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On an annihilation number conjecture*

Vadim E. Levit †

Department of Computer Science, Ariel University, Ariel, Israel

Eugen Mandrescu

Department of Computer Science, Holon Institute of Technology, Holon, Israel

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Abstract

Let $\alpha(G)$ denote the cardinality of a maximum independent set, while $\mu(G)$ be the size of a maximum matching in the graph $G = (V, E)$. If $\alpha(G) + \mu(G) = |V|$, then G is a *König-Egerváry graph*. If $d_1 \leq d_2 \leq \cdots \leq d_n$ is the degree sequence of G, then the *annihilation number* $a\,(G)$ of G is the largest integer k such that $\sum_{i=1}^{k} d_i \leq |E|$. A set $A \subseteq V$ satisfying $\sum_{v \in A} \deg(v) \leq |E|$ is an *annihilation set*; if, in addition, $\deg\,(x) + \sum_{v \in A} \deg(v) > |E|$, for every vertex $x \in V(G) - A$, then A is a *maximal annihilation set* in G.

In 2011, Larson and Pepper conjectured that the following assertions are equivalent:

(i) $\alpha\,(G) = a\,(G)$;

(ii) G is a König-Egerváry graph and every maximum independent set is a maximal annihilating set.

It turns out that the implication "(i) $\Longrightarrow$ (ii)" is correct.

In this paper, we show that the opposite direction is not valid, by providing a series of generic counterexamples.

Keywords: Maximum independent set, maximum matching, König-Egerváry graph, annihilation set, annihilation number.

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†Corresponding author.

E-mail addresses: levitv@ariel.ac.il (Vadim E. Levit), eugen_m@hit.ac.il (Eugen Mandrescu)

1 Introduction

Throughout this paper $G = (V, E)$ is a finite, undirected, loopless graph without multiple edges, with vertex set $V = V(G)$ of cardinality $|V(G)| = n(G)$, and edge set $E = E(G)$ of size $|E(G)| = m(G)$. If $X \subset V(G)$, then $G[X]$ is the subgraph of G induced by X. By $G - v$ we mean the subgraph $G[V(G) - \{v\}]$, for $v \in V(G)$. $K_n, K_{m,n}, P_n, C_n$ denote respectively, the complete graph on $n \geq 1$ vertices, the complete bipartite graph on $m, n \geq 1$ vertices, the path on $n \geq 1$ vertices, and the cycle on $n \geq 3$ vertices, respectively.

The *disjoint union* of the graphs G_1, G_2 is the graph $G_1 \cup G_2$ having the disjoint union of $V(G_1), V(G_2)$ as a vertex set, and the disjoint union of $E(G_1), E(G_2)$ as an edge set. In particular, nG denotes the disjoint union of $n > 1$ copies of the graph G.

A set $S \subseteq V(G)$ is *independent* if no two vertices from S are adjacent, and by $\mathrm{Ind}(G)$ we mean the family of all the independent sets of G. An independent set of maximum size is a *maximum independent set* of G, and $\alpha(G) = \max\{|S| : S \in \mathrm{Ind}(G)\}$. Let $\Omega(G)$ denote the family of all maximum independent sets.

A *matching* in a graph G is a set of edges $M \subseteq E(G)$ such that no two edges of M share a common vertex. A matching of maximum cardinality $\mu(G)$ is a *maximum matching*, and a *perfect matching* is one saturating all vertices of G.

It is known that $\lfloor n(G)/2 \rfloor + 1 \leq \alpha(G) + \mu(G) \leq n(G) \leq \alpha(G) + 2\mu(G)$ hold for every graph G [6]. If $\alpha(G) + \mu(G) = n(G)$, then G is called a König-Egerváry graph [11, 36]. For instance, each bipartite graph is a König-Egerváry graph [13, 20]. Various properties of König-Egerváry graphs can be found in [3, 4, 5, 16, 17, 18, 21, 22, 23, 25, 26, 27, 28, 29, 30, 31, 35].

Let $d_1 \leq d_2 \leq \cdots \leq d_n$ be the degree sequence of a graph G. Pepper [33, 34] defined the annihilation number of G, denoted $a(G)$, to be the largest integer k such that the sum of the first k terms of the degree sequence is at most half the sum of the degrees in the sequence. In other words, $a(G)$ is precisely the largest integer k such that $\sum_{i=1}^{k} d_i \leq m(G)$.

