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Hypergeometric degenerate Bernoulli polynomials and numbers

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Abstract

Carlitz defined the degenerate Bernoulli polynomials $\beta_n(\lambda, x)$ by means of the generating function $t\big((1+\lambda t)^{1/\lambda}-1\big)^{-1}(1+\lambda t)^{x/\lambda}$. In 1875, Glaisher gave several interesting determinant expressions of numbers, including Bernoulli, Cauchy and Euler numbers. In this paper, we show some expressions and properties of hypergeometric degenerate Bernoulli polynomials $\beta_{N,n}(\lambda, x)$ and numbers, in particular, in terms of determinants.

The coefficients of the polynomial $\beta_n(\lambda, 0)$ were completely determined by Howard in 1996. We determine the coefficients of the polynomial $\beta_{N,n}(\lambda, 0)$. Hypergeometric Bernoulli numbers and hypergeometric Cauchy numbers appear in the coefficients.

Keywords: Bernoulli numbers, hypergeometric Bernoulli numbers, hypergeometric Cauchy numbers, hypergeometric functions, degenerate Bernoulli numbers, determinants, recurrence relations.

Math. Subj. Class. (2020): 11B68, 11B37, 11C20, 15A15, 33C15

1 Introduction

Carlitz [7, 8] defined the *degenerate Bernoulli polynomials* $\beta_n(\lambda, x)$ by means of the generating function

$$\left(\frac{t}{(1+\lambda t)^{1/\lambda}-1}\right)(1+\lambda t)^{x/\lambda} = \sum_{n=0}^{\infty} \beta_n(\lambda, x)\frac{t^n}{n!}\,. \tag{1.1}$$

When $\lambda \to 0$ in (1.1), $B_n(x) = \beta_n(0, x)$ are the ordinary Bernoulli polynomials because

$$\lim_{\lambda\to 0}\left(\frac{t}{(1+\lambda t)^{1/\lambda}-1}\right)(1+\lambda t)^{x/\lambda} = \frac{te^{xt}}{e^t-1} = \sum_{n=0}^{\infty} B_n(x)\frac{t^n}{n!}\,.$$

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When $\lambda \to 0$ and $x = 0$ in (1.1), $B_n = \beta_n(0,0)$ are the classical Bernoulli numbers defined by

$$\frac{t}{e^t - 1} = \sum_{n=0}^{\infty} B_n \frac{t^n}{n!} \,. \tag{1.2}$$

The degenerate Bernoulli polynomials in λ and x have rational coefficients. When $x = 0$, $\beta_n(\lambda) = \beta_n(\lambda, 0)$ are called *degenerate Bernoulli numbers*. In [16], explicit formulas for the coefficients of the polynomial $\beta_n(\lambda)$ are found. In [26], a general symmetric identity involving the degenerate Bernoulli polynomials and the sums of generalized falling factorials are proved.

In another direction, *hypergeometric Bernoulli polynomials* $B_{N,n}(x)$ (see, e.g., [17]) are defined by the generating function

$$\frac{e^{tx}}{{}_1F_1(1; N+1; t)} = \sum_{n=0}^{\infty} B_{N,n}(x) \frac{t^n}{n!} \,, \tag{1.3}$$

where ${}_1F_1(a; b; z)$ is the confluent hypergeometric function defined by

$${}_1F_1(a; b; z) = \sum_{n=0}^{\infty} \frac{(a)^{(n)}}{(b)^{(n)}} \frac{z^n}{n!}$$

with the rising factorial $(x)^{(n)} = x(x+1)\cdots(x+n-1)$ $(n \ge 1)$ and $(x)^{(0)} = 1$. When $x = 0$ in (1.3), $B_{N,n} = B_{N,n}(0)$ are the hypergeometric Bernoulli numbers ([12, 13, 14, 15, 19]). When $N = 1$ in (1.3), $B_n(x) = B_{1,n}(x)$ are the ordinary Bernoulli polynomials. When $x = 0$ and $N = 1$ in (1.3), $B_n = B_{1,n}(0)$ are the classical Bernoulli numbers.

Many kinds of generalizations of the Bernoulli numbers have been considered by many authors. For example, such generalizations include poly-Bernoulli numbers, Apostol Bernoulli numbers, various types of q-Bernoulli numbers, Bernoulli Carlitz numbers. One of the advantages of hypergeometric numbers is the natural extension of determinant expressions of the numbers.

A determinant expression of hypergeometric Bernoulli numbers ([2, 18]) is given by

$$B_{N,n} = (-1)^n n! \begin{vmatrix} \frac{N!}{(N+1)!} & 1 & 0 & & \\ \frac{N!}{(N+2)!} & \frac{N!}{(N+1)!} & & & \\ \vdots & \vdots & \ddots & 1 & 0 \\ \frac{N!}{(N+n-1)!} & \frac{N!}{(N+n-2)!} & \cdots & \frac{N!}{(N+1)!} & 1 \\ \frac{N!}{(N+n)!} & \frac{N!}{(N+n-1)!} & \cdots & \frac{N!}{(N+2)!} & \frac{N!}{(N+1)!} \end{vmatrix} . \tag{1.4}$$

The determinant expression for the classical Bernoulli numbers $B_n = B_{1,n}$ was discovered by Glaisher ([11, p. 53]).

In this paper, we introduce and study the hypergeometric degenerate Bernoulli numbers as total generalizations of degenerate Bernoulli numbers and hypergeometric Bernoulli numbers in the aspects of determinants. By applying Trudi's formula and the inversion formula, we show several arithmetical and combinatorial identities. The coefficients of the polynomial $\beta_n(\lambda)$ were completely determined by Howard in 1996. We determine the coefficients of the polynomial $\beta_{N,n}(\lambda)$. The constant term and the leading coefficient are exactly equal to Hypergeometric Bernoulli numbers and hypergeometric Cauchy numbers, respectively.

