$ for each $1 \leq j \leq n$, then there exists a UMSF$(2ms+u, [m_j], k)$ whenever $2s \equiv 2 \pmod{k}$ and $s \geq n$.*

Proof. Let $n \geq 1$, $2s \equiv 2 \pmod{k}$ and $s \geq n$. It follows that $2m(s-1) \equiv 0 \pmod{k}$, $k = 4$ when $s = 3$, and $(k, 2m, s) \neq (6, 6, 3)$. Therefore, Theorem 5.3 guarantees the existence of a k-cycle frame $\mathcal{F}$ of $K_s[2m]$. Let P_i denote the i-th part of $K_s[2m]$, for $1 \leq i \leq s$. Also, let

$$\mathcal{F} = \{F_{ij} : \ 1 \leq i \leq s, 1 \leq j \leq m\},$$

where the F_{ij}s are the holey C_k-factors of $\mathcal{F}$ missing P_i, for $1 \leq i \leq s$. By assumption, there is a UMSF$(2m+u, m_j, k)$ on $P_i \cup \{\infty_1, \infty_u\}$, say $\mathcal{H}_i = \{H_{ij} : \ 1 \leq j \leq m\}$. It follows that $\mathcal{F}^* = \{F_{ij} \cup H_{ij} : \ 1 \leq i \leq s, 1 \leq j \leq m\}$ is a C_k-factorization of K_{2ms+u} with scattered subsystems of sizes $m_1, m_2, \dots, m_n$. Indeed, the factors of the subsystems belong to the H_{ij}s, each of which belongs to exactly one factor of $\mathcal{F}^*$. □

In the UMSF$(2ms+u, [m_j], k)$ constructed in the proof of Lemma 5.4, two subsystems may intersect in 0, 1, or 2 vertices, which are necessarily in the set $\{\infty_1, \infty_u\}$.

Theorem 4.5 provides sufficient conditions for the existence of a UMSF(v, m, k) if m is a divisor of v. From that, we easily obtain the following corollary.

Corollary 5.5. *Let $u \in \{1, 2\}$, $k \geq 3$, and let $k \mid m_j \mid (2m+u)$ for each $1 \leq j \leq n$. Then there exists a UMSF$(2ms+u, [m_j], k)$ whenever the following conditions hold:*

(1) *m_j is even or $(2m+u)/m_j$ is odd,*

(2) *$2n \leq 2s$ and $2s \equiv 2 \pmod{k}$,*

except when $(m_j, k) \in \{(6, 3), (12, 3)\}$, and except possibly when $(2m+u, m_j, k) = (12, 6, 6)$, or k is odd and $2m+u = 2m_j$, for some $j \in \{1, \dots, n\}$.

Note that for values of the triple $(2m+u, m_j, k)$ determining a possible exception in Corollary 5.5 it is possible for a UMSF$(2ms+u, [m_j], k)$ to exist. However, our method cannot construct them because the UMSF$(2m+u, m_j, k)$ to use in the construction does not exist. It is possible that other construction methods would build a UMSF$(2ms+u, [m_j], k)$.

5.3 Nested subsystems

A scenario complementary to the susbsystems being all disjoint is when the subsystems are completely nested, on vertex sets $M_1 \supset M_2 \supset \cdots \supset M_{n-1}$. We modify our notation slightly for this section to make it less cumbersome in this specific context.

Definition 5.6. Let $v = m_0 > m_1 > \cdots > m_{n-1} > m_n = 0$ be non-negative integers. For $1 \le i \le \lfloor \frac{m_0-1}{2} \rfloor$ and $0 \le j < n$, let $U_{i,j}$ be a 2-regular graph of order

$$|V(U_{i,j})| = \begin{cases} m_j - m_{j+1}, & \text{if } 1 \le i \le \lfloor \frac{m_{j+1}-1}{2} \rfloor, \\ m_j, & \text{if } \lfloor \frac{m_{j+1}+1}{2} \rfloor \le i \le \lfloor \frac{m_j-1}{2} \rfloor, \\ 0, & \text{if } \lfloor \frac{m_j+1}{2} \rfloor \le i \le \lfloor \frac{m_0-1}{2} \rfloor. \end{cases}$$

A *nested minisymposium factorization* nMSF$(\{U_{i,j} :\ 1 \le i \le \lfloor \frac{m_0-1}{2} \rfloor,\ 0 \le j < n\})$ is a 2-factorization $\mathcal{F} = \{F_i : 1 \le i \le \lfloor \frac{m_0-1}{2} \rfloor\}$ of K_v^* such that

- $V(K_v^*) = M_0 \supset M_1 \supset \cdots \supset M_{n-1} \supset M_n = \varnothing$ are nested sets with $|M_j| = m_j$;
- each 2-factor $F_i = \bigcup_{j=0}^{n-1} F_{i,j}$, where

$$V(F_{i,j}) = \begin{cases} M_j \setminus M_{j+1} & \text{if } 1 \le i \le \lfloor \frac{m_{j+1}-1}{2} \rfloor \\ M_j & \text{if } \lfloor \frac{m_{j+1}+1}{2} \rfloor \le i \le \lfloor \frac{m_j-1}{2} \rfloor \\ \varnothing & \text{otherwise,} \end{cases}$$

  and each $F_{i,j}$ is isomorphic to $U_{i,j}$;
- for every $0 \le \ell < n$, $\{\bigcup_{j=\ell}^{n-1} F_{i,j} : 1 \le i \le \lfloor \frac{m_\ell-1}{2} \rfloor\}$ is a 2-factorization of a graph isomorphic to $K_{m_\ell}^*$.

In other words, the factorization $\mathcal{F}$ of K_v^* restricted to vertex set M_ℓ factorizes a graph isomorphic to $K_{m_\ell}^*$ into 2-factors whose structure is determined by the $U_{i,j}$s.

Our construction of nested minisymposium factorizations is most tidily expressed by defining holey factorizations.

Definition 5.7. Given positive integers v and m with $v \ge m$, let

$$\mathcal{U} = \left\{ U_i :\ 1 \le i \le \left\lfloor \frac{v-1}{2} \right\rfloor \right\},$$

be a collection of 2-regular graphs on $v - m$ vertices for $1 \le i \le \lfloor \frac{m-1}{2} \rfloor$, and on v vertices for $\lfloor \frac{m+1}{2} \rfloor \le i \le \lfloor \frac{v-1}{2} \rfloor$. A *holey factorization* HF$(\mathcal{U})$ is a decomposition $\mathcal{F} = \{F_i :\ 1 \le i \le \lfloor \frac{v-1}{2} \rfloor\}$ of $K_v^* - G$ (i.e., K_v^* minus the edges of G) where each $F_i \cong U_i$ and $G \cong K_m^*$.

If v is even, then there is a 1-factor, I_v, on the vertices of K_v^* whose edges are not present in $K_v^* - G$. If v and m are both even there is a 1-factor, J_m, on the vertices of G which is a subgraph of I_v. If v is even and m odd, then no edges of I_v are induced on the vertices of G. If v is odd and m is even, then there is a 1-factor J_m on the vertices of G whose edges are present in $K_v^* - G$.

By removing the 2-factors of a subsystem or "filling the hole" with them (making the J_m in the hole coincide with the I_m of the subsystem as required by the parities of v and m) we have an equivalence between the existence of minisymposium factorizations and holey factorizations.

Theorem 5.8. *Let $\mathcal{T} = \{T_i : 1 \le i \le \lfloor \frac{m-1}{2} \rfloor\}$ be a 2-factorization of K_m^* and*

$$\mathcal{U} = \left\{ U_i : 1 \le i \le \left\lfloor \frac{v-1}{2} \right\rfloor \right\},$$

be a collection of 2-regular graphs on $v - m$ vertices for $1 \le i \le \lfloor \frac{m-1}{2} \rfloor$ and on v vertices for $\lfloor \frac{m+1}{2} \rfloor \le i \le \lfloor \frac{v-1}{2} \rfloor$. Then a HF($\mathcal{U}$) exists if and only if a

$$\text{MSF}(\{U_i : \lfloor \tfrac{m+1}{2} \rfloor \le i \le \lfloor \tfrac{v-1}{2} \rfloor\}, \{(T_i, U_i) : 1 \le i \le \lfloor \tfrac{m-1}{2} \rfloor\})$$

exists.

Because in a nested minisymposium factorization the holes are nested and emptying or filling them does not affect the edges outside the hole, this equivalence extends to nested minisymposium factorizations and shows that they can be constructed exactly when the various holey factorizations of $K_{m_j}^*$ with holes of size m_{j+1} exist.

Theorem 5.9. *Let $v = m_0 > m_1 > \cdots > m_{n-1} > m_n = 0$ be positive integers. For $1 \le i \le \lfloor \frac{m_0-1}{2} \rfloor$ and $0 \le j < n$, let $U_{i,j}$ be a 2-regular graph of order*

$$|V(U_{i,j})| = \begin{cases} m_j - m_{j+1}, & \text{if } 1 \le i \le \lfloor \frac{m_{j+1}-1}{2} \rfloor, \\ m_j, & \text{if } \lfloor \frac{m_{j+1}+1}{2} \rfloor \le i \le \lfloor \frac{m_j-1}{2} \rfloor, \\ 0, & \text{if } \lfloor \frac{m_j+1}{2} \rfloor \le i \le \lfloor \frac{m_0-1}{2} \rfloor. \end{cases}$$

A nested minisymposium factorization nMSF($\{U_{i,j} : 1 \le i \le \lfloor \frac{m_0-1}{2} \rfloor,\ 0 \le j < n\}$) exists if and only if a HF($\{U_{i,j} : 1 \le i \le \lfloor \frac{m_j-1}{2} \rfloor\}$) exists for each $0 \le j < n$.

Proof. The forward direction is proved simply by restricting the system to M_j and removing the subsystem on M_{j+1}. The converse is proved by a recursive construction starting with $j = n-1$: in this case, a HF($\{U_{i,n-1} : 1 \le i \le \lfloor \frac{m_{n-1}-1}{2} \rfloor\}$) is an nMSF($\{U_{i,n-1} : 1 \le i \le \lfloor \frac{m_{n-1}-1}{2} \rfloor\}$), say $\mathcal{F}_{n-1}$.

At stage $j < n-1$, use Theorem 5.8 to construct an nMSF($\{U_{i,\ell} : 1 \le i \le \lfloor \frac{m_j-1}{2} \rfloor,\ j \le \ell < n\}$), say $\mathcal{F}_j$, by filling the hole in the HF($\{U_{i,j} : 1 \le i \le \lfloor \frac{m_j-1}{2} \rfloor\}$) with the nMSF($\{U_{i,\ell} : 1 \le i \le \lfloor \frac{m_{j+1}-1}{2} \rfloor,\ j+1 \le \ell < n\}$), denoted by $\mathcal{F}_{j+1}$, built at stage $j+1$. □

Between the extremes of disjoint and nested subsystems, there are factorizations with multiple subsystems with arbitrary intersections. Some structured instances of this much more general problem may be amenable to solution but we leave this to future work.

6 Conclusions and further work

We have introduced the minisymposium problem: a subsystem variant of the generalized Oberwolfach problem. This variant asks for a solution to a generalized Oberwolfach problem that contains a subsystem of a given size. When v, the number of vertices, is even, it is traditional in 2-factor decomposition problems to ask for decompositions of $K_v - I$ where I is a 1-factor. When the number of vertices in the system and the subsystem are both even, then we require that the 1-factor in the subsystem be a subgraph of the 1-factor in the full system. Therefore when the parities of the system and the subsystem agree, the problem becomes more tractable. When the parities are opposite, either the 1-factor of the full system must avoid the subsystem, or the edges of the 1-factor in the subsystem must be in 2-factors of the whole system.

Clearly, this is a very broad statement and we identify some particularly interesting cases, Hamiltonian* and uniform. In the Hamiltonian* minisymposium problem there are as few cycles as possible and in the uniform minisymposium problem all cycles are of the same length. We have shown that the work of Hilton and Johnson provides a complete solution for the Hamiltonian* minisymposium problem in Corollary 3.2. In the case when $v \equiv m \equiv 2 \pmod 4$, Theorem 3.4 uses this Hamiltonian* construction to provide a wide range of solutions when the resulting factors are all bipartite. In particular, a uniform factorization with $k \equiv 2 \pmod 4$, $v \geq 2m$ and $v \equiv m \equiv k \pmod{2k}$ always exists. Corollary 4.10 can be used to extend this to uniform factorizations where $k \equiv 0 \pmod 4$ or $v \equiv m \equiv 0 \pmod{2k}$.

In Section 4 we considered the uniform case. We have solved a large part of the spectrum. In particular, when k is even or v is odd, Corollary 4.6 gives all cases when $m \mid v$ and Corollary 4.7 completely solves all cases when $k = m$ has the same parity as v. Theorem 4.8 gives a powerful recursive construction which is applicable in cases where m does not divide v. By applying it to the case when $k = 3$, we obtain uniform factorizations with cycle lengths divisible by 3. The case when m is odd and v is even seems to be the hardest. Even in the simplest case when $k = 3$, which has been well studied otherwise [18, 19, 20, 22, 34, 35, 36, 37], the case with v and m having opposite parities has not been previously considered and remains open.

While the Hamiltonian* problem is solved and we have made significant inroads into the uniform case, the general problem remains wide open. We expect that when m and v have the same parity solutions will be easier to find. When m and v have opposite parity we expect that v odd with m even is more tractable than the reverse. Considering 2-factorizations where a solution to the Oberwolfach problem is known might be a good starting point. A natural case to consider is the case when all factors are isomorphic i.e. $F_i \cong T_j \cup U_j$ for all i and j. Uniform factorizations are an example of this, but other variations are possible, for example, requiring all factors to be isomorphic to $C_{v-m} \cup C_m$. Indeed, Theorem 3.4 solves all these cases when the factors are bipartite and $v \equiv m \equiv 2 \pmod 4$, but this broader variant remains open.

More complex variants can also be considered. We have briefly considered systems with multiple subsystems. When these subsystems are completely nested the problem essentially reduces to the existence of the necessary ingredients as described in Theorem 5.9. Let $\{m_j\}$ be a multiset of subsystem sizes. When the subsystems are pairwise disjoint, v is divisible by each m_j and either v is odd or at least one subsystem is even, then Lemma 5.1 and Corollary 5.2 use Theorem 1.2 to construct a UMSF$(v, \{m_j\}, k)$ for all but a finite number of admissible v. Even in the seemingly simple case when the subsystems are all disjoint the problem remains generally open even for the uniform case. Further partial results are obtained when the subsystems are scattered, that is, when no two minisymposia have meetings taking place on the same day. Cycle frames in Theorem 5.3 allow us to construct uniform factorizations as described in Lemma 5.4 and Theorem 5.5. The more general case when the intersections of multiple subsystems are arbitrary is completely open.

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Two kinds of partial Motzkin paths with air pockets*

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Abstract

Motzkin paths with air pockets (MAP) are defined as a generalization of Dyck paths with air pockets by allowing some horizontal steps with certain conditions. In this paper, we introduce two generalizations. The first one consists of lattice paths in $\mathbb{N}^2$ starting at the origin, made of steps $U = (1,1)$, $D_k = (1,-k)$, $k \geqslant 1$ and $H = (1,0)$, where two down steps cannot be consecutive, while the second one are lattice paths in $\mathbb{N}^2$ starting at the origin, made of steps U, D_k and H, where each step D_k and H is necessarily followed by an up step, except for the last step of the path. We provide enumerative results for these paths according to the length, the type of the last step, and the height of its end-point. A similar study is made for these paths read from right to left. As a byproduct, we obtain new classes of paths counted by the Motzkin numbers. Finally, we express our results using Riordan arrays.

Keywords: Enumeration, Motzkin paths, kernel method, Riordan array.

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1 Introduction

In a recent paper [2], the authors introduce, study and enumerate special classes of lattice paths, called *Dyck paths with air pockets* (DAP for short). Such paths are non empty lattice paths in the first quadrant of $\mathbb{Z}^2$ starting at the origin, and consisting of up-steps $U = (1,1)$ and down-steps $D_k = (1,-k)$, $k \geqslant 1$, where two down steps cannot be consecutive.

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The *length* of a path is the number of its steps. These paths can be viewed as ordinary Dyck paths where each maximal run of down-steps is condensed into one large down step. As mentioned in [2], they also correspond to a stack evolution with (partial) reset operations that cannot be consecutive (see for instance [7]). Whenever the last point is on the x-axis, they prove that DAP of length n are in one-to-one correspondence with the peakless Motzkin paths of length $n-1$. They also investigate the popularity of many patterns in these paths and they give asymptotic approximations. We also refer to [10] for the enumeration of DAP with respect to the length, the type (up or down) of the last step and the height of the end-point (i.e., the ordinate of the end-point). In a second work [3], the authors make a study for a generalization of these paths by allowing them to go below the x-axis. They call these paths Grand Dyck paths with air pockets (GDAP), and they also yield enumerative results for these paths according to the length and several restrictions on the height.

In this paper, we introduce two generalizations of partial Dyck paths of air pockets by allowing some possible horizontal steps $H=(1,0)$ with some conditions. These two kinds of paths can be viewed as special partial Motzkin paths (lattice paths in $\mathbb{N}^2$ starting at the origin and made of $U=(1,1)$, $D=(1,-1)$, and $H=(1,0)$), where each maximal run of down-steps is condensed into one large down step.

Definition 1.1. A partial Motzkin path with air pocket of the first kind is a lattice path in $\mathbb{N}^2$ starting at the origin, consisting of steps U, D_k and H, where two down steps cannot be consecutive. Let $\mathcal{M}_1$ be the set of these paths.

Definition 1.2. A partial Motzkin path with air pocket of second kind is a lattice path in $\mathbb{N}^2$ starting at the origin, consisting of steps U, D_k and H, where any step H and D_k (except the last step of the path) is immediately followed by an up step U. Let $\mathcal{M}_2$ be the set of these paths.

Whenever these paths end on the x-axis, we call them *Motzkin paths with air pockets* of the first and second kinds, respectively. For short, we denote by PMAP all paths in $\mathcal{M}_i$, $i\in\{1,2\}$, and we denote by MAP all paths ending on the x-axis.

For instance, $UUUDHUD_2\in\mathcal{M}_1$ and $UUUDUHUD_2\in\mathcal{M}_2$ are two PMAP of the first and second kinds, respectively. The paths $UUUDHUD_3UD$ and $UUUDUHUD_4$ UD are MAP of the first and second kinds, respectively. See also Figure 1 and Figure 5 for other examples of PMAP and MAP.

We also consider lattice paths obtained from MAP in $\mathcal{M}_1$ and $\mathcal{M}_2$ by reading them from right to left and by replacing down-steps with up-steps and *vice versa*. More precisely, we define the two sets $\mathcal{M}'_1$ and $\mathcal{M}'_2$ as follows.

Definition 1.3. Let $\mathcal{M}'_1$ be the set of paths in $\mathbb{N}^2$ starting at the origin, consisting of up steps $U_k=(1,k)$, $k\geqslant 1$, horizontal step H and down step D, where two up steps cannot be consecutive.

Definition 1.4. Let $\mathcal{M}'_2$ be the set of paths in $\mathbb{N}^2$ starting at the origin, consisting of up steps $U_k=(1,k)$, $k\geqslant 1$, horizontal step H and down step D, where any H and U_k, $k\geqslant 1$, (except the first step of the path) is preceded by a down step D.

All paths in $\mathcal{M}'_1$ and $\mathcal{M}'_2$ will be be called *PMAP from right to left*, and in the case where they end on the x-axis, we call them *MAP from right to left*. For instance, we have $UDU_3DHUD\in\mathcal{M}'_1$, $UDU_4DHDU\in\mathcal{M}'_2$, and the path $UDU_3DHUDDD$

(resp. $UDU_4DHDUDDD$) is a MAP of the first (resp. second) kind from right to left. See also Figure 3 and Figure 7 for other examples of PMAP and MAP from right to left. Notice that $D_1 = D$ and $U_1 = U$, and we will use both notations.

Throughout the paper, and for each class of paths $\mathcal{M}_i$ and $\mathcal{M}'_i$, $i \in \{1, 2\}$, described above, we will use the following notations. For $k \geqslant 0$, we consider the generating function $f_k = f_k(z)$ (resp. $g_k = g_k(z)$, resp. $h_k = h_k(z)$), where the coefficient of z^n in the series expansion is the number of partial Motzkin paths with air pockets of length n ending at height k with an up-step, (resp. with a down-step, resp. with a horizontal step H).

We introduce the bivariate generating functions

$$F(z,u) = \sum_{k\geqslant 0} u^k f_k(z), \quad G(z,u) = \sum_{k\geqslant 0} u^k g_k(z), \text{ and } H(z,u) = \sum_{k\geqslant 0} u^k h_k(z).$$

For short, we also use the notation $F(u), G(u)$ and $H(u)$ for these functions, and we introduce the generating function

$$\texttt{Total}(z,u) = F(u) + G(u) + H(u).$$

Finally, for any bivariate generating function $H(z,u) = H(u)$, we will use the notation $[u^k]H(u)$ for the coefficient of u^k in $H(u)$.

The outline of this paper is the following. In Section 2, we present enumerative results for partial Motzkin paths with air pockets of the first kind, and for these paths when we read them from right to left. We provide bivariate generating functions that count these paths with respect to the length, the type of the last step (up, down or horizontal step) and the height of the end-point. In Section 3, we make a similar study for PMAP of second kind, and we present new classes of lattice paths counted by the well known Motzkin numbers. All these results are obtained algebraically by using the famous kernel method for solving several systems of functional equations. More precisely, Sections 2.1, 2.2, 3.1 and 3.2 have the same structure: we show how to obtain a system of equations involving f_k, g_k and h_k, and we apply the kernel method in order to provide some expressions of $F(u)$, $G(u)$ and $H(u)$. Finally, in Section 4 we express our results using Riordan arrays and we deduce closed forms for PMAP of length n ending at height k.

2 PMAP of the first kind

In this section, we focus on PMAP of the first kind, i.e. lattices paths in $\mathbb{N}^2$ starting at the origin, made of steps U, D_k and H, such that two down steps cannot be consecutive. The first subsection considers the paths in $\mathcal{M}_1$, while the second subsection handles the paths in $\mathcal{M}'_1$ (see Introduction for the definition of these two sets). We yield enumerative results for these paths according to the length, the type of the last step, and the height of its end-point.

2.1 PMAP in $\mathcal{M}_1$ - From left to right

In this part, we consider PMAP in $\mathcal{M}_1$. Figure 1 shows two examples of such paths.

Let P be a length n PMAP in $\mathcal{M}_1$ ending at height $k \geqslant 0$. If the last step of P is U, then $k \geqslant 1$ and P can be written $P = QU$ where Q is a length $(n-1)$ PMAP ending at height $k-1$. So, we obtain the first relation $f_k = zf_{k-1} + zg_{k-1} + zh_{k-1}$ for $k \geqslant 1$, anchored with $f_0 = 1$ by considering the empty path. If the last step of P is a down step D_a, $a \geqslant 1$, then we have $P = QD_a$ where Q is a length $(n-1)$ PMAP ending at height

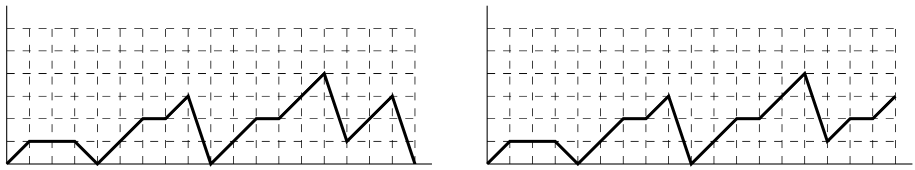

Figure 1: The left drawing shows a Motzkin path with air pockets of length 18 in $\mathcal{M}_1$. The right drawing shows a partial Motzkin path with air pockets of length 18 ending at height 3 in $\mathcal{M}_1$.

$\ell = a + k \geqslant k + 1$ with no down step at its end. So, we obtain the second relation $g_k = z\sum_{\ell\geqslant k+1} f_\ell + z\sum_{\ell\geqslant k+1} h_\ell$. If the last step of P is a horizontal step H, then we have $P = QH$ where Q is a length $(n-1)$ PMAP ending at height k, which implies $h_k = zf_k + zg_k + zh_k$.