Clearly, $a(G) = n(G)$ if and only if $m(G) = 0$. If $m(G) = 1$, then $a(G) = n(G) - 1$. The converse is not true; e.g., the graph $K_{1,p}$ has $a(K_{1,p}) = m(K_{1,p}) = p = n(K_{1,p}) - 1$, while p may be greater than one.

For $A \subseteq V(G)$, let $\deg(A) = \sum_{v \in A} \deg(v)$. Every $A \subseteq V(G)$ satisfying $\deg(A) \leq m(G)$ is an *annihilating set*. Clearly, every independent set is annihilating. An annihilating set A is *maximal* if $\deg(A \cup \{x\}) > m(G)$, for every vertex $x \in V(G) - A$, and it is *maximum* if $|A| = a(G)$ [33]. For example, if $G = K_{p,q} = (A, B, E)$ and $p > q$, then A is a maximum annihilating set, while B is a maximal annihilating set.

Theorem 1.1 ([33]). *For every graph G,*

$$a(G) \geq \max\left\{\left\lfloor \frac{n(G)}{2} \right\rfloor, \alpha(G)\right\}.$$

For instance,

$$a(C_7) = \alpha(C_7) = \left\lfloor \frac{n(C_7)}{2} \right\rfloor, \qquad a(\overline{P_5}) = 3 > \alpha(\overline{P_5}) = \left\lfloor \frac{n(\overline{P_5})}{2} \right\rfloor,$$

$$a(K_{2,3}) = \alpha(K_{2,3}) > \left\lfloor \frac{n(K_{2,3})}{2} \right\rfloor, \quad \text{while} \quad a(\overline{C_6}) = \left\lfloor \frac{n(\overline{C_6})}{2} \right\rfloor > \alpha(\overline{C_6}).$$

The relation between the annihilation number and various parameters of a graph were studied in [1, 2, 7, 8, 9, 10, 12, 14, 15, 19, 32, 33].

Theorem 1.2 ([24]). *For a graph G with $a(G) \geq \frac{n(G)}{2}$, $\alpha(G) = a(G)$ if and only if G is a König-Egerváry graph and every $S \in \Omega(G)$ is a maximum annihilating set.*

All the maximum independent sets of the cycle C_5 are maximum annihilating. Moreover, $a(C_5) = \alpha(C_5)$. Nevertheless, C_5 is not a König-Egerváry graph. In other words, the condition $a(G) \geq \frac{n(G)}{2}$ in Theorem 1.2 is necessary.

Actually, Larson and Pepper [24] proved a stronger result that reads as follows.

Theorem 1.3. *Let G be a graph with $a(G) \geq \frac{n(G)}{2}$. Then the following are equivalent:*

(i) $\alpha(G) = a(G)$*;*

(ii) *G is a König-Egerváry graph and every $S \in \Omega(G)$ is a maximum annihilating set;*

(iii) *G is a König-Egerváry graph and some $S \in \Omega(G)$ is a maximum annihilating set.*

Along these lines, it was conjectured that the impacts of maximum and maximal annihilating sets are the same.

Conjecture 1.4 ([24]). *Let G be a graph with $a(G) \geq \frac{n(G)}{2}$. Then the following assertions are equivalent:*

(i) $\alpha(G) = a(G)$*;*

(ii) *G is a König-Egerváry graph and every $S \in \Omega(G)$ is a maximal annihilating set.*

One can easily infer that every maximum annihilating set is also a maximal annihilating set, since the sum of the $a+1$ smallest entries from the degree sequence $D = (d_1 \leq d_2 \leq \cdots \leq d_n)$ is greater than $m(G)$, then the same is true for every $a+1$ entries of D. Thus the "(i) $\Longrightarrow$ (ii)" part of Conjecture 1.4 is valid, in accordance with Theorem 1.2.

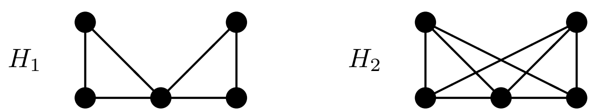

Figure 1: Non-König-Egerváry graphs with $a(H_1) = 3$ and $a(H_2) = 2$.