2 Definition and preliminary results

Denote the generalized falling factorial by for $n \ge 1$

$$(x|\alpha)_n = x(x-\alpha)(x-2\alpha)\cdots\bigl(x-(n-1)\alpha\bigr)$$

with $(x|\alpha)_0 = 1$. When $\alpha = 1$, $(x)_n = (x|1)_n$ is the ordinary falling factorial. Define *hypergeometric degenerate Bernoulli polynomials* $\beta_{N,n}(\lambda, x)$ by

$$\left({}_2F_1\left(1, N-\frac{1}{\lambda}; N+1; -\lambda t\right)\right)^{-1}(1+\lambda t)^{x/\lambda} = \sum_{n=0}^\infty \beta_{N,n}(\lambda, x)\frac{t^n}{n!}, \tag{2.1}$$

where ${}_2F_1(a, b; c; z)$ is the Gauss hypergeometric function defined by

$${}_2F_1(a; b; z) = \sum_{n=0}^\infty \frac{(a)^{(n)}(b)^{(n)}}{(c)^{(n)}}\frac{z^n}{n!}.$$

When $x = 0$ in (2.1), $\beta_{N,n}(\lambda) = \beta_{N,n}(\lambda, 0)$ are the *hypergeometric degenerate Bernoulli numbers*. When $\lambda \to 0$, $B_{N,n}(x) = \lim_{\lambda\to 0}\beta_{N,n}(\lambda, x)$ are the hypergeometric Bernoulli polynomials in (1.3). Since

$$\frac{t}{(1+\lambda t)^{1/\lambda}-1} = t\left(\sum_{n=1}^\infty \frac{(1-\lambda|\lambda)_{n-1}}{n!}t^n\right)^{-1}$$

in (1.1), we can write

$$\begin{aligned}
&{}_2F_1\left(1, N-\frac{1}{\lambda}; N+1; -\lambda t\right)\\
&\qquad = \left(\frac{(1-\lambda|\lambda)_{N-1}}{N!}t^N\right)\left(\sum_{n=N}^\infty \frac{(1-\lambda|\lambda)_{n-1}}{n!}t^n\right)^{-1}\\
&\qquad = 1 + \sum_{n=1}^\infty \frac{(1-\lambda|\lambda)_{N+n-1}N!}{(1-\lambda|\lambda)_{N-1}(N+n)!}t^n\\
&\qquad = 1 + \sum_{n=1}^\infty \frac{(1-N\lambda|\lambda)_n}{(N+n)_n}t^n.
\end{aligned} \tag{2.2}$$

When $N = 1$, $\beta_n(\lambda, x) = \beta_{N,1}(\lambda, x)$ are degenerate Bernoulli polynomials, defined by

$$\left(1 + \sum_{n=1}^\infty \frac{(1-\lambda|\lambda)_n}{(n+1)!}t^n\right)^{-1}(1+\lambda t)^{x/\lambda} = \sum_{n=0}^\infty \beta_n(\lambda, x)\frac{t^n}{n!}.$$

When $N = 1$ and $\lambda \to 0$, $B_n(x) = \lim_{\lambda\to 0}\beta_{N,1}(\lambda, x)$ are the classical Bernoulli polynomials, defined by

$$\left(1 + \sum_{n=1}^\infty \frac{t^n}{(n+1)!}\right)^{-1} e^{xt} = \sum_{n=0}^\infty B_n(x)\frac{t^n}{n!}.$$

The definition (2.1) may be obvious or artificial for the readers with different backgrounds. One of our motivations is mentioned in Section 5.

We have the following recurrence relation of hypergeometric degenerate Bernoulli numbers $\beta_{N,n}(\lambda)$.

Proposition 2.1. *For* $N, n \geq 1$*, we have*

$$\beta_{N,n}(\lambda) = -\sum_{k=0}^{n-1} \frac{n!(1-N\lambda|\lambda)_{n-k}N!}{(N+n-k)!k!}\beta_{N,k}(\lambda)$$

with $\beta_{N,0}(\lambda) = 1$.

Proof. By (2.1) with (2.2), we get

$$\begin{aligned} 1 &= \left(1 + \sum_{l=1}^{\infty} \frac{(1-N\lambda|\lambda)_l N!}{(N+l)!} t^l\right)\left(\sum_{n=0}^{\infty} \beta_{N,n}(\lambda)\frac{t^n}{n!}\right) \\ &= \sum_{n=0}^{\infty} \beta_{N,n}(\lambda)\frac{t^n}{n!} + \sum_{n=1}^{\infty}\sum_{k=0}^{n-1} \frac{(1-N\lambda|\lambda)_{n-k}N!}{(N+n-k)!}\frac{\beta_{N,k}(\lambda)}{k!}t^n . \end{aligned}$$

Comparing the coefficients on both sides, we obtain for $n \geq 1$

$$\frac{\beta_{N,n}(\lambda)}{n!} + \sum_{k=0}^{n-1} \frac{(1-N\lambda|\lambda)_{n-k}N!}{(N+n-k)!}\frac{\beta_{N,k}(\lambda)}{k!} = 0 .$$ □

We can use Proposition 2.1 to give an explicit expression for $\beta_{N,n}(\lambda)$.

Theorem 2.2. *For* $N, n \geq 1$*,*

$$\beta_{N,n}(\lambda) = n!\sum_{k=1}^{n}(-N!)^k \sum_{\substack{i_1+\cdots+i_k=n \\ i_1,\dots,i_k\geq 1}} \frac{(1-N\lambda|\lambda)_{i_1}}{(N+i_1)!}\cdots\frac{(1-N\lambda|\lambda)_{i_k}}{(N+i_k)!} . \tag{2.3}$$

Remark 2.3. When $\lambda \to 0$, Theorem 2.2 is reduced to

$$B_{N,n} = n!\sum_{k=1}^{n} \sum_{\substack{i_1+\cdots+i_k=n \\ i_1,\dots,i_k\geq 1}} \frac{(-N!)^k}{(N+i_1)!\cdots(N+i_k)!} , \tag{2.4}$$

as seen in [2, 18]. When $\lambda \to 0$ and $N = 1$, there is a combinatorial interpretation of Bernoulli numbers in terms of the cardinality of $\mathbb{Z}_2$-graded groupoids [4, Corollary 45].

