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What Comes Beyond the Standard Models

Bled, July 10–19, 2023

Edited by

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Preface in English and Slovenian Language

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This year was 26^{th} time that our series of workshops on "What Comes Beyond the Standard Models?" took place. The series started in 1998 with the idea of organising a workshop in which participants would spend most of the time in discussions, confronting different approaches and ideas. The picturesque town of Bled by the lake of the same name, surrounded by beautiful mountains and offering pleasant walks, was chosen to stimulate the discussions.
The idea was successful and has developed into an annual workshop, which is taking place every year since 1998. Very open-minded and fruitful discussions have become the trademark of our workshops, producing several published works. It took place in the house of Plemelj, which belongs to the Society of Mathematicians, Physicists and Astronomers of Slovenia.
The workshops at Bled changed after the Covid pandemic: For two years, the workshop became almost virtual and correspondingly less open-minded. The discussions, which before asked the speaker to explain and prove each step, can not be done so easily virtually. However, many questions still interrupt the presentations, so that the speakers must often continue their talks several times in the following days. Also, this year, most of the talks were presented virtually.
Most of the contributions present theoretical concepts of solving the open problems of elementary fermion and boson fields and of cosmology.
Recent experiments in astrophysical stochastic gravitational wave background receive comments and suggestions for possible interpretation in contributions.
But there are also the contributions of experimental groups This year the excellent experimental group report the measurements of 22 independent annual cycles with various configurations, which ensures us that they do measure the dark matter.
The origin of the dark matter seems to be the most discussed open problem in the literature in the last few years. Also, most of the authors of the contributions report , beside about other ideas, also on their ideas about the origin of dark matter.
The speakers also defended several concepts on how to solve the open problems in cosmology, all of which are unavoidably connected with the understanding of the elementary fermion and boson fields and the laws of nature.
The participants suggest: The laws of nature are simple and elegant, which requires treating all the fermion and boson fields in an unique way, with their internal spaces and their properties in the ordinary space included. This concept requires that the space-time is not the observed one but much larger, with a unique interaction in all dimensions and the maximal symmetry in both spaces. It is the break of the starting symmetry, caused by phase transitions, which is responsible in $(3+1)$-dimensions for the observed properties of quarks and leptons, of their boson gauge fields, the scalar fields, for explaining the masses of all the fermions with the masses of the fermions forming the dark matter included, explaining the appearance of the matter/antimatter asymmetry as well as all the observations

and measurements. Extending the point particles to strings might take care of the renormalisability of this theory.
All kinds of grand unifying theories are less requiring concepts of the one presented under the first point, but can be included in the above concept and can help to suggest a way of breaking symmetries and to propose measurements and observations.
Among the assumptions that try to explain the cosmological measurements, it is the recently proposed thesis that the smallness of dark energy, known as the cosmological hierarchy, can be explained by assuming that one of the dimensions in $d > (3+1)$ is of the micron size.
Also, these year contributions present the mathematical concepts, which might help to justify the choice of "simple and elegant" laws of nature in maximally symmetric (internal and external) spaces (offering the explanations for all the assumptions of the *standard model*). Such a mathematical point of view can help to find ways of breaking symmetries, "chosen by nature".
There is also the warning about how carefully relativistic quantum mechanics has to be taken into account in the dispersion relations of the wave packets, when measuring the arrival time of high energy cosmic rays.
The authors discuss the possibility that the primordial black holes are responsible for the stochastic gravitational wave background. Several other things are discussed, such as the baryon number non-conservation (at least not in the ordinary matter sector), the mass spectrum of black holes and the properties of domain walls.
The authors pay particular attention to dark matter streaming around our Earth, which might be responsible for many observed but unexplained observations.
Many a presented and developed idea in this proceeding might not be in agreement with the others presented in thhe same proceedings. But yet different ideas, if developed in a consistent way, might help to understand the problems in connection with the measurements and observations, which only can confirm what is the status of the laws in our universe.
The idea, presented in the last New Scientist about the possibility that the dark matter is created in the second Big Bang can hardly be true if the laws of nature are simple and elegant in the space which has more than $(3+1)$=dimensions. However, some authors had similar ideas before.
Looking at the collection of open questions that we set ourselves before starting these Bled workshops and continuously supplementing in each workshop, it shows up that we are all the time mostly looking for an answer to the essential question: How to explain all the assumptions of the *standard model*, which would offer not only the understanding of all the assumed properties for quarks and leptons and all the observed boson fields with the Higgs scalars included, but also for the observed phenomena in cosmology; like it is the understanding of the expansion rate of the universe, of the appearance of the dark matter, of black holes with their (second quantised) quantum nature included, of the necessity of the existence of dark energy and many others.
When trying to understand the quantum nature of fermion and boson fields, we are looking for a theory which is anomaly free and possibly renormalisable so

that we would be able to predict the properties of second quantised fields when proposing measurements, as well as when trying to understand the behaviour of fermion and boson fields within black holes in ordinary matter and dark matter. Many talks are ”unusual” in that they seek to find a new, more trustworthy way of understanding and describing the observed phenomena.

This year was very special: The organisers are asking the University of Ljubljana for the help in arranging the DOI number.
Although the *Society of Mathematicians, Physicists and Astronomers of Slovenia* remain our organiser, for what we are very grateful, yet the Faculty of Mathematics and Physics starts to be our publisher together with the University of Ljubljana. The technical procedure is now different, and the possibility that the participants send the contributions “the last moment” is less available.
Several participants have not succeeded in sending their contributions in time. We publish only abstracts of those who sent in time at least abstracts. Their contributions will be published next year if they want. The same will also happen with contributions, which have not succeeded to send even abstracts in time. From July to November is a short time, in particular, since this period includes vacations.
The organisers are grateful to all the participants for the lively presentations and discussions and an excellent working atmosphere, although most participants appeared virtually, led by Maxim Khlopov.
The reader can find all the talks and soon also the whole Proceedings on the official website of the Workshop: http://bsm.fmf.uni-lj.si/bled2023bsm/presentations.html, and on the Cosmovia Forum https://bit.ly/bled2023bsm ..

Norma Mankoč Borštnik, Holger Bech Nielsen,
Maxim Khlopov, Astri Kleppe *Ljubljana, December 2023*

1 Predgovor (Preface in Slovenian Language)

Letos je serija delavnic z naslovom „Kako preseči oba standardna modela, kozmološkega in elektrošibkega" ("What Comes Beyond the Standard Models?") stekla že šestindvajsetič. Prva delavnica je stekla leta 1998 v želji, da bi udeleženci v izčrpnih diskusijah kritično soočali različne ideje in teorije. Slikovito mestece Bled, ob jezeru z enakim imenom, obkroženo s prijaznimi hribčki, nad katerimi kipijo slikovite gore, ki ponujajo prijetne sprehode in pohode, ponujajo priložnosti za diskusije.
Ideja je bila uspešna, razvila se je v vsakoletno delavnico, ki teče že šestindvajsetič. Zelo odprte, prijateljske in učinkovite diskusije so postale "blagovna znamka" naših delavnic, ideje, ki so se v diskusijah rodile, pa so pogosto botrovale objavljenim člankom. Delavnice domujejo v Plemljevi hiši na Bledu tik ob jezeru. Hišo je Društvu matematikov, fizikom in astronomov zapustil svetovno priznani slovenski matematik Jozef Plemelj.
Delavnice na Bledu so se po pandemiji covida spremenile: Že dve leti je bila delavnica skoraj virtualna in temu primerno manj odprta in tudi manj prijateljska. Razprave, ki govorca prijazno prosijo, da naj vsak korak razloži in predstavi dokaze, v virtualni diskusiji ni mogoče narediti tako učinkovite. Vendar pa še vedno poslušalci postavljajo veliko vprašanj, tako da morajo govorci večkrat nadaljevati svoje predavanje v naslednjih dneh. Tudi v letošnji delavnici je tekla večina predavanj preko spleta.
Večina prispevkov je tudi letos teoretičnih; predstavljajo teoretične koncepte reševanja odprtih problemiov elementarnih fermionskih in bozonskih polj ter kozmologije.
Nedavna merjenja astrofizikalnih stohastičnih gravitacijskih valov odmevajo tudi v letošnjih prispevkih. Ponujajo predloge za interpretacijo.
V zborniku je tudi odlični prispevek eksperimentalne skupine, ki poroča o meritvah dvaindvajsetletnih letnih modulacijah, ki jih experimentalno posodablja in nas prepriča, da je to, kar merijo, lahko samo posledica interakcije temne snovi z merilnimi aparaturami.
Zdi se, da v zadnjih letih največ prispevkov v literaturi poskuša ugotoviti, iz ęsa je temna snov. Tudi ve čina avtorjev prispevkov v zborniku predstavi, med drugimi zamislimi, svoje zamisli o izvoru temne snovi.
Prispevki, ki pojasnjujejo kozmološke meritve, ne morejo mimo dejstva, da so odprta vprašanja v kozmologiji neizogibno povezana z razumevanjem elementarnih fermionskih in bozonskih polj ter zakonov narave.
Udeleženci predlagajo:
Naravni zakoni so preprosti in elegantni, kar pa zahteva obravnavanje vseh fermionskih in bozonskih polj na enoten način; vključno z njihovimi notranjimi prostostnimi stopnjami in njihovimi lastnostmi v prostoru -času. Ta koncept pa zahteva, da je prostor-č večji od $(3+1)$, ki ga opazimo in da je interakcija med fermioni in bozoni v vseh prostorih z maksimalno simetrijo enaka in velja za notranje prostore in za več razsežni prostor-čas. Zlomitev simetrije, ki jo povzrčijo

fazni prehodi, pa privede fermione in bozone v $(3+1)$-razsežnem prostoru-času do kvarkov in leptonov, njihovih bozonskih umeritvenih polj, do skalarnih polj, ki so vzrok masam kvarkov in leptonov in šibkih bozonov, tudi masi skupkov temne snovi iz nove družine kvarkov in leptonov; pojasnijo pojav nesimetrije med barionsko snovjo in antisnovjo in ponudijo razlago tudi za kozmološkaǒpazovanja in meritve. Razširitev točkastih fermionskih in bozonskih polj, tudi skalarnih, v strune bi lahko poskrbela za renormalizabilnost te teorije.
Vse vrste teorij velikega poenotenja so manj zahtevne od omenjenega koncepta, se pa dajo vgraditi v omenjeni koncept. Pomagajo pri iskanju načina zlomitve simetrije, ki ga je uporabila narava in predlagajo nove meritve in opazovanja.
Med predpostavkami, ki poskušajo razložiti kozmološke meritve, se zdi nedavno predlagana teza, da je majhnost temne energije, znana kot kozmološka hierarhija, mogoče razložiti s predpostavko, da je ena od dimenzij v $d > (3+1)$ mikronske velikosti.
Tudi tokratni prispevki predstavljajo matematične koncepte, ki lahko pomagajo utemeljiti izbiro "preprostih in elegantnih" zakonov narave v maksimalno simetričnih (notranjih in zunanjih) prostorih (ki ponujajo razlage za vse predpostavke *standardnega modela*). Takšna matematičn teza lahko pomaga najti nove načine zlomitve simetrij, ki so bližje temu, kar "izbere narava".
Avtor enega prispevka opozarja na nujnost upoštevanja relativistične kvantne mehanike zaradi disperzije valovnih paketov na kozmoloških skalah pri merjenju časa prihoda visokoenergijskih kozmižmičnih žarkov.
Avtorji prispevkov razpravljajo o možnosti, da povzročajo ozadje stohastič nega gravitacijskega valovanja prvotne (primordialne) črne luknje. Pojasnjujejo neohranjanje barionskega števila (v sektorju navadne snovi), masni spekter črnih lukenj in lastnosti domenskih sten.
Avtorji obveščajo bralca, da tokovi temni snovi, ki tečejo okoli našega planeta Zemlje, lahko odgovorni za mnoga doslej nepojasnjena opaženja.
Marsikatera predstavljena in razvita ideja v tem zborniku morda ni skladna z drugimi tezami. A vendarle nove, drugačn in dobro utemeljene ideje lahko pomagajo bolje razumeti težave z razumevanjem meritev in opazovanji.
Ideja, predstavljena v zadnjem New Science, o možnosti, da nastane temna snov ob drugem velikem poku, skoraj ne more biti resnična, če so naravni zakoni preprosti in elegantni v prostoru, ki ima več kot $(3+1)$-dimenzijo. Dodajmo, da so nekateri avtorji teh prispevkov tudi že imeli podobne zamisli, za katere pa se zdi, da niso v skladu vsaj z nekaterimi od meritev.
Pogled na zbirko odprtih vprašanj, ki smo si jih zastavili pred začetek Blejskih delavnic in jih ob vsaki delavnici dopolnjevali. sporoča, da vsa ta leta iščemo odgovor na vprašanje: Kako razložiti vse predpostavke *standardnega modela*, da bi ponudilo ne le razumevanje vseh predpostavljenih lastnosti za kvarke in leptone in vsa bozonska polja, vključno s Higgsovimi skalarji in gravitacijo, ampak bi tudi pojasnilo pojave, ki jih opazimo v vesolju.
Ko poskušamo razumeti kvantno naravo fermionskih in bozonskih polj, iščemo teorijo brez anomalij, ki jo je mogoče renormalizirati, to je oceniti peispevke v vseh redih, tudi znotraj črnih luknenj.

Marsikateri prispevek je “nenavaden”, ker poskuša nov, bolj verodostojen in bolj celosten način razumevanja in opisovanja opaženih pojavov.

Letošnjenje leto je posebno: organizatorji Univerzo v Ljubljani prosijo za pomoč pri ureditvi DOI.
Četudi ostaja *Društvo matematikov, fizikov in astronomov Slovenije* ostaja organizator Blejskih delavnic in smo mu zato hvaležni bo naš založnik postala Fakulteta za matematiko in fiziko skupaj z Univerzo v Ljubljani . Tehnični postopek je zdaj bolj zapleten, posledično pa je možnost, da pošljejo udeleženci prispevke “zadnji trenutek“ manjša.
Kar nekaj udeležencen ni uspelo pravočasno poslati prispevka. Nekateri pa so uspeli poslati povzetke, ki jih objavljamo. Njihove prispevke ostalih, bomo objavili v nasednjem zborniku, če bodo želeli, Od julija do novembra čas hitro steče, poisebej, ker je v tem obdobju tudi časčpočitnic.
Organizatorji se iskreno zahvaljujejo vsem sodelujočim na delavnici za učinkovite predstavitve del, za živahne razprave in dobro delovno vzdušje, kljub temu, da je večina udeležencev sodelovala preko spleta, ki ga je vodil Maxim Yu. Khlopov.
Bralec najde vse pogovore in kmalu tudi celoten Zbornik na uradni spletni strani delavnice: http://bsm.fmf.uni-lj.si/bled2023bsm/presentations.html, in na forumu Cosmovia https://bit.ly/bled2023bsm ..

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Proceedings to the 26th Workshop
What Comes Beyond . . . *(p. 1)*
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1 The Swampland Program, Extra Dimensions, and Supersymmetry

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Abstract. By combining swampland conjectures with observational data, it was recently suggested that the cosmological hierarchy problem (i.e. the smallness of the dark energy in Planck units) could be understood as an asymptotic limit in field space, corresponding to a decompactification of one extra (dark) dimension of a size in the micron range. In these Proceedings we examine the fundamental setting of this framework and discuss general aspects of the effective low energy theory inherited from properties of the overarching string theory. We then explore some novel phenomenology encompassing the dark dimension by looking at potential dark matter candidates, decoding neutrino masses, and digging into new cosmological phenomena.

Povzetek: Med domnevami, ki poskušajo razložiti kozmološke meritve, se zdi sprejemljiva nedavno postavljena teza, da lahko majhnost temne energije (v Planckovih enotah), poznano pod imenom kozmološki hierarhični problem, pojasni domneva, da ima ena od razsežnosti v d > (3 + 1) velikost mikrona. V prispevku avtorjaji raziščjo veljavnost te predpostavke o temni dimenziji in razglabljajo o splošnih lastnostih nizkoenergijske limite teorije strun. Ugotovijo, da lahko predpostavka o temni dimenziji prispeva k pojavu temne snovi v vesolju, napovedo masni spekter nevtrinov ter nove kozmološke pojave.

1.1 Introduction

The challenge for a fundamental theory of nature is to describe both particle physics and cosmology. Accelerator experiments and cosmological observations provide complementary information to constrain the same theory. We have long known that only about 4% of the content of the universe is ordinary baryonic matter; the remainder is dark matter ($\sim$ 22%) and dark energy ($\sim$ 74%). The ΛCDM model, in which the expansion of the universe today is dominated by the cosmological constant Λ and cold dark matter (CDM), is the simplest model that provides a reasonably good account of all astronomical and cosmological observations [1].

The cosmological evolution is described by Einstein's equation,

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + g_{\mu\nu} \Lambda = \frac{8\pi G}{c^4} T_{\mu\nu} \,, \tag{1.1}$$

where $R_{\mu\nu}$ and R are respectively the Ricci tensor and scalar, $g_{\mu\nu}$ is the metric tensor, $T_{\mu\nu}$ is the energy momentum tensor, and $G = 1/(8\pi M_p^2)$ is Newton's gravitational constant. The cosmological constant encapsulates two length scales: the size of the observable Universe $[\Lambda] = L^{-2}$ and of the dark energy $[\Lambda/G \times c^3/\hbar] = L^{-4}$. The observed value of the cosmological constant $\Lambda_{\rm obs} \simeq 0.74 \times 3H_0^2/c^2 \simeq 1.4 \times (10^{26}~{\rm m})^{-2}$ gives a characteristic length of dark energy $\simeq 85~\mu$m, where we have adopted the recent measurement of the Hubble constant $H_0 \simeq 73$ km/s/Mpc by the HST + SH0ES team [2].

At currently achievable collider center-of-mass energies $\sqrt{s} \sim 14$ TeV or, equivalently, at distance scales $< 10^{-21}$ m, the Standard Model (SM) of strong and electroweak interactions, amended with appropriate neutrino masses, provides a successful and predictive theoretical description of all available data [1]. The experimental success of the SM can be considered as the triumph of the gauge symmetry principle to describe particle interactions. Its gauge structure is described by the symmetry group $SU(3)_C \otimes SU(2)_L \otimes U(1)_Y$, with electroweak symmetry breaking at an energy scale of $M_{ew} \sim$ TeV. On the grounds of this, the masses of the weak force carriers ($W^\pm$ and Z^0) are about 16 orders of magnitude smaller than M_p and so the weak force is 10^{24} times stronger than gravity.

A way to connect these hierarchies between particle physics and cosmology is via the size of extra dimensions which are necessary ingredients for consistency of string theory [3]. Indeed, if their size is large compared to the fundamental (string) length, the strength of gravitational interactions becomes strong at distances larger than the actual four-dimensional (4D) Planck length [4,5]. As a result, the string scale is detached from the Planck mass consistently with all experimental bounds if the observable universe is localized in the large compact space [5].

In these Proceedings we summarize the state-of-the-art in this subject area, and discuss future research directions.

1.2 Foundations of the Dark Dimension

The Swampland program seeks to understand which are the "good" low-energy EFTs that can couple to gravity consistently (e.g. the landscape of superstring theory vacua) and distinguish them from the "bad" ones that cannot [6]. In theory space, the frontier discerning the good theories from those downgraded to the swampland is drawn by a family of conjectures classifying the properties that an EFT should call for/avoid to enable a consistent completion into quantum gravity. These conjectures provide a bridge from quantum gravity to astrophysics, cosmology, and particle physics [7–9].

For example, the distance conjecture (DC) forecasts the appearance of infinite towers of states that become exponentially light and trigger the collapse of the

EFT at infinite distance limits in moduli space [10]. Connected to the DC is the anti-de Sitter (AdS) distance conjecture, which correlates the dark energy density to the mass scale m characterizing the infinite tower of states, $m \sim |\Lambda|^{\alpha}$, as the negative AdS vacuum energy $\Lambda \to 0$, with α a positive constant of $\mathcal{O}(1)$ [11]. Besides, under the hypothesis that this scaling behavior holds in dS (or quasi dS) space, an unbounded number of massless modes also pop up in the limit $\Lambda \to 0$. As demonstrated in [12], applying the AdS-DC to dS space could help elucidate the radiative stability of the cosmological hierarchy $\Lambda/M_p^4 \sim 10^{-120}$, because it connects the size of the compact space $R_\perp$ to the dark energy scale $\Lambda^{-1/4}$ via $R_\perp \sim \lambda\ \Lambda^{-1/4}$, where the proportionality factor is estimated to be within the range $10^{-1} < \lambda < 10^{-4}$. Actually, the previous relation between $R_\perp$ and Λ derives from constraints by theory and experiment. On the one hand, since the associated Kaluza-Klein (KK) tower contains massive spin-2 bosons, the Higuchi bound [13] provides an absolute upper limit to α, whereas explicit string calculations of the vacuum energy (see e.g. [14–17]) yield a lower bound on α. All in all, the theoretical constraints lead to $1/4 \leq \alpha \leq 1/2$. On the other hand, experimental arguments (e.g. constraints on deviations from Newton's gravitational inverse-square law [18] and neutron star heating [19]) lead to the conclusion encapsulated in $R_\perp \sim \lambda\ \Lambda^{-1/4}$; namely, that there is one extra dimension of radius $R_\perp$ in the micron range, and that the lower bound for $\alpha = 1/4$ is basically saturated [12]. A theoretical amendment on the connection between the cosmological and KK mass scales confirms $\alpha = 1/4$ [20]. Assembling all this together, we can conclude that the KK tower of the new (dark) dimension opens up at the mass scale $m_{\rm KK} \sim 1/R_\perp$. Within this set-up, the 5-dimensional Planck scale (or species scale where gravity becomes strong [21–24]) is given by $M_* \sim m_{\rm KK}^{1/3} M_p^{2/3}$.

It is of course interesting to explore whether there is a relation between the supersymmetry (SUSY) breaking scale and the measured value of the dark energy density Λ. Such a relation can be derived by combining two quantum gravity consistency swampland constraints, which tie Λ and the gravitino mass $M_{3/2}$, to the mass scale of a light KK tower and, therefore, to the UV cut-off of the EFT [25–27]. One can then use the constraint on $M_{3/2}$ to infer the implications of the dark dimension scenario for the scale of supersymmetry breaking. In general, one can distinguish two situations. In the first case, the gravitino mass and the cosmological constant are related to the same tower of states. This is arguably the simplest scenario, in which the natural scale for SUSY signatures is of order $\Lambda^{1/8} \sim$ TeV, and therefore is within reach of LHC and of the next generation of hadron colliders [28]. In the second case, $M_{3/2}$ and Λ are related to different towers. This scenario requires a decoupling of the gravitino mass from the cosmological constant and is thus more difficult to realize in concrete models.

Possible string theory and effective supergravity realizations of the dark dimension scenario with broken supersymmetry are discussed in [28].

1.3 Dark Matter Candidates

After the big bang, the cosmological energy density scales with time t as $\rho \sim 1/(Gt^2)$ and the density needed for a region of mass $M_{\rm BH}$ to collapse within its

Schwarzschild radius is $\rho \sim c^6/(G^3 M_{BH}^2)$, that being so primordial black holes (PBHs) would initially have around the cosmological horizon mass [29]

$$M_{BH} \sim \frac{c^3 t}{G} \sim 10^{15} \left(\frac{t}{10^{-23}\,\text{s}}\right)\,\text{g}\,. \tag{1.2}$$

This means that a black hole would have the Planck mass ($M_p \sim 10^{-5}$ g) if they formed at the Planck time (10^{-43} s), 1 $M_\odot$ if they formed at the QCD epoch (10^{-5} s), and $10^5 M_\odot$ if they formed at $t \sim 1$ s, comparable to the mass of the holes thought to reside in galactic nuclei. This back-of-the-envelope calculation suggests that PBHs could span an enormous mass range. Despite the fact that the mass spectrum of these PBHs is not set in stone, on cosmological scales they would behave like a typical CDM particle. However, an all-dark-matter interpretation in terms of PBHs is severely constrained by observations [29–31]. The extragalactic γ-ray background [32] and on the CMB spectrum [33] constrain PBH evaporation of black holes with masses $\lesssim 10^{17}$ g, whereas the non-observation of microlensing events from the MACHO [34], EROS [35], Kepler [36], Icarus [37], OGLE [38] and Subaru-HSC [39] collaborations constrain black holes with masses $\gtrsim 10^{21}$ g. Of course it is of interest to see whether new effects associated to the dark dimension could relax these bounds.

It has long been known that microscopic black holes – with Schwarzschild radii smaller than the size of the dark dimension – are quite different: they are bigger, colder, and longer-lived than a usual four-dimensional (4D) black hole of the same mass [40]. Indeed, black holes radiate all particle species lighter than or comparable to their temperature, which in four dimensions is related to the mass of the black hole by

$$T_{BH} = \frac{M_p^2}{8\pi M_{BH}} \sim \left(\frac{M_{BH}}{10^{16}\,\text{g}}\right)^{-1} \text{MeV}\,, \tag{1.3}$$

whereas for five dimensional black holes the temperature mass relation is found to be [41]

$$T_{BH} = \sqrt{\frac{3}{64}}\,\frac{1}{\pi}\,\frac{M_p\,\Lambda^{1/8}}{\lambda^{1/2}\,M_{BH}^{1/2}} \sim \left(\frac{M_{BH}}{10^{10}\,\text{g}}\right)^{-1/2} \text{MeV}\,, \tag{1.4}$$

where we have taken $\lambda \sim 10^{-3}$ as suggested by astrophysical observations [42,43]. It is evident that 5D black holes are colder than 4D black holes of the same mass. The Hawking radiation causes a 4D black hole to lose mass at the following rate [44]

$$\begin{aligned}\left.\frac{dM_{BH}}{dt}\right|_{\text{evap}} &= -\frac{M_p^2}{30720\,\pi\,M_{BH}^2}\sum_i c_i(T_{BH})\,\tilde{f}\,\Gamma_s \\ &\sim -7.5\times 10^{-8}\left(\frac{M_{BH}}{10^{16}\,\text{g}}\right)^{-2}\sum_i c_i(T_{BH})\,\tilde{f}\,\Gamma_s\ \text{g/s}\,,\end{aligned} \tag{1.5}$$

whereas a 5D black hole has an evaporation rate of [41]

$$\left.\frac{dM_{BH}}{dt}\right|_{evap} = -\frac{\Lambda^{1/4}\, M_{Pl}^2}{640\,\pi\,\lambda\, M_{BH}} \sum_i c_i(T_{BH})\,\tilde{f}\,\Gamma_s$$
$$\sim -2.5\times 10^{-13}\,\frac{1}{M_{BH}}\sum_i c_i(T_{BH})\,\tilde{f}\,\Gamma_s\ \mathrm{g/s}\,, \qquad (1.6)$$

where $c_i(T_{BH})$ counts the number of internal degrees of freedom of particle species i of mass m_i satisfying $m_i \ll T_{BH}$, $\tilde{f} = 1$ ($\tilde{f} = 7/8$) for bosons (fermions), and where $\Gamma_{s=1/2} \approx 2/3$ and $\Gamma_{s=1} \approx 1/4$ are the (spin-weighted) dimensionless greybody factors normalized to the black hole surface area [45]. In the spirit of [46], graviton emission can be neglected because the KK modes are excitations in the full transverse space, and so their overlap with the small (higher-dimensional) black holes is suppressed by the geometric factor $(r_s/R_\perp)^2$ relative to the brane fields, where r_s is the Schwarzschild radius [47]. Thus, the geometric suppression precisely compensates for the enormous number of modes, and the total contribution of all KK modes is only the same order as that from a single brane field.

Now, integrating (1.5) and (1.6) it is easily seen that 5D black holes live longer than 4D black holes of the same mass. Armed with this result a straightforward calculation shows that for a species scale of $\mathcal{O}(10^9\ \mathrm{GeV})$, an all-dark-matter interpretation in terms of 5D black holes must be feasible for masses in the range $10^{14} < M_{BH}/\mathrm{g} < 10^{21}$ [41]. This range is extended compared to that in the 4D theory by 3 orders of magnitude in the low mass region.

An astonishing coincidence is that the size of the dark dimension $R_\perp \sim$ wavelength of visible light. This means that the Schwarzschild radius of 5D black holes is well below the wavelength of light. For point-like lenses, this is the critical length where geometric optics breaks down and the effects of wave optics suppress the magnification, obstructing the sensitivity to 5D PBH microlensing signals [39].

It was observed in [48] that the universal coupling of the SM fields to the massive spin-2 KK excitations of the graviton in the dark dimension provides an alternative dark matter candidate. Within this model the cosmic evolution of the hidden sector is primarily dominated by "dark-to-dark" decays, yielding a specific realization of the dynamical dark matter framework [49]. Consider a tower of equally spaced dark gravitons, indexed by an integer l, and with mass $m_l = l\, m_{KK}$. The partial decay width of KK graviton l to SM fields is found to be,

$$\Gamma_{SM}^{l} = \frac{\tilde{\lambda}^2\, m_{KK}^3\, l^3}{80\pi M_{Pl}^2}\,, \qquad (1.7)$$

where $\tilde{\lambda}$ takes into account all the available decay channels and is a function of time [50].

In the absence of isometries in the dark dimension, which is the common expectation, the KK momentum of the dark tower is not conserved. This means that a dark graviton of KK quantum n can decay to two other ones, with quantum numbers n_1 and n_2. If the KK quantum violation can go up to δn, the number of available channels is roughly $l\,\delta n$. In addition, because the decay is almost at threshold, the phase space factor is roughly the velocity of decay products, $v_{\rm r.m.s.} \sim \sqrt{m_{\rm KK}\, \delta n / m_l}$. Putting all this together we obtain the total decay width,

$$\Gamma^l_{\rm tot} \sim \sum_{l'<l} \sum_{0<l''<l-l'} \Gamma^l_{l'l''} \sim \beta^2 \frac{m_l^3}{M_{\rm Pl}^2} \times \frac{m_l}{m_{\rm KK}}\, \delta n \times \sqrt{\frac{m_{\rm KK}\delta_n}{m_l}}$$
$$\sim \beta^2\, \delta n^{3/2} \frac{m_l^{7/2}}{M_{\rm Pl}^2 m_{\rm KK}^{1/2}}\,, \qquad (1.8)$$

where β parametrizes our ignorance of decays in the dark dimension [48].
To estimate the time evolution of the dark matter mass assume that for times larger than $1/\Gamma^l_{\rm tot}$ dark matter which is heavier than the corresponding m_l has already decayed, and so it follows that

$$m_l \sim \left(\frac{M_{\rm Pl}^4\, m_{\rm KK}}{\beta^4\, \delta n^3}\right)^{1/7} t^{-2/7}\,, \qquad (1.9)$$

where t indicates the time elapsed since the big bang [48].
Consistency with CMB anisotropies requires $\Gamma^l_{\gamma\gamma} < 5 \times 10^{-25}$ s^{-1} between the last scattering surface and reionization [51]. Taking $\tilde{\lambda} = 1$ (to set out the decay into photons) and using (1.7) it follows that the CMB requirement is satisfied for $l \lesssim 10^8$ at the time $t_{\rm MR} \sim 6 \times 10^4$ yr of matter-radiation equality. In other words, by setting $\tilde{\lambda} \sim 1$ and $m_l(t_{\rm MR}) \lesssim 1$ MeV, the evolution of m_l with cosmic time given in (1.9) is such that at the last scattering surface the dominant KK state in the dynamical dark matter ensemble has the correct decay width to accommodate the CMB constraints [52].
Now, we have seen that dark matter decay gives the daughter particles a velocity kick. Self-gravitating dark-matter halos that have a virial velocity smaller than this velocity kick may be disrupted by these particle decays. Consistency with existing data requires roughly $\delta n \sim 1$, and $\beta \sim 635$ [53]. For selected fiducial parameters, the cosmic evolution of the incredible bulk predicts via (1.9) a dominant particle mass of ~ 900 keV at CMB, of ~ 500 keV in the Dark Ages, of ~ 150 keV at Cosmic Dawn, and of ~ 50 keV in the local universe. This is in sharp contrast to typical dark matter decay scenarios with one unstable particle (such as sterile neutrinos [54]). Simultaneous observations of signals at Cosmic Dawn and in the local universe could constitute the smoking gun of the incredible bulk [55].
For many purposes, a black hole can be replaced by a bound state of gravitons [56]. As a matter of fact, a correspondence between 5D PBHs and massive KK gravitons as dark matter candidates has been conjectured in [57].
The radion stabilizing the dark dimension could be yet another dark matter contender [58]. This is because in principle the radion could be ultralight, and if this were the case it would serve as a fuzzy dark matter candidate. A simple cosmological production mechanism brings into play unstable KK graviton towers which are fueled by the decay of the inflaton. As in the previous model, the cosmic evolution of the dark sector is mostly driven by "dark-to-dark" decay processes that regulate the decay of KK gravitons within the dark tower, conveying another realization of the dynamical dark matter framework [49]. In the spirit of [59],

within this model it is assumed that the intra-KK decays in the bulk carry a spontaneous breakdown of the translational invariance in the compact space, such that the 5D momenta are not conserved (but now $\delta n \gg 1$). Armed with these two reasonable assumptions it is straightforward to see that the energy the inflaton deposited in the KK tower should have collapsed all into the radion well before BBN.

1.4 Neutrino Masses and Mixing

The dark dimension scenario provides a profitable arena to realize an old idea for explaining the smallness of neutrino masses by introducing the right-handed neutrinos as 5D bulk states with Yukawa couplings to the left-handed lepton and Higgs doublets that are localized states on the SM brane stack [60–62]. The neutrino masses are then suppressed due to the wave function of the bulk states. More indicatively, the generation of neutrino masses originates in 5D bulk-brane interactions of the form

$$\mathcal{L} \supset h_{ij}\, \overline{L}_i\, \tilde{H}\, \Psi_j(y=0)\,, \tag{1.10}$$

where $\tilde{H} = -i\sigma_2 H^*$, L_i denotes the lepton doublets (localized on the SM brane), Ψ_j stands for the 3 bulk (right-handed) R-neutrinos evaluated at the position of the SM brane, $y = 0$ in the fifth-dimension coordinate y, and h_{ij} are coupling constants. This gives a coupling with the L-neutrinos of the form $\langle H \rangle\, \overline{\nu}_{L_i}\, \Psi_j(y=0)$, where $\langle H \rangle = 175$ GeV is the Higgs vacuum expectation value. Expanding Ψ_j into modes canonically normalized leads for each of them to a Yukawa 3×3 matrix suppressed by the square root of the volume of the bulk $\sqrt{\pi R_\perp M_s}$, i.e.,

$$Y_{ij} = \frac{h_{ij}}{\sqrt{\pi R_\perp M_s}} \sim h_{ij}\frac{M_s}{M_p}\,, \tag{1.11}$$

where $M_s \lesssim M_*$ is the string scale, and where in the second rendition we have dropped factors of π's and of the string coupling.

Now, neutrino oscillation data can be well-fitted in terms of two nonzero differences $\Delta m_{ij}^2 = m_i^2 - m_j^2$ between the squares of the masses of the three mass eigenstates; namely, $\Delta m_{21}^2 = (7.53 \pm 0.18) \times 10^{-5}$ eV^2 and $\Delta m_{32}^2 = (2.453 \pm 0.033) \times 10^{-3}$ eV^2 or $\Delta m_{32}^2 = -(2.536 \pm 0.034) \times 10^{-3}$ eV^2 [1]. It is easily seen that to obtain the correct order of magnitude of neutrino masses the coupling h_{ij} should be of order 10^{-4} to 10^{-5} for $10^9 \lesssim M_s/\text{GeV} \lesssim 10^{10}$.

Note that KK modes of the 5D R-neutrino fields behave as an infinite tower of sterile neutrinos, with masses proportional to m_{KK}. However, only the lower mass states of the tower mix with the active SM neutrinos in a pertinent fashion. The non-observation of neutrino disappearance from oscillations into sterile neutrinos at long- and short-baseline experiments places a 90% CL upper limit $R_\perp < 0.4$ μm for the normal neutrino ordering, and $R_\perp < 0.2$ μm for the inverted neutrino ordering [63, 64].[1] This set of parameters corresponds to $\lambda \lesssim 10^{-3}$ and so $m_{KK} \gtrsim 2.5$ eV [55].

[1] We arrived at these upper bounds by looking at the low mass limit of the lightest neutrino state in Fig. 6 of [64] and rounding the numbers to one significant figure.

Before proceeding, it is important to stress that the upper bounds on $R_\perp$ discussed in the previous paragraph are sensitive to assumptions of the 5th dimension geometry. Moreover, in the presence of bulk masses [65, 66], the mixing of the first KK modes to active neutrinos can be suppressed, and therefore the aforementioned bounds on $R_\perp$ can be avoided [67, 68]. It is also worth mentioning that such bulk masses have the potential to increase the relative importance of the higher KK modes, yielding distinct oscillation signatures via neutrino disappearance/appearance effects.
Non-minimal extensions of the dark dimension, in which $M_{3/2}$ and Λ have different KK towers, allow a high-scale SUSY breaking and can therefore host a rather heavy gravitino together with a modulino with a mass of about 50 eV [69]. For a particular example, we note that the modulino could be the fermionic partner of the radion.[2] These models with high-scale SUSY breaking are fully predictive through neutrino-modulino oscillations [70] which can be confronted with data to be collected by experiments at the Forward Physics Facility [71, 72].
A seemingly different, but in fact closely related subject is the the *sharpened* version of the weak gravity conjecture forbidding the presence of non-SUSY AdS vacua supported by fluxes in a consistent quantum gravity theory [73]. This is because (unless the gravitino is very light, with mass in the meV range) *neutrinos have to be Dirac with right-handed states propagating in the bulk so that the KK neutrino towers compensate for the graviton tower to maintain stable dS vacua* [68].

1.5 Mesoscopic Extra Dimension from 5D Inflation

It is unnatural to entertain that the size of the dark dimension would remain fixed during the evolution of the Universe right at the species scale. One possible mechanism to accommodate this hierarchy is to inflate the size of the dark dimension. The required inflationary phase can be described by a 5D dS (or approximate) solution of Einstein equations, with cosmological constant and a 5D Planck scale $M_* \sim 10^9$ GeV [55]. All dimensions (compact and non-compact) expand exponentially in terms of the 5D proper time. It is straightforward to see that this set-up requires about 42 e-folds to expand the 5th dimension from the fundamental length $\mathcal{O}(M_*^{-1})$ to the micron size $\mathcal{O}(R_\perp)$. At the end of 5D inflation, or at any given moment, one can interpret the solution in terms of 4D fields using 4D Planck units from the relation $M_p^2 = M_*^3 R$, which amounts going to the 4D Einstein frame. This implies that if $M_*^{-1} \leq R \leq R_\perp$ expands N e-folds, then the 3D space would expand 3N/2 e-folds as a result of a uniform 5D inflation. Altogether, the 3D space has expanded by about 60 e-folds to solve the horizon problem, while connecting this particular solution to the generation of large size extra dimension.
Besides solving the horizon problem, 4D slow-roll inflation predicts an approximate scale-invariant Harrison-Zel'dovich power spectrum of primordial density

[2] In the standard moduli stabilization by fluxes, all complex structure moduli and the dilaton are stabilized in a supersymmetric way while Kähler class moduli need an input from SUSY breaking. The radion is Kähler class and exists in a model independent fashion within the dark dimension scenario.

perturbations [74, 75] consistent with CMB observations [76]. This is due to the fact that the 2-point function of a massless minimally coupled scalar field in dS space behaves logarithmically at distances larger than the cosmological horizon, a property which is though valid for any spacetime dimensionality [77]. When some dimensions are however compact, this behaviour is expected to hold for distances smaller than the compactification length, while deviating from scale invariance at larger distances, potentially conflicting with observations at large angles. Remarkably, consistency 5D inflation with CMB observations is maintained if the size of the dark dimension is larger than about a micron, implying a change of behaviour in the power spectrum at angles larger than 10 degrees, corresponding to multiple moments $l \lesssim 30$, where experimental errors are getting large [3]. Actually, the scale invariance of the power spectrum is obtained upon summation over the contribution of the inflaton KK-modes' fluctuations that correspond to a tower of scalars from the 4D point of view. The tensor perturbations have been computed in [78]. The tensor-to-scalar ratio is found to be $r = 24\epsilon_V$, and so the 95% CL upper limit $r < 0.032$ (derived using a combination of BICEP/Keck 2018 and *Planck* data) [79, 80] places an experimental constraint on the potential slow-roll parameter: $\epsilon_V < 0.0013$.

Another interesting feature of 5D inflation is that the radion can be stabilized in a local (metastable) dS vacuum, using the contributions of bulk field gradients [81] or of the Casimir energy, assuming a mass for the bulk R-handed neutrinos of the same order of magnitude [82].

1.6 Concluding Remarks

We have seen that the dark dimension scenario carries with it a rich phenomenology:

[noitemsep,topsep=0pt]It provides a profitable arena to accommodate a very light gravitino. It encompasses a framework for primordial black holes, KK gravitons, and a fuzzy radion to emerge as interesting dark matter candidates. It also encompasses an interesting framework for studying cosmology and astroparticle physics. It provides a natural set up for R-neutrinos propagating in the bulk to accommodate neutrino masses in the range $10^{-4} < m_\nu/\text{eV} < 10^{-1}$, despite the lack of any fundamental scale higher than M_*. The suppressed neutrino masses are not the result of a see-saw mechanism, but rather because the bulk modes have couplings suppressed by the volume of the dark dimension (akin to the weakness of gravity at long distances).

We have also seen that uniform 5D inflation can relate the causal size of the observable universe to the present weakness of gravitational interactions by blowing up an extra compact dimension from the microscopic fundamental length of gravity to a large size in the micron range, as required by the dark dimension scenario. Moreover, uniform 5D inflation can lead to an approximate scale invariant power spectrum of primordial density perturbations consistent with observations of CMB anisotropies. The tensor-to-scalar ratio is also consistent with observations. A rough estimate of the magnitude of isocurvature perturbations based on entropy perturbations indicates that they are suppressed. A dedicated investigation along these lines is obviously important to be done.

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References

1. R. L. Workman *et al.* [Particle Data Group], *Review of Particle Physics*, PTEP **2022**, 083C01 (2022) doi:10.1093/ptep/ptac097
2. A. G. Riess, W. Yuan, L. M. Macri, D. Scolnic, D. Brout, S. Casertano, D. O. Jones, Y. Murakami, L. Breuval and T. G. Brink, *et al. A comprehensive measurement of the local value of the Hubble constant with 1 km s^{-1} Mpc^{-1} uncertainty from the Hubble Space Telescope and the SH0ES Team*, Astrophys. J. Lett. **934** (2022) no.1, L7 doi:10.3847/2041-8213/ac5c5b [arXiv:2112.04510 [astro-ph.CO]].
3. L. A. Anchordoqui and I. Antoniadis, *Large extra dimensions from higher-dimensional inflation*, [arXiv:2310.20282 [hep-ph]].
4. N. Arkani-Hamed, S. Dimopoulos and G. R. Dvali, *The Hierarchy problem and new dimensions at a millimeter*, Phys. Lett. B **429** (1998), 263-272 doi:10.1016/S0370-2693(98)00466-3 [arXiv:hep-ph/9803315 [hep-ph]].
5. I. Antoniadis, N. Arkani-Hamed, S. Dimopoulos and G. R. Dvali, *New dimensions at a millimeter to a Fermi and superstrings at a TeV*, Phys. Lett. B **436** (1998), 257-263 doi:10.1016/S0370-2693(98)00860-0 [arXiv:hep-ph/9804398 [hep-ph]].
6. C. Vafa, *The string landscape and the swampland*, [arXiv:hep-th/0509212 [hep-th]].
7. M. van Beest, J. Calderón-Infante, D. Mirfendereski and I. Valenzuela, *Lectures on the Swampland Program in string compactifications*, Phys. Rept. **989**, 1-50 (2022) doi:10.1016/j.physrep.2022.09.002 [arXiv:2102.01111 [hep-th]].
8. E. Palti, *The swampland: introduction and review*, Fortsch. Phys. **67**, no.6, 1900037 (2019) doi:10.1002/prop.201900037 [arXiv:1903.06239 [hep-th]].
9. N. B. Agmon, A. Bedroya, M. J. Kang and C. Vafa, *Lectures on the string landscape and the swampland*, [arXiv:2212.06187 [hep-th]].
10. H. Ooguri and C. Vafa, *On the Geometry of the String Landscape and the Swampland*, Nucl. Phys. B **766**, 21-33 (2007) doi:10.1016/j.nuclphysb.2006.10.033 [arXiv:hep-th/0605264 [hep-th]].
11. D. Lüst, E. Palti and C. Vafa, *AdS and the Swampland*, Phys. Lett. B **797**, 134867 (2019) doi:10.1016/j.physletb.2019.134867 [arXiv:1906.05225 [hep-th]].
12. M. Montero, C. Vafa and I. Valenzuela, *The dark dimension and the Swampland*, JHEP **02**, 022 (2023) doi:10.1007/JHEP02(2023)022 [arXiv:2205.12293 [hep-th]].
13. A. Higuchi, *Forbidden mass range for spin-2 field theory in de Sitter space-time*, Nucl. Phys. B **282** (1987), 397-436 doi:10.1016/0550-3213(87)90691-2
14. H. Itoyama and T. R. Taylor, *Supersymmetry restoration in the compactified* $O(16) \times O(16)$*-prime heterotic string theory*, Phys. Lett. B **186** (1987), 129-133 doi:10.1016/0370-2693(87)90267-X
15. H. Itoyama and T. R. Taylor, *Small cosmological constant in string models*, FERMILAB-CONF-87-129-T.

16. I. Antoniadis and C. Kounnas, *Superstring phase transition at high temperature*, Phys. Lett. B **261** (1991), 369-378 doi:10.1016/0370-2693(91)90442-S
17. Q. Bonnefoy, E. Dudas and S. Lüst, *On the weak gravity conjecture in string theory with broken supersymmetry*, Nucl. Phys. B **947** (2019), 114738 doi:10.1016/j.nuclphysb.2019.114738 [arXiv:1811.11199 [hep-th]].
18. J. G. Lee, E. G. Adelberger, T. S. Cook, S. M. Fleischer and B. R. Heckel, *New test of the gravitational* $1/r^2$ *law at separations down to 52* μm, Phys. Rev. Lett. **124**, no.10, 101101 (2020) doi:10.1103/PhysRevLett.124.101101 [arXiv:2002.11761 [hep-ex]].
19. S. Hannestad and G. G. Raffelt, *Supernova and neutron star limits on large extra dimensions reexamined*, Phys. Rev. D **67**, 125008 (2003) [erratum: Phys. Rev. D **69**, 029901 (2004)] doi:10.1103/PhysRevD.69.029901 [arXiv:hep-ph/0304029 [hep-ph]].
20. L. A. Anchordoqui, I. Antoniadis, D. Lüst and S. Lüst, *On the cosmological constant, the KK mass scale, and the cut-off dependence in the dark dimension scenario*, Eur. Phys. J. C **83**, no.11, 1016 (2023) doi:10.1140/epjc/s10052-023-12206-2 [arXiv:2309.09330 [hep-th]].
21. G. Dvali, *Black holes and large* N *species solution to the hierarchy problem*, Fortsch. Phys. **58**, 528-536 (2010) doi:10.1002/prop.201000009 [arXiv:0706.2050 [hep-th]].
22. G. Dvali and M. Redi, *Black hole bound on the number of species and quantum gravity at LHC*, Phys. Rev. D **77**, 045027 (2008) doi:10.1103/PhysRevD.77.045027 [arXiv:0710.4344 [hep-th]].
23. N. Cribiori, D. Lüst and G. Staudt, *Black hole entropy and moduli-dependent species scale*, Phys. Lett. B **844**, 138113 (2023) doi:10.1016/j.physletb.2023.138113 [arXiv:2212.10286 [hep-th]].
24. D. van de Heisteeg, C. Vafa, M. Wiesner and D. H. Wu, *Species scale in diverse dimensions*, [arXiv:2310.07213 [hep-th]].
25. I. Antoniadis, C. Bachas, D. C. Lewellen and T. N. Tomaras, *On supersymmetry breaking in superstrings*, Phys. Lett. B **207**, 441-446 (1988) doi:10.1016/0370-2693(88)90679-X
26. N. Cribiori, D. Lüst and M. Scalisi, *The gravitino and the swampland*, JHEP **06**, 071 (2021) doi:10.1007/JHEP06(2021)071 [arXiv:2104.08288 [hep-th]].
27. A. Castellano, A. Font, A. Herraez and L. E. Ibáñez, *A gravitino distance conjecture*, JHEP **08**, 092 (2021) doi:10.1007/JHEP08(2021)092 [arXiv:2104.10181 [hep-th]].
28. L. A. Anchordoqui, I. Antoniadis, N. Cribiori, D. Lüst and M. Scalisi, *The Scale of Supersymmetry Breaking and the Dark Dimension*, JHEP **05**, 060 (2023) doi:10.1007/JHEP05(2023)060 [arXiv:2301.07719 [hep-th]].
29. B. Carr and F. Kuhnel, *Primordial black holes as dark matter: Recent developments*, Ann. Rev. Nucl. Part. Sci. **70** (2020), 355-394 doi:10.1146/annurev-nucl-050520-125911 [arXiv:2006.02838 [astro-ph.CO]].
30. A. M. Green and B. J. Kavanagh, *Primordial Black Holes as a dark matter candidate*, J. Phys. G **48** (2021) no.4, 043001 doi:10.1088/1361-6471/abc534 [arXiv:2007.10722 [astro-ph.CO]].
31. P. Villanueva-Domingo, O. Mena and S. Palomares-Ruiz, *A brief review on primordial black holes as dark matter*, Front. Astron. Space Sci. **8**, 87 (2021) doi:10.3389/fspas.2021.681084 [arXiv:2103.12087 [astro-ph.CO]].
32. B. J. Carr, K. Kohri, Y. Sendouda and J. Yokoyama, *New cosmological constraints on primordial black holes*, Phys. Rev. D **81**, 104019 (2010) doi:10.1103/PhysRevD.81.104019 [arXiv:0912.5297 [astro-ph.CO]].
33. S. Clark, B. Dutta, Y. Gao, L. E. Strigari and S. Watson, *Planck constraint on relic primordial black holes*, Phys. Rev. D **95**, no.8, 083006 (2017) doi:10.1103/PhysRevD.95.083006 [arXiv:1612.07738 [astro-ph.CO]].
34. R. A. Allsman *et al.* [MACHO], *MACHO project limits on black hole dark matter in the 1-30 solar mass range*, Astrophys. J. Lett. **550**, L169 (2001) doi:10.1086/319636 [arXiv:astro-ph/0011506 [astro-ph]].

35. P. Tisserand *et al.* [EROS-2], *Limits on the MACHO content of the Galactic halo from the EROS-2 survey of the Magellanic clouds*, Astron. Astrophys. **469**, 387-404 (2007) doi:10.1051/0004-6361:20066017 [arXiv:astro-ph/0607207 [astro-ph]].
36. K. Griest, A. M. Cieplak and M. J. Lehner, *Experimental limits on primordial black hole dark matter from the first 2 yr of Kepler data*, Astrophys. J. **786**, no.2, 158 (2014) doi:10.1088/0004-637X/786/2/158 [arXiv:1307.5798 [astro-ph.CO]].
37. M. Oguri, J. M. Diego, N. Kaiser, P. L. Kelly and T. Broadhurst, *Understanding caustic crossings in giant arcs: characteristic scales, event rates, and constraints on compact dark matter*, Phys. Rev. D **97**, no.2, 023518 (2018) doi:10.1103/PhysRevD.97.023518 [arXiv:1710.00148 [astro-ph.CO]].
38. H. Niikura, M. Takada, S. Yokoyama, T. Sumi and S. Masaki, *Constraints on Earth-mass primordial black holes from OGLE 5-year microlensing events*, Phys. Rev. D **99**, no.8, 083503 (2019) doi:10.1103/PhysRevD.99.083503 [arXiv:1901.07120 [astro-ph.CO]].
39. D. Croon, D. McKeen, N. Raj and Z. Wang, *Subaru-HSC through a different lens: Microlensing by extended dark matter structures*, Phys. Rev. D **102**, no.8, 083021 (2020) doi:10.1103/PhysRevD.102.083021 [arXiv:2007.12697 [astro-ph.CO]].
40. P. C. Argyres, S. Dimopoulos and J. March-Russell, Phys. Lett. B **441** (1998), 96-104 doi:10.1016/S0370-2693(98)01184-8 [arXiv:hep-th/9808138 [hep-th]].
41. L. A. Anchordoqui, I. Antoniadis and D. Lüst, *Dark dimension, the swampland, and the dark matter fraction composed of primordial black holes*, Phys. Rev. D **106**, no.8, 086001 (2022) doi:10.1103/PhysRevD.106.086001 [arXiv:2206.07071 [hep-th]].
42. L. A. Anchordoqui, *Dark dimension, the swampland, and the origin of cosmic rays beyond the Greisen-Zatsepin-Kuzmin barrier*, Phys. Rev. D **106** (2022) no.11, 116022 doi:10.1103/PhysRevD.106.116022 [arXiv:2205.13931 [hep-ph]].
43. N. T. Noble, J. F. Soriano and L. A. Anchordoqui, *Probing the Dark Dimension with Auger data*, Phys. Dark Univ. **42** (2023), 101278 doi:10.1016/j.dark.2023.101278 [arXiv:2306.03666 [hep-ph]].
44. C. Keith and D. Hooper, *511 keV excess and primordial black holes*, Phys. Rev. D **104** (2021) no.6, 063033 doi:10.1103/PhysRevD.104.063033 [arXiv:2103.08611 [astro-ph.CO]].
45. L. Anchordoqui and H. Goldberg, *Black hole chromosphere at the CERN LHC*, Phys. Rev. D **67** (2003), 064010 doi:10.1103/PhysRevD.67.064010 [arXiv:hep-ph/0209337 [hep-ph]].
46. R. Emparan, G. T. Horowitz and R. C. Myers, *Black holes radiate mainly on the brane*, Phys. Rev. Lett. **85**, 499-502 (2000) doi:10.1103/PhysRevLett.85.499 [arXiv:hep-th/0003118 [hep-th]].
47. R. C. Myers and M. J. Perry, *Black holes in higher dimensional space-times*, Annals Phys. **172**, 304 (1986) doi:10.1016/0003-4916(86)90186-7
48. E. Gonzalo, M. Montero, G. Obied and C. Vafa, *Dark Dimension Gravitons as Dark Matter*, JHEP **11**, 109 (2023) doi:10.1007/JHEP11(2023)109 [arXiv:2209.09249 [hep-ph]].
49. K. R. Dienes and B. Thomas, *Dynamical dark matter I: Theoretical overview*, Phys. Rev. D **85**, 083523 (2012) doi:10.1103/PhysRevD.85.083523 [arXiv:1106.4546 [hep-ph]].
50. L. J. Hall and D. Tucker-Smith, *Cosmological constraints on theories with large extra dimensions*, Phys. Rev. D **60**, 085008 (1999) doi:10.1103/PhysRevD.60.085008 [arXiv:hep-ph/9904267 [hep-ph]].
51. T. R. Slatyer and C. L. Wu, *General Constraints on Dark Matter Decay from the Cosmic Microwave Background*, Phys. Rev. D **95**, no.2, 023010 (2017) doi:10.1103/PhysRevD.95.023010 [arXiv:1610.06933 [astro-ph.CO]].
52. J. A. P. Law-Smith, G. Obied, A. Prabhu and C. Vafa, *Astrophysical Constraints on Decaying Dark Gravitons*, [arXiv:2307.11048 [hep-ph]].
53. G. Obied, C. Dvorkin, E. Gonzalo and C. Vafa, *Dark Dimension and Decaying Dark Matter Gravitons*, [arXiv:2311.05318 [astro-ph.CO]].

54. K. N. Abazajian, *Sterile neutrinos in cosmology*, Phys. Rept. **711-712**, 1-28 (2017) doi:10.1016/j.physrep.2017.10.003 [arXiv:1705.01837 [hep-ph]].
55. L. A. Anchordoqui, I. Antoniadis and D. Lüst, *Aspects of the dark dimension in cosmology*, Phys. Rev. D **107**, no.8, 083530 (2023) doi:10.1103/PhysRevD.107.083530 [arXiv:2212.08527 [hep-ph]].
56. G. Dvali and C. Gomez, *Black hole's quantum N-portrait*, Fortsch. Phys. **61**, 742-767 (2013) doi:10.1002/prop.201300001 [arXiv:1112.3359 [hep-th]].
57. L. A. Anchordoqui, I. Antoniadis and D. Lüst, *The dark universe: Primordial black hole* $\leftrightharpoons$ *dark graviton gas connection*, Phys. Lett. B **840**, 137844 (2023) doi:10.1016/j.physletb.2023.137844 [arXiv:2210.02475 [hep-th]].
58. L. A. Anchordoqui, I. Antoniadis and D. Lüst, *Fuzzy Dark Matter, the dark dimension, and the pulsar timing array signal*, [arXiv:2307.01100 [hep-ph]].
59. R. N. Mohapatra, S. Nussinov and A. Perez-Lorenzana, *Large extra dimensions and decaying KK recurrences*, Phys. Rev. D **68**, 116001 (2003) doi:10.1103/PhysRevD.68.116001 [arXiv:hep-ph/0308051 [hep-ph]].
60. K. R. Dienes, E. Dudas and T. Gherghetta, *Neutrino oscillations without neutrino masses or heavy mass scales: A Higher dimensional seesaw mechanism*, Nucl. Phys. B **557**, 25 (1999) doi:10.1016/S0550-3213(99)00377-6 [arXiv:hep-ph/9811428 [hep-ph]].
61. N. Arkani-Hamed, S. Dimopoulos, G. R. Dvali and J. March-Russell, *Neutrino masses from large extra dimensions*, Phys. Rev. D **65**, 024032 (2001) doi:10.1103/PhysRevD.65.024032 [arXiv:hep-ph/9811448 [hep-ph]].
62. G. R. Dvali and A. Y. Smirnov, *Probing large extra dimensions with neutrinos*, Nucl. Phys. B **563**, 63-81 (1999) doi:10.1016/S0550-3213(99)00574-X [arXiv:hep-ph/9904211 [hep-ph]].
63. P. A. N. Machado, H. Nunokawa and R. Zukanovich Funchal, *Testing for Large Extra Dimensions with Neutrino Oscillations*, Phys. Rev. D **84**, 013003 (2011) doi:10.1103/PhysRevD.84.013003 [arXiv:1101.0003 [hep-ph]].
64. D. V. Forero, C. Giunti, C. A. Ternes and O. Tyagi, *Large extra dimensions and neutrino experiments*, Phys. Rev. D **106**, no.3, 035027 (2022) doi:10.1103/PhysRevD.106.035027 [arXiv:2207.02790 [hep-ph]].
65. A. Lukas, P. Ramond, A. Romanino and G. G. Ross, *Solar neutrino oscillation from large extra dimensions*, Phys. Lett. B **495**, 136-146 (2000) doi:10.1016/S0370-2693(00)01206-5 [arXiv:hep-ph/0008049 [hep-ph]].
66. A. Lukas, P. Ramond, A. Romanino and G. G. Ross, *Neutrino masses and mixing in brane world theories*, JHEP **04**, 010 (2001) doi:10.1088/1126-6708/2001/04/010 [arXiv:hep-ph/0011295 [hep-ph]].
67. M. Carena, Y. Y. Li, C. S. Machado, P. A. N. Machado and C. E. M. Wagner, *Neutrinos in Large Extra Dimensions and Short-Baseline* ν_e *Appearance*, Phys. Rev. D **96** (2017) no.9, 095014 doi:10.1103/PhysRevD.96.095014 [arXiv:1708.09548 [hep-ph]].
68. L. A. Anchordoqui, I. Antoniadis and J. Cunat, *The dark dimension and the standard model landscape*, [arXiv:2306.16491 [hep-ph]].
69. L. A. Anchordoqui, I. Antoniadis, K. Benakli, J. Cunat and D. Lüst, *Searching for neutrino-modulino oscillations at the Forward Physics Facility*, [arXiv:2308.11476 [hep-ph]].
70. K. Benakli and A. Y. Smirnov, *Neutrino-modulino mixing*, Phys. Rev. Lett. **79**, 4314-4317 (1997) doi:10.1103/PhysRevLett.79.4314 [arXiv:hep-ph/9703465 [hep-ph]].
71. L. A. Anchordoqui, A. Ariga, T. Ariga, W. Bai, K. Balazs, B. Batell, J. Boyd, J. Bramante, M. Campanelli and A. Carmona, *et al. The Forward Physics Facility: Sites, experiments, and physics potential*," Phys. Rept. **968**, 1-50 (2022) doi:10.1016/j.physrep.2022.04.004 [arXiv:2109.10905 [hep-ph]].
72. J. L. Feng, F. Kling, M. H. Reno, J. Rojo, D. Soldin, L. A. Anchordoqui, J. Boyd, A. Ismail, L. Harland-Lang and K. J. Kelly, *et al. The Forward Physics Facility at the High-Luminosity*

LHC, J. Phys. G **50**, no.3, 030501 (2023) doi:10.1088/1361-6471/ac865e [arXiv:2203.05090 [hep-ex]].
73. H. Ooguri and C. Vafa, *Non-supersymmetric AdS and the Swampland*, Adv. Theor. Math. Phys. **21** (2017), 1787-1801 doi:10.4310/ATMP.2017.v21.n7.a8 [arXiv:1610.01533 [hep-th]].
74. E. R. Harrison, *Fluctuations at the threshold of classical cosmology*, Phys. Rev. D **1**, 2726-2730 (1970) doi:10.1103/PhysRevD.1.2726
75. Y. B. Zeldovich, *A Hypothesis, unifying the structure and the entropy of the universe*, Mon. Not. Roy. Astron. Soc. **160**, 1P-3P (1972) doi:10.1093/mnras/160.1.1P
76. N. Aghanim *et al.* [Planck], *Planck 2018 results VI: Cosmological parameters*, Astron. Astrophys. **641**, A6 (2020) [erratum: Astron. Astrophys. **652**, C4 (2021)] doi:10.1051/0004-6361/201833910 [arXiv:1807.06209 [astro-ph.CO]].
77. B. Ratra, *Restoration of spontaneously broken continuous symmetries in de Sitter space-time*, Phys. Rev. D **31** (1985), 1931-1955 doi:10.1103/PhysRevD.31.1931
78. I. Antoniadis, J. Cunat and A. Guillen, *Cosmological perturbations from five-dimensional inflation*, [arXiv:2311.17680 [hep-ph]].
79. P. A. R. Ade *et al.* [BICEP and Keck], *Improved Constraints on Primordial Gravitational Waves using Planck, WMAP, and BICEP/Keck Observations through the 2018 Observing Season*, Phys. Rev. Lett. **127** (2021) no.15, 151301 doi:10.1103/PhysRevLett.127.151301 [arXiv:2110.00483 [astro-ph.CO]].
80. M. Tristram, A. J. Banday, K. M. Górski, R. Keskitalo, C. R. Lawrence, K. J. Andersen, R. B. Barreiro, J. Borrill, L. P. L. Colombo and H. K. Eriksen, *et al. Improved limits on the tensor-to-scalar ratio using BICEP and Planck data*, Phys. Rev. D **105** (2022) no.8, 083524 doi:10.1103/PhysRevD.105.083524 [arXiv:2112.07961 [astro-ph.CO]].
81. N. Arkani-Hamed, L. J. Hall, D. Tucker-Smith and N. Weiner, *Solving the hierarchy problem with exponentially large dimensions*, Phys. Rev. D **62** (2000), 105002 doi:10.1103/PhysRevD.62.105002 [arXiv:hep-ph/9912453 [hep-ph]].
82. N. Arkani-Hamed, S. Dubovsky, A. Nicolis and G. Villadoro, *Quantum horizons of the standard model landscape*, JHEP **06** (2007), 078 doi:10.1088/1126-6708/2007/06/078 [arXiv:hep-th/0703067 [hep-th]].

2 Status of the DAMA project

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Abstract. The long-standing model-independent annual modulation effect measured by DAMA deep underground at Gran Sasso Laboratory with different experimental configurations is summarized and DAMA/LIBRA–phase2–empowered, presently running, shortly introduced. The evidence of a signal that meets all the requirements of the model-independent Dark Matter annual modulation signature at high C.L. has been confirmed over 22 independent annual cycles with various configurations (DAMA/NaI, DAMA/LIBRA–phase1 and DAMA/LIBRA–phase2); full exposure is 2.86 ton $\times$ yr. The experiment is currently collecting data in the DAMA/LIBRA–phase2–empowered configuration with an even lower software energy threshold.

Povzetek: Avtorji predstavijo meritve letnih modulacij signala, ki ga meri experiment DAMA že 22 let z razli v cnimi eksperimentalnimi konfiguracijami. Laboratorij je postavljen globoko pod zemljo v Gran Sassu. Trenutno teče experiment DAMA/LIBRA–faza2. Poročilo vsebuje meritve experimentov DAMA/NaI, DAMA/LIBRA–faza1 in DAMA/LIBRA–faza2; polna izpostavljenost je tako 2,86 ton x leta. Avtorji, ki vsa ta leta skrbno izključujejo vpliv doslej poznanih delcev na meritve, s tem experimentom, ki ne gradi na nobenem modelu, dokazujejo z visoko stopnjo zaupanja (C.L.), da izmerjeno letno modulacijo povzročajo delci temne snovi. Pri meritvah vseskozi znižujejo nizkoenergijski prag programske opreme.

2.1 Introduction

In 1990 the DAMA project [1] has been proposed as a pioneer in the field of Dark Matter (DM) direct investigation proposing the realization of large mass set-ups (with highly radiopure NaI(Tl) and liquid Xenon set-ups) fully dedicated to the direct detection of Dark Matter particles in the galactic halo mainly by exploiting the model-independent DM signature (originally suggested in the middle of the '80 by Ref. [2,3]). In particular, the DAMA/NaI experiment ($\simeq$ 100 kg of highly

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radiopure NaI(Tl) in a multi-ton multi-component shield) [8] has been a pioneer experiment running deep underground in the Gran Sasso National Laboratory (LNGS) of INFN until 2002. It has been investigated as first the DM signature with suitable exposed mass, sensitivity and control of the running parameters [5,6,8]. In July 2002 – after several years of new developments, carried out during the running of DAMA/NaI, and its dismounting – the installation of the new DAMA/LIBRA experimental set-up (about 250 kg of highly radiopure NaI(Tl) in multi-ton multi-component shield) started. The experimental site as well as many components of the installation were implemented [7]. All the procedures performed during the dismounting of DAMA/NaI and the installation of the DAMA/LIBRA detectors were carried out in high purity (HP) Nitrogen atmosphere. The detectors are also continuously maintained in such an atmosphere in all the operations since then. A significant upgrade of that configuration, named DAMA/LIBRA–phase1, has been performed at the end of 2010 replacing all the PMTs with new ones that have higher quantum efficiency, i.e. lowering the software energy threshold of the experiment; details on the developments and on the reached performances in operative conditions are reported in Ref. [8]. After a period of commissioning, this DAMA/LIBRA–phase2 began data collection. Moreover, at the end of 2012 new preamplifiers and special developed trigger modules were installed, and the apparatus was equipped with more compact electronic modules. Many model-independent results and related corollary analyses have been published [6, 10, 11]. More recently, studies and tests have been carried out to further lower the software energy threshold, below 1 keV (see e.g. preliminary efforts in [11]). The final solution has been to equip the PMTs with new low-background voltage dividers with pre-amps on the same board (named voltage-divider-plus-preamp) and to use Transient Digitizers (TD) with higher vertical resolution (14 bits). This new configuration, we named DAMA/LIBRA–phase2–empowered, is in measurements since December 1, 2021 and data collection is planned to continue until December 2024.
For completeness, we underline that DAMA has been and is working as an observatory for rare processes by developing and also using other low radioactive scintillators deep underground at LNGS. Several low background setups are operative and many different kinds of measurements are carried out [6].

2.2 Already achieved results on DM in short

The main process investigated by the highly radiopure NaI(Tl) DAMA set-ups is the largely model-independent DM annual modulation signature and related properties. This DM signature is due to the Earth's motion with respect to the DM particles constituting the Galactic Dark Halo. In fact, as a consequence of the Earth's revolution around the Sun, which is moving in the Galaxy with respect to the Local Standard of Rest towards the star Vega near the constellation of Hercules, the Earth should be crossed by a larger flux of DM particles around $\simeq$ 2 June and by a smaller one around $\simeq$ 2 December (in the first case the Earth orbital velocity is summed to that of the solar system with respect to the Galaxy, while in the other one the two velocities are subtracted). Thus, this signature has different origin and peculiarities than the seasons on the Earth and than effects correlated with seasons (consider the expected value of the phase as well as the other specific requirements listed below). In particular, this DM annual modulation signature is very distinctive since the effect induced by DM particles must simultaneously satisfy all the following requirements: the rate must contain a component modulated according to a cosine function (1) with one–year period (2) and a phase that peaks roughly $\simeq$ 2 June (3); this modulation must only be found in a well-defined low energy range, where DM particle induced events can be present (4); it must apply only to those events in which just one detector of many actually "fires" (*single-hit* events), since the DM particle multi-interaction probability is negligible (5); the modulation amplitude in the region of maximal sensitivity must be 7% of the constant part of the signal for usually adopted halo distributions (6), but it can be larger in case of some proposed scenarios such as e.g. those in Ref. [12–16] (even up to $\simeq$ 30%). Thus, the DM annual modulation signature depends on Earth's and DM particles' velocities, so does not depend on the Earth hemisphere where it is measured and has a different origin and peculiarities than all the effects correlated with seasons on the Earth. This signature has many peculiarities and, in addition, it allows testing a wide range of parameters in many possible astrophysical, nuclear and particle physics scenarios. It might be mimicked only by systematic effects or side reactions able to account for the whole observed modulation amplitude and to simultaneously satisfy all the requirements given above. Moreover, the NaI(Tl) target nuclei and procedures adopted by DAMA provide sensitivity to large and low mass DM candidates inducing nuclear recoils and/or electromagnetic signals.

So far, the released data of the pioneer DAMA/NaI setup and those of the DAMA/LIBRA–phase1 and of the DAMA/LIBRA–phase2 have given – with high confidence level – positive evidence for the presence of a signal that satisfies all the requirements of the exploited DM annual modulation signature [6, 10, 11]. In fact, the data released so far by DAMA/NaI and by DAMA/LIBRA–phase1 and –phase2 have been analysed with several different and independent analysis strategies, obtaining always consistent results. Any possible systematics or side processes able to mimic the exploited signature has been excluded, both because neither quantitatively significant amplitude can be given nor simultaneous satisfaction of all the specific requirements of the signature [6]. Details on the data and the analyses can be found in DAMA literature [6] and in previous proceedings of this serie of Conferences.

Here we just summarize few of the results [6, 10, 11]. In particular, in Fig. 2.1-top: the (2 – 6)keV residual rates of the *single-hit* scintillation events for the data already released by the former DAMA/NaI, by DAMA/LIBRA–phase1 and by DAMA/LIBRA–phase2 (total exposure 2.86 ton x yr) are shown. The function $A\cos\omega(t-t_0)$ was used to fit the data taking into account a period $T = \frac{2\pi}{\omega} =$ 1 yr and a phase $t_0 = 152.5$ day (June 2^{nd}) as predicted by the DM annual modulation signature. The obtained $\chi^2/d.o.f.$ is 130/155 and the modulation amplitude is $A = (0.00996 \pm 0.00074)$ cpg/kg/keV. For completeness, in Fig. 2.1-bottom: the experimental residual rate of the *single-hit* scintillation events measured by DAMA/LIBRA–phase2 (having lower software energy threshold) in the (1–6) keV energy intervals is shown as a function of the time.

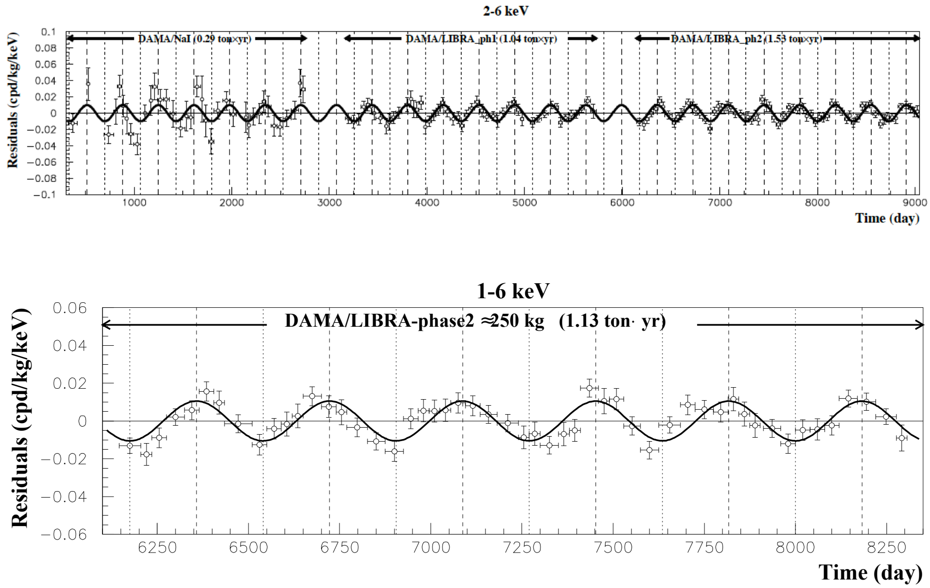

Fig. 2.1: *Top:* Experimental residual rate of the *single-hit* scintillation events measured by DAMA/NaI, DAMA/LIBRA–phase1 and DAMA/LIBRA–phase2 (total exposure 2.86 ton x yr) in the (2–6) keV energy intervals as a function of the time. The superimposed curve is the cosinusoidal functional forms $A \cos \omega(t - t_0)$ with a period $T = \frac{2\pi}{\omega} = 1$ yr, a phase $t_0 = 152.5$ day (June 2^{nd}) and modulation amplitude, A, equal to the central value obtained by best fit. Vertical dashed lines indicate the expected maximum rate, while the dotted lines represent the expected minimum rate. *Bottom:* Experimental residual rate of *single-hit* scintillation events measured by DAMA/LIBRA-phase2, which operates with a lower software energy threshold, in the (1-6) keV energy range, shown as a function of time.