Therefore, we have to solve the following system of equations.

$$\begin{cases} f_0 = 1, \text{ and } f_k = zf_{k-1} + zg_{k-1} + zh_{k-1}, \quad k \geqslant 1, \\ g_k = z \sum\limits_{\ell\geqslant k+1} f_\ell + z \sum\limits_{\ell\geqslant k+1} h_\ell, \quad k \geqslant 0, \\ h_k = zf_k + zg_k + zh_k, \quad k \geqslant 0. \end{cases} \tag{2.1}$$

Summing the recursions in (2.1), we have:

$$\begin{aligned} F(u) &= 1 + z\sum_{k\geqslant 1} u^k f_{k-1} + z\sum_{k\geqslant 1} u^k g_{k-1} + z\sum_{k\geqslant 1} u^k h_{k-1} \\ &= 1 + zuF(u) + zuG(u) + zuH(u), \\ G(u) &= z\sum_{k\geqslant 0} u^k \Big(\sum_{\ell\geqslant k+1} f_\ell\Big) + z\sum_{k\geqslant 0} u^k \Big(\sum_{\ell\geqslant k+1} h_\ell\Big) \\ &= z\sum_{k\geqslant 1} f_k(1 + u + \ldots + u^{k-1}) + z\sum_{k\geqslant 1} h_k(1 + u + \ldots + u^{k-1}) \\ &= z\sum_{k\geqslant 1} \frac{u^k - 1}{u-1} f_k + z\sum_{k\geqslant 1} \frac{u^k - 1}{u-1} h_k \\ &= \frac{z}{u-1}(F(u) - F(1) + H(u) - H(1)), \\ H(u) &= zF(u) + zG(u) + zH(u). \end{aligned}$$

Notice that we have $F(1) - H(1) = 1$ by considering the difference of the first and third equations. Now, setting $a := F(1)$ and solving these functional equations, we obtain

$$\begin{aligned} F(u) &= \frac{2au\,z^2 - u\,z^2 + uz + z^2 - u - z + 1}{u^2 z + u\,z^2 + z^2 - u - z + 1}, \\ G(u) &= -\frac{z\,(2auz + 2az - uz - 2a - z + 2)}{u^2 z + u\,z^2 + z^2 - u - z + 1}, \\ H(u) &= \frac{z\,(2az - u - 2z + 1)}{u^2 z + u\,z^2 + z^2 - u - z + 1}. \end{aligned}$$

In order to compute $a = F(1)$, we use the kernel method (see [1, 9]) on $F(u)$. This method consists in cancelling the denominator by finding u as an algebraic function of z, $s(z)$. So, if we substitute u with $s(z)$ in the numerator, then it necessarily equals zero (in order to counterbalance the cancellation of the denominator), which allows to find $a = F(1)$. Finally, we can deduce the generating function $F(u)$.

We can write the denominator (which is a polynomial in u of degree 2), as $z(u-r)(u-s)$ with

$$r = \frac{1 - z^2 + \sqrt{z^4 - 4z^3 + 2z^2 - 4z + 1}}{2z},$$
$$s = \frac{1 - z^2 - \sqrt{z^4 - 4z^3 + 2z^2 - 4z + 1}}{2z}.$$

Replacing u with s (which have a Taylor expansion at $z = 0$) in order to cancel the numerator of $F(u)$, we obtain the equation

$$2as\,z^2 - s\,z^2 + sz + z^2 - s - z + 1 = 0.$$

Using $zrs = z^2 - z + 1$, a straightforward calculation provides

$$a = F(1) = H(1) + 1 = \frac{r(s-1)}{2z}.$$

Finally, after simplifying by the factor $(u - s)$ in the numerators and denominators, we obtain

$$F(u) = \frac{r}{r-u}, \quad G(u) = \frac{r(s-1)-z}{r-u}, \quad \text{and} \quad H(u) = \frac{1}{r-u}.$$

Extracting the coefficient of u^k in the series expansion of each generating function, we obtain

$$[u^k]F(u) := f_k = \frac{1}{r^k},$$
$$[u^k]G(u) := g_k = \frac{s-1}{r^k} - \frac{z}{r^{k+1}},$$
$$[u^k]H(u) := h_k = \frac{1}{r^{k+1}}.$$

Theorem 2.1. *The bivariate generating function for the total number of PMAP in $\mathcal{M}_1$ with respect to the length and the height of the end-point is given by*

$$\texttt{Total}(z,u) = \frac{1}{z(r-u)},$$

and we have

$$[u^k]\texttt{Total}(z,u) = \frac{1}{zr^{k+1}}.$$

Finally, setting $t(n,k) := [z^n][u^k]\texttt{Total}(z,u)$, we have for $n \geqslant 2$, $k \geqslant 1$,

$$t(n,k) = t(n,k-1) + t(n-1,k) - t(n-1,k-2) - t(n-2,k) - t(n-2,k-1),$$

and setting $t_n := t(n,0)$, then we have $t_0 = t_1 = 1$, and for $n \geqslant 2$,

$$t_n = t_{n-1} + t_{n-2} + \sum_{k=0}^{n-3} t_k t_{n-k-3} + \sum_{k=2}^{n-1} (t_k - t_{k-1})\, t_{n-k-1}.$$

Proof. Since $\texttt{Total}(z,u) = F(u)+G(u)+H(u)$, the first two equalities are immediately deduced from the previous results. The third equality is obtained using the Mathematica package `Guess.m` (written by Manuel Kauers [6]) for guessing recurrence relations. After this, it suffices to check algebraically:

$$\texttt{Total}(z,u) = (u+z-u^2z-z^2-uz^2)\texttt{Total}(z,u) - u + \frac{(z^2-z+1)(1+rs-z)}{r}.$$

Now, let us prove the last equality. Any length n MAP of the first kind is of the form (i) HP, or (ii) UDP, or (iii) $UPHDQ$ where P,Q are some MAP so that the length k of P lies into $[0,n-3]$, and the length of Q is $n-k-3$, or (iv) $P^\sharp Q$ where $P^\sharp = UP'D_\ell$, $\ell \geqslant 1$, and P' is a PMAP such that $P'D_{\ell-1}$ is a MAP of length k lying into $[2,n-1]$, and the length of Q is $n-k-1$. Taking into account all these cases, we obtain the result. $\square$

Let $\mathcal{T}$ be the infinite matrix $\mathcal{T} := [t(n,k)]_{n\geqslant 0, k\geqslant 0}$. The first few rows of the matrix $\mathcal{T}$ are

$$\mathcal{T} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 2 & 2 & 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 5 & 5 & 3 & 1 & 0 & 0 & 0 & 0 & \cdots \\ 13 & 14 & 9 & 4 & 1 & 0 & 0 & 0 & \cdots \\ 36 & 40 & 28 & 14 & 5 & 1 & 0 & 0 & \cdots \\ 105 & 118 & 87 & 48 & 20 & 6 & 1 & 0 & \cdots \\ 317 & 359 & 273 & 161 & 75 & 27 & 7 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Corollary 2.2. *The generating function that counts the PMAP in $\mathcal{M}_1$ with respect to the length is given by*

$$\texttt{Total}(z,1) = \frac{1}{z(r-1)}.$$

The first few terms of the series expansion of $\texttt{Total}(z,1)$ are $1+2z+5z^2+14z^3+41z^4+124z^5+385z^6+1220z^7+3929z^8+12822z^9+O(z^{10})$, which correspond to the sequence A159771 in [13] counting the n-leaf binary trees that do not contain $(()((()())((()())())))$ as a subtree (see [11]). See Figure 2 for an illustration of the 14 PMAP of length 3.

Corollary 2.3. *The generating function that counts the MAP in $\mathcal{M}_1$ with respect to the length is given by*

$$\texttt{Total}(z,0) = \frac{1}{zr}.$$

The first few terms of the series expansion of $\texttt{Total}(z,0)$ are $1+z+2z^2+5z^3+13z^4+36z^5+105z^6+317z^7+982z^8+3105z^9+O(z^{10})$ which correspond to the sequence A114465 in [13] counting Dyck paths of length $2n$ having no ascents of length 2 that start at an odd level. We leave open the question of finding a constructive bijection between these sets.

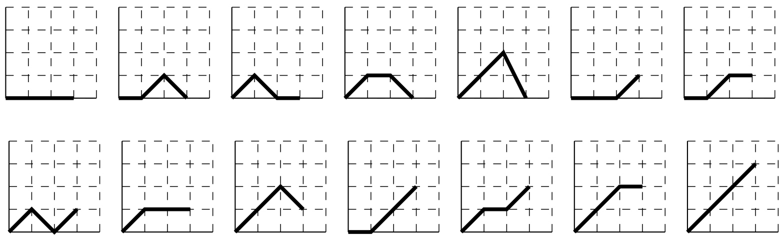

Figure 2: The 14 PMAP of length three in $\mathcal{M}_1$. Notice that five paths end on the x-axis, five paths end at height 1, three paths end at height 2, and one path ends at height 3, which correspond to the fourth row of $\mathcal{T}$.

2.2 PMAP in $\mathcal{M}_1'$ - From right to left

Here, we consider the paths of the previous section, but we read them from right to left. This means that down steps become up steps and *vice versa*, and horizontal steps are unchanged, which implies that two up steps cannot be consecutive now. See Definition 1.3 and Figure 3 for two examples of such paths.

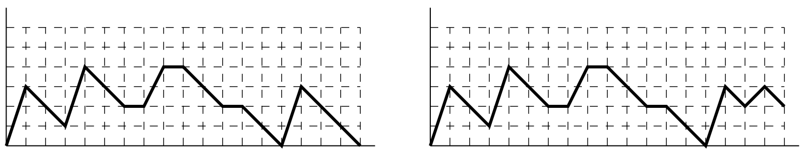

Figure 3: The left drawing shows a Motzkin path with air pockets of length 18 in $\mathcal{M}_1'$. The right drawing shows a partial Motzkin path with air pockets of length 18 ending at height 2 in $\mathcal{M}_1'$.

Let P be a length n PMAP in $\mathcal{M}_1'$ ending at height $k \geqslant 0$. If the last step of P is U_a, $a \geqslant 1$, then $k \geqslant a$ and we have $P = QU_a$ where Q is a length $(n-1)$ PMAP ending at height $\ell = k - a$ with a horizontal step or a down step. So, we obtain the first relation $f_k = z + z(g_0 + g_1 + \ldots + g_{k-1}) + z(h_0 + h_1 + \ldots + h_{k-1})$ for $k \geqslant 1$, anchored with $f_0 = 1$ by considering the empty path. If the last step of P is a down step D, then we have $P = QD$ where Q is a length $(n-1)$ PMAP ending at height $k+1$. So, we obtain the second relation $g_k = zf_{k+1} + zg_{k+1} + zh_{k+1}$. If the last step of P is a horizontal step H, then we have $P = QH$ where Q is a length $(n-1)$ PMAP ending at height k, which implies $h_k = zf_k + zg_k + zh_k$.

Therefore, we have to solve the following system of equations.

$$\begin{cases} f_0 = 1, \\ f_k = z + z(g_0 + g_1 + \ldots + g_{k-1}) + z(h_0 + h_1 + \ldots + h_{k-1}), \quad k \geqslant 1, \\ g_k = zf_{k+1} + zg_{k+1} + zh_{k+1}, \quad k \geqslant 0, \\ h_k = zf_k + zg_k + zh_k, \quad k \geqslant 0. \end{cases} \tag{2.2}$$

Summing the recursions in (2.2), we have:

$$\begin{aligned}
F(u) &= 1 + \frac{zu}{1-u} + z\sum_{k\geqslant 1}(g_0 + \ldots + g_{k-1})u^k + z\sum_{k\geqslant 1}(h_0 + \ldots + h_{k-1})u^k \\
&= 1 + \frac{zu}{1-u} + z\sum_{k\geqslant 0} g_k \frac{u^{k+1}}{1-u} + z\sum_{k\geqslant 0} h_k \frac{u^{k+1}}{1-u} \\
&= 1 + \frac{zu}{1-u}(1 + G(u) + H(u)), \\
G(u) &= z\sum_{k\geqslant 0} f_{k+1}u^k + z\sum_{k\geqslant 0} g_{k+1}u^k + z\sum_{k\geqslant 0} h_{k+1}u^k \\
&= \frac{z}{u}(F(u) - F(0) + G(u) - G(0) + H(u) - H(0)), \\
H(u) &= zF(u) + zG(u) + zH(u).
\end{aligned}$$

Notice that we have $H(0) = \frac{z(1+G(0))}{1-z}$ by considering the third equation and $F(0) = 1$. Solving these functional equations using $a := G(0)$, we obtain

$$\begin{aligned}
F(u) &= \frac{-u^2z^3 + a\,uz^2 + 3\,u^2z^2 - uz^3 - 3\,u^2z + 2\,uz^2 + u^2 + uz - z^2 - u + z}{(1-z)\left(u^2z^2 - u^2z + uz^2 + u^2 - u + z\right)}, \\
G(u) &= \frac{z\left(a\,uz^2 - a\,uz + a\,u + a\,z + uz - a\right)}{(-1+z)\left(u^2z^2 - u^2z + uz^2 + u^2 - u + z\right)}, \\
H(u) &= \frac{\left(a\,uz^2 + u^2z^2 - a\,uz - 2\,u^2z + uz^2 + a\,z + u^2 - u + z\right)z}{(1-z)\left(u^2z^2 - u^2z + uz^2 + u^2 - u + z\right)}.
\end{aligned}$$

In order to compute $a = G(0)$, we use the kernel method on $G(u)$. We can write the denominator (which is a polynomial in u of degree 2), as $(z-1)(z^2-z+1)(u-r)(u-s)$ with

$$r = \frac{1 - z^2 + \sqrt{z^4 - 4\,z^3 + 2\,z^2 - 4\,z + 1}}{2(z^2 - z + 1)},$$

$$s = \frac{1 - z^2 - \sqrt{z^4 - 4\,z^3 + 2\,z^2 - 4\,z + 1}}{2(z^2 - z + 1)}.$$

Replacing u with s (which have a Taylor expansion at $z = 0$) in order to cancel the numerator of $G(u)$, we obtain the equation

$$a(sz^2 - sz + s + z - 1) + sz = 0.$$

Using $sr(z^2 - z + 1) = z$, we deduce

$$a = G(0) = \frac{1-r}{r} - sz.$$

Finally, after simplifying by the factor $(u - s)$ in the numerators and denominators, we obtain

$$F(u) = 1 + \frac{sru}{r-u}, \quad G(u) = \frac{s(1 - r + rz)}{r-u}, \quad \text{and} \quad H(u) = \frac{sr - sru(1-z)}{r-u},$$

which implies that

$$[u^k]F(u) := f_k = [k=0] + [k \neq 0] \cdot \frac{s}{r^{k-1}},$$
$$[u^k]G(u) := g_k = \frac{s(1-r+rz)}{r^{k+1}},$$
$$[u^k]H(u) := h_k = \frac{s}{r^k} - [k \neq 0] \cdot \frac{(1-z)s}{r^{k-1}},$$

where $[k=0]$ (resp. $[k \neq 0]$) equals 1 whenever $k=0$ (resp. $k \neq 0$), and 0 otherwise.

Theorem 2.4. *The bivariate generating function for the total number of PMAP (read from right to left) with respect to the length and the height of the end-point is given by*

$$\texttt{Total}(z,u) = 1 + \frac{s(1+rz+ruz)}{r-u},$$

and we have

$$[u^k]\texttt{Total}(z,u) = [k=0] + \frac{s(rz+1)}{r^{k+1}} + [k \neq 0] \cdot \frac{sz}{r^{k-1}}.$$

Finally, setting $t(n,k) := [z^n][u^k]\texttt{Total}(z,u)$, we have for $n \geqslant 2$, $k \geqslant 1$,

$$t(n,k) = t(n-2,k-1) + t(n-2,k) - t(n-1,k-1) + t(n-1,k+1) + t(n,k-1),$$

and setting $t_n := t(n,0)$, we have $t_0 = t_1 = 1$, and for $n \geqslant 2$,

$$t_n = t_{n-1} + t_{n-2} + \sum_{k=0}^{n-3} t_k t_{n-k-3} + \sum_{k=2}^{n-1} (t_k - t_{k-1})\, t_{n-k-1}.$$

Proof. The first two equalities are directly deduced from the previous results. Since we have

$$(u - u^2z^2 - uz^2 + zu^2 - z - u^2)\texttt{Total}(z,u) + u^2(1-z) - (1+s-sz)u + s = 0,$$

we deduce the third relation. The last equality is already given in Theorem 2.1, since the number of MAP in $\mathcal{M}_1'$ is obviously equal to the number of MAP in $\mathcal{M}_1$. $\square$

Let $\mathcal{T}$ be the infinite matrix $\mathcal{T} := [t(n,k)]_{n \geqslant 0, k \geqslant 0}$. The first few rows of the matrix $\mathcal{T}$ are

$$\mathcal{T} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 1 & 1 & 1 & 1 & 1 & \cdots \\ 2 & 3 & 3 & 3 & 3 & 3 & \cdots \\ 5 & 8 & 10 & 12 & 14 & 16 & \cdots \\ 13 & 23 & 33 & 43 & 53 & 63 & \cdots \\ 36 & 69 & 107 & 149 & 195 & 245 & \cdots \\ 105 & 212 & 348 & 512 & 704 & 924 & \cdots \\ 317 & 665 & 1141 & 1753 & 2509 & 3417 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Since there is an infinite number of PMAP of length n, we do not provide an ordinary generating function (with respect to the length) for these paths. So, we get around this by counting PMAP ending on a point $(x, n-x)$ for a given $n \geqslant 0$.

Corollary 2.5. *The generating function that counts the partial PMAP ending on the line $y = n - x$ is given by*

$$\texttt{Total}(z, z) = 1 + \frac{s(1 + rz + rz^2)}{r - z}.$$

The first few terms of the series expansion of $\texttt{Total}(z, z)$ are $1 + z + 3z^2 + 9z^3 + 25z^4 + 73z^5 + 223z^6 + 697z^7 + 2217z^8 + 7161z^9 + O(z^{10})$, which correspond to the sequence A101499 in [13], which is a Chebyshev transform of the Catalan number that counts peakless Motzkin paths of length n where horizontal steps at level at least one come in 2 colors. See Figure 4 for an illustration of the 9 PMAP in $\mathcal{M}_1'$ ending on the line $y = 3 - x$.

Notice that, from Theorem 2.4 we have

$$\texttt{Total}(z, 0) = 1 + \frac{s(1 + rz)}{r},$$

which is obviously equal to the expression derived in Corollary 2.3 that counts MAP of the first kind with respect to the length.

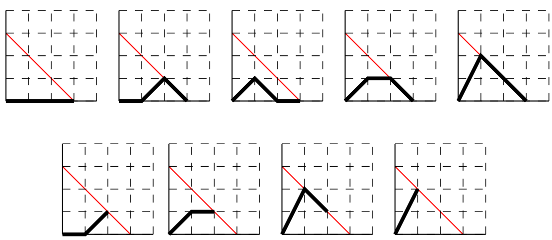

Figure 4: The 9 PMAP ending on the line $y = 3 - x$ in $\mathcal{M}_1'$. Notice that five paths end on the x-axis, three paths end at height 1, and one path ends at height 2, which correspond to the fourth diagonal of $\mathcal{T}$.

3 PMAP of the second kind

In this section, we focus on PMAP of the second kind. See Figure 5 for an illustration of such paths. The first subsection considers paths in $\mathcal{M}_2$, while the second handles paths in $\mathcal{M}_2'$. We yield enumerative results for these paths according to the length, the type of the last step, and the height of the end-point.

3.1 PMAP in $\mathcal{M}_2$ - From left to right

In this part, we consider PMAP in $\mathcal{M}_2$, i.e. lattice paths in $\mathbb{N}^2$ starting at the origin, consisting of steps U, D_k and H, and where any down step or horizontal step (except for the last step of the path) is immediately followed by an up step. Figure 5 shows two examples of such paths.

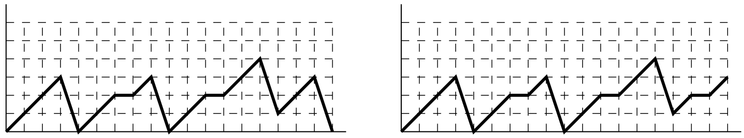

Figure 5: The left drawing shows a MAP of length 18 in $\mathcal{M}_2$. The right drawing shows a PMAP of length 18 ending at height 3 in $\mathcal{M}_2$.

Let P be a length n PMAP in $\mathcal{M}_2$ ending at height $k \geqslant 0$. If the last step of P is U, then $k \geqslant 1$ and we have $P = QU$ where Q is a length $(n-1)$ PMAP ending at height $k-1$. So, we obtain the first relation $f_k = zf_{k-1} + zg_{k-1} + zh_{k-1}$ for $k \geqslant 1$, anchored with $f_0 = 1$ by considering the empty path. If the last step of P is a down step D_a, $a \geqslant 1$, then we have $P = QD_a$ where Q is a length $(n-1)$ PMAP ending at height $\ell = a+k \geqslant k+1$ with an up step. So, we obtain the second relation $g_k = z\sum_{\ell \geqslant k+1} f_\ell$. If the last step of P is a horizontal step H, then we have $P = QH$ where Q is a length $(n-1)$ PMAP ending at height k with an up step, which implies $h_k = zf_k$.

So, we have to solve the following system of equations.

$$\begin{cases} f_0 = 1, \text{ and } f_k = zf_{k-1} + zg_{k-1} + zh_{k-1}, \quad k \geqslant 1, \\ g_k = z \sum\limits_{\ell \geqslant k+1} f_\ell, \quad k \geqslant 0, \\ h_k = zf_k, \quad k \geqslant 0. \end{cases} \tag{3.1}$$

Summing the recursions in (3.1), we have:

$$\begin{aligned} F(u) &= 1 + z\sum_{k\geqslant 1} u^k f_{k-1} + z\sum_{k\geqslant 1} u^k g_{k-1} + z\sum_{k\geqslant 1} u^k h_{k-1} \\ &= 1 + zuF(u) + zuG(u) + zuH(u), \\ G(u) &= z\sum_{k\geqslant 0} u^k \Big(\sum_{\ell\geqslant k+1} f_\ell\Big) = z\sum_{k\geqslant 1} \frac{u^k-1}{u-1} f_k \\ &= \frac{z}{u-1}(F(u) - F(1)), \\ H(u) &= zF(u). \end{aligned}$$

Now, setting $a := F(1)$ and solving these functional equations, we deduce

$$F(u) = \frac{au\,z^2 - u + 1}{u^2z^2 + u^2z - uz - u + 1},$$

$$G(u) = -\frac{z\left(au\,z^2 + auz - a + 1\right)}{u^2z^2 + u^2z - uz - u + 1}, H(u) = \frac{z\left(au\,z^2 - u + 1\right)}{u^2z^2 + u^2z - uz - u + 1}.$$

In order to compute $a = F(1)$, we use the kernel method on $F(u)$. We can write the denominator (which is a polynomial in u of degree 2), as $(z^2+z)(u-r)(u-s)$ with

$$r = \frac{1 + z + \sqrt{-3z^2 - 2z + 1}}{2z\,(z+1)}, \quad \text{and} \quad s = \frac{1 + z - \sqrt{-3z^2 - 2z + 1}}{2z\,(z+1)}.$$

Replacing u with s (which have a Taylor expansion at $z = 0$) in order to cancel the numerator of $F(u)$, we obtain the equation

$$as\,z^2 - s + 1 = 0,$$

and thus

$$a = F(1) = \frac{s-1}{sz^2}.$$

Finally using $z(1+z)rs = 1$ and simplifying by the factor $(u - s)$ in the numerators and denominators, we obtain

$$F(u) = \frac{r}{r-u}, \quad G(u) = \frac{s-1}{sz(r-u)}, \quad \text{and} \quad H(u) = \frac{zr}{r-u},$$

which implies that

$$\begin{aligned}
[u^k]F(u) &:= f_k = \frac{1}{r^k},\\
[u^k]G(u) &:= g_k = (1+z)\cdot\frac{s-1}{r^k},\\
[u^k]H(u) &:= h_k = \frac{z}{r^k}.
\end{aligned}$$

Theorem 3.1. *The bivariate generating function for the total number of PMAP with respect to the length and the height of the end-point is given by*

$$\mathtt{Total}(z,u) = \frac{1}{z(r-u)},$$

and we have

$$[u^k]\mathtt{Total}(z,u) = \frac{1}{zr^{k+1}}.$$

Finally, setting $t(n,k) = [z^n][u^k]\mathtt{Total}(z,u)$, we have for $n \geqslant 2$, $k \geqslant 1$,

$$t(n,k) = t(n,k-1) + t(n-1,k-1) - t(n-1,k-2) - t(n-2,k-2),$$

and setting $t_n := t(n,0)$, we have $t_0 = 1$, and for $n \geqslant 1$,

$$t_n = t_{n-1} + \sum_{k=1}^{n-2} t_k t_{n-1-k}.$$

Proof. The first two equalities are immediately deduced from the previous results. The third equality is obtained using the Mathematica package `Guess.m` ([6]) for guessing recurrence relations. After this, it suffices to check algebraically:

$$\mathtt{Total}(z,u) = (u + uz - u^2z - u^2z^2)\mathtt{Total}(z,u) - u(1+z) + \frac{1}{zr}.$$

For the last equality, it suffices to remark that the generating function of the sequence $(t_n)_{n\geqslant 0}$, that is $1/(zr)$, generates a shift of the well known Motzkin sequence A001006 in [13]. $\square$

Let $\mathcal{T}$ be the infinite matrix $\mathcal{T} := [t(n,k)]_{n\geqslant 0, k\geqslant 0}$. The first few rows of the matrix $\mathcal{T}$ are

$$\mathcal{T} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 2 & 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 2 & 3 & 3 & 1 & 0 & 0 & 0 & 0 & \cdots \\ 4 & 6 & 6 & 4 & 1 & 0 & 0 & 0 & \cdots \\ 9 & 13 & 13 & 10 & 5 & 1 & 0 & 0 & \cdots \\ 21 & 30 & 30 & 24 & 15 & 6 & 1 & 0 & \cdots \\ 51 & 72 & 72 & 59 & 40 & 21 & 7 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Corollary 3.2. *The generating function that counts the PMAP with respect to the length is given by*

$$\texttt{Total}(z,1) = \frac{1}{z(r-1)}.$$

The first few terms of the series expansion of $\texttt{Total}(z,1)$ are $1 + 2z + 4z^2 + 9z^3 + 21z^4 + 51z^5 + 127z^6 + 323z^7 + 835z^8 + 2188z^9 + O(z^{10})$, which correspond to a shift of the sequence A001006 in [13] that counts the Motzkin paths of a given length. See Figure 6 for an illustration of the 9 paths of length 3.