Proof of Theorem 2.2. The proof is by induction on n. From Proposition 2.1 with $n = 1$,

$$\beta_{N,1}(\lambda) = -\frac{(1-N\lambda)N!}{(N+1)!}\beta_{N,0}(\lambda) = -\frac{N!(1-N\lambda)}{(N+1)!} .$$

This matches the expression (2.1) when $n = 1$. Assume that the result is valid up to $n - 1$. For simplicity, put

$$S_k(n) = \sum_{\substack{i_1+\cdots+i_k=n \\ i_1,\dots,i_k\geq 1}} \frac{(1-N\lambda|\lambda)_{i_1}}{(N+i_1)!}\cdots\frac{(1-N\lambda|\lambda)_{i_k}}{(N+i_k)!} . \tag{2.5}$$

Then by Proposition 2.1

$$\begin{aligned}\frac{\beta_{N,n}(\lambda)}{n!} &= -\sum_{l=0}^{n-1}\frac{(1-N\lambda|\lambda)_{n-l}N!}{(N+n-l)!}\frac{\beta_{N,l}(\lambda)}{l!}\\ &= -\frac{(1-N\lambda|\lambda)_n N!}{(N+n)!}-\sum_{l=1}^{n-1}\frac{(1-N\lambda|\lambda)_{n-l}N!}{(N+n-l)!}\sum_{k=1}^{l}(-N!)^k S_k(l)\\ &= -\frac{(1-N\lambda|\lambda)_n N!}{(N+n)!}-\sum_{k=1}^{n-1}(-N!)^k\sum_{l=k}^{n-1}\frac{(1-N\lambda|\lambda)_{n-l}N!}{(N+n-l)!}S_k(l)\\ &= -\frac{(1-N\lambda|\lambda)_n N!}{(N+n)!}-\sum_{k=2}^{n}(-N!)^{k-1}\sum_{l=k-1}^{n-1}\frac{(1-N\lambda|\lambda)_{n-l}N!}{(N+n-l)!}S_{k-1}(l)\\ &= -\frac{(1-N\lambda|\lambda)_n N!}{(N+n)!}+\sum_{k=2}^{n}(-N!)^k S_k(n)\\ &= \sum_{k=1}^{n}(-N!)^k S_k(n)\,.\end{aligned}$$

Here, we put $n-l=i_k$ in the second last equation. □

There is an alternative form of $\beta_{N,n}(\lambda)$ using binomial coefficients. The proof may be similar to that of Theorem 2.2, but a different proof is given.

Theorem 2.4. *For $N,n\ge 1$,*

$$\beta_{N,n}(\lambda)=n!\sum_{k=1}^{n}(-N!)^k\binom{n+1}{k+1}\sum_{\substack{i_1+\cdots+i_k=n\\ i_1,\dots,i_k\ge 0}}\frac{(1-N\lambda|\lambda)_{i_1}}{(N+i_1)!}\cdots\frac{(1-N\lambda|\lambda)_{i_k}}{(N+i_k)!}\,.$$

Proof. Put

$$1+w={}_2F_1\left(1,N-\frac{1}{\lambda};N+1;-\lambda t\right)\,.$$

By the definition (2.1) with $x=0$, we have

$$\begin{aligned}\beta_{N,n}(\lambda) &= \frac{d^n}{dt^n}(1+w)^{-1}\bigg|_{t=0}=\frac{d^n}{dt^n}\left(\sum_{l=0}^{\infty}(-w)^l\right)\bigg|_{t=0}=\sum_{l=0}^{n}\frac{d^n}{dt^n}(-w)^l\bigg|_{t=0}\\ &= \sum_{l=0}^{n}\sum_{k=0}^{l}(-1)^k\binom{l}{k}\frac{d^n}{dt^n}\left({}_2F_1\left(1,N-\frac{1}{\lambda};N+1;-\lambda t\right)\right)^k\bigg|_{t=0}\,.\end{aligned}$$

By (2.2), we get

$$\begin{aligned}\frac{d^n}{dt^n}\left({}_2F_1\left(1,N-\frac{1}{\lambda};N+1;-\lambda t\right)\right)^k\bigg|_{t=0} &= \frac{d^n}{dt^n}\left(\sum_{l=0}^{\infty}\frac{(1-N\lambda|\lambda)_l}{(N+l)_l}t^l\right)^k\bigg|_{t=0}\\ &= n!R_k(n)\,,\end{aligned}$$

where

$$R_k(n) = \sum_{\substack{i_1+\cdots+i_k=n\\ i_1,\dots,i_k\geq 0}} \frac{(1-N\lambda|\lambda)_{i_1}}{(N+i_1)_{i_1}} \cdots \frac{(1-N\lambda|\lambda)_{i_k}}{(N+i_k)_{i_k}}\,.$$

Thus, we have

$$\begin{aligned}
\beta_{N,n}(\lambda) &= \sum_{l=0}^{n}\sum_{k=0}^{l}(-1)^k\binom{l}{k}n!R_k(n)\\
&= n!\sum_{k=0}^{n}(-1)^k R_k(n)\sum_{l=k}^{n}\binom{l}{k}\\
&= n!\sum_{k=0}^{n}(-1)^k R_k(n)\binom{n+1}{k+1}\\
&= n!\sum_{k=1}^{n}(-1)^k\binom{n+1}{k+1}R_k(n)\\
&= n!\sum_{k=1}^{n}(-N!)^k\binom{n+1}{k+1}\sum_{\substack{i_1+\cdots+i_k=n\\ i_1,\dots,i_k\geq 0}} \frac{(1-N\lambda|\lambda)_{i_1}}{(N+i_1)!} \cdots \frac{(1-N\lambda|\lambda)_{i_k}}{(N+i_k)!}\,. \qquad \square
\end{aligned}$$

3 Hypergeometric degenerate Bernoulli polynomials

In this section, a relation between hypergeometric degenerate Bernoulli polynomials and numbers and some more related properties are shown.