Furthermore, we also summarize the results obtained for the same whole exposure: 2.86 ton x yr, when adopting in the maximum likelihood analysis of the *single-hit* events the most general expression for the signal component (i.e. releasing the assumption of a phase value $t_0 = 152.5$ day):

$$\begin{aligned} S_i(E) &= S_0(E) + S_m(E) \cos \omega(t_i - t_0) + Z_m(E) \sin \omega(t_i - t_0) \\ &= S_0(E) + Y_m(E) \cos \omega(t_i - t^*). \end{aligned} \tag{2.1}$$

For signals induced by DM particles, one would have: i) $Z_m \sim 0$ (because of the orthogonality between the cosine and the sine functions); ii) $S_m \simeq Y_m$; iii) $t^* \simeq t_0 = 152.5$ day. In fact, these conditions hold for most of the dark halo models; however, as mentioned above, slight differences can be expected in case of possible contributions from non-thermalized DM components.

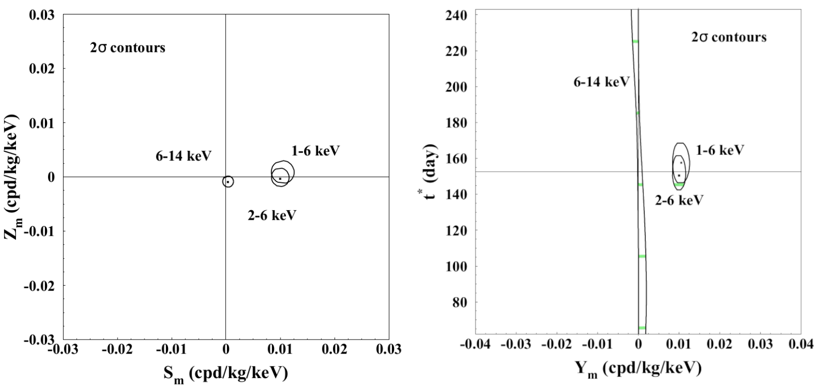

Fig. 2.2: 2σ contours in the plane (S_m, Z_m) (*left*) and in the plane (Y_m, t^*) (*right*) for: i) DAMA/NaI, DAMA/LIBRA–phase1 and DAMA/LIBRA–phase2 in the (2–6) keV and (6–14) keV energy intervals (light areas, green on-line); ii) only DAMA/LIBRA–phase2 in the (1–6) keV energy interval (dark areas, blue on-line). The contours have been obtained by the maximum likelihood method. A modulation amplitude is present in the lower energy intervals and the phase agrees with that expected for DM induced signals.

As shown in Fig. 2 a clear modulation is present in the lower energy intervals and the phase agrees with that expected for DM signal while such a modulation is absent just above.
We invite the reader to see all the details in DAMA literature [6] and in particular in [10, 11] and Refs. therein.
To conduct additional investigations on the nature of the DM particles in given scenarios, model-dependent analyses are necessary [6]; thus, many theoretical and experimental parameters and models are possible and many hypotheses must also be considered. In particular, the DAMA model-independent evidence is compatible with a wide set of astrophysical, nuclear and particle physics scenarios for high and low mass candidates inducing nuclear recoil and/or electromagnetic radiation as also shown at some extent in a wide literature. It is important to note that in these model-dependent corollary analyses, estimating the upper limit on the signal component in the measured rate (see e.g. in Ref. [10]) has to be considered as a prior.
Summarizing: as required by the exploited DM annual modulation signature: 1) the *single-hit* events show a clear cosine-like modulation as expected for the DM signal; 2) the measured period is well compatible with the 1 yr period as expected for the DM signal; 3) the measured phase is compatible with the roughly $\simeq$ 152.5 days expected for the DM signal; 4) the modulation is present only in the low energy (1–6) keV interval and not in other higher energy regions, consistently with expectation for the DM signal; 5) the modulation is present only in the *single-hit* events, while it is absent in the multiple-hit ones as expected for the DM signal; 6) the measured modulation amplitude in NaI(Tl) target of the *single-hit* scintillation

events in the (2–6) keV energy interval, for which data are also available by DAMA/NaI and DAMA/LIBRA–phase1, is: (0.01014 ± 0.00074) cpd/kg/keV (13.7 σ C.L.). And as already mentioned, no systematic or side processes able to mimic the signature, i.e. able to simultaneously satisfy all the many peculiarities of the signature and to account for the whole measured modulation amplitude, has been found or suggested by anyone throughout some decades thus far (for details see in [6]).

2.3 Few comments on some different experiments

By the fact, both the negative results and all the possible positive hints, achieved so-far in the field, can be compatible with the DAMA model-independent DM annual modulation results in many scenarios considering also the existing experimental and theoretical uncertainties; the same holds for indirect approaches. For a discussion see e.g. Ref. [6, 10, 11] and references therein.
Differences and incorrect arguments put forth by others have been addressed in detail in DAMA literature [6] and refs. therein.
For completeness, we remind that in last years some recent activities are using some kinds of NaI(Tl) set-ups which however – with respect to the DAMA ones – present: i) different source and selected materials constituting detectors and set-ups; ii) different powders purification methodologies; iii) different growing procedures with different features in the produced detectors, Tl concentration and uniformity; iv) different additives; v) different protocols; vi) different set-ups design, materials, procedures, etc.; vii) different handling and assembling procedures; viii) different overal radiopurity (materials, purification procedures, protocols, handling, assembling, ...); ix) different PMT, light guide or not, dedicated shield or not; x) different calibration procedures and frequency; xi) different quenching factors for nuclear recoils; xii) different event selection(s) procedures and data qualities; xiii) different analysis strategies; xiv) different exposure; xv) different control/analysis of possible systematics; xvi) different number of years of all set-ups in deep underground; xvii) often different observable exploited with model-dependent approach; xviii) etc. All that plays a relevant role on any claimed comparison.
As an example, let's briefly comment on the case of nuclear quenching factors; in fact, when DM candidates inducing nuclear recoils are considered, the energy scale in keV from γ sources calibrations should be transformed in keV of nuclear recoil by adopting a quenching factor for each target nucleus. This is a source of uncertainty in any comparison among experimental results of different experiments even when nominally the target nuclei are the same, in fact – as known – different quenching factors are expected and measured for different NaI(Tl) crystals – as in case of other detectors as well – since they depend, e.g., on the used growing technique, on the different thallium doping concentration and distribution, etc. A clear evidence is directly offered e.g. by the different α/β light ratio (i.e. α quenching factor) measured with DAMA and COSINE-100 crystals (see Fig. 3). As mentioned also in the ANAIS paper, this effect introduces a systematic uncertainty in the comparison of those recent realizations with the DAMA/LIBRA ones, i.e.

the same energy interval in keV from γ sources calibrations gives instead rise to well different energy interval when expressed in keV of nuclear recoils.

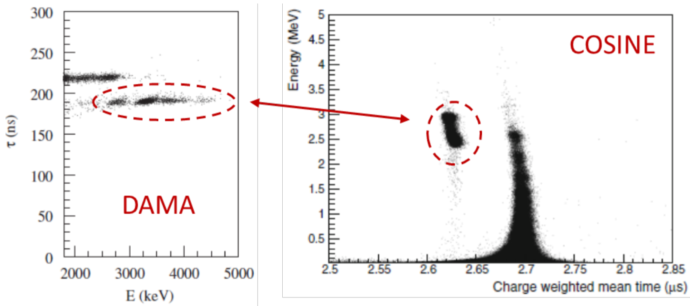

Fig. 2.3: In the dashed closed curve (red online) the identified α particles in DAMA/LIBRA [7] (left) and in COSINE-100 [17] (right), respectively. As it can be seen, an events from ^{238}U and ^{232}Th chains in DAMA crystal span from 2.6 to 4.5 MeVee, while for the COSINE crystal they span from 2.3 to 3.0 MeVee. This directly implies that the quenching factors of COSINE for nuclear recoils should be lower than those of DAMA, as confirmed by the measurements with neutrons as well.

2.4 Status of DAMA/LIBRA–phase2–empowered

A further increase of the experimental sensitivity can allow us to explore more extensively the DM annual modulation signature and to improve the measurement of the modulation parameters such as the phase, which brings important information, and: i) to explore the DM annual modulation signature at lower software energy threshold with high overal efficiency, also offering the possibility to more effectively disentangle among different proposed scenarios; ii) to investigate possible presence of streams in the Galaxy (as we already did for the case of the Sagittarius Dwarf in the past) also in the light of the recent GAIA results; iii) to investigate possible presence of caustics or of effects of gravitational focusing of the Sun; iv) to investigate the nuclear quantities entering in model-dependent corollary analyses; v) to investigate with increased sensitivity the diurnal modulation and other possible diurnal effects due e.g. to Earth shadow and channeling (refer to DAMA literature for detailed discussions); vi) to investigate rare processes other than Dark Matter by analyzing either other parts of the energy spectrum or specific features of the process searched for, as previously done and published with the DAMA/NaI and DAMA/LIBRA–phase1 data (from keV up to tens MeV; see DAMA literature).
Therefore, the DAMA collaboration has been working towards this direction. In particular, the experimental sensitivity to the DM annual modulation signature is connected to the product: $\epsilon \times \Delta E \times M \times T \times (\alpha - \beta^2)$, where ϵ is the overal efficiency, ΔE is the energy region where the DM annual modulation is present,

M is the exposed mass, T is the running time, and $(\alpha - \beta^2)$ shows how the data are collected along each annual cycle; the latter should approach 0.5 for a full year of data taking, and is crucial for a reliable investigation on DM annual modulation signature. Thus, after various R&D's, a safer and cheaper solution has been adopted to improve the signal over noise ratio in the lowest energy bins and thus to lower the software energy threshold and to improve related quantities. In particular, new preamplifiers have been developed to realize a single device with high signal/noise ratio, where the voltage divider and the preamplifier with miniaturized selected components are integrated on the same low background board. The preamplifier is based on the operational amplifier LMH6624 by Texas Instruments working at 5V, with an input bias current of -15 μA at 300 K and a bandwidth of 1.5 GHz. The preamplifier and the voltage divider are printed on the first and third layer of a Pyralux board; between these two layers a second ground layer is placed. The board is directly mounted in the back of the PMT inside a OFHC copper honeycomb structure of the shield. In such a way, the preamplifiers are as close as possible to the input source – that is the anode of the PMT – on the contrary of those used in DAMA/LIBRA–phase2 which are allocated outside the internal part of the set-up's shield obtaining a better signal/noise. The produced boards, allocating the voltage divider and the preamplifier, have also been tested from the radioactivity point of view by using a Germanium detector of the STELLA facility of the LNGS; the measured residual activity of ^{232}Th, ^{235}U, ^{40}K, ^{137}Cs and ^{60}Co is much lower than that of the single PMT. The only concern is for ^{226}Ra, which stays at the level of 13 mBq per piece; this is a typical feature for such devices: the Pyralux support has generally activity about a hundred times lower for ^{226}Ra and about 30 times lower for ^{228}Ra and ^{228}Th. Thus, the small residual activity is mainly due to the electronic components. Considering that the system voltage divider integrated with the preamplifier is placed behind the PMT and shielded in part by the honeycomb Cu structure, its role in the total background of the present DAMA/LIBRA–phase2–empowered is negligible. Summarizing, the features of the new voltage divider plus preamplifier system are: i) signal/noise: $\simeq$ 3.0–9.0; ii) discrimination of single photoelectron from electronic noise: 3 – 8; iii) peak to valley ratio: 4.7 – 11.6; iv) residual radioactivity lower than that of single PMT.

Fig. 4 shows a picture of the voltage divider plus preamplifier system connected to the flying leeds of a PMT.

Further relevant improvements arise from improvements in the electronic chain; in particular, all the Transient Digitizers have been substituted with new ones having higher vertical resolution (14 bits).

Thus, during fall 2021, the DAMA/LIBRA–phase2 set-up was upgraded and the data taking in this new configuration, identified as DAMA/LIBRA–phase2–empowered, started on Dec, 1 2021. The operational features are very stable; in particular, the baseline fluctuations are more than a factor two lower than those of the previous configuration and the RMS of the baseline distributions is around 150 μV, ranging between 110 and 190 μV. The software Trigger Level (STL) is decreased in the offline analysis. The "noise" events due to single photoelectron

Fig. 2.4: Picture of the voltage divider plus preamplifier system connected to the flying leeds of a PMT in DAMA/LIBRA–phase2–empowered during installation in HP Nitrogen atmosphere.

have evident different structures than the scintillation pulses with the same energy, and this feature is used to discriminate them.
Shortly, the data acquisition system (daq) is composed by 5 TD's, CAEN VME VX1730 with dynamic range of 14 bit (which corresponds to a vertical resolution of 0.122 mV/digit), vertical window of 2 V, sampling frequency of 500 MSa/s (2 ns time bin), and 250 MHz bandwidth; each VME module has 16 channels. Thus, the daq acquires three traces for each detector (from the two PMTs, A and B, and the high-energy sum of them, SUM-HE). The read-out of the digitizers is made by a daisy-chain on optical fiber directly connected to the data acquisition computer.
DAMA/LIBRA–phase2–empowered is continuously running; for example, up to March 2023 about 0.28 ton $\times$ yr exposure has been collected with (α - β^2) $\simeq$ 0.488. In the same period, about 3.5Ã$\times$ 10^7 events have been collected from sources for energy calibration and about 1.95 $\times$ 10^7 events ($\simeq$ 7.8 $\times$ 10^5 events/keV) for determining the acceptance window efficiency for all the crystals.

2.5 Conclusions

The DAMA has been a pioneer project in the direct detection of Dark Matter, obtaining the first model-independent evidence for the presence of a particle component of the Dark Matter in the galactic halo on the basis of the exploited DM annual modulation signature.
Three independent experimental set-ups have confirmed the presence of a peculiar annual modulation of the *single–hit* events in the energy region (1–6) keV, that meets all the many requirements of the DM annual modulation signature; the cumulative exposure, considering them all together is 2.86 tons $\times$ yr (over 22 independent annual cycles and with different experimental configurations). No systematic or side processes able to account for the observed signal are available. Corollary investigations on the nature of the DM particle(s) in given scenarios have been performed by corollary model-dependent analyses. Various models

and parameters (experimental and theoretical) are possible, and many hypotheses have to be considered.
To further increase the experimental sensitivity of DAMA/LIBRA and to better disentangle some of the many possible astrophysical, nuclear and particle physics scenarios in the investigation on the DM candidate particle(s), an increase of the exposure in the lowest energy bins and a further decreasing of the software energy threshold have been considered. This is pursued by upgrading DAMA/LIBRA–phase2 to lower the software energy threshold below 1 keV with suitable acceptance efficiency. Preliminarily, particular efforts for lowering the software energy threshold have been done in the already-acquired data of DAMA/LIBRA–phase2 by using the same technique as before with dedicated studies on the efficiencies, obtaining modulation amplitude as a function of energy down to 0.75 keV [11]; a modulation has also been observed below 1 keV. This preliminary result has confirmed the interest in lowering the software energy threshold by a hardware upgrade and an improved exposure at the lowest bins. Thus, DAMA/LIBRA–phase2–empowered was realized and put in operation; it is planned to collect data until December 2024.

References

1. P. Belli, R. Bernabei, C. Bacci, A. Incicchitti, R. Marcovaldi, D. Prosperi: DAMA proposal to INFN Scientific Committee II, April 24th, 1990.
2. A.K. Drukier, K. Freese, D.N. Spergel: Detecting cold dark-matter candidate, Phys. Rev. D **33** (1986) 3495 – 3508.
3. K. Freese, J.A. Frieman, A. Gould: Signal modulation in cold-dark-matter detection, Phys. Rev. D **37** (1988) 3388 – 3405.
4. R. Bernabei et al.: Performances of the 100 kg NaI(Tl) set-up of the DAMA experiment at Gran Sasso, Il Nuovo Cimento A **112** (1999) 545 – 575.
5. R. Bernabei et al.: Dark Matter search, Riv. Nuovo Cimento **26** (2003) 1 – 74; R. Bernabei et al.: Dark Matter particles in the galactic halo: results and implications from DAMA/NaI , Int. J. Mod. Phys. D **13** (2004), 2127 – 2159.; references therein.
6. see publication list in dama.web.roma2.infn.it
7. R. Bernabei et al.: The DAMA/LIBRA apparatus, Nucl. Instr. and Meth. A **592** (2008) 297-315.
8. R. Bernabei et al.: Performances of the new high quantum efficiency PMTs in DAMA/LIBRA, J. Instrum. **7** (2012) P03009-1 – P03009-15.
9. P. Belli et al.: The electronics and DAQ system in DAMA/LIBRA, Int. J. Mod. Phys. A **31** (2016) 1642005-1 – 1642005-14; dedicated issue to DAMA/LIBRA of Int. J. Mod. Phys. A **31** (2016).
10. R. Bernabei et al.: The DAMA project: achievements, implications and perspectives, Progress in Particle and Nuclear Physics **114** (2020) 103810-1 – 103810-72.
11. R. Bernabei et al.: Further results from DAMA/LIBRA-phase2 and perspectives, Nucl. Phys. At. Energy **22** (2021) 329–342; refs. therein.
12. D. Smith, N. Weiner: Inelastic dark matter, Phys. Rev. D **64** (2001) 043502-1 – 043502-11.
13. D. Tucker-Smith, N. Weiner: Status of inelastic dark matter, Phys. Rev. D **72** (2005) 063509-1 – 063509-8.
14. D.P. Finkbeiner et al.: Inelastic dark matter and DAMA/LIBRA: An experimentum crucis, Phys. Rev. D **80** (2009) 115008-1 – 115008-12.

15. K. Freese et al.: Detectability of weakly interacting massive particles in the Sagittarius dwarf tidal stream, Phys. Rev. D **71** (2005) 043516-1 – 043516-15.
16. K. Freese et al.: Effects of the Sagittarius dwarf tidal stream on dark matter detectors, Phys. Rev. Lett. **92** (2004) 111301-1 – 111301-4.
17. G. Adhikari et al.: Initial performance of the COSINE-100 experiment, Eur. Phys. J. C **78** (2018) 107-1 – 107-19.

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3 Balancing baryon and asymmetric dark matter excess

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Abstract. Effect of the electroweak non-conservation of the baryon number could be a key ingredient to explain the ratio of dark and baryonic densities. If dark matter is explained by dark atoms, in which stable $-2n$ charged particles are bound with n nuclei of primordial helium, and this multiple charged particles possess SU(2) electroweak charges, the excess of $-2n$ charged particles over their antiparticles can be related to baryon excess by sphaleron transitions. It provides relationship between the density of asymmetric dark atom dark matter and baryon asymmetry, The cosmological consequences of sphaleron transitions were considered for the minimal walking technicolor (WTC) model, which provides composite Higgs boson solution for the problem of Higgs boson mass divergence in the Standard model. The realisation of multi-component dark atom scenario is possible because the electric charges of new fermions are not fixed and several types of stable multiple charged states are possible. In particular cases the upper limits for the masses of techniparticles could be found, at which dark atom interpretation of dark matter is possible. These limits challenge search for multiple charged stable particles at the LHC.

Povzetek: Avtorji predlagajo model, ki razloži neohranitev barionskega števila v vesolju, če se je to zgodilo pri elektrošibkem faznem prehodu. Predpostavijo, da tvorijo temno snov temni atomi, v katerih so stabilni negatino nabiti delci z nabojem $-2n$ (n je neko celo število) vezani na n jeder primordialnega helija (helija, ki je nastal v prvih nekaj deset sekundah po velikem poku) in nosijo ti stabilni negativno nabiti delci elektrošibke naboje $SU(2)$; tedaj je mogoče presežek ($-2n$) nabitih delcev nad njihovimi antidelci povezati z barionskim presežkom, ki ga povzročijo sfaleronski prehodi za različne vrednosti naboja stabilnih negativno nabitih delcev. Za oceno vpliva teh prehodov na kozmološke pojave so uporabili metodo WTC. Ugotavljajo, da za nekatere vrednosti naboja je masa teh temnih atomov v skladu z opaženimi lastnosti temne snovi. LHC bi te nabite delce lahko izmeril.

3.1 Introduction

The modern cosmological paradigm inevitably involves the dark matter but it's origin and dynamics should be described by physics beyond the Standard Model. In this paper the consequences of the minimal walking technicolor model (WTC) are considered [1–4].
The existing of four new fermions is assumed, namely: heavy techniquarks U, D have a standard electroweak charges and the new one — technicolor charge. These

particles could be observed in the form of technibaryons (UU, UD, DD) which arise as a Goldstone bosons from the global symmetry breaking, $SU(4) \to SO(4)$. In the model, the Higgs boson becomes a composite particle $\frac{1}{\sqrt{2}}\left(U\bar{U} + D\bar{D}\right)$ and technileptons, N and E, are introduced to eliminate anomalies.
Importantly, an (arbitrary) electric charge of techniparticles is not fixed and can be specified by y-parameter: $(y+1,\, y,\, y-1,\, \frac{1}{2}(-3y+1),\, \frac{1}{2}(-3y-1))$ for $(UU,\, UD,\, DD,\, N,\, E)$ states, correspondingly.
In minimal WTC model the bound state $X^{-2n}(^4He^{+2})_n$, where the stable heavy core, X, is $(UU)^{-2n}$ or $(E/N)^{-2n}$, could be considered as a dark matter particle. Such "dark atom", called also X-helium, fulfill conditions of no-go theorem which is formulated in [5,6].
It is assumed that the required excess of techniparticles is generated during nonperturbative electroweak processes violating the laws of baryon and lepton numbers conservation. Such processes are generated by the topology of the $SU(2)$ group and should actively occur in the early Universe. In the literature, these stable classic solutions are called sphaleron transitions [7,8].
The cosmological consequences of this model have already been partially considered in [9–11]. Authors used the thermodynamic approach to balance baryon and dark matter excesses for the case $y = 1$, however, the general case has not been studied.
This paper is organized as follows. In section 3.2 equations for chemical potentials are written out. The solutions for temperatures before and after the electroweak phase transition (EWPT) are considered in sections 3.3.1 and 3.3.2, correspondingly. Some discussion of the results is presented in Conclusions 3.4.

3.2 Equations

The thermodynamic approach for the analysis of sphaleron configurations was developed in [9–12]. In this paper, chemical potentials are introduced in a similar way: $\mu_{iR/L}$, where "i" is a flavor of particle and R/L is the chirality. So the conditions of weak interactions can be written as following:

$$\mu_{iR} = \mu_{iL} \pm \mu_0, \tag{3.1}$$

$$\mu_i = \mu_j + \mu_W, \tag{3.2}$$

where "i"- and "j"-type fermions form an electroweak doublet.
The standard baryon number and lepton number densities are

$$\begin{aligned} B &= \frac{1}{3}\cdot 3\cdot(2+\sigma_t)(\mu_{uL}+\mu_{uR}) + \frac{1}{3}\cdot 3\cdot 3\cdot(\mu_{dL}+\mu_{dR}) = \\ &= (10+2\sigma_t)\mu_{uL} + 6\mu_W, \end{aligned} \tag{3.3}$$

$$\begin{aligned} L &= \sum_i (\mu_{\nu_i L} + \mu_{\nu_i R} + \mu_{iL} + \mu_{iR}) = \\ &= 4\mu + 6\mu_W, \end{aligned} \tag{3.4}$$

where the weight functions for a massive particles are used:

$$\sigma(z) = \begin{cases} \frac{6}{4\pi^2}\int_0^\infty dx\, x^2 \cosh^{-2}\left(\frac{1}{2}\sqrt{x^2+z^2}\right), \text{ for fermions;} \\ \frac{6}{4\pi^2}\int_0^\infty dx\, x^2 \sinh^{-2}\left(\frac{1}{2}\sqrt{x^2+z^2}\right), \text{ for bosons.} \end{cases} \quad (3.5)$$

The number densities for technibaryons and technileptons are similar to the ones, defined in [11]:

$$TB = \frac{2}{3}(\sigma_{UU}\mu_{UU} + \sigma_{UD}\mu_{UD} + \sigma_{DD}\mu_{DD}), \quad (3.6)$$

$$TL = \sigma_E(\mu_{EL} + \mu_{ER}) + \sigma_N(\mu_{NL} + \mu_{NR}), \quad (3.7)$$

The solution of this system of equations is an observable ratio of densities which strongly depends on the sphaleron's freezing out temperature $T_* \sim 200\,\text{GeV}$. If it is higher then the temperature of EWPT T_{EWPT}, the condition of isospin neutrality could be used:

$$\begin{aligned} 0 =& \frac{1}{2}\cdot 3\cdot 3\cdot(\mu_{uL} - \mu_{dL}) + \frac{1}{2}\cdot 3\cdot 3\cdot(\mu_{iL} - \mu_{eL}) + \\ &+ \sigma_{UU}\mu_{UU} - \sigma_{DD}\mu_{DD} + \frac{1}{2}\sigma_N\mu_{NL} - \frac{1}{2}\sigma_E\mu_{EL} - 4\mu_W - \mu_W. \end{aligned} \quad (3.8)$$

The equation of sphaleron transition is also similar to the one, defined in [11]:

$$3(\mu_{uL} + 2\mu_{dL}) + \mu + \frac{1}{2}\mu_{UU} + \mu_{DD} + \mu_{NL} = 0. \quad (3.9)$$

However, the electrical neutrality condition should be slightly modified:

$$\begin{aligned} 0 =& \frac{2}{3}\cdot 3\cdot 3(\mu_{uL} + \mu_{uR}) - \frac{1}{3}\cdot 3\cdot 3(\mu_{dL} + \mu_{dR}) - \frac{1}{3}(\mu_{eL} + \mu_{eR}) + \\ &+ (y+1)\sigma_{UU}\mu_{UU} + y\sigma_{UD}\mu_{UD} + (y-1)\sigma_{DD}\mu_{DD} + \\ &+ \frac{-3y+1}{2}\sigma_N(\mu_{NL} + \mu_{NR}) + \frac{-3y-1}{2}\sigma_E(\mu_{EL} + \mu_{ER}) - 4\mu_W - 2\mu_m, \end{aligned} \quad (3.10)$$

where the charges of the techniparticles are parameterised by y.
The last condition describes the composite nature of Higgs boson and can be used in both cases, because the chemical potentials of particle and antiparticle differ only by sign:

$$\mu_0 = 0. \quad (3.11)$$

This equation was not used in the mentioned papers.
Finally, the definition of the observable ratio of densities is the following:

$$\frac{\Omega_{DM}}{\Omega_b} \approx \frac{3m_{UU}}{2m_p}\left|\frac{TB}{B}\right| + \frac{3m_{E/N}}{m_p}\left|\frac{TL}{B}\right|, \quad (3.12)$$

where it was assumed that the baryonic matter consists of protons only.

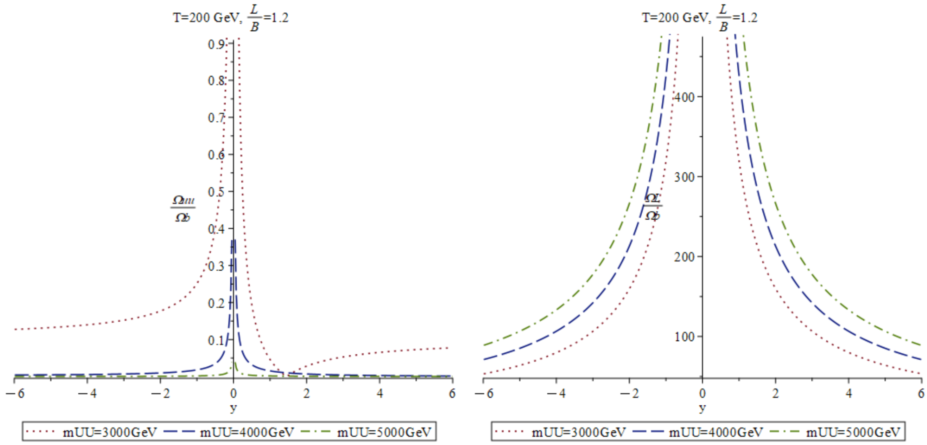

Fig. 3.1: Dependence of technibaryon (left) and technilepton (right) densities on the charge parameter y.

3.3 Baryon and dark matter excesses

3.3.1 Before the electroweak phase transition

Main dependencies To solve the system of equations which was described in previous section, a number of simplifying assumptions should be introduced. The first one has already been done in equation (3.12): the baryonic matter consists of protons only. The second one is an assumption of equal masses: $m_U = m_D$ and $m_N = m_E$. By so, one can find

$$\frac{TB}{B} = -\frac{\sigma_{UU}(3y\sigma_E - 1)}{3y(\sigma_{UU} + 3\sigma_E)}\left(\frac{L}{B} + \frac{9y\sigma_E + 1}{3y\sigma_E - 1}\right), \tag{3.13}$$

$$\frac{TL}{B} = -\frac{\sigma_E(y\sigma_{UU} + 1)}{y(\sigma_{UU} + 3\sigma_E)}\left(\frac{L}{B} + \frac{3y\sigma_{UU} - 1}{y\sigma_{UU} + 1}\right). \tag{3.14}$$

The question on the relationship between the total masses of techniquarks (or technibaryons) and technileptons is more complicated. The simplest way is to set $m_{E/N} = \frac{m_{UU}}{2}$. In this case, the technibaryon component of the DM is exponentially suppressed by factor

$$\frac{\sigma_{UU}(3y\sigma_E - 1)}{3y(\sigma_{UU} + 3\sigma_E)} \sim \frac{\sigma_{UU}}{\sigma_E} \tag{3.15}$$

at high values of total mass. It also can be seen at Fig.3.1. One can see that the ratio of densities $\frac{\Omega_{DM}}{\Omega_b}$ depends on the electric charge parameter y hyperbolically but the value of this ratio for the DM components may differ by several orders of magnitude.

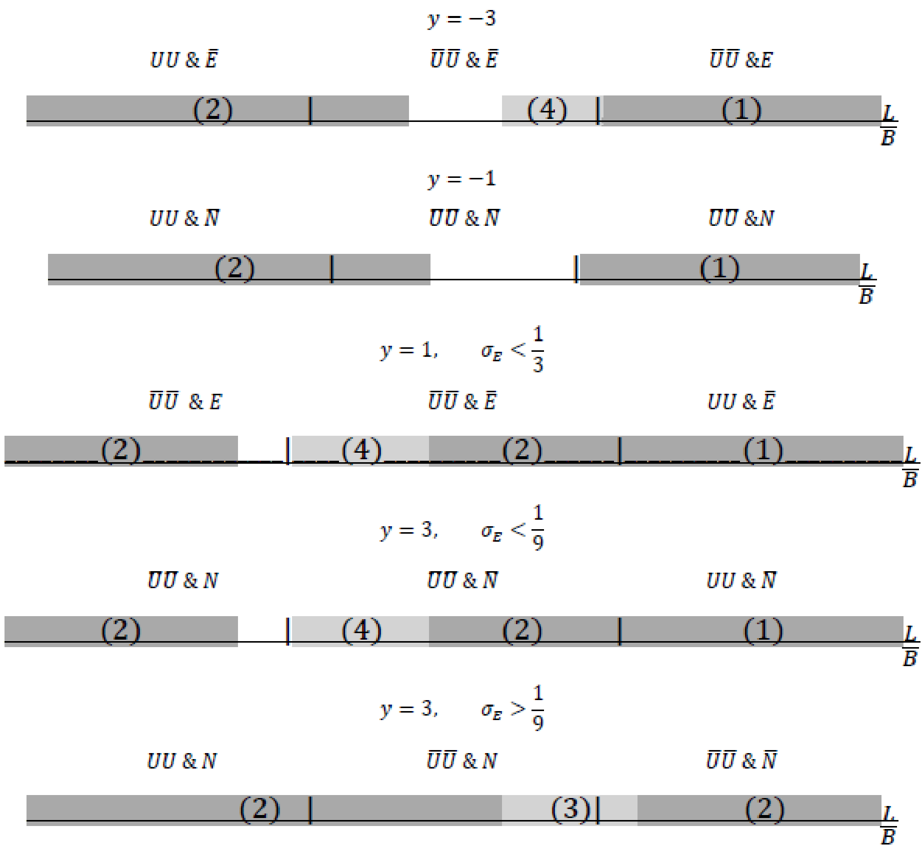

Fig. 3.2: The allowed regions of parameters for different values of charge.

The line break on the left panel of Fig. 3.1 describes the change of the sign of charges in the technibaryons excess (the excess of particles is replaced by an excess of antiparticles). This sign depends on many factors and has a strong impact on the observed physical picture which is shown on Fig.3.2. The vertical lines points zero values of (3.13) and (3.14).
The grey areas on the picture are forbidden because

(1) both the technibaryons and the technileptons have a positive electric charge $+2n$. It leads to the overproduction of anomalous isotopes;
(2) the overproduction of the technileptons: $\frac{\Omega_{E/N}}{\Omega_b} \gg 5$;
(3) the overproduction of the technibaryons: $\frac{\Omega_{UU}}{\Omega_b} \gg 5$;
(4) only the dominant DM component (technileptons) has a positive electric charge, it leads to the overproduction of anomalous isotopes too.

It should be noted, in order to avoid overproduction of the DM, it is necessary to consider the values of $\frac{L}{B} \sim 1$.

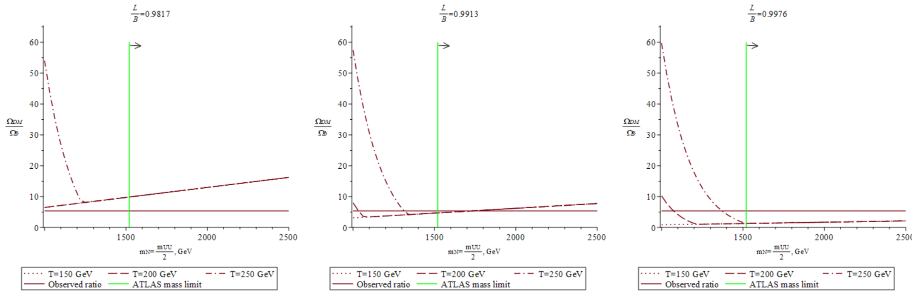

Fig. 3.3: The ratio of densities as the function of the total mass.

The nature of the DM in allowed regions depends on the sign of the charge parameter y. If $y > 0$, the DM should consist of both the technileptonic $(N/E)^{\frac{-3y\pm1}{2}}({}^4He^{+2})_{\frac{-3y\pm1}{4}}$ and the technibaryonic $(\bar{U}\bar{U})^{y+1}({}^4He^{+2})_{\frac{y+1}{2}}$ dark atoms in the X-helium form. But if it has a negative value, the WIMP like bound state $(\bar{U}\bar{U})_m(\bar{N})_n$ arise. Besides, the X-helium technileptonic, $\bar{N}^{-2r}$, and mixed, $\bar{N}^{-2r}(\bar{U}\bar{U})^{+2s}$, cores are possible in this case.
It should be noted also that the case $y = -1$ is specific: the stable technibaryon UU has zero electric charge. So, it becomes the WIMP and the technileptonic X-helium $\bar{E}He$ is a dominant component of the DM.
On Fig.3.3 typical behavior of the densities ratio as a function of the total mass is shown. It strongly depends on the value of ratio $\frac{L}{B}$. The ATLAS limits for the multi-charged particles masses can be seen in [13].
Using the information shown in Figs.3.2 and 3.3, it can be found that the total mass of the techniparticles can not be lighter than 1 TeV. Unfortunately, there are no upper limits for these masses in general case, but in special cases the possibility of the DM overproduction allows to set some restrictions.

Different masses It is possible to study how the mass difference changes the described picture. First of all,assumption on the mass values should be considered more carefully. The equation $m_{E/N} = \frac{m_{UU}}{2}$ which has been used above could be replaced by $m_{E/N} = m_{UU}$. Then, there is no suppression of technibaryonic component of the DM. It is shown on Fig.3.4 how the density of technibaryons depends on the total mass under the new assumption.The allowed parameters regions on Fig.3.2 in this case should decrease.
In general, due to the mass difference, $\Delta m = m_{UU} - m_E$, the sign of the generated excess can change. It is shown in Fig.3.5 for the density of technibaryon component (left panel), density of technilepton component (central panel) and total density of the DM (right panel).

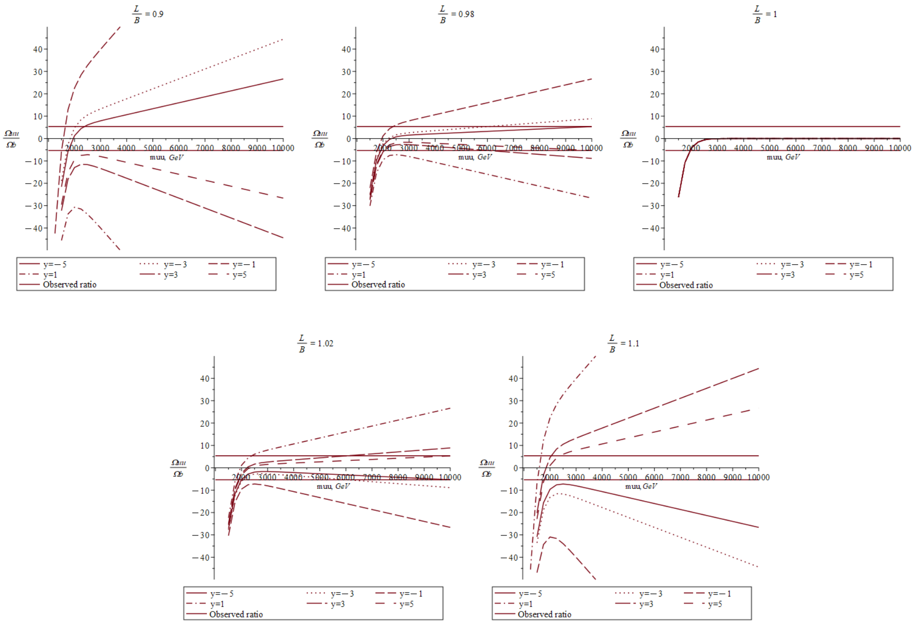

Fig. 3.4: The ratio of densities for technibaryon component of the DM (assuming $m_{E/N} = m_{UU}$) as a function of total mass.

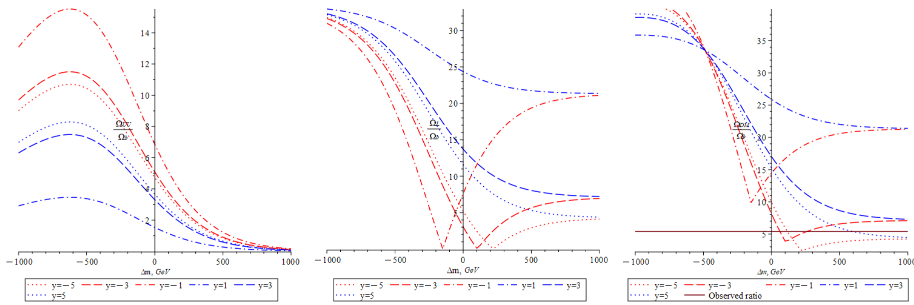

Fig. 3.5: The general dependence of ratios of densities on the mass difference $\Delta m = m_{UU} - m_E$.

To take into account the difference between the masses of technibaryons and technileptons, it is necessary to enter the following parametrization:

$$\sigma_{N,E} = \sigma_f \left(\frac{m}{T}\right) + n, e; \tag{3.16}$$

$$\sigma_{UU} = \sigma_b \left(\frac{m}{T}\right) + 2u; \tag{3.17}$$

$$\sigma_{UD} = \sigma_b \left(\frac{m}{T}\right) + u + d; \tag{3.18}$$

$$\sigma_{DD} = \sigma_b \left(\frac{m}{T}\right) + 2d, \tag{3.19}$$

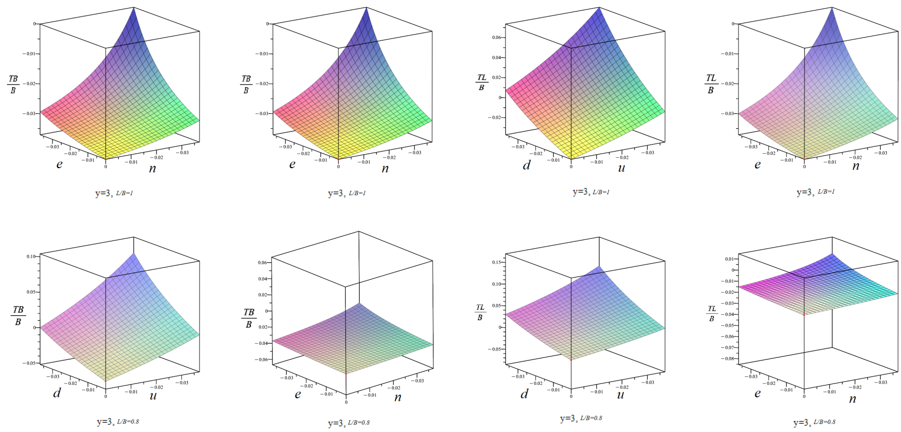

Fig. 3.6: The ratios of densities for different techniparticle masses.

where $m = 1500\,\text{GeV}$ and $T = 250\,\text{GeV}$ are chosen to minimize the physically meaningful value of the weight function.
One can see in Fig.3.6 that the mass difference increasing results to rise of the density ratio for both DM components. Moreover, in some cases the sign of the generated techniparticles excess can be changed.

3.3.2 After the electroweak phase transition

For temperatures below the EWPT temperature, condition of isospin neutrality can not be used, so the system of equations solution is:

$$\frac{TB}{B} = -\alpha\left(\frac{L}{B} + \gamma\,\frac{TL}{B} + \beta\right), \tag{3.20}$$

where the functions α, β and γ are considered assuming an equal masses:

$$\alpha = \frac{\sigma_{UU}}{3}\,\frac{(\sigma_t+5)(2\sigma_{UU}+\sigma_E)+6(\sigma_t+17)}{(9(\sigma_t-1)y+2(\sigma_t+5))\sigma_{UU}+(\sigma_t+5)\sigma_E+3(5\sigma_t+31)}, \tag{3.21}$$

$$\beta = \frac{18(2\sigma_{UU}+\sigma_E+18)}{(\sigma_t+5)(2\sigma_{UU}+\sigma_E)+6(\sigma_t+17)}, \tag{3.22}$$

$$\gamma = \frac{2(\sigma_t+5)\sigma_{UU}+(27(1-\sigma_t)y+\sigma_t+5)\sigma_E+3(5\sigma_t+31)}{\sigma_E((\sigma_t+5)(2\sigma_{UU}+\sigma_E)+6(\sigma_t+17))}. \tag{3.23}$$

For $|y| \lesssim 100$, dependence of the density ratio on the charge can be neglected. So, the result of calculations depends only on two parameters: $\frac{L}{B}$ and $\frac{m}{T_*}$.
If $\Delta m = m_{UU} - m_E \neq 0$ the technibaryon component of the DM should been suppressed by factor

$$\alpha\gamma \xrightarrow{\sigma_t \to 1} \frac{1}{3}\,\frac{\sigma_{UU}}{\sigma_E}. \tag{3.24}$$

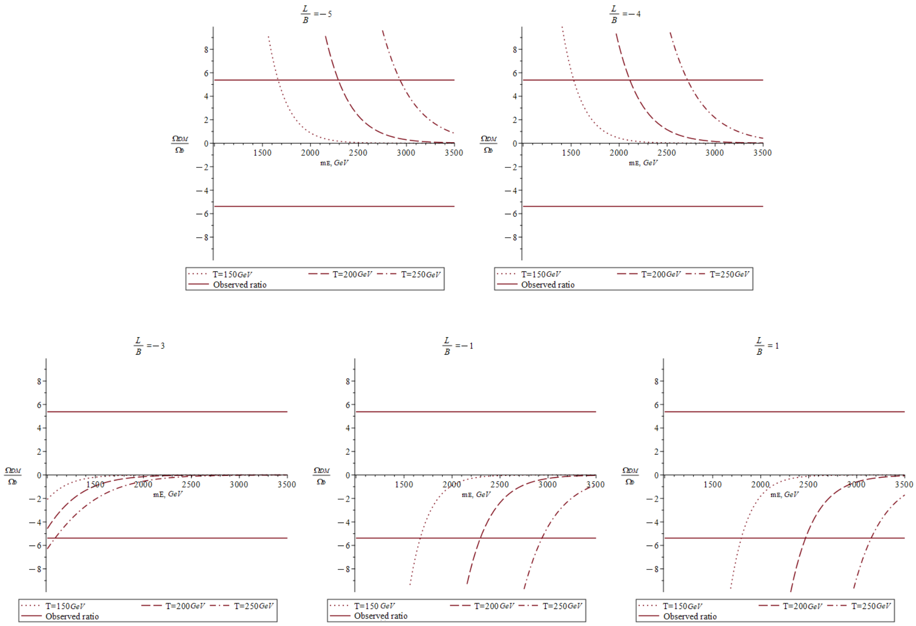

Fig. 3.7: The ratio of densities as a function of two main parameters for $T_* < T_{EWPT}$.

It allows to find the value of densities ratio for the technileptonic DM assuming $\dfrac{\Omega_{UU}}{\Omega_b} \to 0$. In Fig.3.7 one can see how the ratio depends on two main parameters; negative values corresponds to the excess of antitechnileptons.
As it is seen, the expected mass of techniparticles becomes higher with the increasing of $\dfrac{L}{B}$ parameter. Because $\dfrac{L}{B} < 10^8$ (see [14]), the upper limit for the technileptonic mass can be estimated as: $m < 5 - 8\,\text{TeV}$ for different sphaleron freezing out temperatures.

3.4 Conclusion

The cosmological consequences of sphaleron transitions in the minimal walking technicolor model have been considered. The main feature of the model is the nonfixed arbitrary electric charge of new heavy particles. Different values of the charge parameter y lead to different scenarios of multicomponent DM.
It was shown, the ratio of densities $\frac{\Omega_{DM}}{\Omega_b}$ could be explained in both considered cases ($T_* > T_{EWPT}$ and $T_* < T_{EWPT}$). If sphaleron transitions freeze out before the EWPT, the DM should consist of several forms of X-helium and/or WIMP-like bound states. To prevent overproduction of particles, it is necessary to set a lower limit on their mass $m \gtrsim 1\,\text{TeV}$. The upper limit can be found only for some special cases.
If sphaleron transitions freeze out after the EWPT and difference between the masses of technibaryons and technileptons can not be neglected, the DM particle

should not be too heavy ($m < 5 - 8$ TeV). In this case the DM density is provided by the X-helium with technileptonic core.
The found limits for masses could be tested in upcoming accelerator experiments.

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References

1. F. Sannino and K. Tuominen: Orienfold Theory Dynamics and Symmetry Breaking, Phys. Rev. D **71** (2005) 051901
2. D. K. Hong, S. D. Hsu, F. Sannino: Composite Higgs from higher representations, Phys. Lett. B **597** (2004) 89-93
3. D.D. Dietrich, F.Sannino, K.Tuominen: Light composite Higgs from higher representations versus electroweak precision measurements. Predictions for LHC, Phys. Rev. D **72** (2005) 055001
4. D. D. Dietrich, F. Sannino and K. Tuominen: Light composite Higgs and precision electroweak measurements on the Z resonance: An update, Phys. Rev. D **73** (2006) 037701
5. M.Yu. Khlopov: What comes after the Standard Model?, Prog. Part. Nucl. Phys. **116** (2020) 103824
6. K.M. Belotsky, M.Yu. Khlopov, K.I. Shibaev: Composite Dark Matter and its Charged Constituents in: Gravitation and Cosmology. Vol. 12. — Moscow: (KPS 06), 2006.
7. Manton N. S.: Topology in the Weinberg-Salam Theory, Phys. Rev. D. **28** (1983) 2019
8. Klinkhamer F. R., Manton N. S. A saddle-point solution in the Weinberg-Salam theory, hys. Rev. D. **30** (1984) 2212—2220
9. S. B. Gudnason, C. Kouvaris and F. Sannino: Dark matter from new technicolor theories, Phys. Rev. D **74** (2006) 095008
10. M. Yu. Khlopov and C. Kouvaris: Composite dark matter from a model with composite Higgs boson, Phys. Rev. D **78** (2008) 065040
11. M. Yu. Khlopov and C. Kouvaris: Strong interactive massive particles from a strong coupled theory, Phys. Rev. D **77** (2008) 065002
12. J. A. Harvey, M. S. Turner: Cosmological baryon and lepton number in the presence of electroweak fermion-number violation, Phys. Rev. D **42** (1990) 3344
13. The ATLAS Collaboration: Search for heavy long-lived multi-charged particles in the full LHC Run 2 pp collision data at $\sqrt{s}$ = 13 TeV using the ATLAS detector, arXiv: 2023.13613.
14. V. Domcke [et al.]: New Constraint on Primordial Lepton Flavor Asymmetries, Phys. Rev. Lett. **130** (2023) 261803

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4 Some Information-related aspects in fundamental particle modeling

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Abstract. We consider a model of physical vacuum based on a grid of discrete nodes. Each node can have an electric charge of either $+\frac{1}{6}$ or $-\frac{1}{6}$ of the electron charge. Vacuum corresponds to a background pattern of alternation between positive and negative charges, while deviations from the pattern correspond to particles. Using this model, we can represent the propagation, decay, and interaction of particles as the movement of defects according to certain rules. The lack of direct observation of defective nodes leads to uncertainty effects, which can explain quantum effects. We discuss the importance of understanding information storage and movement in different scenarios, including the need for respecting the principle of locality and the limitations of linear sequences for representing complex information. We also examine the behavior of quantum particles, the phenomenon of observation, and the branching of defects in a discrete environment. Through computational experiments, we propose mechanisms to overcome these challenges, such as error corrections, compactification of dimensions and balanced branching and merging, incorporated in the grid.

Povzetek:
Avtor postavi model vakuuma, ki temelji na mreži diskretnih vozlišč. Vsako vozlišče nosi električni naboj $+\frac{1}{6}$ ali $-\frac{1}{6}$ naboja elektrona. Vakuum predstavi z vzorcem ozadja menjavanja pozitivnih in negativnih nabojev, medtem ko odstopanja od vzorca poveže z delci. Avtor opiše gibanje, razpad in interakcijo med delci kot gibanje defektov pod določenimi pravili. Kvantno nedoločenost opiše s pomanjkanjem defektov v vozliščih. Predlaga različne scenarje za razpoznavanje kvantne informacije o dinamiki, lokalnosti, proučuje kvantno obnašanje delcev. Z računalniškimi simulacijami prouči odkrivanje napak, zlomitev simetrij pri kompaktifikaciji prostora, pa tudi razceplenje in združevanje, ki je povezano z dogajanjem v mreži.

4.1 The lattice-based model

We consider a class of models based on an idea that a *lattice* of discrete *nodes* arranged in some specific order may represent the physical vacuum and particles in it. Specific geometry of this arrangement plays the significant role in determining physical laws in the vacuum and spectrum of particles it can contain.
A lattice of this kind might be formed naturally, for example, due to the vacuum phase transition similar to the one occurring in various liquid mixtures [2–5]. When the one-phase two-component mixture is close to this critical point, but is

not yet separated to two phases, the correlation distance of the order parameter (the composition) is finite. This means existence of small areas (domains) in the liquid where one or another component of mixture prevails, which, therefore, are separated from each other by certain walls, where the composition corresponds to the composition of the initial phase.

Assuming vacuum phase transition to be analogous to the phase transition in liquids, we get the interpretation in which the lattice's nodes correspond to miniature vacuum domains or bubbles formed by *two kinds* of vacuum, which are close-packed, filling the space. We estimate the correlation distance that is the characteristic radius of domains to be of order $10^{-21}\,\mathrm{m}$.

Lattice's geometry is supposed to be determined by laws govern phase transition. This aspect is beyond the scope of current work, so here we assume that we already have some lattice for granted, it consists of unified nodes, they and their mutual arrangement (geometry) are mostly stable, and each node can be of one of two kinds. The walls between domains can be approximated by Voronoi cells of lattice nodes.

Each domain, unlike bubbles in ordinary soap foam, has both neighbors of the *same* type as this domain has, and also neighbors of the *another* type. Thus, there must be two kinds of walls between them. Also, domains of the same type can have no strict separation, and a wall (of one kind) may be only between different domains. In the latter case, the physical vacuum would look like two different global mutually penetrating parts formed by neighboring domains of each type, separated by an only non-trivial wall, presumably having gyroid-like [1] shape.

The key and only difference that we assume between these two phases is the sign of their electric charge density.

We prefer to *rephrase* this statement in another way, because when there are two phases, there is a binary difference between them. So, we suppose that this **difference between two vacua *is*** what historically called **the electrical charge**. We do not need any special concept of electrical charge of an unknown origin: the electrical charge emerges as a consequence of splitting vacuum in two phases instead of one, and it labels these phases symmetrically: one of them is labeled positive and one is negative.

We have sufficient motivation for such a rather bold statement. As we showed earlier [6], almost all other charges and quantum numbers can be expressed in terms of electric charge, provided that the relative position of the individual components of this charge is taken into account.

Following this, the vacuum must be electrically neutral everywhere as long as the correct periodic alternation of two types of vacuum domains is maintained. If this order is violated, a defect is presented at the site of this violation.

This anti-structure defect has a discrete electrical charge determined by the difference between counts of two types of domains involved in it, and is proportional to the charge of the single domain. We assume this charge to be $+\frac{1}{6}$ or $-\frac{1}{6}$ of the electron charge, aiming to reproduce correct quantum numbers of the known particles.

Note that the shape of inter-domain wall in the case of defect is also changed with respect to the regular case, including change in count of 'holes' in it. So, the

violation of alternation can be considered both as an *anti-structure* defect in the lattice of nodes and as a *topological* defect of the inter-domain wall.
Since the domain wall is usually considered having some surface tension, we suppose that a change in the shape of the wall must be accompanied by a corresponding change in its energy, more ore less localized at the defect's site.
Following this motivation, and *rephrasing* it in the similar way as we did concerning electric charge, we define that what historically called the **energy is the difference in some geometrical measure** (like the surface area and curvature) **of the domain wall** in respect to its non-disturbed shape. So, in general terms, for a particle in rest the defect energy is its mass, the energy of the distorted but not defected wall is an energy of a distortion field, and the additional energy of propagating particle is defined by the additional violation of the wall's shape according to this propagation.
So, the mass, as the localized energy of defect, defined by this way, must be discrete (since it is determined by the defect's discrete geometry change), but it is not obliged to be proportional to any unit.
Summarizing the above, we believe that the defects in our models can effectively play the role of physical fundamental particles observed in experiments.
Our hypothesis is that the rules for the creation, preservation, propagation, decay and scattering of defects can be derived from the above assumptions under the geometric and topological constraints imposed by a particular lattice.
Guided by the idea of a phase transition in vacuum as the main hypothesis about the cause of lattice formation, we first of all pay attention to the densest packings in three, four or more dimensions as possible variants of its geometry.
At the same time, we are not limited only to flat structures, but also have in mind the possibility of compactification, folding of these lattices, using their periodicity, that keeps the same packing density.
With such compactification, instead of an infinite lattice, we mean only its part or strip, consisting of one or several periods. Corresponding nodes on opposite sides of this strip are identified as the same node. This is made possible by the periodicity of the lattice. It turns out that the edges are seamlessly glued together, and there are no more gluings: the lattice looks internally infinite, the same as without compactification.
A compactified, unidirectionally folded lattice has several advantages over a flat lattice. The main advantage is that the properties of such a folded lattice are determined by its more complex structure in its native higher-dimensional space, while in the lower-dimensional space observed as a result of the folding, only some consequences of these properties appear.
Special interest is given to quasi-periodic lattices. For them, such folding is possible only in special cases when they still have at least one periodic direction.

4.2 Clarification of the concept of information

From the nature of models of this class, as described above, some interesting patterns arise, which we believe it is correct to call *informational* because they can be explained or understood by describing them in terms of information.

Information, in a broad sense, usually refers to several similar, but still different concepts.
Using the term "information" without specifying what is meant mostly does not lead to errors, because this clarification can usually be gleaned from the context. But this does not always happen, so we consider it important, where such confusion can arise, to use different words for different concepts whenever possible.

4.2.1 Entropy

First of all, information often means information entropy, understood as a measure of the uncertainty of the state of some closed system. To calculate the entropy of any system, Shannon's formula

$$H = -\sum_{s \in S} p(s) \log_2 p(s) \tag{4.1}$$

is used, where $p(s)$ is the probability of a state s of this system, and the summation is performed over the complete set S of all possible states of the system.
The closeness of a system here means both the absence (at the time of determining the entropy) of processes influencing the state of the system from the outside (which is necessary for the resulting entropy value to make sense) and the impossibility of somehow knowing the actual state of the system (then this state would be certain with probability $p = 1$, and therefore entropy $H = 0$).
Using the binary logarithm in Shannon's formula, we get entropy measured in bits. Therefore, it can be called "informational" - in contrast to the entropy of statistical mechanics with the Boltzmann constant factor, which is therefore measured in J/K mol.
According to the second law of thermodynamics, the uncertainty of the state of a closed system cannot decrease, that is, its entropy only can increase or remain the same.

4.2.2 Amount of data

The second case of using the word "information" is denoting the amount of specific *data* transmitted to an open system or received from it. For data presented as strings of characters of any alphabet, the amount of data is equal to the length of these strings, that is, the number of these characters in these strings. In this case, to determine the amount of data, it does not matter whether there are any patterns connecting the characters in these strings or not.
In order for the amount of data to be expressed in bits, strings must be made up of bits. If the symbols are not bits, but these symbols can be unambiguously represented as strings of bits, then the amount of data expressed in the number of symbols can be converted into the number of bits and thereby make it compatible in units of measurement with information entropy.
Now let us see how the entropy H and amount of data are connected.

When a hitherto closed system with entropy H opens, specific data in the amount $L <= H$ is transferred to it and it closed again, then the entropy of the system can decrease down to the value $H - L$. This will happen if each data bit from L cancels one of the existing uncertainty bits in H.

4.2.3 Memory size

The third case is the memory size, or the information capacity of the system. This is the maximum amount of data that can be transferred to this system from the outside and received back in undistorted form. The definition given in this way is data dependent, since it is possible to imagine situations where different amounts of data can be recorded in different quantities. Since it is assumed that information capacity is a characteristic of the system itself, we believe that it should be determined through the properties of the system. Apparently, it is adequate to determine the memory size that the system has to be the maximum entropy of this system.

4.3 Principle of local realism in context of model computability

Typically, mathematical modeling of any object or process necessarily includes an abstraction stage. The system is not modeled in its entirety, but only in some aspects that are considered significant. It is assumed that the system still has some degrees of freedom that are not included in the model, but it is also assumed that their influence on its behavior can be neglected and this does not interfere with the achievement of the modeling goal.
We believe that this approach can be applied almost everywhere, except for one special case. Namely, except for modeling at the *fundamental* level, because the goal here is precisely a complete, final description of reality. In our opinion, a model that truly adequately describes nature at this level cannot leave some aspects unimportant - then such a model simply will not be fundamental. The model in this case cannot be less accurate than its prototype.
At the same time, we use the concept of information as a basis for modeling, and computer technology for processing it. Thus, we implicitly claim that a truly fundamental model can be constructed in this way. Hypothetically, the fundamental model is informational, and there is no more primary reality behind such a final model.
This, in our opinion, is equivalent to the assumption that physical reality itself, which is the prototype for such a model, is also purely informational. The result of such reasoning, in our opinion, is a certain list of requirements that must be satisfied by the fundamental information model of objective reality, and which can be called *computability* conditions.

- First, any computation must be completed in a finite number of steps. In particular, the model cannot use infinite summation.
- Secondly, numbers that cannot be accurately represented as information are not suitable for fundamental level modeling. These are real numbers and

more complex constructions that are built on their basis. Usually, to represent real numbers in modern computers, they are approximated by floating-point numbers, that is, rational fractions. The difference between a real number and a fraction is assumed to be practically insignificant. However, for a fundamental model such abstraction is unacceptable. As a consequence of the rejection of real numbers, it turns out to be impossible to apply the concept of continuity and the associated integral and differential calculus and differential equations.

- Thirdly, for each case of information transformation (computation), all input information must be available directly in the location where this transformation occurs. Information from remote locations must travel through intermediate locations before participating in this transformation. The result of the calculation also appears at the transformation location. This limitation corresponds to the physical principle of *locality*.

At the same time, we do not find any basis for the requirement of physical realism, based on informational premises. This requirement usually means that the result of the measurements, that is, the specific data obtained during observation, is really the data that characterizes the observed object (which in this context is a fundamental particle).
In our experience, any data is obtained as a result of a local transformation, the input of which is information from the local environment. Some transformations are irreversible, which means that it is not always possible to get the input having the output information only. Therefore, in our opinion, the principle of realism should not be considered fundamental.
As a result, we come to the conclusion that the principle of local realism for computable fundamental models is reduced only to the principle of locality.
We believe that if we limit ourselves to only computable models, we must accept the need to respect this principle, that is for the lattice can be formulated as follows: *the state of a node can only depend on the state of the nodes that are its immediate neighbors*.

4.4 Influence of symbols' geometrical arrangement on the amount of information

It is well known that information processed in computer technology is usually represented in the form of finite bit strings. The length of such strings is typically fixed because they are stored in hardware memory locations. For definiteness, let's take it as 8 bits.
In strings, each individual bit is in its own place - changing places between them is not allowed. In particular, the computer architecture strictly determines which bit is the least significant and which is the most significant, and for each bit it is known which one is next and which is previous. Therefore, changing each bit in a string leads to a new state of the string, which means there are $2^8 = 256$ such states in total. If there is no reason to consider such states as unequally possible, then the probability of each of them will be equal to $\frac{1}{256}$. One can calculate the maximum entropy (information capacity) of such a string: $H = -\log_2 \frac{1}{256} = 8$ bits.

4.4.1 Copying a string as performing a measurement

Now imagine that we are copying data from one bit string to another one, and in both the location of the bits - low-order, high-order and all the rest - is the same. We will consider this action not from the point of view of a programmer, but in the context of carrying out a measurement in a physical experiment. Let the source string be the object of measurement, and be characterized by a measured physical quantity, determined by its bits. The result of this measurement will be the value placed in the target bit string.
Since the bit arrangement is the same for both the source string and the target, the data can simply be copied bit by bit, and the measurement result will exactly match the data contained in the source string being measured. This is possible due to the one-to-one correspondence between the bits of two strings.

4.4.2 Bit string with undefined ends

Now let's complicate the experiment a little. Let the bit string that we are going to measure be prepared on some another computer, the architecture of which is definitely unknown to us. Even if it is also 8-bit, we cannot say in advance *which end* of the string is the least significant bit and which is the most significant (this situation actually occurred in practice). Now, in order to perform a measurement — a bitwise copy — one needs to decide at which end of the source string to start reading. In other words, to unambiguously read a string — that is, obtain a measurement result — it is necessary to apply 1 additional bit of data, in form of making the choice, to the real data read from the measurement object.
Otherwise, without this additional bit, the read data can be interpreted in *two* ways. For example, if bits 10101010 are read, it could mean either 85 or 170, depending on the direction. In quantum physics, such variants would be called eigenvalues, and the original string would be considered a superposition of them.
Here we need to make a clarification that if the source string is a palindrome, then decisions about the reading order do not need to be made. This corresponds to a degeneracy situation when, due to symmetry, the eigenvalues turn out to be the same. For a string of 8 bits, there are 256 possible options, of which 16 are palindromes, so it turns out that with a probability of $\frac{1}{16}$ a palindrome will be found and 1 bit after reading turns out to be redundant and can be used somewhere else.
An additional data stream will thus come from the measuring device and be mixed with the read data. For each 8-bit string with unknown bit order, an average of $\frac{15}{16}$ bits are needed to restore the correct order. We want to draw special attention to the fact that it is apparently impossible to do without this additional data flow. In order to write a specific number into the target cell, a choice of one of two options must be made, which means the bit for this must be taken from somewhere.

4.4.3 Bit cycles

Let's try another complication of geometry. Let now the same 8 bits be organized in the form of a cyclic string - 8 bits in a ring. With this geometry, it is not only unknown in which direction this ring should be read, but also which of the 8 bits should be read first. Now, to perform a measurement - read 8 bits from the ring and put them into 8 bits in a line - one needs to select one of the 8 bits as the first (this requires an additional $-\log_2 \frac{1}{8} = 3$ data bits) and 1 bit for direction selection. Another thing to consider here is that some data is more symmetrical than others, and for those, selecting a start bit requires fewer additional data bits due to the degeneracy.

4.4.4 Non-string data possess uncertainty

Let's summarize this thought experiment. We draw the following conclusion from it: if the data source has a geometry different from the usual bit string, when reading data from this source into the bit string, it is necessary to apply some additional data to the read data in the amount determined by this geometry. In other words, such a source has additional entropy determined by its geometry.
Now suppose that we have a computer with a standard memory architecture, and we want to represent in this memory a data model with additional entropy (uncertainty) determined by its geometry. It turns out that this is not easy to do: any configuration of data in memory organized as strings of bits *is definite*!
In our opinion, this problem is directly related to the fact that some quantum aspects of nature seem strange from the point of view of both common sense and the bit-string informatics built on the basis of this common sense, focused *only on data without uncertainty*.

4.4.5 A representation of color charge

For example, we could not suggest how to encode the color charge in chromodynamics, using definite bit strings.
Now we believe that the color charge is, in fact, nothing more than a three-bit electric charge having the geometry of a ring, that is, for three bits, an equilateral triangle. All three bits are considered to carry the same charge of $\frac{1}{6}$ of the electron charge, positive (bit 1) or negative (bit 0).
Then only two unambiguous charge options are possible: $+\frac{1}{2}$ (three ones) and $-\frac{1}{2}$ (three zeros), corresponding to leptons [1]. But there are three options of bit triangles with two ones ($+\frac{1}{6}$) and two zeros ($-\frac{1}{6}$) - corresponding to three colors and three anti-colors. And the strong interaction, apparently, is an interaction that depends on the relative arrangement of the color triangles in the interacting gluons and quarks, while the electroweak interaction is one that, for geometric reasons, does not depend on the type and orientation of the color triangle.
Adequately representing such a bit configuration in computer memory, which is string-oriented and therefore deterministic, seems problematic to us. We believe that in order to understand and adequately represent fundamental particles in the form of computer models, a special architecture, focused specifically on this application, is needed.

[1] to get the total electric charge also a weak isospin part must be added

4.4.6 A representation of families

Another example is a geometric combination of defects that could determine the phenomenology of fermion families and the triplet or singlet state of bosons.
Unlike chromodynamic color, these properties have no obvious connection with electrical charge. Therefore, we believe that they are based on charge exchange between two adjacent nodes, because these defects do not change the total charge.
Since the three families are asymmetric in the masses of the corresponding particles, we consider, say, the following combination of five nodes to be suitable.
In this combination there are *three* nodes of the same charge, forming an equilateral triangle, and a *pair* of nodes of different charge, which is inclined relative to the plane of the triangle at some angle, due to which the three nodes of the triangle must be considered as different.
In this configuration, there is only one node (one of the nodes of the pair) that can exchange charge with the other four nodes.
Exchanges with *three* different nodes of the triangle produce *three families*, distinguished from each other by the asymmetrical inclination of the pair with respect to the triangle.
The exchange within the pair corresponds to the transition between the singlet and triplet states of a boson.
We believe that in models [7] where the number of families is equal to powers of two 2^d, (usually $2^2 = 4$ or $2^3 = 8$), this is due to the fact that such models are a priori based on the idea of orthogonal vectors. The lattice constructed on such vectors turns out to be square, cubic or, in the general case, d-hypercubic. In our opinion, the choice of such a geometry is not entirely justified.
The motivation against choosing d-hypercubic lattices as a basis is also, in our opinion, their low efficiency in filling the corresponding d-dimensional spaces.
If our assumption is correct that the measure of the lattice edges connecting neighboring nodes (and the corresponding walls in the domain interpretation) determines the energy, then the hypercubic lattice should be *unstable* and prone to spontaneous decay with the formation of a more efficient lattice which would most likely, have less degrees of freedom than 4 or 8.
As can be seen from our example, in 3-dimensional space a combination of 5 nodes such as an inclined triangular bipyramid is quite possible, providing degrees of freedom sufficient to represent only three families.

4.5 On quantum entanglement

Let's consider an experimental setup for observing quantum entanglement. Let it produce two photons with opposite spins. These two photons are then sent into two different propagation channels and then detected.
Let's calculate the entropy that relates to the spins of these photons at different stages of the experiment, assuming that there is one bit of information per spin degree of freedom.
Two indistinguishable photons with opposite spins can only exist in one way, so the entropy in this case is *zero*.

After separating into two channels, two experimentally different and equally possible situations can occur - spin up in the first channel, spin down in the second, or vice versa - spin down in the first, spin up in the second. So the entropy is *one bit*. This bit results from a split that makes previously indistinguishable photons distinguishable by the number of the channel in which each one propagates.
Now, since the spin measurement is made by two detectors separated by a space-like interval, the result of each individual measurement is one bit of information, for a total of *two bits*.
This situation seems paradoxical because these two bits appear to be connected - there is a correlation between them, despite the space-like interval separating them. From this, a conclusion is sometimes drawn about the alleged non-local mutual influence in this case between photons, which are considered to be in a special entangled state.
We do not see any paradox in such a situation, because the observed result is fully consistent with the result predicted on the basis of the entropy inherent in this situation. This experiment is designed to obtain the measurement result in the form of two bits. But it is performed on a physical system characterized by one bit of entropy. Therefore, it must produce correlated results in these bits.
The obtained measurement results, therefore, should not be considered the real values of the spin that characterized the photon. These results are a combination of information inherent in photons and the influence of the measuring device. Spin itself, if it has non-zero entropy, is not real until it is measured.
If we consider the idea that an individual photon can be in an entangled state with another particle, even located on a space-like interval, then this entails the need to recognize that such a photon is characterized by additional information about which particular particle it is entangled with, and how to find this particle.
In our opinion, this idea arose as a consequence of the development of physical models based on the continuous mathematics of real numbers, which is abstracted from information and therefore fully admits non-local effects.

4.6 Hypothesis about the implementation of spin in a discrete lattice

As a possible explanation of the spin phenomenon, we can propose the idea that the intrinsic angular momentum of particles is due to their propagation not in flat space, but in compactified, that is, curled up, space.
We consider a four-dimensional lattice, which is compactified by folding along one of the crystallographic directions of this lattice with a period T. In this case, the lattice becomes five-dimensional - in the sense that in order to preserve the consideration of a curved lattice in straight Euclidean space, a fifth coordinate has to be introduced to describe it. For example, folding a lattice with period T along the w axis might look like this:

$$(x, y, z, w) \rightarrow (x, y, z, u, v), u = \frac{T}{2\pi} \cos 2\pi \frac{w}{T}, v = \frac{T}{2\pi} \sin 2\pi \frac{w}{T}. \tag{4.2}$$

A cluster of lattice defects, treated as a particle, moves (taking into account discreteness) along a four-dimensional lattice (x, y, z, w) curved by compactification approximately along a geodesic (straight) line, maintaining an inclination with respect to the four-dimensional axes. In a five-dimensional space (x, y, z, u, v), the same movement occurs in a helix, and is decomposed into movement in a straight line in a 3-dimensional subspace (x, y, z) and rotation in a circle with a radius $\frac{T}{2\pi}$ in plane (u, v) perpendicular to it.
The angular momentum normal to the plane (u, v) thus appears to lie in space (x, y, z), but its direction is in principle undefined. The only information it has regarding any vector is the sign of the projection onto it, plus or minus, that is *one bit*. With respect to the direction of propagation this will give two possible helicities, left-handed or right-handed.
In such a scheme, there is no separate intrinsic rotation of the particle - it is determined by the geometry (compactification) of the lattice in which it propagates. We can imagine a natural classification of particles as defect clusters onto bosons, fermions and scalars depending on how many full rotations in (u, v) plane are needed to get them into the original form and orientation.

4.7 Direct propagation requires error correction and compactification

Apparently, propagation along a geodesic, that is, with preservation of direction, is not an obvious or an only possible way of propagation. We believe that the preservation of the original arbitrary direction during propagation in a discrete environment is a consequence of the action of some natural mechanism for correcting errors. This mechanism takes into account the inclination error made in the previous step when selecting a node for the next step. Without it, the spectrum of possible propagation directions would be discrete, or the propagation direction at each step would not be determined at all.
In order for the error correction mechanism, as a kind of computation, to work, it is necessary, as usual, to ensure that the data it uses is local. This condition is satisfied only for a movement in which the lattice is folded, and the movement's component along the folded direction is large enough so that the next node appears to be the same node as the previous node or adjacent to it.
Then the error data from the previous step is made local and may influence the choice of the next step.
Thus, the possible speed of propagation in three-dimensional space turns out to be limited: preservation of the direction, i.e. momentum, occurs only within a certain 4-dimensional cone. The angle between its axis and generatrix depends on the geometry of the lattice. Assuming that the period T in the x, y, z directions is the same as in the w direction, we find that this angle must be within 45 degrees so that in one revolution (T along the w axis) the transition occurs no more than T in the three-dimensional space.
If our assumption is correct, then along with all three-dimensional movements, intense rotations are also always carried out in a plane orthogonal to three-dimensional space. Together they constitute helical motion in 5-dimensional

Euclidean space, and geodesic motion in a curved lattice with its intrinsic four-dimensional geometry.