Corollary 3.3. *The generating function that counts the MAP with respect to the length is given by*

$$\texttt{Total}(z,0) = \frac{1}{zr}.$$

The first few terms of the series expansion of $\texttt{Total}(z,0)$ are $1+z+z^2+2z^3+4z^4+9z^5+21z^6+51z^7+127z^8+323z^9+O(z^{10})$ which correspond to a shift of the sequence A001006 in [13] that counts Motzkin paths of a given length.

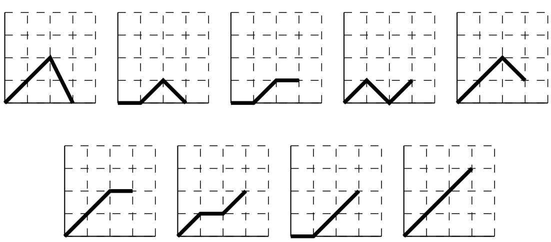

Figure 6: The 9 PMAP of length three in $\mathcal{M}_2$. Notice that two paths end on the x-axis, three paths end at height 1, three paths end at height 2, and one path ends at height 3, which correspond to the fourth row of $\mathcal{T}$.

3.2 PMAP in $\mathcal{M}_2'$ - From right to left

Here, we consider the paths of the previous section, but we read them from right to left. This means that down steps become up steps and *vice versa*, and horizontal steps are unchanged, which implies that any up step or horizontal step (except the first step of the path) is preceded by a down step. See Definition 1.4 and Figure 7 for two examples of such paths.

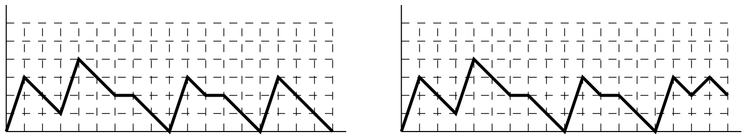

Figure 7: The left drawing shows a Motzkin path with air pockets of length 18 in $\mathcal{M}_2'$ (read from right to left). The right drawing shows a partial Motzkin path with air pockets of length 18 ending at height 2 in $\mathcal{M}_2'$.

Let P be a length n PMAP in $\mathcal{M}_2'$ ending at height $k \geqslant 0$. If the last step of P is U_a, $a \geqslant 1$, then $k \geqslant a$ and we have $P = QU_a$ where Q is a length $(n-1)$ PMAP ending at height $\ell = k-a$ with a down step. So, we obtain the first relation $f_k = z + z(g_0 + g_1 + \ldots + g_{k-1})$ for $k \geqslant 1$, anchored with $f_0 = 1$ by considering the empty path. If the last step of P is a down step D, then we have $P = QD$ where Q is a length $(n-1)$ PMAP ending at height $k+1$. So, we obtain the second relation $g_k = zf_{k+1} + zg_{k+1} + zh_{k+1}$. If the last step of P is a horizontal step H, then we have $P = QH$ where Q is a length $(n-1)$ PMAP ending at height k with a down step whenever it is nonempty. If $k \geqslant 1$, then we have $h_k = zg_k$; if $k = 0$ then we have $h_0 = z + zg_0$ where the monomial z corresponds to the path $P = H$.

So, we have to solve the following system of equations.

$$\begin{cases} f_0 = 1, \text{ and } f_k = z(1 + g_0 + g_1 + \ldots + g_{k-1}), \quad k \geqslant 1, \\ g_k = zf_{k+1} + zg_{k+1} + zh_{k+1}, \quad k \geqslant 0, \\ h_0 = z + zg_0, \text{ and } h_k = zg_k, \quad k \geqslant 1. \end{cases} \tag{3.2}$$

Using the same notations as in the previous sections, and summing the recursions in (3.2), we have:

$$\begin{aligned} F(u) &= 1 + \sum_{k\geqslant 1} u^k f_k = 1 + z\sum_{k\geqslant 1}(1 + g_0 + \ldots + g_{k-1})u^k \\ &= 1 + \frac{zu}{1-u}(1 + G(u)), \\ G(u) &= z\sum_{k\geqslant 0}(f_{k+1} + g_{k+1} + h_{k+1})u^k \\ &= \frac{z}{u}(F(u) - F(0) + G(u) - G(0) + H(u) - H(0)), \\ H(u) &= z + zG(u). \end{aligned}$$

Notice that $F(0) = 1$ and $H(0) = z + zG(0)$ by the third relation. Now, setting

$a := G(0)$ and solving these functional equations, we deduce

$$F(u) = \frac{a\,uz^3 + a\,uz^2 + uz^3 - u^2z + uz^2 + u^2 - uz + z^2 - u + z}{u^2 - uz + z^2 - u + z},$$
$$G(u) = -\frac{z\,(a\,uz + a\,u - a\,z + uz - a)}{u^2 - uz + z^2 - u + z},$$
$$H(u) = -\frac{z\left(a\,uz^2 + a\,uz - a\,z^2 + uz^2 - a\,z - u^2 + uz - z^2 + u - z\right)}{u^2 - uz + z^2 - u + z}.$$

In order to compute $a = G(0)$, we use the kernel method on $F(u)$. We can write the denominator (which is a polynomial in u of degree 2), as $(u - r)(u - s)$ with

$$r = \frac{1 + z + \sqrt{-3\,z^2 - 2\,z + 1}}{2}, \text{ and } s = \frac{1 + z - \sqrt{-3\,z^2 - 2\,z + 1}}{2}.$$

Replacing u with s (which have a Taylor expansion at $z = 0$) in oder to cancel the numerator of $F(u)$, we obtain the equation

$$a\,sz^3 + a\,sz^2 + sz^3 - s^2z + sz^2 + s^2 - sz + z^2 - s + z = 0.$$

Using $rs = z(1 + z)$, we deduce

$$a = G(0) = \frac{1 - r}{r}.$$

Finally, after simplifying by the factor $(u - s)$ in the numerators and denominators, we obtain

$$F(u) = \frac{u(z - 1) + r}{r - u}, \quad G(u) = \frac{1 - r}{r - u}, \quad \text{and} \quad H(u) = \frac{z(1 - u)}{r - u},$$

which implies that

$$[u^k]F(u) := f_k = \frac{1}{r^k} + [k \neq 0] \cdot \frac{z - 1}{r^k},$$
$$[u^k]G(u) := g_k = \frac{1 - r}{r^{k+1}},$$
$$[u^k]H(u) := h_k = \frac{z}{r^{k+1}} - [k \neq 0] \cdot \frac{z}{r^k}.$$

Theorem 3.4. *The bivariate generating function for the total number of PMAP with respect to the length and the height of the end-point is given by*

$$\texttt{Total}(z, u) = \frac{1 - u + z}{r - u}.$$

$$[u^k]\texttt{Total}(z, u) = \frac{z + 1}{r^{k+1}} - [k \neq 0] \cdot \frac{1}{r^k}.$$

Finally, setting $t(n, k) = [z^n][u^k]\texttt{Total}(z, u)$, we have for $n \geqslant 1$, $k \geqslant 1$,

$$t(n, k) = t(n, k - 1) - t(n - 1, k) + t(n - 2, k + 1) + t(n - 1, k + 1),$$

and setting $t_n := t(n, 0)$, we have $t_0 = 1$, and for $n \geqslant 2$,

$$t_n = t_{n-1} + \sum_{k=1}^{n-2} t_k t_{n-1-k}.$$

Proof. The proof are obtained *mutatis mutandis* as for the previous theorems. $\square$

Let $\mathcal{T}$ be the infinite matrix $\mathcal{T} := [t(n,k)]_{n\geqslant 0, k\geqslant 0}$. The first few rows of the matrix $\mathcal{T}$ are

$$\mathcal{T} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & \cdots \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & \cdots \\ 2 & 3 & 4 & 5 & 6 & 7 & 8 & \cdots \\ 4 & 6 & 8 & 10 & 12 & 14 & 16 & \cdots \\ 9 & 15 & 22 & 30 & 39 & 49 & 60 & \cdots \\ 21 & 36 & 54 & 75 & 99 & 126 & 156 & \cdots \\ 51 & 91 & 142 & 205 & 281 & 371 & 476 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Since there is an infinite number of PMAP of length n, we do not provide an ordinary generating function (with respect to the length) for these paths. So, we get around this by counting PMAP ending on a point $(x, n-x)$ for a given $n \geqslant 0$.

Corollary 3.5. *The generating function that counts the partial PMAP ending on the line $y = n - x$ is given by*

$$\texttt{Total}(z, z) = \frac{1}{r - z}.$$

The first few terms of the series expansion of $\texttt{Total}(z, z)$ are $1 + z + 2z^2 + 4z^3 + 9z^4 + 21z^5 + 51z^6 + 127z^7 + 323z^8 + 835z^9 + O(z^{10})$, which correspond to the sequence A001006 in [13] that counts the Motzkin paths with respect to the length. See Figure 8 for the illustration of the 9 PMAP in $\mathcal{M}'_2$ ending on the line $y = 4 - x$.

Notice that we obviously retrieve the results of Corollary 3.3, i.e., the generating function $\texttt{Total}(z, 0)$ that counts the MAP with respect to the length is also a shift of the Motzkin sequence A001006 in [13].

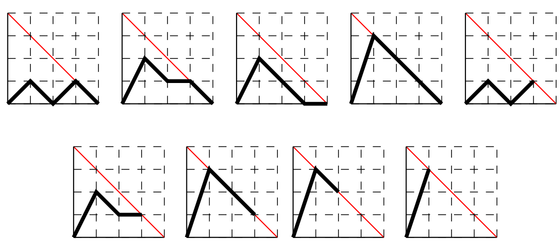

Figure 8: The 9 PMAP ending on the line $y = 4 - x$ in $\mathcal{M}'_2$. Notice that four paths end on the x-axis, three paths end at height 1, one path ends at height 2, and one path ends at height 3, which correspond to the fifth diagonal of $\mathcal{T}$.

4 A Riordan array point of view

In this section, we make links between the previous matrices $\mathcal{T} = [t_{n,k}]_{n\geqslant 0,k\geqslant 0}$ and some Riordan arrays or almost Riordan arrays. We first give a short background on Riordan arrays [4, 5, 12].

An infinite column vector $(a_0, a_1, \dots)^T$ has generating function $f(z)$ if $f(z) = \sum_{n\geqslant 0} a_n z^n$. A *Riordan array* is an infinite lower triangular matrix whose k-th column has generating function $g(z)f(z)^k$ for all $k \geqslant 0$, for some formal power series $g(z)$ and $f(z)$, with $g(0) \neq 0$, $f(0) = 0$, and $f'(0) \neq 0$. Such a Riordan array is denoted by $(g(z), f(z))$. If we multiply this matrix by a column vector $(b_0, b_1, \dots)^T$ having generating function $h(z) = \sum_{n\geqslant 0} b_n z^n$, then the resulting column vector has generating function $g(z)h(f(z))$. This property is known as the *fundamental theorem of Riordan arrays*.

The product of two Riordan arrays $(g(z), f(z))$ and $(h(z), l(z))$ is defined by

$$(g(z), f(z)) * (h(z), l(z)) = (g(z)h(f(z)), l(f(z)))\,.$$

Under the operation "$*$", the set of all Riordan arrays is a group [12]. The identity element is $I = (1, z)$, and the inverse of $(g(z), f(z))$ is

$$(g(z), f(z))^{-1} = \left(1/\left(g \circ \overline{f}\right)(z), \overline{f}(z)\right),$$

where $\overline{f}(z)$ denotes the compositional inverse of $f(z)$.

Moreover, if a matrix $\mathcal{T} = [t_{n,k}]_{n\geqslant 0,k\geqslant 0}$ is a *Riordan array* $(g(z), f(z))$ then $t_{n,k}$ equals the coefficient of $z^n u^k$ in the series expansion of the bivariate generating function

$$\frac{g(z)}{1 - uf(z)},$$

and we say that this is the bivariate generating function of the matrix $\mathcal{T}$.

An *almost Riordan array* $\mathcal{T}'$ is a matrix that consists of an initial column vector $(d_0, d_1, \dots)^T$ with generating function $g_0(z) = \sum_{n\geqslant 0} d_n z^n$, followed by a vertically shifted Riordan array $(g(z), f(z))$ as illustrated below.

$$\mathcal{T}' = \left(\begin{array}{c|ccccc}
d_0 & 0 & \cdots & 0 & 0 & \cdots \\ \hline
d_1 & & & & & \\
d_2 & & & & & \\
d_3 & & & (g(z), f(z)) & & \\
d_4 & & & & & \\
d_5 & & & & & \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots
\end{array}\right)$$

Therefore, the bivariate generating function for this matrix is given by

$$g_0(z) + zu\frac{g(z)}{1 - uf(z)}.$$

Finally, we will say that a matrix $\mathcal{M} = [m_{n,k}]_{n\geqslant 0,k\geqslant 0}$ is the *rectification of the Riordan array* $(g(z), f(z))$ whenever $m_{n,k}$ equals the coefficient of $z^n u^k$ in the series expansion of the bivariate generating function

$$\frac{g(z)}{1 - u\frac{f(z)}{z}}.$$

In this section, $C(z) = \frac{1-\sqrt{1-4z}}{2z}$ will be the generating function where the coefficient of z^n in its series expansion is the Catalan number $c_n = \frac{1}{n+1}\binom{2n}{n}$.

4.1 Comment on Section 2.1

Let $\mathcal{T} = [t(n,k)]_{n\geqslant 0, k\geqslant 0}$ be the matrix given in Section 2.1 where $t(n,k)$ is the number of length n PMAP in $\mathcal{M}_1$ ending at height k.

Proposition 4.1. *The matrix $\mathcal{T} = [t(n,k)]_{n\geqslant 0, k\geqslant 0}$ is a Riordan array defined by*

$$\left(\frac{1}{1-z^2} C\left(\frac{z(1-z+z^2)}{(1-z^2)^2}\right), \frac{z}{1-z^2} C\left(\frac{z(1-z+z^2)}{(1-z^2)^2}\right)\right).$$

Proof. Considering r defined in Section 2.1, we have

$$\begin{aligned}[u^k]\texttt{Total}(z,u) &= \frac{1}{zr^{k+1}} = \left(\frac{1}{zr}\right)\left(\frac{1}{r}\right)^k \\ &= \frac{2}{1-z^2+\sqrt{1-4z+2z^2-4z^3+z^4}}\left(\frac{2z}{1-z^2+\sqrt{1-4z+2z^2-4z^3+z^4}}\right)^k.\end{aligned}$$

Therefore, the array $\mathcal{T}$ satisfies

$$\begin{aligned}\mathcal{T} &= \left(\frac{2}{1-z^2+\sqrt{1-4z+2z^2-4z^3+z^4}}, \frac{2z}{1-z^2+\sqrt{1-4z+2z^2-4z^3+z^4}}\right) \\ &= \left(\frac{1-z^2-\sqrt{1-4z+2z^2-4z^3+z^4}}{2z(1-z+z^2)}, \frac{1-z^2-\sqrt{1-4z+2z^2-4z^3+z^4}}{2(1-z+z^2)}\right) \\ &= \left(\frac{1}{1-z^2} C\left(\frac{z(1-z+z^2)}{(1-z^2)^2}\right), \frac{z}{1-z^2} C\left(\frac{z(1-z+z^2)}{(1-z^2)^2}\right)\right).\end{aligned}$$

□

Proposition 4.2. *The general term $t(n,k)$ equals*

$$\sum_{i=0}^{n-k}\binom{k+\frac{n-k-i}{2}}{\frac{n-k-i}{2}}\frac{1+(-1)^{n-k-i}}{2}\sum_{j=0}^{i} C_{k+j,k}\sum_{m=0}^{j}\binom{j}{m}(-1)^m\sum_{p=0}^{m}\binom{m}{p}(-1)^p\binom{2j-1+\frac{i-j-m-p}{2}}{\frac{i-j-m-p}{2}}\frac{1+(-1)^{i-j-m-p}}{2},$$

*where $C_{n,k} = \frac{k+1}{n+1}\binom{2n-k}{n-k}$ is the general term of the Catalan matrix (*A033184 *in* [13]*).*

Proof. Setting $Z = \frac{z(1-z+z^2)}{(1-z^2)^2}$, we have

$$\begin{aligned}
t(n,k) &= [z^n] z^k \frac{1}{(1-z^2)^{k+1}} C(Z)^{k+1} \\
&= [z^{n-k}] \frac{1}{(1-z^2)^{k+1}} C(Z)^{k+1} \\
&= \sum_{i=0}^{n-k} [z^{n-k-i}] \frac{1}{(1-z^2)^{k+1}} [z^i] C(Z)^{k+1} \qquad \text{(product rule [8])} \\
&= \sum_{i=0}^{n-k} [z^{n-k-i}] \frac{1}{(1-z^2)^{k+1}} \sum_{j=0}^{i} [z^j] C(z)^{k+1} [z^i] Z^j \quad \text{(composition rule [8])} \\
&= \sum_{i=0}^{n-k} [z^{n-k-i}] \frac{1}{(1-z^2)^{k+1}} \sum_{j=0}^{i} [z^j] \frac{1}{z^k} C(z)(zC(z))^k [z^i] Z^j \\
&= \sum_{i=0}^{n-k} \binom{k + \frac{n-k-i}{2}}{\frac{n-k-i}{2}} \frac{1+(-1)^{n-k-i}}{2} \sum_{j=0}^{i} C_{k+j,k} [z^i] Z^j \qquad \text{(see [8]).}
\end{aligned}$$

Since we have

$$\begin{aligned}
[z^i] Z^j &= [z^i] \left(\frac{z(1-z+z^2)}{(1-z^2)^2} \right)^j \\
&= \sum_{m=0}^{j} \binom{j}{m} (-1)^m \sum_{p=0}^{m} \binom{m}{p} (-1)^p \binom{2j-1+\frac{i-j-m-p}{2}}{\frac{i-j-m-p}{2}} \frac{1+(-1)^{i-j-m-p}}{2},
\end{aligned}$$

the result follows. $\square$

4.2 Comment on Section 2.2

Let $\mathcal{T} = [t(n,k)]_{n\geqslant 0, k\geqslant 0}$ be the matrix given in Section 2.2 where $t(n,k)$ is the number of length n PMAP in $\mathcal{M}'_1$ ending at height k.

Proposition 4.3. *The matrix* $\mathcal{T} = [t(n,k)]_{n\geqslant 0, k\geqslant 0}$ *can be written*

$$\mathcal{T} = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & \cdots \\
1 & 1 & 1 & 1 & 1 & 1 & \cdots \\
2 & 3 & 3 & 3 & 3 & 3 & \cdots \\
5 & 8 & 10 & 12 & 14 & 16 & \cdots \\
13 & 23 & 33 & 43 & 53 & 63 & \cdots \\
36 & 69 & 107 & 149 & 195 & 245 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix} = A \cdot B$$

where

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 1 & 0 & 0 & 0 & 0 & \cdots \\ 2 & 3 & 0 & 0 & 0 & 0 & \cdots \\ 5 & 8 & 2 & 0 & 0 & 0 & \cdots \\ 13 & 23 & 10 & 0 & 0 & 0 & \cdots \\ 36 & 69 & 38 & 4 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 1 & 1 & 1 & 1 & \cdots \\ 0 & 0 & 1 & 2 & 3 & 4 & \cdots \\ 0 & 0 & 0 & 1 & 3 & 6 & \cdots \\ 0 & 0 & 0 & 0 & 1 & 4 & \cdots \\ 0 & 0 & 0 & 0 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

are defined as follows:

- *The matrix $B = [b_{n,k}]_{n,k\geqslant 0}$ is defined by $b_{0,0} = 1$, and $b_{n,0} = b_{0,n} = 0$ if $n \geqslant 1$, and $b_{n,k} = \binom{k-1}{n-1}$ otherwise, which is a kind of Pascal matrix.*
- *The matrix $A = [a_{n,k}]_{n,k\geqslant 0}$ is the almost Riordan array with initial column of generating function*

$$g_0(z) = \frac{1 - z^2 - \sqrt{1 - 4z + 2z^2 - 4z^3 + z^4}}{2z(1 - z + z^2)},$$

followed by the shifted Riordan array

$$\left(\frac{1 - 3z + z^2 - z^3 - (1 - z)\sqrt{1 - 4z + 2z^2 - 4z^3 + z^4}}{2z^3(1 - z + z^2)}, \frac{1 - 2z - z^2 - \sqrt{1 - 4z + 2z^2 - 4z^3 + z^4}}{2z}\right).$$

The second column of $\mathcal{T}$ $1, 3, 8, 23, 69, \ldots$ with generating function

$$\frac{1 - 3z + z^2 - z^3 - (1 - z)\sqrt{1 - 4z + 2z^2 - 4z^3 + z^4}}{2z^3(1 - z + z^2)}$$

*is the convolution of the first column of $\mathcal{T}$ $1, 1, 2, 5, 13, 36, \ldots$ (*A114465 *in* [13]*) and the sequence $1, 2, 4, 10, 28, \ldots$ (*A187256 *in* [13]*).*

Proof. An almost Riordan array is represented by an initial column vector with generating function $g_0(z)$, followed by a vertically shifted Riordan array $(g(z), f(z))$. The bivariate generating function of this matrix is then given by $g_0(z) + zu\frac{g(z)}{1-uf(z)}$. In our case, for the almost Riordan array A, we have

$$\begin{aligned} g_0(z) &= \frac{1 - z^2 - \sqrt{1 - 4z + 2z^2 - 4z^3 + z^4}}{2z(1 - z + z^2)}, \\ g(z) &= \frac{1 - 3z + z^2 - z^3 - (1 - z)\sqrt{1 - 4z + 2z^2 - 4z^3 + z^4}}{2z^3(1 - z + z^2)}, \\ f(z) &= \frac{1 - 2z - z^2 - \sqrt{1 - 4z + 2z^2 - 4z^3 + z^4}}{2z}. \end{aligned}$$

We let $G(z,u) = g_0(z) + zu\frac{g(z)}{1-uf(z)}$, the bivariate generating function of the almost Riordan array. Now, it suffices to check that the generating function corresponding to the matrix $A \cdot B$, that is

$$G(z, \frac{u}{1-u}) = \frac{(1-u+uz)(1-2z+2uz-(1+2u)z^2 - \sqrt{1-4z+2z^2-4z^3+z^4}}{2(1-z+z^2)(u(u-1)+(1-u^2)z+u(u+1)z^2)},$$

coincides with the generating function $\texttt{Total}(z,u)$ of the matrix $\mathcal{T}$. □

Proposition 4.4. *The matrix*

$$[t(n,k)]_{n\geqslant 1, k\geqslant 1} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & \cdots \\ 3 & 3 & 3 & 3 & 3 & \cdots \\ 8 & 10 & 12 & 14 & 16 & \cdots \\ 23 & 33 & 43 & 53 & 63 & \cdots \\ 69 & 107 & 149 & 195 & 245 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

is the rectification of the Riordan array $(g(z), h(z))$ *with*

$$h(z) = \frac{1-z^2-\sqrt{1-4z+2z^2-4z^3+z^4}}{2}, \quad \textit{and}$$
$$g(z) = \frac{1-3z+z^2-z^3-(1-z)\sqrt{1-4z+2z^2-4z^3+z^4}}{2z^3(1-z+z^2)}.$$

Proof. It suffices to check that the generating function of $\mathcal{T}$, i.e. $\texttt{Total}(z,u)$, given in Theorem 2.4, equals to

$$\texttt{Total}(z,0) + zu\frac{g(z)}{1-u\frac{h(z)}{z}}.$$ □

We can express $h(z)$ and $g(z)$, respectively, in the following form

$$h(z) = \frac{z(1-z+z^2)}{1-z^2} C\left(\frac{z(1-z+z^2)}{(1-z^2)^2}\right),$$

$$g(z) = \frac{1}{1-3z+z^2-z^3} C\left(\frac{z^3(1-z+z^2)}{(1-3z+z^2-z^3)^2}\right).$$

Then $g(z)$ expands to give the first column $1, 3, 8, 23, \ldots$, whose n-th term v_n can be expressed

$$v_n = \sum_{k=0}^{n}\sum_{j=0}^{k}\binom{k}{j}(-1)^j \sum_{i=0}^{j}\binom{j}{i}(-1)^i \sum_{\ell=0}^{n-3k-j-i}\binom{2k+\ell}{\ell}\sum_{m=0}^{\ell}\binom{\ell}{m}3^{\ell-m}(-1)^m\binom{m}{n-3k-j-i-\ell-m}(-1)^{n-3k-j-i-\ell-m}c_k.$$

Using v_n, we can deduce the following.