Theorem 3.1. *For $N \geq 1$ and $n \geq 0$,*

$$\beta_{N,n}(\lambda, x+y) = \sum_{k=0}^{n}\binom{n}{k}(y|\lambda)_{n-k}\beta_{N,k}(\lambda, x)\,.$$

Proof. By the definition in (2.1),

$$\begin{aligned}
&\sum_{n=0}^{\infty}\beta_{N,n}(\lambda, x+y)\frac{t^n}{n!}\\
&\qquad = \left({}_2F_1\left(1, N-\frac{1}{\lambda}; N+1; -\lambda t\right)\right)^{-1}(1+\lambda t)^{(x+y)/\lambda}\\
&\qquad = \left(\sum_{n=0}^{\infty}\beta_{N,n}(\lambda, x+y)\frac{t^n}{n!}\right)\left(\sum_{l=0}^{\infty}\binom{y/\lambda}{l}(\lambda t)^l\right)\\
&\qquad = \left(\sum_{n=0}^{\infty}\beta_{N,n}(\lambda, x+y)\frac{t^n}{n!}\right)\left(\sum_{l=0}^{\infty}(y|\lambda)_l\frac{t^l}{l!}\right)\\
&\qquad = \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\binom{n}{k}(y|\lambda)_{n-k}\beta_{N,k}(\lambda, x)\right)\frac{t^n}{n!}\,.
\end{aligned}$$

Comparing the coefficients on both sides, we get the desired result. $\square$

By specializing $y = 0$ in Theorem 3.1, we have a relation between the hypergeometric degenerate Bernoulli polynomials and numbers.

Corollary 3.2. *For $N \geq 1$ and $n \geq 0$,*

$$\beta_{N,n}(\lambda, x) = \sum_{k=0}^{n} \binom{n}{k} (x|\lambda)_{n-k} \beta_{N,k}(\lambda) .$$

Theorem 3.3. *For $N \geq 1$ and $n \geq 0$,*

$$\frac{d}{dx}\beta_{N,n}(\lambda, x) = \sum_{k=0}^{n-1} \frac{(-\lambda)^{n-k-1} n!}{(n-k)k!} \beta_{N,k}(\lambda, x) .$$

Proof. By the definition in (2.1),

$$\begin{aligned}
&\frac{d}{dx}\sum_{n=0}^{\infty} \beta_{N,n}(\lambda, x)\frac{t^n}{n!}\\
&= \left({}_2F_1\left(1, N - \frac{1}{\lambda}; N+1; -\lambda t\right)\right)^{-1} \frac{d}{dx}(1+\lambda t)^{(x)/\lambda}\\
&= \log(1+\lambda t)^{1/\lambda} \sum_{n=0}^{\infty} \beta_{N,n}(\lambda, x)\frac{t^n}{n!}\\
&= \left(\frac{1}{\lambda}\sum_{l=1}^{\infty}(-1)^{l-1}\frac{(\lambda t)^l}{l}\right)\left(\sum_{n=0}^{\infty} \beta_{N,n}(\lambda, x)\frac{t^n}{n!}\right)\\
&= \sum_{n=1}^{\infty}\left(\sum_{k=0}^{n-1}\frac{(-\lambda)^{n-k-1}}{(n-k)k!}\beta_{N,k}(\lambda, x)\right) t^n .
\end{aligned}$$

Comparing the coefficients on both sides, we get the desired result. □

4 A determinant expression of hypergeometric degenerate Bernoulli numbers

Theorem 4.1. *For $N, n \geq 1$, we have*

$$\beta_{N,n}(\lambda) = (-1)^n n! \begin{vmatrix}
\frac{(1-N\lambda)N!}{(N+1)!} & 1 & 0 & & \\
\frac{(1-N\lambda|\lambda)_2 N!}{(N+2)!} & \frac{(1-N\lambda)N!}{(N+1)!} & & & \\
\vdots & \vdots & \ddots & 1 & 0 \\
\frac{(1-N\lambda|\lambda)_{n-1} N!}{(N+n-1)!} & \frac{(1-N\lambda|\lambda)_{n-2} N!}{(N+n-2)!} & \cdots & \frac{(1-N\lambda)N!}{(N+1)!} & 1 \\
\frac{(1-N\lambda|\lambda)_n N!}{(N+n)!} & \frac{(1-N\lambda|\lambda)_{n-1} N!}{(N+n-1)!} & \cdots & \frac{(1-N\lambda|\lambda)_2 N!}{(N+2)!} & \frac{(1-N\lambda)N!}{(N+1)!}
\end{vmatrix} .$$

Remark 4.2. When $\lambda \to 0$ in Theorem 4.1, we get a determinant expression of hypergeometric Bernoulli numbers $B_{N,n}$ in (1.4). If $\lambda \to 0$ and $N = 1$ in Theorem 4.1, we recover the classical determinant expression of the Bernoulli numbers B_n ([11, p. 53]).

Proof of Theorem 4.1. For simplicity, we put $\tilde{\beta}_{N,n} = (-1)^n \beta_{N,n}(\lambda)/n!$ and

$$f(i,j) = \begin{cases} \dfrac{(1-N\lambda|\lambda)_{i-j+1} N!}{(N+i-j+1)!} & \text{if } i \ge j; \\ 1 & \text{if } i = j-1; \\ 0 & \text{otherwise} \end{cases}$$

and shall prove that

$$\tilde{\beta}_{N,n} = |f(i,j)|_{1 \le i,j \le n} \,. \tag{4.1}$$

From Proposition 2.1, we have

$$\begin{aligned} \tilde{\beta}_{N,n} &= \sum_{m=0}^{n-1} \frac{(-1)^{n-m-1}(1-N\lambda|\lambda)_{n-m} N!}{(N+n-m)!} \tilde{\beta}_{N,m} \\ &= \sum_{m=0}^{n-1} (-1)^{n-m-1} f(n-m,1) \tilde{\beta}_{N,m} \,. \end{aligned} \tag{4.2}$$