Consider the graphs from Figure 1. The graph H_1 has $a(H_1) > \alpha(H_1)$ and none of its maximum independent sets is a maximal or a maximum annihilating set. The graph H_2 has $a(H_2) = \alpha(H_2)$ and each of its maximum independent sets is both a maximal and a maximum annihilating set. Notice that $a(H_1) > \frac{n(H_1)}{2}$, while $a(H_2) < \frac{n(H_2)}{2}$.

Consider the graphs from Figure 2. The graph G_1 has $\alpha(G_1) = \frac{n(G_1)}{2} < a(G_1)$ and each of its maximum independent sets is neither a maximal nor a maximum annihilating set. The graph G_2 has $a(G_2) = \alpha(G_2) = \frac{n(G_2)}{2}$, every of its maximum independent sets is both a maximal and a maximum annihilating set, and it has a maximal independent set that is a maximal non-maximum annihilating set, namely $\{u, v\}$. The graph G_3 has $a(G_3) = \alpha(G_3) > \frac{n(G_3)}{2}$ and every of its maximum independent sets is both a maximal

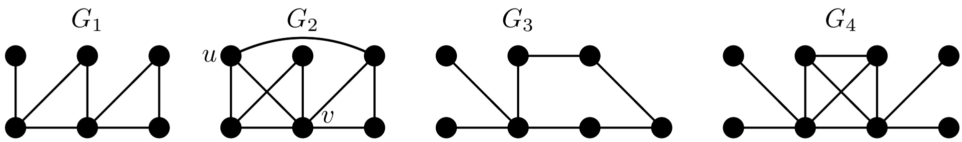

Figure 2: König-Egerváry graphs with $a(G_1) = a(G_3) = 4$, $a(G_2) = 3$, $a(G_4) = 6$.

and a maximum annihilating set. The graph G_4 has $a(G_4) > \alpha(G_4) > \frac{n(G_4)}{2}$ and none of its maximum independent sets is a maximal or a maximum annihilating set.

In this paper we invalidate the "(ii) $\Longrightarrow$ (i)" part of Conjecture 1.4, by providing some generic counterexamples. Let us notice that, if G is a König-Egerváry graph, and $H = qK_1 \cup G$, then H inherits this property. Moreover, the relationship between the independence numbers and annihilation numbers of G and H remains the same, because $\alpha(H) = \alpha(G) + q$ and $a(H) = a(G) + q$. Therefore, it is enough to construct only connected counterexamples. Finally, we prove that Conjecture 1.4 is true for graphs with independence number equal to three.

2 An infinite family of counterexamples

In what follows, we present a series of counterexamples to the opposite direction of Conjecture 1.4. All these graphs have unique maximum independent sets.

Lemma 2.1. *The graph $H_k, k \geq 0$, from Figure 3 is a connected König-Egerváry graph that has a unique maximum independent set, namely, $S_k = \{x_k, \ldots, x_1, a_4, a_3, a_2, a_1\}$, where $H_0 = H_k - \{x_j, y_j : j = 1, 2, \ldots, k\}$ and $S_0 = \{a_4, a_3, a_2, a_1\}$.*

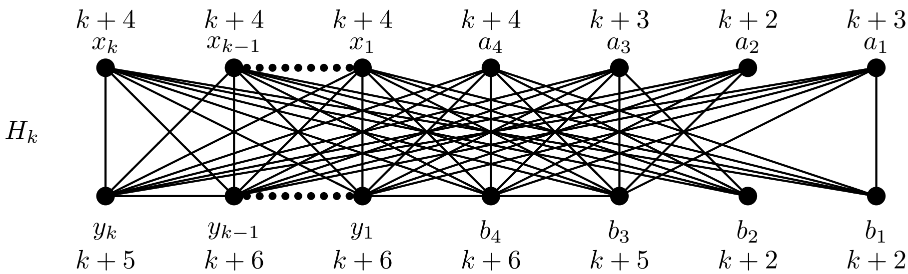

Figure 3: H_k is a König-Egerváry graph with $\alpha(H_k) = k + 4, k \geq 0$.