4.8 Model architectures for various universes

Recently, it has become quite popular to use the concept of the multiverse or multiplicity of worlds to explain some quantum phenomena. And also, there are different opinions about what the past, present and future are.

By experimenting with the representation of the physical world by lattices, we have the opportunity to construct a model lattice so that it corresponds to one or another worldview concept. This gives us the opportunity to experiment and compare these concepts.

We will consider the dynamics in a lattice based on the concept of a cellular automaton. In it, the state of a node at the next step is determined by the state of the same node at the current step, together with the state of its immediate neighbors.

This representation, at first glance, corresponds to the requirement of locality. However, there is still an implicit non-local requirement here, namely the need for synchronization of steps. The calculation of the state of a node at a new step should not begin before the calculation of the state of all nodes at the current step is completed. This kind of synchronization is provided automatically when calculated by a single-threaded computer, when the rule is applied to all sequentially enumerated nodes.

We believe that the single-threaded requirement for a model designed to calculate physical reality is too strict. If we allow calculation in several threads, or even assume that each node is calculated in its own separate thread, then it becomes clear that the condition for synchronizing steps is essentially non-local.

We therefore propose to abandon synchronization and to do this, slightly change the definition of a cellular automaton and remove the concept of steps from it. Now, instead of the state of the same node at different steps, we will say that the state of the node is determined by the state of its immediate neighbors and itself. It turns out that instead of the a priori given direction of calculation from layer to layer, which corresponds to some absolute time, the direction can now only be specified by a rule. Time with this approach turns out to be extraordinary.

Now we can imagine, instead of a sequence of time slices of a d-dimensional lattice, a $d + k$-dimensional lattice in which nodes change their state according to a rule depending on the state of their immediately accessible neighborhood, and this neighborhood covers all neighbors in d+k-dimensional space.

By specifying a $d + k$-dimensional lattice and varying the rule, we get different options for how the simulated world works. Obviously, there can be a lot of options. Let us briefly mention some of them here.

In the case where $k = 1$ and the k-dimensional projection is Z_2 (two points) and the rule depends only on the neighbors from another d-slice, we get an unsynchronized version of the classical cellular automaton. This is a Markov model world in which neither the past nor the future exists, because the nodes lose information about their state, since they are constantly recording the calculated state based on nodes from another slice. When $d = 2$ we can represent the Game of Life.

In order to represent a model with an objectively existing past and/or future, it is sufficient that the 1-dimensional k-projection contains enough d-points to keep information about these past and futures.

We can imagine different versions of the multiverse in this model if we describe a rule leading to the formation of closed components from several d-points, between which the transfer of information occurs. For example, with $k = 1$, the rule in which even d-points depend on their neighbors on the left, and odd ones on their neighbors on the right, leads to a line of classical cellular automata independent of each other, each with its own individual evolution.

4.9 Error correction, making choice and its avoiding

We consider a defect propagation in a discrete environment in which sometimes it becomes necessary to choose one of several equally possible options.

Error accumulation and correction is primarily intended to ensure that the original propagation direction is maintained in a discrete environment. Since a transition in accordance with the principle of locality is possible only between neighboring nodes, movement occurs only in these selected directions. The range of such areas is discrete and very limited.

To prevent this from happening, one can use the following technique. When choosing the next node, somehow save and accumulate the direction deviation error and take it into account when choosing the next node.

It is not difficult to implement such accumulation in a computer model. However, this means that instead of transferring just one bit of data between nodes, much more data must be transferred, namely the accumulated inclination error. This is a significant complication of the original model.

However, there is another problem maintaining the propagation even when error correction is used. Sometimes a situation arises, where there is not enough data to select the next node. This can happen, for example, when propagating in hexagon tilings. There are no straight paths made up of successive edges, and to continue the path it seems obvious that it is necessary to choose the right or left edge.

In order for a choice to be made, any asymmetry is required, some sign on the basis of which such a choice is made. In a computer model, this suggests the use of a random number generator. But in a physical prototype, this will mean that the system is made open and for each decision it requires an influx of data (like a random number or the outcome of tossing a coin) from the *outside*. Without such an influx, it would look like a spontaneous decrease in entropy in a closed system, which, according to the second law, cannot happen.

However, we found out that this problem is 'automatically' solved out in the compactified lattice. When the computation has no enough data to make a choice, it *just does not make a choice*. The propagation branches and then continues in two or three or more threads.

Since for each thread the branching produces an inclination error, the error-correction algorithm tries to correct this error at the next steps, that causes threads to converge back and merge together.
This example, in our opinion, is important because it demonstrates that the defect propagating in the discrete lattice can do that in a various arbitrary directions (inside a cone) being a closed system that does not require any data to be applied to it.

4.10 Conclusion

In general, we find the modeling of the physical vacuum and particles using discrete binary information distributed in a special way very productive, consistent and promising.

References

1. Schoen, Alan H. (May 1970). Infinite periodic minimal surfaces without self-intersections (Technical report). NASA Technical Note. NASA. NASA TN D-5541.
2. S. R. Logan, The Behavior of a Pair of Partially Miscible Liquids, J. Chem. Educ. (1998), 75, 3, 339, https://doi.org/10.1021/ed075p339
3. Hsu, Chia Wei and Starr, Francis W. Interpenetration as a mechanism for liquid-liquid phase transitions. Phys. Rev. E (2009). DOI 10.1103/PhysRevE.79.041502
4. Johan P. Erikson, Partially Miscible Water–Triethylamine Solutions and Their Temperature Dependence, Chem. Educ. (2017), 94, 1, 75–78 https://doi.org/10.1021/acs.jchemed.6b00489
5. A. Gambassi et.al. Critical Casimir effect in classical binary liquid mixtures arXiv:0908.1795v1 (2009)
6. E.G. Dmitrieff: Experience in modeling properties of fundamental particles using binary codes, in: N.S. Mankoč Borštnik, H.B.F. Nielsen, D. Lukman: Proceedings to the 19th Workshop 'What Comes Beyond the Standard Models', Bled, 11. - 19. July 2016.
7. N.S. Mankoč Borštnik: New way of second quantization of fermions and bosons, in: N.S. Mankoč Borštnik, H.B.F. Nielsen, D. Lukman, A. Kleppe: Proceedings to the 24th Workshop 'What Comes Beyond the Standard Models', Bled, July 5–11, 2021.

Proceedings to the 26th Workshop
What Comes Beyond . . . *(p. 50)*
Bled, Slovenia, July 10–19, 2023

5 Dispersion of Nonrelativistic and Ultrarelativistic Wave Packets on Cosmic Scales

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Abstract. For relativistic quantum mechanics, the problem of the dispersion of wave packets on cosmic scales has quite universal significance for the interpretation of the arrival times of high-energy cosmic rays. Indeed, in view of inevitable dispersion of quantum mechanical wave packets describing massive particles, one might ask to which extent the detection time on Earth is influenced by the quantum mechanical dispersion and thus acquires an additional (potentially significant) uncertainty when compared with the creation time of the particle somewhere else in the cosmos. Here, we thus ask the question: Which time variation is to be expected for the arrival of high-energy cosmic rays due to the dispersion of the wave packets, with a special emphasis on the ultrarelativistic limit? We discuss both the nonrelativistic and relativistic limits and find drastic differences.

Povzetek: Pri določanju časa potovaja visokoenergijskih kozmičnih žarkov na kozmičnih razdaljah je pomembno oceniti vlogo disperzije valovnih paketov, s katerimi opisujemo masivne delce. Avtorja ocenjujeta vpliv disperzije na čas potovanja kozmičnih žarkov pri delcih z energijami v nerelativistični limiti in ga primerjata z vplivom disperzije na čas potovanja pri delcih z zelo velikimi, ultrarelativističnimi energijami. Ugotavljata, da je razlika drastična.

5.1 Introduction

Let us start the considerations on the basis of an analogy, namely, that of a quantum-mechanical wave packets and a herd of cows. While this analogy is far from being perfect, the modeling of quantum mechanical wave packets on the basis of an ensemble of classical trajectories has been used as a means of modeling quantum dynamics, notably, for atomic electrons driven by lasers (see Refs. [1–4] and references therein). Also, Bohmian trajectories have been used in order to model atom-surface diffraction effects [5]. Free-particle Bohmian trajectories have recently been studied in Ref. [6]. Within a herd of cows, it is clear that the faster members of the herd will arrive at the stable ahead of the slower cows, even if the entire herd was released at the same time, consistent with a dispersion of the herd. Within a semiclassical approximation, a quantum mechanical wave packet can be thought of as an ensemble of classical particles, which disperse in space on their trajectories much like a herd of cows does.
The question naturally arises: What consequences follow from the dispersion of the wave packet, for particles on their way through the cosmos? How can we be so

sure that a high-energy proton, or, a high-energy neutrino (for example, “Big Bird” detected by IceCube (Refs. [7,8]), really emerged from a distant galaxy at a defined time? This question also is interesting with respect to the “early” arrival of the neutrinos from the 1987A supernova [9], in regard to possible Lorentz-violating effects for high-energy neutrinos [10–12].

5.2 Nonrelativistic Formalism

Cosmic rays consist of relativistic particles. However, for comparison, it is interesting to study quantum mechanical dispersion in a nonrelativistic formalism. Following Ref. [13], we propose a reference wave function, based on a momentum-space envelope, whose time evolution will be studied. One of the simplest envelope functions is a Gaussian one,

$$f(p) = \frac{(2\pi)^{1/4}}{\sqrt{\delta p}} \exp\left(-\frac{(p-p_0)^2}{4\,\delta p^2}\right) , \tag{5.1}$$

where p_0 is the expectation value of the momentum, δp measures the uncertainty (width) in momentum space. The distribution is normalized to unity,

$$\int_{-\infty}^{\infty} \frac{p}{2\pi} |f(p)|^2 = 1 . \tag{5.2}$$

The momentum uncertainty is

$$\delta p = \sqrt{\langle p^2 \rangle - \langle p \rangle^2} . \tag{5.3}$$

A solution of the one-dimensional, free-particle time-dependent Schrödinger equation is

$$\Phi(t,x) = \int_{-\infty}^{\infty} \frac{p}{2\pi} f(p) \exp\left(p\,x - \frac{p^2}{2m} t\right) , \tag{5.4}$$

which satisfies

$$\frac{\partial}{\partial t}\Phi(t,x) = -\frac{\partial^2}{\partial x^2}\Phi(t,x) . \tag{5.5}$$

Here and in the following, we work in a unit system with $\hbar = c = \epsilon_0 = 1$, unless explicitly stated otherwise.
One calculates the expectation values,

$$\int x\,|\Phi(t,x)|^2 = 1 , \tag{5.6}$$

$$\langle x(t) \rangle_{\rm NR} = \int x\,x\,|\Phi(t,x)|^2 , \tag{5.7}$$

$$\langle x(t)^2 \rangle_{\rm NR} = \int x\,x^2\,|\Phi(t,x)|^2 , \tag{5.8}$$

where NR stands for the nonrelativistic limit. The integrals are evaluated by first considering the x-integral, with the help of the formulas

$$\int x\, x \, \exp\left((p - p')x\right) = -\frac{\partial}{\partial p}\delta(p - p')\,, \tag{5.9}$$

$$\int x\, x^2 \, \exp\left((p - p')x\right) = -\frac{\partial^2}{\partial p^2}\delta(p - p')\,. \tag{5.10}$$

The ensuing momentum-space integrations can be carried out using standard formulas, with the result

$$\langle x(t)\rangle^2_{\rm NR} = \frac{p_0^2\, t^2}{m^2}\,, \tag{5.11}$$

$$\langle x(t)^2\rangle_{\rm NR} = \frac{p_0^2\, t^2}{m^2} + \frac{1}{4\delta p^2} + \frac{\delta p^2\, t^2}{m^2}\,. \tag{5.12}$$

This means that

$$\delta x(t)^2_{\rm NR} = \langle x(t)^2\rangle_{\rm NR} - \langle x(t)\rangle^2_{\rm NR} = \frac{1}{4\delta p^2} + \frac{\delta p^2\, t^2}{m^2}\,. \tag{5.13}$$

At the initial time $t = 0$, the nonrelativistic Gaussian wave packet fulfills the minimal uncertainty relation, $\delta x(t)_{\rm NR}\, \delta p = 1/2$. For large propagation times, one can approximate

$$\frac{\delta x(t)_{\rm NR}}{c} \approx \frac{\delta p}{mc}\, t\,, \tag{5.14}$$

i.e., the positional uncertainty grows linearly with time. In Eq. (5.14), we have restored the factor c (speed of light), reverting back to SI (Système International) units temporarily. We have divided out a factor of c in order to facilitate a comparison with the ultrarelativistic limit, which will be considered in the next section.

5.3 Ultrarelativistic Formalism

We consider ultrarelativistic spin-1/2 particles, described by the Dirac equation,

$$(\gamma^\mu \partial_\mu - m)\, \psi(x) = 0\,, \tag{5.15}$$

where $x = x^\mu = (t, \vec{r})$ is the space-time coordinate, and γ^μ are the Dirac gamma matrices, which we use in the Dirac representation,

$$\gamma^0 = \begin{pmatrix} \mathbb{I}_{2\times 2} & 0 \\ 0 & -\mathbb{I}_{2\times 2} \end{pmatrix}, \qquad \vec{\gamma} = \begin{pmatrix} 0 & \vec{\sigma} \\ -\vec{\sigma} & 0 \end{pmatrix}, \tag{5.16}$$

with the Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 0 & - \\ & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{5.17}$$

The solutions which describes negative-helicity, positive-energy particles, for $\vec{p} = p_{xx} = p_x$, is [13]

$$\psi(t,x) = \frac{1}{2}\begin{pmatrix} -\sqrt{(E+m)/E} \\ \sqrt{(E+m)/E} \\ \sqrt{(E-m)/E} \\ -\sqrt{(E-m)/E} \end{pmatrix} \exp\left(-E\,t + p\,x\right)\,, \tag{5.18}$$

and it is normalized to unity, $\int x\,|\psi(t,x)|^2 = 1$. The energy is $E = \sqrt{p^2+m^2}$. A normalizable wave-packet is given by

$$\psi(t,x) = \int \frac{p}{2\pi}\frac{f(p)}{2}\begin{pmatrix} -\sqrt{(E+m)/E} \\ \sqrt{(E+m)/E} \\ \sqrt{(E-m)/E} \\ -\sqrt{(E-m)/E} \end{pmatrix} \exp\left(-E\,t + p\,x\right)\,, \tag{5.19}$$

where the Gaussian weight is given in Eq. (5.1). In the ultrarelativistic limit, the solution takes the form

$$\psi(t,x) \approx \int \frac{p}{2\pi}\frac{f(p)}{2}\,\underline{u}\exp\left(-\sqrt{p^2+m^2}\,t + p\,x\right)\,, \tag{5.20}$$

where $\underline{u} = (-1,1,1,-1)^T$. This means that each component of the bispinor satisfies the Klein-Gordon equation,

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} - m^2\right)\psi(t,x) = 0 \tag{5.21}$$

individually. We can thus derive the dispersion of the wavepacket by only focusing on the scalar wavepacket given by

$$\psi(t,x) = \int \frac{p}{2\pi}\,f(p)\,\exp\left(px - \sqrt{p^2+m^2}\,t\right)\,, \tag{5.22}$$

which is the relativistic form of (5.4). In order to find the dispersion of the wavepacket, by calculating the variance, we need to calculate $\langle x(t)\rangle$ and $\langle x(t)^2\rangle$, defined by

$$\langle x(t)\rangle = \int x\,x\,|\Psi(t,x)|^2\,, \qquad \langle x(t)^2\rangle = \int x\,x^2\,|\Psi(t,x)|^2\,. \tag{5.23}$$

One can perform the integrals for $\langle x(t)\rangle$ and $\langle x(t)^2\rangle$, first by integrating over x on the basis of Eqs. (5.9) and (5.10), and then, assuming a peaked function $f(p)$ around the region $p \approx p_0$, over p. One gets the expectation value of the mean-square position as [13]

$$\begin{aligned}\langle [x(t)]^2\rangle = {} & \frac{1}{4\delta p^2} + t^2 - \frac{m^2\,t^2}{p_0^2} + \frac{m^4 - 3m^2\delta p^2}{p_0^4}\,t^2 \\ & + \frac{10m^4\delta p^2 - 15m^2\delta p^4 - m^6}{p_0^6}\,t^2 \\ & + \frac{m^8 - 21\,m^6\,\delta p^2 + 105\,m^4\,\delta p^4 - 105\,m^2\,\delta p^6}{p_0^8}t^2 + \mathcal{O}(p_0^{-10})\,,\end{aligned} \tag{5.24}$$

and the square of the expectation value of the position as

$$[\langle x(t)\rangle]^2 = t^2 - \frac{m^2\, t^2}{p_0^2} + \frac{m^4 - 3m^2\delta p^2}{p_0^4}\, t^2 + \frac{9m^4\delta p^2 - 15m^2\delta p^4 - m^6}{p_0^6}\, t^2 + \frac{m^8 - 18\, m^6\, \delta p^2 + \frac{177}{2}\, m^4\, \delta p^4 - 105\, m^2\, \delta p^6}{p_0^8} t^2 + \mathcal{O}(p_0^{-10})\,. \quad (5.25)$$

The variance is

$$\delta x(t)^2 = \langle [x(t)^2] \rangle - [\langle x(t)\rangle]^2 = \frac{1}{4\delta p^2} + \frac{m^4\, \delta p^2\, t^2}{p_0^6} + \left(\frac{33m^4\, \delta p^4}{2p_0^8} - \frac{3m^6\, \delta p^2}{p_0^8}\right) t^2 + \mathcal{O}(p_0^{-10})\,. \quad (5.26)$$

This result has been derived in the ultrarelativistic limit, for $m/p_0 \to 0$, and $\delta p/p_0 \to 0$.
One can show [13] that the result also holds if one assumes a large propagation time t in the ultrarelativistic limit,

$$\frac{\delta x(t)_{\mathrm{UR}}}{c} \approx \frac{m^2\, c^2\, \delta p}{p_0^3} t\,, \quad (5.27)$$

where UR stands for the ultrarelativistic limit. Just as in Eq. (5.14), we temporarily restore SI units.

5.4 Cosmic Limit

Let us recall the formulas (5.14) and (5.27) for the nonrelativistic and ultrarelativistic dispersion effects,

$$\frac{\delta x(t)_{\mathrm{NR}}}{c} = \frac{\delta p}{mc}\, t\,, \qquad \frac{\delta x(t)_{\mathrm{UR}}}{c} = \frac{m^2\, c^2\, \delta p}{p_0^3} t\,, \quad (5.28)$$

where we use the equal sign as opposed to the approximate equality, merely as a definition. The ratio is

$$\frac{\delta x(t)_{\mathrm{UR}}}{\delta x(t)_{\mathrm{NR}}} = \frac{m^3\, c^3}{p_0^3}\,, \quad (5.29)$$

expressing the fact that, in the ultrarelativistic limit ($p_0 \gg m\, c$), dispersion effects are significantly suppressed.
For particles traveling at a speed close to the speed of light, the quantity $\delta x(t)/c$ measures the uncertainty in the terrestric arrival time (detection time) of a high-energy particle coming in from the cosmos. The drastic difference between the nonrelativistic and relativistic effects can be illustrated on the basis of a numerical example.
Let us choose as the cosmic travel time an interval of 168,000 light years, which is the distance to the Large Magellanic Could, where the supernova 1987A origi-

nated [9]. One finds

$$\left.\frac{\delta x_{\rm NR}(t)}{c}\right|_{t=168{,}000\,{\rm yr}} \approx 5.529 \times 10^{+21}\,\frac{\delta\xi}{\chi}\,{\rm s}\,, \tag{5.30}$$

$$\left.\frac{\delta x_{\rm UR}(t)}{c}\right|_{t=168{,}000\,{\rm yr}} \approx 5.298 \times 10^{-6}\,\frac{\delta\xi}{\xi}\,\left(\frac{\chi}{\xi}\right)^2\,{\rm s}\,. \tag{5.31}$$

Here, "s" is the symbol for the SI unit "second", $\delta\xi$ is the momentum spread in GeV/c, ξ is equal to the central momentum p_0 in GeV/c, and χ is the mass of the particle, measured in eV/c^2. For the nonrelativistic dispersion relation, extrapolated to large velocities, the result (5.30) implies that if $\delta\xi$, expressed in terms of the eV/c (not GeV/c), has the same numerical value as χ, expressed in terms of the eV/c, then the uncertainty in the arrival time would be 168,000 years, making the attribution of the oncoming cosmic ray to a particular event absolutely impossible. However, the result (5.31), in the ultrarelativistic limit, implies the contrary: Namely, if the particle wave function is centered about a well-defined ultrarelativistic mean momentum $p_0 \gg m$ (i.e., $\delta\xi/\xi \ll 1$ and $\chi/\xi \ll 1$), then the detection time uncertainty amounts to less than a microsecond even for cosmic travel over appreciable distances (here, as an example, the distance to the Large Magellanic Could).

5.5 Conclusions

We have derived the formulas (5.14) and (5.27) for the dispersion of nonrelativistic and ultrarelativistic wave packets on cosmic scales. For cosmic time scales commensurate with the arrival of cosmic rays from the Large Magellanic Cloud, we have obtained the estimates given in Eqs. (5.30) and (5.31). A numerical evaluation shows that, were it not for relativity, then the dispersion of a quantum-mechanical wave packets would make it impossible to associate a cosmic event with a well-defined arrival time of a cosmic ray. Or, in order to pick up the idea formulated in Sec. 5.1, one can say that the speed difference among a herd of "nonrelativistic cows" modeling the nonrelativistic wave packet is simply too pronounced in order to hold the herd together.

By contrast, relativity "shepherds" the cows toward the velocity of light in the ultrarelativistic limit, which prevents a significant dispersion of the wave packet, even on impressive cosmic time scales. In this context, we recall that light pulses traveling through the cosmos (empty space) show no dispersion at all, because their group and phase velocities are both equal to the speed of light. This conclusion is ramified on the basis of the numerical examples considered in Sec. 5.4.

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References

1. G. G. Paulus, F. Grasbon, H. Walther, R. Kopold, and W. Becker, *Identification and Application of Quantum Trajectories in Above-Threshold Ionization*, Laser Phys. **12**, 262–267 (2002).
2. N. Takemoto and A. Becker, *Visualization and interpretation of attosecond electron dynamics in laser-driven hydrogen molecular ion using Bohmian trajectories*, J. Chem. Phys. **134**, 074309 (2011).
3. L. He, Y. Li, Z. Wang, Q. Zhang, P. Lan, and P. Lu, *Quantum trajectories for high-order-harmonic generation from multiple rescattering events in the long-wavelength regime*, Phys. Rev. A **89**, 053417 (2014).
4. A. Jorge, M. Horbatsch, and T. Kirchner, *Classical-trajectory model for ionizing proton-ammonia molecule collisions: the role of multiple ionization*, e-print arXiv:2309.16083v1.
5. R. Guantes, A. S. Sanz, J. Margalef-Roig, and S. Miret-Artés, *Atom–surface diffraction: a trajectory description*, Surface Science Reports **53**, 199–330 (2004).
6. J. S. Briggs, *Trajectories and the perception of classical motion in the free propagation of wave packets*, Nat. Sci. **2**, 20210089 (2022).
7. M. G. Aartsen *et al.* [IceCube Collaboration], *First Observation of PeV-Energy Neutrinos with IceCube*, Phys. Rev. Lett. **111**, 021103 (2013).
8. M. G. Aartsen *et al.* [IceCube Collaboration], *Observation of High-Energy Astrophysical Neutrinos in Three Years of IceCube Data*, Phys. Rev. Lett. **113**, 101101 (2014).
9. V. L. Dadykin, G. T. Zatsepin, V. B. Karchagin, P. V. Korchagin, S. A. Mal'gin, O. G. Ryazhskaya, V. G. Ryasnyi, V. P. Talochkin, F. F. Khalchukov, V. F. Yakushev, M. Aglietta, G. Badino, G. Bologna, C. Castagnoli, A. Castellina, W. Fulgione, P. Galeotti, O. Saavedra, J. Trincero, and S. Vernetto, *Detection of a rare event on 23 February 1987 by the neutrino radiation detector under Mont Blanc*, JETP Lett. **45**, 593–595 (1987), [Pis'ma ZhETF **45**, 464 (1987)].
10. G. Somogyi, I. Nándori, and U. D. Jentschura, *Neutrino Splitting for Lorentz–Violating Neutrinos: Detailed Analysis*, Phys. Rev. D **100**, 035036 (2019).
11. U. D. Jentschura, I. Nándori, and G. Somogyi, *Lorentz Breaking and* $\mathrm{SU}(2)_{\mathrm{L}} \times \mathrm{U}(1)_{\mathrm{Y}}$ *Gauge Invariance for Neutrino Decays*, Int. J. Mod. Phys. E **28**, 1950072 (2019).
12. U. D. Jentschura, *Squeezing the Parameter Space for Lorentz Violation in the Neutrino Sector by Additional Decay Channels*, Particles **3**, 630–641 (2020).
13. J. Nicasio and U. D. Jentschura, *Dispersion of Ultra–Relativistic Tardyonic and Tachyonic Wave Packets on Cosmic Scales*, Symmetry **14**, 2596 (2022).

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6 Properties of fractons

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Abstract. The term 'fracton' was originally proposed to denote free colorless fractionally-charged-massive-particles (FCHAMPs). Later this term has appeared and became widely used in a different meaning in condensed matter and other areas of physics, but here we use it in its original particle physics sense. We discuss the physical nature and particle models, predicting fractons of different type: hadronic fractons (which can be accompanied with the existence of new-long-range interactions), and leptonic fractons, which have no hadronic interaction. Experimental searches for different types of fractons, their methods and results are briefly reviewed.

Povzetek: Izraz "frakton" je bil prvotno predlagan za proste delno nabite masivne delce. Kasneje so isti izraz začeli uporabljati v fiziki trdne snovi, v drugačnem pomenu. Avtorja uporabljata izraz fracton v izvirnem pomenu. Študirata fizikalne lastnosti fraktonov v različnih skupkih: za hadronski frakton (ki pripelje do spoznanja o novih interakcijah dolgega dosega) in za leptonski frakton (ki nima hadronske interakcije). Avtorja predstavita različne vrste fraktonov, modele, s katerimi jih opisujeta in rezulate obravnav.

6.1 Introduction

The term 'fracton' was originally proposed in [1] to denote fractionally charged colorless particles, which can appear in models of mirror matter, if there exist states, which have different mirrority of their electroweak (EW) and QCD charges. If such a state has ordinary EW charge and mirror QCD, or mirror EW charge and ordinary QCD charge, its binding correspondingly by mirror QCD or ordinary QCD confinement would lead to free fractionally charged particles (see [3,3] for details and references). Later the term 'fracton' has independently appeared in other sense in various applications of condensed matter, gauge and lattice theory (see e.g. [4–7] for details and references), but here we us this term in its original particle physics sense of Fractionally Charged Massive Particle (FCHAMP). FCHAMP's can be looked for in Galactical matter, in matter around celestial bodies (i.e. the Sun), in meteoritic matter and within the Earth matter.
The main experimental techniques for the search for FCHAMP's can consists of accelerator experiments, cosmic-rays experiments, mass-spectrometer experiments,

cantilever experiments and experiments involving the gravitational interaction as well. FCHAMP's are inscribed within the searches for new long-range forces. For this aim, the types of FCHAMP's are enumerated; furthermore, FCHAMP's from supersymmetric theories are explained through the mechanisms of symmetry breaking.
The Experimental searches for FCHAMP's are developed through accelerator experiments, cosmic-rays experiments, mass-spectrometer experiments, and gravitational-interaction experiments. FCHAMP's can recast as particles-recombinations products; more in detail, theoretical frameworks of recombination products are introduced; and experimental searches of recombination products are presented.
Prospective investigations are envisaged, from both a theoretical point of view, and an experimental one.
The paper is organised as follows.

6.2 Search for new long-range forces

6.2.1 The G_Y interaction

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Within the G_Y interaction [8], particles can be divided as: standard-model particles ($_o$-particles. i.e. photons, gluons, intermediate vector bosons V_o, and quarks and leptons fermions F_o); and new long-range-interaction $_Y$-particles, i.e. gauge fields V_Y (which do not interact with the $_o$-particles), F_Y $_Y$ fermions. More in detail, $_x$ particles interact with both V fields, and $_o$-particles and $_Y$-particles have the phenomenology of a common gravitational interaction.
The search for $_Y$-matter can be outlined from a host of instances, such as: gravitational radiation from a system of double $_Y$-stars, $_Y$-matter near the Sun, and $_Y$-spheres in the Earth system.
Combination of $_x$-matter and $_Y$-matter lead by forces not weaker than those derived by the usual chemical forces, i.e. such as - $_Y$-matter on the Earth.

6.2.2 Fractionally-charged particles

FCHAMP's [1] arise from the unified gauge symmetry group

$$G_{OXY} \equiv G_{W\ \_S} \times SU(3)_c + G_Y : \tag{6.1}$$

$G_{W\ \_S}$ is defined as $G_{W\ \_S} = SU(2) \times U(1)$, $SU(3)_c$ is the strong color interaction, and the G_Y Y interaction.

6.2.3 Breaking of G_Y

After the breaking of G_Y to G_{Yem}, i.e. the Y electromagnetism, FCHAMP's are X-hadrons produced in

- hadronic processes after the two-gluons mechanism described as in [1] as

$$gg \to x\bar{x}; \tag{6.2}$$

- hadronic processes at high energies;
- XXX hadrons consisting of light 0-quark pairs, for which the contribution of X quark pairs being negligible;
- Y hadrons produced from a symmetry group G_{0XY} as

$$G_{0XY} \equiv G_{W\ \_S} \times SU(3)_c + G_Y \times G_{Yc} \tag{6.3}$$

  which can be looked for after
  - hot phase of the Early Universe; and
  - the position that lower limit of the quark abundance greater than that for relic quarks.

6.3 FCHAMP's from Supersymmetric models: experimental observations

Fractionally-charged lepton) have been proposed also as a probe for supersymmetric theories [9].
In particular, asymptotic freedom is required [10]: after the requirement, fractons arise in supersymmetric GUT's (or GUT's) in the trivial embeddings of $SU(7)$ in $SU(N)$ and in $O(14)$.
The related quarks $\tilde{q}$ with charge $q = \frac{1}{3}e$ arise from models of supersymmetric GUT's such as nontrivial embeddings of $SU(5)$.

It is not possible for E_6 to produce FCHAMP's.

From nontrivial embeddings of $SU(5)$, at least one fracton is expected to be absolutely stable, i.e. as resquested from Earth-based experiments on niobium spheres [10].

6.3.1 Astrophysical observation

From Astrophysical observation, i.e. such as those scrutinised in [11] FCHAMP's can be considered in annihilation processes after the Big Bang, and on the present observed upper bound on their number density in matter; in particular, stable FCHAMP's heavy $q = \frac{1}{3}$ leptons could bind to the p H atoms and produce a shift in the spectral lines of the infrared spectra of Population II (i.e. metal-poor) stars, with concentrations of one part in 10^8.

$\frac{1}{3}e$-charged heavy leptons can annihilate in heavy stars down to an abundance of 10^{-19}.

The masses of FCHAMP's on order of the mass scale of the breaking of the electroweak group and/or at $50 - 100GeV$ ought to be evaluated in $\bar{p}p$ and e^+e^- experiments.

6.4 Electroweak but not strong interactions

There exist potentially unconstrained regions of the parameter space in the (Q_L, m_L) plane available for FCHAMP's with electroweak interaction in the cosmic rays spectra, with Q_L (fractional) electric charge and m_L lepton mass.
As a consequence, constrains on Q_L are obtained in accelerator experiments [12].

6.5 Two-hadrons recombination experimental searches

Two-hadrons recombination experimental searches are outlined within the directions of search of fraction quarks and recombination of two hadrons $(\bar{h}h)$.
Therefore, the processes can be studied also in quark experiments, among which accelerators experiemnts and Cosmic rays ones are chosen in [13].

6.6 Accelerators experiments

Accelerator experiemnts are aimed at studying the limits on fractional-charge particle production from stable-matter search techniques, such as Plastic-detector experimens, Čerenkov-effect experiments, and emulsion experiments (after some symmetry-breaking mechanisms).

6.7 Cosmic-rays experiments

The purpose of Cosmic-rays experiments is to analyse the primary cosmic ray flux; such flux is hypothesised to be produced after interactions of very highly-energetic cosmic rays with the atmosphere.
More in detail, cosmic ray telescope $250m$ underground was designed, to look for magnetic monopoles or for fractionally-charged particles (through the measure of the velocities and energy losses for isolated tracks) [14], [15] [16], [17].
As results, non-relativistic particles $3.5 \cdot 10^{-4} < \beta < 0.4$ are looked for; min particular:

- particles with charge $\frac{2}{3}$: $6 \cdot 10^{13}$ $particles\ cm^{-2}\ sr^{-1}\ s^{-1}$; and
- particles with charge $\frac{1}{3}$: $6 \cdot 10^{-4}$ $particles\ cm^{-2}\ sr^{-1}\ s^{-1}$.

Furthermore, relativistic particles are researched as

- particles with charge $\frac{2}{3}$: $2 \cdot 10^{-12}$ $particles\ cm^{-2}\ sr^{-1}\ s^{-1}$.

6.7.1 Cosmic-rays experiments: more FCHAMP's

The purposes of cosmic-rays experiments are to find isolated quarks unaccompanied by other particles processes; delayed air showers (which can reveal delayed hadrons); and $\frac{4}{3}$e-charged particles which can be detected by drift chambers at zenith angles of 31° to 49° estimated via its energy loss. In the latter case, the flux of $6(4.0 \pm 1.5) \cdot 10^{-9}$ $particles\ cm^{-2}\ sr^{-1}\ s^{-1}$ is looked for: lack of events for $\frac{1}{3}$e-charged particles 90% and for $\frac{2}{3}$e-charged particles is recoreded [18] and related references.

6.7.2 Comparison of cosmic-rays experiments

From [13], the stable matter is searched for by mass spectrometer.
The residual charges $\frac{1}{3}e$ are researched in steel balls: as a reuslt, at the 90% confidence level, $< 10^{-21}$ quarks per nucleon there can exist [19].

6.8 Search for FCHAMP's on Earth matter

The search for stable matter on the Earth conists in looking for probes that, for FCHAMP's, the resulting nucleosynthesis would be different from that of normal nuclei, i.e. FCHAMP's ought to be revealed between; normal Na and heavy Na inside the Earth crust; in the manufacture of the Na sample used; or in the production of the atomic beam(s) [20].

6.9 Fractionally-charged particles in meteors

The experimental validation of Fractionally-charged particles in meteors can be achieved after scrutinizing the results of $< 1.3 \cdot 10^{-21}$ particles per nucleon in meteoritic material; and $< 1.9 \cdot 10^{-23}$ particles per nucleon in meteoritic mineral oil [21].

6.10 Upper limits for the gluon mass

After taking into account an upper limit for the gluon mass too, m_g, $m_g < O(1)\ Mev$, it is found to be
very small in comparison to Λ_{QCD}; furthermore, ultraviolet cut-off of Λ_{UV} in QCD with $m_g \neq 0$, considered as an effective field theory, has a very different form from the ultraviolet cut-off in the electroweak theory with heavy Higgs boson(s) [22].

6.11 Cantilever-type experiments

Cantilever-type experiments can be set for the detection of Sparticles in a weak gravitational field: mass dispersion relation Δm_{ij} for masses m_{ij} are theorised as

$$\frac{\Delta m_{jk}^2}{m_0^2} = \frac{\lambda_j \lambda_k^*}{\pi^2} \ln \frac{M_{Pl}}{M_G}, \tag{6.4}$$

where λ_i factorizes the (requested) coupling constant, m_0 is the mass of the common (standard-model) scalar (normalized to Planck mass M_{Pl}), and M_G is the mass for a (massive) gravitational mode.

The investigated scheme is one in which

- after the breach of higher-dimensional structures, non-perturbative degrees of freedom give rise to Compton-length waves (particles) whose masses M_C are comparable with Planck mass M_{Pl}
- the particles interact very weakly and gravitationally;
- masses M_C are of order $M_C \simeq R/M_{Pl}$;
  with R the lower bound on the compactification (energy) scale:
  their gravitational interaction can modify ordinary Newtonian gravity;
- verifications of M_C can be achieved by cantilever detectors and/or silicon-based microelectromechanical systems.

6.12 Fractional-charges detectors

An instrument aimed at detecting fractional-charge particles is the rotor electrometer [23].
It was designed as a Faraday container with an arbitrary high-impedance amplifier, endowed with copper pads, for which different charges reach the container walls at different velocities, such that the time of flight can be calculated, i.e., after a tuning the impedance suited for the charge to be detected.

6.13 FCHAMP's in Complex orbifolds

The existence of fractional quantum numbers n has also been postulated for complex orbifolds [24].

6.14 $_x$-partilces in matter and in biological samples

After a further assumptions that $_x$-particles should have any kinds of Abelian charges, and that the coupling constant to the corresponding V_Y-photonic fields is of the order of α, then the detection of $_x$-particles in matter and in biological samples should be analyzed by van der Waals energy scales to fix an upper bound for them, which should improve Cavendish-type experiments by and order 10^{-3}.

6.15 FCHAMP's detection after Gravitational-interaction experiments

The possibility of fractionally-charged matter after gravitational interaction can be thoeretized from the early development of Active Galactic Nuclei; from the reconstruct a continuum image of the nucleus as far as the dust emission is concerned for near-infrared baseline-interferometric data; and for the dust-sublimation regions. As a consequence, any discrepancies from the found data could be reconducted to the presence of fractionally-charged dust matter, as form [25].

6.16 Search for FCHAMP's in baryonic matter

The exploration of QCD matter at neutron star core densities (compressed baryonic matter) is evinced from fluctuations and event characterization [28].

6.17 High-energy hadronic processes and quarks quantum numbers

High-energy hadronic processes for quarks quantum numbers can be looked for within resonance production in proton-proton collisions at $\sqrt{s} = 7TeV$ and $\sqrt{s} = 13TeV$
($p - p$ collisions and $p - Pb$ LHC), as described in [27].
The processes to be investigated comprehend

- hadronization in the partonic phase;
- ratio for baryonic resonances to non-resonance baryons having similar quark content; and
- ratios found to be independent of the collision energy of the system.

6.18 Anomalous lepton-lepton scattering

Electron-electron scattering and electron-positron scattering at $0.6 - 1.7MeV$ can be a playground of exploration of anomalous lepton-lepton scattering. For this purposes, at the chosen cross-sections of $0.61MeV$, the observed phenomena were significantly smaller than the predicted values.
The theoretical framing of these results [28] was framed within departure from quantum electrodynamics and/or possibility for a non-Coulombian central force.

6.19 Fractional-electric-charged-particles: ab initio derivation, calculations, experimental verifications

Fractional-electric-charged-particles can be due either to a
- broken color symmetry; or to an
- 'enlarged' GUT theory to which color singlets can be included.
According to these two mechanisms, FCHAMP's are expected to acquire $1/3$ electric charge.
In an **unbroken color symmetry**, an $SU(5)$ model which allows for FCHAMP's in

$S(7)$ non-trivial change of embedding can be considered: such a non-trivial change of embedding consists of
- 2 normal $SU(5)$ families, and
- 2 charge-shifted conjugated families. The latter must be light ($\lesssim 100GeV$) for them to acquire muss only after the breaking of $SU(2) \otimes U(1)$.
Charge-shifted families contain both fractionally-charged leptons and fractionally-charged quarks, with charges q not in sequence

$$q \equiv (\frac{2}{3} + n),$$

with n integer.
Here, **a fracton can be either a lepton or a hadron**.

The **broken** $SU(7)$ split as

$$SU(7) \rightarrow SU(3) \otimes SU(2) \times U(1)$$

- at super-heavy mass scales, is obtained by scalars in the irreps of high dimensionality; and
- cannot couple directly with fermions.
Given the Lagrangian model L_0, which is invariant under super-gauge transfor-

mations in the one-loop approximation, and is written as

$$L_0 = -\frac{1}{2}(\partial_\mu A)^2 - \frac{1}{2}(\partial_\mu B)^2 - \frac{1}{2} i\bar{\psi}\gamma^\mu \partial_\mu \Psi + \frac{1}{2}F^2 + \frac{1}{2}G^2$$

, with A scalar, B pseudoscalar, ψ Majorana particle, F and G auxiliary fields, conformal transformations and γ^5 transformations in the algebra imply that only theories with massless particles can be invariant under supergauge transformations. Accordingly, it is possible to eliminate the pertinent commutators and anticommutators, as from [30].
Differently, it necessary to restrict to the consequent sub-algebra to remove massless particles, i.e. and to obtain super-gauges with constant parameters.
The procedures are discussed in [29].
Given the invariant Λ

$$\Lambda \equiv \lambda F,$$

the auxiliary fields can be eliminated from the Lagrangean: it is this way possible to study of higher order corrections, and to construct more complex and more realistic models, invariant under a combination of super-gauge symmetries and internal ones.

6.20 Broken colour $SU(3)$

A **broken color** $SU(3)$ implies the presence of quarks, and that of (so-called) *'diquarks'*.

The observed charges are looked for as the color singlet $\underline{1}_c$-particles with fractional charges.

• This implies a kind of GUT of electroweak interactions and strong interactions, which equals to minimal extensions of $SU(5)$ to include fractionally-charged color singlets.

• The model $SU(6)$ does not fit the requirements for such constructions: the mixing parameter $\sin^2\theta_w$ implies the presence of many light doublets to be added on purpose.

This procedure is discussed in [31].

6.20.1 Constraints on the obtention of FCHAMP's

Representation constraints (R) and dynamical constraints (D) are made for the obteintion of the FCHAMP's, as • **Representation constraints** (R):
in the simplest case
- R_1: fermionic representations $\underline{1}_c$, $\underline{3}_c$, $\underline{1}_c^*$;
- R_2: reducible fermionic representations:
- complex;
- flavor-chiral;
- free of Adler-Bell-Jackiw anomalies;
- R_3: the found representations of the symmetry group must contain at least $\underline{3}_c$ or $\underline{1}_c{}^*$;
- R_4: the sum of the charges of the found representations should equal zero;
- R_5: $\Delta Q = \pm 1$ for weak currents $\Rightarrow \exists$ two $\underline{1}_c$-fields which differ by ± 1;
- R_7: for pure vector couplings of the unbroken $SU_c(3) \otimes U_{em}(1)$ $SU_c(3) \otimes U_{em}(1)$, Weyl fields must be pairs of fermionic representations $\{F\}$ with $q \neq 0$; and

• **Dynamical constraints** (D)
- D_1: symmetries must break at a mass scale μ, $\mu \simeq 100Gev$
as
$SU_c(3) \otimes U_{em}(1)$;
- D_2: the spectrum must be such that the GUT is at a mass M, $M \gtrsim 10^{14}GeV$ to avoid proton decay;
- $\alpha_s(100GeV)$: $0.1 < \alpha_s < 0.3$;
- $\alpha_M \lesssim 0.3$;
- the mixing angle θ_w: $\sin^2\theta_w \sim 0.20$.

As a result, after combining the constrains, the conditions are obtained:
- R_1, R_2 iff anomaly-free red. repr.'s of $SU(N)$ constructed from irrepr.'s corresponding to single-color Young tables (with all indices desymmetrized);
- R_3, R_5 + request of fractionally-charged color singlets:
- $SU(6)$ is excluded;
- $SU(7)$ must contain a vector coupling $SU_c(3) \otimes U_{em}(1)$;
from spinorial representations of $O(14)$.

Remarks The following remarks are in order.
$Q = \frac{2}{3}$, $Q = -\frac{1}{3}$, $Q = 0$ do not violate the R constraints.

$Q = -\frac{1}{3}$, $Q = 0$ contain only one generation of ordinary quarks and leptons.
As a result, (the values of the charge(s) of) the possible eigenvalues of the charge operator Q can be assigned. For this purpose,
- three copies of the fermionic representations $\{F\}$ are required for the theory to contain the known quarks and leptons;
- all fermions must be light with respect to M;
- Higgs scalars contributing to the fermionic-mass matrix must breaks the weak SU(2) group.
According to D_1 and to D_2, two different predicted values of $\sin^2\theta_w$ for:
- non-exotic Dirac fields;
- exotic Dirac fields.

- *For both cases*:
- Q_5's, i.e. SU(5) charge operators, are defined;
- exotic-charge operators Q_e are conserved;
separately by *any* symmetry-breaking Higgs scalar
(which can be decomposed of two fermions)
for the conservation of the total charge.

6.20.2 Further tentative models 1

The implications of a SU(5) GUT's containing $SU(3)_c \times SU(2)_L \times U(1)$, the mixing angle and the proton lifetime have been analyzed
for a single super-heavy mass scale $\tilde{m}_s$ (unification mass scale).
The lifetime of a proton becomes smaller when superheavy-Higgs-boson-mediated amplitudes become significant (higher-order amplitudes of effective dimension 5 give rise to proton life-times proportional to m_s^2 rather than m_s^4, as from [32].

6.20.3 Further tentative models 2

From [33], [34], the experimental value

$$\sin^2(\hat{\theta}_W(M_W)) = 0.215 \pm 0.014 \tag{6.5}$$

is confirmed; **the proton is nevertheless described as unstable**.
In order to increase the lifetime of the proton, supersymmetric constraints have to be imposed; such constraints require N_H relatively light weak isodoublets and n_g generations of quarks and leptons for the pertinent low-energy limit.
The supersymmetric hypotheses requested about the spectrum of particles are: gauge bosons to have spin $\frac{1}{2}$ fermion partners; the ordinary spin $\frac{1}{2}$ quarks and leptons to have scalar partners; and the Higgs scalars to have spin $\frac{1}{2}$ fermionic partners.
The following assumptions can be done to simplify the analysis: varying $\tilde{\mu}$; and

the request that all the added supersymmetric partners have a mass $\tilde{\mu} \leq \tilde{m}_W$.
For $\tilde{\mu} \leq \tilde{m}_W$, the β function can be investigated.

2-loops β function is used to let the coupling from $\tilde{\mu}$ to $\tilde{m}_s$ evolve.
It is crucial to remark that, in theories involving $SU(2)_L \times U(1)$ involve two parameters, the mixing parameter $\sin^2(\hat{\theta}_W$ and the parameter ρ (obtained from ν_μ deep inelastic scattering) are considered. Such Theories predict the decay of protons. The proton decay can be avoided by: improving and better determining the mixing parameter $\sin^2(\hat{\theta}_W$; and by finding the pertinent agreement of ρ.

6.21 Satisfactory Weinberg angle

It is possible to use a satisfactory Weinberg angle instead of using the VEV's techniques at $\sim 10^9\,GeV$ from irrep.'s for fixing the charge shifts of the two conjugated families [36].

6.22 Remarks on $SU(7)FCHAMP$'s

Differently, the choice of $SU(7)$ allows one to avoid

$$\alpha m_{ss} \lesssim m_W, \tag{6.6}$$

i.e. the mixing Weinberg parameter (which is too high in minimal $SU(5)$ supersymmetric GUT's).
The choice of $SU(7)$ with FCHAMP's allows one to avoid the position of Eq. (6.6) i.e. it allows one to avoid a value of $\sin^2(\theta(m_W))$ too high in minimal $SU(5)$ supersymmetric GUT('s).
In particular, in a minimal $SU(7)$ supersymmetric GUT, which includes FCHAMP's, with the minimum two light Higgs doublets and no intermediate mass scales (not gauge-breaking mass scales and not supersymmetry-breaking mass ones) can give a value for the mixing parameter close to the experimental one.

6.23 Two charges-shifted singlets

By assuming the two charges-shifted singlets as light, the low-energy thresholds from the charges-shifted families and their scalar partners, the errors in the one-loop calculations for the values of m_W and $\sin^2\theta$ are large enough allow for a consistent complete agreement with the experimental standard deviations [37].

6.24 Fractionally-charged-QCD-singlet particles

An $SO(18)$ GUT can contain fractionally-charged-color-singlet fermions and exotic quarks.
The interest in this model is i that it can be broken up to $SU(3)_c \times U(1)_{em}$.

Given T the generators of the $SO(18)$ subgroups, three different ways to define the electric-charge operator Q are possible.
The differences in the ways of the assignments of the charge operator are due to the presence of 'auxiliary' symmetries within the steps of the breach.
The generalized charge operator Q' is defined as

$$Q' = Q + \sqrt{2}aT_3^{L'} + \sqrt{2}bT_3^{R'} + \sqrt{2}cT_3^{L''} + \sqrt{2}dT_3^{R''}, \tag{6.7}$$

with a, b, c, d constants as

$$a = c = 0, b = 1/3, d = 2/3; \tag{6.8a}$$

$$a = c = 0, b = d = 1/3; \tag{6.8b}$$

$$a = b = c = d = 1/3. \tag{6.8c}$$

$$(-a = b = c = d = 0). \tag{6.8d}$$

6.25 Proton stability and neutrino masses

The problem of the proton stability and that of the neutrino masses have been proposed to be discussed unficatedly by hypothesizing the existing of a further single field [39].
A $U(1)_{B-L}$ broken by the presence of a single field of charge 2; this way,
- the remaining $Z-2$ symmetry avoids the proton to decay;
- the charge-2 filed
- can couple with rh ν's and lh ν's;
- endows the lf ν's with a large mass;
- allows fractionally-charged particles to acquire a large mass.

6.26 The model of $SO(10)$ gauge symmetry

The model $SO(10)$ gauge symmetry is considered for the breach at the string level to

$$SO(6) \times SO(4) \times SU(5) \times U(1), \tag{6.9}$$

and

$$SU(3) \times SU(2) \times U(1)^2. \tag{6.10}$$

The investigation is conducted about the *thermal history* of the Universe.
The resulting exotic-matter states can be stable, and are demonstrated to be classified according to the properties of the $SO(10)$ symmetry breaking.
For $SO(10)$, the states have non-standard charges under the $U(1)_{Z'}$ symmetry, as

embedded in SO(10), and are orthogonal to $U(1)_Y$: these states are stable if the

$U(1)_{Z'}$ gauge symmetry
is unbroken down to low energies; or if some residual local discrete symmetry is still unbroken after the $U(1)_{Z'}$ symmetry breaking.

6.27 Fractional-charge particle stable under the $U(1)_{Z'}$ symmetry

It is important to recall that he densities of fractionally-charged hadronic bound states at low temperature are severely constrained and cannot avoid decaying without the hypothesis of an inflationary phase of the Universe. The constraints are aimed at allowing the model with current Astrophysical evidences.

Differently, under a non-Abelian gauge group the fractionally-charged states are confined, s.t. they would form integrally-charged bound states: this way, the conditions established should not be violated.

Furthermore, at lowering temperatures connected with the expansion of the universe, the remaining fractionally-charged hadrons are predicted to give rise to bound states of integer charge with fractionally-charged heavy leptons.

Moreover, the static properties of hadrons, i.e. electromagnetic mass splitting of mesons containing heavy-light quarks are model-dependent; transition and elastic form factors are calculated in the heavy-quark effective theory: this way, it is possible to calculate the mass-splitting relation between the two heavy-light mesons $\tilde{U}_0$ and $\tilde{U}_1$.

6.28 Consequences of conservation of baryon parity

Within the framework of the thermal history of the Universe, high-energy quark from the decay from a heavy particle (i.e. inflatons, modulus or gravitino) can be demonstrated to undergo flavor oscillation and is thermalize after scatterings with the ambient thermal plasma, the scattering being due to the presence of a dimension-nine-baryon-number-violating operator because of the presence of e baryon number symmetry operator, which conserves the baryon parity [42].

6.29 Evolution of barionic matter and of fractionally-charged particles

The evolution of barionic matter and of fractionally-charged particles within the thermal history of the Universe, as well as a description of the pertinent experimental probes and investigation, has been established [43].

6.30 Further models involving Mirror Symmetry

Experimental verifications of the parameter space of supersymmetric GUT theories including also a Yukawa coupling are provided with in [44].
Fractionally-charged branes on orbifold are predicted by including also mirror symmetry in [45].

Anyon quasiparticles with fractional quantum numbers are investigated on orbifolds in [46].

Within the framework of mirror symmetry in $N = 2$ superconformal field theories, charged particles acquire fractional exchange statistics parametrized by the phases; the model exhibits charge non-conservation for the $U(1)$ particles, which applies to quantum Hall effect (as from [47]).

6.31 Higher-dimensional-structures and relic fractionally-charged particles

The $SO(10)$ gauge symmetry has been demonstrated to break at the string level. To this respect, the "Wilsonian particle" named 'uniton' is predicted as heavy down-like quark (standard down-like charge assignment), with fractional charge under the $U(1)_{Z'}$ symmetry, i.e. such that the uniton may be stable [40].

In this case, the symmetry is broken as
- $SO(10) \to SU(3) \times SU(2) \times U(1)^2$;
as a result, the exotic matter states are classified according to the patterns of the $SO(10)$ symmetry breaking.
In $SU(3) \times SU(2) \times U(1)^2$ type models, one also obtains states with regular charges under the Standard Model gauge group, but with fractional charges under the $U(1)_{Z'}$ symmetry;
fractionally charged $SU(3)_C \times SU(2)_L$ singlets.

In $SO_(6) \times SO_(4)$ these states are doublets of $SU(2)_R$ with zero $U(1)_C$ charge and an $SU(3)_C$ singlet in the quartets of $SU(4)$ with zero $U(1)_L$ charge: in standard-like models these states are $SU(3)_C \times SU(2)_L$ singlets with electric charge $Q_{em} = \pm 1/2$.

In $SO(10) \to SO(6) \times SO(4)$,
fractionally-charged standard-like-model states are obtained,
which are $SU(3)_C \times SU(2)_L$ singlets with $Q_{em} = \pm\frac{1}{2}$,and (fractional) charge under $U(1)_Y$: this way, $[(1, 0); (1, \pm 1)]_{(\pm 1/2, \mp 1/2, \pm 1/2)}$ (where the pedices denote the $U(1)_Y$ charge), and $[(1, \pm 3/2); (1, 0)]_{(\pm 1/2, \pm 1/2, \pm 1/2)}$.

In the $SO(10) \to SU(5) \times U(1)$ patterns, exotic states are obtained, - $SU(3)_C \times SU(2)_L$ singlets with $Q_{em} = \pm\frac{1}{2}$, and $[(1, \pm 3/4); (1, \pm 1/2)]_{(\pm 1/2, \pm 1/4, \pm 1/2)}$.

In the $SO(10) \rightarrow SU(3) \times SU(2) \times U(1)^2$ models, standard-model singlets with nontrivial $U(1)_{Z'}$ are obtained.

The heavy particles are ones such as (in the low-energy limit) $SU(2)_L$ doublets and singlets, which are $SU(3)$ singlets: $(1,2)_0, (1,1)_{1/2}$, lepton-like, and fractional electric charge $Q_{em} = \pm 1/2$.

Fractionally charged vector- like quarks The fractionally charged vector- like quarks are obtained as the 'sexton' color triplet), which can interact with u and d quarks and origininate stable baryons and stable mesons; endowed with fractional electric charge of $\pm 1/2$ and $\pm 3/2$; the lower bound for the masses are found as $M \sim \frac{eV}{{Y_0}^{FC}} > 10^{-19} GeV$ [48].

${Y_0}^{FC}$ Relic density of fractionally-charged matter Relic density of fractionally-charged matter is qualified as ${Y_0}^{FC} < 10^{-19} {Y_0}^{B} < 10^{-19} \frac{eV}{m_p} \sim 10^{-28}$; experimental searches for free quarks in various materials allow one to set upper bound on the number density of fractionally-charged-particles smaller than $10^{-19} \sim 10^{-26}$; the experimental constraints are implied: densities of fractionally charged bound states have suppressed densities.

Exotic stable quarks Exotic stable quarks $Q = -1/3$ are found as forbidden: strict constraints arise from cosmology.
Indeed, the constraints are given from superheavy-elements searches; the request that heavy particles, after capture by neutron stars, do not induce the collapse of the object into a black hole require one to look for a mechanism that allows for the decay of the $Q = -1/3$ quarks [49].

6.32 More about the Sexton

The Sexton σ id characterised after a charge 1/6.
CONTROLLA The fractionally charged leptons correspond to charged color singlet bound states, and corresponding antiparticles.
With confinement temperature, the sexton σ can form: neutral color singlet bound states; charged color singlet bound states; $\sigma\sigma\sigma$, $\sigma\sigma q$, $qq\sigma$, $\bar{q}\sigma$ states; and the corresponding antiparticles $\bar{\sigma}\bar{\sigma}\bar{\sigma}$, $\bar{\sigma}\bar{\sigma}\bar{q}$, $\bar{q}\bar{q}\bar{\sigma}$, $q\bar{\sigma}$, with q ordinary quarks.

Integer-charged final states can reconvert into fractionally charged states ,
Integer-charged final states can reconvert into fractionally charged states are due to the presence of a large amount of ordinary particles; the cooling temperature (expansion of the universe) allows for the possibility for the remaining fractional

hadrons to form bound states of integer charge with fractionally-charged-heavy leptons $(1,2)_0$ and $(1,1)_{1/2}$ with $Q_{em} = T_3 + Y$: this way, neutral heavy hydrogen-type bound states $B_{1/2} + L_{-1/2}$, which can capture an electron at a temperature of a few eV, whose cross-section is calculated to be extremely small.

6.32.1 The $SO(10)$ symmetry breaking of $SU(3) \times SU(2) \times U(1)^2$

The $SO(10)$ symmetry breaking of $SU(3) \times SU(2) \times U(1)^2$ conducts to $SU(3)_c \times SU(2)L \times U(1)_Y$ singlet with non-standard $U(1)_{Z'}$ charge, for which the interactions with the Standard Model states vanish to all orders of non-renormalizable terms. This way, if the $SU(3)_H$ is left unbroken, but $U(1)_{Z'}$ should be broken between the weak scale and the Planck scale, the W-singlet W_s is qualified as interaction suppressed by $1/M^2{}_{Z'}$, and can be classified as 'WIMP'. The W-singlet W_s can annihilate into: two light Standard Model fermions, and their superpartners; further conditions on the W-singlet are given as mass (wrt $M_{Z'}$) and inflation; it can be a strongly interacting particle.

6.33 Further analysis of the SO(18) model

The breaking of the SO(18) supersymmetric model has been analyzed to be possible through several mechanisms.

The generalized charge operator Q' for states is obtained as

$$Q' = Q + \sqrt{2}aT_3^{L'} + \sqrt{2}bT_3^{R'} + \sqrt{2}cT_3^{L''} + \sqrt{2}dT_3^{R''}, \qquad (6.11)$$

as fractionally charged color singlet particles where the parameters a, b, c and d appear in the definition of the generalized electric charged operator Q' of the 256-dimensional irreducible representations [38].

Two models are possible.

Model 1 Model 1 is defined after the parameters of the generalized charge operator: $a = c = 0$, $b = 1/3$, $d = 2/3$. The
color singlets with charge $Q = 1/3, 2/3, 4/3, 5/3, 2$, or
with right-handed current appear.

Model 2 Model 2 arises because of the symmetry, which cannot have $SU(3)_c \times SU(2) \times U(1)$ invariant masses. The
parameters of the generalized charge operator are writtten as $a = c = 0$, $b = d = 1/3$.

This way, six generations of ordinary fermions which can not acquire (because of the symmetry breaking) $SU(3)_c \times SU(2) \times U(1)$ invariant masses, and there arise ten generations of non-standard fermions, among which also fractionally-charged color-singlet particles, which acquires charges $Q = 1/3, 2/3, 4/3, 5/3$.

6.34 More about FCHAMP's as leptons

FCHAMP's can be theoretized to exist as leptons with:

- fractional e.m. charge,
- electroweak charge,

- with non-trivial hypercharge $U(1)_Y$,

- no strong interactions, and
- of mass m and charge $Q_L \simeq 2$.

6.34.1 Early-Universe searches

From the phenomenology of the Early Universe, one gains the possibility to retrieve items of information about FCHAMP's as far as [12]

- thermal production,
- annihilation,
- survival,
- cosmological constraints from primordial nucleosynthesis,
- cosmological constraints form microwave-background-radiation, and
- abundance on Earth;

No strong-interaction theoretical investigations are, up to now, present [50].

6.35 Further researches

Further researches of FCHAMP's were proposed in a host of branches. • The search for large samples for fractional charge of any compositions can be lead after the experimental search for particles with fractional charge in free states by means of grounded Faraday cup and high-impedance amplifier [51].

• The search for fractional-charge leptons and quarks in an unconfined state can be pursued [52].
• The search for 4-th-generation integrally-charged quarks can be expected [53].

• The search for fractionally-charged particles in (Anti)-neutrino - Deuterium

Interactions can be persevered [54].

• The search for fractionally-charged-particles with an electric charge $q = e/5$ (and opportune velocities) can be traced [55].
• The search for isolated fractionally-electrically-charged particles with the Mil-

likan method can be tracked [56].

6.36 Conclusions

Free fractionally charged particles is an exotic and profound signature of new physics, which can be related with symmetries (including gauge) beyond the Standard Model (SM). Extension of the SM can also lead to existence of new stable multiple charged particles - constituents of dark atoms of dark matter, or milli-micro-nano-charged particles, originated from mixing of ordinary and dark photon [1]. It makes studies and search for particles with exotic charges an important direction of studies of physics beyond the Standard model.

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References

1. M.Khlopov: Fractionally charged particles and quark confinement. JETP Lett. **33** (1981) 162-164.
2. M.Yu.Khlopov: *Cosmoparticle physics*, World Scientific, Singapore, 1999.
3. M. Khlopov: *Fundamentals of Cosmic particle physics*, CISP-SPRINGER, UK-Germany, 2012.
4. B. C. Rayhaun, D. J. Williamson: Higher-Form Subsystem Symmetry Breaking: Subdimensional Criticality and Fracton Phase Transitions. arXiv:2112.12735v2 [cond-mat.str-el] (2021) to appear in SciPost Phys.
5. F. Peenitez: Fractons, symmetric gauge fields and geometry. arXiv:2107.13884 [cond-mat.str-el]
6. R. F. Sobreiro: An effective gauge theory of fractons: perturbative and non-local aspects. arXiv:2306.08663 [hep-th] (2023)
7. W. B. Fontana, P. R. S. Gomes, C. Chamon: Lattice Clifford fractons and their Chern-Simons-like theory. SciPost Phys. **Core 4** (2021) 012.
8. L. B. Okun': On searches for new long-range forces, Zh. Eksp. Teor. Fiz. 52,(1980) 694–697.
9. P. Frampton and T. Kephart:, Fractionally Charged Particles as Evidence for Supersymmetry, Phys. Rev. Lett. **49** (1982) 1310.
10. G. S. LaRue, J. D. Phillips, W. M. Fairbank: Observation of fractional charge of (1/3)e on matter, Phys. Rev. Lett. **46** (1982) 967.
11. H. Goldberg: Fractionally Charged Heavy Leptons: Cosmological Implications of Their Existence, and a Prediction of Their Abundance, Phys. Rev. Lett. **48** (1982) 1518.
12. P. Langacker, G. Steigman: Requiem for an FCHAMP? Fractionally Charged, Massive Particle, Phys. Rev. D **84**, (2011) 065040.
13. L. Lyons: Quark Search Experiments at Accelerators and in Cosmic Rays, Phys. Rept. **129** (1985) 225.
14. T. Mashimo, S. Orito, K. Kawagoe, S. Nakamura, M. Nozaki: An underground search for anomalous penetrating particles such as massive magnetic monopoles, Phys. Lett. **128** (1983) 327–332.

15. K. Kawagoe, T. Mashimo, S. Nakamura, M. Nozaki, S. Orito: An underground search for anomalous slow penetrating particles, Lett. Nuovo Cim. **41** (1984) 315–319.
16. S.P. Ahien, K. Kinoshita: Calculation of the stopping power of very-low-velocity magnetic monopoles, Phys. Rev. D **26** (1982) 2437.
17. D.M. Ritson: Magnetic monopole energy loss, SLAC-Pub 2950 (1982).
18. T. Wada, Y. Yamashita, I. Yamamoto: Search for $(4/3)e$ charged leptons in cosmic rays, Lett. Nuovo Cim. **40** (1984) 329–335.
19. M. Marinelli, G. Morpurgo: The electric neutrality of matter: A summary, Phys. Lett. B **137** (1984) 439–442.
20. W.J. Dick, G.W. Greenlees, S.L. Kaufman:, Search for Anomalous Isotopes of Sodium, Phys. Rev. Lett. **53** (1984) 431.
21. Peter C. Kim, Eric R. Lee, Irwin T. Lee, Martin L. Perl, V. Halyo, and D. Loomba: Search for fractional-charge particles in meteoritic material, Phys. Rev. Lett. **99** (2007) 161804.
22. S. Nussinov, R. Shrock: Upper Limits on a Possible Gluon Mass, Phys. Rev. D **82** (2010) 034031.
23. J.C. Price, W.R. Innes, S. Klein, M.L. Perl: The rotor electrometer: A new instrument for bulk matter quark search experiments, Rev. Sci. Instrum. **57** (1986) 2691–2698.
24. V. Mathai, G. Wilkin: Fractional quantum numbers via complex orbifolds, arXiv:1811.11748.
25. GRAVITY Collaboration: An image of the dust sublimation region in the nucleus of NGC 1068, Astronomy and Astrophysics **634** (2020) A1.
26. P. Senger: Status of the Compressed Baryonic Matter experiment at FAIR, Int. J. Mod. Phys. E **29** (2020) 2030001.
27. A. Goswami, R. Nayak, B. K. Nandi, S. Dash: Effect of color reconnection and rope formation on resonance production in p−p collisions in Pythia 8, arXiv:1911.00559 [hep-ph].
28. R. Sen: Possible violation of spin-statistics connection in electron-electron scattering at low relativistic energies, arXiv:2004.14481 [hep-ph].
29. J. Wess, B. Zumino: A Lagrangian Model Invariant Under Supergauge Transformations, Phys. Lett. B 4**9** (1974) 52–54.
30. F. A. Berezin, G.I. Kac: Lie groups with commuting and anticommuting parameters, Mathemat. Sbornik (USSR) **82** (1970) 343 [English translation **11** 311–326].
31. H. Goldberg, T. W. Kephart, M. T. Vaughn: Fractionally Charged Color-Singlet Fermions in a Grand Unified Theory, Phys. Rev. Lett. **47** (1981) 1429.
32. W.J. Marciano, G. Senjanovic: Predictions of supersymmetric grand unified theories, Phys. Rev. D **25** (1982) 3092.
33. W.J. Marciano, A. Sirlin: SU (5) Theory vs. Experiment, in: Editors J. Leveille, L. Sulak, and D. Unger: Proceedings of the Second Workshop on Grand Unification,Birkhauser, Boston,1981).
34. A. Sirlin, W. Marciano: Radiative corrections to $\nu_\mu + N \to \mu^? + X$ and their effect on the determination of ρ^2 and $sin^2\theta_W$, Nucl. Phys. B **189** (1981) 442–460.
35. C. H. Llewellyn Smith, J. F. Wheater: Electroweak Radiative Corrections, Phys. Lett. B **105** (1981) 486.
36. L.F. Li, F. Wilczek: Price of Fractionally Charged Particles in a Unified Model, Phys. Lett B **107** (1981) 64.
37. J. Ellis , D. V. Nanopoulos, S. Rudaz: A phenomenological comparison of conventional and supersymmetric guts, Nucl. Phys. B **202** (1982) 43–62.
38. F. X. Dong, T. S. Tu, P. Y. Xue, X. J. Zhou: $SO(18)$ grand unified models with fractionally charged color singlet particles, Annals Phys. **145** (1983), 1–26.
39. J.S Hagelin: Recent progress in 4D string-model building, Editors E. Bonner Billy, Miettinen E. Hannu: , in Proceedings Of The 1990 Meeting Of The Division Of Particles

And Fields Of The Aps, World Scientific Co. Pte. Ltd, Loi Printing Pte. Ltd, Singapore, 1990.
40. S. Chang, C. Coriano, A. E. Faraggi: Stable superstring relics, Nucl. Phys. B **477** (1996) 65–104.
41. A. Gould, B. Draine, R. Romani, S. Nussinov: Neuton stars: Graveyard of charged dark matter, Phys. Lett. Bbf 238 (1990) 337–343.
42. T. Asaka, H. Ishida, W. Yin: Direct baryogenesis in the broken phase, arXiv:1912.08797 [hep-ph].
43. Martin L. Perl, Peter C. Kim, V. Halyo, E. R. Lee, I. T. Lee: The Search for Stable, Massive, Elementary Particles, Int. Journ. Mod. Phys. A **16** (2001) 2137–2164.
44. W. de Boer, R. Ehret, D. Kazakov: Predictions of SUSY masses in the minimal supersymmetric grand unified theory, Z. Phys. C **67** (1995), 647–664.
45. D. E. Diaconescu, J. Gomis: Fractional branes and boundary states in orbifold theories, JHEP **10** (2000), 001–044.
46. M. Atiyah, M. Marcolli: Anyons in geometric models of matter, J. High Energ. Phys. **2017** (2017) 76.
47. L. Cooper, I. I. Kogan, R. J. Szabo: Mirror maps in Chern-Simons gauge theory, Annals Phys. **268** (1998) 61–104.
48. Particle Data Group: Review of Particle Properties, Phys. Rev. D **50** (1994) 1173.
49. E. Nardi, E. Roulet: Are exotic stable quarks cosmologically allowed?, Phys. Lett. B **245** (1990) 105–110.
50. MoEDAL cOLLABORATION: The Physics Programme Of The MoEDAL Experiment At The LHC, Int. J. Mod. Phys. A **29** (2014) 1430050.
51. W.R. Innes, M.L. Perl, J.C. Price: A rotor electrometer for fractional charge searches, SLAC-PUB-3867.
52. M. Perl, E. Lee, D. Loomba: A Brief review of the search for isolatable fractional charge elementary particles, Mod. Phys. Lett. A **19** (2004) 2595.
53. X. He, S. Pakvasa, H. Sugawara: Are 4-th-generation quarks integrally-charged? Annals N. Y. Acad. Sci. **518** (1987) 332.
54. D. Allasia et al.: Search for Fractionally Charged Particles in (Anti)-neutrino - Deuterium Interactions, Phys. Rev. D **37** (1988) 219.
55. MACRO Collaboration: Final search for lightly ionizing particles with the MACRO detector, arXiv:hep-ex/0402006 [hep-ex].
56. E. R. Lee, V. Halyo, I. T. Lee, M. L. Perl: Automated electric charge measurements of fluid microdrops using the Millikan method, Metrologia **41** (2004), S147–S158.
57. M. Khlopov: What comes after the Standard model? Prog. Part. Nucl. Phys. **116** (2021) 103824.

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7 Recent advances of Beyond the Standard model cosmology

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Abstract. BSM physics, underlying the now standard inflationary cosmology with baryosynthesis and dark matter/energy, inevitably leads to features, going beyond this standard cosmological paradigm. The signatures of these features can be considered in the hints for the existence of antihelium component of cosmic rays and massive Primordial Black Holes (PBH), in the discovery of stochastic gravitational wave background (SGWB) and evidence for early galaxy formation as well as in the dark atom solution for the puzzles of direct dark mater searches. We discuss open problems of physics of dark atoms, which can explain positive result of DAMA/NaI and DAMA/LIBRA direct searches for dark matter, of the origin and evolution of antimatter domains in baryon asymmetrical Universe, of the Axion Like Particle (ALP) physics, which may explain the origin of SGWB, discovered by Pulsar Timing Array (PTA) facilities and early galaxy formation, favored by James Webb Space Telescope.

Povzetek: Fizika, ki naj preseže *standardni model*, mora pojasniti vzrok nastanka in širjenja vesolja, potek bariosinteze, nastanek temne snovi in temne energije. Avtor vidi v opazovanjih signale za obstoj antihelijeve komponente v kozmičnih žarkih in za obstoj masivnih prvobitnih črnih lukenj (PBH); meni, da to lahko razloži ozadje stohastičnega gravitacijskega valovanja (SGWB) in zgodnjo tvorbo galaksij, temni atomi pa lahko pojasnijo neskladja med različnimi eksperimentih pri merjeju temne snovi, ki jo je doslej opazille experiment le DAMA/LIBRA. Avtor poskuša razložiti tudi razvoj domen antimaterije in zgodnje nastajanje galaksij, kar morda razloži meritve James-Webbovega teleskopa.

Keywords: axion like particles, antimatter, dark atoms, symmetry breaking, phase transitions, primordial black holes, stochastic gravitational wave background, pulsar timing arrays

7.1 Introduction

The now Standard cosmological paradigm involves inflation, baryosynthesis and dark matter/energy [1–3, 3–5, 7–9], which imply physics Beyond the Standard

Model (BSM) of fundamental interactions. Model dependent cosmological consequences of this physics give its cosmological messengers [10–12], which challenge the concordance of the standard ΛCDM cosmology and can remove the conspiracy of its BSM features [13]. Positive hints to these features need more detailed analysis of physical basis and observable features of BSM messengers. Here we discuss observational signatures of BSM cosmology presented at the XXVI Bled Workshop "What comes beyond the Standard models?" with special emphasis on open problems of their analysis.
Confirmation of these signatures would strongly reduce the set of models and parameters of the physical basis of the modern cosmology. Therefore, mechanisms of inflation and baryosynthesis as well as dark matter candidates proposed in any approach to the unified description of Nature [14, 15] should inevitably include BSM cosmological signatures, which find observational support.
We consider open questions in dark atom messengers of new physics in the direct searches of dark matter (Section 7.2.1), in the footprints of Axion Like Particle (ALP) physics in stochastic gravitational wave background (SGWB) and early galaxy formation, favored by James Webb Space Telescope (JWST) and in the primordial objects of antimatter, which can be sources of antihelium component of cosmic rays (Section 7.3). We discuss is the conclusive Section 7.4 there signatures and their significance in the context of cosmoparticle physics of BSM physics and cosmology.