Proposition 4.5. *The general term $t(n,k)$ equals*

$$\sum_{i=0}^{n+k} v_{n+k-i} \sum_{j=0}^{i}\sum_{m=0}^{j} M_{m,k} \sum_{p=0}^{m}\binom{m}{p}(-1)^p \sum_{q=0}^{p}\binom{p}{q}(-1)^q \binom{2m-1+\frac{j-m-p-q}{2}}{\frac{j-m-p-q}{2}} \frac{1+(-1)^{j-m-p-q}}{2}\binom{k}{\frac{i-j}{2}}(-1)^{\frac{i-j}{2}}\frac{1+(-1)^{i-j}}{2},$$

where

$$M_{n,k} = \begin{cases} [k=0] & \text{if } n=0, \\ \frac{n}{k}\binom{2n-k-1}{n-k} & \text{otherwise,} \end{cases}$$

is the general term of Riordan array $(1, zC(z))$ (see A106566 *in* [13]*).*

4.3 Comment on Section 3.1

Let $\mathcal{T} = [t(n,k)]_{n\geqslant 0, k\geqslant 0}$ be the matrix given in Section 3.1 where $t(n,k)$ is the number of length n PMAP in $\mathcal{M}_2$ ending at height k.

Proposition 4.6. *The matrix $\mathcal{T} = [t(n,k)]_{n\geqslant 0, k\geqslant 0}$ is the Riordan array*

$$\begin{aligned}\mathcal{T} &= (1+zM(z), z(1+zM(z))) \\ &= \left(C\left(\frac{z}{1+z}\right), zC\left(\frac{z}{1+z}\right)\right),\end{aligned}$$

where $M(z) = \frac{1-z-\sqrt{1-2z-3z^2}}{2z^2}$ is the generating function of the Motzkin numbers (see A001006 *in* [13]*).*

Proof. It suffices to check that $\texttt{Total}(z,u)$ given in Theorem 3.1 satisfies

$$\texttt{Total}(z,u) = \frac{C\left(\frac{z}{1+z}\right)}{1-uzC\left(\frac{z}{1+z}\right)},$$

and that $1+zM(z) = C\left(\frac{z}{1+z}\right)$. □

This triangle $\mathcal{T}$ corresponds to A091836 in [13] where the coefficient of row $n-1$ and column k is the number of Motzkin paths of length n having k points on the horizontal axis (besides the first and last point). As mentioned in [13], we obtain

$$t(n,k) = \begin{cases} 1, & \text{if } n=k, \\ \frac{k+1}{n+1}\sum\limits_{j=1}^{n-k} j(-1)^{n-k-j}\binom{n+j}{j}\sum\limits_{i=0}^{n-k}\frac{1}{n-k}\binom{i}{n-k-i+j}\binom{n-k}{i}, & \text{otherwise.} \end{cases}$$

A second expression for $t(n,k)$ is given by the following proposition.

Proposition 4.7. *The general term $t(n,k)$ of the Riordan array $(1+zM(z), z(1+zM(z)))$ is given by*

$$t(n,k) = \begin{cases} 1 & \text{if } n=k, \\ \frac{k+1}{n-k}\sum_{j=0}^{k}\binom{k}{j}\sum_{i=0}^{n-k}\binom{n-k}{i}\binom{i}{n-k-i-j-1} & \text{otherwise.} \end{cases}$$

Proof. We prove this using Lagrange inversion, using the fact that

$$\frac{z}{1+z+z^2} M\left(\frac{z}{1+z+z^2}\right) = z,$$

which means that the compositional inverse $(zM(z))^{-1}$ of $zM(z)$ is

$$(zM(z))^{-1} = \frac{z}{1+z+z^2}.$$

Thus we have

$$\begin{aligned}
t(n,k) &= [z^n](1+zM(z))(z(1+zM(z)))^k \\
&= [z^{n-k}](1+zM(z))^{k+1} \\
&= [z^{n-k}]G(zM(z)), \quad \text{with} \quad G(z) = (1+z)^{k+1} \\
&= \frac{1}{n-k}[z^{n-k-1}]G'(z)\left(\frac{z}{(zM(z))^{-1}}\right)^{n-k} \quad \text{(Lagrange inversion)} \\
&= \frac{1}{n-k}[z^{n-k-1}](k+1)(1+z)^k(1+z+z^2)^{n-k} \\
&= \frac{k+1}{n-k}[z^{n-k}]\sum_{j=0}^{k}\binom{k}{j}z^j\sum_{i=0}^{n-k}\binom{n-k}{i}z^i(1+z)^i \\
&= \frac{k+1}{n-k}[z^{n-k}]\sum_{j=0}^{k}\binom{k}{j}z^j\sum_{i=0}^{n-k}\binom{n-k}{i}z^i\sum_{\ell=0}^{i}\binom{i}{\ell}z^\ell \\
&= \frac{k+1}{n-k}\sum_{j=0}^{k}\binom{k}{j}\sum_{i=0}^{n-k}\binom{n-k}{i}\binom{i}{n-k-i-j-1}.
\end{aligned}$$

□

Remark 4.8. The Riordan array $(1+zM(z), z(1+zM(z)))$ is a pseudo-involution in the Riordan group (see [5, Example 8]), that is, the matrix $[(-1)^k t_{n,k}]_{n,k\geqslant 0}$ is idempotent. Thus, this work yields a significant lattice path interpretation of this array.

4.4 Comment on Section 3.2

Let $\mathcal{T} = [t(n,k)]_{n\geqslant 0,k\geqslant 0}$ be the matrix given in Section 3.2 where $t(n,k)$ is the number of length n PMAP in $\mathcal{M}'_2$ ending at height k.

Proposition 4.9. *The matrix* $\mathcal{T} = [t(n,k)]_{n\geqslant 0,k\geqslant 0}$ *can be written*

$$\mathcal{T} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 1 & 1 & 1 & 1 & 1 & \cdots \\ 1 & 1 & 1 & 1 & 1 & 1 & \cdots \\ 2 & 3 & 4 & 5 & 6 & 7 & \cdots \\ 4 & 6 & 8 & 10 & 12 & 14 & \cdots \\ 9 & 15 & 22 & 30 & 39 & 49 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} = A \cdot B$$

where

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 1 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 1 & 0 & 0 & 0 & 0 & \cdots \\ 2 & 3 & 1 & 0 & 0 & 0 & \cdots \\ 4 & 6 & 2 & 0 & 0 & 0 & \cdots \\ 9 & 15 & 7 & 1 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \quad \textit{and} \quad B = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 1 & 1 & 1 & 1 & \cdots \\ 0 & 0 & 1 & 2 & 3 & 4 & \cdots \\ 0 & 0 & 0 & 1 & 3 & 6 & \cdots \\ 0 & 0 & 0 & 0 & 1 & 4 & \cdots \\ 0 & 0 & 0 & 0 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

are defined as follows:

- *The matrix $B = [b_{n,k}]_{n,k\geqslant 0}$ is defined by $b_{0,0} = 1$, and $b_{n,0} = b_{0,n} = 0$ if $n \geqslant 1$, and $b_{n,k} = \binom{k-1}{n-1}$ otherwise, which is the same as in* Proposition 4.3.
- *The matrix $A = [a_{n,k}]_{n,k\geqslant 0}$ is the almost Riordan array with initial column whose generating function is*

  $$g_0(z) = 1 + zM(z) = \frac{1 + z - \sqrt{1 - 2z - 3z^2}}{2z},$$

  which is followed by the shifted Riordan array $(g(z), z^2 g(z))$ where

  $$g(z) = \frac{1 - z - 2z^2 - \sqrt{1 - 2z - 3z^2}}{2z^3(1 + z)}.$$

Proof. The almost Riordan array A has generating function

$$g_0(z) + zu\frac{g(z)}{1 - z^2 u g(z)}.$$

Setting $G(z,u) = g_0(z) + zu\frac{g(z)}{1-z^2ug(z)}$, it suffices to check that the generating function corresponding to the matrix $A \cdot B$, that is $G(z, \frac{u}{1-u})$, coincides with the generating function $\texttt{Total}(z,u)$ of the matrix $\mathcal{T}$ given in Theorem 3.4. □

Proposition 4.10. *The matrix*

$$[t(n,k)]_{n\geqslant 1, k\geqslant 1} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & \cdots \\ 1 & 1 & 1 & 1 & 1 & \cdots \\ 3 & 4 & 5 & 6 & 7 & \cdots \\ 6 & 8 & 10 & 12 & 14 & \cdots \\ 15 & 22 & 30 & 39 & 49 & \cdots \\ 36 & 54 & 75 & 99 & 126 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

*is the rectification of the Riordan array $(M(z), zR(z))$ where $M(z) = \frac{1-z-\sqrt{1-2z-3z^2}}{2z^2}$ is the generating function of the Motzkin numbers, and $R(z) = \frac{1+z-\sqrt{1-2z-3z^2}}{2z(1+z)}$ is the generating function of the Riordan numbers (*A005043 *in* [13]*).*

Proof. It suffices to check that the generating function of $\mathcal{T}$, i.e. $\texttt{Total}(z,u)$, given in Theorem 3.4, equals to $g_0(z) + zu\frac{M(z)}{1-u\frac{zR(z)}{z}}$. □

We let m_n denote the n-th Motzkin number $m_n = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor}\binom{n}{2k}c_k$ where c_k is the k-th Catalan defined above.

Corollary 4.11. *We have*

$$t(n,k) = \begin{cases} [k=0] & \textit{if } n=0, \\ r(n-1,k) & \textit{otherwise,} \end{cases}$$

where

$$r(n,k) = \sum_{i=0}^{n} \frac{m_i \cdot (k+[n=i])}{n+k-i+[n=i]} \sum_{j=0}^{n-i} (-1)^{n-i-j}\binom{n+k-i+j-1}{j} \sum_{\ell=0}^{n+k-i}\binom{n+k-i}{\ell}\binom{\ell}{n-i-j-\ell}.$$

Proof. We have $(M(z), zR(z))^{-1} = \left(\frac{(1-x)^2}{1-x+x^2}, \frac{x(1-x)}{1-x+x^2}\right)$. If we denote by $(v(z), u(z))$ this inverse Riordan array, then we obtain

$$(M(z), zR(z)) = \left(\frac{1}{v(\bar{u}(z))}, \bar{u}(z)\right),$$

where $\bar{u}$ is the compositional inverse [8] of u. Using the definition of a Riordan array, and Lagrange inversion, we find that the Riordan array $(M(z), zR(z))$ has general term $\tilde{r}(n,k)$ given by

$$\sum_{i=0}^{n} \frac{m_i \cdot (k+[n=k+i])}{n-i+[n=k+i]} \sum_{j=0}^{n-k-i} (-1)^{n-k-i-j}\binom{n-i+j-1}{j} \sum_{\ell=0}^{n-i}\binom{n-i}{\ell}\binom{\ell}{n-k-i-j-\ell}.$$

This array begins as follows:

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 1 & 0 & 0 & 0 & 0 & \cdots \\ 2 & 1 & 1 & 0 & 0 & 0 & \cdots \\ 4 & 3 & 1 & 1 & 0 & 0 & \cdots \\ 9 & 6 & 4 & 1 & 1 & 0 & \cdots \\ 21 & 15 & 8 & 5 & 1 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

To rectify this array and thus to obtain the array $(M(z), R(z))$, we change n to $n+k$, and we obtain $r(n,k)$ above. To this array we must now prepend the row $(1,0,0,0,\ldots)$, and the result follows. □

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Independent coalition in graphs: existence and characterization

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Abstract

An independent coalition in a graph G consists of two disjoint sets of vertices V_1 and V_2 neither of which is an independent dominating set but whose union $V_1 \cup V_2$ is an independent dominating set. An independent coalition partition, abbreviated, ic-partition, in a graph G is a vertex partition $\pi = \{V_1, V_2, \ldots, V_k\}$ such that each set V_i of π either is a singleton dominating set, or is not an independent dominating set but forms an independent coalition with another set $V_j \in \pi$. The maximum number of classes of an ic-partition of G is the independent coalition number of G, denoted by $IC(G)$. In this paper, we study the concept of ic-partition. In particular, we discuss the possibility of the existence of ic-partitions in graphs and introduce a family of graphs for which no ic-partition exists. We also determine the independent coalition number of some classes of graphs and investigate graphs G of order n with $IC(G) \in \{1, 2, 3, 4, n\}$ and the trees T of order n with $IC(T) = n - 1$.

Keywords: Independent coalition, independent coalition partition, independent dominating set, idomatic partition.

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1 Introduction

Let $G = (V, E)$ denote a simple graph of order n with vertex set $V = V(G)$ and edge set $E = E(G)$. The *open neighborhood* of a vertex $v \in V$ is the set $N(v) = \{u | \{u, v\} \in E\}$, and its *closed neighborhood* is the set $N[v] = N(v) \cup \{v\}$. Each vertex of $N(v)$ is called a *neighbor* of v, and the cardinality of $N(v)$ is called the *degree* of v, denoted by $\deg(v)$ or

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$\deg_G(v)$. A vertex v of degree 1 is called a *pendant vertex* or *leaf*, and its neighbor is called a *support vertex*. A vertex of degree $n-1$ is called a *full vertex* while a vertex of degree 0 is called an *isolated vertex*. The *minimum* and *maximum degree* of G are denoted by $\delta(G)$ and $\Delta(G)$, respectively. For a set S of vertices of G, the subgraph induced by S is denoted by $G[S]$. For two sets X and Y of vertices, let $[X, Y]$ denote the set of edges between X and Y. If every vertex of X is adjacent to every vertex of Y, we say that $[X, Y]$ is *full*, while if there are no edges between them, we say that $[X, Y]$ is *empty*. A subset $V_i \subseteq V$ is called a *singleton set* if $|V_i| = 1$, and is called a *non-singleton set* if $|V_i| \geq 2$. The *join* $G+H$ of two disjoint graphs G and H is the graph obtained from the union of G and H by adding every possible edge between the vertices of G and the vertices of H. We denote the family of paths, cycles, complete graphs and stars of order n by P_n, C_n, K_n and $K_{1,n-1}$, respectively, and the complete k-partite graph with partite sets of order $n_1, n_2, \ldots, n_k$, by $K_{n_1,\ldots,n_k}$. A double star with respectively p and q leaves connected to each support vertex is denoted by $S_{p,q}$. The complete graph K_3 is called a *triangle*, and a graph is *triangle-free* if it has no K_3 as an induced subgraph. The *girth* of a graph with a cycle is the length of its shortest cycle. For a graph G, the girth of G is denoted by $g(G)$. For a graph G of order n, let $\overline{G}$ denote the complement of G with $V(\overline{G}) = V(G)$ and $E(\overline{G}) = E(K_n) - E(G)$ [13].

A set $S \subseteq V$ in a graph $G = (V, E)$ is called a *dominating set* if every vertex $v \in V$ is either an element of S or is adjacent to an element of S. A set $S \subseteq V$ is called an *independent set* if its vertices are pairwise nonadjacent. The *vertex independence number*, denoted by $\alpha(G)$, is the maximum cardinality of an independent set of G. An *independent dominating set* in a graph G is a set which is both independent and dominating.

A partition of the vertices of G into dominating sets (independent dominating sets) is called a domatic partition (idomatic partition). The maximum number of classes of a domatic partition (idomatic partition) of G is called the domatic number (idomatic number) of G, denoted by $d(G)$ ($id(G)$). The concepts of domination and domatic partition and their variations have been studied widely in the literature. See, for example, [1, 2, 3, 4, 5, 6, 12].

The term *coalition* was introduced by Haynes et al, [7] and has been studied further in [8, 9, 10, 11].

Definition 1.1 ([7])**.** A coalition in a graph G consists of two disjoint sets of vertices $V_1, V_2 \subset V$, neither of which is a dominating set but whose union $V_1 \cup V_2$ is a dominating set. We say that the sets V_1 and V_2 form a coalition, and are *coalition partners*.

Definition 1.2 ([7])**.** A *coalition partition*, henceforth called a *c-partition*, in a graph G is a *vertex partition* $\pi = \{V_1, V_2, \ldots, V_k\}$ such that every set V_i of π is either a *singleton dominating set*, or is not a dominating set but forms a coalition with another set V_j in π. The *coalition number* $C(G)$ equals the maximum order k of a c-partition of G, and a c-partition of G having order $C(G)$ is called a *$C(G)$-partition*.

Herein we will focus on coalitions involving independent dominating sets in graphs. In other words, we will study the concepts of independent coalition and independent coalition partition which have been introduced in [7] as an area for future research. We begin with the following definitions.

Definition 1.3. An *independent coalition* in a graph G consists of two disjoint sets of independent vertices V_1 and V_2, neither of which is an independent dominating set but whose union $V_1 \cup V_2$ is an independent dominating set. We say the sets V_1 and V_2 form an independent coalition, and are *independent coalition partners* (or *ic-partners*).

Definition 1.4. An *independent coalition partition*, abbreviated *ic-partition*, in a graph G is a vertex partition $\pi = \{V_1, V_2, \ldots, V_k\}$ such that every set V_i of π is either a singleton dominating set, or is not an independent dominating set but forms an independent coalition with another set $V_j \in \pi$. The *independent coalition number* $IC(G)$ equals the maximum number of classes of an ic-partition of G, and an ic-partition of G having order $IC(G)$ is called an *$IC(G)$-partition*.

Definition 1.5 ([8])**.** Let G be a graph of order n with vertex set $V = \{v_1, v_2, \ldots, v_n\}$. The *singleton partition*, denoted π_1, of G is the partition of V into n singleton sets, that is, $\pi_1 = \{\{v_1\}, \{v_2\}, \ldots, \{v_n\}\}$.

This paper is organized as follows. Section 1 is devoted to terminology and definitions. We discuss the possibility of the existence of ic-partitions in graphs and derive some bounds on independent coalition number in Section 2. In Section 3, we determine the independent coalition number of some classes of graphs. The graphs G with $IC(G) \in \{1, 2, 3, 4\}$ are investigated in Section 4. In Section 5, we characterize triangle-free graphs G with $IC(G) = n$ and trees T with $IC(T) = n - 1$. Finally, we end the paper with some research problems.

2 Independent coalition partition: existence and bound

This section is divided into two subsections. In the first subsection, we show that not all graphs admit an ic-partition, and in the second subsection, we present some bounds on $IC(G)$ whenever the graph G admits an ic-partition.

2.1 Existence

In the following definition, we construct graphs with arbitrarily large order for which no ic-partition exists.

Definition 2.1. Let $\mathcal{B}$ be the set of all graphs obtained from the complete graph K_n, $(n \geq 4)$ with the vertices v_i, $(1 \leq i \leq n)$, and two additional vertices v_{n+1}, v_{n+2} such that v_{n+1} and v_{n+2} are adjacent to v_n, and v_{n+1} is adjacent to v_{n-1}. Figure 1 illustrates such a graph for $n = 4$.

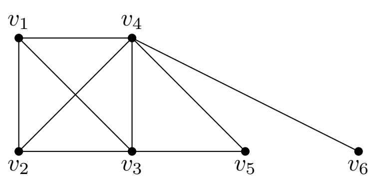

Figure 1: The graph G in $\mathcal{B}$ for $n = 4$.

Proposition 2.2. *Let G be a graph. If $G \in \mathcal{B}$, then G has no ic-partition.*

Proof. Suppose, to the contrary, that G has an ic-partition π. The vertices $v_1, v_2, \ldots, v_{n-1}$ are pairwise adjacent, so they must be in different classes. Further, v_n is a full vertex, so it must be in a singleton class. Since v_{n-1} is adjacent to all vertices except v_{n+2}, and

$\{v_{n-1}, v_{n+2}\}$ dominates G, it follows that $\{v_{n-1}\} \in \pi$. Further, since $\{v_{n-1}\}$ can only form an independent coalition with $\{v_{n+2}\}$, it follows that $\{v_{n+2}\} \in \pi$. If $\{v_{n+1}\} \in \pi$, then π is a singleton partition. In this case, $\{v_{n+1}\}$ has no ic-partner, a contradiction. Hence, $\{v_{n+1}\} \notin \pi$. It follows that π consists of a non-singleton set $\{v_{n+1}, v_i\}$ such that $v_i \in \{v_1, v_2, \ldots, v_{n-2}\}$, and n singleton sets. Assume, without loss of generality, that $\{v_{n+1}, v_1\} \in \pi$. Now for each $2 \leq i \leq n-2$, the set $\{v_i\}$ has no ic-partner, a contradiction. □

2.2 Bounds

Definition 1.4 implies that an ic-partition of a graph G is also a c-partition. Further, we note that an ic-partition of G is a proper coloring as well. Hence, we have the following two sharp bounds on $IC(G)$. To see the sharpness of them, consider the complete graph K_n.

Observation 2.3. Let G be a graph. If G has an ic-partition, then $IC(G) \leq C(G)$. Furthermore, this bound is sharp.

Observation 2.4. Let G be a graph. If G has an ic-partition, then $IC(G) \geq \chi(G)$. Furthermore, this bound is sharp.

Given a connected graph G and an ic-partition π of it, the following theorem shows that each set in π admits at most $\Delta(G)$ ic-partners.