When $n = 1$, it is trivial because by Theorem 2.2

$$\tilde{\beta}_{N,1} = -\frac{(1-N\lambda)N!}{(N+1)!} \,.$$

Assume that (4.1) is valid up to $n-1$. By expanding along the first row, the right-hand side of (4.1) is equal to

$$\begin{aligned} & f(1,1)\tilde{\beta}_{N,n-1} - \begin{vmatrix} f(2,1) & 1 & 0 & & \\ f(3,1) & 1 & & & \\ \vdots & \vdots & \ddots & 1 & 0 \\ f(n-1,1) & f(n-1,3) & \cdots & f(n-1,n-1) & 1 \\ f(n,1) & f(n,3) & \cdots & f(n,n-1) & f(n,n) \end{vmatrix} \\ &= f(1,1)\tilde{\beta}_{N,n-1} - f(2,1)\tilde{\beta}_{N,n-2} + \cdots + (-1)^{n-2} \begin{vmatrix} f(n-1,1) & 1 \\ f(n,1) & f(n,n) \end{vmatrix} \\ &= \sum_{m=0}^{n-1} (-1)^{n-m-1} f(n-m,1)\tilde{\beta}_{N,m} = \tilde{\beta}_{N,n} \,. \end{aligned}$$

Here, we used the relation (4.2) with $\tilde{\beta}_{N,0} = 1$. □

5 Applications of Trudi's formula and inversion relations

One motivation of this paper comes from a 1989 paper of Cameron [6], in which he considered the operator A defined on the set of sequences of non-negative integers as follows: for $\mathbf{x} = \{x_n\}_{n \ge 1}$ and $\mathbf{z} = \{z_n\}_{n \ge 1}$, set $A\mathbf{x} = \mathbf{z}$, where

$$1 + \sum_{n=1}^{\infty} z_n t^n = \left(1 - \sum_{n=1}^{\infty} x_n t^n\right)^{-1} \,. \tag{5.1}$$

Suppose that x enumerates a class C. Then Ax enumerates the class of disjoint unions of members of C, where the order of the "component" members of C is significant. The operator A also plays an important role for free associative (non-commutative) algebras. More motivations and background together with many concrete examples (in particular, in the aspects of graph theory) by this operator can be seen in [6].

Though only nonnegative numbers in the sequence are treated with combinatorial interpretations in [6], the transformation in (5.1) can be extended to negative or rational numbers too. Some combinatorial interpretations for rational numbers can be found in [3, 4], where a categorical setting is proposed. In the sense of Cameron's operator A, we have the following relations.

$$A\left\{-\frac{1}{(n+1)!}\right\}=\left\{\frac{B_n}{n!}\right\}$$
$$A\left\{-\frac{1}{(N+n)_n}\right\}=\left\{\frac{B_{N,n}}{n!}\right\}$$
$$A\left\{-\frac{(1-\lambda|\lambda)_n}{(n+1)!}\right\}=\left\{\frac{\beta_n(\lambda)}{n!}\right\}$$
$$A\left\{-\frac{(1-N\lambda|\lambda)_n}{(N+n)_n}\right\}=\left\{\frac{\beta_{N,n}(\lambda)}{n!}\right\}$$

These relations are interchangeable in the sense of determinants too.

We shall use Trudi's formula to obtain different explicit expressions and inversion relations for the numbers $\beta_{N,n}(j)$.

Lemma 5.1. *For $n \ge 1$, we have*

$$\begin{vmatrix} a_1 & a_0 & 0 & \cdots & \\ a_2 & a_1 & \ddots & & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ a_{n-1} & & \cdots & a_1 & a_0 \\ a_n & a_{n-1} & \cdots & a_2 & a_1 \end{vmatrix}$$
$$=\sum_{t_1+2t_2+\cdots+nt_n=n}\binom{t_1+\cdots+t_n}{t_1,\dots,t_n}(-a_0)^{n-t_1-\cdots-t_n}a_1^{t_1}a_2^{t_2}\cdots a_n^{t_n},$$

where $\binom{t_1+\cdots+t_n}{t_1,\dots,t_n}=\frac{(t_1+\cdots+t_n)!}{t_1!\cdots t_n!}$ are the multinomial coefficients.

This relation is known as Trudi's formula [24, Vol. 3, p. 214], [25] and the case $a_0=1$ of this formula is known as Brioschi's formula [5], [24, Vol. 3, pp. 208–209].

In addition, there exists an inversion formula (see, e.g. [22]). From Cameron's operator $A\mathbf{x}=\mathbf{z}$ in (5.1),

$$\sum_{n=0}^\infty\sum_{k=0}^n(-1)^{n-k}x_{n-k}z_k=1\,.$$

Hence, for $n\ge 1$

$$\sum_{k=0}^n(-1)^{n-k}x_{n-k}z_k=0\,.$$

When $x_0=z_0=1$, we have the following inversion formula.

Lemma 5.2.

$$\textit{If } x_n = \begin{vmatrix} z_1 & 1 & & \\ z_2 & \ddots & \ddots & \\ \vdots & \ddots & \ddots & 1 \\ z_n & \cdots & z_2 & z_1 \end{vmatrix}, \textit{ then } z_n = \begin{vmatrix} x_1 & 1 & & \\ x_2 & \ddots & \ddots & \\ \vdots & \ddots & \ddots & 1 \\ x_n & \cdots & x_2 & x_1 \end{vmatrix}.$$

From Trudi's formula, it is possible to give the combinatorial expression

$$x_n = \sum_{t_1+2t_2+\cdots+nt_n=n} \binom{t_1+\cdots+t_n}{t_1,\dots,t_n} (-1)^{n-t_1-\cdots-t_n} z_1^{t_1} z_2^{t_2} \cdots z_n^{t_n} .$$

By applying these lemmas to Theorem 4.1, we obtain an explicit expression for the hypergeometric degenerate Bernoulli numbers.