7.2 Open problems of dark matter physics

7.2.1 Dark atom signature in direct dark matter searches

The increasingly high statistics of positive result of underground direct dark matter search in DAMA/NaI and DAMA/LIBRA experiments [16] challenges its Weakly Interacting Massive Particles (WIMP) interpretation with the account for negative results of direct WIMP searches by other groups (see [1] for review and references). Though the difference in experimental strategy may leave some room for WIMP interpretation of this positive result [16], its non-WIMP interpretation deserves serious attention and can make this result an experimental evidence for dark atom nature of dark matter [1, 12, 17, 18].
The dark atom hypothesis assumes existence of stable particles with negative even electric charge $-2n$, which bind with n primordial helium-4 nuclei in neutral nuclear interacting atom like states.
The motivation for the existence of such multiple charged stable states can come from the composite nature of Higgs boson. Indeed, the lack of positive evidence for supersymmetric (SUSY) particles at the LHC can indicate very high energy SUSY scale [19]. It makes hardly possible to use SUSY for solution of the problem of the divergence of Higgs boson mass and the origin of the energy scale of the electroweak symmetry breaking and implies a non-SUSY solution of this problem. Such a non-SUSY solution can be provided by composite Higgs boson and if Higgs constituents are charged, their composite multiple charged states can naturally provide $-2n$ charged constituents of dark atoms, as it takes place in Walking Technicolor model (WTC). Nontrivial electroweak charges of techniparticles provide

due to electroweak sphaleron transitions balance between their charge asymmetry (excess of $-2n$ charged particles over their antiparticles) and baryon asymmetry, thus predicting relationship between dark matter density in the form of dark atoms and baryon density. The model involves only one parameter of BSM physics - mass of the stable $-2n$ charged particles and reproduction of the observed dark matter density makes possible to determine this parameter [20, 21]. It is shown in [20, 21] that the excess of $-2n$ charged techniparticles and -2 charged ($\bar{U}\bar{U}\bar{U}$) clusters of stable antiquarks $\bar{U}$ with the charge $-2/3$ of new stable generation like the 5th generation in the approach [15], can explain the dark matter. The excess generated by such balance depends on the mass of multiple charged particles. It can put upper limit on the mass of stable multiple charged particles, at which dark atoms can explain the observed density of dark matter [21].

Dark atom structure depends on the value of $-2n$ charge. Double charged particles ($n = 1$) form with primordial helium nucleus Bohr-like OHe atom, in which radius of helium Bohr orbit is nearly equal to the sizes of this nucleus. At $n > 1$ particle with the charge $-2n$ forms with n helium nuclei Thomson-like XHe atom, in which $-2n$ charged lepton is situated within an n-α-particle nucleus. In the both cases dark atoms structure strongly differs from the usual atoms. Instead of small nuclear interacting core and large electroweakly interacting shell they have a heavy lepton or lepton-like core and nuclear interacting helium shell. The latter determines their interaction with baryonic matter.

Dark atom hypothesis can qualitatively explain negative results of direct WIMP searches. Owing to their nuclear interaction they are slowed down in the terrestrial matter. It leads to negligible nuclear recoil in the underground detectors [1, 12, 17]. At each level of terrestrial matter dark atom concentration is determined by the balance between the incoming cosmic flux of dark atoms and their diffusion towards the center of Earth. At the 1 km depth this equilibrium concentration is adjusted to the incoming cosmic flux at the timescale of less than 1 hour. It leads to annual modulation of this concentration within the underground detector. If dark atoms can form low energy (few keV) bound states with nuclei of detector, the energy release in such binding should possess annual modulation. It can explain the signal, detected in DAMA/NaI and DAMA/LIBRA experiments.

The crucial point in this explanation is the existence of a potential barrier and a shallow well in the interaction of a nucleus and dark atom. Qualitatively the origin of this barrier can be related with the specific interplay of Coulomb repulsion and nuclear attraction between the nucleus of detector and helium shell of dark atom. However in the lack of usual approximations of atomic physics (small ratio of nuclear to atomic sizes and electroweak coupling of electronic shell) quantitative description of dark atom interaction implies development of special numerical methods. In this direction numerical methods of continuous extension of a classical three body problem to realistic quantum-mechanical description were developed [22] both for Bohr-like and Thomson-like dark atoms interaction with nuclei [23]. Presently numerical solutions for Schrodinger equation for dark-atom -nucleus quantum system are under way to make possible interpretation of the results [16] in terms of signature of dark atoms.

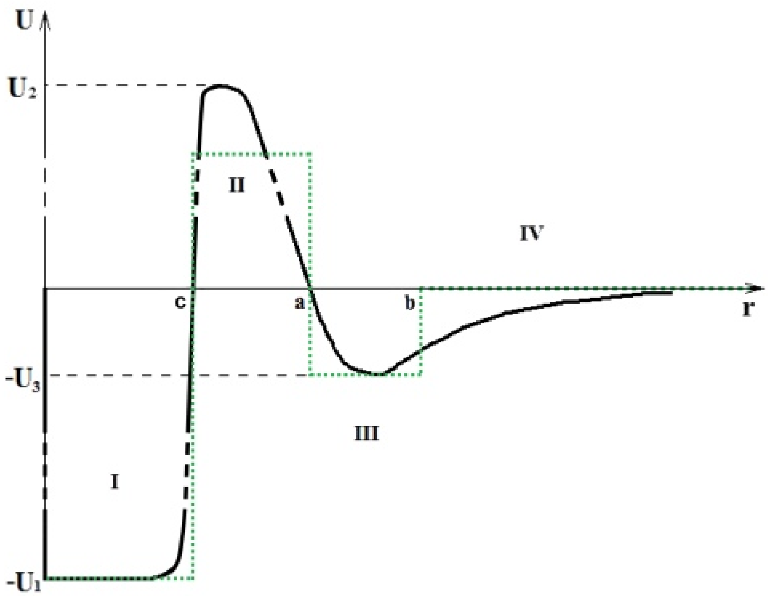

Fig. 7.1: Potential barrier in dark atom- nucleus interaction can lead to a shallow potential well in which a low energy bound state can exist

7.2.2 Axion-like particle models of dark matter

Axion-Like Particle (ALP) models can be reduced to a simple model of a complex field $\Psi = \psi \exp i\theta$ with broken global $U(1)$ symmetry [1,11]. The potential

$$V = V_0 + \delta V$$

contains the term $V_0 = \frac{\lambda}{2}(\Psi^*\Psi - f^2)^2 thatleadstospontaneousbreakingoftheU(1)symmetrywith$ $f \exp(i\theta) and the term \delta V(\theta) = \Lambda^4(1 - \cos\theta)\ with \Lambda \ll f$ that leads to manifest breaking of the residual symmetry, leading to a discrete set of degenerated ground states, corresponding to

$$\theta_{vac} = 0, 2\pi, 4\pi, ...$$

In the result of the second step of symmetry breaking an ALP field $\phi = f\theta$ is generated with the mass $m_\phi = \Lambda^2/f$.

The term (7.2.2) can be present in the theory initially. Then coherent oscillations of the ALP field start, when Hubble parameter $H = m_\phi = \Lambda^2/f$. This term can be generated by instanton transitions, as it is the case in the axion models. Then these oscillations are switched on, when this term is generated at $T \sim \Lambda$ (i.e. at $H = \Lambda^2/m_{Pl}$). In spite of a very small mass of ALP, they are created initially nonrelativistic and thus behave like Cold Dark matter in the process of growth of small density fluctuations at the matter dominated stage. However, as we can see in the next section 7.3, ALP field evolution may be accompanied by creation of strong primordial inhomogeneities of different kind.

7.3 Strong primordial inhomogeneities from ALP physics

If the first phase transition takes place during inflation with Hubble parameter H_{infl}, quantum fluctuations within the scalar field Ψ led to variations in phase between disconnected regions, quantified by $\delta\theta = H_{infl}/(2\pi f)$. In regions where inflationary dynamics have resulted in phase values greater than π, the field will oscillate around the minimum at $\theta_{vac} = 2\pi$. Conversely, in the surrounding space where phase values are less than π, the oscillations will favor the minimum at $\theta_{vac} = 0$. This led to the formation of closed domain walls with $\theta = \pi$, which are closed surfaces in space. The phase variation process can be compared to a one-dimensional Brownian motion, where the field's phase undergoes a random walk with step lengths of approximately $\delta\theta$ per Hubble time. If the field Ψ possess interactions with quarks and leptons, in which baryon and lepton numbers are not conserved, decay of the field $f\theta$ in its motion to ground state should generate baryon asymmetry, if $\theta < \pi$, or excess of antibaryons, if $\theta > \pi$ (see Fig. 7.2).

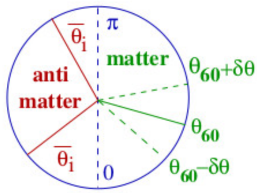

Fig. 7.2: Phase fluctuations at inflationary stage can cross π, leading to formation of closed domain walls. In the model of spontaneous baryosynthesis, based on ALP physics phase fluctuations, crossing π and 0, result in antibaryon domains in baryon asymmetric Universe [1]

7.3.1 ALP signatures in PBH, SGWB and JWST

Formation of closed domain walls can lead to formation of primordial black holes (PBH) [19, 24]. Their mass M is determined by the two fundamental scales of the ALP physics, f and Λ, as given by $M_{min} = f(m_{Pl}/\Lambda)^2 \leq M \leq M_{max} = f(m_{Pl}/f)^2(m_{Pl}/\Lambda)^2$.Here$M_{min}$ is determined that the gravitational radius of wall $r_g = 2M/m_{Pl}^2$ exceeds the width of wall $d \sim f/\Lambda^2$, while the maxiomal mass is determined by the condition that the wall can enter horizon before it starts to dominate in the energy density within horizon. This mechanisms of PBH formation can provide formation of PBHs with stellar mass, and even larger than stellar up to the seeds for Active Galactic nuclei (AGN) [25–27]. LIGO/VIRGO detected gravitational wave signal from coalescence of black holes with masses ($M > 50M_\odot$), which exceed the limit of pair instability, and put forward the question on their primordial origin [28, 29].
Recent discovery of Stochastic Gravitational Wave Background (SGWB) by Pulsar Timing Arrays (PTA) [30] can be another evidence of ALP physics. Collapse of

large closed domain walls with mass $M > M_{max}$ leads to separation of the region from the surrounding Universe with formation of a wormhole and then baby Universe. This process is accompanied by gravitational wave background radiation, which can reproduce the PTA data. Simultaneously, the values of phase at the stages of inflation, preceding the stage of crossing π, at which the contour of the future domain wall is created, approach to π and result in the energy density much larger, than the average one for the ALP field. It leads to the high ALP density in the regions, surrounding the wall and even, if the wall disappears in the baby Universe the ALP density in the surrounding region is much higher than the average in the Universe. It strongly facilitates galaxy formation at the redshifts $z > 10$, indicated by the data of JWST, in the regions surrounding large closed domain walls. In that way ALP physics can simultaneously explain the PTA and JWST data [31,32]. The possibilities to probe such ALP physics are illustrated on Fig. 17.9. The open question in this scenario is the evolution of the regions of the enhanced ALP density, and, in particular, whether black hole formation is possible in their central part, or the enhanced ALP density itself plays the role of AGN seeds of early galaxies.

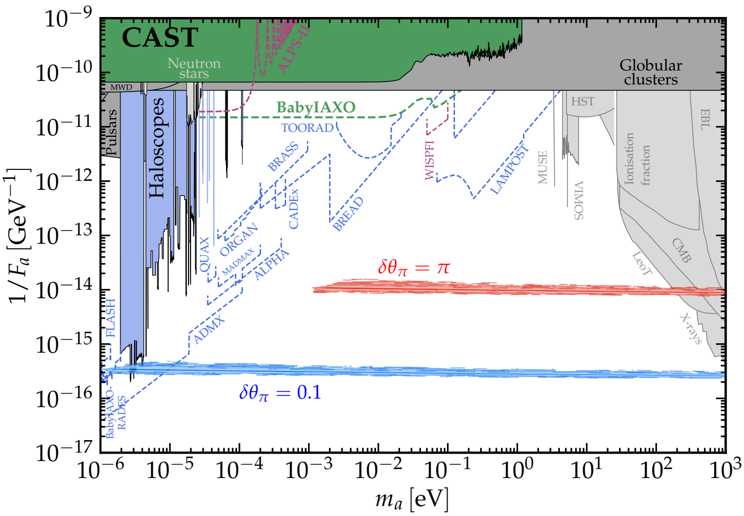

Fig. 7.3: The possibility to explain SGWB and JWST data in ALP model (taken from [31,32]).

7.3.2 Antimatter domains in baryon asymmetric Universe

Baryon asymmetry of the Universe, reflecting the absence of the comparable with baryonic matter amount of macroscopic antimatter in the observed Universe. Its origin is ascribed to the mechanism of baryosynthesis, in which baryon excess is created in very early Universe. Inhomogeneous baryosynthesis can lead in the extreme case to the change of sign of this excess, giving rise to antimatter domains, produced in the same process, in which the baryonic matter was created [33–38]. Surrounded by matter, antimatter domains should be sufficiently large to survive to the present time and to give rise to antimatter objects in the Galaxy. It implies also effect of inflation in addition to nonhomogeneous baryosynthesis. Such combination of inflation and nonhomogeneous baryosynthesis can take place in spontaneous baryosynthesis at the specific choice of ALP parameters.

This choice should determine the initial properties of antimatter domains and the forms of macroscopic antimatter, which can be formed in them. The estimated minimal mass of surviving domain together with upper limit on possible amount of antimatter in our Galaxy from the observed gamma radiation gave the interval of mass $10^3 M_\odot \leq M \leq 10^5 M_\odot$, which is typical for globular clusters. Symmetry of electromagnetic and nuclear interactions of matter and antimatter, space distribution of globular clusters in galactic halo with low gas density seemed to favor the hypothesis of antimatter globular cluster in our Galaxy [38,39]. The antibaryon density may be much higher, than the baryonic density and then specific ultra-dense antibaryon stars can be formed [40],

The predicted fraction of antihelium nuclei in cosmic rays from astrophysical sources is far below the sensitivity of AMS02 experiment [41] . It makes the unpublished suspected antihelium-4 event, registered in 2017, a strong signature of macroscopic antimatter in our Galaxy. The unprecedented sensitivity of AMS02 experiment to the cosmic ray fluxes makes this collaboration especially responsible for presentation of its results, which cannot be tested by any other experimental group. That is why the collaboration continues to gain more and more statistics and check all the possible background interpretation before the discovery of cosmic antihelium-4 is announced.

With the hope that such an announcement will be made, studies of possible forms of macroscopic antimatter objects in our Galaxy are challenging.

Such analysis should involve evolution of antibaryon domains in baryon asymmetrical universe [42,43] in the context of models of nonhomogeneous baryosynthesis. The earlier idea to use the observed properties of the M4 globular cluster as possible prototype of antimatter object [44] should be strongly corrected, since chemical evolution of isolated antimatter domain should strongly differ from such evolution of the ordinary matter. One can expect that primordial nucleosynthesis should lead to production of primordial antihelium at rather wide range of antibaryon density in the domain, but the products of stellar anti-nucleosynthesis cannot come to the domain from other parts of the Galaxy, while heavy elements produced by antimatter stars within it leave domain and annihilate with the matter in the Galaxy. It makes hardly possible enrichment of antimatter object by antinuclei heavier than antihelium-4, while the observed metallicity in all the galactic globular clusters is close to the Solar one, favoring the income of products of stellar nucleosynthesis from other parts of the Galaxy.

Primordial metallicity can appear in domains with high antibaryon density, in which antinuclei much heavier than helium-4 can be produced. In the context of nonhomogeneous nucleosynthesis based on ALP physics such high density antibaryon domains can appear after crossing π and should have massive do-

main walls at their border. Collapse of such walls with the mass $M > M_{max} = f(m_{Pl}/f)^2(m_{Pl}/\Lambda)^2$ puts such domains within baby Universes, so that only domains surrounded by walls with $M < M_{max}$ can be observable and there is an open question, whether such domains are sufficiently large to survive in the matter surrounding. .
In any case to confront AMS02 searches the composition and spectrum of cosmic antinuclei from antimatter objects in our Galaxy should be predicted with the account for propagation in galactic magnetic fields of antinuclei from local source in the Galaxy [45]

7.4 Conclusions

The increasing hints to new physics phenomena in DAMA experiments, LIGO-VIRGO-KAGRA, PTA and JWST data, possible existence of antihelium component of cosmic rays can indicate not only effects of BSM physics, but also lead to the BSM cosmology, involving such deviations from the standard cosmological model as Warmer-than-Cold dark matter scenario of nuclear interacting dark atoms of dark matter, or primordial strong nonhomogeneities of energy and/or baryon density, giving rise to new scenarios of galaxy formation and evolution. We have outlined here the open questions in the proposed BSM models and scenarios, which can explain these deviations and deserve special studies and discussion at future Bled Workshops.
In the context of cosmoparticle physics, studying fundamental relationship of macro- and micro- worlds in the cross-disciplinary studies of of its physical, astrophysical and cosmological signatures, confirmation of these cosmological messengers of new physics would provide a sensitive probe for BSM cosmology based on the proper choice of parameters of proper class of BSM models, since only such models, which predict these deviations from the standard cosmological paradigm can pretend to be realistic in this case.

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References

1. M.Khlopov: What comes after the Standard model? Progress in Particle and Nuclear Physics **116**, 103824 (2021).
2. A.D. Linde: *Particle Physics and Inflationary Cosmology*, Harwood, Chur, 1990.
3. E.W. Kolb and M.S. Turner: *The Early Universe*, Addison-Wesley, Boston, MA,USA, 1990.
4. D.S. Gorbunov and V.A. Rubakov: *Introduction to the Theory of the Early Universe Hot Big Bang Theory. Cosmological Perturbations and Inflationary Theory*, World Scientific, Singapore, 2011.

5. D.S. Gorbunov and V.A. Rubakov: *Introduction to the Theory of the Early Universe Hot Big Bang Theory*, World Scientific, Singapore, 2011.
6. M.Y. Khlopov: *Cosmoparticle Physics*, World Scientific, Singapore, 1999.
7. M.Y. Khlopov: *Fundamentals of Cosmoparticle Physics*, CISP-Springer, Cambridge, UK: 2012.
8. M. Khlopov: Cosmological Reflection of Particle Symmetry, Symmetry **8**, 81 (2016).
9. M. Khlopov: Fundamental particle structure in the cosmological dark matter, Int. J. Mod. Phys. A **28**, 1330042 (2013).
10. M.Khlopov: BSM Cosmology from BSM Physics. Bled Workshops in Physics. **22**, 153-160 (2021).
11. M.Khlopov: Physics and Cosmology Beyond the Standard Models. Physics of Particles and Nuclei **54**, 896–901 (2023).
12. M.Khlopov: Multimessenger Probes for New Physics in Light of A. Sakharov's Legacy in Cosmoparticle Physics. Universe **7**, 222 2021. https://doi.org/10.3390/universe7070222
13. M.Yu. Khlopov: Removing the conspiracy of BSM physics and BSM cosmology, Int. J. Mod. Phys. D **28**, 1941012 (2019).
14. N. Mankoc-Borstnik: Unification of spins and charges in Grassmann space? Mod. Phys. Lett.A **10**, 587-596 (1995).
15. N. Mankoc-Borstnik: Achievements of spin-charge family theory so far. Bled Workshops in Physics. **22**, 202-232 (2021).
16. R.Bernabei, et al: Recent efforts in the DAMA project. Bled Workshops in Physics. **24**, (2023), This issue.
17. M. Y. Khlopov: Composite dark matter from 4th generation, JETP Letters **83**, 1 –4 (2006).
18. V. Beylin, M. Khlopov, V .Kuksa, N. Volchaanskiy: New physics of strong interaction and Dark Universe, Universe **6**, 196 (2020).
19. S.V. Ketov, M.Yu. Khlopov: Cosmological Probes of Supersymmetric Field Theory Models at Superhigh Energy Scales, Symmetry **11**, 511 (2019).
20. A.Chaudhuri, M.Khlopov. Balancing asymmetric dark matter with baryon asymmetry and dilution of frozen dark matter by sphaleron transition. Universe **7**, 275 (2021) https://doi.org/10.3390/universe7080275. arXiv:2106.11646
21. V.A.Beylin, M.Yu.Khlopov, D.O. Sopin: Balancing multiple charge particle excesses with baryon asymmetry. International Journal of Modern Physics D 2340005 (14 pages)(2023) DOI: 10.1142/S0218271823400059
22. T.E. Bikbaev, M.Yu. Khlopov, A.G. Mayorov: Numerical simulation of dark atom interaction with nuclei, Bled Workshops in Physics, **21**, 105-117 (2020), arXiv:2011.01362 [hep-ph].
23. T.E.Bikbaev, M.Yu.Khlopov and A.G. Mayorov: Numerical simulation of Bohr-like and Thomson-like dark atoms with nuclei. Bled Workshops in Physics. **22**, 65-77 (2021), arXiv:2111.09042. This issue
24. M.Yu. Khlopov: Primordial black holes, Res. Astron. Astrophys.**10**, 495 (2010).
25. S.G. Rubin, A.S. Sakharov, M.Y. Khlopov: Formation of primordial galactic nuclei at phase transitions in the early Universe, JETP **92**, 921 (2001).
26. K.M. Belotsky, V.I. Dokuchaev, Y.N. Eroshenko, E.A. Esipova, M.Y. Khlopov, L.A. Khromykh, A.A. Kirillov, V.V. Nikulin, S.G. Rubin, I.V. Svadkovsky: Clusters of primordial black holes, Eur. Phys. J. C **79**, 246 (2019).
27. A.D. Dolgov: Massive primordial black holes in contemporary and young universe (old predictions and new data) Int.J.Mod.Phys. A **33**, 1844029 (2018).
28. The LIGO Scientific Collaboration; the Virgo Collaboration; R. Abbott *et al*: GW190521: A Binary Black Hole Merger with a Total Mass of 150 $M_\odot$, Phys. Rev. Lett. **125**, 101102 (2020)

29. The LIGO Scientific Collaboration; the Virgo Collaboration; R. Abbott *et al*: Properties and astrophysical implications of the 150 Msun binary black hole merger GW190521, Astrophys J. Lett. **900**, L13 (2020)
30. NANOGrav Collaboration, G. Agazie et al.; The NANOGrav 15-year Data Set: Evidence for a Gravitational-Wave Background, 2023 Astrophys J. Lett. **951**, L8 (2023).
31. Shu-Yuan Guo, Maxim Khlopov, Xuewen Liu, Lei Wu, Yongcheng Wu, Bin Zhu: Footprints of Axion-Like Particle in Pulsar Timing Array Data and JWST Observations. arXiv:2306.17022 [hep-ph]. https://doi.org/10.48550/arXiv.2306.17022.
32. Shu-Yuan Guo, Talk at this Workshop.
33. V.M. Chechetkin, M.Yu. Khlopov, M.G. Sapozhnikov, Y.B. Zeldovich: Astrophysical aspects of antiproton interaction with He (Antimatter in the Universe), Phys. Lett. B **118**, 329 (1982).
34. A.D. Dolgov: Matter and antimatter in the universe, Nucl. Phys. Proc. Suppl. **113**, 40 (2002)
35. A. Dolgov, J. Silk: Baryon isocurvature fluctuations at small scales and baryonic dark matter, Phys. Rev. D **47**, 4244 (1993).
36. A.D. Dolgov, M. Kawasaki, N. Kevlishvili: Inhomogeneous baryogenesis, cosmic antimatter, and dark matter, Nucl. Phys. B **807**, 229 (2009).
37. M.Y. Khlopov, S.G. Rubin, A.S. Sakharov: Possible origin of antimatter regions in the baryon dominated Universe, Phys. Rev. D **62**, 083505 (2000).
38. K.M. Belotsky, Y.A. Golubkov, M.Y. Khlopov, R.V. Konoplich, A.S. Sakharov: Antihelium flux as a signature for antimatter globular cluster in our Galaxy, Phys. Atom. Nucl. **63**, 233 (2000).
39. M.Yu. Khlopov: An antimatter globular cluster in our Galaxy - a probe for the origin of the matter, Gravitation and Cosmology, **4**, 69-72 (1998).
40. S.I. Blinnikov, A.D. Dolgov, K.A. Postnov: Antimatter and antistars in the universe and in the Galaxy, Phys. Rev. D **92**, 023516 (2015).
41. V. Poulin, P. Salati, I. Cholis, M. Kamionkowski, J. Silk: Where do the AMS-02 antihelium events come from? Phys. Rev. D **99**, 023016 (2019).
42. M.Yu.Khlopov, O.M.Lecian: Analyses of Specific Aspects of the Evolution of Antimatter Glubular Clusters Domains. Astronomy Reports, **65**, 967–972 (2021) DOI: 10.1134/S106377292110019X
43. M.Khlopov, O.M.Lecian: The Formalism of Milky-Way Antimatter-Domains Evolution. Galaxies, **11**, 50 (2023) https://doi.org/10.3390/galaxies11020050
44. M.Yu. Khlopov, A.O. Kirichenko, A.G. Mayorov: Anihelium flux from antimatter globular cluster, Bled Workshops in Physics, **21**, 118-127 (2020), arXiv:2011.06973 [astro-ph.HE].
45. A.O.Kirichenko, A.V.Kravtsova, M.Yu.Khlopov and A.G. Mayorov Researching of magnetic cutoff for local sources of charged particles in the halo of the Galaxy. Bled Workshops in Physics, **22** 171-177 (2021)

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8 Multidimensional $f(R)-$gravity as the source of primordial black holes

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Abstract. In this paper we suggest mechanism. We calculate black holes mass spectrum arising in quadratic $f(R)-$gravity model with tensor corrections. In the limit of effective field theory for an extra space' scalar curvature it is possible to form dense domain walls. Formed walls occured to be supercritical, thus we follow the results obtained by A. Vilenkin in his papers in order to calculate the mass spectrum of primordial black holes (PBHs) arising via dynamics of domain walls. Taking into account accretion process, considered mechanism could produce black holes which aren't constrainted as dark matter candidates.

Povzetek: Avtorji študirajo masni spekter zgodnjih (primordial) črnih lukenj, ki nastajajo ob mejah med domenami pod superkritičnimi pogoji. Pri tem se opirajo na objavljena dela A. Vilenkina. V akciji za gravitacijsko polje dopustijo kvadratične člene s tenzorskimi popravki in dodajo skalarno polje. Ob upoštevanju nastanka novih tvorb napovedujejo tudi nastanek črnih lukenj, ki bi bile lahko tudi kandidati za temno snov.

8.1 Introduction

$f(R)-$gravity is a class of theories that consider more than just scalar curvature of space-time. Lagrangians of such theories may contain non-linear scalar curvature terms as well as tensor corrections to the scalar curvature.

It is known that one of the central problems of theories with compact extra dimensions is to ensure their compactification and stabilization of [1] during cosmological evolution. This can be done, for example, by $f(R)-$modifying the gravity [2, 3] or by introducing additional scalar fields [4]. The latter approach is particularly promising because the quadratic $f(R)-$gravity Starobinsky [5, 6] gives the best fit to observational constraints on the parameters of cosmological inflation [7]. Moreover, in multidimensional $f(R)-$gravity, the processes of cosmological inflation and compactification are manifestations of the overall gravitational dynamics in different [8] subspaces.

The possibilities of $f(R)-$gravity are widely studied [9, 10], they offer solutions to many cosmological problems [11–14]. One of the problems that $f(R)-$gravity can solve is the existence of PBHs [15].

The idea of the proposed mechanism is based on the known possibility of formation of domain walls in the process of cosmological inflation with their subsequent

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collapse into PBHs [16, 17]. The initial size of the walls u_0 is within the interval:

$$H_{\inf}^{-1} \lesssim u_0 \lesssim H_{\inf}^{-1} \cdot e^{N_{\inf}}, \tag{8.1}$$

where $N_{\inf}$ — number of e–folds of inflation and $H_{\inf}$ is the scale of cosmological inflation. The formation of such domain walls requires a scalar field with a non-trivial potential containing several minima. Exactly such an effective scalar field appears in multidimensional $f(R)-$models in the Einstein frame [3, 8, 18].
The fate of the domain wall formed as a result of quantum fluctuations at the stage of cosmological inflation depends on its surface energy density σ. Domain wall has its own gravitational field. The timescale then its field may become dominant within the Hubble horizon is given by the expression obtained in [19, 20]:

$$t_\sigma = \frac{1}{2\pi G\sigma}, \tag{8.2}$$

where G is a gravitational constant.
Initially, the domain wall is at rest with the Hubble expansion, i.e., its radius grows as a scale factor before it crosses the Hubble horizon. The time t_H when the domain wall crosses the cosmological horizon depends on the parameter of the equation of state ω of the stage of cosmological evolution. For example in case $\omega = 1/3$:

$$t_H = H_{\inf} u_0^2/4 \tag{8.3}$$

If $t_\sigma > t_H$, in that case domain wall crosses the horizon and never could be dominant in the Universe. Such configuration is considered, e.g. in [17, 21].
If $t_\sigma < t_H$, physics changes significantly. Such relation between these timescales is considered in [22, 23] and we will use their results in this paper in order to calculate black holes mass spectrum.

8.2 $f(R)-$gravity model

In this section we briefly discuss the considered model and obtained results. We considered multidimensional quadratic $f(R)-$gravity with tensor corrections. It is given by action in the Jordan frame [1]

$$S[g_{\mu\nu}] = \frac{m_D^{D-2}}{2} \int d^{4+n}x\sqrt{|g_D|}\,\left[f(R) + c_1 R_{AB}R^{AB} + c_2 R_{ABCD}R^{ABCD}\right],$$

$$f(R) = a_2 R^2 + R - 2\Lambda_D\,, \tag{8.4}$$

where m_D — multidimensional Planck mass.Multidimensional space is set to be a direct product $\mathbb{M} = \mathbb{M}_4 \times \mathbb{M}_n$, where $\mathbb{M}_4$ — four-dimensional space, $\mathbb{M}_n$ — compact extra space with n dimensions:

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu - e^{2\beta(t)}d\Omega_n^2\,, \tag{8.5}$$

[1] In this paper we use the following conventions for the Riemann curvature tensor $R^\beta_{\mu\nu\alpha} = \partial_\alpha\Gamma^\beta_{\mu\nu} - \partial_\nu\Gamma^\beta_{\mu\alpha} + \Gamma^\beta_{\sigma\alpha}\Gamma^\sigma_{\nu\mu} - \Gamma^\beta_{\sigma\nu}\Gamma^\sigma_{\mu\alpha}$, and Ricci tensor as follows: $R_{\mu\nu} = R^\alpha_{\mu\alpha\nu}$

where $g_{\mu\nu}$ — four-dimensional metric $\mathbb{M}_4$, β — length function and $d\Omega_n^2$ — volume of a maximally symmetric compact extra space with positive curvature $\mathbb{M}_n$. Let R_4 be the Ricci scalar for $\mathbb{M}_4$ and R_n for $\mathbb{M}_n$. Then we can rewrite R as follows:

$$R = R_4 + R_n + P_k\,, \quad P_k = 2n\,\partial^2\beta + n(n+1)(\partial\beta)^2\,,$$
$$R_n >> R_4, P_k. \tag{8.6}$$

After integrating out extra dimensions we can calculate an effective lagrangian using (8.6) (for details see [3,15]).

$$S = \frac{m_4^2}{2}\int d^4x\sqrt{-g_4}\,\text{sign}(f')\left[R_4 + K(\phi)(\partial\phi)^2 - 2V(\phi)\right]\,, \tag{8.7}$$

where effective Planck mass in Eisnstein frame: $m_4 = \sqrt{2\pi^{\frac{n+1}{2}}/\Gamma(\frac{n+1}{2})}$, and $g_4^{\mu\nu}$ — observable four-dimensional metric. The scalar field ϕ is scalar curvature of extra space: $\phi \equiv R_n$. Action (8.7) contain potential and non-trivial kinetic term, which are expressed by parameters of lagrangian (8.4) (see [8]):

$$K(\phi) = \frac{1}{4\phi^2}\left[6\phi^2\left(\frac{f''}{f'}\right)^2 - 2n\phi\left(\frac{f''}{f'}\right) + \frac{n(n+2)}{2}\right] + \frac{c_1+c_2}{f'\phi}\,, \tag{8.8}$$

$$V(\phi) = -\frac{\text{sign}\,(f')}{2(f')^2}\left[\frac{|\phi|}{n(n-1)}\right]^{n/2}\left[f(\phi) + \frac{c_1 + 2c_2/(n-1)}{n}\phi^2\right]\,. \tag{8.9}$$

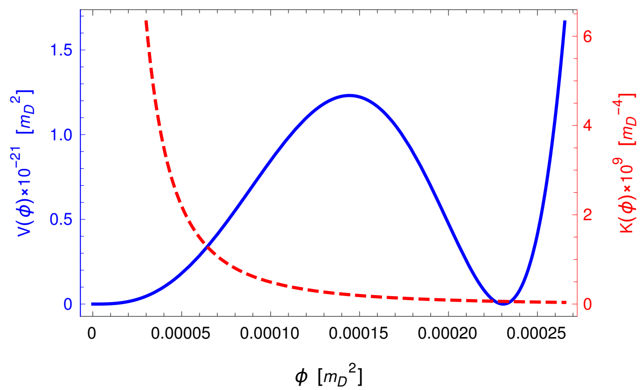

Fig. 8.1: Graphs of potential and kinetic terms (8.8), (8.9) for chosen in [15] parameters: $n = 6$, $c_1 = -8000$, $c_2 = -5000$, $a_2 = -500$. Left minimum is in $\phi = 0$, but it is unreachable because of kinetic term, right minimum is at $\phi_{min} = 3/13000$.

To simplify the lagrangian we make a substitution $d\phi/d\psi = 1/m_4\sqrt{K(\phi)}$ and then obtain:

$$S = \frac{m_4^2}{2}\int d^4x\sqrt{-g_4}\,R_4 + \int d^4x\sqrt{-g_4}\left[\frac{1}{2}(\partial\psi)^2 - V(\psi)\right]\,, \tag{8.10}$$

In [15] we have also calculated characteristic parameters of the domain walls. Surface energy density:

$$\sigma = 5 \cdot 10^{-9} m_D^3 \tag{8.11}$$

and characteristic width of the walls:

$$\delta \approx 1.2 \cdot 10^{11} m_D^{-1} \tag{8.12}$$

One of the result's of that previous paper is relation between fomain wall's radius (u_w) and its gravitational radius (u_g):

$$\frac{u_g}{u_w} = 8\pi G\sigma u_w > m_4^{-2}\sigma\delta \approx 16\,. \tag{8.13}$$

Later in this paper we will show this condition holds for supercritical domain walls.

8.3 Generation of the domain walls

The mass spectrum of the PBHs will be determined by the spectrum of the size of the scalar field fluctuations during the cosmological inflation process. To calculate this spectrum, the section uses the approach developed by Linde [24], in which quantum fluctuations of the scalar field in the cosmological inflation metric are described as random walks [25]. In this case, the density distribution of the values of this field $f(\psi, t)$ is a solution of the Fokker-Planck equation. We follow this approach. At the time of inflation the field ψ evolves in the slow rolling regime, so, neglecting the shape of the potential, the probability density of the field taking the value ψ at time t at the cosmological inflation stage is given by the expression [24,26–29]:

$$f(\psi, t) = \frac{1}{\sqrt{2\pi}\sigma(t)} \exp\left(-\frac{(\psi - \psi_u)^2}{2\sigma^2(t)}\right),\ \sigma(t) = \frac{H_{inf}}{2\pi}\sqrt{H_{inf}t}. \tag{8.14}$$

In this expression, ψ_u — the initial value of the field.
The number of fluctuations leading to the formation of black holes is very sensitive to the choice of initial parameters. The probability of such fluctuation — is the probability that the field ψ will fall into the alternative (left) vacuum of the potential. It's given by the expression:

$$P(t) = \int_{-\infty}^{\psi_{cr}} f(\psi, t)\, d\psi, \tag{8.15}$$

where ψ_{cr} corresponds to the maximum value of the potential potential.
Let us put $\psi_{cr} = \psi_u - \Delta$, $\Delta > 0$, then we obtain the number of critical fluctuations in the form (after some transformations):

$$n_c(t) = P(t)e^{3H_{inf}t} = \frac{1}{2}\text{erfc}\left(\frac{\Delta}{\sqrt{2}\sigma(t)}\right) e^{3H_{inf}t}. \tag{8.16}$$

8.4 PBH mass spectrum

For supercritical domain walls it is true that:

$$t_\sigma < t_H. \tag{8.17}$$

We could rewrite it and involve characteristic width of the walls as follows:

$$\frac{t_\sigma}{2G} < \frac{\delta}{2G} < \frac{t_H}{2G}. \tag{8.18}$$

From (10.4) follows:

$$2\pi G\delta\sigma > 1, \tag{8.19}$$

which is the same as (8.13).
The only observational consequence of collapse of domain walls or wormhole overlapping is the spectrum of masses of formed black holes. In the case of existence of supercritical domain walls, the final mass of the black hole is determined by the initial size of the wall, namely by the comoving volume corresponding to the wall radius.
When a wormhole appears, one of its ends is directed to our Universe and the other to the daughter Universe. The region of our Universe in which the black hole is formed will in this case be external to the domain wall, since it has moved to the daughter Universe.
In the works [22, 23] initial and final masses are calculated taking into account the accretion at the relativistic stage of black holes appearing in the case of supercritical domain walls in the Universe. Following the results of [23], we obtain that the black holes formed at the relativistic stage, taking into account accretion at this stage, have masses:

$$M_{bh} = 5.6 t_H/G. \tag{8.20}$$

According to [22, 23], the mass of a black hole due to the accretion of radiation increases by no more than a factor of two.
Assuming that the inflaton decays almost instantaneously, that is, assuming that the rechiting stage following inflation occurs in a time much shorter than the cosmological inflation, we can put the parameter of the equation of state of matter in the Universe $\omega = 1/3$ at the time of wall stretching. The expression for the radius of the comoving volume at the moment it crosses the horizon at the [17] radiation-dominated stage (for the wall formed at time t during inflation):

$$r(t) = \frac{e^{2(N_{inf} - H_{inf} t)}}{2H_{inf}N_{inf}}. \tag{8.21}$$

By definition, this radius is equal to the Hubble radius at time t_H. The Hubble radius at the RD stage changes with time as follows:

$$r_H = 2t_H. \tag{8.22}$$

The time moment t_H is related to the final mass of the black hole (taking into account accretion) at the RD stage only through the gravitational constant and the numerical coefficient (8.20):

$$t_H = \frac{GM_{bh}}{5.6} \to r_H = \frac{GM_{bh}}{2.8}. \tag{8.23}$$

From (8.21), express the time and substitute (8.23):

$$t = \frac{N_{inf}}{H_{inf}} - \frac{\ln\left(\sqrt{2N_{inf}H_{inf}r_H}\right)}{H_{inf}} = \frac{N_{inf}}{H_{inf}} - \frac{\ln\left(\sqrt{N_{inf}H_{inf}GM_{bh}/1.4}\right)}{H_{inf}}. \tag{8.24}$$

Now, we obtain the PBH mass spectrum by substituting (8.24) into (8.16):

$$n_c(M_{bh}) = \frac{1}{2}\mathrm{erfc}\left(\frac{\sqrt{2}\pi\Delta}{H_{inf}\sqrt{N_{inf} - \ln\left(\sqrt{N_{inf}H_{inf}GM_{bh}/1.4}\right)}}\right) \times \\ \times \frac{1.4^{3/2}\, e^{3N_{inf}}}{(N_{inf}GH_{inf}M_{bh})^{3/2}}. \tag{8.25}$$

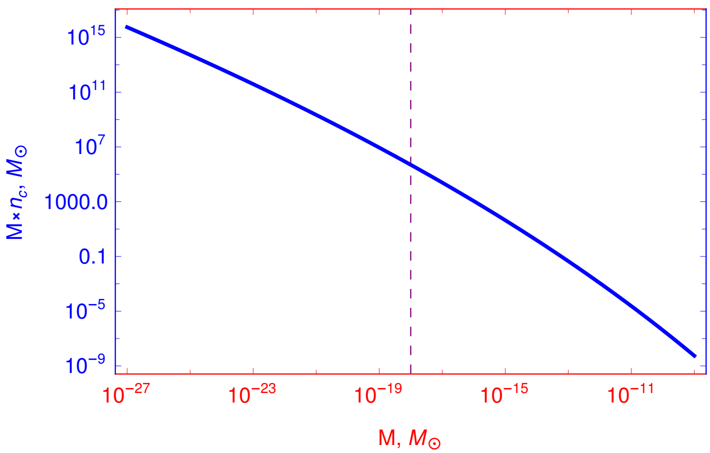

Fig. 8.2: Integral mass spectrum given by (8.25) with accretion of radiation. $\Delta = 9\sigma(t_{inf})$. The minimal mass is determined as: $M_{min} = \delta/2G \sim 10^{-27}\, M_\odot$. Dashed line crosses the black holes which should be evaporated by now.

Note that the expression (8.25) does not explicitly contain the parameters of the original multidimensional modified gravity model —no quantity included in the mass spectrum expression depends on the parameters of the model (8.4). Minimal mass of PBH is determined by characteristic width of domain wall.

8.5 PBH mass spectrum in modern Universe

In order to estimate the form of the spectrum in modern Universe we will use marginal estimate — Eddington limited accretion, although cases of accretion beyond this rate are observed [30], which can be achieved by the lack of spherical symmetry. Time dependence of mass during accretion in the Eddington regime:

$$M(t) = M_0 \exp\left(\frac{4\pi G m_p t}{\varepsilon c \sigma_{th}}\right), \tag{8.26}$$

where m_p is the proton mass, σ_{th} is the Thomson scattering cross section for the electron, M_0 — the initial mass, ε is the radiative efficiency (for Schwarzschild black holes this value is 0.1). It is convenient to rewrite the formula (8.26) as

$$M(t) = M_0 \exp\left(\frac{t}{t_{Edd}}\right), \tag{8.27}$$

where $t_{Edd} \approx 45.1\cdot 10^6$ years. In the early Universe at $z \gtrsim 6$ (the age of the Universe is ≈ 900 million years), supermassive black holes are observed with masses as high as $10^{10}\,M_\odot$ [31–33]. Their origin remains unclear. Attempting to explain their existence by accretion of matter onto black holes of stellar origin requires that these black holes increase their mass at a limiting rate — Eddingtonian.
We assume that black holes in our mechanism would accrete in Eddingtonian regime up to $z \approx 6$, thus their masses increasing by a factor:

$$M_f/M_i = e^{900\cdot 10^6 \text{ years}/t_{Edd}} = e^{900\cdot 10^6 \text{ years}/45.1\cdot 10^6 \text{ years}} \approx e^{20} \sim 5\cdot 10^8 \tag{8.28}$$

To take into account accretion factor we make a substitution $M_{bh} \to M_{bh}/k$ in (8.25)"

$$n_c(M_{bh}) = \frac{1}{2}\text{erfc}\left(\frac{\sqrt{2}\pi\Delta}{H_{\text{inf}}\sqrt{N_{\text{inf}} - \ln\left(\sqrt{N_{\text{inf}}H_{\text{inf}}GM_{bh}/1.4\cdot k}\right)}}\right)\times \\ \times \frac{(1.4\cdot k)^{3/2}\, e^{3N_{\text{inf}}}}{(N_{\text{inf}}GH_{\text{inf}}M_{bh})^{3/2}}. \tag{8.29}$$

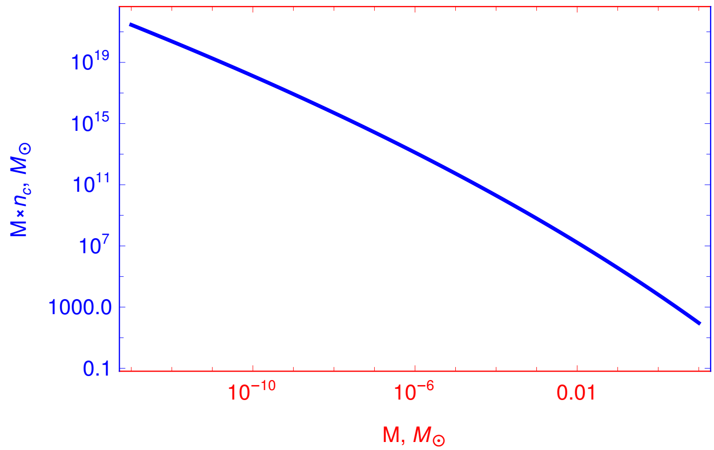

Fig. 8.3: Eddington limited mass spectrum on $z = 6$ (this value is chosen because there is observational data about quasars). $\Delta = 8.5\sigma(t_{\text{inf}})$. Minimal mass is determined by accretion factor $k \approx 5 \cdot 10^8$, according to (8.28). Minimal mass $M_{\text{min}} \sim 10^{-13}\, M_\odot = 10^{20}$ g.

Values for Δ are chosen so that the mass spectrum does not contradict the observations.
According to [34] masses of black holes obtained in presented $f(R)$−gravity model combined with Eddington limited accretion are unconstrainted as dark matter candidates.

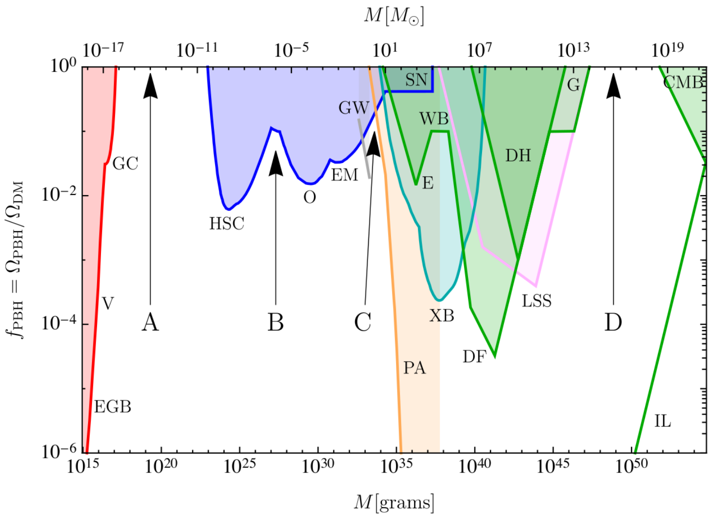

Fig. 8.4: Constraints on black holes as dark matter candidates from [34].

If we assume Eddington limited accretion for initial mass spectrum (8.25) for each PBH, one may notice that spectrum illustrated on Figure 8.3 lies within the interval on Figure 8.4, which is unconstrained. That means the whole dark matter could possibly consist of black holes with such masses.

8.6 Conclusion

The domain walls obtained in the framework of this model are supercritical - their gravitational field becomes very significant within the Hubble horizon long before the walls could cross the cosmological horizon. The results of A. Vilenkin's papers are used here. A distinctive feature of this model is the absence of dependence of the shape of the black hole mass spectrum on the parameters of multidimensional modified gravity. Only the left boundary (minimal mass) depends on the parameters of the initial model. We assume that black holes would accrete in Eddington regime and in that case it is possible to explain a significant part of the dark matter, if it consists of primordial black holes.

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References

1. E. Witten : Instability of the Kaluza-Klein vacuum, Nuclear Physics B **195** (1982) 481–492.
2. R. Tonguç : Acceleration of the Universe via f(R) gravities and the stability of extra dimensions, Physical Review D **75** (2007) 064033.
3. S. Rubin, K. Bronnikov : Self-stabilization of extra dimensions, Physical Review D **73** (2006) 124019.
4. S. Carroll, J. Geddes, M. Hoffman, R. Wald : Classical stabilization of homogeneous extra dimensions, Physical Review D **66** (2002) 024036.
5. A. Starobinsky : A New Type of Isotropic Cosmological Models Without Singularity, Physics Letters B **91** (1980) 99–102.
6. A. Vilenkin : Classical and quantum cosmology of the Starobinsky inflationary model, Physical Review D **32** (1985) 2511.
7. Planck Collaboration : Planck 2018 results. X. Constraints on inflation, Astronomy and Astrophysics **641** (2018) A10.
8. J. Fabris, A. Popov, S. Rubin : Multidimensional gravity with higher derivatives and inflation, Physics Letters B **806** (2020) 135458
9. A. De Felice, S. Tsujikawa : f(R) Theories, Living Reviews in Relativity **13** (2010) 156.
10. S. Capozziello, M. De Laurentis : Extended Theories of Gravity, Physics Reports **509** (2011) 167–321.
11. S. Capozziello, M. De Laurentis : The Dark Matter problem from f(R) gravity viewpoint, Annalen der Physik **524** (2012) 545–578.
12. K. Bronnikov, R. Konoplich, S. Rubin : The diversity of universes created by pure gravity, Classical and Quantum Gravity **24** (2007) 1261–1277.

13. K. Bronnikov, R. Budaev, A. Grobov, A. Dmitriev, S. Rubin : Inhomogeneous compact extra dimensions, Journal of Cosmology and Astroparticle Physics **2017** (2017) 001.
14. K. Bronnikov, A. Popov, S. Rubin : Inhomogeneous compact extra dimensions and de Sitter cosmology, The European Physical Journal C **80** (2020) 970.
15. V. Nikulin, M. Krasnov, S. Rubin : Compact extra dimensions as the source of primordial black holes, Frontiers in Astronomy and Space Sciences **9** (2022) 8.
16. S. Rubin, M. Khlopov, A. Sakharov : Primordial Black Holes from Non-Equilibrium Second Order Phase Transition, Gravitation and Cosmology **6** (2000) 51–58.
17. K. Belotsky, V. Dokuchaev, Y. Eroshenko, E. Esipova, M. Khlopov, L. Khromykh, A. Kirillov, V. Nikulin, S. Rubin, I. Svadkovsky : Clusters of Primordial Black Holes, The European Physical Journal C **79** (2019) 246.
18. Y. Lyakhova, A. Popov, S. Rubin : Classical evolution of subspaces, The European Physical Journal C **78** (2018) 1–13.
19. A. Vilenkin : Gravitational field of vacuum domain walls and strings, Physical Review D **23** (1981) 852–857.
20. J. Ipser, P. Sikivie : The Gravitationally Repulsive Domain Wall, Physical Review D **30** (1984) 712.
21. M. Khlopov, S. Rubin : *Cosmological Pattern of Microphysics in the Inflationary Universe*, Springer Netherlands, 2004.
22. J. Garriga, A. Vilenkin, J. Zhang : Black holes and the multiverse, Journal of Cosmology and Astroparticle Physics **2016** (2016) 064-064.
23. H. Deng, J. Garriga, A. Vilenkin : Primordial black hole and wormhole formation by domain walls, Journal of Cosmology and Astroparticle Physics **2017** (2017) 050–050.
24. A. Linde : Scalar field fluctuations in the expanding universe and the new inflationary universe scenario, Physics Letters B **116** (1982) 335-339.
25. A. Starobinsky : Spectrum of relict gravitational radiation and the early state of the universe, Journal of Experimental and Theoretical Physics Letters **30** (1979) 682-685.
26. V. Dokuchaev, Y. Eroshenko, S. Rubin, D. Samarchenko : Mechanism for the suppression of intermediate-mass black holes, Astronomy Letters **36** (2010) 773–779.
27. A. Vilenkin, L. Ford : Gravitational effects upon cosmological phase transitions, Physical Review D **26** (1982) 1231–1241.
28. A. Starobinsky : Dynamics of phase transition in the new inflationary universe scenario and generation of perturbations, Physics Letters B **117** (1982) 175-178.
29. R. Hardwick, V. Vennin, C. Byrnes, J. Torrado, D. Wands : The stochastic spectator, Journal of Cosmology and Astroparticle Physics **2017** 2017 018–018.
30. K. Inayoshi, Z. Haiman, J. Ostriker : Hyper-Eddington accretion flows on to massive black holes, Monthly Notices of the Royal Astronomical Society **459** (2016) 3738–3755.
31. V. Paliya and others : Supermassive black holes at high redshifts, arXiv e-prints (2019).
32. C. Willott, L. Albert, D. Arzoumanian, J. Bergeron, D. Crampton, P. Delorme, J. Hutchings, A. Omont, C. Reylé, D. Schade : Eddington–limited accretion and the black hole mass funstion at redshift 6, The Astronomical Journal **140** (2010) 546–560.
33. E. Bañados, B. Venemans, C. Mazzucchelli, E. Farina, F. Walter, F. Wang, R. Decarli, D. Stern, X. Fan, F. Davies, J. Hennawi, R. Simcoe, M. Turner, H. Rix, J. Yang, D. Kelson, G. Rudie, J. Winters : An 800-million-solar-mass black hole in a significantly neutral Universe at a redshift of 7.5, Nature **553** (2017) 473–476.
34. B. Carr, F. Kühnel : Primordial black holes as dark matter candidates, Physics Lecture Notes **2017** (2017) 018–018.

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9 How far has so far the Spin-Charge-Family theory succeeded to offer the explanation for the observed phenomena in elementary particle physics and cosmology

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Abstract. This talk discusses the achievements of the *spin-charge-family* theory. The project started in the year 1993 when trying to understand the internal spaces of fermions and bosons with the Grassmann algebra [1]. Recognizing that the Grassmann algebra suggests the existence of the anticommuting fermion states with the integer spin (and commuting boson states with integer spin) and that Grassmann algebra is expressible with two kinds of the Clifford algebras [1–3, 10], both offering a description of the anticommuting half spin fermion states ([6] and references therein), it became obvious that the Clifford odd algebra (the superposition of odd products of γ^a) offers a way for describing the internal spaces for fermions, while $\tilde{\gamma}^a$ can be chosen to determine quantum numbers of families of fermions ([6] and references therein). Three years ago, it become evident that the Clifford even algebra (the superposition of even products of γ^a) offers a way to describe the internal spaces of the corresponding boson gauge fields ([7, 8] and references therein). Can the *spin-charge-family* theory, extending the point fields to strings, be related to *string theories*?

Povzetek: Prispevek predstavi dosežke teorije *spin-charge-family*. Projekt teče od leta 1993 s poskusom opisati notranje prostore fermionov in bozonov z Grassmannovo algebro [1]. Ob spoznanju, da Grassmannova algebra ponudi opis antikomutirajočih fermionskih stanj s celoštevilčnim spinom (in komutirajočih bozonskih stanj s celoštevilčnim spinom) in da je mogoče izraziti Grassmannovo algebro z dvema vrstama Cliffordovih algeber [1–3, 10], ki obe ponujata opis antikomutirajočih fermionskih stanj s polštevilskim spinom ([6] in v vsebovanih referencah), je postalo očitno, da Cliffordova liha algebra (superpozicija lihih produktov γ^a) ponuja opis notranjih prostorov za fermione, $\tilde{\gamma}^a$ pa kvantna števila za družine fermionov [6]. Pred tremi leti je postalo očitno, da ponuja Cliffordova soda algebra (superpozicija sodih produktov γ^a) način za opis notranjih prostorov ustreznih bozonskih umeritvenih polj ([7, 8] in v vsebovanih referencah). Ali lahko teorijo *spin-charge-family*, ki opisuje doslej točkasta polja, če jih razširimo v strune, povežemo s *teorijami strun*?

Keywords: Second quantization of fermion and boson fields with Clifford algebra; Beyond the standard model; Kaluza-Klein-like theories in higher dimensional spaces; Clifford algebra in odd dimensional spaces; Ghosts in quantum field theories

9.1 Introduction

The *standard model* (corrected with the right-handed neutrinos) has been experimentally confirmed without raising any severe doubts so far on its assumptions, which, however, remain unexplained.
The *standard model* assumptions have several explanations in the literature, mostly with several new, not explained assumptions. The most popular are the grand unifying theories ([7–11] and many others).

In a long series of works ([1,2,9,12], and the references there in) the author has found, together with the collaborators ([3–6,6,19,41,42] and the references therein), the phenomenological success with the model named the *spin-charge-family* theory with the properties:

a. In $d \geq (13+1)$ the creation operators manifest in $d = (3+1)$ the properties of all the observed quarks and leptons, with the families included, and of their gauge boson fields, with the scalar fields included, making several predictions.
a.i. The internal space of fermions are in each even dimension ($d = 2(2n+1)$, $d = 4n$) described by the "basis vectors", $\hat{b}^{m\dagger}_{f}$, which are superposition of odd products of anti-commuting objects (operators) γ^a's, appearing in $2^{\frac{d}{2}-1}$ families, each family with $2^{\frac{d}{2}-1}$ members. Correspondingly the "basis vectors" of one Lorentz irreducible representation in internal space of fermions, together with their Hermitian conjugated partners, anti-commute, fulfilling (on the vacuum state) all the requirements for the second quantized fermion fields ([3, 6] and references therein).
a.i.*i.* The second kind of anti-commuting objects, $\tilde{\gamma}^a$, Sect. 9.2.1, equip each irreducible representation of odd "basis vectors" with the family quantum number [3,5]. Correspondingly each of $2^{\frac{d}{2}-1}$ families carries $2^{\frac{d}{2}-1}$ members.
a.i.*ii.* Creation operators for single fermion states, inheriting anti-commutativity of "basis vectors", are tensor products, $*_T$, of a finite number of odd "basis vectors", and the (continuously) infinite momentum/coordinate basis. Applying on the vacuum state [5,6], the creation operators and their Hermitian conjugated partners fulfil the anticommutation relations for the second quantized fermion states.
a.i.*iii.* The Hilbert space of second quantized fermion field is represented by the tensor products, $*_{T_H}$, of all possible numbers of creation operators, from zero to infinity [6], applying on a vacuum state.
a.i.*iv***.** Spins from higher dimensions, $d > (3+1)$, described by the eigenvectors of the superposition of the Cartan subalgebra of S^{ab}, manifest in $d = (3+1)$ all the charges of the *standard model* quarks and leptons and antiquarks and antileptons, as the reader can see in Table 9.3 for one, anyone, of families.
a.i.*v***.** Let be pointed out that in even $d = (13+1)$ one irreducible representation of the Clifford odd "basis vectors" contains fermions and antifermions, quarks and leptons and antiquarks and antileptons.
a.ii. The internal space of bosons are described by the "basis vectors" which are superposition of even products of anti-commuting objects (operators) γ^a's: They appear in two (orthogonal) groups, each with $2^{\frac{d}{2}-1} \times 2^{\frac{d}{2}-1}$ members, called ${}^{I}\mathcal{A}^{m\dagger}_{f}$

and ${}^{II}\mathcal{A}_f^{m\dagger}$, respectively [1].

a.ii.*i.* The Clifford even "basis vectors" have the properties of the gauge fields of the Clifford odd "basis vectors"; They do not appear in families, and have their Hermitian conjugated partners within the same group.

a.ii.*ii.* One group of the Clifford even "basis vectors" transforms, when applying algebraically on the Clifford odd "basis vector", this Clifford odd "basis vector" into other members of the same family. The other group of the Clifford even "basis vectors" transform, when applying algebraically on the Clifford odd "basis vector", this Clifford odd "basis vector" into the same member of another family; what also the spin connection fields, $\omega_{ab\alpha}$ and $\tilde{\omega}_{ab\alpha}$, presented in the action, Eq. (9.1), do.

a.ii.*iii.* Creation operators for boson states, inheriting commutativity of "basis vectors", are tensor products, $*_T$, of a finite number of even "basis vectors", and the (continuously) infinite momentum/coordinate basis.

a.ii.*iv***.** Spins from higher dimensions, $d > (3+1)$, described by the eigenvectors of the superposition of the Cartan subalgebra of $\mathcal{S}^{ab} = S^{ab} + \tilde{S}^{ab}$ [2], manifest as charges (in adjoint representations) of the gauge fields — the vector ones with the space index $\alpha = (0,1,2,3)$, and the scalar ones with the space index $\alpha \geq 5$).

a.ii.*v***.** Besides the internal space, the Clifford even "basis vectors" must carry also the space index α, to be able to represent the vector and scalar gauge fields; all the scalar fields carry in the *spin-charge-family* theory the space index, assumed as ${}^{i}\mathcal{A}_f^{m\dagger i}\mathcal{C}^m_{f\alpha}, i = (I, II)$[3]. Both groups, ${}^{I}\mathcal{A}_f^{m\dagger I}\mathcal{C}^m_{f\alpha}$ and ${}^{II}\mathcal{A}_f^{m\dagger II}\mathcal{C}^m_{f\alpha}$ can be related to $\omega_{ab\alpha}$ and $\tilde{\omega}_{ab\alpha}$, presented in Eq. (9.1), respectively. These relations can be found in Eq. (37) of Ref. [7].

a.iii. In odd dimensional spaces, $d = (2n+1)$, the properties of the internal space of fermions and bosons differ essentially from those in even dimensional spaces; one half of the "basis vectors" have the properties as those from $d = 2n$, the second half, following from those of $d = 2n$ by the application of $S^{0\,2n+1}$, behave as the Fadeev-Popov ghost — anticommuting appear in two orthogonal groups with the Hermitian conjugated partners within the same group, commuting appear in families and have their Hermitian conjugated partners in a separate group [8].

a.iii.*i***.** The theory offers a new understanding of the second quantized fermion fields, as mentioned in **a.** and it is explained in Refs. [5, 6], it also enables a new understanding of the second quantization of boson fields as presented in Refs. [7,8,11,21].

The properties of the Clifford odd and even "basis vectors" wait to be studied.

[1] The Clifford odd "basis vectors", appearing in $2^{\frac{d}{2}-1}$ families, each family having $2^{\frac{d}{2}-1}$ members, contain together with their Hermitian conjugated partners twice $2^{\frac{d}{2}-1} \times 2^{\frac{d}{2}-1}$ objects, the same as the two groups of the Clifford even "basis vectors".

[2] The definition of $S^{ab} = \frac{i}{4}(\gamma^a\gamma^b - \gamma^b\gamma^a)$, and $\tilde{S}^{ab} = \frac{i}{4}(\tilde{\gamma}^a\tilde{\gamma}^b - \tilde{\gamma}^b\tilde{\gamma}^a)$.

[3] The reader can find the demonstration of the properties of the fermion and boson gauge fields for $d = (5+1)$ case in Subsect. 2.3 of Ref. [7]

b. In a simple starting action, Eq. (9.1), in $d = 2(2n+1)$-dimensional space

$$\begin{aligned}
\mathcal{A} &= \int d^d x\, E\, \frac{1}{2}\, (\bar{\psi}\, \gamma^a p_{0a} \psi) + h.c. + \\
&\quad \int d^d x\, E\, (\alpha\, R + \tilde{\alpha}\, \tilde{R})\,, \\
p_{0\alpha} &= p_\alpha - \frac{1}{2} S^{ab} \omega_{ab\alpha} - \frac{1}{2} \tilde{S}^{ab} \tilde{\omega}_{ab\alpha}\,, \\
p_{0a} &= f^\alpha{}_a p_{0\alpha} + \frac{1}{2E} \{p_\alpha, E f^\alpha{}_a\}_-\,, \\
R &= \frac{1}{2} \{f^{\alpha[a} f^{\beta b]}\, (\omega_{ab\alpha,\beta} - \omega_{ca\alpha}\, \omega^c{}_{b\beta})\} + h.c.\,, \\
\tilde{R} &= \frac{1}{2} \{f^{\alpha[a} f^{\beta b]}\, (\tilde{\omega}_{ab\alpha,\beta} - \tilde{\omega}_{ca\alpha}\, \tilde{\omega}^c{}_{b\beta})\} + h.c.\,,
\end{aligned} \tag{9.1}$$

with [4] $f^{\alpha[a} f^{\beta b]} = f^{\alpha a} f^{\beta b} - f^{\alpha b} f^{\beta a}$ [5] [6], massless fermions carry only spins and interact with only gravity — with the vielbeins and the two kinds of spin connection fields (the gauge fields of momenta, of $S^{ab} = \frac{i}{4}(\gamma^a \gamma^b - \gamma^b \gamma^a)$ and of $\tilde{S}^{ab} = \frac{1}{4}(\tilde{\gamma}^a \tilde{\gamma}^b - \tilde{\gamma}^b \tilde{\gamma}^a)$, respectively [7]). The starting action includes only even products of γ^a's and $\tilde{\gamma^a}$'s ([6] and references therein).
b.i. Vielbeins can be expressed by ${}^i\mathcal{A}^{m\dagger}_f {}^i\mathcal{C}^m_{f\alpha}$, $i = (I, II)$. Also this way of representing vielbeins needs further studies.
b.ii. Gravity fields in d, which are the gauge fields of S^{ab}, $((a,b) = (5,6,....,d))$, with the space index $\alpha = m = (0,1,2,3)$, manifest as the *standard model* vector gauge fields [4], those with the space index $\sigma = 5,6,7,...,d$, manifest as the *standard model* scalar gauge fields [4,6], those with $(a,b) = (0,1,2,3))$, and with the space index $\alpha = (0,1,2,3)$ manifest as ordinary gravity. The supersymmetric transformations for all three kinds of boson fields deserve further studies. In the gravity case the expression of graviton with superposition of two ${}^i\mathcal{A}^{m\dagger}_f {}^i\mathcal{C}^m_{f\alpha}$, $i = (I, II)$, and

[4] $f^\alpha{}_a$ are inverted vielbeins to $e^a{}_\alpha$ with the properties $e^a{}_\alpha f^\alpha{}_b = \delta^a{}_b$, $e^a{}_\alpha f^\beta{}_a = \delta^\beta_\alpha$, $E = \det(e^a{}_\alpha)$. Latin indices $a, b, .., m, n, .., s, t, ..$ denote a tangent space (a flat index), while Greek indices $\alpha, \beta, .., \mu, \nu, ..\sigma, \tau, ..$ denote an Einstein index (a curved index). Letters from the beginning of both the alphabets indicate a general index ($a, b, c, ..$ and $\alpha, \beta, \gamma, ..$), from the middle of both the alphabets the observed dimensions $0, 1, 2, 3$ ($m, n, ..$ and $\mu, \nu, ..$), indexes from the bottom of the alphabets indicate the compactified dimensions ($s, t, ..$ and $\sigma, \tau, ..$). We assume the signature $\eta^{ab} = diag\{1, -1, -1, \cdots, -1\}$.

[5] The vielbeins, f^a_α, and the two kinds of the spin connection fields, $\omega_{ab\alpha}$ (the gauge fields of S^{ab}) and $\tilde{\omega}_{ab\alpha}$ (the gauge fields of $\tilde{S}^{ab}$), manifest in $d = (3+1)$ as the known vector gauge fields and the scalar gauge fields taking care of masses of quarks and leptons and antiquarks and antileptons and of the weak boson fields [4,9,12,19]

[6] Since the multiplication with either γ^a's or $\tilde{\gamma}^a$'s changes the Clifford odd "basis vectors" into the Clifford even objects, and even "basis vectors" commute, the action for fermions can not include an odd numbers of γ^a's or $\tilde{\gamma}^a$'s, what the simple starting action of Eq. (9.1) does not. In the starting action γ^a's and $\tilde{\gamma}^a$'s appear as $\gamma^0 \gamma^a \hat{p}_{0a}$ or as $\gamma^0 \gamma^c\, S^{ab} \omega_{abc}$ and as $\gamma^0 \gamma^c\, \tilde{S}^{ab} \tilde{\omega}_{abc}$.

[7] If no fermions are present, the two kinds of spin connection fields are uniquely expressible by the vielbeins.

gravitino with the superposition of ${}^{i}\mathcal{A}^{m\dagger}_{f}{}^{i}\mathcal{C}^{m}_{f\alpha}, i = (I, II)$ and $\hat{b}^{m\dagger}_{f}$ seems promising.

b.ii.*i.* The scalar gauge fields of $\tilde{S}^{ab}$, and of particular superposition of S^{ab}, with the space index $s = (7, 8)$ manifest as the scalar higgs and Yukawa couplings [6,9], determining mass matrices (of $\widetilde{SU}(2) \times \widetilde{SU}(2) \times U(1)$ symmetry) and correspondingly the masses of quarks and leptons (predicting the fourth families to the observed three) and of the weak boson fields after (some of) the scalar fields with the space index $(7, 8)$ gain constant values.
The theory predicts at low energy two groups with four families. To the lower group of four families the so far observed three belong [39–41, 43, 44], and the stable of the upper four families, the fifth family of (heavy) quarks and leptons, offers the explanation for the appearance of dark matter. Due to the heavy masses of the fifth family quarks, the nuclear interaction among hadrons of the fifth family members is very different than the ones so far observed [42, 45].
b.ii.*ii.* The scalar gauge fields of $\tilde{S}^{ab}$ and of S^{ab} with the space index $s = (9, 10, ..., 14)$ and $(a, b) = (5, 6, ..., d)$ offer the explanation for the observed matter/antimatter asymmetry [6, 9, 12, 19] in the universe.

The theory seems very promising to offer a new insight into the second quantization of fermion and boson fields and to show the next step beyond the *standard model*.
The more work is put into the theory, the more phenomena the theory can explain.
Let me add: Other references use a different approach by trying to make the next step with Clifford algebra to the second quantized fermion, which might also be a boson field [17, 47].

In Sect. 10.2, creation and annihilation operators for fermions and bosons in even and odd dimensional spaces are presented.
Subsect. 9.2.1 starts with relating the Grassmann algebra with the two Clifford subalgebras.
In Subsect. 10.2.2, "basis vectors" in even and odd-dimensional spaces are presented.
In Subsect. 9.2.3 creation and annihilation operators are described as tensor products of the "basis vectors" and basis in ordinary space.
Sect. 9.3 the *spin-charge-family* theory so far are shortly overviewed.
Sect. 10.3 presents what the reader could learn new from this article.
In App. 9.5, the properties of the Clifford even "basis vectors" are demonstrated in the toy model in $d = (5 + 1)$.
In App. 9.6, the reader can find concrete examples for $d = (3 + 1)$, taken from Ref. [7].
In App. 9.7, some useful formulas and relations are presented.
In App. 9.8 one irreducible representation (one family) of $SO(13, 1)$, group, analysed with respect to $SO(3, 1)$, $SU(2)_{I}$, $SU(2)_{II}$, $SU(3)$, and $U(1)$, representing "basis vectors" of quarks and leptons and antiquarks and antilepons is discussed.

9.2 Creation and annihilation operators for fermions and bosons in even and odd dimensional spaces

This section reviews the Refs. [1,5–7].
Refs. [1, 3, 5, 6, 12], describing the internal space of fermion fields by the superposition of odd products of γ^a in even dimensional spaces ($d = 2(2n+1)$, or $d = 4n$, in $d = (13+1)$ indeed), demonstrate that the odd Clifford algebra offers the description of families of quarks and leptons and antiquarks and antileptons as assumed by the *standard model* before the electroweak phase transitions. Table 9.3 demonstrates the "basis vectors" of one family of quarks and leptons and antiquarks and antileptons, Table Table III represents the basis vectors of eight families of quarks u and neutrinos. Describing the internal space of fermions with the Clifford odd objects enables us to explain the second quantization postulates for fermions [25–27].
The assumed simple action, Eq. (9.1), in which fermions interact with only the gravitational fields in $d = (13+1)$-dimensional space, contains from the point of view of $d = (3+1)$ all the vector gauge fields and the scalar gauge fields, assumed by the *standard model* [4, 6, 28, 40, 43, 44]. This simple action explains also several observed phenomena in cosmology [12, 30, 42, 45].
Although it was expected all the time, it became evident only three years ago that while the Clifford odd algebra offer the description of the internal space of fermions, offers Clifford even algebra the explanation of the internal space of the corresponding boson gauge fields, offering the understanding of the second quantized boson fields [25–27].
In even dimensional spaces, the number of "basis vectors" and their Hermitian conjugated partners is the same for fermion and boson fields: $2^{\frac{d}{2}-1} \times 2^{\frac{d}{2}-1} \times 2$.
Fermion "basis vectors" appear in $2^{\frac{d}{2}-1}$ families (irreducible representations), each family has $2^{\frac{d}{2}-1}$ members. Their Hermitian conjugated partners appear in a separate group.
Boson "basis vectors" appear in two groups, with the Hermitian conjugated partners within the same group, each group has $2^{\frac{d}{2}-1} \times 2^{\frac{d}{2}-1}$ members.