Theorem 2.5. *Let G be a connected graph with maximum degree $\Delta(G)$, and let π be an ic-partition of G. If $X \in \pi$, then X is in at most $\Delta(G)$ independent coalitions. Furthermore, this bound is sharp.*

Proof. Let π be an ic-partition of G, and let X be a set in π. If X is a dominating set, then it has no ic-partner. Hence, we may assume that X does not dominate G. Let x be a vertex that is not dominated by X. Now every ic-partner of X must dominate x, that is, it must contain a vertex in $N[x]$. Hence, there are at most $|N[x]| \leq \Delta(G) + 1$ sets in π that can form an independent coalition with X. Now we show that X cannot form an independent coalition with $\Delta(G) + 1$ sets. Suppose, to the contrary, that X has $\Delta(G) + 1$ ic-partners (name $V_1, V_2, \ldots, V_{\Delta+1}$). Consequently, $[X, V_i]$ is empty for each $1 \leq i \leq \Delta(G) + 1$. Let $U = \bigcup_{i=1}^{\Delta(G)+1} V_i$, and $G' = G[U]$. Consider an arbitrary vertex $v \in U$ (say $v \in V_i$) and an arbitrary set V_j such that $1 \leq j \leq \Delta(G) + 1$ and $j \neq i$. Since $X \cup V_j$ dominates G and $[X, V_i]$ is empty, it follows that v has a neighbor in V_j. Choosing V_j arbitrarily, we conclude that $\deg_{G'}(v) \geq \Delta(G)$, and so $\deg_{G'}(v) = \Delta(G)$. Hence, for each $v \in U$, we have $\deg_{G'}(v) = \Delta(G)$. Now since G is connected, there is a path $P = (v_0, v_1, \ldots, v_k)$ connecting U to X such that $v_0 \in U$ and $v_k \in X$. Note that $[U, X]$ is empty, and so $V(P) \setminus (U \cup X) \neq \emptyset$. Let i be the smallest index for which $v_i \notin U \cup X$. It follows that $v_{i-1} \in U$, and so $\deg_{G'}(v_{i-1}) = \Delta(G)$. Thus, we have $\deg_G(v_{i-1}) \geq \deg_{G'}(v_{i-1}) + 1 = \Delta(G) + 1$, a contradiction.

To prove the sharpness, let G be the graph that is obtained from the complete graph K_n with vertices v_i, ($1 \leq i \leq n$), and a path $P_2 = (a, b)$, where b is adjacent to v_1. Let $A = \{a\}$, $B = \{b\}$ and $V_i = \{v_i\}$, for $1 \leq i \leq n$. One can observe that $\Delta(G) = n$ and that the singleton partition $\pi_1 = \{V_1, V_2, \ldots, V_n, A, B\}$ is an ic-partition of G such that A forms an independent coalition with V_i, for each $1 \leq i \leq n$. This completes the proof. □

Note that the bound presented in Theorem 2.5 does not hold for disconnected graphs. As a counterexample, consider the graph $G = K_2 \cup K_2$ and the singleton partition π_1 of it. On can verify that π_1 is an ic-partition of G such that each set in π_1 has two ic-partners, while $\Delta(G) = 1$.

The next bound relates independent coalition number of a graph to its idomatic number. As we will see in the proof of Theorem 2.6, any graph admitting an idomatic partition has an ic-partition. However, the converse is not necessarily true. For example, the singleton partition of the cycle C_5 is an ic-partition of it, while C_5 has no idomatic partition. Or the cycle C_7 has the ic-partition $\pi = \{\{v_1, v_5\}, \{v_2, v_4\}, \{v_3\}, \{v_6\}, \{v_7\}\}$, while it has no idomatic partition.

Theorem 2.6. *Let G be a connected graph, and let $r \geq 0$ be the number of full vertices of G. If G admits an idomatic partition, then $IC(G) \geq 2id(G) - r$.*

Proof. Let $F = \{v_1, v_2, \ldots, v_r\}$ be the set of full vertices of G, and let $\pi = \{V_1, V_2, \ldots, V_{id(G)}\}$ be an idomatic partition of G of order $id(G)$. Note that each full vertex must be in a singleton set of π. Without loss of generality, assume that $v_i \in V_i$, for each $1 \leq i \leq r$. It follows that for each $r + 1 \leq i \leq k$, we have $|V_i| \geq 2$. Now for each $r + 1 \leq i \leq k$, we partition V_i into two nonempty subsets $V_{i,1}$ and $V_{i,2}$. Note that no proper subset of V_i is a dominating set. Thus, neither $V_{i,1}$ nor $V_{i,2}$ is an independent dominating set, and so $V_{i,1}$ and $V_{i,2}$ are ic-partners. It follows that the partition $\pi' = \{V_1, V_2, \ldots, V_r, V_{r+1,1}, V_{r+1,2}, V_{r+2,1}, V_{r+2,2}, \ldots, V_{id(G),1}, V_{id(G),2}\}$ is an ic-partition of G of order $2id(G) - r$. Hence, $IC(G) \geq 2id(G) - r$. $\square$

3 Independent coalition number for some classes of graphs

Let us begin this section with some routine results.

Observation 3.1. For $n \geq 1$, we have $IC(K_n) = n$.

Observation 3.2. For $n \geq 3$, we have $IC(K_{1,n-1}) = 3$.

Observation 3.3. For $p, q \geq 1$, we have $IC(S_{p,q}) = 4$.

For complete multipartite graphs, the following result is obtained.

Proposition 3.4. *Let $G = K_{n_1,n_2,\ldots,n_k}$ be a complete k-partite graph with $m \geq 0$ full vertices (m partite sets of cardinality 1). Then $IC(G) = 2k - m$.*

Proof. Let $\pi = \{V_1, V_2, \ldots, V_k\}$ be the partition of G into its partite sets. Assume, without loss of generality, that the sets V_i, for $1 \leq i \leq m$, are those containing full vertices. Now for each $m + 1 \leq i \leq k$, we partition V_i into two sets $V_{i,1}$ and $V_{i,2}$. Observe that $V_{i,1}$ and $V_{i,2}$ are ic-partners, and so the partition $\pi' = \{\{V_1\}, \{V_2\}, \ldots, \{V_m\}, \{V_{m+1,1}, V_{m+1,2}\}, \{V_{m+2,1}, V_{m+2,2}\}, \ldots, \{V_{k,1}, V_{k,2}\}\}$ is an ic-partition of G of order $2k - m$. Thus, $IC(G) \geq 2k - m$. Now let π'' be an ic-partition of G. We note that π'' has the following properties:

- For any set $S \in \pi''$, all vertices in S are in the same partite set of G.
- For any set $V_i \in \pi$, the vertices in V_i are in at most two sets of π''.

Hence, we have $IC(G) \leq 2k - m$, and so $IC(G) = 2k - m$. $\square$

Next we determine the independent coalition number of all paths and cycles.

Lemma 3.5 ([7]). *For any path P_n, $C(P_n) \leq 6$.*

Theorem 3.6. *For the path P_n,*

$$IC(P_n) = \begin{cases} n & \text{if } n \leq 4; \\ 4 & \text{if } n = 5; \\ 5 & \text{if } n = 6, 7, 8, 9; \\ 6 & \text{if } n \geq 10. \end{cases}$$

Proof. It is clear that for $1 \leq n \leq 4$, we have $IC(P_n) = n$. Now let $n = 5$. Consider the path P_5 with $V(P_5) = \{v_1, v_2, v_3, v_4, v_5\}$. It is easily seen that $IC(P_5) \neq 5$. Thus, $IC(P_5) \leq 4$. The partition $\{\{v_1, v_3\}, \{v_2\}, \{v_4\}, \{v_5\}\}$ is an ic-partition of P_5, so $IC(P_5) = 4$. Now assume $n = 6$. Consider the path P_6 with $V(P_6) = \{v_1, v_2, v_3, v_4, v_5, v_6\}$. It is clear that $IC(P_6) \neq 6$. The partition $\{\{v_1, v_6\}, \{v_2\}, \{v_3\}, \{v_4\}, \{v_5\}\}$ is an ic-partition of P_6, so $IC(P_6) = 5$. Next assume $n = 7$. Consider the path P_7 with $V(P_7) = \{v_1, v_2, v_3, v_4, v_5, v_6, v_7\}$. By Lemma 3.5 and Observation 2.3, we have $IC(P_n) \leq 6$. Now we show that $IC(P_7) \neq 6$. Suppose that $IC(P_7) = 6$. Let π be an $IC(P_7)$-partition. We note that π consists of a set (name A) of cardinality 2 and five singleton sets. Since $\gamma_i(P_7) = 3$, each singleton set must be an ic-partner of A. On the other hand, Theorem 2.5 implies that A has at most two ic-partners, a contradiction. The partition $\{\{v_1, v_6\}, \{v_2, v_7\}, \{v_3\}, \{v_4\}, \{v_5\}\}$ is an ic-partition of P_7. Therefore, $IC(P_7) = 5$. Next we assume $n = 8$. Consider the path P_8 with $V(P_8) = \{v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8\}$. By Lemma 3.5 and Observation 2.3, we have $IC(P_n) \leq 6$. Now we show that $IC(P_8) \neq 6$. Suppose that $IC(P_8) = 6$. Let π be an $IC(P_8)$-partition. We consider two cases.

Case 1. π consists of a set (name A) of cardinality 3 and five singleton sets. Since $\gamma_i(P_8) = 3$, each singleton set must be an ic-partner of A. On the other hand, Theorem 2.5 implies that A has at most two ic-partners, a contradiction.

Case 2. π consists of two sets of cardinality 2 and four singleton sets. Since $\gamma_i(P_8) = 3$, each singleton set must be an ic-partner of a set of cardinality 2. Therefore, using Theorem 2.5, we deduce that for any two ic-partners C and D, it holds that $|C \cup D| = 3$. On the other hand, v_3 and v_6 are not present in any independent dominating set of cardinality 3, a contradiction.

The partition $\{\{v_1, v_3, v_6\}, \{v_2, v_7\}, \{v_8\}, \{v_4\}, \{v_5\}\}$ is an ic-partition of P_8. Therefore, $IC(P_8) = 5$.

Now let $n = 9$. Consider the path P_9 with $V(P_9) = \{v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8, v_9\}$. By Lemma 3.5 and Observation 2.3, we have $IC(P_n) \leq 6$. Now we show that $IC(P_9) \neq 6$. Suppose that $IC(P_9) = 6$. Let π be an $IC(P_9)$-partition. There exist three cases.

Case 1. π consists of a set (name A) of cardinality 4 and five singleton sets. Since $\gamma_i(P_9) = 3$, each singleton set must be an ic-partner of A. On the other hand, by Theorem 2.5, A has at most two ic-partners, a contradiction.

Case 2. π consists of a set (name A) of cardinality 3, a set (name B) of cardinality 2 and four singleton sets. Since $\gamma_i(P_9) = 3$, no two singleton sets in π are ic-partners. Furthermore, by Theorem 2.5, A has at most two ic-partners, so at least two singleton sets of π must be ic-partners of B, which is impossible, as P_9 has a unique independent dominating set of cardinality 3.

Case 3. π consists of three sets of cardinality 2, and three singleton sets. We note that each singleton set in π must be an ic-partner of a set of cardinality 2, which is impossible, as P_9 has a unique independent dominating set of cardinality 3.

The partition $\{\{v_1, v_3, v_5\}, \{v_2, v_4, v_9\}, \{v_6\}, \{v_7\}, \{v_8\}\}$ is an ic-partition of P_9. Therefore, $IC(P_9) = 5$.

Finally, let $n \geq 10$. Consider the path P_n with $V(P_n) = \{v_1, v_2, \dots, v_n\}$. By Lemma 3.5 and Observation 2.3, we have $IC(P_n) \leq 6$. Now we consider the sets $V_1 = \{v_1, v_6\} \cup \{v_{2n-1} : n \geq 5\}$, $V_2 = \{v_2, v_5\} \cup \{v_{2n} : n \geq 5\}$, $V_3 = \{v_3\}$, $V_4 = \{v_4\}$, $V_5 = \{v_7\}$, $V_6 = \{v_8\}$. Then $\pi = \{V_1, V_2, V_3, V_4, V_5, V_6\}$ is an ic-partition of P_n, where V_3 and V_4 are ic-partners of V_1, and V_5 and V_6 are ic-partners of V_2. So the proof is complete. □

Lemma 3.7 ([7])**.** *For any cycle C_n, $C(C_n) \leq 6$.*

Lemma 3.7 and Observation 2.3 imply the following result.

Lemma 3.8. *For any cycle C_n, $IC(C_n) \leq 6$.*

Lemma 3.9. *For any cycle C_n with $n \geq 8$ and $n \equiv 0 \pmod 2$, it holds that $IC(C_n) = 6$.*

Proof. Let $V(C_n) = \{v_1, v_2, \dots, v_n\}$. Consider the sets $V_1 = \{v_1, v_6\} \cup \{v_{2n-1} : n \geq 5\}$, $V_2 = \{v_2, v_5\} \cup \{v_{2n} : n \geq 5\}$, $V_3 = \{v_3\}$, $V_4 = \{v_4\}$, $V_5 = \{v_7\}$, $V_6 = \{v_8\}$. Then $\pi = \{V_1, V_2, V_3, V_4, V_5, V_6\}$ is an ic-partition of C_n, for $n \geq 8$, where V_3 and V_4 are ic-partners of V_1, and V_5 and V_6 are ic-partners of V_2. Hence, by Lemma 3.8 and Observation 2.3, we have $IC(C_n) = 6$. □

Lemma 3.10. *For any cycle C_n with $n \geq 8$ and $n \equiv 0 \pmod 3$, it holds that $IC(C_n) = 6$.*

Proof. Let $V(C_n) = \{v_1, v_2, \dots, v_{3k}\}$. Consider the sets $V_1 = \{v_{3i+1}\}$, $V_2 = \{v_{3i+2}\}$ and $V_3 = \{v_{3i+3}\}$, for $0 \leq i \leq k-1$. Now for each $1 \leq i \leq 3$, we partition V_i into two nonempty sets $V_{i,1}$ and $V_{i,2}$. Observe that $V_{i,1}$ and $V_{i,2}$ are ic-partners. Hence, by Lemma 3.8 and Observation 2.3, we have $IC(C_n) = 6$. □

Lemma 3.11. *For any cycle C_n with $n \geq 8$ and $n \equiv 5 \pmod 6$, it holds that $IC(C_n) = 6$.*

Proof. Assume $n = 6k-1$, ($k \geq 2$). Let $V(C_n) = \{v_1, v_2, \dots, v_n\}$. Consider the sets

$$A = \bigcup_{i=0}^{k-1} \{v_{3i+1}\},\ A_1 = \bigcup_{i=k}^{2k-1} \{v_{3i+1}\},\ A_2 = \bigcup_{i=k}^{2k-1} \{v_{3i}\},$$

$$B = \bigcup_{i=k}^{2k} \{v_{3i-1}\},\ B_1 = \bigcup_{i=1}^{k-1} \{v_{3i-1}\},\ B_2 = \bigcup_{i=1}^{k-1} \{v_{3i}\}.$$

Let $\pi = \{A, A_1, A_2, B, B_1, B_2\}$. One can observe that π is an ic-partition of C_n, where A_1 and A_2 are ic-partners of A, and B_1 and B_2 are ic-partners of B. Now using Lemma 3.8 and Observation 2.3, we have $IC(C_n) = 6$. □

Lemma 3.12. *For any cycle C_n with $n \geq 8$ and $n \equiv 1 \pmod 6$, it holds that $IC(C_n) = 6$.*

Proof. Assume $n = 6k + 1$, $(k \geq 2)$. Let $V(C_n) = \{v_1, v_2, \dots, v_n\}$. Consider the sets

$$A = \left(\bigcup_{i=0}^{k}\{v_{3i+1}\}\right) \cup \{v_{3k+3}\},\ A_1 = \bigcup_{i=k+2}^{2k}\{v_{3i}\},\ A_2 = \bigcup_{i=k+2}^{2k}\{v_{3i-1}\},$$

$$B = \left(\bigcup_{i=k+1}^{2k}\{v_{3i+1}\}\right) \cup \{v_{3k+2}\},\ B_1 = \bigcup_{i=1}^{k}\{v_{3i-1}\},\ B_2 = \bigcup_{i=1}^{k}\{v_{3i}\}.$$

Let $\pi = \{A, A_1, A_2, B, B_1, B_2\}$. One can observe that π is an ic-partition of C_n, where A_1 and A_2 are ic-partners of A, and B_1 and B_2 are ic-partners of B. Now using Lemma 3.8 and Observation 2.3, we have $IC(C_n) = 6$. $\square$

Theorem 3.13. *For the cycle C_n,*

$$IC(C_n) = \begin{cases} n & \text{if } n \leq 6; \\ 5 & \text{if } n = 7; \\ 6 & \text{if } n \geq 8. \end{cases}$$

Proof. If $1 \leq n \leq 6$, then it is easy to check that $IC(C_n) = n$. Now assume $n = 7$. Consider the cycle C_7 with $V(C_7) = \{v_1, v_2, v_3, v_4, v_5, v_6, v_7\}$. First we show that $IC(C_7) \neq 6$. Suppose, to the contrary, that $IC(C_7) = 6$. Let π be an $IC(C_7)$-partition. We note that π consists of five singleton sets and a set of cardinality 2 (name A). By Theorem 2.5, A has at most two ic-partners. Hence, π contains two singleton sets that are ic-partners, which contradicts the fact that $\gamma_i(C_7) = 3$. The partition $\{\{v_1, v_3\}, \{v_5\}, \{v_6\}, \{v_4, v_7\}, \{v_2\}\}$ is an ic-partition of C_7, so $IC(C_7) = 5$. Furthermore, by Lemmas 3.9, 3.10, 3.11 and 3.12 we have $IC(C_n) = 6$, for $n \geq 8$. $\square$

4 Graphs with small independent coalition number

In this section we investigate graphs G with $IC(G) \in \{1, 2, 3, 4\}$. We will make use of the following two lemmas.

Lemma 4.1. *Let G be a graph of order n containing $r \geq 1$ full vertices, and let $F = \{v_1, v_2, \dots, v_r\}$ be the set of full vertices of G. Then $IC(G) = k$, if and only if $IC(G[V \setminus F]) = k - r$, where $r < k \leq n$.*

Proof. Assume first that $IC(G[V \setminus F]) = k - r$. Let $\pi = \{V_1, V_2, \dots, V_{k-r}\}$ be an $IC(G[V \setminus F])$-partition. Now the partition $\pi' = \{V_1, V_2, \dots, V_{k-r}, \{v_1\}, \{v_2\}, \dots, \{v_r\}\}$, is an ic-partition of G, so $IC(G) \geq k$. Now we prove that $IC(G) = k$. Suppose, to the contrary, that $IC(G) > k$. Let π be an $IC(G)$-partition. Now the partition $\pi' = \pi \setminus \{\{v_1\}, \{v_2\}, \dots, \{v_r\}\}$ is an ic-partition of $G[V \setminus F]$ such that $|\pi'| > k - r$, a contradiction. Hence, $IC(G) = k$. Conversely, assume that $IC(G) = k$. Let π be an $IC(G)$-partition. Now the partition $\pi' = \pi \setminus \{\{v_1\}, \{v_2\}, \dots, \{v_r\}\}$ is an ic-partition of $G[V \setminus F]$, so $IC(G[V \setminus F]) \geq k - r$. Now we prove that $IC(G[V \setminus F]) = k - r$. Suppose, to the contrary, that $IC(G[V \setminus F]) > k - r$. Let π be an $IC(G[V \setminus F])$-partition. Now the partition $\pi' = \pi \cup \{\{v_1\}, \{v_2\}, \dots, \{v_r\}\}$ is an ic-partition of G such that $|\pi'| > k$, a contradiction. Hence, $IC(G[V \setminus F]) = k - r$. $\square$

Lemma 4.2. *Let G be a graph containing a nonempty set of isolated vertices I. If $IC(G) \geq 3$, then for any $IC(G)$-partition π, there is a set $V_r \in \pi$ such that $V_r = I$.*

Proof. First we show that all vertices in I are in the same set of π. Suppose, to the contrary, that there are sets $V_i \in \pi$ and $V_j \in \pi$ such that both V_i and V_j contain isolated vertices. Let $V_k \in \pi$ be an arbitrary set in π such that $V_k \notin \{V_i, V_j\}$. (Since $IC(G) \geq 3$, such a set exists). Then V_k has no ic-partner, a contradiction. Now let V_r be the set in π containing isolated vertices. Further, let v be an arbitrary vertex in V_r, and let $u \in V(G)$ be an arbitrary vertex such that $u \neq v$. If $u \in V_r$, then u is not adjacent to v. Otherwise, the set in π containing u is an ic-partner of V_r, which again implies that u is not adjacent to v. Hence, we have $\deg(v) = 0$. Choosing v arbitrarily, we conclude that $V_r = I$. □

Proposition 4.3. *Let G be a graph of order n. Then*

(1) $IC(G) = 1$ *if and only if* $G \simeq K_1$.

(2) $IC(G) = 2$ *if and only if* $G \simeq K_2$ *or* $G \simeq \overline{K}_n$, *for some* $n \geq 2$.

Proof. (1) It is clear that $IC(G) = 1$ if and only if $G \simeq K_1$.
(2) If $G \simeq K_2$, then we clearly have $IC(G) = 2$. Now assume $G \simeq \overline{K}_n$, for some $n \geq 2$. Let π be an ic-partition of G. Note that no more than two sets in π contain isolated vertices, for otherwise, no two sets in π are ic-partners. Thus, $|\pi| \leq 2$. Partitioning vertices of G into two nonempty sets yields an ic-partition of G. Hence, $IC(G) = 2$. Conversely, suppose that $IC(G) = 2$. Let $\pi = \{V_1, V_2\}$ be an $IC(G)$-partition. If both V_1 and V_2 are singleton dominating sets, then $G \simeq K_2$. Hence, we may assume that at least one of them (say V_1) is not a singleton dominating set. It follows that V_2 is not a singleton dominating set either, for otherwise, G is a star, and so by Observation 3.2, we have $IC(G) = 3$. Hence, V_1 and V_2 are ic-partners, and so $V = V_1 \cup V_2$ is an independent set. Hence, $G \simeq \overline{K}_n$, for some $n \geq 2$. □

Definition 4.4. Let $\mathcal{B}_1$ represent the family of bipartite graphs H with partite sets H_1 and H_2 such that $|H_1| \geq 2$, $|H_2| \geq 2$, $\delta(H) \geq 1$ and $id(H) = 2$.

Definition 4.5. For $m \geq 1$, let $\mathcal{B}_2$ represent the family of graphs $H \cup \overline{K}_m$, where H is a bipartite graph with $\delta(H) \geq 1$ and $id(H) = 2$.

Definition 4.6. For $m \geq 1$, let $\mathcal{B}_3$ represent the family of graphs $H \cup \overline{K}_m$, where H is a 3-partite graph with $\delta(H) \geq 1$ and $id(H) = 3$.

Proposition 4.7. *Let G be a graph. Then $IC(G) = 3$, if and only if $G \in \{K_3, K_{1,n-1}\} \cup \mathcal{B}_2$.*

Proof. Observations 3.1 and 3.2 imply that $IC(K_3) = 3$ and that $IC(K_{1,n-1}) = 3$, respectively. Now let $G \in \mathcal{B}_2$. Let I be the set of isolated vertices of G, and let $\{H_1, H_2\}$ be a partition of $G - I$ into its partite sets. We observe that the partition $\{I, H_1, H_2\}$ is an ic-partition of G, so $IC(G) \geq 3$. Now we show that $IC(G) = 3$. Suppose, to the contrary, that $IC(G) \geq 4$. Let $\pi = \{V_1, V_2, \ldots, V_k\}$ be an $IC(G)$-partition. By Lemma 4.2, we have $I \in \{V_1, V_2, \ldots, V_k\}$. Assume, without loss of generality, that $I = V_1$. Now for each $2 \leq i \leq k$, V_i forms an independent coalition with V_1, and so V_i dominates H. Hence, the partition $\{V_2, \ldots, V_k\}$ is an idomatic partition of H, which contradicts the assumption. Hence, $IC(G) = 3$. Conversely, let G be a graph with $IC(G) = 3$, and let

$\pi = \{V_1, V_2, V_3\}$ be an $IC(G)$-partition. We consider four cases depending on the number of full vertices of G.