Proof. Notice that the graph H_k from Figure 3 can be defined as follows:

$$V(H_k) = V(K_{k+4,k+4}) = \{x_i, y_i : i = 1, \ldots, k\} \cup \{a_1, a_2, a_3, a_4\} \cup \{b_1, b_2, b_3, b_4\},$$
$$E(H_k) = E(K_{k+4,k+4}) \cup \{y_k y_{k-1}, \ldots, y_2 y_1, y_1 b_4, b_4 b_3\} - \{a_3 b_1, a_2 b_2, a_2 b_1, a_1 b_2\}.$$

Clearly, $S_k = \{x_k, \ldots, x_1, a_4, a_3, a_2, a_1\}$ is an independent set and

$$\{x_j y_j : j = 1, 2, ..., k\} \cup \{a_4 b_4, a_3 b_2, a_3 b_3, a_1 b_1\}$$

is a perfect matching of H_k. Hence, we get

$$|V_k| = 2\mu(H_k) = |S_k| + \mu(H_k) \leq \alpha(H_k) + \mu(H_k) \leq |V_k|,$$

which implies $\alpha(H_k) + \mu(H_k) = |V_k|$, i.e., H_k is a König-Egerváry graph, and $\alpha(H_k) = k + 4 = |S_k|$.

Let $L_k = H_k[X_k \cup Y_k]$, $k \geq 1$, and $L_0 = H_k[A \cup B]$, where

$$X_k = \{x_j : j = 1, \ldots, k\}, \qquad Y_k = \{y_j : j = 1, \ldots, k\},$$
$$A = \{a_1, a_2, a_3, a_4\} \quad \text{and} \quad B = \{b_1, b_2, b_3, b_4\}.$$

Since L_k has, on the one hand, $K_{k,k}$ as a subgraph, and, on the other hand,

$$y_k y_{k-1}, y_{k-1} y_{k-2}, \ldots, y_2 y_1 \in E(L_k),$$

it follows that X_k is the unique maximum independent set of L_k.

The graph L_0 has A as a unique independent set, because

$$C_8 + b_3 b_4 = (A \cup B, \{a_1 b_4, b_4 a_2, a_2 b_3, b_3 a_3, a_3 b_2, b_2 a_4, a_4 b_1, b_1 a_1, b_3 b_4\})$$

has A as a unique maximum independent set, and L_0 can be obtained from $C_8 + b_3 b_4$ by adding a number of edges.

Since H_k can be obtained from the union of L_k and L_0 by adding some edges, and $S_k = X_k \cup A$ is independent in H_k, it follows that H_k has S_k as a unique maximum independent set. □

Corollary 2.2. *The graph $G_k, k \geq 0$, from Figure 4 is a connected König-Egerváry graph that has a unique independent set, namely, $S_k = \{x_i : i = 1, \ldots, k\} \cup \{a_i : i = 1, \ldots, 5\}$, where $G_0 = G_k - \{x_j, y_j : j = 1, 2, \ldots, k\}$ and $S_0 = \{a_i : i = 1, \ldots, 5\}$.*

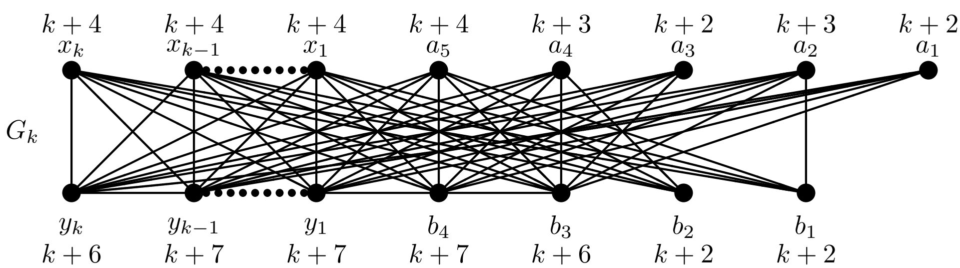

Figure 4: G_k is a König-Egerváry graph with $\alpha(G_k) = k + 5, k \geq 0$.

Proof. Notice that the graph G_k from Figure 4 can be defined as follows:

$$\begin{aligned} V(G_k) &= V(K_{k+5,k+4}) \\ &= \{x_i, y_i : i = 1, \ldots, k\} \cup \{a_1, a_2, a_3, a_4, a_5\} \cup \{b_1, b_2, b_3, b_4\}, \\ E(G_k) &= E(K_{k+4,k+4}) \cup \{y_k y_{k-1}, \ldots, y_2 y_1, y_1 b_4, b_4 b_3\} \\ &\qquad - \{a_3 b_2, a_3 b_1, a_2 b_2, a_1 b_2, a_1 b_1\}. \end{aligned}$$