Theorem 5.3. *For* $N, n \geq 1$,

$$\beta_{N,n}(\lambda) = n! \sum_{t_1+2t_2+\cdots+nt_n=n} \binom{t_1+\cdots+t_n}{t_1,\dots,t_n} (-1)^{t_1+\cdots+t_n} \times \left(\frac{(1-N\lambda)N!}{(N+1)!}\right)^{t_1} \left(\frac{(1-N\lambda|\lambda)_2 N!}{(N+2)!}\right)^{t_2} \cdots \left(\frac{(1-N\lambda|\lambda)_n N!}{(N+n)!}\right)^{t_n} .$$

Theorem 5.4. *For* $N, n \geq 1$,

$$\frac{(-1)^n(1-N\lambda|\lambda)_n N!}{(N+n)!} = \begin{vmatrix} \beta_{N,1}(\lambda) & 1 & 0 & & \\ \frac{\beta_{N,2}(\lambda)}{2!} & \beta_{N,1}(\lambda) & & & \\ \vdots & \vdots & \ddots & 1 & 0 \\ \frac{\beta_{N,n-1}(\lambda)}{(n-1)!} & \frac{\beta_{N,n-2}(\lambda)}{(n-2)!} & \cdots & \beta_{N,1}(\lambda) & 1 \\ \frac{\beta_{N,n}(\lambda)}{n!} & \frac{\beta_{N,n-1}(\lambda)}{(n-1)!} & \cdots & \frac{\beta_{N,2}(\lambda)}{2!} & \beta_{N,1}(\lambda) \end{vmatrix}.$$

Applying the Trudi's formula in Lemma 5.1 to Theorem 5.4, we get the inversion relation of Theorem 5.3.

Theorem 5.5. *For* $N, n \geq 1$,

$$\frac{(1-N\lambda|\lambda)_n N!}{(N+n)!} = \sum_{t_1+2t_2+\cdots+nt_n=n} \binom{t_1+\cdots+t_n}{t_1,\dots,t_n} (-1)^{t_1+\cdots+t_n} \times (\beta_{N,1}(\lambda))^{t_1} \left(\frac{\beta_{N,2}(\lambda)}{2!}\right)^{t_2} \cdots \left(\frac{\beta_{N,n}(\lambda)}{n!}\right)^{t_n} .$$

6 Coefficients of hypergeometric degenerate Bernoulli numbers

Hypergeometric Cauchy polynomials $c_{N,n}(x)$ ([20]) have similar properties. The generating function is given by

$$\frac{1}{(1+t)^x {}_2F_1(1,N;N+1;-t)} = \sum_{n=0}^{\infty} c_{N,n}(x)\frac{t^n}{n!}\,. \tag{6.1}$$

When $x=0$ in (6.1), $c_{N,n}=c_{N,n}(0)$ are the hypergeometric Cauchy numbers ([12, 13, 14, 15, 19]). When $N=1$ in (6.1), $c_n(x)=c_{1,n}(x)$ are the ordinary Cauchy polynomials (e.g., [9]). When $x=0$ and $N=1$ in (6.1), $c_n=c_{1,n}(0)$ are the classical Cauchy numbers (see, e.g., [10, Chapter VII]), defined by

$$\frac{t}{\log(1+t)} = \sum_{n=0}^{\infty} c_n\frac{t^n}{n!}\,. \tag{6.2}$$

The number $c_n/n!$ is sometimes referred to as the Bernoulli number of the second kind (see, e.g. [16]). A determinant expression of hypergeometric Cauchy numbers ([1, 23]) is given by

$$c_{N,n} = n!\begin{vmatrix} \frac{N}{N+1} & 1 & 0 & & \\ \frac{N}{N+2} & \frac{N}{N+1} & & & \\ \vdots & \vdots & \ddots & 1 & 0 \\ \frac{N}{N+n-1} & \frac{N}{N+n-2} & \cdots & \frac{N}{N+1} & 1 \\ \frac{N}{N+n} & \frac{N}{N+n-1} & \cdots & \frac{N}{N+2} & \frac{N}{N+1} \end{vmatrix}\,. \tag{6.3}$$

The determinant expression for the classical Cauchy numbers $c_n=c_{1,n}$ was discovered by Glaisher ([11, p. 50]). A more general case is considered in [21].

From the expression in Theorem 5.3, the hypergeometric degenerate Bernoulli number $\beta_{N,n}$ is a polynomial in λ with rational coefficients and degree at most n. Thus, we can write

$$\beta_{N,n}(\lambda) = d_{n,n}\lambda^n + d_{n,n-1}\lambda^{n-1} + \cdots + d_{n,1}\lambda + d_{n,0}\,. \tag{6.4}$$

In this section, we give some coefficients explicitly. By this theorem, we can see that hypergeometric degenerate Bernoulli numbers are closely related with both hypergeometric Bernoulli numbers and hypergeometric Cauchy numbers.

Theorem 6.1. *For $N\ge 1$ and $n\ge 0$, we have*

$$d_{n,n} = c_{N,n} \quad \textit{and} \quad d_{n,0} = B_{N,n}\,.$$

Remark 6.2. When $N=1$, Theorem 6.1 is reduced to [16, Theorem 3.1]. This implies that the leading coefficient of $\beta_n(\lambda)$ is equal to the n-th Cauchy number c_n and the constant term is equal to the n-th Bernoulli number B_n.

Proof of Theorem 6.1. Since

$$\begin{aligned}(1-N\lambda|\lambda)_{n-k} &= \lambda^{n-k}\sum_{l=0}^{n-k}(-1)^{n-k-l}\begin{bmatrix} n-k \\ l \end{bmatrix}\left(\frac{1}{\lambda}-N\right)^l \\ &= \sum_{l=0}^{n-k}\begin{bmatrix} n-k \\ l \end{bmatrix}\sum_{i=0}^{l}(-1)^{n-k-i}\binom{l}{i}\lambda^{n-k-i}N^{l-i}\,,\end{aligned}$$

by Proposition 2.1 we obtain for $n \ge 1$

$$\begin{aligned} &\frac{\beta_{N,n}(\lambda)}{n!} \\ &= -\sum_{k=0}^{n-1} \frac{(1-N\lambda|\lambda)_{n-k} N!}{(N+n-k)!k!} \beta_{N,k}(\lambda) \\ &= -\sum_{k=0}^{n-1} \frac{N!\beta_{N,k}(\lambda)}{(N+n-k)!k!} \sum_{l=0}^{n-k} \begin{bmatrix} n-k \\ l \end{bmatrix} \sum_{i=0}^{l} (-1)^{n-k-i} \binom{l}{i} \lambda^{n-k-i} N^{l-i} . \end{aligned} \tag{6.5}$$

Note that $\beta_{N,0}(\lambda) = 1$.