Case 1. G has three full vertices. In this case, the sets V_1, V_2 and V_3 are all singleton dominating sets, so $G \simeq K_3$.

Case 2. G has two full vertices. Note that this case never occurs.

Case 3. G has one full vertex. Let v_1 be the full vertex of G. Lemma 4.1 implies that $IC(G - v_1) = 2$. Thus, by Proposition 4.3, either $G - v_1 \simeq K_2$, implying that $G \simeq K_3$, or $G - v_1 \simeq \overline{K}_n$, for some $n \geq 2$, which implies that $G \simeq K_{1,n-1}$, for some $n \geq 3$.

Case 4. G has no full vertex. Let I be the set of isolated vertices of G. First we note that V_1, V_2 and V_3 are not pairwise ic-partners, for otherwise, we have $G \simeq \overline{K}_n$, and so by Proposition 4.3, we have $IC(G) = 2$, a contradiction. Hence, π contains a set (say V_1) that forms an independent coalition with V_2 and V_3, while V_2 and V_3 are not ic-partners. Therefore, each vertex in V_1 is an isolated vertex, so it follows from Lemma 4.2 that $I = V_1$. Further, the sets V_2 and V_3 are independent dominating sets of $G[V_2 \cup V_3]$, implying that $id(G[V_2 \cup V_3]) \geq 2$. It remains to show that $id(G[V_2 \cup V_3]) = 2$. Suppose, to the contrary, that $id(G[V_2 \cup V_3]) \geq 3$. Let $\pi' = \{U_1, U_2, \ldots, U_k\}$, ($k \geq 3$), be an idomatic partition of $G[V_2 \cup V_3]$. Then the partition $\pi'' = \{U_1, U_2, \ldots, U_k, V_1\}$ is clearly an ic-partition of G, implying that $IC(G) \geq 4$, a contradiction. Hence, $G \in \mathcal{B}_2$. □

Proposition 4.8. *Let G be a graph. If $IC(G) = 4$, then $G \in \{K_4, K_2 + \overline{K}_n, K_1 + B\} \cup \mathcal{B}_1 \cup \mathcal{B}_3$, where $n \geq 2$ and $B \in \mathcal{B}_2$.*

Proof. Let $\pi = \{V_1, V_2, V_3, V_4\}$ be an ic-partition of G. We consider two cases.

Case 1. G has a full vertex. Let v_1 be a full vertex of G. Lemma 4.1 implies that $IC(G - v_1) = 3$. Thus, by Proposition 4.7, we have $G - v_1 \simeq K_3$, implying that $G \simeq K_4$, or $G - v_1 \simeq K_{1,n}$, for some $n \geq 2$, implying that $G \simeq K_2 + \overline{K}_n$, for some $n \geq 2$, or $G - v_1 \in \mathcal{B}_2$, which implies that $G \simeq K_1 + B$, where $B \in \mathcal{B}_2$.

Case 2. G has no full vertex. First assume that G contains a nonempty set I of isolated vertices. Then by Lemma 4.2, we have $I \in \pi$. Without loss of generality, assume that $I = V_4$. Now for each $1 \leq i \leq 3$, V_i must form an independent coalition with I. Thus, $U = G[V_1 \cup V_2 \cup V_3]$ is a 3-partite graph with $id(U) \geq 3$. Since $IC(G) = 4$, the case $id(U) > 3$ is impossible. Hence, $id(U) = 3$, and so $G \in \mathcal{B}_3$. Now assume that G contains no isolated vertex. Since G has neither full vertices nor isolated vertices, each set of π has either one or two ic-partners. If there is a set of π, (say V_1) having one ic-partner, (say V_2), then it follows that V_3 and V_4 are ic-partners, and so G is a bipartitie graph with partite sets $V_1 \cup V_2$ and $V_3 \cup V_4$. Otherwise, assume, without loss of generality, that V_2 and V_3 are ic-partners of V_1. It follows that V_4 has an ic-partner in $\{V_2, V_3\}$. By symmetry, we may assume that V_4 and V_3 are ic-partners. Then G is again a bipartitie graph with partite sets $V_1 \cup V_2$ and $V_3 \cup V_4$. Now using Theorem 2.6, we have $id(G) = 2$, and so $G \in \mathcal{B}_1$. □

5 Graphs with large independent coalition number

Our main goal in this section is to investigate structure of graphs G of order n with $IC(G) = n$, under specified conditions. In addition, we will characterize all trees T of order n with $IC(T) = n - 1$. Let us begin with an observation that characterizes all disconnected graphs G of order n with $IC(G) = n$.

Observation 5.1. Let G be a disconnected graph of order n. Then $IC(G) = n$ if and only if $G \simeq K_s \cup K_r$, for some $s \geq 1$, and $r \geq 1$.

Now we introduce two sufficient conditions for a graph G of order n to have independent coalition number n.

Observation 5.2. If G is a graph of order n with $\alpha(G) = 2$, then $IC(G) = n$.

Proof. Let G be a graph of order n such that $\alpha(G) = 2$. Consider the singleton partition π_1 of G. Note that for any two non-adjacent vertices v and u in $V(G)$, the sets $\{v\}$ and $\{u\}$ in π_1, are ic-partners. Hence, π_1 is an ic-partition of G, and so $IC(G) = n$. $\square$

Observation 5.3. Let G be a graph of order n. If G admits a partition of its vertices into two maximal cliques, then $IC(G) = n$.

5.1 Graphs G with $\delta(G) = 1$ and $IC(G) = n$

In this subsection, we characterize graphs G of order n with $\delta(G) = 1$ and $IC(G) = n$. We need the following definition.

Definition 5.4. Let G be a graph of order n, $(n \geq 3)$, and let $\delta(G) = 1$. Furthermore, let x be a pendant vertex of G, and let y be the support vertex of x. Then $G \in \mathcal{F}$ if and only if $V(G) \setminus \{x, y\}$ induces a clique.

Theorem 5.5. *Let G be a graph of order n with $\delta(G) = 1$. Then $IC(G) = n$ if and only if either $G \simeq K_2$, or $G \in \mathcal{F}$.*

Proof. Obviously, $IC(K_2) = 2$. Now assume that $G \in \mathcal{F}$. Let x be a pendant vertex of G, and let y be the support vertex of x. Further, let $U = V(G) \setminus \{x, y\}$. Note that U contains no full vertex. If y is a full vertex, then G is obtained from the complete graph K_{n-1}, where one of its vertices is adjacent to a leaf. In this case, we clearly have $IC(G) = n$. Thus, we may assume that y is not a full vertex, that is, there is a vertex $u \in U$ such that u is not adjacent to y. Then it is easy to verify that the sets $\{y\}$ and $\{u\}$ are ic-partners, and that each vertex in $U \setminus \{u\}$ forms an independent coalition with $\{x\}$. Therefore, $IC(G) = n$. Conversely, suppose that G is a graph with $\delta(G) = 1$ and $IC(G) = n$. Let x be a leaf of G, and let y be the support vertex of x. Consider the singleton partition π_1 of G. Note that each set in $\pi_1 \setminus \{\{x\}, \{y\}\}$ must be an ic-partner of $\{x\}$ or $\{y\}$, to dominate x. Let $A = N(y) \setminus \{x\}$, and $B = V(G) \setminus (\{x, y\} \cup A)$. We consider four cases.

Case 1. $A = \emptyset$ and $B = \emptyset$. In this case, we have $G \simeq K_2$.

Case 2. $A = \emptyset$ and $B \neq \emptyset$. By Observation 5.1, we have $G \simeq K_2 \cup K_r$, for some $r \geq 1$. Thus, $G \in \mathcal{F}$.

Case 3. $A \neq \emptyset$ and $B = \emptyset$. For each $v \in A$, the set $\{v\}$ cannot be an ic-partner of $\{y\}$, so it must be an ic-partner of $\{x\}$. This implies that A induces a clique. Hence, $G \in \mathcal{F}$.

Case 4. $A \neq \emptyset$ and $B \neq \emptyset$. For each $v \in A$, the set $\{v\}$ cannot be an ic-partner of $\{y\}$, so it must be an ic-partner of $\{x\}$. This implies that $[A, B]$ is full and that A induces a clique. Now for each vertex $u \in B$, in order for the set $\{u\}$ to be an ic-partner of $\{x\}$ or $\{y\}$, u must be adjacent to all other vertices in B. Hence, B induces a clique, and so $G \in \mathcal{F}$, which completes the proof. $\square$

As an immediate result from Theorem 5.5 we have:

Corollary 5.6. *Let T be a tree of order n. Then $IC(T) = n$ if and only if $T \in \{P_1, P_2, P_3, P_4\}$.*

5.2 Triangle-free graphs G with $IC(G) = n$

In this subsection, we characterize graphs G of order n with $g(G) = 4$ and $IC(G) = n$. This will lead to characterization of all triangle-free graphs G of order n with $IC(G) = n$. We will make use the following lemmas.

Lemma 5.7. *Let G be a triangle-free graph of order n with $IC(G) = n$. Then $g(G) \leq 6$.*

Proof. Let G be a graph of order n with $IC(G) = n$, and suppose, to the contrary, that $g(G) \geq 7$. Let $C \subseteq G$ be a cycle of order $g(G)$. Consider an arbitrary vertex $v \in V(C)$. Note that $\gamma_i(C) \geq 3$, and so $\{v\}$ is not an ic-partner of any set $\{u\} \subset V(C)$. Therefore, it must be an ic-partner of a set $\{u\} \subseteq V(G) \setminus V(C)$. It follows that, $\{u\}$ dominates $V(C) \setminus N_c[v]$, which implies that G contains triangles, a contradiction. □

Lemma 5.8. *Let G be a graph of order n with $g(G) = 6$. Then $IC(G) = n$ if and only if $G \simeq C_6$.*

Proof. Let G be a graph of order n with $g(G) = 6$. If $G \simeq C_6$, then by Theorem 3.13, we have $IC(G) = 6$. Conversely, assume that $IC(G) = n$. Let $C \subseteq G$ be a cycle of order 6, and suppose, to the contrary, that $V(G) \setminus V(C) \neq \emptyset$. Consider an arbitrary vertex $v \in V(G) \setminus V(C)$. If $\{v\}$ is an ic-partner of a set $\{u\} \subset V(C)$, then $\{v\}$ must dominate $V(C) \setminus N_c[u]$, which implies that G contains triangles, a contradiction. Otherwise, $\{v\}$ must be an ic-partner of a set $\{u\} \subset V(G) \setminus V(C)$. Now since $\{u, v\}$ dominates C, it follows that G contains triangles, or induces cycles of order 4, a contradiction. □

Our next result can be established almost the same way as Lemma 5.8, so we state it without proof.

Lemma 5.9. *Let G be a graph of order n with $g(G) = 5$. Then $IC(G) = n$ if and only if $G \simeq C_5$.*

In order to characterize graphs G of order n with $IC(G) = n$ and $g(G) = 4$, we need the following definitions.

Definition 5.10. Let $\mathcal{K}_0$ represent a bipartite graph with partite sets $H_1 = \{v_1, v_2, v_3, v_4\}$ and $H_2 = \{u_1, u_2, u_3, u_4\}$ such that for each $1 \leq i \leq 4$, v_i is adjacent to all vertices in H_2, except u_i (see Figure 2).

Definition 5.11. Let $\mathcal{K}$ represent a family of 4-partite graphs with partite sets $H_1 = \{v_1, v_2, v_3, v_4\}$, $H_2 = \{u_1, u_2, u_3, u_4\}$, $H_3 = \{n_1, n_2, \ldots, n_k\}$ and $H_4 = \{m_1, m_2, \ldots, m_k\}$, for $k \geq 1$, with the following properties:

- $[H_1, H_3]$ is full and $[H_2, H_4]$ is full,
- $[H_1, H_4]$ is empty and $[H_2, H_3]$ is empty,
- For each $1 \leq i \leq 4$, v_i is adjacent to all vertices in H_2, except u_i,
- For each $1 \leq i \leq k$, n_i is adjacent to all vertices in H_4, except m_i.

Figure 3 illustrates such a graph for $k = 3$.

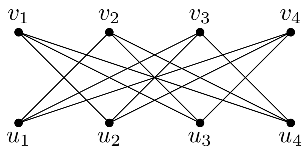

Figure 2: The graph $\mathcal{K}_0$.

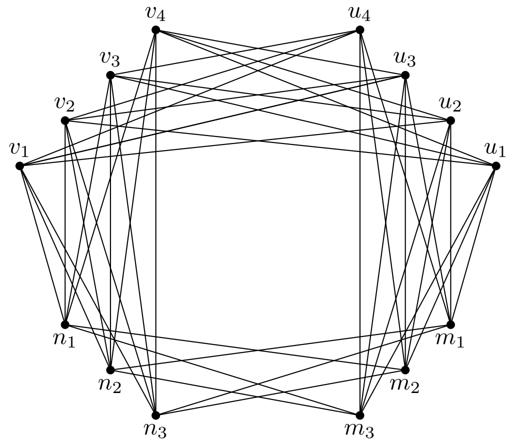

Figure 3: The graph in $\mathcal{K}$ for $k = 3$.

Theorem 5.12. *Let G be a graph of order n with $g(G) = 4$. Then $IC(G) = n$ if and only if $G \in \{C_4, \mathcal{K}_0\} \cup \mathcal{K}$.*

Proof. It is easy to check that $IC(C_4) = 4$ and that $IC(\mathcal{K}_0) = 8$. Now let $G \in \mathcal{K}$. We observe that for each $1 \leq i \leq 4$, $\{v_i\}$ and $\{u_i\}$ are ic-partners, and that for each $1 \leq i \leq k$, $\{n_i\}$ and $\{m_i\}$ are ic-partners. Thus, $IC(G) = n$. Conversely, let G be a graph of order n with $g(G) = 4$ and $IC(G) = n$, and let C be a cycle of G of order 4 with $V(C) = \{x, y, z, t\}$ and $E(C) = \{xy, yz, zt, tx\}$. If $G = C$, then the desired result follows. Hence, we assume that $G \neq C$. Since x is adjacent to y and t, neither $\{y\}$ nor $\{t\}$ is an ic-partner of $\{x\}$. Now consider two cases.

Case 1. $\{x\}$ and $\{z\}$ are ic-partners. In this case, G is dominated by $\{x, z\}$. Let $A = N(x) \setminus \{y, t\}$ and $B = N(z) \setminus \{y, t\}$. If $A \neq \emptyset$, (say $v \in A$), then it is not difficult to check that $\{v\}$ has no ic-partner. Thus, $A = \emptyset$, and so by symmetry, we have $B = \emptyset$. Hence, $G \simeq C_4$.

Case 2. $\{x\}$ and $\{z\}$ are not ic-partners. Let $\{e\}$ be an ic-partner of $\{x\}$. Since $\{x, e\}$ dominates G and z is not adjacent to x, it must be adjacent to e. Let $A = N(x) \setminus \{y, t\}$ and $B = N(e) \setminus \{z\}$. It is not difficult to verify that $A \cap B = \emptyset$. Now if $A = \emptyset$, then $\{z\}$ cannot form an independent coalition with any other set, so $A \neq \emptyset$. Let $\{f\} \subseteq A$ be an ic-partner of $\{z\}$. We note that if a set $\{g\}$ forms an independent coalition with $\{y\}$, then $g \in B$. Further, if a set $\{h\}$ forms an independent coalition with $\{t\}$ then $h \in B$. Let $\{g\}$ and $\{h\}$ be ic-partners of $\{y\}$ and $\{t\}$, respectively. Observe that $\{g\} \neq \{h\}$. Now let $A' = A \setminus \{f\}$ and $B' = B \setminus \{g, h\}$. There exist the following subcases.

Subcase 2.1. $A' = \emptyset$ and $B' = \emptyset$. In this case, we have $G \simeq \mathcal{K}_0$.

Subcase 2.2. $A' = \emptyset$ and $B' \neq \emptyset$. Let $v \in B'$. One can verify that $\{v\}$ cannot form an independent coalition with any other set. Thus, this case is impossible.

Subcase 2.3. $A' \neq \emptyset$ and $B' = \emptyset$. Let $v \in A'$. One can verify that $\{v\}$ cannot form an independent coalition with any other set. Thus, this case is impossible.

Subcase 2.4. $A' \neq \emptyset$ and $B' \neq \emptyset$. Let $v \in A'$. If a set $\{u\}$ forms an independent coalition with $\{v\}$, then $u \in B'$. Furthermore, for each vertex $u \in B'$, $\{u\}$ cannot form an independent coalition with more than one sets $\{v\} \subseteq A'$. Thus, $|A'| \leq |B'|$. Using a similar argument, we deduce that $|B'| \leq |A'|$, and so $|A'| = |B'|$. Consequently, the following statements hold in the graph G:

- $G[\{x, y, z, t, e, f, g, h\}]$ is a bipartite graph with partite sets $V_1 = \{x, z, g, h\}$ and $V_2 = \{y, t, e, f\}$, which is isomorphic to $\mathcal{K}_0$,
- $[V_1, A']$ is full and $[V_2, B']$ is full,
- $[V_1, B']$ is empty and $[V_2, A']$ is empty,
- $G[A' \cup B']$ is a bipartite graph with partite sets A' and B' such that $\deg_{G[A' \cup B']}(v) = |A'| - 1 = |B'| - 1$, for each $v \in A' \cup B'$.

Hence, $G \in \mathcal{K}$ and the proof is complete. □

Using Observation 5.1, Corollary 5.6, Lemmas 5.7, 5.8 and 5.9, and Theorem 5.12, we infer the following result.

Corollary 5.13. *Let G be a triangle-free graph of order n. Then $IC(G) = n$ if and only if $G \in \{C_4, C_5, C_6, P_1, P_2, P_3, P_4, \overline{K}_2, K_1 \cup K_2, K_2 \cup K_2, \mathcal{K}_0\} \cup \mathcal{K}$.*

5.3 Trees T with $IC(T) = n - 1$

The following theorem characterizes all trees T of order n with $IC(T) = n - 1$.

Theorem 5.14. *Let T be a tree of order n. Then $IC(T) = n - 1$ if and only if $T \in \{P_5, P_6, S_{1,2}, K_{1,3}\}$.*

Proof. By Theorem 3.6, we have $IC(P_5) = 4$ and $IC(P_6) = 5$. Further, by Observation 3.2, we have $IC(K_{1,3}) = 3$ and by Observation 3.3, we have $IC(S_{1,2}) = 4$. Conversely, let T be a tree of order n with $IC(T) = n - 1$, where x is a leaf, and y is the support vertex of x. Define $A = N(y) \setminus \{x\}$ and $B = V(G) \setminus (\{x, y\} \cup A)$. Further, let π be an $IC(T)$-partition. Note that π contains a set of cardinality 2 (say $V_1 = \{u, v\}$) and $n - 2$ singleton sets. Since x and y are adjacent, we have $V_1 \neq \{x, y\}$. Note as well that any set in π must be an ic-partner of the set containing x, or the set containing y, to dominate x. We consider two cases.

Case 1. $B = \emptyset$. If $A = \emptyset$, then we have $T \simeq K_2$, and so $IC(T) = 2 \neq n - 1$. Hence, $A \neq \emptyset$, and so $T \simeq K_{1,n-1}$, for some $n \geq 3$. Now by Lemma 3.7, we have $IC(T) = 3$. Hence, $T \simeq K_{1,3}$.

Case 2. $B \neq \emptyset$. Since T is connected, we have $A \neq \emptyset$. We divide this case into some subcases.

Subcase 2.1. $u \in A$ and $v \in B$. We first show that $|A| = 1$. Suppose, to the contrary, that $|A| \geq 2$. Then there is a vertex $z \in A$ such that $z \neq u$. Since z and y are adjacent,

$\{z\}$ cannot be an ic-partner of $\{y\}$, so it must be an ic-partner of $\{x\}$. Since $\{x\}$ does not dominate u, u must be adjacent to z, which is a contradiction, since y, z and u induce a triangle. Now $\{y\}$ cannot be an ic-partner of $\{x\}$ or $\{u, v\}$, so it must have an ic-partner in B. This implies that $|B| \geq 2$. Let $\{t\} \subset B$ be an ic-partner of $\{y\}$. we show that $B \setminus \{v, t\} = \emptyset$. Suppose that $B \setminus \{v, t\} \neq \emptyset$. Let $z \in B \setminus \{v, t\}$. Note that v and t are adjacent. Now $\{z\}$ must be an ic-partner of $\{x\}$ or $\{y\}$, so z must be adjacent to t and v, which is a contradiction, since z, t and v induce a triangle. Hence, $B = \{v, t\}$ and so $T \simeq P_5$.

Subcase 2.2. $\{u, v\} \subseteq B$. An argument similar to the one presented above implies that $|A| = 1$. Now we show that $B \setminus \{u, v\} = \emptyset$. Suppose that $B \setminus \{u, v\} \neq \emptyset$. Let $z \in B \setminus \{u, v\}$, and let $A = \{t\}$. Since t and y are adjacent, $\{t\}$ cannot be an ic-partner of $\{y\}$, so it must be an ic-partner of $\{x\}$. Thus, t must be adjacent to u, v and z. Now $\{z\}$ must be an ic-partner of $\{x\}$ or $\{y\}$, so z must be adjacent to u and v, which is a contradiction, since z, u and t induce a triangle. Hence, $T \simeq S_{1,2}$.

Subcase 2.3. $\{u, v\} \subseteq A$. We first show that $A \setminus \{u, v\} = \emptyset$. Suppose that $A \setminus \{u, v\} \neq \emptyset$. Let $z \in A \setminus \{u, v\}$. Since z is adjacent to y, $\{z\}$ must be an ic-partner of $\{x\}$, so z must be adjacent to u and v, which is a contradiction, since z, u and y induce a triangle. Now we show that $|B| = 1$. Suppose that $|B| \neq 1$. First assume $|B| \geq 3$. Let $z, t, w \in B$. Now z, t and w induce a triangle, since the sets containing each of them, must be an ic-partner of $\{x\}$ or $\{y\}$, a contradiction. Now assume $|B| = 2$. Let $B = \{z, t\}$. Each of the sets $\{z\}$ and $\{t\}$ must be an ic-partner of $\{x\}$ or $\{y\}$. Thus, z must be adjacent to t. Now $\{u, v\}$ must be an ic-partner of $\{x\}$, so z and t must be dominated by $\{u, v\}$. Now the induced subgraph $T[\{u, v, z, t\}]$ contains at least one cycle, a contradiction. Hence, we have $T \simeq S_{1,2}$.

Subcase 2.4. $u = y$ and $v \in B$. we first show that $|A| = 1$. Suppose that $|A| \geq 2$. Let $z, t \in A$. Since z and t are adjacent to y, $\{z\}$ and $\{t\}$ cannot be an ic-partner of y, so each of them must be an ic-partner of $\{x\}$. Thus, z must be adjacent to t, which is a contradiction, since z, t and y induce a triangle. Now we show that $|B| = 2$. Suppose that $|B| \neq 2$. If $|B| = 1$, then $T \simeq P_4$, a contradiction. Otherwise, let $\{v, z, t\} \subseteq B$ and $A = \{w\}$. Now $\{w\}$ cannot be an ic-partner of $\{u, v\}$, so it must be an ic-partner of $\{x\}$. Thus, w must be adjacent to v, z and t. Now observe that $\{u, v\}$ must have an ic-partner in B. Assume, without loss of generality, that $\{u, v\}$ and $\{z\}$ are ic-partners. This implies that t is adjacent to z or v, which is impossible, since both cases lead to existence of an induced triangle. Hence, $T \simeq S_{1,2}$.

Subcase 2.5. $u = x$ and $v \in A$. We first show that $|B| \leq 2$. Suppose that $|B| \geq 3$. Let $\{z, t, w\} \subseteq B$. Note that $\{y\}$ must have an ic-partner in B. Assume, without loss of generality, that $\{y\}$ and $\{z\}$ are ic-partners. It follows that z is adjacent to t and w. Now if $\{y\}$ is an ic-partner of $\{t\}$ or $\{w\}$, then t must be adjacent to w, which is impossible, since z, t and w induce a triangle. Hence, both t and w must be ic-partners of $\{u, v\}$, which implies that t is adjacent to w. Now z, t and w induce a triangle, a contradiction. Now we show that $|A| \leq 2$. Suppose that $|A| \geq 3$. Let $\{z, t, v\} \subseteq A$. The sets $\{z\}$ and $\{t\}$ must be ic-partners of $\{u, v\}$. This implies that z is adjacent to t. Now z, t and y induce a triangle, a contradiction. Further, we observe that the case $|A| = |B| = 2$ is impossible. Hence, either $|A| = 2$ and $|B| = 1$, which implies that $T \simeq S_{1,2}$, or $|A| = 1$ and $|B| = 2$, which implies that $T \simeq P_5$.