9.2.1 Grassmann and Clifford algebras and representations of Clifford subalgebras

This part is a short overview of several references, cited in Ref. ([6], Subsects. 3.2,3.3), also appearing in Ref. [5,7,8,12].
In Grassmann space the infinitesimal generators of the Lorentz transformations $\mathcal{S}^{ab}$ are expressible with anticommuting coordinates θ^a and their conjugate momenta $p^{\theta a} = i \frac{\partial}{\partial \theta_a}$ [1],

$$\{\theta^a, \theta^b\}_+ = 0\,, \quad \{p^{\theta a}, p^{\theta b}\}_+ = 0\,, \quad \{p^{\theta a}, \theta^b\}_+ = i\eta^{ab}\,,$$
$$\mathcal{S}^{ab} = \theta^a p^{\theta b} - \theta^b p^{\theta a}\,. \tag{9.2}$$

Grassmann space offers the description of the internal degrees of freedom of fermions and bosons in the second quantized procedure [10]. In both cases there

exist the creation and annihilation operators, which fulfil the anticommutation relations required for fermions, and commutation relations for bosons [10].
Making a choice [19]

$$(\theta^a)^\dagger = \eta^{aa} \frac{\partial}{\partial\theta_a}\,, \quad \text{leads to} \quad (\frac{\partial}{\partial\theta_a})^\dagger = \eta^{aa}\theta^a\,, \tag{9.3}$$

with $\eta^{ab} = diag\{1, -1, -1, \cdots, -1\}$.
θ^a and $\frac{\partial}{\partial\theta_a}$ are, up to the sign, Hermitian conjugated to each other. The identity is the self adjoint member of the algebra. The choice for the following complex properties of θ^a

$$\{\theta^a\}^* = (\theta^0, \theta^1, -\theta^2, \theta^3, -\theta^5, \theta^6, ..., -\theta^{d-1}, \theta^d)\,, \tag{9.4}$$

correspondingly requires $\{\frac{\partial}{\partial\theta_a}\}^* = (\frac{\partial}{\partial\theta_0}, \frac{\partial}{\partial\theta_1}, -\frac{\partial}{\partial\theta_2}, \frac{\partial}{\partial\theta_3}, -\frac{\partial}{\partial\theta_5}, \frac{\partial}{\partial\theta_6}, ..., -\frac{\partial}{\partial\theta_{d-1}}, \frac{\partial}{\partial\theta_d})\,.$
There are 2^d superposition of products of θ^a, the Hermitian conjugated partners of which are the corresponding 2^d superposition of products of $\frac{\partial}{\partial\theta_a}$.
There exist two kinds of the Clifford algebra elements (operators), γ^a and $\tilde{\gamma}^a$, expressible with θ^a's and their conjugate momenta $p^{\theta a} = i\,\frac{\partial}{\partial\theta_a}$ [1,6],

$$\begin{aligned}
&\gamma^a = (\theta^a + \frac{\partial}{\partial\theta_a})\,, \quad \tilde{\gamma}^a = i\,(\theta^a - \frac{\partial}{\partial\theta_a})\,,\\
&\qquad\qquad \text{from where it follows}\\
&\theta^a = \frac{1}{2}\,(\gamma^a - i\tilde{\gamma}^a)\,, \quad \frac{\partial}{\partial\theta_a} = \frac{1}{2}\,(\gamma^a + i\tilde{\gamma}^a)\,,
\end{aligned} \tag{9.5}$$

offering together $2 \cdot 2^d$ operators: 2^d are superposition of products of γ^a and 2^d superposition of products of $\tilde{\gamma}^a$. It is easy to prove if taking into account Eqs. (10.2, 10.4), that they form two anti-commuting Clifford subalgebras, $\{\gamma^a, \tilde{\gamma}^b\}_+ = 0$, Refs. ([6] and references therein)

$$\begin{aligned}
&\{\gamma^a, \gamma^b\}_+ = 2\eta^{ab} = \{\tilde{\gamma}^a, \tilde{\gamma}^b\}_+\,,\\
&\{\gamma^a, \tilde{\gamma}^b\}_+ = 0\,, \quad (a, b) = (0, 1, 2, 3, 5, \cdots, d)\,,\\
&\quad (\gamma^a)^\dagger = \eta^{aa}\,\gamma^a\,, \quad (\tilde{\gamma}^a)^\dagger = \eta^{aa}\,\tilde{\gamma}^a\,.
\end{aligned} \tag{9.6}$$

The Grassmann algebra offers the description of the "anti-commuting integer spin second quantized fields" and of the "commuting integer spin second quantized fields", the reader is invited to read [5, 6, 10], which offer the representations and equations of motion when using Grassmann algebra to describe internal spaces of fermions and bosons.
Each of the two Clifford algebras which are superposition of odd products of either γ^a's or $\tilde{\gamma}^a$'s offers the description of the second quantized half integer spin fermion fields in the fundamental representations of the group and subgroups, Table 9.3.
The superposition of even products of either γ^a's or $\tilde{\gamma}^a$'s offer the description of the commuting second quantized boson fields with integer spins [11,21] which from the point of the subgroups of the $SO(d-1, 1)$ group manifest spins and

charges in the adjoint representations of the group and subgroups, Table (9.3, Tables (1, 2, 3, 4) in Ref. [7], Table 9.3.

There is no fermions with the integer spin observed and there is only one kind of half spin fermions and their integer spin gauge fields observed so far.
The *postulate*, which determines how does the operator $\tilde{\gamma}^a$ operate on polynomial of γ^a
$A = \sum_{k=0}^{d} a_{a_1 a_2 \ldots a_k} \gamma^{a_1}\gamma^{a_2}\ldots\gamma^{a_k}$, $a_i \le a_{i+1}$ [1,3,9,19]

$$\tilde{\gamma}^a A = (-)^A \, i \, A\gamma^a \,, \tag{9.7}$$

with $(-)^A = -1$, if A is (a function of) odd products of γ^a's, otherwise $(-)^A = 1$, reduces the two Clifford sub algebras to only one. $\tilde{\gamma}^a$, indeed $\tilde{S}^{ab} = \frac{i}{4}(\tilde{\gamma}^a\tilde{\gamma}^b - \tilde{\gamma}^b\tilde{\gamma}^a)$, equip each irreducible representation with the family quantum numbers. The subalgebra, determined by $\tilde{\gamma}^a$'s, looses its meaning.

The "basis vectors" for either fermions or bosons will be defined in Subsect. 10.2.2 as products of eigenvectors of each of the chosen Cartan subalgebra member

$$\begin{aligned} &S^{03}, S^{12}, S^{56}, \cdots, S^{d-1\; d}\,, \\ &\tilde{S}^{03}, \tilde{S}^{12}, \tilde{S}^{56}, \cdots, \tilde{S}^{d-1\; d}\,, \\ &\mathcal{S}^{ab} = S^{ab} + \tilde{S}^{ab} = i\,(\theta^a \frac{\partial}{\partial\theta_b} - \theta^b \frac{\partial}{\partial\theta_a})\,. \end{aligned} \tag{9.8}$$

Each eigenvector of S^{ab}, $\tilde{S}^{ab}$or $\mathcal{S}^{ab}$, chosen to be Cartan subalgebra members of the Lorentz algebra in the internal space of fermions and bosons, Eq. (9.8), can be superposition of either an odd or an even number of γ^a's: either as $(\alpha\gamma^a + \beta\gamma^b)$ or as $(\alpha + \beta\gamma^a\gamma^b)$, respectively.

9.2.2 "Basis vectors" describing internal spaces of fermions and bosons in even and odd dimensional spaces

We use the technique [1,3] which makes "basis vectors" products of nilpotents and projectors which are eigenvectors of the chosen Cartan subalgebra members, Eq. (9.8), of the Lorentz algebra in the space of γ^a's, either in the case of the Clifford odd or in the case of the Clifford even products of γ^a's.
There are in even-dimensional spaces $\frac{d}{2}$ members of the Cartan subalgebra, Eq. (9.8). In odd-dimensional spaces there are $\frac{d-1}{2}$ members of the Cartan subalgebra.
In even dimensional spaces, one can define for any of the $\frac{d}{2}$ Cartan subalgebra members S^{ab} or $\tilde{S}^{ab}$ or of both kinds the nilpotent $\stackrel{ab}{(k)}$ and the projector $\stackrel{ab}{[k]}$

$$\begin{aligned} &\stackrel{ab}{(k)}:= \frac{1}{2}(\gamma^a + \frac{\eta^{aa}}{ik}\gamma^b)\,, \quad (\stackrel{ab}{(k)})^2 = 0\,, \\ &\stackrel{ab}{[k]}:= \frac{1}{2}(1 + \frac{i}{k}\gamma^a\gamma^b)\,, \quad (\stackrel{ab}{[k]})^2 = \stackrel{ab}{[k]}\,. \end{aligned} \tag{9.9}$$

It follows, if taking into account the relations, Eqs. (10.5, 10.6)

$$
\begin{aligned}
S^{ab} \stackrel{ab}{(k)} &= \frac{k}{2} \stackrel{ab}{(k)}, \qquad & \tilde{S}^{ab} \stackrel{ab}{(k)} &= \frac{k}{2} \stackrel{ab}{(k)}, \\
S^{ab} \stackrel{ab}{[k]} &= \frac{k}{2} \stackrel{ab}{[k]}, \qquad & \tilde{S}^{ab} \stackrel{ab}{[k]} &= -\frac{k}{2} \stackrel{ab}{[k]},
\end{aligned}
\tag{9.10}
$$

with $k^2 = \eta^{aa}\eta^{bb}$ [8], demonstrating that the eigenvalues of S^{ab} (determining the spin) on nilpotents and projectors expressed with γ^a differ from the eigenvalues of $\tilde{S}^{ab}$ (determining the family quantum number) on nilpotents and projectors expressed with γ^a [9]
Let us point out that eigenvalues of S^{ab}, determining the spin, and the eigenvalues of $\tilde{S}^{ab}$, determining the family quantum numbers, are half integer, $\pm\frac{1}{2}$ or $\pm\frac{i}{2}$, while the eigenvalues of $\mathcal{S}^{ab}$, expressible with $(S^{ab} + \tilde{S}^{ab})$ are integers, ± 1 and zero or $\pm i$ and zero.

In even dimensional spaces, the "basis vectors" can be defined as algebraic, $*_A$, products of nilpotents and projectors so that each product is the eigenvector of all the $\frac{d}{2}$ Cartan subalgebra members of Eq.(9.8).

The fermion "basis vectors" can be chosen as the algebraic, $*_A$, products of an odd number of the nilpotents and the rest of the projectors; each of them is the eigenvector of one of the Cartan subalgebra members.
The boson "basis vectors" are the algebraic, $*_A$ products of an even number of nilpotents and the rest of the projectors. (In App. 9.6, the reader can find concrete examples.)
It follows that the Clifford odd "basis vectors", which are the superposition of odd products of γ^a, must include an odd number of nilpotents, at least one, while the superposition of an even products of γ^a, that is Clifford even "basis vectors", must include an even number of nilpotents or only projectors.
Correspondingly the Clifford odd "basis vectors" have in even d properties appropriate to describe the internal space of the second quantized fermion fields while the Clifford even "basis vectors" have properties appropriate to describe the internal space of the second quantized boson fields.

Taking into account Eqs. (10.5, 10.6) one finds [7]

$$
\begin{aligned}
&\gamma^a \stackrel{ab}{(k)} = \eta^{aa} \stackrel{ab}{[-k]}, \quad \gamma^b \stackrel{ab}{(k)} = -ik \stackrel{ab}{[-k]}, \quad \gamma^a \stackrel{ab}{[k]} = \stackrel{ab}{(-k)}, \quad \gamma^b \stackrel{ab}{[k]} = -ik\eta^{aa} \stackrel{ab}{(-k)}, \\
&\tilde{\gamma}^a \stackrel{ab}{(k)} = -i\eta^{aa} \stackrel{ab}{[k]}, \quad \tilde{\gamma}^b \stackrel{ab}{(k)} = -k \stackrel{ab}{[k]}, \quad \tilde{\gamma}^a \stackrel{ab}{[k]} = i \stackrel{ab}{(k)}, \quad \tilde{\gamma}^b \stackrel{ab}{[k]} = -k\eta^{aa} \stackrel{ab}{(k)},
\end{aligned}
\tag{9.11}
$$

More relations are presented in App. 9.7.
The relations of Eq. (10.10) demonstrate that the properties of "basis vectors" which include an odd number of nilpotents, differ essentially from the "basis vectors",

---

[8] Let us prove one of the relations in Eq. (10.9): $S^{ab} \stackrel{ab}{(k)} = \frac{i}{2}\gamma^a\gamma^b\frac{1}{2}(\gamma^a + \frac{\eta^{aa}}{ik}\gamma^b) = \frac{1}{2^2}\{-i(\gamma^a)^2\gamma^b + i(\gamma^b)^2\gamma^a\frac{\eta^{aa}}{ik}\} = \frac{1}{2}\frac{\eta^{aa}\eta^{bb}}{k}\frac{1}{2}\{\gamma^a + \frac{k^2}{\eta^{bb}ik}\gamma^b\}$. For $k^2 = \eta^{aa}\eta^{bb}$ the first relation follows.

[9] The reader can find the proof of Eq. (10.9) also in Ref. [6], App. (I).

which include an even number of nilpotents. One namely recognizes [7]:

i. The Hermitian conjugated partner of a nilpotent $\stackrel{ab}{(k)}^{\dagger}$ is $\eta^{aa} \stackrel{ab}{(-k)}$; correspondingly neither S^{ab} nor $\tilde{S}^{ab}$ nor both can transform odd products of nilpotents to be one of the $2^{\frac{d}{2}-1}$ members of one of $2^{\frac{d}{2}-1}$ irreducible representations (families). The Hermitian conjugated partners of the Clifford odd "basis vectors" must belong to a different group of $2^{\frac{d}{2}-1}$ members of $2^{\frac{d}{2}-1}$ families.

Since S^{ac} transforms $\stackrel{ab}{(k)} *_A \stackrel{cd}{(k')}$ into $\stackrel{ab}{[-k]} *_A \stackrel{cd}{[-k']}$, and $\tilde{S}^{ab}$ transforms $\stackrel{ab}{(k)} *_A \stackrel{cd}{(k')}$ into $\stackrel{ab}{[k]} *_A \stackrel{cd}{[k']}$, the Hermitian conjugated partners of the Clifford even "basis vectors" must belong to the same group of $2^{\frac{d}{2}-1} \times 2^{\frac{d}{2}-1}$ members. Projectors are self-adjoint.

ii. While an odd product of γ^a anti-commute with another odd product of γ^a, the Clifford odd "basis vectors" anti-commute. An even product of γ^a commute with another even (or odd) product of γ^a, therefore the Clifford even "basis vectors" commute.

In the tensor product, $*_T$, with basis in ordinary space, Clifford odd and Clifford even "basis vectors", form creation and annihilation operators that inherit anti-commutation or commutation relations from basis vectors. The Clifford odd creation operators manifest, together with their Hermitian conjugated annihilation operators on the vacuum state, Eq. (10.14), the properties of the anti-commutation relations postulated by Dirac for the second quantized fermion fields. The Clifford even creation operators manifest correspondingly the commutation relations for the second quantized boson fields.

iii. The Clifford odd "basis vectors" have all the eigenvalues of the Cartan subalgebra members equal to either $\pm\frac{1}{2}$ or to $\pm\frac{i}{2}$.

The Clifford even "basis vectors" have all the eigenvalues of the Cartan subalgebra members $\mathcal{S}^{ab} = S^{ab} + \tilde{S}^{ab}$ equal to either ± 1 and zero or to $\pm i$ and zero.

In odd dimensional spaces, $d = (2n+1)$, *the properties of the Clifford odd and the Clifford even "basis vectors" differ essentially from their properties in even dimensional spaces* ([8] in Subsect. 9.2.2, [7] in Sect. 2.2.2.).

.Half of the "basis vectors" have properties as those Clifford odd and even "basis vectors" of $d' = 2n$. The second half, appearing by applying on these half of "basis vectors" by $S^{0(2n+1)}$, although anti-commuting, the Clifford odd "basis vectors" manifest properties of the Clifford even "basis vectors" in even dimensional spaces; they appear in two separate groups, each group having their Hermitian conjugated within their own group. And the Clifford even "basis vectors", although commuting, manifest properties of the Clifford odd "basis vectors" in even dimensional spaces; they appear in $2^{\frac{d}{2}-1}$ families, each family with $2^{\frac{d}{2}-1}$ members, their Hermitian conjugated partners appear in a separate group.

Clifford odd and even "basis vectors" in even d This subsection shortly reviews Subsect. 2.2.1 of Ref. [7].

Let us start with the *Clifford odd "basis vectors"* .

The Clifford odd "basis vectors" must be products of an odd number of nilpotents, and the rest, up to $\frac{d}{2}$, of projectors, each nilpotent and each projector must be the "eigenvector" of one of the members of the Cartan subalgebra, Eq. (9.8), correspondingly are the "basis vectors" eigenvectors of all the members of the Cartan subalgebra of the Lorentz algebra: S^{ab}'s determine $2^{\frac{d}{2}-1}$ members of one family, $\tilde{S}^{ab}$'s transform each member of one family to the same member of the rest of $2^{\frac{d}{2}-1}$ families.
Let us call the Clifford odd "basis vectors" $\hat{b}^{m\dagger}_{f}$, if it is the m^{th} membership of the family f. The Hermitian conjugated partner of $\hat{b}^{m\dagger}_{f}$ is called $\hat{b}^{m}_{f}$ $(= (\hat{b}^{m\dagger}_{f})^{\dagger}$.
In $d = 2(2n+1)$ the "basis vector" $\hat{b}^{1\dagger}_{1}$ is chosen to be the product of only nilpotents, all the rest members belonging to the $f = 1$ family follow by the application of $S^{01}, S^{03}, \ldots, S^{0d}, S^{15}, \ldots, S^{1d}, S^{5d} \ldots, S^{d-2\,d}$. They are presented on the left-hand side of Eq. (10.11). Their Hermitian conjugated partners [10] are presented on the right-hand side. The algebraic product mark $*_A$ among nilpotents and projectors is skipped.

$$
\begin{aligned}
&d = 2(2n+1)\,, \\
&\hat{b}^{1\dagger}_{1} = \stackrel{03}{(+i)}\stackrel{12}{(+)}\stackrel{56}{(+)}\cdots\stackrel{d-1\,d}{(+)}\,, \qquad && \hat{b}^{1}_{1} = \stackrel{03}{(-i)}\stackrel{12}{(-)}\cdots\stackrel{d-1\,d}{(-)}\,, \\
&\hat{b}^{2\dagger}_{1} = \stackrel{03}{[-i]}\stackrel{12}{[-]}\stackrel{56}{(+)}\cdots\stackrel{d-1\,d}{(+)}\,, && \hat{b}^{2}_{1} = \stackrel{03}{[-i]}\stackrel{12}{[-]}\stackrel{56}{(-)}\cdots\stackrel{d-1\,d}{(-)}\,, \\
&\cdots && \cdots \\
&\hat{b}^{2^{\frac{d}{2}-1}\dagger}_{1} = \stackrel{03}{[-i]}\stackrel{12}{[-]}\stackrel{56}{(+)}\ldots\stackrel{d-3\,d-2}{[-]}\;\stackrel{d-1\,d}{[-]}\,, && \hat{b}^{2^{\frac{d}{2}-1}\dagger}_{1} = \stackrel{03}{[-i]}\stackrel{12}{[-]}\stackrel{56}{(-)}\stackrel{78}{[-]}\ldots\stackrel{d-3\,d-2}{[-]}\;\stackrel{d-1\,d}{[-]}\,, \\
&\cdots\,, && \cdots\,.
\end{aligned}
\tag{9.12}
$$

In $d = 4n$ the choice of the starting "basis vector" with maximal number of nilpotents must have one projector, all the rest follows in equivalent way as in the case of $d = 2(2n+1)$.
The reader can notice that all the "basis vectors" within any family, as well as the "basis vectors" among families, are orthogonal; that is, their mutual algebraic products are zero. The same is true within their Hermitian conjugated partners.

$$
\hat{b}^{m\dagger}_{f} *_A \hat{b}^{m'\dagger}_{f'} = 0\,, \quad \hat{b}^{m}_{f} *_A \hat{b}^{m'}_{f'} = 0\,, \quad \forall m, m', f, f'. \tag{9.13}
$$

Choosing the vacuum state equal to

$$
|\psi_{oc}> = \sum_{f=1}^{2^{\frac{d}{2}-1}} \hat{b}^{m}_{f}{}_{*_A} \hat{b}^{m\dagger}_{f}\,|\,1\,>\,, \tag{9.14}
$$

[10] Taking into account that $(\gamma^a)^\dagger = \eta^{aa}\gamma^a$, $k^2 = \eta^{aa}\eta^{bb}$, and $k^\dagger k = 1$, it follows: $(\stackrel{ab}{(k)})^\dagger = \frac{1}{2}\eta^{aa}(\gamma^a + \frac{\eta^{bb}kk}{ik^\dagger k(-k)}\gamma^b) = \frac{1}{2}\eta^{aa}(\gamma^a + \frac{\eta^{aa}}{i(-k)}\gamma^b) = \stackrel{ab}{(-k)}$, and $(\stackrel{ab}{[k]})^\dagger = \frac{1}{2}(1 + \frac{i}{k}\gamma^a\gamma^b)^\dagger = \frac{1}{2}(1 + \frac{-i}{k^\dagger}\eta^{aa}\eta^{bb}\gamma^b\gamma^a) = \stackrel{ab}{[k]}$

for one of members m, anyone of the odd irreducible representations f it follows that the Clifford odd "basis vectors" obey the relations

$$\begin{aligned} \hat{b}^{m}_{f} {}_{*A} |\psi_{oc}> &= 0.\, |\psi_{oc}>\,, \\ \hat{b}^{m\dagger}_{f} {}_{*A} |\psi_{oc}> &= |\psi^{m}_{f}>\,, \\ \{\hat{b}^{m}_{f}, \hat{b}^{m'}_{f`}\}_{*A+} |\psi_{oc}> &= 0.\, |\psi_{oc}>\,, \\ \{\hat{b}^{m\dagger}_{f}, \hat{b}^{m'\dagger}_{f`}\}_{*A+} |\psi_{oc}> &= 0.\, |\psi_{oc}>\,, \\ \{\hat{b}^{m}_{f}, \hat{b}^{m'\dagger}_{f`}\}_{*A+} |\psi_{oc}> &= \delta^{mm'}\, \delta_{ff`} |\psi_{oc}>\,. \end{aligned} \tag{9.15}$$

In above equation the normalization $<\psi_{oc}|\hat{b}^{m\dagger}_{f} *_A \hat{b}^{m\dagger}_{f} *_A |\psi_{oc}>= 1$ is used. The anti-commutation relation is defined: $\{\hat{b}^{m\dagger}_{f}, \hat{b}^{m'\dagger}_{f`}\}_{*A+} = \hat{b}^{m\dagger}_{f} *_A \hat{b}^{m'\dagger}_{f`} + \hat{b}^{m'\dagger}_{f`} *_A \hat{b}^{m\dagger}_{f}$.
The reader can find in tables 1. n 2. of Ref. [7] the properties of the Clifford odd "basis vectors" for $d = (5+1)$ case. Assuming that this case represents $SO(6)$ broken into $SU(3) \times U(1)$, Fig. 1. manifests the triplet and singlet of any of $2^{\frac{6}{2}-1} = 4$ families.

We shall see that the *Clifford even "basis vectors"* manifest in even dimensional spaces properties of the internal spaces of the gauge fields of the corresponding *Clifford odd "basis vectors"* , which manifest properties of the fermion second quantized fields [7,11].

The Clifford even "basis vectors" must be products of an even number of nilpotents and the rest, up to $\frac{d}{2}$, of projectors. Again, like in the Clifford odd "basis vectors", each nilpotent and each projector is the "eigenvector" of one of the members of the Cartan subalgebra of the Lorentz algebra, $\mathcal{S}^{ab} = S^{ab} + \tilde{S}^{ab}$, Eq. (9.8). Correspondingly the "basis vectors" are the eigenvectors of all the members of the Cartan subalgebra of the Lorentz algebra.
The Clifford even "basis vectors", called ${}^{i}\hat{\mathcal{A}}^{m\dagger}_{f}$, where $i = (I, II)$, appear in two groups, each group has $2^{\frac{d}{2}-1} \times 2^{\frac{d}{2}-1}$ members, each member is either self adjoint or has his Hermitian conjugated partner within the same group.
S^{ab} and $\tilde{S}^{ab}$ generate from the starting "basis vector" of each group all the $2^{\frac{d}{2}-1} \times 2^{\frac{d}{2}-1}$ members. ¡et us present the members in $d = 2(2n+1)$ dimensions. In the case of $d = 4n$ the building from the starting "basis vectors" all the members of each of the two groups goes in equivalent way. In $d = 4n$ we can start with only nilpotents, in $d = 2(2n+1)$ case one must be a projector.

$$d = 2(2n+1)$$

$$\begin{aligned}
&{}^{I}\hat{\mathcal{A}}^{1\dagger}_{1} = \overset{03}{(+i)}\overset{12}{(+)} \cdots \overset{d-1\, d}{[+]}\,, && {}^{II}\hat{\mathcal{A}}^{1\dagger}_{1} = \overset{03}{(-i)}\overset{12}{(+)} \cdots \overset{d-1\, d}{[+]}\,, \\
&{}^{I}\hat{\mathcal{A}}^{2\dagger}_{1} = \overset{03}{[-i]}\overset{12}{[-]}\overset{56}{(+)} \cdots \overset{d-1\, d}{[+]}\,, && {}^{II}\hat{\mathcal{A}}^{2\dagger}_{1} = \overset{03}{[+i]}\overset{12}{[-]}\overset{56}{(+)} \cdots \overset{d-1\, d}{[+]}\,, \\
&{}^{I}\hat{\mathcal{A}}^{3\dagger}_{1} = \overset{03}{(+i)}\overset{12}{(+)}\overset{56}{(+)} \cdots \overset{d-3\, d-2}{[-]}\;\overset{d-1\, d}{(-)}\,, && {}^{II}\hat{\mathcal{A}}^{3\dagger}_{1} = \overset{03}{(-i)}\overset{12}{(+)}\overset{56}{(+)} \cdots \overset{d-3\, d-2}{[-]}\;\overset{d-1\, d}{(-)}\,, \\
&\qquad\qquad \cdots && \cdots
\end{aligned} \tag{9.16}$$

There are $2^{\frac{d}{2}-1} \times 2^{\frac{d}{2}-1}$ Clifford even "basis vectors" of the kind ${}^{I}\hat{\mathcal{A}}^{m\dagger}_{f}$ and the same number of the Clifford even "basis vectors" of the kind ${}^{II}\hat{\mathcal{A}}^{m\dagger}_{f}$.

Table 1, presented in Ref. ([7], Subsect. 2.3) illustrates properties of the Clifford odd and Clifford even "basis vectors" on the case of $d = (5+1)$. We shall discuss in here only the general case by carefully inspecting properties of both kinds of "basis vectors".
The Clifford even "basis vectors" belonging to two different groups are orthogonal due to the fact that they differ in the sign of one nilpotent or one projector, or the algebraic product of a member of one group with a member of another group gives zero according to the first two lines of Eq. (10.25): $\stackrel{ab}{(k)}\stackrel{ab}{[k]}= 0$, $\stackrel{ab}{[k]}\stackrel{ab}{(-k)}= 0$, $\stackrel{ab}{[k]}\stackrel{ab}{[-k]}= 0$.

$$ {}^{I}\hat{\mathcal{A}}^{m\dagger}_{f} *_{A} {}^{II}\hat{\mathcal{A}}^{m\dagger}_{f} = 0 = {}^{II}\hat{\mathcal{A}}^{m\dagger}{}_{\cdot f} *_{A} {}^{I}\hat{\mathcal{A}}^{m\dagger}_{f} \,. \tag{9.17} $$

The members of each of these two groups have the property

$$ {}^{i}\hat{\mathcal{A}}^{m\dagger}_{f} *_{A} {}^{i}\hat{\mathcal{A}}^{m'\dagger}_{f`} \rightarrow \begin{cases} {}^{i}\hat{\mathcal{A}}^{m\dagger}_{f`}\,, i = (I, II) \\ \qquad\qquad \text{or zero}\,. \end{cases} \tag{9.18} $$

For a chosen $(m, f, f`)$ there is only one $m'$ (out of $2^{\frac{d}{2}-1}$) which gives nonzero contribution. The reader should pay attention on the repetition of the index $m$ and $f`$ on the left and on the right hand side of Eq. (10.18).
Two "basis vectors", ${}^{i}\hat{\mathcal{A}}^{m\dagger}_{f}$ and ${}^{i}\hat{\mathcal{A}}^{m'\dagger}_{f'}$, the algebraic product, $*_A$, of which gives non zero contribution, "scatter" into the third one ${}^{i}\hat{\mathcal{A}}^{m\dagger}_{f`}$ of the same kind, for $i = (I, II)$.

It remains to evaluate the algebraic application, $*_A$, of the Clifford even "basis vectors" ${}^{I,II}\hat{\mathcal{A}}^{m\dagger}_{f}$ on the Clifford odd "basis vectors" $\hat{b}^{m'\dagger}_{f`}$. One finds, taking into account Eq. (10.25), for ${}^{I}\hat{\mathcal{A}}^{m\dagger}_{f}$

$$ {}^{I}\hat{\mathcal{A}}^{m\dagger}_{f} *_{A} \hat{b}^{m'\dagger}_{f`} \rightarrow \begin{cases} \hat{b}^{m\dagger}_{f`}\,, \\ \text{or zero}\,, \end{cases} \tag{9.19} $$

The reader should pay attention on the repetition of the index m and $f`$ on the left and on the right hand side in the above equation.
For each ${}^{I}\hat{\mathcal{A}}^{m\dagger}_{f}$ there are among $2^{\frac{d}{2}-1} \times 2^{\frac{d}{2}-1}$ members of the Clifford odd "basis vectors" (describing the internal space of fermion fields) $2^{\frac{d}{2}-1}$ members, $\hat{b}^{m'\dagger}_{f`}$, fulfilling the relation of Eq. (10.19). All the rest $(2^{\frac{d}{2}-1} \times (2^{\frac{d}{2}-1} - 1))$ Clifford odd "basis vectors" give zero contributions. Or equivalently, there are $2^{\frac{d}{2}-1}$ pairs of quantum numbers $(f, m')$ for which $\hat{b}^{m\dagger}_{f`} \neq 0$ [11] .

[11] Let us treat a particular case in $d = 2(2n+1)$-dimensional space: ${}^{I}\hat{\mathcal{A}}^{m\dagger}_{f}(\equiv\stackrel{03}{(+i)}\stackrel{12}{(+)}\stackrel{56}{(+)} \ldots \stackrel{d-3\,d-2}{(+)}\stackrel{d-1\,d}{[+]}\,)*_A\, \hat{b}^{m'\dagger}_{f`}(\equiv\stackrel{03}{(-i)}\stackrel{12}{(-)}\stackrel{56}{(-)} \ldots \stackrel{d-3\,d-2}{(-)}\stackrel{d-1\,d}{(+)}\,) \rightarrow \hat{b}^{m\dagger}_{f`}(\equiv\stackrel{03}{[+i]}\stackrel{12}{[+]}\stackrel{56}{[+]} \ldots \stackrel{d-3\,d-2}{[+]}\stackrel{d-1\,d}{(+)}$. The $\mathcal{S}^{ab}$ (meaning $\mathcal{S}^{03}, \mathcal{S}^{12}, \mathcal{S}^{56}, \ldots \mathcal{S}^{d-1\,d}$) say for

Taking into account Eq. (10.25) one finds

$$\hat{b}^{m\dagger}_{f} *_A {}^{I}\hat{\mathcal{A}}^{m'\dagger}_{f`} = 0\,, \quad \forall (m, m`, f, f`)\,. \tag{9.20}$$

Eqs. (10.19, 10.20) demonstrates that ${}^{I}\hat{\mathcal{A}}^{m\dagger}_{f}$, applying on $\hat{b}^{m'\dagger}_{f`}$, transforms the Clifford odd "basis vector" into another Clifford odd "basis vector" of the same family, transferring to the Clifford odd "basis vector" integer spins, or gives zero. For "scattering" the Cifford even "basis vectors" ${}^{II}\hat{\mathcal{A}}^{m\dagger}_{f}$ on the Clifford odd "basis vectors" $\hat{b}^{m'\dagger}_{f`}$ it follows

$${}^{II}\hat{\mathcal{A}}^{m\dagger}_{f} *_A \hat{b}^{m'\dagger}_{f`} = 0\,, \quad \forall (m, m', f, f`)\,, \tag{9.21}$$

while we get

$$\hat{b}^{m\dagger}_{f} *_A {}^{II}\hat{\mathcal{A}}^{m'\dagger}_{f`} \rightarrow \begin{cases} \hat{b}^{m\dagger}_{f``}\,, \\ \text{or zero}\,, \end{cases} \tag{9.22}$$

For each $\hat{b}^{m\dagger}_{f}$ there are among $2^{\frac{d}{2}-1} \times 2^{\frac{d}{2}-1}$ members of the Clifford even "basis vectors" (describing the internal space of boson fields) , ${}^{II}\hat{\mathcal{A}}^{m'\dagger}_{f`}$, $2^{\frac{d}{2}-1}$ members (with appropriate f` and m') fulfilling the relation of Eq. (10.22) while f`` runs over $(1 - 2^{\frac{d}{2}-1})$.
All the rest ($2^{\frac{d}{2}-1} \times \; (2^{\frac{d}{2}-1} - 1)$) Clifford even"basis vectors" give zero contributions.
Or equivalently, there are $2^{\frac{d}{2}-1}$ pairs of quantum numbers (f', m') for which $\hat{b}^{m\dagger}_{f}$ and ${}^{II}\hat{\mathcal{A}}^{m'\dagger}_{f`}$ give non zero contribution [12].

Eqs. (10.21, 10.22) demonstrate that ${}^{II}\hat{\mathcal{A}}^{m'\dagger}_{f'}$, "absorbed" by $\hat{b}^{m\dagger}_{f}$, transforms the Clifford odd "basis vector" into the Clifford odd "basis vector" of the same family member and of another family, or gives zero.
Tables 1, 2, 3, presented in Subsect. 2.3 in Ref. [7], and Table 9.2, presented in App. **??** illustrate properties of the Clifford odd and Clifford even "basis vectors"

---

the above case that the boson field with the quantum numbers $(i, 1, 1, \ldots, 1, 0)$ when "scattering" on the fermion field with the Cartan subalgebra quantum numbers $(S^{03}, S^{1,2}, S^{56} \ldots S^{d-3\,d-2}, S^{d-1\,d}) = (-\frac{i}{2}, -\frac{1}{2}, -\frac{1}{2}, \ldots, -\frac{1}{2}, \frac{1}{2})$, and the family quantum numbers $(-\frac{i}{2}, -\frac{1}{2}, -\frac{1}{2}, \ldots, \; -\frac{1}{2}, \frac{1}{2})$ transfers to the fermion field its quantum numbers $(i, 1, 1, \ldots, 1, 0)$, transforming fermion family members quantum numbers to $(\frac{i}{2}, \frac{1}{2}, \frac{1}{2}, \ldots, \frac{1}{2}, \frac{1}{2})$, leaving family quantum numbers unchanged.

12 Let us treat a particular case in $d = 2(2n+1)$-dimensional space:
$\hat{b}^{m\dagger}_{f}(\equiv \overset{03}{(-i)}\overset{12}{(-)}\overset{56}{(-)} \cdots \overset{d-3\,d-2}{(-)}\overset{d-1\,d}{(+)}) *_A {}^{II}\hat{\mathcal{A}}^{m`\dagger}_{f`}(\equiv \overset{03}{(+i)}\overset{12}{(+)}\overset{56}{(+)} \cdots \overset{d-3\,d-2}{(+)}\overset{d-1\,d}{[-]}) \rightarrow$
$\hat{b}^{m\dagger}_{f`'}(\equiv \overset{03}{[-i]}\overset{12}{[-]}\overset{56}{[-]} \cdots \overset{d-3\,d-2}{[-]}\overset{d-1\,d}{(+)})$ When the fermion field with the Cartan subalgebra family members quantum numbers $(S^{03}, S^{12}, S^{56} \ldots \; S^{d-3\,d-2}, S^{d-1\,d}) = (-\frac{i}{2}, -\frac{1}{2}, -\frac{1}{2}, \ldots, -\frac{1}{2}, \frac{1}{2})$ and family quantum numbers $(\tilde{S}^{03}, \tilde{S}^{12}, \tilde{S}^{56} \ldots \tilde{S}^{d-3\,d-2}, \tilde{S}^{d-1\,d})$ $(-\frac{i}{2}, -\frac{1}{2}, -\frac{1}{2}, \ldots, -\frac{1}{2}, \frac{1}{2})$ "absorbs" a boson field with the Cartan subalgebra quantum numbers $\mathcal{S}^{ab}$ (meaning $\mathcal{S}^{03}, \mathcal{S}^{12}, \mathcal{S}^{56}, \ldots \mathcal{S}^{d-1\,d}$) equal to $(i, 1, 1, \ldots, 1, 0)$, the fermion field changes the family quantum numbers $(\tilde{S}^{03}, \tilde{S}^{1,2}, \tilde{S}^{56} \ldots \tilde{S}^{d-3\,d-2}, \tilde{S}^{d-1\,d})$ to $(\frac{i}{2}, \frac{1}{2}, \frac{1}{2}, \ldots, \frac{1}{2}, \frac{1}{2})$, keeping family members quantum numbers unchanged.

on the case of $d = (5+1)$. Looking at this case, the reader can easily evaluate properties of either even or odd "basis vectors". We discuss in this subsection the general case by carefully inspecting the properties of both kinds of "basis vectors".

While the Clifford odd "basis vectors", $\hat{b}^{m\dagger}_{f}$, offer the description of the internal space of the second quantized anti-commuting fermion fields, appearing in families, the Clifford even "basis vectors", ${}^{I,II}\hat{\mathcal{A}}^{m\dagger}_{f}$, offer the description of the internal space of the second quantized commuting boson fields, having no families and appearing in two groups. One of the two groups, ${}^{I}\hat{\mathcal{A}}^{m\dagger}_{f}$, transferring their integer quantum numbers to the Clifford odd "basis vectors", $\hat{b}^{m\dagger}_{f}$, changes the family members quantum numbers leaving the family quantum numbers unchanged. The second group, transferring their integer quantum numbers to the Clifford odd "basis vector", changes the family quantum numbers leaving the family members quantum numbers unchanged.
Both groups of Clifford even "basis vectors" manifest as the gauge fields of the corresponding fermion fields: One concerning the family members quantum numbers, the other concerning the family quantum numbers.
It is shown in several articles [4,6,12,39–44] that the group ${}^{I}\hat{\mathcal{A}}^{m\dagger}_{f}$ determine (mostly) properties of the observed vector gauge fields, while the group ${}^{II}\hat{\mathcal{A}}^{m\dagger}_{f}$ determine the properties of the scalar gauge fields, at least the Higgs's scalars and Yukawa couplings, determining masses of quarks and lepton for three so far observed of four (predicted by the *spin-charge-family*) families [39–41,43,44] and possibly also for the masses of the second group of four families of quarks and leptons and antiquarks and antileptons, the stable of which determines the dark matter [42]. There are additional scalar fields with the space index $\alpha = (9, 10, ..., 14)$ offering the explanation for the matter/antimatter in the universe [6,12].

Clifford odd and even "basis vectors" in d odd Let us shortly overview properties of the fermion and boson "basis vectors" in odd dimensional spaces, as presented in Ref. [8], Subsect. 2.2.
In even dimensional spaces the Clifford odd "basis vectors" fulfil the postulates for the second quantized fermion fields, Eq. (10.15), and the Clifford even "basis vectors" have the properties of the internal spaces of their corresponding gauge fields, Eqs. (10.18, 10.19, 10.22). In odd dimensional spaces, the Clifford odd and even "basis vectors" have unusual properties resembling properties of the internal spaces of the Faddeev–Popov ghosts, as we described in [8].
In $d = (2n+1)$-dimensional cases, $n = 1, 2, \ldots$, half of the "basis vectors", $2^{\frac{2n}{2}-1} \times 2^{\frac{2n}{2}-1}$, can be taken from the $2n$-dimensional part of space, presented in Eqs. (10.11, 10.16, 10.18).
The rest of the "basis vectors" in odd dimensional spaces, $2^{\frac{2n}{2}-1} \times 2^{\frac{2n}{2}-1}$, follow if $S^{0\,2n+1}$ is applied on these half of the "basis vectors". Since $S^{0\,2n+1}$ are Clifford even operators, they do not change the oddness or evenness of the "basis vectors".
For the Clifford odd "basis vectors", the $2^{\frac{d-1}{2}-1}$ members appearing in $2^{\frac{d-1}{2}-1}$ families and representing the part which is the same as in even, $d = 2n$, dimensional space are present on the left-hand side of Eq. (9.23), the part obtained by applying $S^{0\,2n+1}$ on the one of the left-hand side is presented on the right

hand side. Below the "basis vectors" and their Hermitian conjugated partners are presented.

$$
\begin{array}{llll}
 & d = & 2(2n+1)+1 & \\
\hat{b}^{1\dagger}_{1} = \stackrel{03}{(+i)}\stackrel{12}{(+)}\stackrel{56}{(+)}\cdots \stackrel{d-2\; d-1}{(+)}\,, & & \hat{b}^{1\dagger}_{2^{\frac{d-1}{2}-1}+1} = \stackrel{03}{[-i]}\stackrel{12}{(+)}\stackrel{56}{(+)}\cdots \stackrel{d-2\; d-1}{(+)}\;\gamma^{d}\,, & \\
\cdots & & \cdots & \\
\hat{b}^{2^{\frac{d-1}{2}-1}\dagger}_{1} = \stackrel{03}{[-i]}\stackrel{12}{[-]}\stackrel{56}{(+)}\ldots \stackrel{d-2\; d-1}{[-]}\,, & & \hat{b}^{2^{\frac{d-1}{2}-1}\dagger}_{2^{\frac{d-1}{2}-1}+1} = \stackrel{03}{(+i)}\stackrel{12}{[-]}\stackrel{56}{(+)}\ldots \stackrel{d-2\; d-1}{[-]}\;\gamma^{d}\,, & \\
\cdots & & \cdots\,, & \\
 & & \cdots\,, & \\
\hat{b}^{1}_{1} = \stackrel{03}{(-i)}\stackrel{12}{(-)}\stackrel{56}{(-)}\cdots \stackrel{d-2\; d-1}{(-)}\,, & & \hat{b}^{1}_{2^{\frac{d-1}{2}-1}+1} = \stackrel{03}{[+i]}\stackrel{12}{(-)}\stackrel{56}{(-)}\cdots \stackrel{d-2\; d-1}{(-)}\;\gamma^{d}\,, & \\
\cdots & & \cdots\,. &
\end{array}
\tag{9.23}
$$

The application of S^{0d} or $\tilde{S}^{0d}$ on the left-hand side of the "basis vectors" (and the Hermitian conjugated partners of both) generate the whole set of $2\times 2^{d-2}$ members of the Clifford odd "basis vectors" and their Hermitian conjugated partners in $d=(2n+1)$- dimensional space appearing on the left-hand side and the right-hand sides of Eq. (9.23).

It is not difficult to see that $\hat{b}^{m\dagger}_{2^{\frac{d-1}{2}-1}+k}$ and $\hat{b}^{m'}_{2^{\frac{d-1}{2}-1}+k'}$ on the right-hand side of Eq. (9.23) obtain properties of the two groups (they are orthogonal to each other; the algebraic products, $*_A$, of a member from one group, and any member of another group give zero) with the Hermitian conjugated partners within the same group; they have properties of the Clifford even "basis vectors" from the point of view of the Hermiticity property: The operators γ^a are up to a constant the self-adjoint operators, while S^{0d} transforms one nilpotent into a projector.
S^{ab} do not change the Clifford oddness of $\hat{b}^{m\dagger}_{f}$, and $\hat{b}^{m}_{f}$; $\hat{b}^{m\dagger}_{f}$ remain to be Clifford odd objects, however, with the properties of boson fields.

Let us find the Clifford even "basis vectors" in odd dimensional space $d=2(2n+1)+1$.

$$
\begin{array}{llll}
 & d = & 2(2n+1)+1 & \\
{}^{I}\mathcal{A}^{1\dagger}_{1} = \stackrel{03}{(+i)}\stackrel{12}{(+)}\stackrel{56}{(+)}\cdots \stackrel{d-2\; d-1}{[+]}\,, & & {}^{I}\mathcal{A}^{1\dagger}_{2^{\frac{d-1}{2}-1}+1} = \stackrel{03}{[-i]}\stackrel{12}{(+)}\stackrel{56}{(+)}\cdots \stackrel{d-2\; d-1}{[+]}\;\gamma^{d}\,, & \\
\cdots & & \cdots & \\
{}^{I}\mathcal{A}^{2^{\frac{d-1}{2}-1}\dagger}_{1} = \stackrel{03}{[-i]}\stackrel{12}{[-]}\stackrel{56}{[-]}\ldots \stackrel{d-2\; d-1}{[+]}\,, & & {}^{I}\mathcal{A}^{2^{\frac{d-1}{2}-1}\dagger}_{2^{\frac{d-1}{2}-1}+1} = \stackrel{03}{(+i)}\stackrel{12}{[-]}\stackrel{56}{[-]}\ldots \stackrel{d-2\; d-1}{[+]}\;\gamma^{d}\,, & \\
\cdots & & \cdots\,, & \\
\cdots & & \cdots & \\
{}^{II}\mathcal{A}^{1\dagger}_{1} = \stackrel{03}{(-i)}\stackrel{12}{(+)}\stackrel{56}{(+)}\cdots \stackrel{d-2\; d-1}{[+]}\,, & & {}^{II}\mathcal{A}^{1\dagger}_{2^{\frac{d-1}{2}-1}+1} = \stackrel{03}{[+i]}\stackrel{12}{(+)}\stackrel{56}{(+)}\cdots \stackrel{d-2\; d-1}{[+]}\;\gamma^{d}\,, & \\
\cdots & & \cdots\,. &
\end{array}
\tag{9.24}
$$

The right hand side of Eq. (9.23), although anti-commuting, is resembling the properties of the Clifford even "basis vectors" on the left hand side of Eq. (9.24),

while the right-hand side of Eq. (9.24), although commuting, resembles the properties of the Clifford odd “basis vectors”, from the left hand side of Eq. (9.23): γ^a are up to a constant the self adjoint operators, while S^{0d} transform one nilpotent into a projector (or one projector into a nilpotent). However, S^{ab} do not change Clifford eveness of ${}^{I}\mathcal{A}^{m\dagger}_{f}$, $i = (I, II)$.
The reader can see the illustration with the special case for $d = (4+1)$ in Subsect. 3.2.2. of Ref. [8].

9.2.3 Second quantized fermion and boson fields with internal spaces described by Clifford “basis vectors” in even dimensional spaces

We learn in Subsect. 10.2.2 (See also Sect. 2.3 in Ref. [7]) that in even dimensional spaces the Clifford odd and the Clifford even “basis vectors”, which are the superposition of the Clifford odd and the Clifford even products of γ^a's, respectively, offer the description of the internal spaces of fermion and boson fields:
The Clifford odd “basis vectors” explain the second quantization postulates for fermions ([6,7] and the references therein), the Clifford even “basis vectors” explain the second quantization postulates for bosons, the gauge fields of fermions. The Clifford odd algebra offers $2^{\frac{d}{2}-1}$ “basis vectors” $\hat{b}^{m\dagger}_{f}$, appearing in $2^{\frac{d}{2}-1}$ families, with the family quantum numbers determined by $\tilde{S}^{ab} = \frac{i}{2}\{\tilde{\gamma}^a, \tilde{\gamma}^b\}_-$, which, together with their $2^{\frac{d}{2}-1} \times 2^{\frac{d}{2}-1}$ Hermitian conjugated partners $\hat{b}^{m}_{f}$ fulfil the postulates for the second quantized fermion fields, Eq. (10.15) in this paper, Eq.(26) in Ref. [6], explaining the second quantization postulate of Dirac.
The Clifford even algebra offers $2^{\frac{d}{2}-1} \times 2^{\frac{d}{2}-1}$ “basis vectors” of ${}^{I}\hat{\mathcal{A}}^{m\dagger}_{f}$, and the same number of ${}^{II}\hat{\mathcal{A}}^{m\dagger}_{f}$, with the properties of the second quantized boson fields manifesting the gauge fields of fermion fields described by the Clifford odd “basis vectors” $\hat{b}^{m\dagger}_{f}$, as we can see in Figs. (10.1, 10.2).
The first figure represents the “basis vectors” of fermions in $d = (5+1)$-dimensional space for any of the $2^{\frac{d}{2}-1}$ families, analysed with respect to the subgroups $SU(3)$ and $U(1)$ of the group $SO(5,1)$, their “basis vectors” are presented in Table 2 of Ref. [7]. The reader can see in the figure three members of a colour charged triplet, and one colourless singlet, manifesting the colour part of quarks and the colourless leptons. The same figure represent any of the families.
The second figure represents the “basis vectors” for anyone of the two corresponding Clifford even groups, ${}^{I}\hat{\mathcal{A}}^{m\dagger}_{f}$, or ${}^{II}\hat{\mathcal{A}}^{m\dagger}_{f}$. The corresponding “basis vectors” for ${}^{I}\hat{\mathcal{A}}^{m\dagger}_{f}$ are presented in Table 3 of Ref. [7], the “basis vectors” of ${}^{II}\hat{\mathcal{A}}^{m\dagger}_{f}$ can be found in App. 9.5 in Table 9.2. The reader can see in Fig. 10.2 four “basis vectors” in the centre of the coordinate system, one sextet, one triplet and one antitriplet. These Clifford even “basis vectors” manifest the colour octet (the sextet and two of four “singlets”), the gauge fields of the Clifford odd triplet “basis vectors”. The triplets and antitriplets, if applied as presented in Eq. (10.22) on the Clifford odd “basis vector”, representing the singlet, transform the singlet to one member of the triplet.

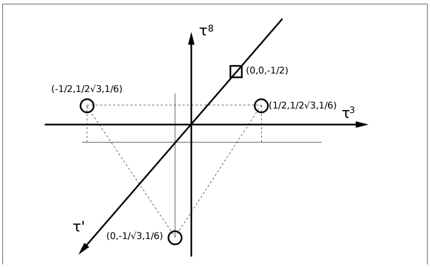

Fig. 9.1: The representations for the Clifford odd "basis vectors" of the subgroups $SU(3)$ and $U(1)$ of the group $SO(5,1)$, the properties of which appear in Tables 1 and 2 of Ref. [7], are presented, taken from Ref. [7] ($\tau^3 = \frac{1}{2}(-S^{12} - iS^{03})$, $\tau^8 = \frac{1}{2\sqrt{3}}(S^{12} - iS^{03} - 2S^{56})$, $\tau' = -\frac{1}{3}(S^{12} - iS^{03} + S^{56})$. On the abscissa axis, on the ordinate axis and on the third axis, are the eigenvalues of τ^3, τ^8 and τ'), One notices one triplet, denoted by ◯ with the values $\tau' = \frac{1}{6}$, ($\tau^3 = -\frac{1}{2}, \tau^8 = \frac{1}{2\sqrt{3}}, \tau' = \frac{1}{6}$), ($\tau^3 = \frac{1}{2}, \tau^8 = \frac{1}{2\sqrt{3}}, \tau' = \frac{1}{6}$), ($\tau^3 = 0, \tau^8 = -\frac{1}{\sqrt{3}}, \tau' = \frac{1}{6}$), respectively, and one singlet denoted by □ ($\tau^3 = 0, \tau^8 = 0, \tau' = -\frac{1}{2}$). The triplet and the singlet appear in four families, with the family quantum numbers presented in the last three columns of Table 2 of Ref. [7]

.

As we learn in Subsect. 10.2.2, the Clifford odd and the Clifford even "basis vectors" are chosen to be products of nilpotents, $\stackrel{ab}{(k)}$ (with the odd number of nilpotents if describing fermions and the even number of nilpotents if describing bosons), and projectors, $\stackrel{ab}{[k]}$. Nilpotents and projectors are (chosen to be) eigenvectors of the Cartan subalgebra members of the Lorentz algebra in the internal space of S^{ab} for the Clifford odd "basis vectors" and of $\mathcal{S}^{ab} (= S^{ab} + \tilde{S}^{ab})$ for the Clifford even "basis vectors".

To define the creation operators, for fermions or bosons, besides the "basis vectors" defining the internal space of fermions and bosons, the basis in ordinary space in momentum or coordinate representation is needed. Here Ref. [6], Subsect. 3.3 and App. J is overviewed.

Let us introduce the momentum part of the single-particle states. (The extended version is presented in Ref. [6] in Subsect. 3.3 and App. J.)

$$
\begin{aligned}
|\vec{p}> &= \hat{b}^{\dagger}_{\vec{p}}\,|\,0_p\;>\,, \quad <\vec{p}\,| = <\,0_p\,|\,\hat{b}_{\vec{p}}\,,\\
<\vec{p}\,|\,\vec{p}'> &= \delta(\vec{p}-\vec{p}') = <\,0_p\,|\hat{b}_{\vec{p}}\;\hat{b}^{\dagger}_{\vec{p}'}|\,0_p\;>\,,\\
&\text{pointing out}\\
<\,0_p\,|\hat{b}_{\vec{p'}}\,\hat{b}^{\dagger}_{\vec{p}}\,|\,0_p\;> &= \delta(\vec{p'}-\vec{p})\,,
\end{aligned}
\tag{9.25}
$$

with the normalization $<\,0_p\,|\,0_p\;>= 1$. While the quantized operators $\hat{\vec{p}}$ and $\hat{\vec{x}}$ commute $\{\hat{p}^i\,, \hat{p}^j\}_- = 0$ and $\{\hat{x}^k\,, \hat{x}^l\}_- = 0$, it follows for $\{\hat{p}^i\,, \hat{x}^j\}_- = i\eta^{ij}$. One

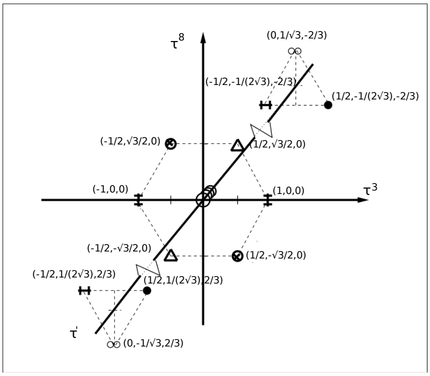

Fig. 9.2: The Clifford even "basis vectors" ${}^{II}\hat{\mathcal{A}}^{m\dagger}_{f}$ in the case that $d = (5+1)$ are presented, with respect to the eigenvalues of the commuting operators of the subgroups $SU(3)$ and $U(1)$ of the group $SO(5,1)$, τ^3, , Eq. (**??**): ($\tau^3 = \frac{1}{2}(-\mathcal{S}^{12} - i\mathcal{S}^{03})$, $\tau^8 = \frac{1}{2\sqrt{3}}(\mathcal{S}^{12} - i\mathcal{S}^{03} - 2\mathcal{S}^{56})$ and $\tau' = -\frac{1}{3}(\mathcal{S}^{12} - i\mathcal{S}^{03} + \mathcal{S}^{56})$). Their properties appear in Table 9.2. The abscissa axis carries the eigenvalues of τ^3, the ordinate axis carries the eigenvalues of τ^8, and the third axis carries the eigenvalues of τ'. One notices four "singlets" in the origin of the coordinate system with ($\tau^3 = 0, \tau^8 = 0, \tau' = 0$), denoted by $\bigcirc$, representing four self adjoint Clifford even "basis vectors" ${}^{II}\hat{\mathcal{A}}^{m\dagger}_{f}$, with ($f = 1, m = 4$), ($f = 2, m = 3$), ($f = 3, m = 1$), ($f = 4, m = 2$), one sextet of three pairs, Hermitian conjugated to each other, with $\tau' = 0$, denoted by $\triangle$ (${}^{II}\hat{\mathcal{A}}^{2\dagger}_{1}$ with ($\tau' = 0, \tau^3 = -\frac{1}{2}, \tau^8 = -\frac{3}{2\sqrt{3}}$) and ${}^{II}\hat{\mathcal{A}}^{4\dagger}_{4}$ with ($\tau' = 0, \tau^3 = \frac{1}{2}, \tau^8 = \frac{3}{2\sqrt{3}}$)), by $\ddagger$ (${}^{II}\hat{\mathcal{A}}^{3\dagger}_{1}$ with ($\tau' = 0, \tau^3 = -1, \tau^8 = 0$) and ${}^{I}\hat{\mathcal{A}}^{4\dagger}_{2}$ with $\tau' = 0, \tau^3 = 1, \tau^8 = 0$)), and by $\otimes$ (${}^{II}\hat{\mathcal{A}}^{2\dagger}_{2}$ with ($\tau' = 0, \tau^3 = \frac{1}{2}, \tau^8 = -\frac{3}{2\sqrt{3}}$) and ${}^{II}\hat{\mathcal{A}}^{3\dagger}_{4}$ with ($\tau' = 0, \tau^3 = -\frac{1}{2}, \tau^8 = \frac{3}{2\sqrt{3}}$)), and one triplet, denoted by $\star\star$ (${}^{II}\hat{\mathcal{A}}^{4\dagger}_{3}$ with ($\tau' = \frac{2}{3}, \tau^3 = \frac{1}{2}, \tau^8 = \frac{1}{2\sqrt{3}}$)), by $\bullet$ (${}^{II}\hat{\mathcal{A}}^{3\dagger}_{3}$ with ($\tau' = \frac{2}{3}, \tau^3 = -\frac{1}{2}, \tau^8 = \frac{1}{2\sqrt{3}}$)), and by $\odot\odot$ (${}^{II}\hat{\mathcal{A}}^{2\dagger}_{3}$ with ($\tau' = \frac{2}{3}, \tau^3 = 0, \tau^8 = -\frac{1}{\sqrt{3}}$)), as well as one antitriplet, Hermitian conjugated to triplet, denoted by $\star\star$ (${}^{II}\hat{\mathcal{A}}^{1\dagger}_{1}$ with ($\tau' = -\frac{2}{3}, \tau^3 = -\frac{1}{2}, \tau^8 = -\frac{1}{2\sqrt{3}}$)), by $\bullet$ (${}^{II}\hat{\mathcal{A}}^{1\dagger}_{2}$ with ($\tau' = -\frac{2}{3}, \tau^3 = \frac{1}{2}, \tau^8 = -\frac{1}{2\sqrt{3}}$)), and by $\odot\odot$ (${}^{II}\hat{\mathcal{A}}^{4\dagger}_{1}$ with ($\tau' = -\frac{2}{3}, \tau^3 = 0, \tau^8 = \frac{1}{\sqrt{3}}$)).

correspondingly finds

$$
\begin{gathered}
< \vec{p}\,|\,\vec{x} > = < 0_{\vec{p}}\,|\,\hat{b}_{\vec{p}}\;\hat{b}^{\dagger}_{\vec{x}}|0_{\vec{x}} >= (< 0_{\vec{x}}\,|\,\hat{b}_{\vec{x}}\;\hat{b}^{\dagger}_{\vec{p}}\;|0_{\vec{p}} >)^{\dagger} \\
< 0_{\vec{p}}\,|\{\hat{b}^{\dagger}_{\vec{p}}, \hat{b}^{\dagger}_{\vec{p}\,'}\}_{-}|0_{\vec{p}} > = 0\,, \qquad < 0_{\vec{p}}\,|\{\hat{b}_{\vec{p}}, \hat{b}_{\vec{p}\,'}\}_{-}|0_{\vec{p}} >= 0\,, \qquad < 0_{\vec{p}}\,|\{\hat{b}_{\vec{p}}, \hat{b}^{\dagger}_{\vec{p}\,'}\}_{-}|0_{\vec{p}} >= 0\,, \\
< 0_{\vec{x}}\,|\{\hat{b}^{\dagger}_{\vec{x}}, \hat{b}^{\dagger}_{\vec{x}\,'}\}_{-}|0_{\vec{x}} > = 0\,, \qquad < 0_{\vec{x}}\,|\{\hat{b}_{\vec{x}}, \hat{b}_{\vec{x}\,'}\}_{-}|0_{\vec{x}} >= 0\,, \qquad < 0_{\vec{x}}\,|\{\hat{b}_{\vec{x}}, \hat{b}^{\dagger}_{\vec{x}\,'}\}_{-}|0_{\vec{x}} >= 0\,, \\
< 0_{\vec{p}}\,|\{\hat{b}_{\vec{p}}, \hat{b}^{\dagger}_{\vec{x}}\}_{-}|0_{\vec{x}} > = e^{i\vec{p}\cdot\vec{x}}\frac{1}{\sqrt{(2\pi)^{d-1}}}\,, \quad < 0_{\vec{x}}\,|\{\hat{b}_{\vec{x}}, \hat{b}^{\dagger}_{\vec{p}}\}_{-}|0_{\vec{p}} >= e^{-i\vec{p}\cdot\vec{x}}\frac{1}{\sqrt{(2\pi)^{d-1}}}\,. \quad (9.26)
\end{gathered}
$$

The internal space of either fermion or boson fields has the finite number of "basis vectors", $2^{\frac{d}{2}-1} \times 2^{\frac{d}{2}-1}$ for fermions (and the same number of their Hermitian conjugated partners), and twice $2^{\frac{d}{2}-1} \times 2^{\frac{d}{2}-1}$ for bosons. The momentum basis is continuously infinite.

The creation operators for either fermions or bosons must be tensor products, $*_T$, of both contributions, the "basis vectors" describing the internal space of fermions or bosons and the basis in ordinary momentum or coordinate space.
The creation operators for a free massless fermion of the energy $p^0 = |\vec{p}|$, belonging to a family f and to a superposition of family members m applying on the vacuum state $|\psi_{oc}> *_T |0_{\vec{p}}>$ can be written as ([6], Subsect.3.3.2, and the references therein, [7])

$$\hat{b}^{s\dagger}_{f}(\vec{p}) = \sum_{m} c^{sm}{}_{f}(\vec{p})\, \hat{b}^{\dagger}_{\vec{p}} *_T \hat{b}^{m\dagger}_{f} \,, \tag{9.27}$$

where the vacuum state for fermions $|\psi_{oc}> *_T |0_{\vec{p}}>$ includes both spaces, the internal part, Eq.(10.14), and the momentum part, Eq. (9.25) (in a tensor product for a starting single particle state with zero momentum, from which one obtains the other single fermion states of the same "basis vector" by the operator $\hat{b}^{\dagger}_{\vec{p}}$ which pushes the momentum by an amount $\vec{p}$ [13]).
The creation operators and annihilation operators for fermion fields fulfil the anti-commutation relations for the second quantized fermion fields [14]

$$\begin{aligned}
< 0_{\vec{p}}\, |\{\hat{b}^{s'}_{f`}(\vec{p'})\,, \hat{b}^{s\dagger}_{f}(\vec{p})\}_{+} |\psi_{oc}> |0_{\vec{p}}> &= \delta^{ss'}\delta_{ff'}\, \delta(\vec{p}' - \vec{p})\cdot |\psi_{oc}>\,,\\
\{\hat{b}^{s'}_{f`}(\vec{p'})\,, \hat{b}^{s}_{f}(\vec{p})\}_{+} |\psi_{oc}> |0_{\vec{p}}> &= 0\cdot |\psi_{oc}> |0_{\vec{p}}>\,,\\
\{\hat{b}^{s'\dagger}_{f'}(\vec{p'})\,, \hat{b}^{s\dagger}_{f}(\vec{p})\}_{+} |\psi_{oc}> |0_{\vec{p}}> &= 0\cdot |\psi_{oc}> |0_{\vec{p}}>\,,\\
\hat{b}^{s\dagger}_{f}(\vec{p})\, |\psi_{oc}> |0_{\vec{p}}> &= |\psi^{s}_{f}(\vec{p})>\,,\\
\hat{b}^{s}_{f}(\vec{p})\, |\psi_{oc}> |0_{\vec{p}}> &= 0\cdot |\psi_{oc}> |0_{\vec{p}}>\,,\\
|p^0| &= |\vec{p}|\,.
\end{aligned} \tag{9.28}$$

The creation operators $\hat{b}^{s\dagger}_{f}(\vec{p})$ and their Hermitian conjugated partners annihilation operators $\hat{b}^{s}_{f}(\vec{p})$, creating and annihilating the single fermion states, respectively, fulfil when applying the vacuum state, $|\psi_{oc}> *_T|0_{\vec{p}}>$, the anti-commutation relations for the second quantized fermions, postulated by Dirac (Ref. [6], Subsect. 3.3.1, Sect. 5). [15]

---

[13] The creation operators and their Hermitian conjugated annihilation operators in the coordinate representation can be read in [6] and the references therein: $\hat{b}^{s\dagger}_{f}(\vec{x}, x^0) = \sum_m \hat{b}^{m\dagger}_{f} *_T \int_{-\infty}^{+\infty} \frac{d^{d-1}p}{(\sqrt{2\pi})^{d-1}}\, c^{sm}{}_{f}\,(\vec{p})\, \hat{b}^{\dagger}_{\vec{p}}\, e^{-i(p^0x^0 - \varepsilon\vec{p}\cdot\vec{x})}$ ([6], subsect. 3.3.2., Eqs. (55,57,64) and the references therein).

[14] Let us evaluate: $< 0_{\vec{p}}\, |\{\hat{b}^{s'}_{f`}(\vec{p'})\,, \hat{b}^{s\dagger}_{f}(\vec{p})\}_{+} |\psi_{oc}> |0_{\vec{p}}> = \delta^{ss'}\delta_{ff'}\, \delta(\vec{p}' - \vec{p})\cdot |\psi_{oc}> = < 0_{\vec{p}}\, |\hat{b}^{s'}_{f`}\, \hat{b}^{s\dagger}_{f}\, \hat{b}_{\vec{p}'} \hat{b}^{\dagger}_{\vec{p}} + \hat{b}^{\dagger}_{\vec{p}} \hat{b}_{\vec{p}'}\, \hat{b}^{s\dagger}_{f}\, \hat{b}^{s'}_{f`}\, |\psi_{oc}> |0_{\vec{p}}> = < 0_{\vec{p}}\, |\hat{b}^{s'}_{f`}\, \hat{b}^{s\dagger}_{f}\, \hat{b}_{\vec{p}'} \hat{b}^{\dagger}_{\vec{p}}\, |\psi_{oc}> |0_{\vec{p}}> = \delta^{ss'}\delta_{ff'}\, \delta(\vec{p} - \vec{p}')\, |\psi_{oc}>$, since, according to Eq. (10.15), $\hat{b}^{s'}_{f`}\, |\psi_{oc}> = 0$.

[15] The anti-commutation relations of Eq. (9.28) are valid also if we replace the vacuum state, $|\psi_{oc}> |0_{\vec{p}}>$, by the Hilbert space of the Clifford fermions generated by the tensor products multiplication, $*_{T_H}$, of any number of the Clifford odd fermion states of all possible internal quantum numbers and all possible momenta (that is, of any number of $\hat{b}^{s\,\dagger}_{f}(\vec{p})$ of any $(s, f, \vec{p})$), Ref. ([6], Sect. 5.).

To write the creation operators for boson fields, we must take into account that boson gauge fields have the space index α, describing the α component of the boson field in the ordinary space [16]. We, therefore, add the space index α as follows ([7] and references therein)

$$ {}^{i}\hat{\mathcal{A}}^{m\dagger}_{f\alpha}(\vec{p}) = \hat{b}^{\dagger}_{\vec{p}} *_{T} {}^{i}\mathcal{C}^{m}{}_{f\alpha}\, {}^{i}\hat{\mathcal{A}}^{m\dagger}_{f} \,, i = (I, II)\,. \tag{9.29} $$

We treat free massless bosons of momentum $\vec{p}$ and energy $p^0 = |\vec{p}|$ and of particular "basis vectors" ${}^{i}\hat{\mathcal{A}}^{m\dagger}_{f}$'s which are eigenvectors of all the Cartan subalgebra members [17], ${}^{i}\mathcal{C}^{m}{}_{f\alpha}$ carry the space index α of the boson field. Creation operators operate on the vacuum state $|\psi_{oc_{ev}}> *_{T} |0_{\vec{p}}>$ with the internal space part just a constant, $|\psi_{oc_{ev}}>= |\,1>$, and for a starting single boson state with zero momentum from which one obtains the other single boson states with the same "basis vector" by the operators $\hat{b}^{\dagger}_{\vec{p}}$ which push the momentum by an amount $\vec{p}$, making also ${}^{i}\mathcal{C}^{m}{}_{f\alpha}$ depending on $\vec{p}$.
For the creation operators for boson fields in a coordinate representation one finds using Eqs. (9.25, 9.26)

$$ {}^{i}\hat{\mathcal{A}}^{m\dagger}_{f\alpha}(\vec{x}, x^0) = \int_{-\infty}^{+\infty} \frac{d^{d-1}p}{(\sqrt{2\pi})^{d-1}}\, {}^{i}\hat{\mathcal{A}}^{m\dagger}_{f\alpha}(\vec{p})\, e^{-i(p^0 x^0 - \varepsilon \vec{p}\cdot\vec{x})}|_{p^0=|\vec{p}|} \,, i = (I, II) \tag{9.30} $$

To understand what new the Clifford algebra description of the internal space of fermion and boson fields, Eqs. (9.29, 9.30, 9.27), bring to our understanding of the second quantized fermion and boson fields and what new can we learn from this offer, we need to relate $\sum_{ab} c^{ab}\omega_{ab\alpha}$ and $\sum_{mf} {}^{I}\hat{\mathcal{A}}^{m\dagger}_{f}\, {}^{I}\mathcal{C}^{m}{}_{f\alpha}$, recognizing that ${}^{I}\hat{\mathcal{A}}^{m\dagger}_{f}\, {}^{I}\mathcal{C}^{m}{}_{f\alpha}$ are eigenstates of the Cartan subalgebra members, while $\omega_{ab\alpha}$ are not. And, equivalently, we need to relate $\sum_{ab} \tilde{c}^{ab}\tilde{\omega}_{ab\alpha}$ and $\sum_{mf} {}^{II}\hat{\mathcal{A}}^{m\dagger}_{f}\, {}^{II}\mathcal{C}^{m}{}_{f\alpha}$.
The gravity fields, the vielbeins and the two kinds of spin connection fields, $f^{a}{}_{\alpha}$, $\omega_{ab\alpha}$, $\tilde{\omega}_{ab\alpha}$, respectively, are in the *spin-charge-family* theory (unifying spins, charges and families of fermions and offering not only the explanation for all the assumptions of the *standard model* but also for the increasing number of phenomena observed so far) the only boson fields in $d = (13+1)$, observed in $d = (3+1)$ besides as gravity also as all the other boson fields with the Higgs's scalars included [4].

[16] In the *spin-charge-family* theory the Higgs's scalars origin in the boson gauge fields with the vector index $(7, 8)$, Ref. ([6], Sect. 7.4.1, and the references therein).

[17] In the general case, the energy eigenstates of bosons are in a superposition of ${}^{i}\hat{\mathcal{A}}^{m\dagger}_{f}$, for either $i = I$ or $i = II$. One example, which uses the superposition of the Cartan subalgebra eigenstates manifesting the $SU(3) \times U(1)$ subgroups of the group $SO(5, 1)$, is presented in Fig. 10.2.

We, therefore, need to relate:

$$\{\frac{1}{2}\sum_{ab} S^{ab}\,\omega_{ab\alpha}\}\sum_{m}\beta^{mf}\,\hat{b}^{m\dagger}_{f}(\vec{p}) \text{ related to } \{\sum_{m'f'}{}^{I}\hat{\mathcal{A}}^{m'\dagger}_{f'}\,\mathcal{C}^{m'f'}_{\alpha}\}\sum_{m}\beta^{mf}\,\hat{b}^{m\dagger}_{f}(\vec{p})\,,$$
$$\forall f \text{ and } \forall\,\beta^{mf}\,,$$
$$\mathcal{S}^{cd}\sum_{ab}(c^{ab}{}_{mf}\,\omega_{ab\alpha}) \text{ related to } \mathcal{S}^{cd}\,({}^{I}\hat{\mathcal{A}}^{m\dagger}_{f}\,\mathcal{C}^{mf}_{\alpha})\,,$$
$$\forall\,(m,f),$$
$$\forall \text{ Cartan subalgebra member } \mathcal{S}^{cd}\,. \tag{9.31}$$

Let be repeated that ${}^{I}\hat{\mathcal{A}}^{m\dagger}_{f}$ are chosen to be the eigenvectors of the Cartan subalgebra members, Eq. (9.8). Correspondingly we can relate a particular ${}^{I}\hat{\mathcal{A}}^{m\dagger}_{f}\,{}^{I}\mathcal{C}^{m}{}_{f\alpha}$ with such a superposition of $\omega_{ab\alpha}$'s, which is the eigenvector with the same values of the Cartan subalgebra members as there is a particular ${}^{I}\hat{\mathcal{A}}^{m\dagger}_{f}\mathcal{C}^{mf}_{\alpha}$. We can do this in two ways:

i. Using the first relation in Eq. (9.31). On the left hand side of this relation S^{ab}'s apply on $\hat{b}^{m\dagger}_{f}$ part of $\hat{b}^{m\dagger}_{f}(\vec{p})$. On the right hand side ${}^{I}\hat{\mathcal{A}}^{m\dagger}_{f}$ apply as well on the same "basis vector" $\hat{b}^{m\dagger}_{f}$.

ii. Using the second relation, in which $\mathcal{S}^{cd}$ apply on the left hand side on $\omega_{ab\alpha}$'s,

$$\mathcal{S}^{cd}\sum_{ab} c^{ab}{}_{mf}\,\omega_{ab\alpha} = \sum_{ab} c^{ab}{}_{mf}\,i\,(\omega_{cb\alpha}\eta^{ad} - \omega_{db\alpha}\eta^{ac} + \omega_{ac\alpha}\eta^{bd} - \omega_{ad\alpha}\eta^{bc})\,, \tag{9.32}$$

on each $\omega_{ab\alpha}$ separately; $c^{ab}{}_{mf}$ are constants to be determined from the second relation, where on the right-hand side of this relation $\mathcal{S}^{cd}(= S^{cd} + \tilde{S}^{cd})$ apply on the "basis vector" ${}^{I}\hat{\mathcal{A}}^{m\dagger}_{f}$ of the corresponding gauge field [18].

While ${}^{I}\hat{\mathcal{A}}^{m\dagger}_{f}\,{}^{I}\mathcal{C}^{m}{}_{f\alpha}$ determine the observed vector gauge fields [4,6] determine ${}^{II}\hat{\mathcal{A}}^{m\dagger}_{f}\,{}^{II}\mathcal{C}^{m}{}_{f\alpha}$ the observed scalar gauge fields, those which determine masses of quarks and leptons and antiquarks and antileptons and of weak bosons, after the electroweak break ([6], Subsect. 6.2.2) [19].
The fields ${}^{II}\hat{\mathcal{A}}^{m\dagger}_{f}\,{}^{II}\mathcal{C}^{m}{}_{f\alpha}$ must correspondingly be related with the fields $\tilde{\omega}_{ab\alpha}$.