For constant term of the polynomial in λ, as $i = n-k$ in (6.5)

$$\begin{aligned} \frac{d_{0,0}}{n!} &= -\sum_{k=0}^{n-1} \frac{(1-N\lambda|\lambda)_{n-k} N!}{(N+n-k)!k!} d_{k,0}(\lambda) \\ &= -\sum_{k=0}^{n-1} \frac{N! d_{k,0}}{(N+n-k)!k!} \sum_{l=0}^{n-k} \begin{bmatrix} n-k \\ l \end{bmatrix} \binom{l}{n-k} N^{l-n+k} \\ &= -\sum_{k=0}^{n-1} \frac{N! d_{k,0}}{(N+n-k)!k!} . \end{aligned}$$

Hence,

$$\sum_{k=0}^{n} \binom{N+n}{k} d_{k,0} = 0$$

with $d_{0,0} = 1$. Since the hypergeometric Bernoulli numbers $B_{N,n}$ satisfies the same recurrence relation, namely,

$$\sum_{k=0}^{n} \binom{N+n}{k} B_{N,k} = 0$$

with $B_{N,0} = 1$ ([2, Proposition 1], [18, (6)]), we can conclude that

$$d_{n,0} = B_{N,n} .$$

For the leading coefficient, that is, the coefficient of λ^n of the polynomial in λ, as $i = 0$ in (6.5)

$$\begin{aligned} \frac{d_{n,n}}{n!} &= -\sum_{k=0}^{n-1} \frac{(-1)^{n-k} N! d_{k,k}}{(N+n-k)!k!} \sum_{l=0}^{n-k} \begin{bmatrix} n-k \\ l \end{bmatrix} N^l \\ &= -\sum_{k=0}^{n-1} \frac{(-1)^{n-k} N! (N)^{(n-k)}}{(N+n-k)!k!} d_{k,k} \\ &= -\sum_{k=0}^{n-1} \frac{(-1)^{n-k} N}{(N+n-k)k!} d_{k,k} . \end{aligned}$$

Thus, for $n \ge 1$

$$\sum_{k=0}^{n} \frac{(-1)^{n-k} N}{(N+n-k)k!} d_{k,k} = 0$$

or

$$\sum_{k=0}^{n}\frac{(-1)^k}{(N+n-k)k!}d_{k,k}=0$$

with $d_{0,0}=1$. Since the hypergeometric Cauchy numbers $c_{N,n}$ satisfies the same recurrence relation, namely,

$$\sum_{k=0}^{n}\frac{(-1)^k}{(N+n-k)k!}c_{N,k}=0$$

with $c_{N,0}=1$ ([20, Proposition 1]), we can conclude that

$$d_{n,n}=c_{N,n}\,. \qquad \square$$

6.1 Another method

Howard [16] found explicit formulas for all the coefficients by proving the following. For $n\ge 2$

$$\beta_n(\lambda)=c_n\lambda^n+\sum_{j=1}^{n}(-1)^{n-j}\frac{n}{j}B_j\begin{bmatrix}n-1\\j-1\end{bmatrix}\lambda^{n-j}\,, \tag{6.6}$$

where $\begin{bmatrix}n\\k\end{bmatrix}$ are the Stirling numbers of the first kind, determined by

$$x(x-1)\cdots(x-n+1)=\sum_{k=0}^{n}(-1)^{n-k}\begin{bmatrix}n\\k\end{bmatrix}x^k\,. \tag{6.7}$$

and c_n are the Cauchy numbers (of the firs kind), defined by the generating function

$$\frac{t}{\log(1+t)}=\sum_{n=0}^{\infty}c_n\frac{t^n}{n!}\,. \tag{6.8}$$

We have another expression of the coefficients of $\beta_{N,n}(\lambda)$ directly from Theorem 2.2. Since by (6.7)

$$\begin{aligned}(1-N\lambda|\lambda)_i&=\sum_{j=0}^{i}(-1)^{i-j}\sum_{l=1}^{i}\begin{bmatrix}i\\l\end{bmatrix}\binom{l}{j}N^{l-j}\cdot\lambda^{i-j}\\&=\sum_{j=0}^{i}(-1)^{i-j}\sum_{l=j}^{i}\begin{bmatrix}i\\l\end{bmatrix}\binom{l}{j}N^{l-j}\cdot\lambda^{i-j}\,,\end{aligned}$$

we have for $j=0,1,\dots,n$

$$\begin{aligned}d_{n,n-j}&=n!\sum_{k=1}^{n}\frac{(-N!)^k(-1)^{n-j}}{j!}\sum_{\substack{i_1+\cdots+i_k=n\\i_1,\dots,i_k\ge 1}}\frac{1}{(N+i_1)!\cdots(N+i_k)!}\\&\quad\times\frac{d^j}{dx^j}\left(\left(\sum_{l_1=1}^{i_1}\begin{bmatrix}i_1\\l_1\end{bmatrix}x^{l_1}\right)\cdots\left(\sum_{l_k=1}^{i_k}\begin{bmatrix}i_k\\l_k\end{bmatrix}x^{l_k}\right)\right)\Bigg|_{x=N}\,.\end{aligned} \tag{6.9}$$