According to Lemma 2.1, $G_k - a_1$ is a König-Egerváry graph with a unique maximum independent set, namely, $W_k = \{x_i : i = 1, \ldots, k\} \cup \{a_i : i = 1, \ldots, 4\}$. Since $S_k = W_k \cup \{a_1\}$ is an independent set and $\mu(G_k) = \mu(G_k - a_1) = k + 4$, it follows that G_k is a König-Egerváry graph and S_k is its unique maximum independent set. □

Theorem 2.3. *For every $k \geq 0$, there exists a connected non-bipartite König-Egerváry graph $H_k = (V_k, E_k)$, of order $2k+8$, satisfying the following:*

- $a(H_k) > \frac{n(H_k)}{2} = \alpha(H_k)$,
- *each $S \in \Omega(H_k)$ is a maximal annihilating set.*

Proof. Let $H_k = (V_k, E_k), k \geq 0$, be the graph from Figure 3 (in the bottom and the top lines are written the degrees of its vertices), where $H_0 = H_k - \{x_1, \ldots, x_k, y_1, \ldots, y_k\}$. Clearly, every H_k is non-bipartite.

By Lemma 2.1, each $H_k, k \geq 0$, is a König-Egerváry graph with a unique maximum independent set, namely, $S_k = \{x_k, \ldots, x_1, a_4, a_3, a_2, a_1\}$, where $S_0 = \{a_4, a_3, a_2, a_1\}$.

Case 1. $k = 0$. Since $m(H_0) = 13$ and the degree sequence $(2, 2, 2, 3, 3, 4, 5, 5)$, we infer that $a(H_0) = 5 > 4 = \alpha(H_0)$. In addition, $\deg(S_0) = m(H_0) - 1$, i.e., each maximum independent set of H_0 is a maximal non-maximum annihilating set.

Case 2. $k \geq 1$. Clearly, H_k has $m(G_k) = k^2 + 9k + 13$ and its degree sequence is

$$k+2, k+2, k+2, k+3, k+3, \underbrace{k+4, \ldots, k+4}_{k+1}, k+5, k+5, \underbrace{k+6, \ldots, k+6}_{k}.$$

Since the sum of the first $k+6$ degrees of the sequence satisfies

$$k^2 + 10k + 16 > m(H_k),$$

we infer that the annihilation number $a(H_k) \leq k+6$. The sum $12 + 4(x-5) + kx$ of the first $x \geq 5$ degrees of the sequence satisfies $12 + 4(x-5) + kx \leq m(H_k)$ for $x \leq \frac{k^2+9k+21}{k+4}$. This implies

$$a(H_k) = \left\lfloor \frac{k^2 + 9k + 21}{k+4} \right\rfloor = k+5 > k+4 = \alpha(H_k),$$

i.e., H_k has no maximum annihilating set belonging to $\Omega(H_k)$. Since its unique maximum independent set $S_k = \{a_1, a_2, a_3, a_4, x_1, x_2, \ldots, x_k\}$ has

$$\deg(S_k) = k^2 + 8k + 12 < m(H_k),$$

while

$$\deg(S_k) + \min\{\deg(v) : v \in V_k - S\} = \left(k^2 + 8k + 12\right) + (k+2) > m(H_k),$$

we infer that S_k is a maximal annihilating set. $\square$

Theorem 2.4. *For every $k \geq 0$, there exists a connected non-bipartite König-Egerváry graph $G_k = (V_k, E_k)$, of order $2k+9$, satisfying the following:*

- $a(G_k) > \left\lceil \frac{n(G_k)}{2} \right\rceil = \alpha(G_k)$,
- *each $S \in \Omega(G_k)$ is a maximal annihilating set.*

Proof. Let $G_k = (V_k, E_k), k \geq 1$, be the graph from Figure 4 (in the bottom and the top lines are written the degrees of its vertices), and $G_0 = G_k - \{x_1, \ldots, x_k, y_1, \ldots, y_k\}$.

Corollary 2.2 claims that $G_k, k \geq 0$, is a König-Egerváry graph with a unique maximum independent set, namely $S_k = \{x_1, \ldots, x_k, a_1, \ldots, a_5\}, k \geq 1$, and $S_0 = \{a_1, \ldots, a_5\}$.