9.3 Short overview of some of achievements of *spin-charge-family* theory

The *spin-charge-family* theory ([1–5,7–9,12,19,41,42], and the references therein) assumes in $d = (13+1)$-dimensional space a simple action, Eq. (9.1), for massless fermions and for massless vielbeins and two kinds of spin connection fields, with

[18] The reader can find the relation of Eq. (9.31) demonstrated for the case $d = 3+1$ in Ref. [11] at the end of Sect. 3.

[19] There are ${}^{I}\hat{\mathcal{A}}^{m\dagger}_{f}\,{}^{I}\mathcal{C}^{m}{}_{f\alpha}$ fields, carrying the space index $\alpha = (7,8)$, and indices (m,f) which manifest the charges Q, Y, τ, Eq. (108) of Ref. [6], determining together with ${}^{II}\hat{\mathcal{A}}^{m'\dagger}_{f'}\,{}^{II}\mathcal{C}^{m'}{}_{f'\alpha}$ the Higgs scalar and Yukawa couplings.

which fermions interact. Description of the internal degrees of fermions by the odd Clifford "basis vectors" $\hat{b}^{m\dagger}_{f}$, Eq. (10.11), offer the unique explanation of spins, charges and families from the point of view of $d = (3+1)$, explaining the assumptions of the *standard model* for fermions before the electroweak break. Defining the creation operators for fermions as tensor products of Clifford odd "basis vectors" with the basis in ordinary space, explain as well the second quantisation postulates for fermions, assumed by Dirac [25–27].

In the last three years ([7,8], and references therein) the author recognised that the description of the internal degrees of boson fields by the Clifford even "basis vectors", Eq. (10.16), offer the unique explanation of spins and charges from the point of view of $d = (3+1)$ for the vector gauge fields as assumed by the standard model. Fig. 10.1 and Fig. 10.2 in Subsect. 10.2.2, representing the fermion "basis vectors" $\hat{b}^{m\dagger}_{f}$ of the colour triplet quarks and the colourless singlet leptons, four members of any of four families [20], and the "basis vectors" of the corresponding vector gauge fields, ${}^{I}\mathcal{A}^{m\dagger}_{f}\,{}^{I}\mathcal{C}^{m}{}_{f\alpha}$ with 16 members, demonstrate the relation between the fermion "basis vectors" and the "basis vectors" of the vector gauge fields. Besides the vector gauge fields carrying the vector index $\alpha = (0,1,2,3)$ [21], explaining the assumed vector gauge fields of the *standard model*, there are the scalar gauge fields with the space index $\alpha \geq 5$, explaining the scalar gauge fields of the *standard model* — scalar Higgs and Yukawa couplings. Defining the creation operators for bosons as tensor products of the Clifford even "basis vectors" , ${}^{i}\mathcal{A}^{m\dagger}_{f}$, $i = (I, II)$, Eq. (10.16), and the basis in ordinary space, explain the second quantisation postulates for bosons.

Extending the point second quantised fields for fermion and their boson gauge fields to strings moving in ordinary space promise to understand gravity as the second quantised field.

The presentation above is the introduction into the next step, in which the point second quantized fermion and boson fields have to be extended by strings to assure renormalisability of fermion and boson second quantized fields. This next step is just starting.

The action in Eq. (9.1) includes besides vielbeins f^{α}_{a} two kinds of the spin connection fields, $\omega_{ab\alpha}$ and $\tilde{\omega}_{ab\alpha}$.

It is shown in Ref. ([4], Eq. (21)) that the vector gauge fields with the space index $\alpha = (0,1,2,3)$ are expressible by the spin connection fields $\omega_{ab\alpha}$, while spin connection fields can be expressed by vielbein fields f^{α}_{a} and the fermion fields,

[20] Assuming $d = (5+1)$ it follows that there are $2^{\frac{6}{2}-1}$ members in any of $2^{\frac{6}{2}-1}$ families [7].

[21] Index α is explained in Subsect. 9.2.3.

Eq. (4) of Ref. [4] [22]. Eq. (20) of Ref. [4]) relates for a particular symmetry in $d \geq 5$ vielbeins with vector gauge fields.
Eq. (9.31) relates ${}^{I}\mathcal{A}^{m\dagger}_{f}\,{}^{I}\mathcal{C}^{m}_{f\alpha}$ and ${}^{II}\mathcal{A}^{m\dagger}_{f}\,{}^{II}\mathcal{C}^{m}_{f\alpha}$ with $\omega_{ab\alpha}$ and $\tilde{\omega}_{ab\alpha}$, respectively, from where it follows that vielbeins are expressible by ${}^{I}\mathcal{A}^{m\dagger}_{f}\,{}^{I}\mathcal{C}^{m}_{f\alpha}$ and ${}^{II}\mathcal{A}^{m\dagger}_{f}\,{}^{II}\mathcal{C}^{m}_{f\alpha}$.
Description of the internal spaces of boson gauge fields with the Clifford even "basis vectors" suggest to replace $\omega_{ab\alpha}$ and $\tilde{\omega}_{ab\alpha}$ in Eq. (9.1) with ${}^{I}\mathcal{A}^{m\dagger}_{f}\,{}^{I}\mathcal{C}^{m}_{f\alpha}$ and ${}^{II}\mathcal{A}^{m\dagger}_{f}\,{}^{II}\mathcal{C}^{m}_{f\alpha}$.
In the case that there are no fermion fields present $\omega_{ab\alpha}$ and $\tilde{\omega}_{ab\alpha}$ are uniquely described by vielbeins, as we can see in Eq. (9.33) (after neglecting terms determined by fermion fields ψ). In this case $\omega_{ab\alpha}$ is equal to $\tilde{\omega}_{ab\alpha}$ [23].

Most of the work in the *spin-charge-family* theory is done so far by using the action of Eq. (9.1) with fermion fields described by tensor products of the Clifford odd "basis vectors" and basis in ordinary space applying on the vacuum state and in interaction with only gravity, the vielbeins and two kinds of spin connection fields presented in this action. The review of achievements so far is presented in Ref. [6].
To manifest that the assumed simple starting action, Eq. (9.1), in $d = (13+1)$, together with the description of the internal space of fermions with the Clifford odd "basis vectors", offer the explanation of all the assumptions of the *standard model* for quarks and leptons and antiquarks and antileptons, their vector gauge fields and scalar gauge fields (scalar Higgs and Yukawa couplings), for dark matter, for matter/antimatter asymmetry in the universe, making several predictions, some of the achievements of the *spin-charge-family* theory are very shortly presented in what follows.
The reader can find more in Ref. [6] and the references presented in this reference.

Let us present a few of the achievements.

A. *Each irreducible representation of the Lorentz group* $SO(13,1)$, S^{ab}, analysed with respect to the *standard model* groups, $SO(3,1), SU(2)_{I}, SU(2)_{II}, SU(3), U1$ as presented in Table 9.3, includes quarks and leptons (with the right handed neutrino included) and antiquarks and antileptons related to handedness as required by the *standard model*. (The $SO(10)$ unifying theories must relate charges and handedness "by hand".)

[22] Varying the action of Eq. (9.1) with respect to the spin connection fields, the expression for the spin connection fields ${\omega_{ab}}^{e}$ follows

$$\begin{aligned}
{\omega_{ab}}^{e} &= \frac{1}{2E}\{e^{e}{}_{\alpha}\,\partial_{\beta}(Ef^{\alpha}{}_{[a}f^{\beta}{}_{b]}) - e_{a\alpha}\,\partial_{\beta}(Ef^{\alpha}{}_{[b}f^{\beta e]}) - e_{b\alpha}\partial_{\beta}(Ef^{\alpha[e}f^{\beta}{}_{a]})\} \\
&+ \frac{1}{4}\{\bar{\Psi}(\gamma^{e}\,S_{ab} - \gamma_{[a}{S_{b]}}^{e})\Psi\} \\
&- \frac{1}{d-2}\{\delta^{e}_{a}[\frac{1}{E}\,e^{d}{}_{\alpha}\partial_{\beta}(Ef^{\alpha}{}_{[d}f^{\beta}{}_{b]}) + \bar{\Psi}\gamma_{d}S^{d}{}_{b}\,\Psi] - \delta^{e}_{b}[\frac{1}{E}e^{d}{}_{\alpha}\partial_{\beta}(Ef^{\alpha}{}_{[d}f^{\beta}{}_{a]}) + \bar{\Psi}\gamma_{d}S^{d}{}_{a}\ldots
\end{aligned} \tag{9.33}$$

If replacing S^{ab} in Eq. (9.33) with $\tilde{S}^{ab}$, the expression for the spin connection fields ${\tilde{\omega}_{ab}}^{e}$ follows.

[23] This demonstrates that there is the relation between the coordinate dependence of ${}^{I}\mathcal{A}^{m\dagger}_{f}\,{}^{I}\mathcal{C}^{m}_{f\alpha}$ and ${}^{II}\mathcal{A}^{m\dagger}_{f}\,{}^{II}\mathcal{C}^{m}_{f\alpha}$, although their "basis vectors" remain different.

A.i. *The irreducible representations of* $SO(13,1)$ represent families of fermions, with quantum numbers determined by $\tilde{S}^{ab}$, Eqs. (9.8, 10.9).

B. *Fermions interact in* $d = (13+1)$ *with the gravity only*, manifesting in $d = (3+1)$ all the *observed vector and scalar gauge fields, as well as gravity*, Ref. [4].
B.i. Gravity is represented by the *vielbeins* (the gauge fields of momenta) and the *two kinds of the spin connection fields* (the gauge fields of S^{ab} and $\tilde{S}^{ab}$).
B.ii. Eq. (9.34), Subsect. 6.1 of Ref. [6], represents the interaction of massless fermions with the vector gauge fields (first line in Eq. (9.34)), and with the scalar gauge fields carrying the space index $(7,8)$, the space index $(7,8)$ determines the weak charge, $SU(2)_I$, and the hyper charge of scalar fields ([6], Eq. (109,110,111)), (second line of in Eq. (9.34)), before these scalar fields gain non zero vacuum expectation values causing the electroweak break, the last line determines scalar triplets with respect to the space index $(9,10,11,12,13,14)$ offering the explanation for matter/antimatter asymmetry in our universe [12]

$$\mathcal{L}_f = \bar{\psi}\gamma^m(p_m - \sum_{A,i} g^{Ai}\tau^{Ai}A^{Ai}_m)\psi + \\ \{\sum_{s=7,8} \bar{\psi}\gamma^s p_{0s}\,\psi\} + \\ \{\sum_{t=5,6,9,\dots,14} \bar{\psi}\gamma^t p_{0t}\,\psi\}, \tag{9.34}$$

where $p_{0s} = p_s - \frac{1}{2}S^{s's"}\omega_{s's"s} - \frac{1}{2}\tilde{S}^{ab}\tilde{\omega}_{abs}$, $p_{0t} = p_t - \frac{1}{2}S^{t't"}\omega_{t't"t} - \frac{1}{2}\tilde{S}^{ab}\tilde{\omega}_{abt}$, with $m \in (0,1,2,3)$, $s \in (7,8)$, $(s',s") \in (5,6,7,8)$, (a,b) (appearing in $\tilde{S}^{ab}$) run within either $(0,1,2,3)$ or $(5,6,7,8)$, t runs $\in (5,\dots,14)$, $(t',t")$ run either $\in (5,6,7,8)$ or $\in (9,10,\dots,14)$. The appearance of the scalar condensate (so far just assumed, Sect. 6 in Ref. [6]) breaks the manifold $\mathcal{M}^{13,1}$ to $\mathcal{M}^{7,1} \times \mathcal{M}^6$, brings masses of the scale $\propto 10^{16}$ GeV or higher to all the vector and scalar gauge fields, which interact with the condensate [12], *leaving the weak, colour, electromagnetic and gravitational fields massless*. Fermions ψ correspondingly appear in $2^{\frac{7+1}{2}-1} = 8$ massless families, divided in two groups, each group interacting with different scalar fields as discussed in Subsect. 6.2.2 in Ref. [6], remaining massless up to the electroweak break.
Table 8 and Eqs. (108,109,110) of [6] point out the properties of two groups of four families of quarks and leptons, explaining that both groups of four families have the same symmetry of 4×4 mass matrices for quarks and leptons.
The *spin-charge-family* theory obviously predicts at observable energies two groups of four families. To the lower four families the observed three families belong. The stable (at low energies) of the upper four families of quarks, clustered into neutrons, contributes to the dark matter.
How strong is the influence of scalar fields on the masses of quarks and leptons, depends on the coupling constants and the masses of the scalar fields. The *spn-charge-family* predicts that in both groups of four families, the mass matrices 4×4

have the symmetry $SU(2) \times SU(2) \times U(1)$ of the form [24]

$$\mathcal{M}^{\alpha} = \begin{pmatrix} -a_1 - a & e & d & b \\ e* & -a_2 - a & b & d \\ d* & b* & a_2 - a & e \\ b* & d* & e* & a_1 - a \end{pmatrix}^{\alpha}, \tag{9.35}$$

with α representing family members — quarks and leptons [39–41, 43, 44, 48]. The symmetry of mass matrices allows to calculate properties of the fourth family from the known masses of the observed three families and from the mixing matrices of quarks and leptons: Knowing the values of the 3×3 submatrix of the unitary 4×4 matrix allows to calculate all the remaining matrix elements of the 4×4 matrix. The masses of the lower three families do not make a problem. There are measured elements of the 3×3 submatrix of the unitary 4×4 mixing matrix which are even for quarks far to be known accurately enough to allow prediction of the masses of the fourth family of quarks and correspondingly to calculate the rest elements of the mixing matrix. Fitting the experimental data (and the meson decays evaluations in the literature, as well as our own evaluations) the authors of the paper [44] very roughly estimate that the fourth family quarks masses might be pretty above 1 TeV.

Since the matrix elements of the 3×3 submatrix of the 4×4 mixing matrix depend weakly on the fourth family masses, the calculated mixing matrix offers the prediction to what values will more accurate measurements move the present experimental data and also the fourth family mixing matrix elements in dependence of the fourth family masses, Eq. (9.36):

In Eq. (9.36) the matrix elements, taken from Ref. [44], of the 4×4 mixing matrix for quarks obtained when the 4×4 mass matrices respect the symmetry of Eq. (9.35) while the parameters of the mass matrices are fitted to the (exp) experimental data [49] are presented for two choices of the fourth family quark masses: $m_{u_4} = m_{d_4} = 700$ GeV (scf_1) and $m_{u_4} = m_{d_4} = 1\,200$ GeV (scf_2). In parentheses, ( ) and [ ], the changes of the matrix elements are presented, which are due to the changes of the top mass within the experimental inaccuracies: with the $m_t = (172 + 3 \times 0.76)$ GeV and $m_t = (172 - 3 \times 0.76)$, respectively (if there are one, two or more numbers in parentheses the last one or more numbers are different, if there is no parentheses no numbers are different) .

[24] The symmetry $SU(2) \times SU(2) \times U(1)$ of the mass matrices, Eq. (9.35), is expected to remain in all loop corrections [48].

$$|V_{(ud)}| = \left(\begin{array}{l|llll}
\text{exp} & 0.97425 \pm 0.00022 & 0.2253 \pm 0.0008 & 0.00413 \pm 0.00049 & \\ \hline
\text{scf}_1 & 0.97423(4) & 0.22539(7) & 0.00299 & 0.00776(1) \\
\text{scf}_2 & 0.97423[5] & 0.22538[42] & 0.00299 & 0.00793[466] \\ \hline
\text{exp} & 0.225 \pm 0.008 & 0.986 \pm 0.016 & 0.0411 \pm 0.0013 & \\ \hline
\text{scf}_1 & 0.22534(3) & 0.97335 & 0.04245(6) & 0.00349(60) \\
\text{scf}_2 & 0.22531[5] & 0.97336[5] & 0.04248 & 0.00002[216] \\ \hline
\text{exp} & 0.0084 \pm 0.0006 & 0.0400 \pm 0.0027 & 1.021 \pm 0.032 & \\ \hline
\text{scf}_1 & 0.00667(6) & 0.04203(4) & 0.99909 & 0.00038 \\
\text{scf}_2 & 0.00667 & 0.04206[5] & 0.99909 & 0.00024[21] \\ \hline
\text{scf}_1 & 0.00677(60) & 0.00517(26) & 0.00020 & 0.99996 \\
\text{scf}_2 & 0.00773 & 0.00178 & 0.00022 & 0.99997[9]
\end{array}\right) . \tag{9.36}$$

Let us conclude that according to Ref. [44] the masses of the fourth family lie much above the known three. The larger are masses of the fourth family the larger are $V_{u_1 d_4}$ in comparison with $V_{u_1 d_3}$ and the more is valid that $V_{u_2 d_4} < V_{u_1 d_4}, V_{u_3 d_4} < V_{u_1 d_4}$. The flavour changing neutral currents are correspondingly weaker.

Although the results of Ref. [44] are old and that new evaluations are needed, the accuracy of the measured mixing matrices for quarks has not improve in meantime enough to predict masses of the fourth families of quarks.

The *spin-charge-family theory predicts the existence* of besides the lower group of four families of quarks and leptons and antiquarks and antileptons also *the upper group of four families*; quarks and leptons carry the same charges as the lower group of four families, the members of one family of which are presented Table 9.3. There are, however, different scalar fields with the space index $(7, 8)$, which determine mass matrices of the upper four families, although demonstrating the same $\widetilde{SU}(2) \times \widetilde{SU}(2) \times U(1)$ symmetry, discussed in Subsect. 6.2.2 of Ref. [6], in particular in Eq. (108,111).

Different scalar fields are responsible for much higher masses of quarks and leptons than those of the lower four families. Correspondingly, the "nuclear" force among the baryons and mesons of these quarks and antiquarks differ a lot from the nuclear force of the baryons and mesons of the lower four families [25].

The stable of the upper four families offers an explanation for the appearance of the *dark matter* in our universe [26].

[25] In Ref. [45] the weak and "nuclear" scattering of such very heavy baryons by ordinary nucleons is studied, showing that the cross-section for such scattering is very small and therefore consistent with the observation of experiments so far, provided that the quark mass of this baryon is about 100 TeV or above.

[26] In Ref. [42] a simple hydrogen-like model is used to evaluate properties of baryons of these heavy quarks, with one gluon exchange determining the force among the constituents of the fifth family baryons. The weak force and the electromagnetic force start to be at small distances due to heavy masses of quarks of the same order of magnitude as the colour force.

A rough estimation of properties of baryons of the stable fifth family members, of their behaviour during the evolution of the universe and when scattering on the ordinary matter, as well as a study of possible limitations on the family properties due to the cosmological and direct experimental evidences are done in Ref. [42]. The authors of Ref. [42] study the freeze out procedure of the fifth family quarks and anti-quarks and the formation of baryons and anti-baryons up to the temperature $k_b T = 1$ GeV, when the colour phase transition starts which depletes almost all the fifth family quarks and antiquarks, while the colourless fifth family neutrons with very small scattering cross section decouples long before (at $k_b T = 100$ GeV), Fig. **??**. The cosmological evolution suggests for the mass limits

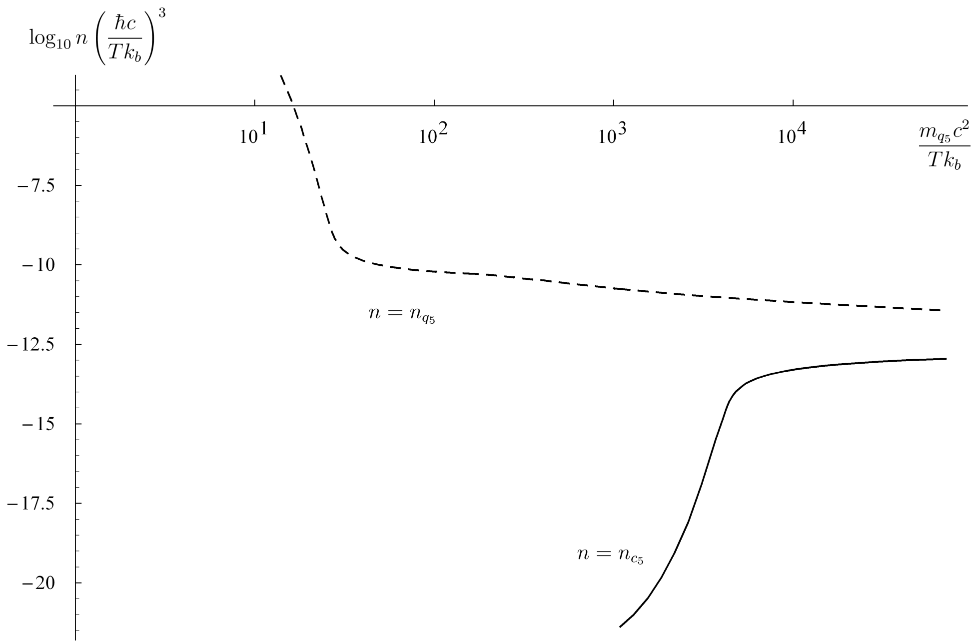

Fig. 9.3: The dependence of the two number densities, n_{q_5} of the fifth family quarks and n_{c_5} of the fifth family clusters of quarks, as functions of $\frac{m_{q_5} c^2}{k_b T}$ is presented for the special values $m_{q_5} = 71$ TeV. The estimated scattering cross sections, entering into Boltzmann equation, are presented in Ref. [42], Eqs. (2,3,4,5). In the treated energy (temperature $k_b T$) interval the one gluon exchange gives the main contribution to the scattering cross sections entering into the Boltzmann equations for n_{q_5} and n_{c_5}. The figure is taken from Ref. [42].

the range 10 TeV $< m_{q_5} <$ a few $\cdot\, 10^2$ TeV and for the scattering cross sections

$10^{-8}\,\mathrm{fm}^2 < \sigma_{c_5} < 10^{-6}\,\mathrm{fm}^2$. The measured density of the dark matter does not put much limitation on the properties of heavy enough clusters [27].

The DAMA/LIBRA experiments [56] limit (provided that they measure the heavy fifth family clusters) the quark mass in the interval: $200\,\mathrm{TeV} < m_{q_5} < 10^5\,\mathrm{TeV}$, Ref. [42].
Masses of the stable fifth family of quarks and leptons are much above the fourth family members [28].

The masses of quarks and leptons of two groups of four families are spread from 10^{-11}GeV (ν of the first family) to 10^{15}GeV (u and d of the fourth family of the upper four families).

There are additional scalar fields in the *spin-charge-family* theory, carrying the space index $s = (9, 10, 11, 12, 13, 14)$: The reader can find the properties of these scalar fields in Eqs. (113,114) and Table 9 in Subsect. 6.2.2 of Ref. [6]. They are triplets or antitriplets with respect to the space index s, carrying additional quantum numbers in adjoint representations determined by S^{ab} or $\tilde{S}^{ab}$, causing transitions of antileptons into quarks and back and leptons into antiquarks and back, what might be responsible in the expanding universe for the matter/antimatter asymmetry and also for the proton decay [12].

If the antiquark $\bar{u}_L^{\bar{c}2}$, from the line 43 presented in Table 9.3, with the "fermion" charge $\tau^4 = -\frac{1}{6}$, the weak charge $\tau^{13} = 0$, the second $SU(2)_{II}$ charge $\tau^{23} = -\frac{1}{2}$, the colour charge $(\tau^{33}, \tau^{38}) = (\frac{1}{2}, -\frac{1}{2\sqrt{3}})$, the hyper charge $Y (= \tau^4 + \tau^{23} =) -\frac{2}{3}$ and the electromagnetic charge $Q (= Y + \tau^{13} =) -\frac{2}{3}$ submits the scalar field $A^2_{\substack{9\,10\\(\oplus)}}$ with $\tau^4 = 2 \times (-\frac{1}{6})$, $\tau^{13} = 0$, $\tau^{23} = -1$, $(\tau^{33}, \tau^{38}) = (\frac{1}{2}, \frac{1}{2\sqrt{3}})$, $Y = -\frac{4}{3}$ and $Q = -\frac{4}{3}$, it transforms into u_R^{c3} from the line 17 of Table 9.3, carrying the quantum numbers $\tau^4 = \frac{1}{6}$, $\tau^{13} = 0$, $\tau^{23} = \frac{1}{2}$, $(\tau^{33}, \tau^{38}) = (0, -\frac{1}{\sqrt{3}})$, $Y = \frac{2}{3}$ and $Q = \frac{2}{3}$. If this scalar field $A^2_{\substack{9\,10\\(\oplus)}}$ is absorbed by the colourless antielectron, $\bar{e}_L$, presented in Table 9.3 in the line 57, carrying the "fermion" charge $\tau^4 = \frac{1}{2}$, the weak charge $\tau^{13} = 0$, the second $SU(2)_{II}$ charge $\tau^{23} = \frac{1}{2}$, $Y = 1$, $Q = 1$, this antielectron $\bar{e}_L$ transforms into $\bar{d}_R^{\bar{c}1}$ quark from the line 3 in Table 9.3, with the "fermion" charge $\frac{1}{6}$, the weak charge $\tau^{13} = 0$, the second $SU(2)_{II}$ charge $\tau^{23} = -\frac{1}{2}$, the colour charge $(\tau^{33}, \tau^{38}) = (\frac{1}{2}, \frac{1}{2\sqrt{3}})$, the hyper charge $Y = -\frac{1}{3}$ and the electromagnetic charge $Q = -\frac{1}{3}$.
These two quarks, d_R^{c1} and u_R^{c3} can bind together with u_R^{c2} from the 9^{th} line of the same table (at low enough energy, after the electroweak transition), into the colour chargeless baryon - a proton. This transition is presented in Fig. 9.4.
The opposite transition at low energies would make the proton decay.

[27] In the case that the weak interaction determines the cross section of the neutron n_5, the interval for the fifth family quarks would be $10\ \mathrm{TeV} < m_{q_5}\, c^2 < 10^5\ \mathrm{TeV}$.

[28] Although the upper four families carry the weak (of two kinds) and the colour charge, these group of four families are completely decoupled from the lower four families up to the $< 10^{16}$ GeV, unless the breaks of symmetries recover.

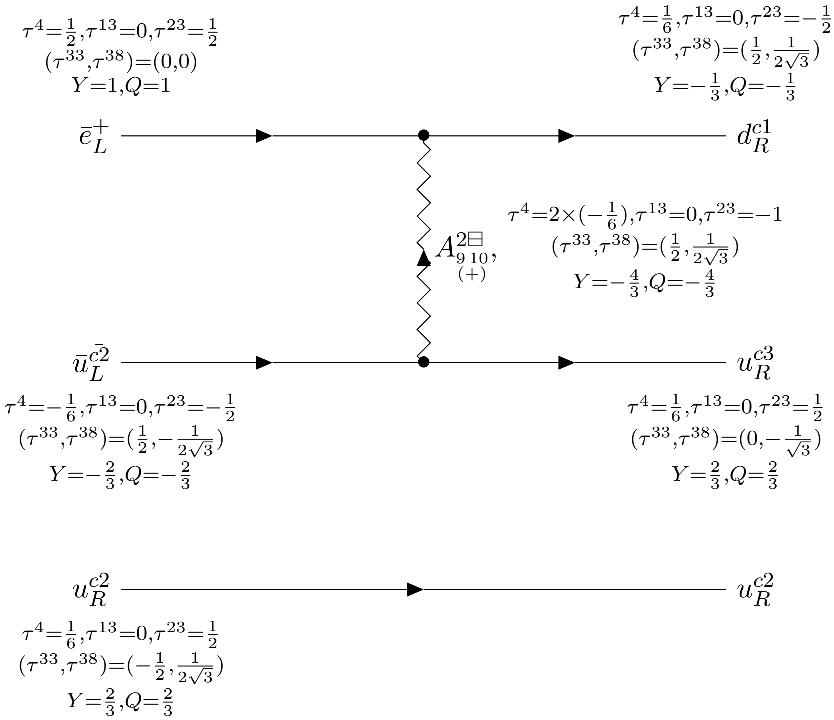

Fig. 9.4: The birth of a "right handed proton" out of an positron $\bar{e}_L^{+}$, antiquark $\bar{u}_L^{\bar{c}2}$ and quark (spectator) u_R^{c2}. This can happen for any of the members of the lower four families, presented in Table 9.1.

Table 9.1 presents the two groups of four families of "basis vectors" for two of the members of quarks and leptons and antiquarks and antileptons, presented in Table 9.3, that is for $\hat{u}_R^{c1\dagger}$ — the right handed u-quark with spin $\frac{1}{2}$ and the colour charge ($\tau^{33} = 1/2, \tau^{38} = 1/(2\sqrt{3})$), appearing in the first line of Table 9.3 — and of the colourless right handed neutrino $\hat{\nu}_R^{\dagger}$ of spin $\frac{1}{2}$, appearing in the 25th line of Table 9.3. The reader can noticed that the "basis vectors" of the $SO(7,1)$ are completely the same for any of the eight families for quarks and neutrinos (leptons indeed). They differ in the $SU(3)$ and $U(1)$ part of the $SO(13,1)$.

To reproduce the observed fermion and boson fields the spontaneous breaks are needed, first from $SO(13,1)$ (and $\widetilde{SO}(13+1)$) to $SO(7,1) \times SU(3) \times U(1)$ (and to $\widetilde{SO}(7+$

Table 9.1: Eight families of the "basis vectors", $\hat{u}^{c1\dagger}_{R}$ — the right handed u-quark with spin $\frac{1}{2}$ and the colour charge ($\tau^{33} = 1/2$, $\tau^{38} = 1/(2\sqrt{3})$), appearing in the first line of Table 9.3 — and of the colourless right handed neutrino $\hat{\nu}^{\dagger}_{R}$ of spin $\frac{1}{2}$, appearing in the 25^{th} line of Table 9.3 — are presented in the left and in the right part of this table, respectively. Table is taken from [51]. Families belong to two groups of four families, the group (I) is a doublet with respect to ($\vec{\tilde{N}}_L$ and $\vec{\tilde{\tau}}^{(1)}$) and a singlet with respect to ($\vec{\tilde{N}}_R$ and $\vec{\tilde{\tau}}^{(2)}$), Eqs. (85,86) in Subsect. 4.3.2 of Ref. [6]), the other (II) is a singlet with respect to ($\vec{\tilde{N}}_L$ and $\vec{\tilde{\tau}}^{(1)}$) and a doublet with respect to ($\vec{\tilde{N}}_R$ and $\vec{\tilde{\tau}}^{(2)}$), Eqs. (85,86), in Subsect. 4.3.2 of Ref. [6]). All the families follow from the starting one by the application of the operators ($\tilde{N}^{\pm}_{R,L}$, $\tilde{\tau}^{(2,1)\pm}$).

					$\tilde{\tau}^{13}$	$\tilde{\tau}^{23}$	$\tilde{N}^3_L$	$\tilde{N}^3_R$	$\tilde{\tau}^4$
I	$\hat{u}^{c1\dagger}_{R\,1}$	$\overset{03}{(+i)}\overset{12}{[+]}\vert\overset{56}{[+]}\overset{78}{(+)}\vert\vert\overset{9\,10}{(+)}\overset{11\,12}{[-]}\overset{13\,14}{[-]}$	$\hat{\nu}^{\dagger}_{R\,1}$	$\overset{03}{(+i)}\overset{12}{[+]}\vert\overset{56}{[+]}\overset{78}{(+)}\vert\vert\overset{9\,10}{(+)}\overset{11\,12}{(+)}\overset{13\,14}{(+)}$	$-\frac{1}{2}$	0	$-\frac{1}{2}$	0	$-\frac{1}{2}$
I	$\hat{u}^{c1\dagger}_{R\,2}$	$\overset{03}{[+i]}\overset{12}{(+)}\vert\overset{56}{[+]}\overset{78}{(+)}\vert\vert\overset{9\,10}{(+)}\overset{11\,12}{[-]}\overset{13\,14}{[-]}$	$\hat{\nu}^{\dagger}_{R\,2}$	$\overset{03}{[+i]}\overset{12}{(+)}\vert\overset{56}{[+]}\overset{78}{(+)}\vert\vert\overset{9\,10}{(+)}\overset{11\,12}{(+)}\overset{13\,14}{(+)}$	$-\frac{1}{2}$	0	$\frac{1}{2}$	0	$-\frac{1}{2}$
I	$\hat{u}^{c1\dagger}_{R\,3}$	$\overset{03}{(+i)}\overset{12}{[+]}\vert\overset{56}{(+)}\overset{78}{[+]}\vert\vert\overset{9\,10}{(+)}\overset{11\,12}{[-]}\overset{13\,14}{[-]}$	$\hat{\nu}^{\dagger}_{R\,3}$	$\overset{03}{(+i)}\overset{12}{[+]}\vert\overset{56}{(+)}\overset{78}{[+]}\vert\vert\overset{9\,10}{(+)}\overset{11\,12}{(+)}\overset{13\,14}{(+)}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	0	$-\frac{1}{2}$
I	$\hat{u}^{c1\dagger}_{R\,4}$	$\overset{03}{[+i]}\overset{12}{(+)}\vert\overset{56}{(+)}\overset{78}{[+]}\vert\vert\overset{9\,10}{(+)}\overset{11\,12}{[-]}\overset{13\,14}{[-]}$	$\hat{\nu}^{\dagger}_{R\,4}$	$\overset{03}{[+i]}\overset{12}{(+)}\vert\overset{56}{(+)}\overset{78}{[+]}\vert\vert\overset{9\,10}{(+)}\overset{11\,12}{(+)}\overset{13\,14}{(+)}$	$\frac{1}{2}$	0	$\frac{1}{2}$	0	$-\frac{1}{2}$
II	$\hat{u}^{c1\dagger}_{R\,5}$	$\overset{03}{[+i]}\overset{12}{[+]}\vert\overset{56}{[+]}\overset{78}{[+]}\vert\vert\overset{9\,10}{(+)}\overset{11\,12}{[-]}\overset{13\,14}{[-]}$	$\hat{\nu}^{\dagger}_{R\,5}$	$\overset{03}{[+i]}\overset{12}{[+]}\vert\overset{56}{[+]}\overset{78}{[+]}\vert\vert\overset{9\,10}{(+)}\overset{11\,12}{(+)}\overset{13\,14}{(+)}$	0	$-\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{1}{2}$
II	$\hat{u}^{c1\dagger}_{R\,6}$	$\overset{03}{(+i)}\overset{12}{(+)}\vert\overset{56}{[+]}\overset{78}{[+]}\vert\vert\overset{9\,10}{(+)}\overset{11\,12}{[-]}\overset{13\,14}{[-]}$	$\hat{\nu}^{\dagger}_{R\,6}$	$\overset{03}{(+i)}\overset{12}{(+)}\vert\overset{56}{[+]}\overset{78}{[+]}\vert\vert\overset{9\,10}{(+)}\overset{11\,12}{(+)}\overset{13\,14}{(+)}$	0	$-\frac{1}{2}$	0	$\frac{1}{2}$	$-\frac{1}{2}$
II	$\hat{u}^{c1\dagger}_{R\,7}$	$\overset{03}{[+i]}\overset{12}{[+]}\vert\overset{56}{(+)}\overset{78}{(+)}\vert\vert\overset{9\,10}{(+)}\overset{11\,12}{[-]}\overset{13\,14}{[-]}$	$\hat{\nu}^{\dagger}_{R\,7}$	$\overset{03}{[+i]}\overset{12}{[+]}\vert\overset{56}{(+)}\overset{78}{(+)}\vert\vert\overset{9\,10}{(+)}\overset{11\,12}{(+)}\overset{13\,14}{(+)}$	0	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{1}{2}$
II	$\hat{u}^{c1\dagger}_{R\,8}$	$\overset{03}{(+i)}\overset{12}{(+)}\vert\overset{56}{(+)}\overset{78}{(+)}\vert\vert\overset{9\,10}{(+)}\overset{11\,12}{[-]}\overset{13\,14}{[-]}$	$\hat{\nu}^{\dagger}_{R\,8}$	$\overset{03}{(+i)}\overset{12}{(+)}\vert\overset{56}{(+)}\overset{78}{(+)}\vert\vert\overset{9\,10}{(+)}\overset{11\,12}{(+)}\overset{13\,14}{(+)}$	0	$\frac{1}{2}$	0	$\frac{1}{2}$	$-\frac{1}{3}$

$1) \times \widetilde{SU}(3) \times \widetilde{U}(1))$ [29], and then further to $SO(3,1) \times SU(2) \times SU(3) \times U(1)$ (and to $\widetilde{SO}(3,1) \times \widetilde{SU}(2) \times \widetilde{SU}(3) \times \widetilde{U}(1))$ [30].

In Refs. [30, 52–55] we study the toy model of $d = (5+1)$, in which the spin connection fields with the space index $s = (5,6)$ force the infinite surface in the fifth and the sixth dimension to form an almost S^2 sphere, keeping rotational symmetry of the surface around one point, while fermions in $d = (3+1)$ of particular handedness keep their masslessness. These happens for all the families of fermions of particular handedness [55].

We have not yet successfully repeated the $d = (5+1)$ toy model of the spontaneous "compactification" in the case of $SO(13,1)$ (we put compactification into quotation

29 The assumed break from the starting symmetry $SO(13,1)$ to $SO(7,1) \times SU(3) \times U(1)$ is supposed to be caused by the appearance of the condensate of two right handed neutrinos with the family quantum numbers of the upper four families (that is of the four families, which do not contain the three so far observed families) at the energy of $\geq 10^{16}$ GeV. This condensate is presented in Table 6 in Subsect. 4.3.2 of Ref. [6]. The condensate makes massive the $SU(2)_{II}$ vector gauge fields and $U(1)_{\tau^4}$ vector gauge fields, as well as all the scalar gauge fields, leaving massless only the $SU(3)$ colour, $U(1)_Y$ and $SU(2)$ weak vector gauge fields, as well as the gravity in $d = (3+1)$.

30 The reader can notice in Table 9.3 that the $SO(7,1)$ content of $SO(13,1)$ is identical for quarks and leptons and identical for antiquarks and antileptons. Quarks distinguish from leptons and antiquarks from antileptons only in the $SO(6)$ content of $SO(13,1)$, that is in the colour charge and in the "fermion" charge.

marks, since an almost S^n sphere, as also S^2 sphere in the toy model of $d = (5+1)$, has the singular points in all infinities).

Fermions — quarks and leptons and antiquarks and antileptons — remain massless and mass protected, with the spin, handedness, $SU(3)$ triplet or singlet charges, weak $SU(2)$ charge, hyper charge and family charge as presented in Tables 9.1, 9.3, "waiting for" spontaneous break of mass protection at the electroweak break. The simple starting action of the *spin-charge-family* theory, Eqs. (9.1, 9.34), offers three singlet and twice two triplet scalar gauge fields, with the space index $s = (7, 8)$ (carrying with respect to the space index the weak and the hyper charge as required for Higgs's scalar in the *standard model*) which break the mass protection of fermions and cause the electroweak break.

9.4 Conclusions

This conclusions reviews the similar Sect. 3 of the Ref. [7], adding new recognitions. In the *spin-charge-family* theory [1–6, 9, 12, 19, 41, 42], and the references therein, the Clifford odd algebra describes the internal space of fermion fields.
The Clifford odd "basis vectors", they are the superposition of odd products of γ^a's, in a tensor product with the basis in ordinary space form the creation and annihilation operators, in which the anti-commutativity of the "basis vectors" is transferred to the creation and annihilation operators for fermions, explaining the second quantization postulates for fermion fields.
The Clifford odd "basis vectors" have all the properties of fermions: Half integer spins concerning the Cartan subalgebra members of the Lorentz algebra in the internal space of fermions in even dimensional spaces ($d = 2(2n+1)$ or $d = 4n$), Subsects. (10.2.2, 9.2.3, App 9.6). The Clifford odd "basis vectors" appear in families and have their Hermitian conjugated partners in a different group.
The Clifford even "basis vectors" are offering the description of the internal space of boson fields. The Clifford even "basis vectors" are the superposition of even products of γ^a's. In a tensor product with the basis in ordinary space the Clifford even "basis vectors" form the creation and annihilation operators which manifest the commuting properties of the second quantized boson fields, offering the explanation for the second quantization postulates for boson fields [11, 21].
The Clifford even "basis vectors" have all the properties of boson fields: Integer spins for the Cartan subalgebra members of the Lorentz algebra in the internal space of bosons, as discussed in Subsects. 10.2.2.
With respect to the subgroups of the $SO(d-1, 1)$ group the Clifford even "basis vectors" manifest the adjoint representations, as illustrated in Figs. (10.1, 10.2). The Clifford even"basis vectors" appear in two groups. All the members of each group have their Hermitian conjugated partners within the same group, or they are self adjoint. (They do not form families.) There are the same number of the Clifford odd "basis vectors", appearing in families, and of their Hermitian conjugated partners, as there are the sum of the members of the two groups of the Clifford even "basis vectors".

There are two kinds of anti-commuting algebras [1]: The Grassmann algebra, offering in d-dimensional space $2\,.\,2^d$ operators (2^d θ^a's and 2^d $\frac{\partial}{\partial\theta_a}$'s, Hermitian conjugated to each other, Eq. (10.2)), and the two Clifford subalgebras, each with 2^d operators named γ^a's and $\tilde{\gamma}^a$'s, respectively, [1,3], Eq. (10.5) [31].
Either the Grassmann algebra [19] or the two Clifford subalgebras can be used to describe the internal space of anti-commuting objects, if the superposition of odd products of operators (θ^a's or γ^a's, or $\tilde{\gamma}^a$'s) are used to describe the internal space of these objects. The commuting objects must be a superposition of even products of operators (θ^a's or γ^a's or $\tilde{\gamma}^a$'s).

No integer spin anti-commuting objects have been observed so far, and to describe the internal space of the so far observed fermions only one of the two Clifford odd subalgebras are needed.
The problem can be solved by reducing the two Clifford subalgebras to only one, the one (chosen to be) determined by γ^a's, as presented in Eq. (10.6), what enables that $2^{\frac{d}{2}-1}$ irreducible representations of $S^{ab} = \frac{i}{2}\{\gamma^a\,,\,\gamma^b\}_-$ (each with the $2^{\frac{d}{2}-1}$ members) obtain the family quantum numbers determined by $\tilde{S}^{ab} = \frac{i}{2}\{\tilde{\gamma}^a\,,\,\tilde{\gamma}^b\}_-$.

The decision to use in the *spin-charge-family* theory in $d = 2(2n+1)$, $n \geq 3$ ($d \geq (13+1)$ indeed), the superposition of the odd products of the Clifford algebra elements γ^a's to describe the internal space of fermions which interact with gravity only (with the vielbeins, the gauge fields of momenta, and the two kinds of the spin connection fields, the gauge fields of S^{ab} and $\tilde{S}^{ab}$, respectively), Eq. (9.1), offers not only the explanation for all the assumed properties of fermions and bosons in the *standard model*, with the appearance of the families of quarks and leptons and antiquarks and antileptons ([6] and the references therein) and of the corresponding vector gauge fields and the Higgs's scalars included [4], but also for the appearance of the dark matter [42] in the universe, for the explanation of the matter/antimatter asymmetry in the universe [12], and for several other observed phenomena, making several predictions [16,40,41,43].

The recognition that the use of the superposition of the even products of the Clifford algebra elements γ^a's to describe the internal space of boson fields, what appears to manifest all the properties of the observed boson fields, as demonstrated

[31] The operators in each of the two Clifford subalgebras appear in even-dimensional spaces in two groups of $2^{\frac{d}{2}-1}\times 2^{\frac{d}{2}-1}$ of the Clifford odd operators (the odd products of either γ^a's in one subalgebra or of $\tilde{\gamma}^a$'s in the other subalgebra), which are Hermitian conjugated to each other: In each Clifford odd group of any of the two subalgebras, there appear $2^{\frac{d}{2}-1}$ irreducible representation each with the $2^{\frac{d}{2}-1}$ members and the group of their Hermitian conjugated partners.

There are as well the Clifford even operators (the even products of either γ^a's in one subalgebra or of $\tilde{\gamma}^a$'s in another subalgebra) which again appear in two groups of $2^{\frac{d}{2}-1}\times 2^{\frac{d}{2}-1}$ members each. In the case of the Clifford even objects, the members of each group of $2^{\frac{d}{2}-1}\times 2^{\frac{d}{2}-1}$ members have the Hermitian conjugated partners within the same group, Subsect. 10.2.2.

The Grassmann algebra operators are expressible with the operators of the two Clifford subalgebras and opposite, Eq. (10.4). The two Clifford sub-algebras are independent of each other, Eq. (10.5), forming two independent spaces.

in this article, makes clear that the Clifford algebra offers not only the explanation for the postulates of the second quantized anti-commuting fermion fields but also for the postulates of the second quantized boson fields.
This recognition, however, offers the possibility to relate

$$
\begin{aligned}
&\{\frac{1}{2}\sum_{ab} S^{ab}\,\omega_{ab\alpha}\}\sum_{m}\beta^{mf}\,\hat{b}^{m\dagger}_{f}(\vec{p}) \;\; \text{to}\; \{\sum_{m'f'}{}^{I}\hat{\mathcal{A}}^{m'\dagger}_{f'}\,{}^{I}\mathcal{C}^{m'}{}_{f'\alpha}\}\sum_{m}\beta^{mf}\,\hat{b}^{m\dagger}_{f}(\vec{p})\,,\\
&\qquad\qquad\qquad\qquad\qquad\qquad \forall f \text{ and } \forall\,\beta^{mf}\,,\\
&\mathcal{S}^{cd}\sum_{ab}(c^{ab}{}_{mf}\,\omega_{ab\alpha}) \;\; \text{to}\; \mathcal{S}^{cd}\,({}^{I}\hat{\mathcal{A}}^{m\dagger}_{f}\,{}^{I}\mathcal{C}^{m}{}_{f\alpha})\,,\\
&\qquad\qquad\qquad\qquad\qquad\qquad \forall\,(m,f),\\
&\qquad\qquad\qquad\qquad\qquad\qquad \forall \text{ Cartan subalgebra member } \mathcal{S}^{cd}\,,
\end{aligned}
$$

and equivalently for ${}^{II}\hat{\mathcal{A}}^{m\dagger}_{f}\,{}^{II}\mathcal{C}^{m}{}_{f\alpha}$ and $\tilde{\omega}_{ab\alpha}$, what offers the possibility to replace the covariant derivative $p_{0\alpha}$

$$
p_{0\alpha} = p_{\alpha} - \frac{1}{2}S^{ab}\omega_{ab\alpha} - \frac{1}{2}\tilde{S}^{ab}\tilde{\omega}_{ab\alpha}
$$

in Eq. (9.1) with

$$
p_{0\alpha} = p_{\alpha} - \sum_{mf}{}^{I}\hat{\mathcal{A}}^{m\dagger}_{f}\,{}^{I}\mathcal{C}^{m}{}_{f\alpha} - \sum_{mf}{}^{II}\hat{\mathcal{A}}^{m\dagger}_{f}\,{}^{II}\mathcal{C}^{m}{}_{f\alpha}\,,
$$

where the relations among ${}^{I}\hat{\mathcal{A}}^{m\dagger}_{f}{}^{I}\mathcal{C}^{m}_{f\alpha}$ and ${}^{II}\hat{\mathcal{A}}^{m\dagger}_{f}\,{}^{II}\mathcal{C}^{m}_{f\alpha}$ with respect to $\omega_{ab\alpha}$ and $\tilde{\omega}_{ab\alpha}$, need additional study.
But let us point out that the "basis vectors" describing the internal spaces of either fermion or boson fields remain the same independent of the choice of the basis of ordinary space. Transformation of the basis in ordinary space

$$
\begin{aligned}
\hat{b}^{s\dagger}_{f}(\vec{x},x^0) &= \sum_{m}\hat{b}^{m\dagger}_{f}\;*_T\int_{-\infty}^{+\infty}\frac{d^{d-1}p}{(\sqrt{2\pi})^{d-1}}\,c^{sm}{}_{f}\;(\vec{p})\;\hat{b}^{\dagger}_{\vec{p}}\;e^{-i(p^0x^0-\varepsilon\vec{p}\cdot\vec{x})}|_{p^0=|\vec{p}|}\,,\\
{}^{i}\hat{\mathcal{A}}^{s\dagger}_{g\alpha}(\vec{x},x^0) &= \sum_{m,f}{}^{I}\hat{\mathcal{A}}^{m\dagger}_{f}\;*_T\int_{-\infty}^{+\infty}\frac{d^{d-1}p}{(\sqrt{2\pi})^{d-1}}\,{}^{i}\mathcal{C}^{m\dagger}_{f\alpha}(\vec{p})\,e^{-i(p^0x^0-\varepsilon\vec{p}\cdot\vec{x})}|_{p^0=|\vec{p}|}\,,i=(I,II)
\end{aligned}
\qquad (9.137)
$$

does not influence the internal spaces of either the Clifford odd or the Clifford even "basis vectors".

Let us add that in odd dimensional spaces, $d=(2n+1), n=$ integer, the properties of the internal spaces of fermion and boson fields differ essentially from the properties of the internal spaces of fermion and boson fields in even dimensional spaces, $d=2(2n+1), d=4n$ [8]: One half of the "basis vectors" have the properties of those of $d=2n$, the other half, following from the first half by the application of $S^{0\,2n+1}$, behave as the Fadeev-Popov ghosts. The anticommuting ones, remaining the superposition of odd products of γ^a's appear in two orthogonal groups with their Hermitian conjugated partners within the same group. The commuting one, still remaining superposition of even products of γ^a's appear in families and have their Hermitian conjugated partners in a separate group, suggesting that taking into account odd dimensional spaces might help to make the theory renormalisable.

We need to make the *spin-charge-family* theory renormalisable. This weak point of the proposal is shared with all the Kaluza-Klein-like theories [13].

Since the *strings theories* [57] seem promising to solve this problem, the fermion and boson fields, described in the *spin-charge-family* with the tensor products of the Clifford odd (for fermions) and the Clifford even (for bosons) "basis vectors" and the points in ordinary space, should the points in the ordinary space be extended to strings.

We see in the two above equations, Eq. (9.37), that the internal spaces described by the "basis vectors" remain the same if transforming the external basis from momentum to coordinate representation. This means that we must relate the *spin-charge-family* theory with the *string theories* by extension of the points in the coordinate space to strings, while keeping the "basis vectors" as the odd products of nilpotents and the rest of projectors for fermions, and as the even products of nilpotents and the rest of projectors for bosons.

Let us recognize as the first step that multiplying algebraically $\hat{b}^{m}_{f}$ by $\hat{b}^{m'\dagger}_{f`}$ one reproduces ${}^{II}\hat{\mathcal{A}}^{m''\dagger}_{f``}$, what the *strings theories* call the left and the right movers forming the boson strings while multiplying algebraically $\hat{b}^{m\dagger}_{f}$ by $\hat{b}^{m'}_{f`}$ one reproduces ${}^{II}\hat{\mathcal{A}}^{m''\dagger}_{f''}$, what the *strings theories* could call the right and the left movers forming the boson strings.

But the *strings theories* usually do not present two kinds of boson fields.

This recognition is the very starting trial to extend the *spin-charge-family* theory from the point second quantised fields to strings in collaboration with Holger beck Nielsen. We hope to write the first trial for this proceeding.

9.5 Properties of boson fields

In Ref. ([7], Table 3), the Clifford even "basis vectors" for one kind of ${}^{i}\hat{\mathcal{A}}^{m\dagger}_{f}, i = (I, II)$ is presented, namely for $i = I$ in the case that $d = (5+1)$. Table 9.2 represents the second kind of the Clifford even "basis vectors", ${}^{II}\hat{\mathcal{A}}^{m\dagger}_{f}$, for the same particular case $d = (5+1)$. Comparing both tables we see that the Clifford even "basis vector", which are products of even number of nilpotents, and the rest of projectors, contain different nilpotents and projectors. Correspondingly, their application on the Clifford odd "basis vectors" $\hat{b}^{m\dagger}_{f}$ differ from the application of ${}^{I}\hat{\mathcal{A}}^{m\dagger}_{f}$, as we notice in Eqs. (10.19, 10.20, 10.21, 10.22). However, to both the same Fig. 10.2.

Comparing Table 3 in Ref. [7] and Table 9.2 we see that both Clifford even "basis vectors", ${}^{I}\hat{\mathcal{A}}^{m\dagger}_{f}$ and ${}^{II}\hat{\mathcal{A}}^{m\dagger}_{f}$, have the same properties with respect to the Cartan subalgebra $(\mathcal{S}^{03}, \mathcal{S}^{12}, \mathcal{S}^{56})$ or with respect to the superposition of $(\mathcal{S}^{03}, \mathcal{S}^{12}, \mathcal{S}^{56})$, manifesting the subgroups $SU(2) \times SU(2) \times U(1)$ or $SU(3) \times U(1)$. To point out this fact the same symbols are used in both tables to denote either selfadjoint members ($\bigcirc$), or Hermitian conjugated partners with the same quantum numbers ($\star\star$, $\triangle$, e.t.c.).

Since the nilpotents and projectors are not the same, the algebraic application of ${}^{II}\hat{\mathcal{A}}^{m\dagger}_{f}$ on Clifford odd "basis vectors" and their Hermitian conjugated partners differ as well.

Table 9.2: The Clifford even "basis vectors" ${}^{II}\hat{\mathcal{A}}^{m\dagger}_{f}$, each of them is the product of projectors and an even number of nilpotents, and each is the eigenvector of all the Cartan subalgebra members, $\mathcal{S}^{03}$, $\mathcal{S}^{12}$, $\mathcal{S}^{56}$, Eq. (9.8), are presented for $d = (5+1)$-dimensional case. Indexes m and f determine $2^{\frac{d}{2}-1} \times 2^{\frac{d}{2}-1}$ different members ${}^{II}\hat{\mathcal{A}}^{m\dagger}_{f}$. In the third column the "basis vectors" ${}^{II}\hat{\mathcal{A}}^{m\dagger}_{f}$ which are Hermitian conjugated partners to each other are pointed out with the same symbol. For example, with $\star\star$ are equipped the first member with $m = 1$ and $f = 1$ and the last member with $m = 4$ and $f = 3$. The sign $\bigcirc$ denotes the Clifford even "basis vectors" which are self adjoint $(({}^{II}\hat{\mathcal{A}}^{m\dagger}_{f})^{\dagger} = {}^{II}\hat{\mathcal{A}}^{m'\dagger}_{f`})$. It is obvious that $^{\dagger}$ has no meaning, since ${}^{II}\hat{\mathcal{A}}^{m\dagger}_{f}$ are self adjoint or are Hermitian conjugated partner to another ${}^{II}\hat{\mathcal{A}}^{m'\dagger}_{f`}$. This table represents also the eigenvalues of the three commuting operators $\mathcal{N}^{3}_{L,R}$ and $\mathcal{S}^{56}$ of the subgroups $SU(2) \times SU(2) \times U(1)$ of the group $SO(5,1)$ and the eigenvalues of the three commuting operators τ^{3}, τ^{8} and τ' of the subgroups $SU(3) \times U(1)$.

f	m	*	${}^{II}\hat{\mathcal{A}}^{m\dagger}_{f}$	$\mathcal{S}^{03}$	$\mathcal{S}^{12}$	$\mathcal{S}^{56}$	$\mathcal{N}^{3}_{L}$	$\mathcal{N}^{3}_{R}$	τ^{3}	τ^{8}	τ'
I	1	$\star\star$	$\stackrel{03}{[-i]}\stackrel{12}{(+)}\stackrel{56}{(+)}$	0	1	1	$\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2\sqrt{3}}$	$-\frac{2}{3}$
	2	$\bullet$	$\stackrel{03}{(+i)}\stackrel{12}{[-]}\stackrel{56}{(+)}$	i	0	1	$-\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$-\frac{3}{2\sqrt{3}}$	$-\frac{2}{3}$
	3	$\odot\odot$	$\stackrel{03}{(+i)}\stackrel{12}{(+)}\stackrel{56}{[-]}$	i	1	0	0	1	0	$\frac{1}{\sqrt{3}}$	$-\frac{2}{3}$
	4	$\bigcirc$	$\stackrel{03}{[-i]}\stackrel{12}{[-]}\stackrel{56}{[-]}$	0	0	0	0	0	0	0	0
II	1	$\triangle$	$\stackrel{03}{(-i)}\stackrel{12}{[+]}\stackrel{56}{(+)}$	−i	0	1	$\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{3}{2\sqrt{3}}$	0
	2	$\otimes$	$\stackrel{03}{[+i]}\stackrel{12}{(-)}\stackrel{56}{(+)}$	0	−1	1	$-\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	$-\frac{3}{2\sqrt{3}}$	0
	3	$\bigcirc$	$\stackrel{03}{[+i]}\stackrel{12}{[+]}\stackrel{56}{[-]}$	0	0	0	0	0	0	0	0
	4	$\odot\odot$	$\stackrel{03}{(-i)}\stackrel{12}{(-)}\stackrel{56}{[-]}$	−i	−1	0	−0	−1	0	$-\frac{1}{\sqrt{3}}$	$\frac{2}{3}$
III	1	$\bigcirc$	$\stackrel{03}{[-i]}\stackrel{12}{[+]}\stackrel{56}{[+]}$	0	0	0	0	0	0	0	0
	2	$\ddagger$	$\stackrel{03}{(+i)}\stackrel{12}{(-)}\stackrel{56}{[+]}$	i	−1	0	−1	0	1	0	0
	3	$\triangle$	$\stackrel{03}{(+i)}\stackrel{12}{[+]}\stackrel{56}{(-)}$	i	0	−1	$-\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{3}{2\sqrt{3}}$	0
	4	$\star\star$	$\stackrel{03}{[-i]}\stackrel{12}{(-)}\stackrel{56}{(-)}$	0	−1	−1	$-\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2\sqrt{3}}$	$\frac{2}{3}$
IV	1	$\ddagger$	$\stackrel{03}{(-i)}\stackrel{12}{(+)}\stackrel{56}{[+]}$	−i	1	0	1	0	−1	0	0
	2	$\bigcirc$	$\stackrel{03}{[+i]}\stackrel{12}{[-]}\stackrel{56}{[+]}$	0	0	0	0	0	0	0	0
	3	$\otimes$	$\stackrel{03}{[+i]}\stackrel{12}{(+)}\stackrel{56}{(-)}$	0	1	−1	$\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{3}{2\sqrt{3}}$	0
	4	$\bullet$	$\stackrel{03}{(-i)}\stackrel{12}{[-]}\stackrel{56}{(-)}$	−i	0	−1	$\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2\sqrt{3}}$	$\frac{2}{3}$

9.6 "Basis vectors" in $d = (3+1)$

This section is the copy of the one in Ref. [7]. It was suggested by the referee of Ref. [7], to illustrate on a simple case of $d = (3+1)$ the properties of "basis vectors" when describing internal spaces of fermions and bosons by the Clifford algebra: i. The way of constructing the"basis vectors" for fermions which appear in families and for bosons which have no families. ii. The manifestation of anti-commutativity of the second quantized fermion fields and commutativity of the second quantized boson fields. iii. The creation and annihilation operators, described by a tensor product, $*_T$, of the "basis vectors" and their Hermitian conjugated partners with the basis in ordinary space-time.

This section is a short overview of some sections presented in the article [8], equipped by concrete examples of "basis vectors" for fermions and bosons in $d = (3+1)$.

"Basis vectors"

Let us start by arranging the "basis vectors" as a superposition of products of (operators [32]) γ^a, each "basis vector" is the eigenvector of all the Cartan subalgebra members, Eq. (10.7). To achieve this, we arrange "basis vectors" to be products of nilpotents and projectors, Eqs. (10.8, 10.9), so that every nilpotent and every projector is the eigenvector of one of the Cartan subalgebra members.
Example 1.
Let us notice that, for example, two nilpotents anti-commute, while one nilpotent and one projector (or two projectors) commute due to Eq. (10.5):
$\frac{1}{2}(\gamma^0 - \gamma^3)\frac{1}{2}(\gamma^1 - i\gamma^2) = -\frac{1}{2}(\gamma^1 - i\gamma^2)\frac{1}{2}(\gamma^0 - \gamma^3)$, while $\frac{1}{2}(\gamma^0 - \gamma^3)\frac{1}{2}(1 + i\gamma^1\gamma^2) = \frac{1}{2}(1 + i\gamma^1\gamma^2)\frac{1}{2}(\gamma^0 - \gamma^3)$.

In $d = (3+1)$ there are 16 $(2^{d=4})$ "eigenvectors" of the Cartan subalgebra members (S^{03}, S^{12}) and $(\mathcal{S}^{03}, \mathcal{S}^{12})$ of the Lorentz algebras S^{ab} and $\mathcal{S}^{ab}$, Eq. (10.7).

Half of them are the Clifford odd "basis vectors", appearing in two irreducible representations, in two "families" $(2^{\frac{4}{2}-1}$, $f = (1,2))$, each with two $(2^{\frac{4}{2}-1}$, $m = (1,2))$ members, $\hat{b}^{m\dagger}_f$, Eq. (9.38).
There is a separate group of $2^{\frac{4}{2}-1} \times 2^{\frac{4}{2}-1}$ (Clifford odd) Hermitian conjugated partners $\hat{b}^{m}_f = (\hat{b}^{m\dagger}_f)^\dagger$ appearing in a separate group which is not reachable by S^{ab}, Eq. (9.39).
There are two separate groups of $2^{\frac{4}{2}-1} \times 2^{\frac{4}{2}-1}$ Clifford even "basis vectors", ${}^{i}\mathcal{A}^{m\dagger}_f, i = (I, II)$, the $2^{\frac{4}{2}-1}$ members of each are self-adjoint, the rest have their Hermitian conjugated partners within the same group, Eqs. (9.41, 9.42).
All the members of each group are reachable by S^{ab} or $\tilde{S}^{ab}$ from any starting "basis vector" ${}^{i}\mathcal{A}^{1\dagger}_1$.
Example 2.
$\hat{b}^{m=1\dagger}_{f=1} = \stackrel{03}{(+i)}\stackrel{12}{[+]}$ $(= \frac{1}{2}(\gamma^0 - \gamma^3)\frac{1}{2}(1 + i\gamma^1\gamma^2))$ is a Clifford odd "basis vector", its Hermitian conjugated partner, Eq. (10.5), is $\hat{b}^{m=1}_{f=1} = \stackrel{03}{(-i)}\stackrel{12}{[+]}$ $(= \frac{1}{2}(\gamma^0 + \gamma^3)\frac{1}{2}(1 + i\gamma^1\gamma^2)$, not reachable by either S^{ab} or by $\tilde{S}^{ab}$ from any of two members in any of two "families" of the group of $\hat{b}^{m\dagger}_f$, presented in Eq. (9.38).
${}^{I}\mathcal{A}^{m=1\dagger}_{f=1} (= \stackrel{03}{[+i]}\stackrel{12}{[+]} = \frac{1}{2}(1 + \gamma^0\gamma^3)\frac{1}{2}(1 + i\gamma^1\gamma^2)$ is self-adjoint, ${}^{I}\mathcal{A}^{m=2\dagger}_{f=1} (= \stackrel{03}{(-i)}\stackrel{12}{(-)} = \frac{1}{2}(\gamma^0 - \gamma^3)(\gamma^1 - i\gamma^2)$. Its Hermitian conjugated partner, belonging to the same group, is ${}^{I}\mathcal{A}^{m=1\dagger}_{f=2}$

[32] We repeat that we treat γ^a as operators, not as matrices. We write "basis vectors" as the superposition of products of γ^a. If we want to look for a matrix representation of any operator, say S^{ab}, we arrange the "basis vectors" into a series and write a matrix of transformations caused by the operator. However, we do not need to look for the matrix representations of the operators since we can directly calculate the application of any operators on "basis vectors".

and is reachable from ${}^{I}\mathcal{A}^{m=1\dagger}_{f=1}$ by the application of $\tilde{S}^{01}$, since $\tilde{\gamma}^0 *_A \stackrel{03}{[+i]} = i \stackrel{03}{(+i)}$ and $\tilde{\gamma}^1 *_A \stackrel{12}{[+]} = i \stackrel{12}{(+)}$.

Clifford odd "basis vectors"

Let us first present the Clifford odd anti-commuting "basis vectors", appearing in two "families" $\hat{b}^{m\dagger}_f$, and their Hermitian conjugated partners $(\hat{b}^{m\dagger}_f)^\dagger$. Each member of the two groups is a product of one nilpotent and one projector. We choose the right-handed Clifford odd "basis vectors" [33]. Clifford odd "basis vectors" appear in two families, each family has two members [34]. Let us notice that members of each of two families have the same quantum numbers (S^{03}, S^{12}). They distinguish in "family" quantum numbers $(\tilde{S}^{03}, \tilde{S}^{12})$.

$$\begin{array}{cccc}
f=1 & f=2 & & \\
\tilde{S}^{03}=\frac{i}{2}, \tilde{S}^{12}=-\frac{1}{2} & \tilde{S}^{03}=-\frac{i}{2}, \tilde{S}^{12}=\frac{1}{2} & S^{03} & S^{12} \\
\hat{b}^{1\dagger}_1 = \stackrel{03}{(+i)}\stackrel{12}{[+]} & \hat{b}^{1\dagger}_2 = \stackrel{03}{[+i]}\stackrel{12}{(+)} & \frac{i}{2} & \frac{1}{2} \\
\hat{b}^{2\dagger}_1 = \stackrel{03}{[-i]}\stackrel{12}{(-)} & \hat{b}^{2\dagger}_2 = \stackrel{03}{(-i)}\stackrel{12}{[-]} & -\frac{i}{2} & -\frac{1}{2}\,.
\end{array} \tag{9.38}$$

We find for their Hermitian conjugated partners

$$\begin{array}{cccc}
S^{03}=-\frac{i}{2}, S^{12}=\frac{1}{2} & S^{03}=\frac{i}{2}, S^{12}=-\frac{1}{2} & \tilde{S}^{03} & \tilde{S}^{12} \\
\hat{b}^{1}_1 = \stackrel{03}{(-i)}\stackrel{12}{[+]} & \hat{b}^{1}_2 = \stackrel{03}{[+i]}\stackrel{12}{(-)} & -\frac{i}{2} & -\frac{1}{2} \\
\hat{b}^{2}_1 = \stackrel{03}{[-i]}\stackrel{12}{(+)} & \hat{b}^{2}_2 = \stackrel{03}{(+i)}\stackrel{12}{[-]} & \frac{i}{2} & \frac{1}{2}\,.
\end{array} \tag{9.39}$$

The vacuum state $|\psi_{oc}>$, Eq. (10.14), on which the Clifford odd "basis vectors" apply is equal to: $|\psi_{oc}> = \frac{1}{\sqrt{2}}(\stackrel{03}{[-i]}\stackrel{12}{[+]} + \stackrel{03}{[+i]}\stackrel{12}{[+]})$.
Let us recognize that the Clifford odd "basis vectors" anti-commute due to the odd number of nilpotents, Example 1. And they are orthogonal according to Eqs. (10.25, 10.26, 10.27): $\hat{b}^{m\dagger}_f *_A \hat{b}^{m'\dagger}_{f'} = 0$.
Example 3.

According to the vacuum state presented above, one finds that, for example, $\hat{b}^{1\dagger}_1 (= \stackrel{03}{(+i)}\stackrel{12}{[+]})|\psi_{oc}>$ is $\hat{b}^{1\dagger}_1$ back, since $\stackrel{03}{(+i)}\stackrel{12}{[+]} *_A \stackrel{03}{[-i]}\stackrel{12}{[+]} = \stackrel{03}{(+i)}\stackrel{12}{[+]}$, according to Eq. (10.25), while $\stackrel{03}{(-i)}\stackrel{12}{[+]} *_A \stackrel{03}{[-i]}\stackrel{12}{[+]} = 0$ (due to $(\gamma^0+\gamma^3)(1-\gamma^0\gamma^3)=0$).
Let us apply S^{01} and $\tilde{S}^{01}$ on some of the "basis vectors" $\hat{b}^{m\dagger}_f$, say $\hat{b}^{1\dagger}_1$.
When applying $S^{01} = \frac{i}{2}\gamma^0\gamma^1$ on $\frac{1}{2}(\gamma^0-\gamma^3)\frac{1}{2}(1+i\gamma^1\gamma^2)(\equiv \stackrel{03}{(+i)}\stackrel{12}{[+]})$ we get $-\frac{i}{2}\frac{1}{2}(1-\gamma^0\gamma^3)\frac{1}{2}(\gamma^1-i\gamma^2)(\equiv (-\frac{i}{2}\stackrel{03}{[-i]}\stackrel{12}{(-)})$.
When applying $\tilde{S}^{01} = \frac{i}{2}\tilde{\gamma}^0\tilde{\gamma}^1$ on $\frac{1}{2}(\gamma^0-\gamma^3)\frac{1}{2}(1+i\gamma^1\gamma^2)(\equiv \stackrel{03}{(+i)}\stackrel{12}{[+]})$ we get, according to Eq. (10.6), or if using Eq. (10.10), $-\frac{i}{2}\frac{1}{2}(1+\gamma^0\gamma^3)\frac{1}{2}(\gamma^1+i\gamma^2)(\equiv (-\frac{i}{2}\stackrel{03}{[+i]}\stackrel{12}{(+)})$.

---

[33] We could choose the left-handed Clifford odd "basis vectors" by exchanging the role of 'basis vectors" and their Hermitian conjugated partners.

[34] In the case of $d=(1+1)$, we would have one family with one member only, which must be nilpotent.

It then follows, after using Eqs. (10.10, 10.25, 10.26, 10.27) or just the starting relation, Eq. (10.5), and taking into account the above concrete evaluations, the relations of Eq. (10.15) for our particular case

$$\begin{aligned}
\hat{b}^{m\dagger}_{f} *_A |\psi_{oc}> &= |\psi^m_f>\,,\\
\hat{b}^{m}_{f} *_A |\psi_{oc}> &= 0\cdot|\psi_{oc}>\,,\\
\{\hat{b}^{m\dagger}_{f}, \hat{b}^{m'\dagger}_{f'}\}_{-} *_A |\psi_{oc}> &= 0\cdot|\psi_{oc}>\,,\\
\{\hat{b}^{m}_{f}, \hat{b}^{m'}_{f'}\}_{-} *_A |\psi_{oc}> &= 0\cdot|\psi_{oc}>\,,\\
\{\hat{b}^{m}_{f}, \hat{b}^{m'\dagger}_{f'}\}_{-} *_A |\psi_{oc}> &= \delta^{mm'}\delta_{ff`}|\psi_{oc}>\,. \qquad (9.40)
\end{aligned}$$

The last relation of Eq. (9.40) takes into account that each "basis vector" carries the "family" quantum number, determined by $\tilde{S}^{ab}$ of the Cartan subalgebra members, Eq. (10.7), and the appropriate normalization of "basis vectors", Eqs. (9.38, 9.39).

Clifford even "basis vectors"

Besides $2^{\frac{4}{2}-1}\times 2^{\frac{4}{2}-1}$ Clifford odd "basis vectors" and the same number of their Hermitian conjugated partners, Eqs. (9.38, 9.39), the Clifford algebra objects offer two groups of $2^{\frac{4}{2}-1}\times 2^{\frac{4}{2}-1}$ Clifford even "basis vectors", the members of the group ${}^{I}\mathcal{A}^{m\dagger}_{f}$ and ${}^{II}\mathcal{A}^{m\dagger}_{f}$, which have Hermitian conjugated partners within the same group or are self-adjoint [35]. We have the group ${}^{I}\mathcal{A}^{m\dagger}_{f}$, $m=(1,2), f=(1,2)$, the members of which are Hermitian conjugated to each other or are self-adjoint,

$$\begin{array}{llcclllcc}
 & & \mathcal{S}^{03} & \mathcal{S}^{12} & & & & \mathcal{S}^{03} & \mathcal{S}^{12}\\
{}^{I}\mathcal{A}^{1\dagger}_{1} = & \stackrel{03}{[+i]}\stackrel{12}{[+]} & 0 & 0\,, & & {}^{I}\mathcal{A}^{1\dagger}_{2} = & \stackrel{03}{(+i)}\stackrel{12}{(+)} & i & 1\\
{}^{I}\mathcal{A}^{2\dagger}_{1} = & \stackrel{03}{(-i)}\stackrel{12}{(-)} & -i & -1\,, & & {}^{I}\mathcal{A}^{2\dagger}_{2} = & \stackrel{03}{[-i]}\stackrel{12}{[-]} & 0 & 0\,,
\end{array} \qquad (9.41)$$

and the group ${}^{II}\mathcal{A}^{m\dagger}_{f}$, $m=(1,2), f=(1,2)$, the members of which are either Hermitian conjugated to each other or are self adjoint

$$\begin{array}{llcclllcc}
 & & \mathcal{S}^{03} & \mathcal{S}^{12} & & & & \mathcal{S}^{03} & \mathcal{S}^{12}\\
{}^{II}\mathcal{A}^{1\dagger}_{1} = & \stackrel{03}{[+i]}\stackrel{12}{[-]} & 0 & 0\,, & & {}^{II}\mathcal{A}^{1\dagger}_{2} = & \stackrel{03}{(+i)}\stackrel{12}{(-)} & i & -1\\
{}^{II}\mathcal{A}^{2\dagger}_{1} = & \stackrel{03}{(-i)}\stackrel{12}{(+)} & -i & 1\,, & & {}^{II}\mathcal{A}^{2\dagger}_{2} = & \stackrel{03}{[-i]}\stackrel{12}{[+]} & 0 & 0\,.
\end{array} \qquad (9.42)$$

The Clifford even "basis vectors" have no families. The two groups, ${}^{I}\mathcal{A}^{m\dagger}_{f}$ and ${}^{II}\mathcal{A}^{m\dagger}_{f}$ (they are not reachable from one another by $\mathcal{S}^{ab}$), are orthogonal (which can easily be checked, since $\stackrel{ab}{(\pm k)} *_A \stackrel{ab}{(\pm k)}=0$, and $\stackrel{ab}{[\pm k]} *_A \stackrel{ab}{[\mp k]}=0$).

$$ {}^{I}\mathcal{A}^{m\dagger}_{f} *_A {}^{II}\mathcal{A}^{m'\dagger}_{f`} = 0\,, \quad \text{for any } (m, m', f, f`)\,. \qquad (9.43)$$

Application of ${}^{i}\mathcal{A}^{m\dagger}_{f}, i=(I,II)$ on $\hat{b}^{m\dagger}_{f}$

Let us demonstrate the application of ${}^{i}\mathcal{A}^{m\dagger}_{f}, i=(I,II)$, on the Clifford odd "basis vectors" $\hat{b}^{m\dagger}_{f}$, Eqs. (10.19, 10.22), for our particular case $d=(3+1)$ and compare the result with the result of application S^{ab} and $\tilde{S}^{ab}$ on $\hat{b}^{m\dagger}_{f}$ evaluated above in

[35] Let be repeated that $\mathcal{S}^{ab}=S^{ab}+\tilde{S}^{ab}$ [11].