If $j = n$ in (6.9), we get the coefficient of the constant in λ as

$$d_{n,0} = n! \sum_{k=1}^{n} (-N!)^k \sum_{\substack{i_1+\cdots+i_k=n \\ i_1,\dots,i_k \ge 1}} \frac{n!}{n!(N+i_1)! \cdots (N+i_k)!},$$

which is equal to $B_{N,n}$ by (2.4). If $j = 0$ in (6.9), we get the leading coefficient in λ as

$$\begin{aligned} d_{n,n} &= n! \sum_{k=1}^{n} (-N!)^k (-1)^n \sum_{\substack{i_1+\cdots+i_k=n \\ i_1,\dots,i_k \ge 1}} \frac{(N)^{(i_1)}}{(N+i_1)!} \cdots \frac{(N)^{(i_k)}}{(N+i_k)!} \\ &= n! \sum_{k=1}^{n} (-1)^{n-k} \sum_{\substack{i_1+\cdots+i_k=n \\ i_1,\dots,i_k \ge 1}} \frac{N^k}{(N+i_1)\cdots(N+i_k)}, \end{aligned}$$

which is equal to $c_{N,n}$ in [1, 23].

However, it seems difficult to express other terms of $\beta_{N,n}(\lambda)$ in any explicit form, except the leading coefficient and the constant.

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References

[1] M. Aoki and T. Komatsu, Remarks on hypergeometric Cauchy numbers, *Math. Rep. (Bucur.)* **23** (2021), in press.

[2] M. Aoki, T. Komatsu and G. K. Panda, Several properties of hypergeometric Bernoulli numbers, *J. Inequal. Appl.* **2019** (2019), Paper No. 113, doi:10.1186/s13660-019-2066-y.

[3] H. Blandín and R. Díaz, Compositional Bernoulli numbers, *Afr. Diaspora J. Math.* **7** (2008), 119–134.

[4] H. Blandín and R. Díaz, Rational combinatorics, *Adv. Appl. Math.* **40** (2008), 107–126, doi:10.1016/j.aam.2006.12.006.

[5] F. Brioschi, Sulle funzioni Bernoulliane ed Euleriane, *Ann. Mat. Pura Appl.* **1** (1858), 260–263, doi:10.1007/bf03197335.

[6] P. J. Cameron, Some sequences of integers, *Discrete Math.* **75** (1989), 89–102, doi:10.1016/0012-365x(89)90081-2.

[7] L. Carlitz, A degenerate Staudt-Clausen theorem, *Arch. Math. (Basel)* **7** (1956), 28–33, doi:10.1007/bf01900520.

[8] L. Carlitz, Degenerate Stirling, Bernoulli and Eulerian numbers, *Utilitas Math.* **15** (1979), 51–88.

[9] G.-S. Cheon, S.-G. Hwang and S.-G. Lee, Several polynomials associated with the harmonic numbers, *Discrete Appl. Math* **155** (2007), 2573–2584, doi:10.1016/j.dam.2007.07.008.

[10] L. Comtet, *Advanced Combinatorics: The Art of Finite and Infinite Expansions*, D. Reidel Publishing, Dordrecht, 1974.

[11] J. W. L. Glaisher, Expressions for Laplace's coefficients, Bernoullian and Eulerian numbers etc. as determinants, *Messenger Math.* **6** (1875), 49–63.

[12] A. Hassen and H. D. Nguyen, Hypergeometric Bernoulli polynomials and Appell sequences, *Int. J. Number Theory* **4** (2008), 767–774, doi:10.1142/s1793042108001754.

[13] A. Hassen and H. D. Nguyen, Hypergeometric zeta functions, *Int. J. Number Theory* **6** (2010), 99–126, doi:10.1142/s179304211000282x.

[14] F. T. Howard, A sequence of numbers related to the exponential function, *Duke Math. J.* **34** (1967), 599–615, doi:10.1215/s0012-7094-67-03465-5.

[15] F. T. Howard, Some sequences of rational numbers related to the exponential function, *Duke Math. J.* **34** (1967), 701–716, doi:10.1215/s0012-7094-67-03473-4.

[16] F. T. Howard, Explicit formulas for degenerate Bernoulli numbers, *Discrete Math.* **162** (1996), 175–185, doi:10.1016/0012-365x(95)00284-4.

[17] S. Hu and M.-S. Kim, On hypergeometric Bernoulli numbers and polynomials, *Acta Math. Hungar.* **154** (2018), 134–146, doi:10.1007/s10474-017-0767-6.

[18] S. Hu and T. Komatsu, Explicit expressions for the related numbers of higher order Appell polynomials, *Quaest. Math.* **43** (2020), 1019–1029, doi:10.2989/16073606.2019.1596174.

[19] K. Kamano, Sums of products of hypergeometric Bernoulli numbers, *J. Number Theory* **130** (2010), 2259–2271, doi:10.1016/j.jnt.2010.04.005.

[20] T. Komatsu, Hypergeometric Cauchy numbers, *Int. J. Number Theory* **9** (2013), 545–560, doi:10.1142/s1793042112501473.

[21] T. Komatsu, Two types of hypergeometric degenerate Cauchy numbers, *Open Math.* **18** (2020), 417–433, doi:10.1515/math-2020-0030.

[22] T. Komatsu and J. L. Ramírez, Some determinants involving incomplete Fubini numbers, *An. Ştiinţ. Univ. "Ovidius" Constanţa Ser. Mat.* **26** (2018), 143–170, doi:10.2478/auom-2018-0038.

[23] T. Komatsu and P. Yuan, Hypergeometric Cauchy numbers and polynomials, *Acta Math. Hungar.* **153** (2017), 382–400, doi:10.1007/s10474-017-0744-0.

[24] T. Muir, *The Theory of Determinants in the Historical Order of Development (Volumes 1–4)*, Dover Publications, New York, 4th edition, 1960.

[25] N. Trudi, Intorno ad alcune formole di sviluppo, *Rendic. dell' Accad. Napoli* (1862), 135–143.

[26] P. T. Young, Degenerate Bernoulli polynomials, generalized factorial sums, and their applications, *J. Number Theory* **128** (2008), 738–758, doi:10.1016/j.jnt.2007.02.007.