Subcase 2.6. $u = x$ and $v \in B$. We first show that $|A| = 1$. Suppose that $|A| \geq 2$. Let $\{z, t\} \subseteq A$. Now each of the sets $\{z\}$ and $\{t\}$ must be an ic-partner of $\{u, v\}$. This implies that z is adjacent to t, which is a contradiction, since y, z and t induce a triangle. Now we show that $|B| \leq 3$. Suppose that $|B| \geq 4$. Let $\{v, t, w, h\} \subseteq B$ and $A = \{z\}$. Note that $\{y\}$ must have an ic-partner in B. Assume, without loss of generality, that $\{y\}$ and $\{t\}$ are ic-partners. It follows that t is adjacent to w, h and v. Now $\{w\}$ must be an ic-partner of $\{y\}$ or $\{u, v\}$. One can observe that both cases lead to contradiction. Hence, either $|B| = 2$, which implies that $T \simeq P_5$, or $|B| = 3$, which implies that $T \simeq P_6$. □

6 Discussion and conclusions

In Proposition 2.2, we introduced a family of graphs admitting no ic-partition. This result motivates the following problem:

Problem 6.1. Characterize graphs admitting an ic-partition.

In Observations 2.3 and 2.4, we presented the sharp inequalities $IC(G) \leq C(G)$ and $IC(G) \geq \chi(G)$. This raises the following problems:

Problem 6.2. Characterize graphs G in which the equality $IC(G) = C(G)$ holds.

Problem 6.3. Characterize graphs G in which the equality $IC(G) = \chi(G)$ holds.

In Theorem 5.14, trees T of order n, with $IC(T) = n - 1$ have been characterized. This raises the following problem:

Problem 6.4. Characterize graphs G of order n with $IC(G) = n - 1$.

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Spectra of signed graphs and related oriented graphs

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Abstract

For every oriented graph G', there exists a bipartite signed graph $\dot{H}$ such that the spectrum of $\dot{H}$ contains the full information about the spectrum of the skew adjacency matrix of G'. This allows us to transfer some problems concerning the skew eigenvalues of oriented graphs to the framework of signed graphs, where the theory of real symmetric matrices can be employed. In this paper, we continue the previous research by relating the characteristic polynomials, eigenspaces and the energy of G' to those of $\dot{H}$. Simultaneously, we address some open problems concerning the skew eigenvalues of oriented graphs.

Keywords: Oriented graph, signed graph, eigenvalues, characteristic polynomial, eigenspaces, energy.

Math. Subj. Class. (2020): 05C20, 05C22, 05C50

1 Introduction

For a finite simple undirected graph $G = (V, E)$, an *oriented graph* G' is a pair (G, σ'), where σ' is the edge *orientation* satisfying $\sigma'(ij) \in \{i, j\}$, for every $ij \in E$. Similarly, a *signed graph* $\dot{G}$ is a pair $(G, \dot{\sigma})$, where $\dot{\sigma}$ is the edge *signature* satisfying $\dot{\sigma}(ij) \in \{+1, -1\}$, for every $ij \in E$. In both cases, G is referred to as the *underlying graph*. The *order* n is the number of vertices of G. The edge set of G' consists of oriented edges, where the edge ij is oriented from i to j if $\sigma'(ij) = j$; this is designated by $i \to j$ (or $j \leftarrow i$). The edge set of $\dot{G}$ consists of positive and negative edges.

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The *skew adjacency matrix* $S_{G'} = (s_{ij})$ of G' is the $n \times n$ matrix such that $s_{ij} = 0$ if ij is not an edge of G', $s_{ij} = 1$ if $i \to j$, and $s_{ij} = -1$ otherwise. This matrix is skew symmetric and differs from the adjacency matrix of G' whose (i, j)-entry is 0 whenever $i \leftarrow j$. The characteristic polynomial, the eigenvalues and the spectrum of $S_{G'}$ are known as the *skew characteristic polynomial*, the *skew eigenvalues*, the *skew spectrum* of G', respectively. To easy language and be consistent with the forthcoming terminology for signed graphs, in this article we omit the prefix 'skew'. The spectrum of G' consists of purely imaginary numbers, the non-zero eigenvalues come as complex conjugates, and thus the rank of $S_{G'}$ is even. Also, $S^{\intercal}_{G'}S_{G'} = -S^2_{G'}$.

The *adjacency matrix* $A_{\dot{G}}$ of $\dot{G}$ is obtained from the standard $(0, 1)$-adjacency matrix of G by reversing the sign of all edges mapped to -1 by $\dot{\sigma}$. By the *characteristic polynomial*, the *eigenvalues* and the *spectrum* of $\dot{G}$ we mean the characteristic polynomial, the eigenvalues and the spectrum of $A_{\dot{G}}$, respectively. Since $A_{\dot{G}}$ is symmetric, its eigenvalues are real.

An *r-cube* or a *hypercube* Q_r is the r-regular graph of order 2^r with vertex set $\{0, 1\}^r$ (all possible binary r-tuples) in which two vertices are adjacent if and only if they differ in exactly one coordinate. Accordingly, an oriented (resp. signed) r-cube is an oriented (signed) graph underlined by Q_r. If Γ is either an oriented graph or a signed graph, then its *energy* $\mathcal{E}(\Gamma)$ is the sum of modulus of its eigenvalues.

It follows from [19] that every oriented graph G' is related to a bipartite signed graph $\dot{H}$ in such a way that the spectrum of $\dot{H}$ contains the full information about the spectrum of G'. All necessary details about this relation are given in the next section. This means that the theory of spectra of oriented graphs is strongly connected to the theory of spectra of signed graphs, and that many problems concerning spectral parameters of oriented graphs can be transferred to the framework of signed graphs, where the entire theory of real symmetric matrices can be employed. In this way, some known results on oriented graphs are proved in an elementary way [19], some open problems are resolved [15] and some known results concerning signed graphs are transferred to the context of oriented graphs [18].

In this article we continue the research by expressing the coefficients of the characteristic polynomial of G' in terms of the coefficients of the characteristic polynomial of $\dot{H}$. We also generate the eigenspaces of G' on the basis of the eigenvectors of $\dot{H}$ and relate the energies of both graphs. The results on characteristic polynomials and energies are of particular interest since they address some open problems posed in literature. The results on eigenspaces are followed by an immediate application in the engineering domain.

Section 2 contains additional terminology and notation, along with a short review of results of [19]. In particular, oriented graphs are related to signed graphs in the forthcoming Theorem 2.1. Some comments and results that arise directly from this theorem are given in Section 3. Characteristic polynomials, eigenspaces and energies are considered in Sections 4–6, respectively. Some notes on particular (oriented or signed) hypercubes are given in Section 7.

2 Preliminaries

We say that an oriented or a signed graph is connected, regular, or bipartite if the same holds for its underlying graph. Similarly, a matching (or a perfect matching) refers to the matching in the underlying graph.

Two oriented (resp. signed) graphs with the same underlying graph are switching equivalent if there is a subset U of the vertex set V, such that one of them is obtained by reversing the orientation (sign) of every edge located between U and $V \setminus U$. In matrix terminology, G_1' and G_2' are switching equivalent if there exists a diagonal matrix S with ± 1 on the main diagonal, such that $S_{G_2'} = S^{-1} S_{G_1'} S$, and similarly for signed graphs. S is referred to as the *switching matrix*. Observe that the spectrum remains unchanged under the switching operation.

We say that an oriented even cycle $C_{2\ell}'$ is *oriented uniformly* if by traversing along the cycle we pass through an odd (resp. even) number of edges oriented in the route direction for ℓ odd (even), where the 'route direction' refers to any of two possible directions: clockwise or counterclockwise. A *canonical orientation* in a bipartite graph G is the orientation which orients all the edges from one colour class to the other. Clearly, in this orientation every cycle is oriented uniformly.

A cycle $\dot{C}$ in a signed graph is *positive* if the product of its edge signs $\dot{\sigma}(\dot{C})$ is 1. Otherwise, it is *negative*. A signed graph is said to be *homogeneous* if all edges have the same sign, e.g., if its edge set is empty. It is *balanced* if it switches to its underlying graph; equivalently, it does not contain negative cycles. The negation $-\dot{G}$ of a signed graph is obtained by reversing the sign of every edge of $\dot{G}$. Observe that if $\dot{G}$ is bipartite, then it is switching equivalent to $-\dot{G}$ and they share the same spectrum.

We proceed with results of [19]. The signature $\dot{\sigma}$ is *associated* with the orientation σ' (and also $\dot{G}$ is *associated* with G') if

$$\dot{\sigma}(ik)\dot{\sigma}(jk) = s_{ik}s_{jk} \text{ holds for every pair of edges } ik \text{ and } jk. \tag{2.1}$$

Being associated is a symmetric relation. We write $\dot{\sigma} \sim \sigma'$ to indicate that $\dot{\sigma}$ and σ' are mutually associated. The following results hold: If $\dot{\sigma} \sim \sigma'$, then $-S_{G'}^2 = A_{\dot{G}}^2$. For a graph G and an orientation σ', there exists a signature $\dot{\sigma}$ associated with σ' if and only if G is bipartite.

The next result gives a crucial relation between the spectrum of an oriented graph and the spectrum of a related signed graph. If G' is an oriented graph, then its *bipartite double* $\mathrm{bd}(G')$ is the oriented graph whose skew adjacency matrix is the Kronecker product $S_{\mathrm{bd}(G')} = A_{K_2} \otimes S_{G'}$, where K_2 is the complete graph with 2 vertices. This definition extends the definition of a bipartite double of an ordinary graph, where $\mathrm{bd}(G)$ has $A_{\mathrm{bd}(G)} = A_{K_2} \otimes A_G$ as the adjacency matrix. Evidently, if G underlies G', then $\mathrm{bd}(G)$ underlies $\mathrm{bd}(G')$. In our notation, exponents denote multiplicities of the eigenvalues.

Theorem 2.1. *Let $G' = (G, \sigma')$ be an oriented graph with $\mathrm{rank}(S_{G'}) = 2k$. The following statements hold true:*

(i) *If G' is bipartite and $\dot{\sigma} \sim \sigma'$, then $\pm i\lambda_1, \pm i\lambda_2, \dots, \pm i\lambda_k, 0^{(n-2k)}$ are the eigenvalues of G' if and only if $\pm\lambda_1, \pm\lambda_2, \dots, \pm\lambda_k, 0^{(n-2k)}$ are the eigenvalues of $\dot{G} = (G, \dot{\sigma})$.*

(ii) *If G' is non-bipartite, $H' = (\mathrm{bd}(G), \sigma'')$ is a bipartite double of G' and $\dot{\sigma} \sim \sigma''$, then $\pm i\lambda_1, \pm i\lambda_2, \dots, \pm i\lambda_k, 0^{(n-2k)}$ are the eigenvalues of G' if and only if $(\pm\lambda_1)^{(2)}, (\pm\lambda_2)^{(2)}, \dots, (\pm\lambda_k)^{(2)}, 0^{(2n-4k)}$ are the eigenvalues of $\dot{H} = (\mathrm{bd}(G), \dot{\sigma})$.*

Therefore, G' is related to a bipartite signed graph whose spectrum gives the full information on the spectrum of G'. If G' is bipartite, a required signed graph is its associate. If G' is non-bipartite, then a required signed graph is associated with $\mathrm{bd}(G')$. One may

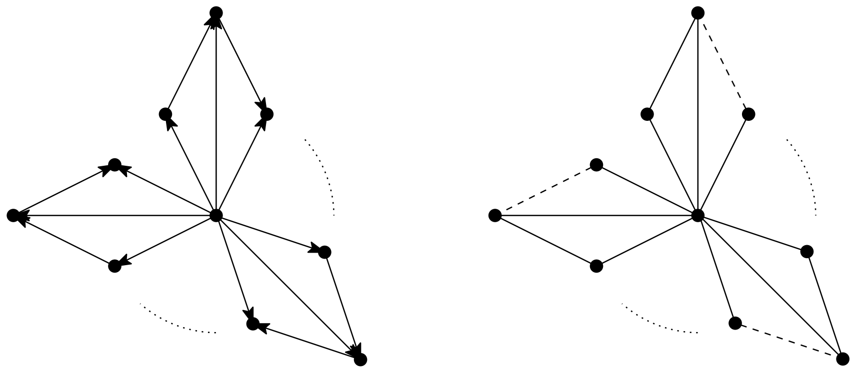

Figure 1: An infinite family of oriented graphs G' and signed graphs $\dot{G}$ such that $-S_{G'}^2 = A_{\dot{G}}^2$. Negative edges are dashed.

notice that item (ii) of the previous theorem covers the bipartite case in the sense that, if G' is bipartite, then $\mathrm{bd}(G')$ consists of two copies of G' and $\dot{H}$ also has two identical copies, each associated with G'. However, this would lead to unnecessary complicating, as there is no need to deal with a bipartite double if G' is already bipartite. Thus, the bipartite case is separated in item (i) of the same theorem.

3 Comments to Theorem 2.1

In the bipartite case, G' is associated with $\dot{G}$ if and only if it is associated with $-\dot{G}$. Hence, G' actually has two associates (with the same spectrum). Similarly, $\dot{G}$ is associated with G' if and only if it is associated with the oriented graph obtained by reversing the orientation of every edge of G'. Again, $\dot{G}$ has two associates which share the same spectrum. Henceforth, when we say 'let $\dot{G}$ be a signed graph associated with G'' (or something similar), we always mean that $\dot{G}$ is allowed to take any of the two options (that differ up to negation).

We have pointed out in the previous section that $-S_{G'}^2 = A_{\dot{G}}^2$ holds whenever G' and $\dot{G}$ are associated. However, this identity is not exclusively reserved for associated graphs. For example, Figure 1 illustrates infinite families of graphs that are not mutually associated in the sense of the equality (2.1) (since they are non-bipartite), but satisfy the previous matrix identity. In this case, they share the same underlying graph, but the identity can occur even if they do not. In fact, the identity occurs if and only if for every pair i, j of vertices, $-s_{ij}^{(2)} = a_{ij}^{(2)}$ holds, where an exponent indicates that we deal with the entry of matrix square. This leads to the following result.

Theorem 3.1. *If an oriented graph G' and a signed graph $\dot{G}$ are defined on the same vertex set, then $-S_{G'}^2 = A_{\dot{G}}^2$ holds if and only if for every pair of vertices i, j*

$$\big|\{k\,:\,(i\to k\wedge j\to k)\vee(i\leftarrow k\wedge j\leftarrow k)\}\big|-\big|\{k\,:\,(i\to k\wedge j\leftarrow k)\vee(i\leftarrow k\wedge j\to k)\}\big|$$

in G' is equal to

$$\big|\{k\,:\,\dot{\sigma}(ik)\dot{\sigma}(jk)=1\}\big|-\big|\{k\,:\,\dot{\sigma}(ik)\dot{\sigma}(jk)=-1\}\big|$$

in $\dot{G}$.

To demonstrate an application of Theorem 2.1, we deduce a known result. Observe that a bipartite canonically oriented graph is associated with a homogeneous signed graph, necessarily switching equivalent to its underlying graph, and so sharing the spectrum with it. Since bipartite oriented graphs are switching equivalent if and only if associated signed graphs are switching equivalent (where the equivalence is realized by the same switching matrix), we deduce the following result (conjectured in [7], and proved in [3]): If $\pm\mathrm{i}\lambda_1, \pm\mathrm{i}\lambda_2, \dots, \pm\mathrm{i}\lambda_n$ are the skew eigenvalues of a bipartite oriented graph $G' = (G, \sigma')$, then the eigenvalues of G are $\pm\lambda_1, \pm\lambda_2, \dots, \pm\lambda_n$ if and only if G' switches to a canonically oriented graph.

Here are more details on the cycle structure of $\dot{H}$ of Theorem 2.1(ii). It will be used in the forthcoming sections.

Theorem 3.2. *Let G' and $\dot{H}$ be as in* Theorem 2.1(ii). *For every odd cycle C' of G', the signed cycle $\dot{C}$ of $\dot{H}$ associated with* $\mathrm{bd}(C')$ *is negative.*

Proof. If the vertices of C', labelled in the natural order, are $1, 2, \dots, \ell$, then the vertices of its bipartite double $\mathrm{bd}(C')$ are divided into the colour classes, say $A = \{a_1, a_2, \dots, a_\ell\}$ and $B\{b_1, b_2, \dots, b_\ell\}$, the edges of $\mathrm{bd}(C')$ are $a_1b_2, b_2a_3, a_3b_4, \dots, b_{\ell-1}a_\ell, a_\ell b_1$, $b_1a_2, \dots, a_{\ell-1}b_\ell, b_\ell a_1$, and these edges inherit the orientation from G' in the sense that $i \to j$ implies $a_i \to b_j$ and $b_i \to a_j$.

Now, if the edges of G' are oriented in the route direction, so are the edges of $\mathrm{bd}(G')$. In this case, the edges of the associated signed cycle $\dot{C}$ alternate in sign, which in particular means that $\dot{C}$ is negative (as it counts exactly ℓ negative edges and ℓ is odd) and the edges a_ib_j and b_ia_j differ in sign (again, since ℓ is odd). If the edges of G' are not oriented in the route direction, then C' is obtained from the previous cycle by reversing the orientation of some edges. The desired conclusion follows since changing the orientation of a single edge ij changes the orientation of both a_ib_j and b_ia_j (in $\mathrm{bd}(C')$) and changes the sign of both a_ib_j and b_ia_j (in $\dot{C}$), so it does not change the signature of $\dot{C}$. □

Corollary 3.3. *Let G' and $\dot{H}$ be as in* Theorem 2.1(ii). *Then $\dot{H}$ is unbalanced and* $\mathrm{bd}(G')$ *contains a cycle that is not oriented uniformly.*

Proof. The first statement follows from the previous theorem. The second one follows from an easy observation that a negative cycle in $\dot{H}$ corresponds to a cycle in $\mathrm{bd}(G')$ that is not oriented uniformly. □

4 Characteristic polynomials

Here are relations between the coefficients of the characteristic polynomials.

Theorem 4.1. *If $\sum_{i=0}^{n} s_ix^{n-i}$ is the characteristic polynomial of an oriented graph G', then $s_i = 0$ for i odd and $s_i \geq 0$ for i even. If G' is bipartite and $\sum_{i=0}^{\lfloor n/2 \rfloor} a_{2i}x^{n-2i}$ is the characteristic polynomial of an associated signed graph, then*

$$a_{2i} = (-1)^i s_{2i}. \tag{4.1}$$

If G' is non-bipartite and $\sum_{i=0}^{n} a_{2i}x^{n-2i}$ is the characteristic polynomial of $\dot{H}$ (where $\dot{H}$ is as in the formulation of Theorem 2.1(ii)*), then*

$$a_{2i} = (-1)^i \sum_{\substack{\ell = -\min\{i, n-i\} \\ i \equiv \ell \pmod 2}}^{\min\{i,n-i\}} s_{i-\ell} s_{i+\ell}. \tag{4.2}$$

Proof. Under the assumptions of Theorem 2.1, the characteristic polynomial of the skew adjacency matrix of G' is

$$x^n + s_1 x^{n-1} + \cdots + s_{n-1}x + s_n = x^{n-2k}(x^2 + \lambda_1^2)(x^2 + \lambda_2^2) \cdots (x^2 + \lambda_k^2). \tag{4.3}$$

We immediately obtain $s_i = 0$ for i odd (since s_i is the ith elementary symmetric polynomial in eigenvalues), and $s_i \geq 0$ for i even.

If G' is bipartite, then the characteristic polynomial of an associated signed graph reads $\sum_{i=0}^{\lfloor n/2 \rfloor} a_{2i} x^{n-2i} = x^{n-2k}(x^2 - \lambda_1^2)(x^2 - \lambda_2^2) \cdots (x^2 - \lambda_k^2)$, which yields that even coefficients alternate in sign, and $|a_{2i}| = s_{2i}$. This implies the equality (4.1).

If G' is non-bipartite, then the characteristic polynomial of $\dot{H}$ is

$$\sum_{i=0}^{n} a_{2i} x^{n-2i} = \big(x^{n-2k}(x^2 - \lambda_1^2)(x^2 - \lambda_2^2) \cdots (x^2 - \lambda_k^2)\big)^2.$$

Comparing it with (4.3), we arrive at

$$a_{2i} = (-1)^i \sum_{\ell = -\min\{i,n-i\}}^{\min\{i,n-i\}} s_{i-\ell} s_{i+\ell},$$

but we know that odd coefficients under the sum are zero, so the summation reduces to even coefficients, which leads to (4.2). □

For $2i \leq n$, we note that the formula (4.2) is simplified to

$$a_{2i} = (-1)^i \sum_{\ell=0}^{i} s_{2\ell} s_{2(i-\ell)}.$$

We visualize this in the following example.

Example 4.2. Let G' be the oriented graph with 10 vertices illustrated in Figure 1. The coefficients of its characteristic polynomial are $(s_0, s_2, \ldots, s_{10}) = (1, 15, 60, 92, 48, 0)$. The coefficients of the characteristic polynomial of $\dot{H}$ of Theorem 2.1(ii) are

$$(a_0, a_2, \ldots, a_{20}) = (1, -30, 345, -1984, 6456, -12480, 14224, -8832, 2304, 0, 0).$$

Say, $345 = a_4 = s_0 s_4 + s_2^2 + s_4 s_0 = 60 + 225 + 60$ or $-8832 = a_{14} = (-1)^7(s_{10}s_4 + s_8 s_6 + s_6 s_8 + s_4 s_{10}) = 0 - 2 \cdot 92 \cdot 48 + 0$, as in Theorem 4.1.

We recall that a basic figure in a graph is a disjoint union of edges and cycles. If $\sum_{i=0}^{n} a_i x^i$ is the characteristic polynomial of the adjacency matrix of a signed graph $\dot{G} = (G, \dot{\sigma})$, then from [4]

$$a_i = \sum_{B \in \mathcal{B}_i} (-1)^{p(B)} \dot{\sigma}(B) 2^{|c(B)|},$$

where $\mathcal{B}_i$ is the set of basic figures on i vertices in G, $p(B)$ is the number of components of B, $c(B)$ is the set of cycles in B and $\dot\sigma(B) = \prod_{\dot C \in c(B)} \dot\sigma(\dot C)$. This result can be extended to oriented graphs on the basis of Theorems 2.1 and 4.1. However, this has already been performed in [5, pages 4516–4517] and [9, Theorem 2.3], and so we just refer the reader to these references. It is worth mentioning that these results address the research problem of [1, Section 6], asking for an interpretation of the coefficients s_i in terms of G'.

We conclude this section by the following observation. If the characteristic polynomials of G (the underlying graph), $\dot G = (G, \dot\sigma)$ (a signed graph) and $G' = (G, \sigma')$ (an oriented graph) are $\sum_{i=0}^n a_i(G)x^{n-i}$, $\sum_{i=0}^n a_i(\dot G)x^{n-i}$ and $\sum_{i=0}^n s_i(G')x^{n-i}$, respectively, then $a_i(G) = a_i(\dot G) = s_i(G') \pmod 2$, for $1 \le i \le n$, as $A_G = A_{\dot G} = S_{G'} \pmod 2$. Since $s_i(G') = 0$ for i odd, this in particular means that $a_i(G)$ and $a_i(\dot G)$ are even, whenever i is odd. Moreover, we have the following consequence.