Example 3. We found, for example, that $S^{01}(=\frac{i}{2}\gamma^0\gamma^1) *_A \hat{b}^{1\dagger}_1(=\frac{1}{2}(\gamma^0-\gamma^3)\frac{1}{2}(1+i\gamma^1\gamma^2)(=\stackrel{03}{(+i)}\stackrel{12}{[+]})=-\frac{i}{2}\frac{1}{2}(1-\gamma^0\gamma^3)\frac{1}{2}(\gamma^1-i\gamma^2)(=(-\frac{i}{2}\stackrel{03}{[-i]}\stackrel{12}{(-i)})=\hat{b}^{2\dagger}_1$.
Applying ${}^{I}\mathcal{A}^{2\dagger}_1(=\stackrel{03}{(-i)}\stackrel{12}{(-)}) *_A \hat{b}^{1\dagger}_1(=\stackrel{03}{(+i)}\stackrel{12}{[+]})=-\stackrel{03}{[-i]}\stackrel{12}{(-)}$, which is $\hat{b}^{2\dagger}_1$, presented in Eq. (9.38). We obtain in both cases the same result, up to the factor $\frac{i}{2}$ (in front of $\gamma^0\gamma^1$ in S^{01}). In the second case one sees that ${}^{I}\mathcal{A}^{2\dagger}_1$ (carrying $\mathcal{S}^{03}=-i, \mathcal{S}^{12}=-1$) transfers these quantum numbers to $\hat{b}^{1\dagger}_1$ (carrying $S^{03}=\frac{i}{2}, S^{12}=\frac{1}{2}$) what results in $\hat{b}^{2\dagger}_1$ (carrying $S^{03}=\frac{-i}{2}, S^{12}=\frac{-1}{2}$).
We can check what the application of the rest three ${}^{I}\mathcal{A}^{m\dagger}_f$, do when applying on $\hat{b}^{m\dagger}_f$. The self-adjoint member carrying $\mathcal{S}^{03}=0, \mathcal{S}^{12}=0$, either gives $\hat{b}^{m\dagger}_f$ back, or gives zero, according to Eq. (10.25). The Clifford even "basis vectors", carrying non zero $\mathcal{S}^{03}$ and $\mathcal{S}^{12}$ transfer their internal values to $\hat{b}^{m\dagger}_f$ or give zero. In all cases ${}^{I}\mathcal{A}^{m\dagger}_f$ transform a "family" member to another or the same "family" member of the same "family".
Example 4.:
${}^{I}\mathcal{A}^{1\dagger}_1(=\stackrel{03}{[+i]}\stackrel{12}{[+]}) *_A \hat{b}^{1\dagger}_1(=\stackrel{03}{(+i)}\stackrel{12}{[+]})=\hat{b}^{1\dagger}_1(=\stackrel{03}{(+i)}\stackrel{12}{[+]})$, ${}^{I}\mathcal{A}^{1\dagger}_1(=\stackrel{03}{[+i]}\stackrel{12}{[+]}) *_A \hat{b}^{1\dagger}_2(=\stackrel{03}{[+i]}\stackrel{12}{(+)})=\hat{b}^{1\dagger}_2(=\stackrel{03}{[+i]}\stackrel{12}{(+)})$,
${}^{I}\mathcal{A}^{2\dagger}_1(=\stackrel{03}{(-i)}\stackrel{12}{(-)}) *_A \hat{b}^{1\dagger}_2(=\stackrel{03}{[+i]}\stackrel{12}{(+)})=\hat{b}^{2\dagger}_2(=\stackrel{03}{(-i)}\stackrel{12}{[-]})$, ${}^{I}\mathcal{A}^{2\dagger}_1(=\stackrel{03}{(-i)}\stackrel{12}{(-)}) *_A \hat{b}^{2\dagger}_2(=\stackrel{03}{(-i)}\stackrel{12}{[-]})=0$.

One easily sees that the application of ${}^{II}\mathcal{A}^{m\dagger}_f$ on $\hat{b}^{m'\dagger}_{f'}$ give zero for all (m, m', f, f') (due to $\stackrel{ab}{[\pm k]} *_A \stackrel{ab}{[\mp k]}=0$, $\stackrel{ab}{[\pm k]} *_A \stackrel{ab}{(\mp k)}=0$, and similar applications).
We realised in Example 3. that the application of $\tilde{S}^{01}=\frac{i}{2}\tilde{\gamma}^0\tilde{\gamma}^1$ on $\hat{b}^{1\dagger}_1$ gives $(-\frac{i}{2}\stackrel{03}{[+i]}\stackrel{12}{(+i)})=-\frac{i}{2}\hat{b}^{1\dagger}_2$.

Let us algebraically, $*_A$, apply ${}^{II}\mathcal{A}^{2\dagger}_1(=\stackrel{03}{(-i)}\stackrel{12}{(+)})$, with quantum numbers $(\mathcal{S}^{03}, \mathcal{S}^{12})=(-i, 1)$, from the right hand side the Clifford odd "basis vector" $\hat{b}^{1\dagger}_1$. This application causes the transition of $\hat{b}^{1\dagger}_1$ (with quantum numbers $(\tilde{S}^{03}, \tilde{S}^{12})=(\frac{i}{2}, -\frac{1}{2})$ (see Eq. (10.9)) into $\hat{b}^{1\dagger}_2$ (with quantum numbers $(\tilde{S}^{03}, \tilde{S}^{12})=(-\frac{i}{2}, \frac{1}{2})$). ${}^{II}\mathcal{A}^{2\dagger}_1$ obviously transfers its quantum numbers to Clifford odd "basis vectors", keeping m unchanged, and changing the "family" quantum number: $\hat{b}^{1\dagger}_1 *_A {}^{II}\mathcal{A}^{2\dagger}_1=\hat{b}^{1\dagger}_2$.

We can conclude: The internal space of the Clifford even "basis vectors" has properties of the gauge fields of the Clifford odd "basis vectors"; ${}^{I}\mathcal{A}^{m\dagger}_f$ transform "family" members of the Clifford odd "basis vectors" among themselves, keeping the "family" quantum number unchanged, ${}^{II}\mathcal{A}^{m\dagger}_f$ transform a particular "family" member into the same "family" member of another "family".

Creation and annihilation operators

To define creation and annihilation operators for fermion and boson fields, we must include besides the internal space, the ordinary space, presented in Eq. (9.25), which defines the momentum or coordinate part of fermion and boson fields.

We define the creation operators for the single particle fermion states as a tensor product, $*_T$, of the Clifford odd "basis vectors" and the basis in ordinary space, Eq. (9.27):
$\hat{b}^{s\dagger}_f(\vec{p}) = \sum_m c^{sm}{}_f(\vec{p})\, \hat{b}^\dagger_{\vec{p}} *_T \hat{b}^{m\dagger}_f$. The annihilation operators are their Hermitian conjugated partners.
We have seen in Example 1. that Clifford odd "basis vectors" (having odd products of nilpotents) anti-commute. The commuting objects $\hat{b}^\dagger_{\vec{p}}$ (multiplying the "basis vectors") do not change the Clifford oddness of $\hat{b}^{s\dagger}_f(\vec{p})$. The two Clifford odd objects, $\hat{b}^{s\dagger}_f(\vec{p})$ and $\hat{b}^{s'\dagger}_{f'}(\vec{p'})$, keep their anti-commutativity, fulfilling the anti-commutation relations as presented in Eq. (9.28). Correspondingly we do not need to postulate anti-commutation relations of Dirac. The Clifford odd "basis vectors" in a tensor product with the basis in ordinary space explain the second quantized postulates for fermion fields.
The Clifford odd "basis vectors" contribute for each $\vec{p}$ a finite number of $\hat{b}^{s\dagger}_f(\vec{p})$, the ordinary basis offers infinite possibilities [36].

Recognizing that internal spaces of fermion fields and their corresponding boson gauge fields are describable in even dimensional spaces by the Clifford odd and even "basis vectors", respectively, it becomes evidently that when including the basis in ordinary space, we must take into account that boson gauge fields have the space index α, which describes the α component of the boson fields in ordinary space.
We multiply, therefore, as presented in Eq. (9.29), the Clifford even "basis vectors" with the coefficient ${}^i\mathcal{C}^m{}_{f\alpha}$ carrying the space index α so that the creation operators ${}^i\hat{\mathcal{A}}^{m\dagger}_{f\alpha}(\vec{p}) = \hat{b}^\dagger_{\vec{p}} *_T {}^i\mathcal{C}^m{}_{f\alpha}\, {}^i\hat{\mathcal{A}}^{m\dagger}_f$, $i = (I, II)$ carry the space index α [37]. The self-adjoint "basis vectors", like $({}^i\hat{\mathcal{A}}^{1\dagger}_{1\alpha}, {}^i\hat{\mathcal{A}}^{2\dagger}_{2\alpha}, i = (I, II))$, do not change quantum numbers of the Clifford odd "basis vectors", since they have internal quantum numbers equal to zero.
In higher dimensional space, like in $d = (5 + 1)$, ${}^I\hat{\mathcal{A}}^{1\dagger}_3$, presented in Table 10.1, could represent the internal space of a photon field, which transfers to, for example, a fermion and anti-fermion pair with the internal space described by $(\hat{b}^{1\dagger}_1, \hat{b}^{3\dagger}_1)$, presented in Table 10.2, the momentum in ordinary space.
The subgroup structure of $SU(3)$ gauge fields can be recognized in Fig. 10.2.

Properties of the gauge fields ${}^i\hat{\mathcal{A}}^{m\dagger}_{f\alpha}$ need further studies.
In even dimensional spaces, the Clifford odd and even "basis vectors", describing internal spaces of fermion and boson fields, offer the explanation for the second quantized postulates for fermion and boson fields [17].

[36] An infinitesimally small difference between $\vec{p}$ and $\vec{p'}$ makes two creation operators $\hat{b}^{s\dagger}_f(\vec{p})$ and $\hat{b}^{s\dagger}_f(\vec{p'})$ with the same "basis vector" describing the internal space of fermion fields still fulfilling the anti-commutation relations (as we learn from atomic physics; two electrons can carry the same spin if they distinguish in the coordinate part of the state).

[37] Requiring the local phase symmetry for the fermion part of the action, Eq. (9.1), would lead to the requirement of the existence of the boson fields with the space index α.

9.7 Some useful relations in Grassmann and Clifford algebras, needed also in App. 9.8

This appendix contains some helpful relations. For more detailed explanations and for proofs, the reader is kindly asked to read [6,7] and the references therein. For fermions, the operator of handedness Γ^d is determined as follows:

$$\Gamma^{(d)} = \prod_a (\sqrt{\eta^{aa}}\gamma^a) \cdot \begin{cases} (i)^{\frac{d}{2}}\,, & \text{for } d \text{ even}\,,\\ (i)^{\frac{d-1}{2}}\,, & \text{for } d \text{ odd}\,, \end{cases} \tag{9.44}$$

The vacuum state for the Clifford odd "basis vectors", $|\psi_{oc}>$, is defined as

$$|\psi_{oc}>= \sum_{f=1}^{2^{\frac{d}{2}-1}} \hat{b}^{m}_{f}{}_{*_A} \hat{b}^{m\dagger}_{f} |1> \,. \tag{9.45}$$

Taking into account that the Clifford objects γ^a and $\tilde{\gamma}^a$ fulfil relations of Eq. 10.5, one obtains beside the relations presented in Eq. (10.10) the following once where $i = (I, II)$ denotes the two groups of Clifford even "basis vectors", while m and f determine membership of "basis vectors" in any of the two groups I or II.

$$\begin{aligned}
&\stackrel{ab}{(k)}\stackrel{ab}{(-k)} = \eta^{aa}\stackrel{ab}{[k]}\,, \quad \stackrel{ab}{(-k)}\stackrel{ab}{(k)} = \eta^{aa}\stackrel{ab}{[-k]}\,, \quad \stackrel{ab}{(k)}\stackrel{ab}{[k]} = 0\,, \quad \stackrel{ab}{(k)}\stackrel{ab}{[-k]} = \stackrel{ab}{(k)}\,,\\
&\stackrel{ab}{(-k)}\stackrel{ab}{[k]} = \stackrel{ab}{(-k)}\,, \quad \stackrel{ab}{[k]}\stackrel{ab}{(k)} = \stackrel{ab}{(k)}\,, \quad \stackrel{ab}{[k]}\stackrel{ab}{(-k)} = 0\,, \quad \stackrel{ab}{[k]}\stackrel{ab}{[-k]} = 0\,,\\
&\stackrel{ab}{(k)}^{\dagger} = \eta^{aa}\stackrel{ab}{(-k)}\,, \quad (\stackrel{ab}{(k)})^2 = 0\,, \quad \stackrel{ab}{(k)}\stackrel{ab}{(-k)} = \eta^{aa}\stackrel{ab}{[k]}\,,\\
&\stackrel{ab}{[k]} := \frac{1}{2}(1 + \frac{i}{k}\gamma^a\gamma^b)\,, \quad (\stackrel{ab}{[k]})^2 = \stackrel{ab}{[k]}\,, \quad \stackrel{ab}{[k]}\stackrel{ab}{[-k]} = 0\,,\\
&\stackrel{ab}{(\tilde{k})}\stackrel{ab}{(k)} = 0\,, \quad \stackrel{ab}{(\tilde{k})}\stackrel{ab}{(-k)} = -i\eta^{aa}\stackrel{ab}{[-k]}\,, \quad \stackrel{ab}{\widetilde{(-k)}}\stackrel{ab}{(k)} = -i\eta^{aa}\stackrel{ab}{[k]}\,, \quad \stackrel{ab}{(\tilde{k})}\stackrel{ab}{[k]} = i\stackrel{ab}{(k)}\,,\\
&\stackrel{ab}{(\tilde{k})}\stackrel{ab}{[-k]} = 0\,, \quad \stackrel{ab}{\widetilde{(-k)}}\stackrel{ab}{[k]} = 0\,, \quad \stackrel{ab}{\widetilde{(-k)}}\stackrel{ab}{[-k]} = i\stackrel{ab}{(-k)}\,, \quad \stackrel{ab}{[\tilde{k}]}\stackrel{ab}{(k)} = \stackrel{ab}{(k)}\,,\\
&\stackrel{ab}{[\tilde{k}]}\stackrel{ab}{(-k)} = 0\,, \quad \stackrel{ab}{[\tilde{k}]}\stackrel{ab}{[k]} = 0\,, \quad \stackrel{ab}{\widetilde{[-k]}}\stackrel{ab}{[k]} = \stackrel{ab}{[k]}\,, \quad \stackrel{ab}{[\tilde{k}]}\stackrel{ab}{[-k]} = \stackrel{ab}{[-k]}\,,
\end{aligned} \tag{9.46}$$

The algebraic multiplication among $\stackrel{ab}{(\tilde{k})}$ and $\stackrel{ab}{[\tilde{k}]}$ goes as in the case of $\stackrel{ab}{(k)}$ and $\stackrel{ab}{[k]}$

$$\begin{aligned}
&\stackrel{ab}{(\tilde{k})}\stackrel{ab}{[\tilde{k}]} = 0\,, \quad \stackrel{ab}{[\tilde{k}]}\stackrel{ab}{(\tilde{k})} = \stackrel{ab}{(\tilde{k})}\,, \quad \stackrel{ab}{(\tilde{k})}\stackrel{ab}{[-\tilde{k}]} = \stackrel{ab}{(\tilde{k})}\,, \quad \stackrel{ab}{[\tilde{k}]}\stackrel{ab}{(-\tilde{k})} = 0\,,\\
&\stackrel{ab}{\widetilde{(-k)}}\stackrel{ab}{(\tilde{k})} = \eta^{aa}\stackrel{ab}{\widetilde{[-k]}}\,, \quad \stackrel{ab}{\widetilde{(-k)}}\stackrel{ab}{\widetilde{[-k]}} = 0\,,
\end{aligned} \tag{9.47}$$

One can further find that

$$\begin{aligned}
&S^{ac}\stackrel{ab}{(k)}\stackrel{cd}{(k)} = -\frac{i}{2}\eta^{aa}\eta^{cc}\stackrel{ab}{[-k]}\stackrel{cd}{[-k]}\,, \quad S^{ac}\stackrel{ab}{[k]}\stackrel{cd}{[k]} = \frac{i}{2}\stackrel{ab}{(-k)}\stackrel{cd}{(-k)}\,,\\
&S^{ac}\stackrel{ab}{(k)}\stackrel{cd}{[k]} = -\frac{i}{2}\eta^{aa}\stackrel{ab}{[-k]}\stackrel{cd}{(-k)}\,, \quad S^{ac}\stackrel{ab}{[k]}\stackrel{cd}{(k)} = \frac{i}{2}\eta^{cc}\stackrel{ab}{(-k)}\stackrel{cd}{[-k]}\,.
\end{aligned} \tag{9.48}$$

9.8 One family representation of Clifford odd "basis vectors" in $d = (13 + 1)$

This appendix, is following App. D of Ref. [8], with a short comment on the corresponding gauge vector and scalar fields and fermion and boson representations in $d = (14 + 1)$-dimensional space included.
In even dimensional space $d = (13 + 1)$ ([11], App. A), one irreducible representation of the Clifford odd "basis vectors", analysed from the point of view of the subgroups $SO(3, 1) \times SO(4)$ (included in $SO(7, 1)$) and $SO(7, 1) \times SO(6)$ (included in $SO(13, 1)$, while $SO(6)$ breaks into $SU(3) \times U(1)$), contains the Clifford odd "basis vectors" describing internal spaces of quarks and leptons and antiquarks, and antileptons with the quantum numbers assumed by the *standard model* before the electroweak break. Since $SO(4)$ contains two $SU(2)$ groups, $Y = \tau^{23} + \tau^{4}$, one irreducible representation includes the right-handed neutrinos and the left-handed antineutrinos, which are not in the *standard model* scheme.

The Clifford even "basis vectors", analysed to the same subgroups, offer the description of the internal spaces of the corresponding vector and scalar fields, appearing in the *standard model* before the electroweak break [11,21]; as explained in Subsect. 9.2.2.
For an overview of the properties of the vector and scalar gauge fields in the *spin-charge-family* theory, the reader is invited to see Refs. ([4,6] and the references therein). The vector gauge fields, expressed as the superposition of spin connections and vielbeins, carrying the space index $m = (0, 1, 2, 3)$, manifest properties of the observed boson fields. The scalar gauge fields, causing the electroweak break, carry the space index $s = (7, 8)$ and determine the symmetry of mass matrices of quarks and leptons.
In this Table 9.3, one can check the quantum numbers of the Clifford odd "basis vectors" representing quarks and leptons *and antiquarks and antileptons* if taking into account that all the nilpotents and projectors are eigenvectors of one of the Cartan subalgebra members, $(S^{03}, S^{12}, S^{56}, \dots, S^{13\,14})$, with the eigenvalues $\pm\frac{i}{2}$ for $\stackrel{ab}{(\pm i)}$ and $\stackrel{ab}{[\pm i]}$, and with the eigenvalues $\pm\frac{1}{2}$ for $\stackrel{ab}{(\pm 1)}$ and $\stackrel{ab}{[\pm 1]}$.
Taking into account that the third component of the weak charge, $\tau^{13} = \frac{1}{2}(S^{56} - S^{78})$, for the second $SU(2)$ charge, $\tau^{23} = \frac{1}{2}(S^{56} + S^{78})$, for the colour charge $[\tau^{33} = \frac{1}{2}(S^{9\,10} - S^{11\,12})$ and $\tau^{38} = \frac{1}{2\sqrt{3}}(S^{9\,10} + S^{11\,12} - 2S^{13\,14})]$, for the "fermion charge" $\tau^{4} = -\frac{1}{3}(S^{9\,10} + S^{11\,12} + S^{13\,14})$, for the hyper charge $Y = \tau^{23} + \tau^{4}$, and electromagnetic charge $Q = Y + \tau^{13}$, one reproduces all the quantum numbers of quarks, leptons, and *antiquarks, and antileptons*. One notices that the $SO(7, 1)$ part is the same for quarks and leptons and the same for antiquarks and antileptons. Quarks distinguish from leptons only in the colour and "fermion" quantum numbers and antiquarks distinguish from antileptons only in the anti-colour and "anti-fermion" quantum numbers.

In odd dimensional space, $d = (14 + 1)$, the eigenstates of handedness are the superposition of one irreducible representation of $SO(13, 1)$, presented in Table 9.3,

and the one obtained if on each "basis vector" appearing in $SO(13,1)$ the operator $S^{0\,(14+1)}$ applies, Subsect. 9.2.2, Ref. [8].
Let me point out that in addition to the electroweak break of the *standard model* the break at $\geq 10^{16}$ GeV is needed ([6], and references therein). The condensate of the two right-handed neutrinos causes this break (Ref. [6], Table 6); it interacts with all the scalar and vector gauge fields, except the weak, $U(1), SU(3)$ and the gravitational field in $d = (3+1)$, leaving these gauge fields massless up to the electroweak break, when the scalar fields, leaving massless only the electromagnetic, colour and gravitational fields, cause masses of fermions and weak bosons.
The theory predicts two groups of four families: To the lower group of four families, the three so far observed contribute. The theory predicts the symmetry of both groups to be $SU(2) \times SU(2) \times U(1)$, Ref. ([6], Sect. 7.3), which enable to calculate mixing matrices of quarks and leptons for the accurately enough measured 3×3 sub-matrix of the 4×4 unitary matrix. No sterile neutrinos are needed, and no symmetry of the mass matrices must be guessed [43].
In the literature, one finds a lot of papers trying to reproduce mass matrices and measured mixing matrices for quarks and leptons [58–62, 64].
The stable of the upper four families predicted by the *spin-charge-family* theory is a candidate for the dark matter, as discussed in Refs. [6, 42]. In the literature, there are several works suggesting candidates for the dark matter and also for matter/antimatter asymmetry [65, 66].

i		$\vert ^a\psi_i>$	$\Gamma^{(3,1)}$	S^{12}	τ^{13}	τ^{23}	τ^{33}	τ^{38}	τ^4	Y	Q
		(Anti)octet, $\Gamma^{(7,1)} = (-1)\,1\,,\ \Gamma^{(6)} = (1)\,-1$ of (anti)quarks and (anti)leptons									
1	u_R^{c1}	$\stackrel{03}{(+i)}\stackrel{12}{[+]}\vert\stackrel{56}{[+]}\stackrel{78}{(+)}\vert\vert\stackrel{9\,10}{(+)}\;\stackrel{11\,12}{[-]}\;\stackrel{13\,14}{[-]}$	1	$\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2\sqrt{3}}$	$\frac{1}{6}$	$\frac{2}{3}$	$\frac{2}{3}$
2	u_R^{c1}	$\stackrel{03}{[-i]}\stackrel{12}{(-)}\vert\stackrel{56}{[+]}\stackrel{78}{(+)}\vert\vert\stackrel{9\,10}{(+)}\;\stackrel{11\,12}{[-]}\;\stackrel{13\,14}{[-]}$	1	$-\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2\sqrt{3}}$	$\frac{1}{6}$	$\frac{2}{3}$	$\frac{2}{3}$
3	d_R^{c1}	$\stackrel{03}{(+i)}\stackrel{12}{[+]}\vert\stackrel{56}{(-)}\stackrel{78}{[-]}\vert\vert\stackrel{9\,10}{(+)}\;\stackrel{11\,12}{[-]}\;\stackrel{13\,14}{[-]}$	1	$\frac{1}{2}$	0	$-\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2\sqrt{3}}$	$\frac{1}{6}$	$-\frac{1}{3}$	$-\frac{1}{3}$
4	d_R^{c1}	$\stackrel{03}{[-i]}\stackrel{12}{(-)}\vert\stackrel{56}{(-)}\stackrel{78}{[-]}\vert\vert\stackrel{9\,10}{(+)}\;\stackrel{11\,12}{[-]}\;\stackrel{13\,14}{[-]}$	1	$-\frac{1}{2}$	0	$-\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2\sqrt{3}}$	$\frac{1}{6}$	$-\frac{1}{3}$	$-\frac{1}{3}$
5	d_L^{c1}	$\stackrel{03}{[-i]}\stackrel{12}{[+]}\vert\stackrel{56}{(-)}\stackrel{78}{(+)}\vert\vert\stackrel{9\,10}{(+)}\;\stackrel{11\,12}{[-]}\;\stackrel{13\,14}{[-]}$	-1	$\frac{1}{2}$	$-\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{1}{2\sqrt{3}}$	$\frac{1}{6}$	$\frac{1}{6}$	$-\frac{1}{3}$
6	d_L^{c1}	$-\stackrel{03}{(+i)}\stackrel{12}{(-)}\vert\stackrel{56}{(-)}\stackrel{78}{(+)}\vert\vert\stackrel{9\,10}{(+)}\;\stackrel{11\,12}{[-]}\;\stackrel{13\,14}{[-]}$	-1	$-\frac{1}{2}$	$-\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{1}{2\sqrt{3}}$	$\frac{1}{6}$	$\frac{1}{6}$	$-\frac{1}{3}$
7	u_L^{c1}	$-\stackrel{03}{[-i]}\stackrel{12}{[+]}\vert\stackrel{56}{[+]}\stackrel{78}{[-]}\vert\vert\stackrel{9\,10}{(+)}\;\stackrel{11\,12}{[-]}\;\stackrel{13\,14}{[-]}$	-1	$\frac{1}{2}$	$\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{1}{2\sqrt{3}}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{2}{3}$
8	u_L^{c1}	$\stackrel{03}{(+i)}\stackrel{12}{(-)}\vert\stackrel{56}{[+]}\stackrel{78}{[-]}\vert\vert\stackrel{9\,10}{(+)}\;\stackrel{11\,12}{[-]}\;\stackrel{13\,14}{[-]}$	-1	$-\frac{1}{2}$	$\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{1}{2\sqrt{3}}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{2}{3}$
9	u_R^{c2}	$\stackrel{03}{(+i)}\stackrel{12}{[+]}\vert\stackrel{56}{[+]}\stackrel{78}{(+)}\vert\vert\stackrel{9\,10}{[-]}\;\stackrel{11\,12}{(+)}\;\stackrel{13\,14}{[-]}$	1	$\frac{1}{2}$	0	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2\sqrt{3}}$	$\frac{1}{6}$	$\frac{2}{3}$	$\frac{2}{3}$
10	u_R^{c2}	$\stackrel{03}{[-i]}\stackrel{12}{(-)}\vert\stackrel{56}{[+]}\stackrel{78}{(+)}\vert\vert\stackrel{9\,10}{[-]}\;\stackrel{11\,12}{(+)}\;\stackrel{13\,14}{[-]}$	1	$-\frac{1}{2}$	0	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2\sqrt{3}}$	$\frac{1}{6}$	$\frac{2}{3}$	$\frac{2}{3}$
11	d_R^{c2}	$\stackrel{03}{(+i)}\stackrel{12}{[+]}\vert\stackrel{56}{(-)}\stackrel{78}{[-]}\vert\vert\stackrel{9\,10}{[-]}\;\stackrel{11\,12}{(+)}\;\stackrel{13\,14}{[-]}$	1	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2\sqrt{3}}$	$\frac{1}{6}$	$-\frac{1}{3}$	$-\frac{1}{3}$
12	d_R^{c2}	$\stackrel{03}{[-i]}\stackrel{12}{(-)}\vert\stackrel{56}{(-)}\stackrel{78}{[-]}\vert\vert\stackrel{9\,10}{[-]}\;\stackrel{11\,12}{(+)}\;\stackrel{13\,14}{[-]}$	1	$-\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2\sqrt{3}}$	$\frac{1}{6}$	$-\frac{1}{3}$	$-\frac{1}{3}$
13	d_L^{c2}	$\stackrel{03}{[-i]}\stackrel{12}{[+]}\vert\stackrel{56}{(-)}\stackrel{78}{(+)}\vert\vert\stackrel{9\,10}{[-]}\;\stackrel{11\,12}{(+)}\;\stackrel{13\,14}{[-]}$	-1	$\frac{1}{2}$	$-\frac{1}{2}$	0	$-\frac{1}{2}$	$\frac{1}{2\sqrt{3}}$	$\frac{1}{6}$	$\frac{1}{6}$	$-\frac{1}{3}$
14	d_L^{c2}	$-\stackrel{03}{(+i)}\stackrel{12}{(-)}\vert\stackrel{56}{(-)}\stackrel{78}{(+)}\vert\vert\stackrel{9\,10}{[-]}\;\stackrel{11\,12}{(+)}\;\stackrel{13\,14}{[-]}$	-1	$-\frac{1}{2}$	$-\frac{1}{2}$	0	$-\frac{1}{2}$	$\frac{1}{2\sqrt{3}}$	$\frac{1}{6}$	$\frac{1}{6}$	$-\frac{1}{3}$
15	u_L^{c2}	$-\stackrel{03}{[-i]}\stackrel{12}{[+]}\vert\stackrel{56}{[+]}\stackrel{78}{[-]}\vert\vert\stackrel{9\,10}{[-]}\;\stackrel{11\,12}{(+)}\;\stackrel{13\,14}{[-]}$	-1	$\frac{1}{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$\frac{1}{2\sqrt{3}}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{2}{3}$
16	u_L^{c2}	$\stackrel{03}{(+i)}\stackrel{12}{(-)}\vert\stackrel{56}{[+]}\stackrel{78}{[-]}\vert\vert\stackrel{9\,10}{[-]}\;\stackrel{11\,12}{(+)}\;\stackrel{13\,14}{[-]}$	-1	$-\frac{1}{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$\frac{1}{2\sqrt{3}}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{2}{3}$
17	u_R^{c3}	$\stackrel{03}{(+i)}\stackrel{12}{[+]}\vert\stackrel{56}{[+]}\stackrel{78}{(+)}\vert\vert\stackrel{9\,10}{[-]}\;\stackrel{11\,12}{[-]}\;\stackrel{13\,14}{(+)}$	1	$\frac{1}{2}$	0	$\frac{1}{2}$	0	$-\frac{1}{\sqrt{3}}$	$\frac{1}{6}$	$\frac{2}{3}$	$\frac{2}{3}$
18	u_R^{c3}	$\stackrel{03}{[-i]}\stackrel{12}{(-)}\vert\stackrel{56}{[+]}\stackrel{78}{(+)}\vert\vert\stackrel{9\,10}{[-]}\;\stackrel{11\,12}{[-]}\;\stackrel{13\,14}{(+)}$	1	$-\frac{1}{2}$	0	$\frac{1}{2}$	0	$-\frac{1}{\sqrt{3}}$	$\frac{1}{6}$	$\frac{2}{3}$	$\frac{2}{3}$
19	d_R^{c3}	$\stackrel{03}{(+i)}\stackrel{12}{[+]}\vert\stackrel{56}{(-)}\stackrel{78}{[-]}\vert\vert\stackrel{9\,10}{[-]}\;\stackrel{11\,12}{[-]}\;\stackrel{13\,14}{(+)}$	1	$\frac{1}{2}$	0	$-\frac{1}{2}$	0	$-\frac{1}{\sqrt{3}}$	$\frac{1}{6}$	$-\frac{1}{3}$	$-\frac{1}{3}$
20	d_R^{c3}	$\stackrel{03}{[-i]}\stackrel{12}{(-)}\vert\stackrel{56}{(-)}\stackrel{78}{[-]}\vert\vert\stackrel{9\,10}{[-]}\;\stackrel{11\,12}{[-]}\;\stackrel{13\,14}{(+)}$	1	$-\frac{1}{2}$	0	$-\frac{1}{2}$	0	$-\frac{1}{\sqrt{3}}$	$\frac{1}{6}$	$-\frac{1}{3}$	$-\frac{1}{3}$
21	d_L^{c3}	$\stackrel{03}{[-i]}\stackrel{12}{[+]}\vert\stackrel{56}{(-)}\stackrel{78}{(+)}\vert\vert\stackrel{9\,10}{[-]}\;\stackrel{11\,12}{[-]}\;\stackrel{13\,14}{(+)}$	-1	$\frac{1}{2}$	$-\frac{1}{2}$	0	0	$-\frac{1}{\sqrt{3}}$	$\frac{1}{6}$	$\frac{1}{6}$	$-\frac{1}{3}$

Continued on next page

i		$\vert^a\psi_i>$ (Anti)octet, $\Gamma^{(7,1)} = (-1)\,1$, $\Gamma^{(6)} = (1)\,-1$ of (anti)quarks and (anti)leptons	$\Gamma^{(3,1)}$	S^{12}	τ^{13}	τ^{23}	τ^{33}	τ^{38}	τ^{4}	Y	Q
22	d_L^{c3}	$-\stackrel{03}{(+i)}\stackrel{12}{(-)}\vert\stackrel{56}{(-)}\stackrel{78}{(+)}\Vert\stackrel{9\,10}{[-]}\stackrel{11\,12}{[-]}\stackrel{13\,14}{(+)}$	-1	$-\frac{1}{2}$	$-\frac{1}{2}$	0	0	$-\frac{1}{\sqrt{3}}$	$\frac{1}{6}$	$\frac{1}{6}$	$-\frac{1}{3}$
23	u_L^{c3}	$-\stackrel{03}{[-i]}\stackrel{12}{[+]}\vert\stackrel{56}{[+]}\stackrel{78}{[-]}\Vert\stackrel{9\,10}{[-]}\stackrel{11\,12}{[-]}\stackrel{13\,14}{(+)}$	-1	$\frac{1}{2}$	$\frac{1}{2}$	0	0	$-\frac{1}{\sqrt{3}}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{2}{3}$
24	u_L^{c3}	$\stackrel{03}{(+i)}\stackrel{12}{(-)}\vert\stackrel{56}{[+]}\stackrel{78}{[-]}\Vert\stackrel{9\,10}{[-]}\stackrel{11\,12}{[-]}\stackrel{13\,14}{(+)}$	-1	$-\frac{1}{2}$	$\frac{1}{2}$	0	0	$-\frac{1}{\sqrt{3}}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{2}{3}$
25	ν_R	$\stackrel{03}{(+i)}\stackrel{12}{[+]}\vert\stackrel{56}{[+]}\stackrel{78}{(+)}\Vert\stackrel{9\,10}{(+)}\stackrel{11\,12}{(+)}\stackrel{13\,14}{(+)}$	1	$\frac{1}{2}$	0	$\frac{1}{2}$	0	0	$-\frac{1}{2}$	0	0
26	ν_R	$\stackrel{03}{[-i]}\stackrel{12}{(-)}\vert\stackrel{56}{[+]}\stackrel{78}{(+)}\Vert\stackrel{9\,10}{(+)}\stackrel{11\,12}{(+)}\stackrel{13\,14}{(+)}$	1	$-\frac{1}{2}$	0	$\frac{1}{2}$	0	0	$-\frac{1}{2}$	0	0
27	e_R	$\stackrel{03}{(+i)}\stackrel{12}{[+]}\vert\stackrel{56}{(-)}\stackrel{78}{[-]}\Vert\stackrel{9\,10}{(+)}\stackrel{11\,12}{(+)}\stackrel{13\,14}{(+)}$	1	$\frac{1}{2}$	0	$-\frac{1}{2}$	0	0	$-\frac{1}{2}$	-1	-1
28	e_R	$\stackrel{03}{[-i]}\stackrel{12}{(-)}\vert\stackrel{56}{(-)}\stackrel{78}{[-]}\Vert\stackrel{9\,10}{(+)}\stackrel{11\,12}{(+)}\stackrel{13\,14}{(+)}$	1	$-\frac{1}{2}$	0	$-\frac{1}{2}$	0	0	$-\frac{1}{2}$	-1	-1
29	e_L	$\stackrel{03}{[-i]}\stackrel{12}{[+]}\vert\stackrel{56}{(-)}\stackrel{78}{(+)}\Vert\stackrel{9\,10}{(+)}\stackrel{11\,12}{(+)}\stackrel{13\,14}{(+)}$	-1	$\frac{1}{2}$	$-\frac{1}{2}$	0	0	0	$-\frac{1}{2}$	$-\frac{1}{2}$	-1
30	e_L	$-\stackrel{03}{(+i)}\stackrel{12}{(-)}\vert\stackrel{56}{(-)}\stackrel{78}{(+)}\Vert\stackrel{9\,10}{(+)}\stackrel{11\,12}{(+)}\stackrel{13\,14}{(+)}$	-1	$-\frac{1}{2}$	$-\frac{1}{2}$	0	0	0	$-\frac{1}{2}$	$-\frac{1}{2}$	-1
31	ν_L	$-\stackrel{03}{[-i]}\stackrel{12}{[+]}\vert\stackrel{56}{[+]}\stackrel{78}{[-]}\Vert\stackrel{9\,10}{(+)}\stackrel{11\,12}{(+)}\stackrel{13\,14}{(+)}$	-1	$\frac{1}{2}$	$\frac{1}{2}$	0	0	0	$-\frac{1}{2}$	$-\frac{1}{2}$	0
32	ν_L	$\stackrel{03}{(+i)}\stackrel{12}{(-)}\vert\stackrel{56}{[+]}\stackrel{78}{[-]}\Vert\stackrel{9\,10}{(+)}\stackrel{11\,12}{(+)}\stackrel{13\,14}{(+)}$	-1	$-\frac{1}{2}$	$\frac{1}{2}$	0	0	0	$-\frac{1}{2}$	$-\frac{1}{2}$	0
33	$\bar{d}_L^{\bar{c}1}$	$\stackrel{03}{[-i]}\stackrel{12}{[+]}\vert\stackrel{56}{[+]}\stackrel{78}{(+)}\Vert\stackrel{9\,10}{[-]}\stackrel{11\,12}{(+)}\stackrel{13\,14}{(+)}$	-1	$\frac{1}{2}$	0	$\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2\sqrt{3}}$	$-\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{3}$
34	$\bar{d}_L^{\bar{c}1}$	$\stackrel{03}{(+i)}\stackrel{12}{(-)}\vert\stackrel{56}{[+]}\stackrel{78}{(+)}\Vert\stackrel{9\,10}{[-]}\stackrel{11\,12}{(+)}\stackrel{13\,14}{(+)}$	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2\sqrt{3}}$	$-\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{3}$
35	$\bar{u}_L^{\bar{c}1}$	$-\stackrel{03}{[-i]}\stackrel{12}{[+]}\vert\stackrel{56}{(-)}\stackrel{78}{[-]}\Vert\stackrel{9\,10}{[-]}\stackrel{11\,12}{(+)}\stackrel{13\,14}{(+)}$	-1	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2\sqrt{3}}$	$-\frac{1}{6}$	$-\frac{2}{3}$	$-\frac{2}{3}$
36	$\bar{u}_L^{\bar{c}1}$	$-\stackrel{03}{(+i)}\stackrel{12}{(-)}\vert\stackrel{56}{(-)}\stackrel{78}{[-]}\Vert\stackrel{9\,10}{[-]}\stackrel{11\,12}{(+)}\stackrel{13\,14}{(+)}$	-1	$-\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2\sqrt{3}}$	$-\frac{1}{6}$	$-\frac{2}{3}$	$-\frac{2}{3}$
37	$\bar{d}_R^{\bar{c}1}$	$\stackrel{03}{(+i)}\stackrel{12}{[+]}\vert\stackrel{56}{[+]}\stackrel{78}{[-]}\Vert\stackrel{9\,10}{[-]}\stackrel{11\,12}{(+)}\stackrel{13\,14}{(+)}$	1	$\frac{1}{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{1}{2\sqrt{3}}$	$-\frac{1}{6}$	$-\frac{1}{6}$	$\frac{1}{3}$
38	$\bar{d}_R^{\bar{c}1}$	$-\stackrel{03}{[-i]}\stackrel{12}{(-)}\vert\stackrel{56}{[+]}\stackrel{78}{[-]}\Vert\stackrel{9\,10}{[-]}\stackrel{11\,12}{(+)}\stackrel{13\,14}{(+)}$	1	$-\frac{1}{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{1}{2\sqrt{3}}$	$-\frac{1}{6}$	$-\frac{1}{6}$	$\frac{1}{3}$
39	$\bar{u}_R^{\bar{c}1}$	$\stackrel{03}{(+i)}\stackrel{12}{[+]}\vert\stackrel{56}{(-)}\stackrel{78}{(+)}\Vert\stackrel{9\,10}{[-]}\stackrel{11\,12}{(+)}\stackrel{13\,14}{(+)}$	1	$\frac{1}{2}$	$-\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{1}{2\sqrt{3}}$	$-\frac{1}{6}$	$-\frac{1}{6}$	$-\frac{2}{3}$
40	$\bar{u}_R^{\bar{c}1}$	$\stackrel{03}{[-i]}\stackrel{12}{(-)}\vert\stackrel{56}{(-)}\stackrel{78}{(+)}\Vert\stackrel{9\,10}{[-]}\stackrel{11\,12}{(+)}\stackrel{13\,14}{(+)}$	1	$-\frac{1}{2}$	$-\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{1}{2\sqrt{3}}$	$-\frac{1}{6}$	$-\frac{1}{6}$	$-\frac{2}{3}$
41	$\bar{d}_L^{\bar{c}2}$	$\stackrel{03}{[-i]}\stackrel{12}{[+]}\vert\stackrel{56}{[+]}\stackrel{78}{(+)}\Vert\stackrel{9\,10}{(+)}\stackrel{11\,12}{[-]}\stackrel{13\,14}{(+)}$	-1	$\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2\sqrt{3}}$	$-\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{3}$
42	$\bar{d}_L^{\bar{c}2}$	$\stackrel{03}{(+i)}\stackrel{12}{(-)}\vert\stackrel{56}{[+]}\stackrel{78}{(+)}\Vert\stackrel{9\,10}{(+)}\stackrel{11\,12}{[-]}\stackrel{13\,14}{(+)}$	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2\sqrt{3}}$	$-\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{3}$
43	$\bar{u}_L^{\bar{c}2}$	$-\stackrel{03}{[-i]}\stackrel{12}{[+]}\vert\stackrel{56}{(-)}\stackrel{78}{[-]}\Vert\stackrel{9\,10}{(+)}\stackrel{11\,12}{[-]}\stackrel{13\,14}{(+)}$	-1	$\frac{1}{2}$	0	$-\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2\sqrt{3}}$	$-\frac{1}{6}$	$-\frac{2}{3}$	$-\frac{2}{3}$
44	$\bar{u}_L^{\bar{c}2}$	$-\stackrel{03}{(+i)}\stackrel{12}{(-)}\vert\stackrel{56}{(-)}\stackrel{78}{[-]}\Vert\stackrel{9\,10}{(+)}\stackrel{11\,12}{[-]}\stackrel{13\,14}{(+)}$	-1	$-\frac{1}{2}$	0	$-\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2\sqrt{3}}$	$-\frac{1}{6}$	$-\frac{2}{3}$	$-\frac{2}{3}$
45	$\bar{d}_R^{\bar{c}2}$	$\stackrel{03}{(+i)}\stackrel{12}{[+]}\vert\stackrel{56}{[+]}\stackrel{78}{[-]}\Vert\stackrel{9\,10}{(+)}\stackrel{11\,12}{[-]}\stackrel{13\,14}{(+)}$	1	$\frac{1}{2}$	$\frac{1}{2}$	0	$\frac{1}{2}$	$-\frac{1}{2\sqrt{3}}$	$-\frac{1}{6}$	$-\frac{1}{6}$	$\frac{1}{3}$
46	$\bar{d}_R^{\bar{c}2}$	$-\stackrel{03}{[-i]}\stackrel{12}{(-)}\vert\stackrel{56}{[+]}\stackrel{78}{[-]}\Vert\stackrel{9\,10}{(+)}\stackrel{11\,12}{[-]}\stackrel{13\,14}{(+)}$	1	$-\frac{1}{2}$	$\frac{1}{2}$	0	$\frac{1}{2}$	$-\frac{1}{2\sqrt{3}}$	$-\frac{1}{6}$	$-\frac{1}{6}$	$\frac{1}{3}$
47	$\bar{u}_R^{\bar{c}2}$	$\stackrel{03}{(+i)}\stackrel{12}{[+]}\vert\stackrel{56}{(-)}\stackrel{78}{(+)}\Vert\stackrel{9\,10}{(+)}\stackrel{11\,12}{[-]}\stackrel{13\,14}{(+)}$	1	$\frac{1}{2}$	$-\frac{1}{2}$	0	$\frac{1}{2}$	$-\frac{1}{2\sqrt{3}}$	$-\frac{1}{6}$	$-\frac{1}{6}$	$-\frac{2}{3}$
48	$\bar{u}_R^{\bar{c}2}$	$\stackrel{03}{[-i]}\stackrel{12}{(-)}\vert\stackrel{56}{(-)}\stackrel{78}{(+)}\Vert\stackrel{9\,10}{(+)}\stackrel{11\,12}{[-]}\stackrel{13\,14}{(+)}$	1	$-\frac{1}{2}$	$-\frac{1}{2}$	0	$\frac{1}{2}$	$-\frac{1}{2\sqrt{3}}$	$-\frac{1}{6}$	$-\frac{1}{6}$	$-\frac{2}{3}$
49	$\bar{d}_L^{\bar{c}3}$	$\stackrel{03}{[-i]}\stackrel{12}{[+]}\vert\stackrel{56}{[+]}\stackrel{78}{(+)}\Vert\stackrel{9\,10}{(+)}\stackrel{11\,12}{(+)}\stackrel{13\,14}{[-]}$	-1	$\frac{1}{2}$	0	$\frac{1}{2}$	0	$\frac{1}{\sqrt{3}}$	$-\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{3}$
50	$\bar{d}_L^{\bar{c}3}$	$\stackrel{03}{(+i)}\stackrel{12}{(-)}\vert\stackrel{56}{[+]}\stackrel{78}{(+)}\Vert\stackrel{9\,10}{(+)}\stackrel{11\,12}{(+)}\stackrel{13\,14}{[-]}$	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	0	$\frac{1}{\sqrt{3}}$	$-\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{3}$
51	$\bar{u}_L^{\bar{c}3}$	$-\stackrel{03}{[-i]}\stackrel{12}{[+]}\vert\stackrel{56}{(-)}\stackrel{78}{[-]}\Vert\stackrel{9\,10}{(+)}\stackrel{11\,12}{(+)}\stackrel{13\,14}{[-]}$	-1	$\frac{1}{2}$	0	$-\frac{1}{2}$	0	$\frac{1}{\sqrt{3}}$	$-\frac{1}{6}$	$-\frac{2}{3}$	$-\frac{2}{3}$
52	$\bar{u}_L^{\bar{c}3}$	$-\stackrel{03}{(+i)}\stackrel{12}{(-)}\vert\stackrel{56}{(-)}\stackrel{78}{[-]}\Vert\stackrel{9\,10}{(+)}\stackrel{11\,12}{(+)}\stackrel{13\,14}{[-]}$	-1	$-\frac{1}{2}$	0	$-\frac{1}{2}$	0	$\frac{1}{\sqrt{3}}$	$-\frac{1}{6}$	$-\frac{2}{3}$	$-\frac{2}{3}$
53	$\bar{d}_R^{\bar{c}3}$	$\stackrel{03}{(+i)}\stackrel{12}{[+]}\vert\stackrel{56}{[+]}\stackrel{78}{[-]}\Vert\stackrel{9\,10}{(+)}\stackrel{11\,12}{(+)}\stackrel{13\,14}{[-]}$	1	$\frac{1}{2}$	$\frac{1}{2}$	0	0	$\frac{1}{\sqrt{3}}$	$-\frac{1}{6}$	$-\frac{1}{6}$	$\frac{1}{3}$
54	$\bar{d}_R^{\bar{c}3}$	$-\stackrel{03}{[-i]}\stackrel{12}{(-)}\vert\stackrel{56}{[+]}\stackrel{78}{[-]}\Vert\stackrel{9\,10}{(+)}\stackrel{11\,12}{(+)}\stackrel{13\,14}{[-]}$	1	$-\frac{1}{2}$	$\frac{1}{2}$	0	0	$\frac{1}{\sqrt{3}}$	$-\frac{1}{6}$	$-\frac{1}{6}$	$\frac{1}{3}$
55	$\bar{u}_R^{\bar{c}3}$	$\stackrel{03}{(+i)}\stackrel{12}{[+]}\vert\stackrel{56}{(-)}\stackrel{78}{(+)}\Vert\stackrel{9\,10}{(+)}\stackrel{11\,12}{(+)}\stackrel{13\,14}{[-]}$	1	$\frac{1}{2}$	$-\frac{1}{2}$	0	0	$\frac{1}{\sqrt{3}}$	$-\frac{1}{6}$	$-\frac{1}{6}$	$-\frac{2}{3}$
56	$\bar{u}_R^{\bar{c}3}$	$\stackrel{03}{[-i]}\stackrel{12}{(-)}\vert\stackrel{56}{(-)}\stackrel{78}{(+)}\Vert\stackrel{9\,10}{(+)}\stackrel{11\,12}{(+)}\stackrel{13\,14}{[-]}$	1	$-\frac{1}{2}$	$-\frac{1}{2}$	0	0	$\frac{1}{\sqrt{3}}$	$-\frac{1}{6}$	$-\frac{1}{6}$	$-\frac{2}{3}$
57	$\bar{e}_L$	$\stackrel{03}{[-i]}\stackrel{12}{[+]}\vert\stackrel{56}{[+]}\stackrel{78}{(+)}\Vert\stackrel{9\,10}{[-]}\stackrel{11\,12}{[-]}\stackrel{13\,14}{[-]}$	-1	$\frac{1}{2}$	0	$\frac{1}{2}$	0	0	$\frac{1}{2}$	1	1
58	$\bar{e}_L$	$\stackrel{03}{(+i)}\stackrel{12}{(-)}\vert\stackrel{56}{[+]}\stackrel{78}{(+)}\Vert\stackrel{9\,10}{[-]}\stackrel{11\,12}{[-]}\stackrel{13\,14}{[-]}$	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	0	0	$\frac{1}{2}$	1	1
59	$\bar{\nu}_L$	$-\stackrel{03}{[-i]}\stackrel{12}{[+]}\vert\stackrel{56}{(-)}\stackrel{78}{[-]}\Vert\stackrel{9\,10}{[-]}\stackrel{11\,12}{[-]}\stackrel{13\,14}{[-]}$	-1	$\frac{1}{2}$	0	$-\frac{1}{2}$	0	0	$\frac{1}{2}$	0	0
60	$\bar{\nu}_L$	$-\stackrel{03}{(+i)}\stackrel{12}{(-)}\vert\stackrel{56}{(-)}\stackrel{78}{[-]}\Vert\stackrel{9\,10}{[-]}\stackrel{11\,12}{[-]}\stackrel{13\,14}{[-]}$	-1	$-\frac{1}{2}$	0	$-\frac{1}{2}$	0	0	$\frac{1}{2}$	0	0
61	$\bar{\nu}_R$	$\stackrel{03}{(+i)}\stackrel{12}{[+]}\vert\stackrel{56}{(-)}\stackrel{78}{(+)}\Vert\stackrel{9\,10}{[-]}\stackrel{11\,12}{[-]}\stackrel{13\,14}{[-]}$	1	$\frac{1}{2}$	$-\frac{1}{2}$	0	0	0	$\frac{1}{2}$	$\frac{1}{2}$	0
62	$\bar{\nu}_R$	$-\stackrel{03}{[-i]}\stackrel{12}{(-)}\vert\stackrel{56}{(-)}\stackrel{78}{(+)}\Vert\stackrel{9\,10}{[-]}\stackrel{11\,12}{[-]}\stackrel{13\,14}{[-]}$	1	$-\frac{1}{2}$	$-\frac{1}{2}$	0	0	0	$\frac{1}{2}$	$\frac{1}{2}$	0
63	$\bar{e}_R$	$\stackrel{03}{(+i)}\stackrel{12}{[+]}\vert\stackrel{56}{[+]}\stackrel{78}{[-]}\Vert\stackrel{9\,10}{[-]}\stackrel{11\,12}{[-]}\stackrel{13\,14}{[-]}$	1	$\frac{1}{2}$	$\frac{1}{2}$	0	0	0	$\frac{1}{2}$	$\frac{1}{2}$	1
64	$\bar{e}_R$	$\stackrel{03}{[-i]}\stackrel{12}{(-)}\vert\stackrel{56}{[+]}\stackrel{78}{[-]}\Vert\stackrel{9\,10}{[-]}\stackrel{11\,12}{[-]}\stackrel{13\,14}{[-]}$	1	$-\frac{1}{2}$	$\frac{1}{2}$	0	0	0	$\frac{1}{2}$	$\frac{1}{2}$	1

Table 9.3: The left-handed ($\Gamma^{(13,1)} = -1$, Eq. (10.23)) irreducible representation of one family of spinors — the product of the odd number of nilpotents and of projectors, which are eigenvectors of the Cartan subalgebra of the $SO(13,1)$ group [3,12], manifesting the subgroup $SO(7,1)$ of the colour charged quarks and antiquarks and the colourless leptons and antileptons — is presented. It contains the left-handed ($\Gamma^{(3,1)} = -1$) weak ($SU(2)_{I}$) charged ($\tau^{13} = \pm\frac{1}{2}$, and $SU(2)_{II}$ chargeless ($\tau^{23} = 0$ quarks and leptons, and the right-handed ($\Gamma^{(3,1)} = 1$) weak ($SU(2)_{I}$) chargeless and $SU(2)_{II}$ charged ($\tau^{23} = \pm\frac{1}{2}$) quarks and leptons, both with the spin S^{12} up and down ($\pm\frac{1}{2}$, respectively). Quarks distinguish from leptons only in the $SU(3) \times U(1)$ part: Quarks are triplets of three colours ($c^{i} = (\tau^{33}, \tau^{38})$ $= [(\frac{1}{2}, \frac{1}{2\sqrt{3}}), (-\frac{1}{2}, \frac{1}{2\sqrt{3}}), (0, -\frac{1}{\sqrt{3}})$, carrying the "fermion charge" ($\tau^{4} = \frac{1}{6}$). The colourless leptons carry the "fermion charge" ($\tau^{4} = -\frac{1}{2}$). The same multiplet contains also the left handed weak ($SU(2)_{I}$) chargeless and $SU(2)_{II}$ charged antiquarks and antileptons and the right handed weak ($SU(2)_{I}$) charged and $SU(2)_{II}$ chargeless antiquarks and antileptons. Antiquarks distinguish from antileptons again only in the $SU(3) \times U(1)$ part: Antiquarks are anti-triplets carrying the "fermion charge" ($\tau^{4} = -\frac{1}{6}$). The anti-colourless antileptons carry the "fermion charge" ($\tau^{4} = \frac{1}{2}$). $Y = (\tau^{23} + \tau^{4})$ is the hyper charge, the electromagnetic charge is $Q = (\tau^{13} + Y)$.

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References

1. N. Mankoč Borštnik, "Spinor and vector representations in four dimensional Grassmann space", *J. of Math. Phys.* **34** (1993) 3731-3745, "Unification of spin and charges in Grassmann space?", hep-th 9408002, IJS.TP.94/22, Mod. Phys. Lett.**A (10)** No.7 (1995) 587-595;
2. N. S. Mankoč Borštnik, "New way of second quantized theory of fermions with either Clifford or Grassmann coordinates and *spin-charge-family* theory " [arXiv:1802.05554v4, arXiv:1902.10628], N. S. Mankoč Borštnik, "How Clifford algebra can help understand second quantization of fermion and boson fields", [arXiv: 2210.06256. physics.gen-ph V1] .
3. A. Borštnik, N.S. Mankoč Borštnik, "Left and right handedness of fermions and bosons", J. of Phys. G: Nucl. Part. Phys.**24**(1998)963-977, hep-th/9707218.
4. A. Borštnik Bračič, N. S. Mankoč Borštnik, "On the origin of families of fermions and their mass matrices", hep-ph/0512062, Phys Rev. **D 74** 073013-28 (2006).
5. N. S. Mankoč Borštnik, "How Clifford algebra helps understand second quantized quarks and leptons and corresponding vector and scalar boson fields, opening a new step beyond the standard model", Reference: NUPHB 994 (2023) 116326 , [arXiv: 2210.06256, physics.gen-ph V2].

6. N. S. Mankoč Borštnik, "Clifford odd and even objects in even and odd dimensional spaces", Symmetry 2023,15,818-12-V2 94818, https:doi.org/10.3390/sym15040818, [arxiv.org/abs/2301.04466] , https://www.mdpi.com/2073-8994/15/4/818 Manuscript ID: symmetry-2179313.
7. H. Georgi, in *Particles and Fields* (edited by C. E. Carlson), A.I.P., 1975; Google Scholar.
8. H. Fritzsch and P. Minkowski, *Ann. Phys.* **93** (1975) 193.
9. J. Pati and A. Salam, *Phys.Rev.* **D 8** (1973) 1240.
10. H. Georgy and S.L. Glashow, *Phys. Rev. Lett.* **32** (1974) 438.
11. Y. M. Cho, *J. Math. Phys.* **16** (1975) 2029.
12. N.S. Mankoč Borštnik, "Matter-antimatter asymmetry in the *spin-charge-family* theory", *Phys. Rev.* **D 91** (2015) 065004 [arXiv:1409.7791].
13. N.S. Mankoč Borštnik N S, "The spin-charge-family theory is explaining the origin of families, of the Higgs and the Yukawa couplings", *J. of Modern Phys.* **4** (2013) 823 [arXiv:1312.1542].
14. N. S. Mankoč Borštnik, H. B. Nielsen, "How does Clifford algebra show the way to the second quantized fermions with unified spins, charges and families, and with vector and scalar gauge fields beyond the *standard model*", Progress in Particle and Nuclear Physics, http://doi.org/10.1016.j.ppnp.2021.103890 .
15. "Representations in Grassmann space and fermion degrees of freedom", Proceedings to the 20^{th} Workshop "What comes beyond the standard models", Bled, 9-17 of July, 2017, Ed. N.S. Mankoč Borštnik, H.B. Nielsen, D. Lukman, DMFA Založništvo, Ljubljana, December 2017 [arxiv:1805.06318, arXiv:1806.01629]
16. A. Borštnik Bračič, N. S. Mankoč Borštnik, "On the origin of families of fermions and their mass matrices", hep-ph/0512062, Phys. Rev. **D 74** 073013-28 (2006).
17. N.S. Mankoč Borštnik, H.B.F. Nielsen, *J. of Math. Phys.* **43**, 5782 (2002) [arXiv:hep-th/0111257], "How to generate families of spinors", *J. of Math. Phys.* **44** 4817 (2003) [arXiv:hep-th/0303224].
18. N.S. Mankoč Borštnik, D. Lukman, "Vector and scalar gauge fields with respect to $d = (3 + 1)$ in Kaluza-Klein theories and in the *spin-charge-family theory*", *Eur. Phys. J. C* **77** (2017) 231.
19. N.S. Mankoč Borštnik and H.B. Nielsen, "Why nature made a choice of Clifford and not Grassmann coordinates", Proceedings to the 20^{th} Workshop "What comes beyond the standard models", Bled, 9-17 of July, 2017, Ed. N.S. Mankoč Borštnik, H.B. Nielsen, D. Lukman, DMFA Založništvo, Ljubljana, December 2017, p. 89-120 [arXiv:1802.05554v1v2].
20. N.S. Mankoč Borštnik, H.B.F. Nielsen, "Understanding the second quantization of fermions in Clifford and in Grassmann space", "New way of second quantization of fermions — Part I and Part II proceedings [arXiv:2007.03517, arXiv:2007.03516], "New way of second quantized theory of fermions with either Clifford or Grassmann coordinates and *spin-charge-family* theory " [arXiv:1802.05554v4,arXiv:1902.10628].
21. N. S. Mankoč Borštnik, "How do Clifford algebras show the way to the second quantized fermions with unified spins, charges and families, and to the corresponding second quantized vector and scalar gauge field ", Proceedings to the 24^{rd} Workshop "What comes beyond the standard models", 5 - 11 of July, 2021, Ed. N.S. Mankoč Borštnik, H.B. Nielsen, D. Lukman, A. Kleppe, DMFA Založništvo, Ljubljana, December 2021, [arXiv:2112.04378] .
22. N. S. Mankoč Borštnik, "How Clifford algebra can help understand second quantization of fermion and boson fields", [arXiv: 2210.06256. physics.gen-ph V1] ,
23. N. S. Mankoč Borštnik, "Clifford odd and even objects offer description of internal space of fermions and bosons, respectively, opening new insight into the second quantization of fields", The 13^{th} Bienal Conference on Classical and Quantum Relativistic Dynamics

of Particles and Fields IARD 2022, Prague, 6−9 June, [http://arxiv.org/abs/2210.07004] .

24. N.S. Mankoč Borštnik, H.B.F. Nielsen, "Understanding the second quantization of fermions in Clifford and in Grassmann space: *New way of second quantization of fermions—Part I and Part II*. (2020), [arXiv:2007.03516], [arXiv:1802.05554v4, arXiv:1902.10628].
25. P.A.M. Dirac *Proc. Roy. Soc. (London)*, **A 117** (1928) 610.
26. H.A. Bethe, R.W. Jackiw, "Intermediate quantum mechanics", New York : W.A. Benjamin, 1968.
27. S. Weinberg, "The quantum theory of fields", Cambridge, Cambridge University Press, 2015.
28. N.S. Mankoč Borštnik, H.B.F. Nielsen, "The spin-charge-family theory offers understanding of the triangle anomalies cancellation in the standard model", *Fortschritte der Physik, Progress of Physics* (2017) 1700046.
29. N.S. Mankoč Borštnik, "The explanation for the origin of the Higgs scalar and for the Yukawa couplings by the *spin-charge-family* theory", *J. of Mod. Physics* **6** (2015) 2244-2274, http://dx.org./10.4236/jmp.2015.615230 [arXiv:1409.4981].
30. D. Lukman, N.S. Mankoč Borštnik and H.B. Nielsen, "An effective two dimensionality cases bring a new hope to the Kaluza-Klein-like theories", *New J. Phys.* 13:103027, 2011.
31. T. Kaluza, "On the unification problem in Physics", *Sitzungsber. d. Berl. Acad.* (1918) 204, O. Klein, "Quantum theory and five-dimensional relativity", *Zeit. Phys.* **37**(1926) 895.
32. E. Witten, "Search for realistic Kaluza-Klein theory",*Nucl. Phys.* **B 186** (1981) 412.
33. M. Duff, B. Nilsson, C. Pope, *Phys. Rep.* **C 130** (1984)1, M. Duff, B. Nilsson, C. Pope, N. Warner, *Phys. Lett.* **B 149** (1984) 60.
34. T. Appelquist, H. C. Cheng, B. A. Dobrescu, *Phys. Rev.* **D 64** (2001) 035002.
35. M. Saposhnikov, P. TinyakovP 2001 *Phys. Lett.* **B 515** (2001) 442 [arXiv:hep-th/0102161v2].
36. C. Wetterich,*Nucl. Phys.* **B 253** (1985) 366.
37. The authors of the works presented in *An introduction to Kaluza-Klein theories*, Ed. by H. C. Lee, World Scientific, Singapore 1983.
38. M. Blagojević, *Gravitation and gauge symmetries*, IoP Publishing, Bristol 2002.
39. M. Breskvar, D. Lukman, N. S. Mankoč Borštnik, "On the Origin of Families of Fermions and Their Mass Matrices — Approximate Analyses of Properties of Four Families Within Approach Unifying Spins and Charges", Proceedings to the 9th Workshop "What Comes Beyond the Standard Models", Bled, Sept. 16 - 26, 2006, Ed. by Norma Mankoč Borštnik, Holger Bech Nielsen, Colin Froggatt, Dragan Lukman, DMFA Založništvo, Ljubljana December 2006, p.25-50, hep-ph/0612250.
40. G. Bregar, M. Breskvar, D. Lukman, N.S. Mankoč Borštnik, "Families of Quarks and Leptons and Their Mass Matrices", Proceedings to the 10^{th} international workshop "What Comes Beyond the Standard Model", 17 -27 of July, 2007, Ed. Norma Mankoč Borštnik, Holger Bech Nielsen, Colin Froggatt, Dragan Lukman, DMFA Založništvo, Ljubljana December 2007, p.53-70, hep-ph/0711.4681.
41. G. Bregar, M. Breskvar, D. Lukman, N.S. Mankoč Borštnik, "Predictions for four families by the Approach unifying spins and charges" *New J. of Phys.* **10** (2008) 093002, hep-ph/0606159, hep/ph-07082846.
42. G. Bregar, N.S. Mankoč Borštnik, "Does dark matter consist of baryons of new stable family quarks?", *Phys. Rev. D* **80**, 083534 (2009), 1-16.
43. G. Bregar, N.S. Mankoč Borštnik, "Can we predict the fourth family masses for quarks and leptons?", Proceedings (arxiv:1403.4441) to the 16 th Workshop "What comes beyond the standard models", Bled, 14-21 of July, 2013, Ed. N.S. Mankoč Borštnik, H.B. Nielsen, D. Lukman, DMFA Založništvo, Ljubljana December 2013, p. 31-51, http://arxiv.org/abs/1212.4055.

44. G. Bregar, N.S. Mankoč Borštnik, ”The new experimental data for the quarks mixing matrix are in better agreement with the *spin-charge-family* theory predictions”, Proceedings to the 17^th Workshop ”What comes beyond the standard models”, Bled, 20-28 of July, 2014, Ed. N.S. Mankoč Borštnik, H.B. Nielsen, D. Lukman, DMFA Založništvo, Ljubljana December 2014, p.20-45 [arXiv:1502.06786v1] [arxiv:1412.5866].
45. N.S. Mankoč Borštnik, M. Rosina, ”Are superheavy stable quark clusters viable candidates for the dark matter?”, International Journal of Modern Physics D (IJMPD) **24** (No. 13) (2015) 1545003.
46. Pavšič, M. *The Landscape of Theoretical Physics: Global View*; van der Merwe, A., Ed.; Kluwer Academic Publishers: New York, NY, USA, 2001.
47. Pavšič, M. Quantized fields á la Clifford and unification. *arXiv* **2017**,[arXiv:1707.05695].
48. A. Hernandez-Galeana and N.S. Mankoč Borštnik, ”The symmetry of 4×4 mass matrices predicted by the *spin-charge-family* theory — $SU(2) \times SU(2) \times U(1)$ — remains in all loop corrections”, Proceedings to the 21^{st} Workshop ”What comes beyond the standard models”, 23 of June - 1 of July, 2018, Ed. N.S. Mankoč Borštnik, H.B. Nielsen, D. Lukman, DMFA Založništvo, Ljubljana, December 2017 [arXiv:1902.02691, arXiv:1902.10628].
49. A. Ceccucci (CERN), Z. Ligeti (LBNL), Y. Sakai (KEK), Particle Data Group, Aug. 29, 2014 [http://pdg.lbl.gov/2014/reviews/rpp2014-rev-ckm-matrix.pdf].
50. Review of Particle, Particle Data Group, P.A. Zyla, R.M. Barnett, J. Beringer, O. Dahl, D.A. Dwyer, D.E. Groom, C -J. Lin, K.S. Lugovsky, E. Pianori ...., Author Notes, Progress of Theoretical and Experimental Physics, Volume 2020, Issue 8, August 2020, 083C01, https://doi.org/10.1093/ptep/ptaa104, 14 August 2020.
51. N.S. Mankoč Borštnik, ”The explanation for the origin of the Higgs scalar and for the Yukawa couplings by the *spin-charge-family* theory”, *J. of Mod. Physics* **6** (2015) 2244-2274, http://dx.org./10.4236/jmp.2015.615230 [arXiv:1409.4981].
52. N.S. Mankoč Borštnik, H.B. Nielsen, “Particular boundary condition ensures that a fermion in d=1+5, compactified on a finite disk, manifests in d=1+3 as massless spinor with a charge 1/2, mass protected and chirally coupled to the gauge field”, hep-th/0612126, arxiv:0710.1956, *Phys. Lett.* **B** 663, Issue 3, 22 May 2008, Pages 265-269.
53. 160. D. Lukman, N.S. Mankoč Borštnik, H.B. Nielsen, ”An effective two dimensionality” cases bring a new hope to the Kaluza-Klein-like theories”, http://arxiv.org/abs/1001.4679v5, *New J. Phys.* **13** (2011) 103027, 1-25.
54. D. Lukman, N.S. Mankoč Borštnik, H.B. Nielsen, ”Families of Spinors in $d = (1 + 5)$ Compactified on an Infinite Disc with the Zweibein Which Makes a Disc Curved on S^2 and a Possibility for Masslessness”, Proceedings to the 13^{th} international workshop ”What Comes Beyond the Standard Models”, 13 -23 of July, 2010, Ed. N.S. Mankoč Borštnik, H.B. Nielsen, D. Lukman, DMFA Založništvo, Ljubljana December 2010, p. 193-202, arXiv:1012.0224.

55. D. Lukman and N.S. Mankoč Borštnik, ”Spinor states on a curved infinite disc with non-zero spin-connection fields”, http://arxiv.org/abs/1205.1714, *J. Phys. A: Math. Theor.* **45** (2012) 465401 (19pp).

56. R. Bernabei, P. Belli, A. Bussolotti, V. Caracciolo, R. Cerulli, N. Ferrari, A. Leoncini, V. Merlo, F. Montecchia, F. Cappella, A. d’Angelo, A. Incicchitti, A. Mattei, C.J. Dai, X.H. Ma, X.D. Sheng, Z.P. Ye, “New and recent results, and perspectives from DAMA/LIBRA–phase2, [arXiv:2209.00882], Proceedings to the 25^{rd} Workshop ”What comes beyond the standard models”, 6 - 12 of July, 2022, Ed. N.S. Mankoč Borštnik, H.B. Nielsen, A. Kleppe, DMFA Založništvo, Ljubljana, December 2022, [arXiv:2303.17040 in physics.gen-ph]. The direct link is: http://bsm.fmf.uni-lj.si/bled2022bsm/talks/bled22.pd.

57. Hawking, Stephen W. (28 February 2006). The Theory of Everything: The Origin and Fate of the Universe. Phoenix Books; Special Anniversary. ISBN 978-1-59777-508-3.
58. H. Fritzsch, Weak-interaction mixing in the six-quark theory. Fritzsch, H. *Phys. Lett.* **1978**, *73B*, 317.
59. Frogatt, C.D.; Nielsen, H.B. Hierarchy of quark masses, cabibbo angles and CP violation. *Nucl. Phys.* **1979**, *B147*, 277.
60. Jarlskog, C. Commutator of the Quark Mass Matrices in the Standard Electroweak Model and a Measure of Maximal CP Nonconservation. *Phys. Rev. Lett.* **1985**, *55*, 1039.
61. Gustavo C. ; Branco C.; LavouraL. Ansatz for the quark mass matrices allowing for a high top-quark mass.*Phys. Rev.* **1991**, *D 44, R582(R)* .
62. Harari, H.; Nir, Y. B-anti B mixing and relations among quark masses, angles and phases. *Phys. Lett.* **1987**, *B195*, 586.
63. Stech, B. Are the neutrino masses and mixings closely related to the masses and mixings of quarks? *Phys. Lett.* **1997**, *B403*, 114.
64. Altarelli, G.; Feruglio, F. Models of neutrino masses and mixings. *New J. Phys.* **2004**, *6*, 106.
65. Mohapatra, R.N. A unified solution to the big problems of the standard model. *arXiv* 2022 arXiv:2207.10619.
66. Frampton, P. Predictions of additional baryons and mesons. *arXiv* **2022**, [arXiv:2209.05349]. [arxiv:2211.09579.]

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10 Clifford algebra, internal spaces of fermions and bosons, extened to strings

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Abstract. Abstract: The *string theory* seems to be a mathematically consistent way for explaining so far observed fermion and boson second quantized fields, with gravity included, by offering the renormalizability of the theory by extending the point fermions and bosons into strings and by offering the supersymmetry among fermions and bosons. In a long series of works [1–8] one of the authors in collaboration with another author and other collaborators, has found the phenomenological success with the model named the *spin-charge-family* theory with the properties: The creation and annihilation operators for fermions and bosons fields are described as tensor products of the Clifford odd (for fermions) and the Clifford even (for bosons) "basis vectors" and basis in ordinary space, explaining the second quantization postulates. The theory offers the explanation for the observed properties of fermion and bosons and for several cosmological observations. Since the number of creation and annihilation operators for fermions and bosons is in this theory the same, manifesting correspondingly a kind of supersymmetry, the authors start to study in this contribution the properties of the creation and annihilation operators if extending the point fermions and bosons into strings [13, 14], expecting that this theory offers the low energy limit for the *string theory*.