Case 1. The non-bipartite König-Egerváry graph G_0 has $m(G_0) = 15$ and the degree sequence $(2, 2, 2, 2, 3, 3, 4, 6, 6)$. Hence, $a(G_0) = 6 > 5 = \alpha(G_0)$. In addition, $\Omega(G_0) = \{S_0\}$, and $\deg(S_0) = 14$, i.e., each maximum independent set of G_0 is a maximal non-maximum annihilating set.

Case 2. $k \geq 1$. Clearly, G_k has $m(G_k) = k^2 + 10k + 15$ and its degree sequence is

$$k+2, k+2, k+2, k+2, k+3, k+3, \underbrace{k+4, \ldots, k+4}_{k+1}, k+6, k+6, \underbrace{k+7, \ldots, k+7}_{k}.$$

Since the sum of the first $k + 7$ degrees of the sequence satisfies

$$k^2 + 11k + 18 > m(G_k),$$

we infer that the annihilation number $a(G_k) \leq k + 6$. The sum $14 + 4(x - 5) + kx$ of the first $x \geq 6$ degrees of the sequence satisfies $14 + 4(x - 6) + kx \leq m(G_k)$ for $x \leq \frac{k^2+10k+25}{k+4}$. This implies

$$a(G_k) = \left\lfloor \frac{k^2 + 10k + 25}{k + 4} \right\rfloor = k + 6 > k + 5 = \alpha(G_k),$$

i.e., G_k has no maximum annihilating set belonging to $\Omega(G_k)$. Since its unique maximum independent set S_k has

$$\deg(S_k) = k^2 + 9k + 14 < m(G_k),$$

while

$$\deg(S_k) + \min\{\deg(v) : v \in V_k - S_k\} = \left(k^2 + 9k + 14\right) + (k + 2) > m(G_k),$$

we infer that S_k is a maximal annihilating set. $\square$

3 Conclusions

If G is a König-Egerváry graph with $\alpha(G) \in \{1, 2\}$, then $\alpha(G) = a(G)$ and each maximum independent set is maximal annihilating, since the list of such König-Egerváry graphs reads as follows:

$$\{K_1, K_2, K_1 \cup K_1, K_1 \cup K_2, K_2 \cup K_2, P_3, P_4, C_4, K_3 + e, K_4 - e\}.$$

Consequently, Conjecture 1.4 is correct for König-Egerváry graphs with $\alpha(G) \leq 2$.

Let G be a disconnected König-Egerváry graph with $\alpha(G) = 3$.

- If $\alpha(G) = a(G)$, then

$$G \in \left\{ \begin{array}{c} 3K_1, 2K_1 \cup K_2, K_1 \cup 2K_2, 3K_2, K_1 \cup P_3, K_1 \cup P_4, \\ K_1 \cup C_4, K_1 \cup (K_3 + e), K_1 \cup (K_4 - e), K_2 \cup P_3, K_2 \cup C_4 \end{array} \right\},$$

  while every $S \in \Omega(G)$ is a maximal annihilating set.

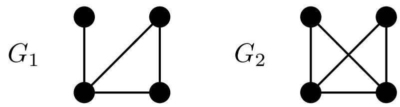

Figure 5: $G_1 = K_3 + e$ and $G_2 = K_4 - e$.

- If $\alpha(G) < a(G)$, then $G \in \{K_2 \cup P_4, K_2 \cup (K_3 + e), K_2 \cup (K_4 - e)\}$, while for every such G, there exists a maximum independent set, which is a not a maximal annihilating set. Moreover, for $K_2 \cup (K_3 + e)$ and $K_2 \cup (K_4 - e)$ all maximum independent sets are not maximal annihilating.

Thus Conjecture 1.4 is true for disconnected König-Egerváry graphs with $\alpha(G) = 3$.

We have already mentioned in Introduction that the "(i) $\Longrightarrow$ (ii)" part of Conjecture 1.4 is true.

Proposition 3.1. *Let G be a graph with $a(G) \geq \frac{n(G)}{2}$. If G is a connected König-Egerváry graph with $\alpha(G) = 3$, and every $S \in \Omega(G)$ is a maximal annihilating set, then $\alpha(G) = a(G)$.*

Proof. In Figure 6 we present all connected König-Egerváry graphs with $\alpha(G) = 3$ having $n(G) \in \{4, 5\}$. For these graphs $\alpha(G) = a(G)$, which means that Conjecture 1.4 is true.