Theorem 4.3. *For a graph G, let $\dot{\mathcal{G}}$ (resp. $\mathcal{G}'$) consist of all signed graphs (oriented graphs) having G as the underlying graph. If there is at least one $\Gamma \in \{G\} \cup \dot{\mathcal{G}} \cup \mathcal{G}'$, such that the determinant of Γ is odd, then G, all signed graphs of $\dot{\mathcal{G}}$ and all oriented graphs of $\mathcal{G}'$ are non-singular (i.e. their determinant is non-zero).*

Proof. This result follows from the previous observation applied to $a_0(G), a_0(\dot G)$ and $s_0(G')$. Indeed, if for example $a_0(\dot G)$ is odd, then $a_0(\dot G) = a_0(G) = 1 \pmod 2$, and the same holds for the elements of $\dot{\mathcal{G}} \cup \mathcal{G}'$ in the role of G. □

5 Eigenspaces

Let $\mathcal{E}(\lambda)$ and $\mathcal{E}(-\lambda)$ be the eigenspaces of eigenvalues λ ($\neq 0$) and $-\lambda$ of a bipartite signed graph $\dot G$, and $\mathcal{E}(\mathrm{i}\lambda)$, $\mathcal{E}(-\mathrm{i}\lambda)$ the eigenspaces of $\mathrm{i}\lambda$ and $-\mathrm{i}\lambda$ belonging to the spectrum of an associated oriented graph G'. Then $\mathcal{E}(\mathrm{i}\lambda) \cup \mathcal{E}(-\mathrm{i}\lambda)$ is spanned (in $\mathbb{C}^n$) by the union of bases of $\mathcal{E}(\lambda)$ and $\mathcal{E}(-\lambda)$. Indeed, since $-S_{G'}^2 = A_{\dot G}^2$, the eigenspace of λ^2 for $A_{\dot G}^2$ coincides with the eigenspace of $-\lambda^2$ for S^2, and thus the former eigenspace is spanned by the union of eigenbases of λ and $-\lambda$ (for $A_{\dot G}$), and the latter one is spanned by the union of eigenbases of $\mathrm{i}\lambda$ and $-\mathrm{i}\lambda$ (for $S_{G'}$). By the same reasoning, if 0 is an eigenvalue of $\dot G$, then it has the same eigenspace for $\dot G$ and G'. However, we can say more.

We first note the following, probably known, result. If $\mathbf{x}$ is an eigenvector associated with the eigenvalue λ ($\neq 0$) of a bipartite signed graph $\dot G$, then by negating the entries on one colour class, we get an eigenvector associated with $-\lambda$. Indeed, with an appropriate vertex labelling, the adjacency matrix has the form

$$A_{\dot G} = \begin{pmatrix} O & B \\ B^\intercal & O \end{pmatrix} \tag{5.1}$$

If $\mathbf{x} = \begin{pmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{pmatrix}$, then $A_{\dot G}\begin{pmatrix} -\mathbf{x}_1 \\ \mathbf{x}_2 \end{pmatrix} = \begin{pmatrix} B\mathbf{x}_2 \\ -B^\intercal\mathbf{x}_1 \end{pmatrix} = -\lambda\begin{pmatrix} -\mathbf{x}_1 \\ \mathbf{x}_2 \end{pmatrix}$, as desired. If $A_{\dot G}$ has no the previous form, the eigenvectors are permutationally equal, and the result follows. Henceforth, we assume that the adjacency matrix of a bipartite signed graph is as in (5.1).

Theorem 5.1. *Let $\mathbf{x}_j = \begin{pmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{pmatrix}$ and $\mathbf{x}_{-j} = \begin{pmatrix} -\mathbf{x}_1 \\ \mathbf{x}_2 \end{pmatrix}$ be eigenvectors associated with distinct eigenvalues λ_j and $-\lambda_j$ of a bipartite signed graph $\dot G$, and let, with a consistent vertex labelling, G' be an associated oriented graph. Then $\mathbf{x}_j + \mathrm{i}\mathbf{x}_{-j}$ and $\mathbf{x}_j - \mathrm{i}\mathbf{x}_{-j}$ are eigenvectors associated with $\mathrm{i}\lambda_j$ and $-\mathrm{i}\lambda_j$.*

Proof. If the adjacency matrix of $\dot{G}$ has the form (5.1), then the skew adjacency matrix of G' is $S_{G'} = \begin{pmatrix} O & B \\ -B^\intercal & O \end{pmatrix}$. Indeed, in the adjacency matrix of the underlying graph G, B has no negative entries, and reversing the sign of an edge uv changes the sign of both a_{uv}, a_{vu} in $A_{\dot{G}}$ and both s_{uv}, s_{vu} in $S_{G'}$.

We compute

$$\begin{aligned} S_{G'}(\mathbf{x}_j \pm \mathrm{i}\mathbf{x}_{-j}) &= \begin{pmatrix} B\mathbf{x}_2 \\ -B^\intercal\mathbf{x}_1 \end{pmatrix} \pm \mathrm{i} \begin{pmatrix} B\mathbf{x}_2 \\ B^\intercal\mathbf{x}_1 \end{pmatrix} = \begin{pmatrix} \lambda_j\mathbf{x}_1 \\ -\lambda_j\mathbf{x}_2 \end{pmatrix} \pm \mathrm{i} \begin{pmatrix} \lambda_j\mathbf{x}_1 \\ \lambda_j\mathbf{x}_2 \end{pmatrix} \\ &= \pm\,\mathrm{i}\lambda_j \left(\begin{pmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{pmatrix} \pm \mathrm{i} \begin{pmatrix} -\mathbf{x}_1 \\ \mathbf{x}_2 \end{pmatrix} \right) = \pm\mathrm{i}\lambda_j(\mathbf{x}_j \pm \mathrm{i}\mathbf{x}_{-j}), \end{aligned}$$

as desired. □

The previous theorem tells us that, in the bipartite case, the eigenspaces of $\mathrm{i}\lambda_j$ and $-\mathrm{i}\lambda_j$ of G' contain vectors of the form

$$\mathbf{y}_{\mathrm{i}j} = \begin{pmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{pmatrix} + \mathrm{i} \begin{pmatrix} -\mathbf{x}_1 \\ \mathbf{x}_2 \end{pmatrix} \text{ and } \mathbf{y}_{-\mathrm{i}j} = \overline{\mathbf{y}}_{\mathrm{i}j} = \begin{pmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{pmatrix} - \mathrm{i} \begin{pmatrix} -\mathbf{x}_1 \\ \mathbf{x}_2 \end{pmatrix},$$

respectively. Conversely, if these eigenvectors are given, then the eigenvectors associated with λ_j and $-\lambda_j$ of $\dot{G}$ are easily computed. We proceed with the non-bipartite case.

Theorem 5.2. *Let G' be a non-bipartite oriented graph and $\mathbf{y}$ and $\overline{\mathbf{y}}$ eigenvectors associated with the eigenvalues $\mathrm{i}\lambda_j$ and $-\mathrm{i}\lambda_j$, respectively. Then the pair $\begin{pmatrix} \mathbf{y} \\ \mathbf{y} \end{pmatrix}, \begin{pmatrix} \mathbf{y} \\ -\mathbf{y} \end{pmatrix}$ (resp. $\begin{pmatrix} \overline{\mathbf{y}} \\ \overline{\mathbf{y}} \end{pmatrix}, \begin{pmatrix} \overline{\mathbf{y}} \\ -\overline{\mathbf{y}} \end{pmatrix}$) is associated with $\mathrm{i}\lambda_j$ (resp. $-\mathrm{i}\lambda_j$) in the bipartite double $\mathrm{bd}(G')$, where $S_{\mathrm{bd}(G')} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes S_{G'}$. Let $\dot{H}$ be a signed graph associated with $\mathrm{bd}(G')$, and k the multiplicity of λ_j in the spectrum of $\dot{H}$. The first (resp. second) pair is spanned by $\mathbf{x}_{j\ell} + \mathrm{i}\mathbf{x}_{-j\ell}$ (resp. $\mathbf{x}_{j\ell} - \mathrm{i}\mathbf{x}_{-j\ell}$), for $1 \leq \ell \leq k$, where $\mathbf{x}_{j\ell}$ and $\mathbf{x}_{-j\ell}$ span the eigenspaces of λ_j and $-\lambda_j$, respectively.*

Proof. Since $S_{\mathrm{bd}(G')}$ is the tensor product, the eiegnvectors of $\mathrm{i}\lambda_j$ for $\mathrm{bd}(G')$ are $(1,1)^\intercal \otimes \mathbf{y}$ and $(1,-1)^\intercal \otimes \mathbf{y}$, and similarly for $-\mathrm{i}\lambda_j$. This establishes the proof of the first part of the statement.

By Theorem 5.1, the eigenvectors of $\mathrm{i}\lambda_j$ for $\mathrm{bd}(G')$ are $\mathbf{x}_{j\ell} + \mathrm{i}\mathbf{x}_{-j\ell}$, for $1 \leq \ell \leq k$. They are obviously linearly independent (because $\mathbf{x}_{j\ell}$ and $\mathbf{x}_{-j\ell}$ are). Since the multiplicity of $\mathrm{i}\lambda_j$ in the spectrum of $\mathrm{bd}(G')$ is k (cf. Theorem 2.1(ii)), its geometric multiplicity is at most k and the desired conclusion follows. □

We proceed with an application in control theory. The following differential equation is the standard model of multi-agent single-input linear control systems:

$$\frac{d\mathbf{x}}{dt} = M\mathbf{x} + \mathbf{b}u, \tag{5.2}$$

where the scalar $u = u(t)$ is the control input, $M \in \mathbb{R}^n \times \mathbb{R}^n$ and $\mathbf{x}, \mathbf{b} \in \mathbb{R}^n$. The system is controllable if for any $\mathbf{x}^\star$ and time $t^\star$, there exists a control function $u(t)$, $0 \leq t \leq t^\star$, such that the solution of the differential equation gives $\mathbf{x}^\star = \mathbf{x}(t^\star)$ irrespective of $\mathbf{x}(0)$. There are many controllable criteria, one of which is the Popov-Belevitch-Hautus Test to be found in any of [6, 11]. Accordingly, the system is controllable if and only if there is no $\mathbf{z} \in \mathbb{C}^n \setminus \{\mathbf{0}\}$ such that $\mathbf{z}^* M = \lambda \mathbf{z}^*$ and $\mathbf{z}^*\mathbf{b} = 0$, where $*$ denotes the complex conjugate.

Theorem 5.3. *Let A be the adjacency matrix of a bipartite signed graph and S the skew adjacency matrix of an associated oriented graph. If the system* (5.2) *with $M = A$ is controllable, then it is controllable with $M = S$.*

Proof. Since the system with $M = A$ is controllable, we have $\mathbf{x}^\intercal\mathbf{b} \neq 0$ for every eigenvector of A. Moreover A has no repeated eigenvalues. Indeed, if $\mathbf{x}_1$ and $\mathbf{x}_2$ are linearly independent eigenvectors associated with the same eigenvalue, with $\mathbf{x}_1^\intercal\mathbf{b} = b_1 \neq 0$ and $\mathbf{x}_2^\intercal\mathbf{b} = b_2 \neq 0$, then $(b_2\mathbf{x}_1 - b_1\mathbf{x}_2)^\intercal\mathbf{b} = 0$.

As $\mathbf{z}^* S = \lambda\mathbf{z}^*$ is equivalent to $S\mathbf{z} = \lambda\mathbf{z}$, and $\mathbf{z}$ is an eigenvector for S if and only if $\mathbf{z}^*$ is an eigenvector for the same matrix, we have to show that for every eigenvector $\mathbf{y}$ of S, $\mathbf{y}^*\mathbf{b} = 0$ holds.

Since A and S share the same eigenspace for 0, the claim holds for eigenvectors associated with 0 (if any). Every eigenvector associated with $\mathrm{i}\lambda_j$ has the form $z(\mathbf{x}_j + \mathrm{i}\mathbf{x}_{-j})$, where $z = z_1 + \mathrm{i}z_2 \neq 0$, and $\mathbf{x}_j$ and $\mathbf{x}_{-j}$ are as in the formulation of Theorem 5.1. As before, we may take $\mathbf{x}_j^\intercal\mathbf{b} = b_1 \neq 0$ and $\mathbf{x}_{-j}^\intercal\mathbf{b} = b_2 \neq 0$. Then

$$\begin{aligned}(z(\mathbf{x}_j + \mathrm{i}\mathbf{x}_{-j}))^*\mathbf{b} &= \big(z_1\mathbf{x}_j - z_2\mathbf{x}_{-j} + \mathrm{i}(z_1\mathbf{x}_{-j} + z_2\mathbf{x}_j)\big)^*\mathbf{b}\\ &= (z_1\mathbf{x}_j - z_2\mathbf{x}_{-j})^\intercal\mathbf{b} - \mathrm{i}(z_1\mathbf{x}_{-j} + z_2\mathbf{x}_j)^\intercal\mathbf{b}\\ &= z1b_1 - z_2b_2 + \mathrm{i}(z_1b_2 + z_2b_1).\end{aligned}$$

Equating with 0, we obtain $z_1 = \frac{b_2}{b_1}z_2$ and $z_2 = -\frac{b_2}{b_1}z_1$, which leads to the impossible scenario $z_1 = z_2 = 0$. Hence, the system with $M = S$ is controllable. $\square$

6 Energy

We start with the following result concerning signed graphs. There is a similar result for oriented graphs reported in [1].

Theorem 6.1. *Let $\mathcal{E}(\dot{G})$ be the energy of a signed graph $\dot{G}$ having n vertices, m edges, average vertex degree $\overline{d} = \frac{2m}{n}$ and eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$. Then*

$$\sqrt{2\bigg(m + \binom{n}{2}\Big(\prod_{i=1}^{n}|\lambda_i|\Big)^{\frac{2}{n}}\bigg)} \leq \mathcal{E}(\dot{G}) \leq n\sqrt{\overline{d}}. \tag{6.1}$$

Both equalities hold if and only if $\dot{G}$ is either edgeless or has exactly two distinct eigenvalues and these eigenvalues are equal in absolute value.

Proof. The proof of inequalities of (6.1) is an imitation of the proof of [1, Theorem 2.5] (concerning oriented graphs). Namely, both follow from the next chain of inequalities and equalities:

$$2\Big(m+\binom{n}{2}\Big(\prod_{i=1}^{n}|\lambda_i|\Big)^{\frac{2}{n}}\Big)\leq\sum_{i=1}^{n}|\lambda_i|^2+2\sum_{1\leq i<j\leq n}|\lambda_i||\lambda_j|=\Big(\sum_{i=1}^{n}|\lambda_i|\Big)^2$$
$$\leq n\sum_{i=1}^{n}|\lambda_i|^2=2mn=n^2\overline{d}.$$

To clarify, the first inequality in above chain follows from the inequality between the geometric mean and the arithmetic mean, in our case:

$$\Big(\prod_{i=1}^{n}|\lambda_i|\Big)^{\frac{2}{n}}=\sqrt[n(n-1)]{\prod_{i=1}^{n}|\lambda_i|^{2(n-1)}}=\sqrt[n(n-1)]{\prod_{1\leq i<j\leq n}(|\lambda_i||\lambda_j|)^2}$$
$$\leq\frac{2\sum_{1\leq i<j\leq n}|\lambda_i||\lambda_j|}{n(n-1)}.$$

The second inequality in the chain is a consequence of the Cauchy-Schwarz inequality.

Consider now the equality cases. The first equality in (6.1) holds if and only if $|\lambda_i||\lambda_j|$ is a constant for every $i\neq j$. This occurs if either $\lambda_i=0$, for all i, or $\lambda_i=\pm\lambda_j\neq 0$, for all $i\neq j$. The second equality holds if and only if $(|\lambda_1|,|\lambda_2|,\dots,|\lambda_n|)$ is a constant vector (possibly zero). Hence, both inequalities hold if and only if $\dot{G}$ is as in the statement formulation. □

We observe that, in the particular case of graphs, the equalities in (6.1) are attained if and only if G a disjoint union of either isolated vertices or isolated edges. In the 'signed' case, there are many other examples. All of them are regular with an even number of vertices, say $2n$, and a symmetric spectrum of the form $[\lambda^n,(-\lambda)^n]$. These signed graphs belong to the class of strongly regular signed graphs in the sense of definition of [13]; we note in passing that this definition generalizes the concept of strongly regular graphs. Considering the minimal polynomial, we deduce that the common vertex degree is equal to λ^2. Therefore, λ is the square root of an integer. Disconnected examples are not of interest, since each of them is a disjoint union of connected ones. We know from [14] that a signed r-cube without positive quadrangles has the spectrum $\big[\sqrt{r}^{2^{r-1}},(-\sqrt{r})^{2^{r-1}}\big]$. There are no other examples for $\lambda\leq\sqrt{3}$. Signed graphs with spectrum $[2^n,(-2)^n]$ are completely determined in [10, 16] (see also [12]). For those with $[\sqrt{5}^n,(-\sqrt{5})^n]$, see [17]. Some other constructions can be found in [8]. We also remark that all signed line graphs with the required spectrum are constructed in [16].

Observe next that the upper bound (6.1) does not depend on the eigenvalues of $\dot{G}$. This, in particular, means that it simultaneously holds for a signed graph $\dot{G}$ and its underlying graph G. According to the previous discussion, in the connected case, this bound is simultaneously attained for $\dot{G}$ and G if and only if $G\cong K_1$ or $G\cong K_2$; so, in exactly two simple cases, and in both $\dot{G}$ switches to G. For the remaining connected signed graphs that attain this bound, their underlying graphs do not attain it. This leads to an assumption that the energy of a signed graph could often be larger than the energy of its underlying graph. In this context we have experimented with a large number of connected graphs having small order, or small number of edges (i.e., obtained by inserting few edges to a tree), or large number of edges (i.e., obtained by deleting few edges of a complete graph). Our conclusions are

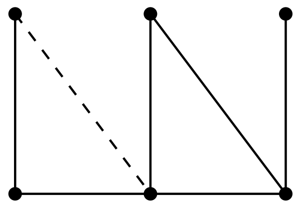

Figure 2: The signed graph $\dot{G}$ with $7.814 \approx \mathcal{E}(\dot{G}) < \mathcal{E}(G) \approx 7.996$.

summarized as follows: $\mathcal{E}(G) = \mathcal{E}(\dot{G})$ holds if $\dot{G}$ switches to G, or G is non-bipartite and $\dot{G}$ switches to $-G$. (For example, the latter occurs for every unicyclic graph with an odd cycle). We have encountered hundreds of examples in which $\mathcal{E}(G) < \mathcal{E}(\dot{G})$. An example in which $\mathcal{E}(G) > \mathcal{E}(\dot{G})$ is illustrated in Figure 2. Motivated by these experiments, we formulate the following problems.

Problem 6.2. Determine graphs G such that $\mathcal{E}(G) \leq \mathcal{E}((G, \dot{\sigma}))$ holds for every signature $\dot{\sigma}$ defined on the edge set of G.

Problem 6.3. Determine (or, at least, characterize) signed graphs $\dot{G}$ with $\mathcal{E}(\dot{G}) < \mathcal{E}(G)$.

We proceed with oriented graphs. If G' is associated with $\dot{G}$, then $\mathcal{E}(G') = \mathcal{E}(\dot{G})$, by Theorem 2.1(i). If $\dot{H}$ is as in Theorem 2.1(ii), then $\mathcal{E}(G') = \frac{\mathcal{E}(\dot{H})}{2}$. Accordingly, if the underlying graph G is regular of degree r, then $\mathcal{E}(G') = n\sqrt{r}$ (the upper bound (6.1)) if and only if G' has exactly two eigenvalues (complex conjugates); this is established in [1], as well. Some examples are constructed in the mentioned reference. Here we note that the search on regular oriented graphs attaining the upper bound (6.1) is reduced to the search on signed graphs with the same spectral property: An associate of a bipartite signed graph with $\mathcal{E}(\dot{G}) = n\sqrt{r}$ has the required spectral property, and if it figures as a bipartite double, then a corresponding constituent has the same property. In this context it is worth mentioning that, on the basis of the results of [10, 16], all oriented graphs with n vertices and spectrum $[\mathrm{i}2^{n/2}, (-\mathrm{i}2)^{n/2}]$ are determined in [15] (their energy attains $2n$); they include infinite families of both bipartite and non-bipartite oriented graphs. More examples can be extracted from known signed graphs with two eigenvalues obtained in the foregoing references.

We also observe that an oriented graph shares the energy with its underlying graph G whenever it is associated with a balanced signed graph. In this context, we point out that, due to Corollary 3.3, $\dot{H}$ of Theorem 2.1(ii) is never balanced. It also shares the energy with G whenever it is associated with a signed graph that switches to $-G$.

This section is concluded with the following result. A conference matrix A is an $n \times n$ matrix with diagonal entries 0 and off-diagonal entries ± 1, satisfying $A^{\intercal}A = (n-1)I$.

Theorem 6.4. *An oriented graph G' (resp. a signed graph $\dot{G}$) with n vertices attains $\mathcal{E}(G') = n\sqrt{n-1}$ ($\mathcal{E}(\dot{G}) = n\sqrt{n-1}$) if and only if its adjacency matrix is the skew-symmetric conference matrix (symmetric conference matrix).*

Proof. If $\mathcal{E}(G') = n\sqrt{n-1}$, by Theorems 2.1(ii) and 6.1, G' is regular of degree $n-1$ and its spectrum is $\big[\mathrm{i}\sqrt{n-1}^{n/2}, -(\mathrm{i}\sqrt{n-1})^{n/2}\big]$. Considering the minimal polynomial of $A_{G'}$, we obtain $A_{G'}^2 = (n-1)I$, which means that $A_{G'}$ is a skew-symmetric conference matrix. Conversely, if $A_{G'}$ is a skew-symmetric conference matrix then, by definition, we have $A_{G'}^2 = (n-1)I$, which gives $\mathcal{E}(G') = n\sqrt{n-1}$.

The proof for $\dot{G}$ is analogous. □

The previous result addresses the open problem (6) of [1, Section 6] related to the existence of skew-symmetric conference matrices that do not give the maximum energy of the corresponding oriented graph. It also partially addresses the problem (2) of the same reference related to determination of oriented graphs with maximum energy, as it gives their characterization via the matrix structure. However, their determination remains a difficult research problem, since it is equivalent to the complete determination of skew-symmetric conference matrices.

7 Notes on hypercubes

Due to [22, Theorem 2.1(iv)] (see also [21, Theorem 2]) a signed graph is balanced if its cycle basis has all positive cycles. Since the induced quadrangles of a signed r-cube contain a cycle basis, we arrive at the following result.

Lemma 7.1 (cf. [22, Theorem 2.1(iv)]). *For $r \geq 2$, a signed r-cube $\dot{Q}_r$ is balanced if and only if it has no negative quadrangles.*

In [2, 3, 20], the authors gave several algorithms which iteratively construct an oriented r-cube such that its energy is either equal to the energy of the underlying graph or attains the upper bound (6.1). (This upper bound reduces to $n\sqrt{r}$.) These algorithms are useful since they give explicit orientations for which the previous settings occur. In this context we offer the following contribution (see also the text below the theorem).

Theorem 7.2. *The following statements hold true:*

(i) *A signed r-cube without negative quadrangle is associated with an oriented r-cube whose energy is equal to the energy of its underlying r-cube.*

(ii) *A signed r-cube is associated with an oriented r-cube whose energy attains the upper bound of* (6.1) *if and only if it has no positive quadrangle.*

Proof. (i): If every quadrangle (if any) in $\dot{Q}_r$ is positive then $\dot{Q}_r$ is balanced, by Lemma 7.1, and therefore it switches to the underlying graph Q_r. Consequently, $\dot{Q}_r$ and Q_r have the same energy. By Theorem 2.1, $\dot{Q}_r$ shares the energy with an associated oriented cube, and the result follows.

(ii): Since a signed graph and an associated oriented graph share the same energy, it is sufficient to show that the energy $n\sqrt{r}$ is exclusive to a signed cube without positive quadrangles. First, such a cube has this energy (as mentioned in the previous section). Conversely, assume that, for $r \geq 3$, a signed r-cube with the adjacency matrix A has a positive quadrangle, say $uvwz$ (where the vertices are in the natural order). It follows that the (u, w)-entry of A^2 is 2, and so A^2 is not a multiple of the identity matrix, but then the spectrum of A deviates the equality condition of Theorem 6.1, as follows by considering the minimal polynomial. □

In other words, to construct an oriented r-cube whose energy is $\mathcal{E}(Q_r)$ (resp. $n\sqrt{r}$), it is sufficient to take any signed r-cube without negative quadrangle (without positive quadrangle), and construct an oriented associate.

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