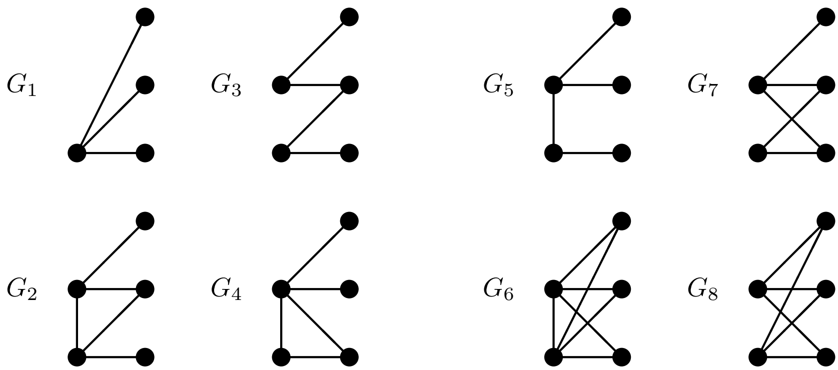

Figure 6: König-Egerváry graphs with $\alpha(G) = 3 = a(G)$ and $n(G) \leq 5$.

Now, we may assume that $n(G) = 6$, since $\alpha(G) \geq \mu(G)$ holds for each König-Egerváry graph.

Let $d_1 \leq d_2 \leq \cdots \leq d_6$ be the degree sequence of G.

It is known that $\alpha(G) \leq a(G)$ (Theorem 1.1). Thus we have only three cases with $3 = \alpha(G) < a(G)$ to cover, namely, $a(G) \in \{4, 5, 6\}$.

Case 1. $a(G) = 4$. Then, by definition, $d_1 + d_2 + d_3 + d_4 \leq m(G) \leq d_5 + d_6$ and $d_1 + d_2 + d_3 + d_4 + d_5 > m(G) > d_6$.

Let q be the number of edges in G joining the vertices v_5, v_6 with the vertices v_1, v_2, v_3, v_4. At least two vertices from the set $\{v_1, v_2, v_3, v_4\}$ must be joined by an edge, otherwise, $\alpha(G) \geq 4 > 3$. Assume that $v_3v_4 \in E(G)$. Hence, $v_5v_6 \in E(G)$, otherwise,

$$d_5 + d_6 = q < q + 2 \leq d_1 + d_2 + d_3 + d_4,$$

in contradiction with $d_1 + d_2 + d_3 + d_4 \leq d_5 + d_6$. Similarly, there are no more edges but v_3v_4 joining vertices from the set $\{v_1, v_2, v_3, v_4\}$, otherwise

$$d_5 + d_6 = q + 2 < q + 4 \leq d_1 + d_2 + d_3 + d_4,$$

in contradiction with $d_1 + d_2 + d_3 + d_4 \leq d_5 + d_6$. Therefore, $\{v_1, v_2, v_3\}$ is a maximum independent set of G, since $\alpha(G) = 3$. On the other hand, $\{v_1, v_2, v_3\}$ is not a maximal annihilating set, because $d_1 + d_2 + d_3 + d_4 \leq m(G)$.

Case 2. $a(G) = 5$. By definition, it follows that $d_1 + d_2 + d_3 + d_4 + d_5 \leq m(G) \leq d_6$. Hence, the set $\{v_1, v_2, v_3, v_4, v_5\}$ is independent, in contradiction with the fact that $\alpha(G) = 3$.

Case 3. $a(G) = 6$. This means that G has no edges, which is not possible, because $\alpha(G) = 3$. □

To complete the picture, Theorems 2.3 and 2.4 present various counterexamples to the "(ii) $\Longrightarrow$ (i)" part of Conjecture 1.4 for every independence number greater than three. Our intuition tells us that the real obstacle for the "(i) $\Longrightarrow$ (ii)" part Conjecture 1.4 not to be true is the size of the annihilation number. It motivates the following.

Conjecture 3.2. *If G is a König-Egerváry graph with $a(G) \geq \frac{3}{4}n(G)$, and every $S \in \Omega(G)$ is a maximal annihilating set, then $\alpha(G) = a(G)$.*

ORCID iDs

Vadim E. Levit https://orcid.org/0000-0002-4190-7050
Eugen Mandrescu https://orcid.org/0000-0003-3533-9728

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