Povzetek: Zdi se, da ponuja *teorija strun* matematično konsistentno obravnavo v drugi kvantizaciji za doslej opažena fermionska in bozonska polja, tudi za gravitacijo, ko jim zagotovi renormalizabilnost (končnost prispevkov Feynmanovih grafov) z razširitvijo točkastih delcev v strune in s ponujeno simetrijo med fermioni in bozoni. V dolgem nizu del je uspelo enemu od avtorjev (N.S.M.B,), v sodelovanju z drugim avtorjem in sodelavci, z modelom, imenovanim teorija *spin-charge-family*, ki opiše notranji prostor fermionov in bozonov z lihim (za fermione) in sodim (za bozone) produktom Cliffordovih objektov, razložiti postulate druge kvantizacije polj, lastnosti opaženih fermionov in bosonov in nekatere kozmološke meritve (doslej). Ker je število kreacijskih in anihilacijskih operatorjev za fermionska polja enako številu teh operatorjev za bozonska polja, začenjata avtorja s tem prispevkom študij, ki naj pokaže, če razširita doslej točkaste delce v strune, ali ponudi teorija v limiti opazljivih energij to, kar poskuša pokazati *teorija strun*.

10.1 Introduction

In this contribution we start to study how far can the description of the internal space of fermion and boson fields with the Clifford odd and even "basis vectors" be able to reproduce the predictions of the *string theories*, provided that the creation

and annihilation operators are the tensor products of the "basis vectors" and the basis in ordinary space, and if extending the points in the ordinary space to strings. We are making the first steps in this study: We try to reproduce the internal wave function for the boson fields, represented in the "string theories" with the tensor products of the left and right movers, with the algebraic products of the Clifford odd "basis vectors" and their Hermitian conjugated partners.
Having in our case two kinds of the Clifford even "basis vectors", ${}^{I}\hat{\mathcal{A}}^{m\dagger}_{f}$ and ${}^{II}\hat{\mathcal{A}}^{m\dagger}_{f}$, we reproduce both:
i. ${}^{I}\hat{\mathcal{A}}^{m\dagger}_{f}$ as the algebraic product of the Clifford odd "basis vectors" $\hat{b}^{m'\dagger}_{f`} *_A (\hat{b}^{m'\dagger}_{f``})^\dagger$ for each of $2^{\frac{d}{2}-1}$ families of $\hat{b}^{m'\dagger}_{f`}$ and $(\hat{b}^{m'\dagger}_{f``})^\dagger$, f is kept unchanged;
ii. ${}^{II}\hat{\mathcal{A}}^{m\dagger}_{f}$ as the algebraic product of the Clifford odd "basis vectors" $(\hat{b}^{m\dagger}_{f})^\dagger *_A \hat{b}^{m\dagger}_{f`}$, for each of $2^{\frac{d}{2}-1}$ members, with m kept unchanged.
In **i.** each of $2^{\frac{d}{2}-1}$ families reproduces the same ${}^{I}\hat{\mathcal{A}}^{m\dagger}_{f}$, in **ii.** each of $2^{\frac{d}{2}-1}$ family members reproduces the same ${}^{II}\hat{\mathcal{A}}^{m\dagger}_{f}$.
Since in the *string theory* only bosons with the transverse momentum are considered, we study ${}^{I}\hat{\mathcal{A}}^{m\dagger}_{f}$ and ${}^{II}\hat{\mathcal{A}}^{m\dagger}_{f}$ carrying only the transfer momenta.
To manifest properties of both kinds of boson field for $d = (9+1)$, we first discuss $d = (5+1)$ and then comment the case $d = (9+1)$.
We make first a short overview about the Clifford odd and even "basis vectors", in any even dimensional space, describing the internal space of fermions by the Clifford odd "basis vectors" appearing in $2^{\frac{d}{2}-1}$ families, each family having $2^{\frac{d}{2}-1}$ members, and the same number $2^{\frac{d}{2}-1} \times 2^{\frac{d}{2}-1}$ of their Hermitian conjugated partners appearing in a separate group, and the internal space of bosons by the Clifford even "basis vectors" appearing in two groups each one with $2^{\frac{d}{2}-1} \times 2^{\frac{d}{2}-1}$ members), having their Hermitian conjugated partners within each group.
If the internal spaces of fermions and bosons, described by the Clifford odd and even "basis vectors", respectively, are what the "nature" is using, then the internal space of *string theories* can be correlated to the one suggested by the Clifford algebra.
All the sections and subsections up to Subsect. 10.2.4 are shortly overviewing Refs. ([6–8] and the references therein) in the way needed in Subsects. (10.2.4, 10.2.5) and Sect. 10.3, where the authors start to present internal spaces of fermions and bosons, used in the *spin-charge-family* theory, looking for the parallelism with the internal spaces of boson strings. Presenting the boson internal space as the algebraic products of Clifford odd "basis vectors" and their Hermitian conjugated partners, they simulate left and right movers of IIA and IIB super *string theory*.

10.2 Creation and annihilation operators for fermions and bosons

The Clifford odd and even "basis vectors" and the creation and annihilation operators for fermions and bosons are shortly presented, overviewing Ref. [7], Sect. 2. up to 2.2.

10.2.1 Grassmann and Clifford algebras

The internal spaces of anti-commuting or commuting second quantized fields can be described by using either the Grassmann or the Clifford algebras [1,6]
In Grassmann d-dimensional space there are d anti-commuting (operators) θ^a, and d anti-commuting operators which are derivatives with respect to θ^a, $\frac{\partial}{\partial\theta_a}$,

$$\{\theta^a,\theta^b\}_+ = 0\,, \quad \{\frac{\partial}{\partial\theta_a},\frac{\partial}{\partial\theta_b}\}_+ = 0\,,$$
$$\{\theta_a,\frac{\partial}{\partial\theta_b}\}_+ = \delta_{ab}\,,\ (a,b) = (0,1,2,3,5,\cdots,d)\,. \tag{10.1}$$

Making a choice

$$(\theta^a)^\dagger = \eta^{aa}\frac{\partial}{\partial\theta_a}\,, \quad \text{leads to} \quad (\frac{\partial}{\partial\theta_a})^\dagger = \eta^{aa}\theta^a\,, \tag{10.2}$$

with $\eta^{ab} = diag\{1,-1,-1,\cdots,-1\}$.
θ^a and $\frac{\partial}{\partial\theta_a}$ are, up to the sign, Hermitian conjugated to each other. The identity is a self adjoint member of the algebra. The choice for the following complex properties of θ^a

$$\{\theta^a\}^* = (\theta^0,\theta^1,-\theta^2,\theta^3,-\theta^5,\theta^6,...,-\theta^{d-1},\theta^d)\,, \tag{10.3}$$

correspondingly requires $\{\frac{\partial}{\partial\theta_a}\}^* = (\frac{\partial}{\partial\theta_0},\frac{\partial}{\partial\theta_1},-\frac{\partial}{\partial\theta_2},\frac{\partial}{\partial\theta_3},-\frac{\partial}{\partial\theta_5},\frac{\partial}{\partial\theta_6},...,-\frac{\partial}{\partial\theta_{d-1}},\frac{\partial}{\partial\theta_d})\,.$
There are 2^d superposition of products of θ^a, the Hermitian conjugated partners of which are the corresponding superposition of products of $\frac{\partial}{\partial\theta_a}$ [10].
There exist two kinds of the Clifford algebra elements (operators), γ^a and $\tilde\gamma^a$, expressible with θ^a's and their conjugate momenta $p^{\theta a} = i\,\frac{\partial}{\partial\theta_a}$ [1], Eqs. (10.1, 10.2),

$$\gamma^a = (\theta^a + \frac{\partial}{\partial\theta_a})\,, \quad \tilde\gamma^a = i\,(\theta^a - \frac{\partial}{\partial\theta_a})\,,$$
$$\theta^a = \frac{1}{2}\,(\gamma^a - i\tilde\gamma^a)\,, \quad \frac{\partial}{\partial\theta_a} = \frac{1}{2}\,(\gamma^a + i\tilde\gamma^a)\,, \tag{10.4}$$

offering together $2\cdot 2^d$ operators: 2^d are superposition of products of γ^a and 2^d of $\tilde\gamma^a$. It is easy to prove if taking into account Eqs. (10.2, 10.4), that they form two anti-commuting Clifford subalgebras, $\{\gamma^a,\tilde\gamma^b\}_+ = 0$, Refs. ([6] and references therein)

$$\{\gamma^a,\gamma^b\}_+ = 2\eta^{ab} = \{\tilde\gamma^a,\tilde\gamma^b\}_+\,,$$
$$\{\gamma^a,\tilde\gamma^b\}_+ = 0\,, \quad (a,b) = (0,1,2,3,5,\cdots,d)\,,$$
$$(\gamma^a)^\dagger = \eta^{aa}\gamma^a\,, \quad (\tilde\gamma^a)^\dagger = \eta^{aa}\tilde\gamma^a\,. \tag{10.5}$$

While the Grassmann algebra offers the description of the "anti-commuting integer spin second quantized fields" and of the "commuting integer spin second quantized fields" [5,6], the Clifford algebras which are superposition of odd products

of either γ^a's or $\tilde{\gamma}^a$'s offer the description of the second quantized half integer spin fermion fields, which from the point of the subgroups of the $SO(d-1,1)$ group manifest spins and charges of fermions and antifermions in the fundamental representations of the group and subgroups . [6].
The superposition of even products of either γ^a's or $\tilde{\gamma}^a$'s offer the description of the commuting second quantized boson fields with integer spins (as we can see in [11] and shall see in this contribution) which from the point of the subgroups of the $SO(d-1,1)$ group manifest spins and charges in the adjoint representations of the group and subgroups.
The following *postulate*, which determines how does $\tilde{\gamma}^a$ operate on γ^a, reduces the two Clifford subalgebras, γ^a and $\tilde{\gamma}^a$, to one, to the one described by γ^a [1,3,9]

$$\{\tilde{\gamma}^a B = (-)^B\, i\, B\gamma^a\}|\psi_{oc}>\,, \tag{10.6}$$

with $(-)^B = -1$, if B is (a function of) odd products of γ^a's, otherwise $(-)^B = 1$ [3], the vacuum state $|\psi_{oc}>$ is defined in Eq. (10.14) of Subsect. 10.2.2.

After the postulate of Eq. (10.6) no vector space of $\tilde{\gamma}^a$'s needs to be taken into account for the description of the internal space of either fermions or bosons, in agreement with the observed properties of fermions and bosons. Also the Grassmann algebra is reduced to only one of the Clifford subalgebras.
The operator $\tilde{\gamma}^a$ will from now on be used to describe the properties of fermion "basis vectors", determining by $\tilde{S}^{ab} = \frac{i}{4}(\tilde{\gamma}^a\tilde{\gamma}^b - \tilde{\gamma}^b\tilde{\gamma}^a)$ the "family" quantum numbers of the irreducible representations of the Lorentz group in internal space of fermions, S^{ab}, and the properties of bosons "basis vectors" determined by $\mathcal{S}^{ab} = S^{ab} + \tilde{S}^{ab}$. We shall see that while the fermion "basis vectors" appear in "families", the boson "basis vectors" have no "families" and manifest properties of the gauge fields of the corresponding fermion fields.
Each irreducible representation of the Clifford odd "basis vectors" described by γ^a's are chosen to be equipped by the quantum numbers of the Cartan subalgebra members of $\tilde{S}^{ab}$, chosen in Eq. (10.7), as follows

$$\begin{aligned}
&S^{03}, S^{12}, S^{56}, \cdots, S^{d-1\; d}\,,\\
&\tilde{S}^{03}, \tilde{S}^{12}, \tilde{S}^{56}, \cdots, \tilde{S}^{d-1\; d}\,,\\
&\mathcal{S}^{ab} = S^{ab} + \tilde{S}^{ab} = i\,(\theta^a \frac{\partial}{\partial\theta_b} - \theta^b \frac{\partial}{\partial\theta_a})\,.
\end{aligned} \tag{10.7}$$

10.2.2 "Basis vectors" and relations among Clifford even and Clifford odd "basis vectors"

This subsection is a short overview of similar sections of several articles of the author, like [5,8,11,12].
After the reduction of the two Clifford subalgebras to only one, Eq. (10.6), we only need to define "basis vectors" for the case that the internal space of second quantized fields is described by superposition of odd or even products γ^a's [1].

[1] In Ref. [6], the reader can find in Subsects. (3.2.1 and 3.2.2) definitions for the "basis vectors" for the Grassmann and the two Clifford subalgebras, which are products of

Let us use the technique which makes "basis vectors" products of nilpotents and projectors [1,3] which are eigenvectors of the (chosen) Cartan subalgebra members, Eq. (10.7), of the Lorentz algebra in the space of γ^a's, either in the case of the Clifford odd or in the case of the Clifford even products of γ^a's.
There are in even-dimensional spaces $\frac{d}{2}$ members of the Cartan subalgebra, Eq. (10.7). In odd-dimensional spaces there are $\frac{d-1}{2}$ members of the Cartan subalgebra.
One finds in even dimensional spaces for any of the $\frac{d}{2}$ Cartan subalgebra member, S^{ab}, applying on a nilpotent $\stackrel{ab}{(k)}$ or on projector $\stackrel{ab}{[k]}$

$$\begin{aligned} \stackrel{ab}{(k)}: &= \frac{1}{2}(\gamma^a + \frac{\eta^{aa}}{ik}\gamma^b)\,, \quad (\stackrel{ab}{(k)})^2 = 0\,, \\ \stackrel{ab}{[k]}: &= \frac{1}{2}(1 + \frac{i}{k}\gamma^a\gamma^b)\,, \quad (\stackrel{ab}{[k]})^2 = \stackrel{ab}{[k]}, \end{aligned} \tag{10.8}$$

the relations

$$\begin{aligned} S^{ab}\; \stackrel{ab}{(k)} &= \frac{k}{2}\; \stackrel{ab}{(k)}\,, \qquad \tilde{S}^{ab}\; \stackrel{ab}{(k)} = \frac{k}{2}\; \stackrel{ab}{(k)}\,, \\ S^{ab}\; \stackrel{ab}{[k]} &= \frac{k}{2}\; \stackrel{ab}{[k]}\,, \qquad \tilde{S}^{ab}\; \stackrel{ab}{[k]} = -\frac{k}{2}\; \stackrel{ab}{[k]}\,, \end{aligned} \tag{10.9}$$

with $k^2 = \eta^{aa}\eta^{bb}$ [2], demonstrating that the eigenvalues of S^{ab} on nilpotents and projectors expressed with γ^a differ from the eigenvalues of $\tilde{S}^{ab}$ on nilpotents and projectors expressed with γ^a, so that $\tilde{S}^{ab}$ can be used to equip each irreducible representation of S^{ab} with the "family" quantum number. [3]
Taking into account Eq. (10.5) one finds

$$\begin{aligned}
&\gamma^a\; \stackrel{ab}{(k)} = \eta^{aa}\; \stackrel{ab}{[-k]}, \quad \gamma^b\; \stackrel{ab}{(k)} = -ik\; \stackrel{ab}{[-k]}, \quad \gamma^a\; \stackrel{ab}{[k]} = \stackrel{ab}{(-k)}, \quad \gamma^b\; \stackrel{ab}{[k]} = -ik\eta^{aa}\; \stackrel{ab}{(-k)}\,, \\
&\tilde{\gamma^a}\; \stackrel{ab}{(k)} = -i\eta^{aa}\; \stackrel{ab}{[k]}, \quad \tilde{\gamma^b}\; \stackrel{ab}{(k)} = -k\; \stackrel{ab}{[k]}, \quad \tilde{\gamma^a}\; \stackrel{ab}{[k]} = \; i\; \stackrel{ab}{(k)}, \quad \tilde{\gamma^b}\; \stackrel{ab}{[k]} = -k\eta^{aa}\; \stackrel{ab}{(k)}\,, \\
&\stackrel{ab}{(k)}\stackrel{ab}{(-k)} = \eta^{aa}\; \stackrel{ab}{[k]}\,, \quad \stackrel{ab}{(-k)}\stackrel{ab}{(k)} = \eta^{aa}\; \stackrel{ab}{[-k]}\,, \quad \stackrel{ab}{(k)}\stackrel{ab}{[k]} = 0\,, \quad \stackrel{ab}{(k)}\stackrel{ab}{[-k]} = \stackrel{ab}{(k)}\,, \\
&\stackrel{ab}{(-k)}\stackrel{ab}{[k]} = \stackrel{ab}{(-k)}\,, \quad \stackrel{ab}{[k]}\stackrel{ab}{(k)} = \stackrel{ab}{(k)}\,, \quad \stackrel{ab}{[k]}\stackrel{ab}{(-k)} = 0\,, \quad \stackrel{ab}{[k]}\stackrel{ab}{[-k]} = 0\,, \\
&\stackrel{ab}{(k)}^{\dagger} = \eta^{aa}\; \stackrel{ab}{(-k)}\,, \quad (\stackrel{ab}{(k)})^2 = 0\,, \quad \stackrel{ab}{(k)}\stackrel{ab}{(-k)} = \eta^{aa}\; \stackrel{ab}{[k]}\,, \\
&\stackrel{ab}{[k]}^{\dagger} = \stackrel{ab}{[k]}\,, \quad (\stackrel{ab}{[k]})^2 = \stackrel{ab}{[k]}\,, \quad \stackrel{ab}{[k]}\stackrel{ab}{[-k]} = 0\,.
\end{aligned} \tag{10.10}$$

More relations are presented in App. 10.4.
The relations in Eq. (10.10) demonstrate that the properties of "basis vectors" which include an odd number of nilpotents, differ essentially from the "basis vectors", which include an even number of nilpotents.

---

nilpotents and projectors chosen to be the eigenvectors of the corresponding Cartan subalgebra members of the Lorentz algebras presented in Eq. (10.7).

[2] Let us prove one of the relations in Eq. (10.9): $S^{ab}\; \stackrel{ab}{(k)} = \frac{i}{2}\gamma^a\gamma^b\frac{1}{2}(\gamma^a + \frac{\eta^{aa}}{ik}\gamma^b) = \frac{1}{2^2}\{-i(\gamma^a)^2\gamma^b + i(\gamma^b)^2\gamma^a\frac{\eta^{aa}}{ik}\} = \frac{1}{2}\frac{\eta^{aa}\eta^{bb}}{k}\frac{1}{2}\{\gamma^a + \frac{k^2}{\eta^{bb}ik}\gamma^b\}$. For $k^2 = \eta^{aa}\eta^{bb}$ the first relation follows.

[3] The reader can find the proof of Eq. (10.9) also in Ref. [6], App. (I).

10.2.3 Cifford odd and even "basis vectors"

Let us define Clifford odd and even "basis vectors" as products of nilpotents and projectors in even-dimensional spaces.

a. *Clifford odd "basis vectors"*

This part overviews several papers with the same topic ([6, 8] and references therein).
The Clifford odd "basis vectors" are chosen to be products of an odd number of nilpotents, and the rest, up to $\frac{d}{2}$, of projectors, each nilpotent and each projector are chosen to be the "eigenstate" of one of the members of the Cartan subalgebra, Eq. (10.7), correspondingly are the "basis vectors" eigenstates of all the members of the Lorentz algebra: S^{ab}'s determine $2^{\frac{d}{2}-1}$ members of one family, $\tilde{S}^{ab}$'s transform each member of one family to the same member of the rest of $2^{\frac{d}{2}-1}$ families.
Let us call the Clifford odd "basis vectors" $\hat{b}^{m\dagger}_{f}$, if it is the m^{th} membership of the family f. The Hermitian conjugated partner of $\hat{b}^{m\dagger}_{f}$ is called $\hat{b}^{m}_{f}$ $(= (\hat{b}^{m\dagger}_{f})^{\dagger}$.
Let us start in $d = 2(2n+1)$ with the "basis vector" $\hat{b}^{1\dagger}_{1}$ which is the product of only nilpotents, all the rest members belonging to the $f = 1$ family follow by the application of $S^{01}, S^{03}, \ldots, S^{0d}, S^{15}, \ldots, S^{1d}, S^{5d} \ldots, S^{d-2\,d}$. They are presented on the left-hand side. Their Hermitian conjugated partners are presented on the right-hand side. The algebraic product mark $*_A$ among nilpotents and projectors is skipped.

$$
\begin{array}{ll}
\multicolumn{2}{c}{d = 2(2n+1)\,,} \\
\hat{b}^{1\dagger}_{1} = \stackrel{03}{(+i)}\stackrel{12}{(+)}\stackrel{56}{(+)}\cdots\stackrel{d-1\,d}{(+)}\,, & \hat{b}^{1}_{1} = \stackrel{03}{(-i)}\stackrel{12}{(-)}\cdots\stackrel{d-1\,d}{(-)}\,, \\
\hat{b}^{2\dagger}_{1} = \stackrel{03}{[-i]}\stackrel{12}{[-]}\stackrel{56}{(+)}\cdots\stackrel{d-1\,d}{(+)}\,, & \hat{b}^{2}_{1} = \stackrel{03}{[-i]}\stackrel{12}{[-]}\stackrel{56}{(-)}\cdots\stackrel{d-1\,d}{(-)}\,, \\
\cdots & \cdots \\
\hat{b}^{2^{\frac{d}{2}-1}\dagger}_{1} = \stackrel{03}{[-i]}\stackrel{12}{[-]}\stackrel{56}{(+)}\ldots\stackrel{d-3\,d-2}{[-]}\;\stackrel{d-1\,d}{[-]}\,, & \hat{b}^{2^{\frac{d}{2}-1}}_{1}{}^{\dagger} = \stackrel{03}{[-i]}\stackrel{12}{[-]}\stackrel{56}{(-)}\stackrel{78}{[-]}\ldots\stackrel{d-3\,d-2}{[-]}\;\stackrel{d-1\,d}{[-]}\,, \\
\cdots\,, & \cdots\,.
\end{array}
\tag{10.11}
$$

In $d = 4n$ the choice of the starting "basis vector" with maximal number of nilpotents must have one projector

$$
\begin{array}{ll}
\multicolumn{2}{c}{d = 4n\,,} \\
\hat{b}^{1\dagger}_{1} = \stackrel{03}{(+i)}\stackrel{12}{(+)}\cdots\stackrel{d-1\,d}{[+]}\,, & \hat{b}^{1}_{1} = \stackrel{03}{(-i)}\stackrel{12}{(-)}\cdots\stackrel{d-1\,d}{[+]} \\
\hat{b}^{2\dagger}_{1} = \stackrel{03}{[-i]}\stackrel{12}{[-]}\stackrel{56}{(+)}\cdots\stackrel{d-1\,d}{[+]}\,, & \hat{b}^{2}_{1} = \stackrel{03}{[-i]}\stackrel{12}{[-]}\stackrel{56}{(-)}\cdots\stackrel{d-1\,d}{[+]}\,, \\
\cdots\,, & \cdots\,, \\
\hat{b}^{2^{\frac{d}{2}-1}\dagger}_{1} = \stackrel{03}{[-i]}\stackrel{12}{[-]}\stackrel{56}{(-)}\ldots\stackrel{d-3\,d-2}{[-]}\;\stackrel{d-1\,d}{[+]}\,, & \hat{b}^{2^{\frac{d}{2}-1}}_{1} = \stackrel{03}{[-i]}\stackrel{12}{[-]}\stackrel{56}{(-)}\ldots\stackrel{d-3\,d-2}{[-]}\;\stackrel{d-1\,d}{[+]}\,, \\
\cdots\,, & \cdots\,.
\end{array}
\tag{10.12}
$$

The Hermitian conjugated partners of the Clifford odd "basis vectors" $\hat{b}^{m\dagger}_{1}$, presented in Eq. (10.12) on the right-hand side, follow if all nilpotents $\stackrel{ab}{(k)}$ of $\hat{b}^{m\dagger}_{1}$ are transformed into $\eta^{aa}\stackrel{ab}{(-k)}$.

For either $d = 2(2n+1)$ or for $d = 4n$ all the $2^{\frac{d}{2}-1}$ families follow by applying $\tilde{S}^{ab}$'s on all the members of the starting family. (Or one can find the starting $\hat{b}^{1\dagger}_{f}$ for all families f and then generate all the members $\hat{b}^{m}_{f}$ from $\hat{b}^{1\dagger}_{f}$ by the application of S^{ab} on the starting member.)
It is not difficult to see that all the "basis vectors" within any family, as well as the "basis vectors" among families, are orthogonal; that is, their algebraic product is zero. The same is true within their Hermitian conjugated partners. Both can be proved by the algebraic multiplication using Eqs. (10.10, 10.25).

$$\hat{b}^{m\dagger}_{f} *_A \hat{b}^{m`\dagger}_{f`} = 0\,, \quad \hat{b}^{m}_{f} *_A \hat{b}^{m`}_{f`} = 0\,, \quad \forall m, m', f, f`\,. \tag{10.13}$$

When we choose the vacuum state equal to

$$|\psi_{oc}>= \sum_{f=1}^{2^{\frac{d}{2}-1}} \hat{b}^{m}_{f}{}_{*_A} \hat{b}^{m\dagger}_{f} | 1 \;>\,, \tag{10.14}$$

for one of members m, which can be anyone of the odd irreducible representations f it follows that the Clifford odd "basis vectors" obey the relations

$$\begin{aligned}
\hat{b}^{m}_{f}{}_{*_A}|\psi_{oc}> &= 0.\,|\psi_{oc}>\,,\\
\hat{b}^{m\dagger}_{f}{}_{*_A}|\psi_{oc}> &= |\psi^{m}_{f}>\,,\\
\{\hat{b}^{m}_{f}, \hat{b}^{m'}_{f`}\}_{*_A +}|\psi_{oc}> &= 0.\,|\psi_{oc}>\,,\\
\{\hat{b}^{m\dagger}_{f}, \hat{b}^{m'\dagger}_{f`}\}_{*_A +}|\psi_{oc}> &= 0.\,|\psi_{oc}>\,,\\
\{\hat{b}^{m}_{f}, \hat{b}^{m'\dagger}_{f`}\}_{*_A +}|\psi_{oc}> &= \delta^{m m'}\,\delta_{f f`}|\psi_{oc}>\,,
\end{aligned} \tag{10.15}$$

b. *Clifford even "basis vectors"*

This part proves that the Clifford even "basis vectors" are in even-dimensional spaces offering the description of the internal spaces of boson fields — the gauge fields of the corresponding Clifford odd "basis vectors": It is a new recognition, offering a new understanding of the second quantized fermion and **boson** fields [11].
The Clifford even "basis vectors" must be products of an even number of nilpotents and the rest, up to $\frac{d}{2}$, of projectors; each nilpotent and each projector is chosen to be the "eigenstate" of one of the members of the Cartan subalgebra of the Lorentz algebra, $\mathcal{S}^{ab} = S^{ab} + \tilde{S}^{ab}$, Eq. (10.7). Correspondingly the "basis vectors" are the eigenstates of all the members of the Cartan subalgebra of the Lorentz algebra.
The Clifford even "basis vectors" appear in two groups, each group has $2^{\frac{d}{2}-1} \times 2^{\frac{d}{2}-1}$ members. The members of one group can not be reached from the members of another group by either S^{ab}'s or $\tilde{S}^{ab}$'s or both.
S^{ab} and $\tilde{S}^{ab}$ generate from the starting "basis vector" of each group all the $2^{\frac{d}{2}-1} \times 2^{\frac{d}{2}-1}$ members. Each group contains the Hermitian conjugated partner of any member; $2^{\frac{d}{2}-1}$ members of each group are products of only (self adjoint) projectors.
Let us call the Clifford even "basis vectors" ${}^{i}\hat{\mathcal{A}}^{m\dagger}_{f}$, where $i = (I, II)$ denotes the two groups of Clifford even "basis vectors", while m and f determine membership

of "basis vectors" in any of the two groups, I or II.

$$
\begin{array}{ll}
 & d = 2(2n+1) \\
{}^{I}\hat{\mathcal{A}}^{1\dagger}_{1} = \stackrel{03}{(+i)}\stackrel{12}{(+)}\cdots \stackrel{d-1\,d}{[+]}\,, & {}^{II}\hat{\mathcal{A}}^{1\dagger}_{1} = \stackrel{03}{(-i)}\stackrel{12}{(+)}\cdots \stackrel{d-1\,d}{[+]}\,, \\
{}^{I}\hat{\mathcal{A}}^{2\dagger}_{1} = \stackrel{03}{[-i]}\stackrel{12}{[-]}\stackrel{56}{(+)}\cdots \stackrel{d-1\,d}{[+]}\,, & {}^{II}\hat{\mathcal{A}}^{2\dagger}_{1} = \stackrel{03}{[+i]}\stackrel{12}{[-]}\stackrel{56}{(+)}\cdots \stackrel{d-1\,d}{[+]}\,, \\
{}^{I}\hat{\mathcal{A}}^{3\dagger}_{1} = \stackrel{03}{(+i)}\stackrel{12}{(+)}\stackrel{56}{(+)}\cdots \stackrel{d-3\,d-2}{[-]}\;\stackrel{d-1\,d}{(-)}\,, & {}^{II}\hat{\mathcal{A}}^{3\dagger}_{1} = \stackrel{03}{(-i)}\stackrel{12}{(+)}\stackrel{56}{(+)}\cdots \stackrel{d-3\,d-2}{[-]}\;\stackrel{d-1\,d}{(-)}\,, \\
\cdots & \cdots \\
 & d = 4n \\
{}^{I}\hat{\mathcal{A}}^{1\dagger}_{1} = \stackrel{03}{(+i)}\stackrel{12}{(+)}\cdots \stackrel{d-1\,d}{(+)}\,, & {}^{II}\hat{\mathcal{A}}^{1\dagger}_{1} = \stackrel{03}{(-i)}\stackrel{12}{(+)}\cdots \stackrel{d-1\,d}{(+)}\,, \\
{}^{I}\hat{\mathcal{A}}^{2\dagger}_{1} = \stackrel{03}{[-i]}\stackrel{12}{[-i]}\stackrel{56}{(+)}\cdots \stackrel{d-1\,d}{(+)}\,, & {}^{II}\hat{\mathcal{A}}^{2\dagger}_{1} = \stackrel{03}{[+i]}\stackrel{12}{[-i]}\stackrel{56}{(+)}\cdots \stackrel{d-1\,d}{(+)}\,, \\
{}^{I}\hat{\mathcal{A}}^{3\dagger}_{1} = \stackrel{03}{(+i)}\stackrel{12}{(+)}\stackrel{56}{(+)}\cdots \stackrel{d-3\,d-2}{[-]}\;\stackrel{d-1\,d}{[-]}\,, & {}^{II}\hat{\mathcal{A}}^{3\dagger}_{1} = \stackrel{03}{(-i)}\stackrel{12}{(+)}\stackrel{56}{(+)}\cdots \stackrel{d-3\,d-2}{[-]}\;\stackrel{d-1\,d}{[-]} \\
\cdots & \cdots
\end{array}
\tag{10.16}
$$

There are $2^{\frac{d}{2}-1} \times 2^{\frac{d}{2}-1}$ Clifford even "basis vectors" of the kind ${}^{I}\hat{\mathcal{A}}^{m\dagger}_{f}$ and there are $2^{\frac{d}{2}-1} \times 2^{\frac{d}{2}-1}$ Clifford even "basis vectors" of the kind ${}^{II}\hat{\mathcal{A}}^{m\dagger}_{f}$.

Table 10.5, presented in App. 10.5 illustrates properties of the Clifford odd and Clifford even "basis vectors" on the case of $d = (5+1)$. Looking at this case it is easy to evaluate properties of either even or odd "basis vectors". We shall discuss in this subsection the general case by carefully inspecting properties of both kinds of "basis vectors".

The Clifford even "basis vectors" belonging to two different groups are orthogonal due to the fact that they differ in the sign of one nilpotent or one projector, or the algebraic product of a member of one group with a member of another group gives zero according to the first two lines of Eq. (10.25): $\stackrel{ab}{(k)}\stackrel{ab}{[k]}= 0$, $\stackrel{ab}{[k]}\stackrel{ab}{(-k)}= 0$, $\stackrel{ab}{[k]}\stackrel{ab}{[-k]}= 0$.

$$
{}^{I}\hat{\mathcal{A}}^{m\dagger}_{f} *_A {}^{II}\hat{\mathcal{A}}^{m\dagger}_{f} = 0 = {}^{II}\hat{\mathcal{A}}^{m\dagger}{}_{\cdot f} *_A {}^{I}\hat{\mathcal{A}}^{m\dagger}_{f}\,. \tag{10.17}
$$

The members of each of these two groups have the property

$$
{}^{i}\hat{\mathcal{A}}^{m\dagger}_{f} *_A {}^{i}\hat{\mathcal{A}}^{m'\dagger}_{f`} \rightarrow \left\{ \begin{array}{r} {}^{i}\hat{\mathcal{A}}^{m\dagger}_{f`}\,, i = (I, II) \\ \text{or zero}\,. \end{array} \right. \tag{10.18}
$$

For a chosen $(m, f, f`)$ there is only one m' (out of $2^{\frac{d}{2}-1}$) which gives nonzero contribution.

Two "basis vectors", ${}^{i}\hat{\mathcal{A}}^{m\dagger}_{f}$ and ${}^{i}\hat{\mathcal{A}}^{m'\dagger}_{f'}$, the algebraic product, $*_A$, of which gives non zero contribution, "scatter" into the third one ${}^{i}\hat{\mathcal{A}}^{m\dagger}_{f`}$, for $i = (I, II)$.

It remains to evaluate the algebraic application, $*_A$, of the Clifford even "basis vectors" ${}^{I,II}\hat{\mathcal{A}}^{m\dagger}_{f}$ on the Clifford odd "basis vectors" $\hat{b}^{m'\dagger}_{f`}$. One finds, taking into account Eq. (10.25), for ${}^{I}\hat{\mathcal{A}}^{m\dagger}_{f}$

$$
{}^{I}\hat{\mathcal{A}}^{m\dagger}_{f} *_A \hat{b}^{m'\dagger}_{f`} \rightarrow \left\{ \begin{array}{c} \hat{b}^{m\dagger}_{f`}\,, \\ \text{or zero}\,, \end{array} \right. \tag{10.19}
$$

For each ${}^{I}\hat{\mathcal{A}}^{m\dagger}_{f}$ there are among $2^{\frac{d}{2}-1}\times\, 2^{\frac{d}{2}-1}$ members of the Clifford odd "basis vectors" (describing the internal space of fermion fields) $2^{\frac{d}{2}-1}$ members, $\hat{b}^{m'\dagger}_{f`}$, fulfilling the relation of Eq. (10.19). All the rest ($2^{\frac{d}{2}-1}\times\, (2^{\frac{d}{2}-1}-1)$) Clifford odd "basis vectors" give zero contributions. Or equivalently, there are $2^{\frac{d}{2}-1}$ pairs of quantum numbers $(f, m')$ for which $\hat{b}^{m\dagger}_{f`}\neq 0$.

Taking into account Eq. (10.25) one finds

$$\hat{b}^{m\dagger}_{f} *_A {}^{I}\hat{\mathcal{A}}^{m'\dagger}_{f`} = 0\,, \quad \forall (m, m`, f, f`)\,. \tag{10.20}$$

For "scattering" the Clifford even "basis vectors" ${}^{II}\hat{\mathcal{A}}^{m\dagger}_{f}$ on the Clifford odd "basis vectors" $\hat{b}^{m'\dagger}_{f`}$ it follows

$${}^{II}\hat{\mathcal{A}}^{m\dagger}_{f} *_A \hat{b}^{m'\dagger}_{f`} = 0\,, \;\; \forall (m, m', f, f`)\,, \tag{10.21}$$

while we get

$$\hat{b}^{m\dagger}_{f} *_A {}^{II}\hat{\mathcal{A}}^{m'\dagger}_{f`} \rightarrow \begin{cases} \hat{b}^{m\dagger}_{f``}\,, \\ \text{or zero}\,, \end{cases} \tag{10.22}$$

For each $\hat{b}^{m\dagger}_{f}$ there are among $2^{\frac{d}{2}-1}\times 2^{\frac{d}{2}-1}$ members of the Clifford even "basis vectors" (describing the internal space of boson fields) , ${}^{II}\hat{\mathcal{A}}^{m'\dagger}_{f`}$, $2^{\frac{d}{2}-1}$ members (with appropriate f' and m') fulfilling the relation of Eq. (10.22).
All the rest ($2^{\frac{d}{2}-1}\times\, (2^{\frac{d}{2}-1}-1)$) Clifford even"basis vectors" give zero contributions.

10.2.4 Symmetry relations between Clifford even and Clifford odd "basis vectors"

This part is, together with Subsect. 10.2.5 and Sect. 10.3, the only new part of this contribution. All the rest is the repetition of Refs. [6–8] and the references therein, needed for Subsects. (10.2.4, 10.2.5) and Sect. 10.3.
In Subsect. 10.2.2 the relations among the Clifford even "basis vectors", ${}^{i}\hat{\mathcal{A}}^{m\dagger}_{f}, i = (I, II)$, as well as among the Clifford even ${}^{i}\hat{\mathcal{A}}^{m\dagger}_{f}, i = (I, II)$ and odd "basis vectors", $\hat{b}^{m\dagger}_{f}$ and $\hat{b}^{m}_{f}$, are discussed.
This, the only new part up to now, looks for relations among the properties of *boson strings* (IIA and IIB) description of the internal space and the description of the internal space with the Clifford even "basis vectors". We mostly treat the particular case of $d = (5 + 1)$. The case with $d = (9 + 1)$ is comment in the next subsection 10.2.5.
Let us look for the Clifford even "basis vectors", ${}^{I}\hat{\mathcal{A}}^{m\dagger}_{f}$ and ${}^{II}\hat{\mathcal{A}}^{m\dagger}_{f}$, which have in the internal space spin in transverse dimensions only, in order to relate "basis vectors" to the closed IIA and IIB *boson strings*. One finds ${}^{I}\hat{\mathcal{A}}^{m\dagger}_{f}$ as the algebraic products of the Clifford odd "basis vectors" and their Hermitian conjugated

partners, $\hat{b}^{m\dagger}_{f} *_A (\hat{b}^{m'\dagger}_{f})^\dagger$, for each of $2^{\frac{d}{2}-1}$ families, obtaining the same group of ${}^{I}\hat{\mathcal{A}}^{m\dagger}_{f}$ for any f, while ${}^{II}\hat{\mathcal{A}}^{m\dagger}_{f}$ can be written as the algebraic products of the Clifford odd "basis vectors" and their hermitian conjugated partners in the opposite order, $(\hat{b}^{m'\dagger}_{f`})^\dagger *_A \hat{b}^{m'\dagger}_{f``}$, for each of $2^{\frac{d}{2}-1}$ family members, while $f'$ and $f``$ run over all the families. Again any m gives the same group of ${}^{II}\hat{\mathcal{A}}^{m\dagger}_{f}$.
We look for the relations among the internal spaces defined in *strings theory* and the internal spaces defined with the Clifford odd and even "basis vectors", used in the *spin-charge-family* theory for only those fields ${}^{I}\hat{\mathcal{A}}^{m\dagger}_{f}$ and ${}^{II}\hat{\mathcal{A}}^{m\dagger}_{f}$ with $\mathcal{S}^{12} = \pm 1$. In this Subsect. we pay attention on:

A. How one can generate Clifford even "basis vectors" from the algebraic products, $*_A$, of the Clifford odd $\hat{b}^{m\dagger}_{f}$ and their Hermitian conjugated partners $\hat{b}^{m'}_{f`} = (\hat{b}^{m'\dagger}_{f`})^\dagger$.

B. How γ^a and $\tilde{\gamma}^a$ apply on the Clifford even and odd basis vectors".

Generating Clifford even "basis vectors" from the algebraic products, $*_A$, of the Clifford odd "basis vectors", $\hat{b}^{m\dagger}_{f}$, and their Hermitian conjugated partners, $\hat{b}^{m'}_{f`}$. Let us start with the generation of the Clifford even "basis vectors" for ${}^{I}\hat{\mathcal{A}}^{m\dagger}_{f}$ from the algebraic products of the Clifford odd "basis vectors" and their Hermitian conjugated partners, $\hat{b}^{m\dagger}_{f} *_A (\hat{b}^{m'\dagger}_{f})^\dagger$, for a particular case when $d = (5+1)$ to learn what we can expect after comparing these relations with the similar relations in the *string theory*.
And let us treat ${}^{II}\hat{\mathcal{A}}^{m\dagger}_{f}$ as the algebraic product, $(\hat{b}^{m\dagger}_{f})^\dagger *_A \hat{b}^{m\dagger}_{f}$, again for the case when $d = (5+1)$.
The generalization to any even d is not difficult.
We shall comment our way of describing internal space and the *string theory* way for $d = (9+1)$ in Subsect. 10.2.5.
We need in this $d = (5+1)$ case Tables 1, 2, 3 from Ref. [7] and Table 2 from the contribution of N.S.M.B. in this proceedings. Table 1 is repeated in App. 10.5, Tables 2, 3 are repeated in this subsection.
In Table 10.1, and Table 2 (presented in the contribution of N.S.M.B. in this proceeding), the Clifford even "basis vectors" are presented as products of an even number of nilpotents ($\stackrel{ab}{(k)}$), two or zero, and the rest of projectors ($\stackrel{ab}{[k]}$), one or three, Eqs. (10.8, 10.10).
The "basis vectors" have in $d = (5+1)$ 16 members ($2^{\frac{d}{2}-1} \times 2^{\frac{d}{2}-1}$), which are chosen to be eigenvectors of all the Cartan subalgebra members, Eq. (10.7). These Clifford even "basis vectors" are either self adjoint (when they are products of only projectors, Eq. (10.10)), or they have their Hermitian conjugated partners within the same group.
We see in both above mentioned tables (10.1 and 2, presented in the contribution of the N.S.M.B. in this proceeding) that we can split the Clifford even "basis vectors", either ${}^{I}\hat{\mathcal{A}}^{m\dagger}_{f}$ or ${}^{II}\hat{\mathcal{A}}^{m\dagger}_{f}$, into two groups:

One group has the eigenvalues of $\mathcal{S}^{12} = \pm 1$. This represents bosons with internal momentum in transverse dimensions, $\mathcal{S}^{12} = \pm 1$. One notices that the members with $\mathcal{S}^{12} = 1$ have their Hermitian conjugated partners among those with $\mathcal{S}^{12} = -1$, and opposite. These "basis vectors" can transfer to fermions, the internal space of which is described by the Clifford odd "basis vectors", their $\mathcal{S}^{12} = 1$ or -1, respectively.

The second group has the eigenvalues of $\mathcal{S}^{12} = 0$. They have their Hermitian conjugated partners among those with $\mathcal{S}^{12} = 0$ (and $\mathcal{S}^{03}$ and $\mathcal{S}^{56}$ equal to ± 1) or are self adjoint (they are products of projectors only).

Table 10.1: This table is taken from Ref. [7] and the references therein. The Clifford even "basis vectors" ${}^{I}\hat{\mathcal{A}}^{m\dagger}_{f}$, each of them is the product of projectors and an even number of nilpotents, and each is the eigenvector of all the Cartan subalgebra members, $\mathcal{S}^{03}$, $\mathcal{S}^{12}$, $\mathcal{S}^{56}$, Eq. (10.7), are presented for $d = (5+1)$-dimensional case. Indexes m and f determine $2^{\frac{d}{2}-1} \times 2^{\frac{d}{2}-1}$ different members ${}^{I}\hat{\mathcal{A}}^{m\dagger}_{f}$. In the third column the "basis vectors" ${}^{I}\hat{\mathcal{A}}^{m\dagger}_{f}$ which are Hermitian conjugated partners to each other (and can therefore annihilate each other) are pointed out with the same symbol. For example, with $\star\star$ are equipped the first member with $m = 1$ and $f = 1$ and the last member of $f = 3$ with $m = 4$. The sign $\bigcirc$ denotes the Clifford even "basis vectors" which are self-adjoint $({}^{I}\hat{\mathcal{A}}^{m\dagger}_{f})^{\dagger} = {}^{I}\hat{\mathcal{A}}^{m'\dagger}_{f`}$. (It is obvious that $^{\dagger}$ has no meaning, since ${}^{I}\hat{\mathcal{A}}^{m\dagger}_{f}$ are self adjoint or are Hermitian conjugated partner to another ${}^{I}\hat{\mathcal{A}}^{m'\dagger}_{f`}$.) This table also presents the eigenvalues of the three commuting operators τ^3, τ^8 and τ' of the subgroups $SU(3) \times U(1)$.

f	m	*	${}^{I}\hat{\mathcal{A}}^{m\dagger}_{f}$	$\mathcal{S}^{03}$	$\mathcal{S}^{12}$	$\mathcal{S}^{56}$	τ^3	τ^8	τ'
I	1	$\star\star$	$\stackrel{03}{[+i]}\stackrel{12}{(+)}\stackrel{56}{(+)}$	0	1	1	$-\frac{1}{2}$	$-\frac{1}{2\sqrt{3}}$	$-\frac{2}{3}$
	2	$\triangle$	$\stackrel{03}{(-i)}\stackrel{12}{[-]}\stackrel{56}{(+)}$	$-i$	0	1	$-\frac{1}{2}$	$-\frac{3}{2\sqrt{3}}$	0
	3	$\ddagger$	$\stackrel{03}{(-i)}\stackrel{12}{(+)}\stackrel{56}{[-]}$	$-i$	1	0	-1	0	0
	4	$\bigcirc$	$\stackrel{03}{[+i]}\stackrel{12}{[-]}\stackrel{56}{[-]}$	0	0	0	0	0	0
II	1	$\bullet$	$\stackrel{03}{(+i)}\stackrel{12}{[+]}\stackrel{56}{(+)}$	i	0	1	$\frac{1}{2}$	$-\frac{1}{2\sqrt{3}}$	$-\frac{2}{3}$
	2	$\otimes$	$\stackrel{03}{[-i]}\stackrel{12}{(-)}\stackrel{56}{(+)}$	0	-1	1	$\frac{1}{2}$	$-\frac{3}{2\sqrt{3}}$	0
	3	$\bigcirc$	$\stackrel{03}{[-i]}\stackrel{12}{[+]}\stackrel{56}{[-]}$	0	0	0	0	0	0
	4	$\ddagger$	$\stackrel{03}{(+i)}\stackrel{12}{(-)}\stackrel{56}{[-]}$	i	-1	0	1	0	0
III	1	$\bigcirc$	$\stackrel{03}{[+i]}\stackrel{12}{[+]}\stackrel{56}{[+]}$	0	0	0	0	0	0
	2	$\odot\odot$	$\stackrel{03}{(-i)}\stackrel{12}{(-)}\stackrel{56}{[+]}$	$-i$	-1	0	0	$-\frac{1}{\sqrt{3}}$	$\frac{2}{3}$
	3	$\bullet$	$\stackrel{03}{(-i)}\stackrel{12}{[+]}\stackrel{56}{(-)}$	$-i$	0	-1	$-\frac{1}{2}$	$\frac{1}{2\sqrt{3}}$	$\frac{2}{3}$
	4	$\star\star$	$\stackrel{03}{[+i]}\stackrel{12}{(-)}\stackrel{56}{(-)}$	0	-1	-1	$\frac{1}{2}$	$\frac{1}{2\sqrt{3}}$	$\frac{2}{3}$
IV	1	$\odot\odot$	$\stackrel{03}{(+i)}\stackrel{12}{(+)}\stackrel{56}{[+]}$	i	1	0	0	$\frac{1}{\sqrt{3}}$	$-\frac{2}{3}$
	2	$\bigcirc$	$\stackrel{03}{[-i]}\stackrel{12}{[-]}\stackrel{56}{[+]}$	0	0	0	0	0	0
	3	$\otimes$	$\stackrel{03}{[-i]}\stackrel{12}{(+)}\stackrel{56}{(-)}$	0	1	-1	$-\frac{1}{2}$	$\frac{3}{2\sqrt{3}}$	0
	4	$\triangle$	$\stackrel{03}{(+i)}\stackrel{12}{[-]}\stackrel{56}{(-)}$	i	0	-1	$\frac{1}{2}$	$\frac{3}{2\sqrt{3}}$	0

Table 10.2 represets the Clifford odd "basis vectors", used in the *spin-charge-family* theory to describe internal space of fermions. The Clifford odd "basis vectors" have in $d = (5+1)$ 4 members $(2^{\frac{d}{2}-1})$ in each of 4 $(2^{\frac{d}{2}-1})$ families, presented in Table (10.2, 10.5) which are chosen to be the eigenvectors of all the Cartan subalgebra members, Eq. (10.7). Their Hermitian conjugated partners are presented (together with the Clifford odd and both Clifford even "basis vectors") in Table 10.5. The Clifford odd"basis vectors", and their Hermitian conjugated partners — they appear in a separate group — are presented as products of an odd number of nilpotents $(\stackrel{ab}{(k)})$, three or one, and the rest of projectors $(\stackrel{ab}{[k]})$, zero or two, Eqs. (10.8, 10.10).
We need to know the structure of the Clifford odd "basis vectors" and their Hermitian conjugated partners to be able to understand the description of the boson "basis vectors", ${}^{I}\hat{\mathcal{A}}^{m\dagger}_{f}$ and ${}^{II}\hat{\mathcal{A}}^{m\dagger}_{f}$, as the algebraic products of the Clifford odd "basis vectors" $\hat{b}^{m\dagger}_{f}$ and their Hermitian conjugated partners $\hat{b}^{m'}_{f'} = (\hat{b}^{m'\dagger}_{f'})^{\dagger}$. One can obtain the Clifford even "basis vectors", ${}^{I}\hat{\mathcal{A}}^{m\dagger}_{f}$ and ${}^{II}\hat{\mathcal{A}}^{m\dagger}_{f}$, as algebraic products of the Clifford odd "basis vectors" and their Hermitian conjugated partners, ${}^{I}\hat{\mathcal{A}}^{m\dagger}_{f} = \hat{b}^{m'\dagger}_{f`} *_A (\hat{b}^{m''\dagger}_{f`})^{\dagger}$, while ${}^{II}\hat{\mathcal{A}}^{m\dagger}_{f} = (\hat{b}^{m'\dagger}_{f`})^{\dagger} *_A \hat{b}^{m'\dagger}_{f``}$. One can check that $\hat{b}^{m'\dagger}_{f`} *_A (\hat{b}^{m''\dagger}_{f`})^{\dagger}$ applying on $\hat{b}^{m'''\dagger}_{f``}$ obey Eq. (10.19) and that $\hat{b}^{m'''\dagger}_{f``}$ applying on $(\hat{b}^{m'\dagger}_{f`})^{\dagger} *_A \hat{b}^{m'\dagger}_{f``}$ obey Eq. (10.22).
We see in Table 10.3 and in Table 10.4 the same number, $8 = \frac{1}{2} 2^{\frac{d=6}{2}-1} \times 2^{\frac{d=6}{2}-1}$, of the Clifford even "basis vectors" ${}^{I}\hat{\mathcal{A}}^{m\dagger}_{f}$ and ${}^{II}\hat{\mathcal{A}}^{m\dagger}_{f}$, each with $\mathcal{S}^{12} = \pm 1$: Four of each of them have $\mathcal{S}^{12} = +1$ and four have $\mathcal{S}^{12} = -1$.
If the handedness of the Clifford odd "basis vectors" is chosen to be the right handedness, $\Gamma^{(d)} = \prod_a (\sqrt{\eta^{aa}} \gamma^a) \cdot \begin{cases} (i)^{\frac{d}{2}}\,, & \text{for d even}\,, \\ (i)^{\frac{d-1}{2}}\,, & \text{for d odd}\,, \end{cases}$ then their Hermitian conjugated parters have left handedness for either $\mathcal{S}^{12} = +1$ and $\mathcal{S}^{12} = -1$, resembling left and right movers contributing to boson strings in *string theories* AII and BII.

Let us point out that to both groups of the Clifford even "basis vectors" appearing in Table 10.3 and in Table 10.4 all family members m *and all families* f *contribute*:
a. For ${}^{I}\hat{\mathcal{A}}^{m\dagger}_{f}$, all the family members m for a particular family f and their Hermitian conjugated partners contribute in $\hat{b}^{m'\dagger}_{f`} *_A (\hat{b}^{m''\dagger}_{f`})^{\dagger}$, using only half of possibilities, the other half contribute to $\mathcal{S}^{12} = 0$. Each family f' of $\hat{b}^{m'\dagger}_{f`} *_A (\hat{b}^{m''\dagger}_{f`})^{\dagger}$ generates the same eight Clifford even ${}^{I}\hat{\mathcal{A}}^{m\dagger}_{f}$ as are the ones presented in Table 10.3 for $f` = 1$.
b. For ${}^{II}\hat{\mathcal{A}}^{m\dagger}_{f}$, all the families f of a particular member m and their Hermitian conjugated partners contribute in $(\hat{b}^{m'\dagger}_{f`})^{\dagger} *_A \hat{b}^{m'\dagger}_{f`'}$, using only half of possibilities, the other half contribute to $\mathcal{S}^{12} = 0$. Each family member m' generates in $(\hat{b}^{m'\dagger}_{f`})^{\dagger} *_A \hat{b}^{m'\dagger}_{f`'}$ the same eight Clifford even ${}^{II}\hat{\mathcal{A}}^{m\dagger}_{f}$ as are the ones presented in Table 10.4 for $m' = 1$.

Table 10.2: This table is taken from Ref. [7]. The "basis vectors" $\hat{b}^{m\dagger}_{f}$ are presented for $d = (5+1)$-dimensional case. Each $\hat{b}^{m\dagger}_{f}$ is a product of projectors and of an odd number of nilpotents and is the "eigenvector" of all the Cartan subalgebra members, (S^{03}, S^{12}, S^{56}) and $(\tilde{S}^{03}, \tilde{S}^{12}, \tilde{S}^{56})$, Eq. (10.7), m marks the members of each family, while f determines the family quantum numbers (the eigenvalues of $(\tilde{S}^{03}, \tilde{S}^{12}, \tilde{S}^{56})$). This table also presents in the columns $(8^{th}, 9^{th}, 10^{th})$ the eigenvalues of the three commuting operators (τ^3, τ^8 and τ') of the subgroups $SU(3) \times U(1)$, $(\tau^3 := \frac{1}{2}(-S^{12} - iS^{03}), \tau^8 = \frac{1}{2\sqrt{3}}(-iS^{03} + S^{12} - 2S^{56}), \tau' = -\frac{1}{3}(-iS^{03} + S^{12} + S^{56}))$, as well as (in the last three columns) the corresponding $(\tilde{\tau}^3, \tilde{\tau}^8, \tilde{\tau}')$. $\Gamma^{(3+1)} = i\gamma^0\gamma^1\gamma^2\gamma^3$ is written in the 7^{th} column. $\Gamma^{(5+1)} = -1$ $(= -\gamma^0\gamma^1\gamma^2\gamma^3\gamma^5\gamma^6)$. Operators $\hat{b}^{m\dagger}_{f}$ and $\hat{b}^{m}_{f}$ fulfil the anti-commutation relations for the second quantized fermion fields.

f	m	$\hat{b}^{m\dagger}_{f}$	S^{03}	S^{12}	S^{56}	Γ^{3+1}	τ^3	τ^8	τ'	$\tilde{S}^{03}$	$\tilde{S}^{12}$	$\tilde{S}^{56}$	$\tilde{\tau}^3$	$\tilde{\tau}^8$	$\tilde{\tau}'$
I	1	$\stackrel{03}{(+i)}\stackrel{12}{[+]}\mid\stackrel{56}{[+]}$	$\frac{i}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	1	0	0	$-\frac{1}{2}$	$\frac{i}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2\sqrt{3}}$	$\frac{1}{6}$
	2	$\stackrel{03}{[-i]}\stackrel{12}{(-)}\mid\stackrel{56}{[+]}$	$-\frac{i}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	1	0	$-\frac{1}{\sqrt{3}}$	$\frac{1}{6}$	$\frac{i}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2\sqrt{3}}$	$\frac{1}{6}$
	3	$\stackrel{03}{[-i]}\stackrel{12}{[+]}\mid\stackrel{56}{(-)}$	$-\frac{i}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	-1	$-\frac{1}{2}$	$\frac{1}{2\sqrt{3}}$	$\frac{1}{6}$	$\frac{i}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2\sqrt{3}}$	$\frac{1}{6}$
	4	$\stackrel{03}{(+i)}\stackrel{12}{(-)}\mid\stackrel{56}{(-)}$	$\frac{i}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	-1	$\frac{1}{2}$	$\frac{1}{2\sqrt{3}}$	$\frac{1}{6}$	$\frac{i}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2\sqrt{3}}$	$\frac{1}{6}$
II	1	$\stackrel{03}{[+i]}\stackrel{12}{[+]}\mid\stackrel{56}{(+)}$	$\frac{i}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	1	0	0	$-\frac{1}{2}$	$-\frac{i}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	0	$-\frac{1}{\sqrt{3}}$	$\frac{1}{6}$
	2	$\stackrel{03}{(-i)}\stackrel{12}{(-)}\mid\stackrel{56}{(+)}$	$-\frac{i}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	1	0	$-\frac{1}{\sqrt{3}}$	$\frac{1}{6}$	$-\frac{i}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	0	$-\frac{1}{\sqrt{3}}$	$\frac{1}{6}$
	3	$\stackrel{03}{(-i)}\stackrel{12}{[+]}\mid\stackrel{56}{[-]}$	$-\frac{i}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	-1	$-\frac{1}{2}$	$\frac{1}{2\sqrt{3}}$	$\frac{1}{6}$	$-\frac{i}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	0	$-\frac{1}{\sqrt{3}}$	$\frac{1}{6}$
	4	$\stackrel{03}{[+i]}\stackrel{12}{(-)}\mid\stackrel{56}{[-]}$	$\frac{i}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	-1	$\frac{1}{2}$	$\frac{1}{2\sqrt{3}}$	$\frac{1}{6}$	$-\frac{i}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	0	$-\frac{1}{\sqrt{3}}$	$\frac{1}{6}$
III	1	$\stackrel{03}{[+i]}\stackrel{12}{(+)}\mid\stackrel{56}{[+]}$	$\frac{i}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	1	0	0	$-\frac{1}{2}$	$-\frac{i}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2\sqrt{3}}$	$\frac{1}{6}$
	2	$\stackrel{03}{(-i)}\stackrel{12}{[-]}\mid\stackrel{56}{[+]}$	$-\frac{i}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	1	0	$-\frac{1}{\sqrt{3}}$	$\frac{1}{6}$	$-\frac{i}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2\sqrt{3}}$	$\frac{1}{6}$
	3	$\stackrel{03}{(-i)}\stackrel{12}{(+)}\mid\stackrel{56}{(-)}$	$-\frac{i}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	-1	$-\frac{1}{2}$	$\frac{1}{2\sqrt{3}}$	$\frac{1}{6}$	$-\frac{i}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2\sqrt{3}}$	$\frac{1}{6}$
	4	$\stackrel{03}{[+i]}\stackrel{12}{[-]}\mid\stackrel{56}{(-)}$	$\frac{i}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	-1	$\frac{1}{2}$	$\frac{1}{2\sqrt{3}}$	$\frac{1}{6}$	$-\frac{i}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2\sqrt{3}}$	$\frac{1}{6}$
IV	1	$\stackrel{03}{(+i)}\stackrel{12}{(+)}\mid\stackrel{56}{(+)}$	$\frac{i}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	1	0	0	$-\frac{1}{2}$	$\frac{i}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	0	0	$-\frac{1}{2}$
	2	$\stackrel{03}{[-i]}\stackrel{12}{[-]}\mid\stackrel{56}{(+)}$	$-\frac{i}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	1	0	$-\frac{1}{\sqrt{3}}$	$\frac{1}{6}$	$\frac{i}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	0	0	$-\frac{1}{2}$
	3	$\stackrel{03}{[-i]}\stackrel{12}{(+)}\mid\stackrel{56}{[-]}$	$-\frac{i}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	-1	$-\frac{1}{2}$	$\frac{1}{2\sqrt{3}}$	$\frac{1}{6}$	$\frac{i}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	0	0	$-\frac{1}{2}$
	4	$\stackrel{03}{(+i)}\stackrel{12}{[-]}\mid\stackrel{56}{[-]}$	$\frac{i}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	-1	$\frac{1}{2}$	$\frac{1}{2\sqrt{3}}$	$\frac{1}{6}$	$\frac{i}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	0	0	$-\frac{1}{2}$

How γ^a and $\tilde{\gamma}^a$ apply on Clifford odd and even "basis vectors" Table 10.5, taken from Ref. [6], represents Clifford odd and Clifford even "basis vectors". One easily sees that the algebraic multiplication of any Clifford odd "basis vector" by γ^a leads to the corresponding Clifford even "basis vector" ${}^{II}\hat{\mathcal{A}}^{m\dagger}_{f}$, in agreement with Eq. (10.10). The algebraic multiplication of any Clifford odd "basis vector" by $\tilde{\gamma}^a$ leads to the corresponding Clifford even "basis vector" ${}^{I}\hat{\mathcal{A}}^{m\dagger}_{f}$.
Multiplying the first column of odd I $\hat{b}^{m\dagger}_{1}$ in Table 10.5 by γ^0: $\gamma^0 *_A \hat{b}^{m\dagger}_{1}$ leads to the third column of *even* II ${}^{II}\hat{\mathcal{A}}^{m\dagger}_{3}$, according to Eq. (10.10).
Multiplying the first column of odd I $\hat{b}^{m\dagger}_{1}$ in Table 10.5 by $\tilde{\gamma}^0$: $\tilde{\gamma^0} *_A \hat{b}^{m\dagger}_{1}$ leads to the third column of *even* I ${}^{I}\hat{\mathcal{A}}^{m\dagger}_{3}$, according to Eq. (10.10).

Table 10.3: The Clifford even "basis vectors" ${}^{I}\hat{\mathcal{A}}^{m\dagger}_{f}$, belonging to transverse momentum in internal space, $\mathcal{S}^{12}$ equal to 1, the first half ${}^{I}\hat{\mathcal{A}}^{m\dagger}_{f}$, and -1, the second half ${}^{I}\hat{\mathcal{A}}^{m\dagger}_{f}$, for $d = (5+1)$, are presented as algebraic products of the $f = 1$ family "basis vectors" $\hat{b}^{m'\dagger}_{1}$ and their Hermitian conjugated partners $(\hat{b}^{m''\dagger}_{1})^{\dagger}$: $\hat{b}^{m'\dagger}_{1} *_A (\hat{b}^{m''\dagger}_{1})^{\dagger}$. The Hermitian conjugated partners of two ${}^{I}\hat{\mathcal{A}}^{m\dagger}_{f}$ are marked with the same symbol. The Clifford even "basis vectors" ${}^{I}\hat{\mathcal{A}}^{m\dagger}_{f}$ are the product of one projector and two nilpotents, the Clifford odd "basis vectors" and their Hermitian conjugated partners are products of one nilpotent and two projectors or of three nilpotents. The Clifford even and Clifford odd objects are eigenvectors of all the corresponding Cartan subalgebra members, Eq. (10.7). There are $2^{\frac{6}{2}-1} \times 2^{\frac{6}{2}-1}$ algebraic products of $\hat{b}^{m'\dagger}_{1} *_A (\hat{b}^{m''\dagger}_{1})^{\dagger}$. The rest of 16 members present ${}^{I}\hat{\mathcal{A}}^{m\dagger}_{f}$ with $\mathcal{S}^{12} = 0$. The members $\hat{b}^{m'\dagger}_{f}$ together with their Hermitian conjugated partners of each of the four families, $f = (1,2,3,4)$, offers the same ${}^{I}\hat{\mathcal{A}}^{m\dagger}_{f}$ with $\mathcal{S}^{12} = \pm 1$ as the ones presented in this table. (And equivalently for $\mathcal{S}^{12} = 0$.)

$\mathcal{S}^{12}$	symbol	${}^{I}\hat{\mathcal{A}}^{m\dagger}_{f} =$ $\hat{b}^{m'\dagger}_{f`} *_A (\hat{b}^{m''\dagger}_{f`})^{\dagger}$
1	⋆⋆	${}^{I}\hat{\mathcal{A}}^{1\dagger}_{1} = \hat{b}^{1\dagger}_{1} *_A (\hat{b}^{4\dagger}_{1})^{\dagger}$ $\overset{03}{[+i]}\overset{12}{(+)}\overset{56}{(+)}$ $\overset{03}{(+i)}\overset{12}{[+]}\overset{56}{[+]} *_A \overset{03}{(-i)}\overset{12}{(+)}\overset{56}{(+)}$
1	‡	${}^{I}\hat{\mathcal{A}}^{3\dagger}_{1} = \hat{b}^{3\dagger}_{1} *_A (\hat{b}^{4\dagger}_{1})^{\dagger}$ $\overset{03}{(-i)}\overset{12}{(+)}\overset{56}{[-]}$ $\overset{03}{[-i]}\overset{12}{[+]}\overset{56}{(-)} *_A \overset{03}{(-i)}\overset{12}{(+)}\overset{56}{(+)}$
1	⊙⊙	${}^{I}\hat{\mathcal{A}}^{1\dagger}_{4} = \hat{b}^{1\dagger}_{1} *_A (\hat{b}^{2\dagger}_{1})^{\dagger}$ $\overset{03}{(+i)}\overset{12}{(+)}\overset{56}{[+]}$ $\overset{03}{(+i)}\overset{12}{[+]}\overset{56}{[+]} *_A \overset{03}{[-i]}\overset{12}{(+)}\overset{56}{[+]}$
1	⊗	${}^{I}\hat{\mathcal{A}}^{3\dagger}_{4} = \hat{b}^{3\dagger}_{1} *_A (\hat{b}^{2\dagger}_{1})^{\dagger}$ $\overset{03}{[-i]}\overset{12}{(+)}\overset{56}{(-)}$ $\overset{03}{[-i]}\overset{12}{[+]}\overset{56}{(-)} *_A \overset{03}{[-i]}\overset{12}{(+)}\overset{56}{[+]}$
−1	⊗	${}^{I}\hat{\mathcal{A}}^{2\dagger}_{2} = \hat{b}^{2\dagger}_{1} *_A (\hat{b}^{3\dagger}_{1})^{\dagger}$ $\overset{03}{[-i]}\overset{12}{(-)}\overset{56}{(+)}$ $\overset{03}{[-i]}\overset{12}{(-)}\overset{56}{[+]} *_A \overset{03}{[-i]}\overset{12}{[+]}\overset{56}{(+)}$
−1	‡	${}^{I}\hat{\mathcal{A}}^{4\dagger}_{2} = \hat{b}^{4\dagger}_{1} *_A (\hat{b}^{3\dagger}_{1})^{\dagger}$ $\overset{03}{(+i)}\overset{12}{(-)}\overset{56}{[-]}$ $\overset{03}{(+i)}\overset{12}{(-)}\overset{56}{(-)} *_A \overset{03}{[-i]}\overset{12}{[+]}\overset{56}{(+)}$
−1	⊙⊙	${}^{I}\hat{\mathcal{A}}^{2\dagger}_{3} = \hat{b}^{2\dagger}_{1} *_A (\hat{b}^{1\dagger}_{1})^{\dagger}$ $\overset{03}{(-i)}\overset{12}{(-)}\overset{56}{[+]}$ $\overset{03}{[-i]}\overset{12}{(-)}\overset{56}{[+]} *_A \overset{03}{(-i)}\overset{12}{[+]}\overset{56}{[+]}$
−1	⋆⋆	${}^{I}\hat{\mathcal{A}}^{4\dagger}_{3} = \hat{b}^{4\dagger}_{1} *_A (\hat{b}^{1\dagger}_{1})^{\dagger}$ $\overset{03}{[+i]}\overset{12}{(-)}\overset{56}{(-)}$ $\overset{03}{(+i)}\overset{12}{(-)}\overset{56}{(-)} *_A \overset{03}{(-i)}\overset{12}{[+]}\overset{56}{[+]}$

Table 10.4: The Clifford even "basis vectors" ${}^{II}\hat{\mathcal{A}}^{m\dagger}_{f}$, belonging to transverse momentum in internal space, $\mathcal{S}^{12}$ equal to 1, the first half ${}^{II}\hat{\mathcal{A}}^{m\dagger}_{f}$, and -1, the second half ${}^{II}\hat{\mathcal{A}}^{m\dagger}_{f}$, for $d = (5+1)$, are presented as algebraic products of the first, $m = 1$, member of "basis vectors" $\hat{b}^{m'=1\dagger}_{f'}$ and the Hermitian conjugated partners $(\hat{b}^{m'=1\dagger}_{f''})^{\dagger}$. The Hermitian conjugated partners of two ${}^{II}\hat{\mathcal{A}}^{m\dagger}_{f}$ are marked with the same symbol. The Clifford even "basis vectors" ${}^{II}\hat{\mathcal{A}}^{m\dagger}_{f}$ are the product of one projector and two nilpotents, the Clifford odd "basis vectors" and the Hermitian conjugated partners are products of one nilpotent and two projectors or of three nilpotents. Clifford even and Clifford odd objects are eigenvectors all the corresponding Cartan subalgebra members, Eq. (10.7). There are $2^{\frac{6}{2}-1} \times 2^{\frac{6}{2}-1}$ algebraic products of $\hat{b}^{m'\dagger}_{f'} *_A (\hat{b}^{m'\dagger}_{f''})^{\dagger}$, f' and f'' run over all four families. The rest of 16 members presents ${}^{II}\hat{\mathcal{A}}^{m\dagger}_{f}$ with $\mathcal{S}^{12} = 0$. The members $\hat{b}^{m'\dagger}_{f'}$ together with $(\hat{b}^{m'\dagger}_{f'}$ $m' = (1,2,3,4)$, offers the same ${}^{I}\hat{\mathcal{A}}^{m\dagger}_{f}$ with $\mathcal{S}^{12} = \pm 1$ as the ones presented in this table. (And equivalently for $\mathcal{S}^{12} = 0$.)

$\mathcal{S}^{12}$	symbol	${}^{II}\hat{\mathcal{A}}^{m\dagger}_{f} = (\hat{b}^{1\dagger}_{f'})^{\dagger} *_A \hat{b}^{1\dagger}_{f''}$
1	★★	${}^{II}\hat{\mathcal{A}}^{1\dagger}_{1} = (\hat{b}^{1\dagger}_{1})^{\dagger} *_A \hat{b}^{1\dagger}_{4}$ $\stackrel{03}{[-i]}\stackrel{12}{(+)}\stackrel{56}{(+)}$ $\stackrel{03}{(-i)}\stackrel{12}{[+]}\stackrel{56}{[+]} *_A \stackrel{03}{(+i)}\stackrel{12}{(+)}\stackrel{56}{(+)}$
1	⊙⊙	${}^{II}\hat{\mathcal{A}}^{3\dagger}_{1} = (\hat{b}^{1\dagger}_{2})^{\dagger} *_A \hat{b}^{1\dagger}_{4}$ $\stackrel{03}{(+i)}\stackrel{12}{(+)}\stackrel{56}{[-]}$ $\stackrel{03}{[+i]}\stackrel{12}{[+]}\stackrel{56}{(-)} *_A \stackrel{03}{(+i)}\stackrel{12}{(+)}\stackrel{56}{(+)}$
1	‡	${}^{II}\hat{\mathcal{A}}^{1\dagger}_{4} = (\hat{b}^{1\dagger}_{1})^{\dagger} *_A \hat{b}^{1\dagger}_{3}$ $\stackrel{03}{(-i)}\stackrel{12}{(+)}\stackrel{56}{[+]}$ $\stackrel{03}{(-i)}\stackrel{12}{[+]}\stackrel{56}{[+]} *_A \stackrel{03}{[+i]}\stackrel{12}{(+)}\stackrel{56}{[+]}$
1	⊗	${}^{II}\hat{\mathcal{A}}^{3\dagger}_{4} = (\hat{b}^{1\dagger}_{2})^{\dagger} *_A \hat{b}^{1\dagger}_{3}$ $\stackrel{03}{[+i]}\stackrel{12}{(+)}\stackrel{56}{(-)}$ $\stackrel{03}{[+i]}\stackrel{12}{[+]}\stackrel{56}{(-)} *_A \stackrel{03}{[+i]}\stackrel{12}{(+)}\stackrel{56}{[+]}$
−1	⊗	${}^{II}\hat{\mathcal{A}}^{2\dagger}_{2} = (\hat{b}^{1\dagger}_{3})^{\dagger} *_A \hat{b}^{1\dagger}_{2}$ $\stackrel{03}{[+i]}\stackrel{12}{(-)}\stackrel{56}{(+)}$ $\stackrel{03}{[+i]}\stackrel{12}{(-)}\stackrel{56}{[+]} *_A \stackrel{03}{[+i]}\stackrel{12}{[+]}\stackrel{56}{(+)}$
−1	⊗⊗	${}^{II}\hat{\mathcal{A}}^{4\dagger}_{2} = (\hat{b}^{1\dagger}_{4})^{\dagger} *_A \hat{b}^{1\dagger}_{2}$ $\stackrel{03}{(-i)}\stackrel{12}{(-)}\stackrel{56}{[-]}$ $\stackrel{03}{(-i)}\stackrel{12}{(-)}\stackrel{56}{(-)} *_A \stackrel{03}{[+i]}\stackrel{12}{[+]}\stackrel{56}{(+)}$
−1	‡	${}^{II}\hat{\mathcal{A}}^{2\dagger}_{3} = (\hat{b}^{1\dagger}_{3})^{\dagger} *_A \hat{b}^{1\dagger}_{1}$ $\stackrel{03}{(+i)}\stackrel{12}{(-)}\stackrel{56}{[+]}$ $\stackrel{03}{[+i]}\stackrel{12}{(-)}\stackrel{56}{[+]} *_A \stackrel{03}{(+i)}\stackrel{12}{[+]}\stackrel{56}{[+]}$
−1	★★	${}^{II}\hat{\mathcal{A}}^{4\dagger}_{3} = (\hat{b}^{1\dagger}_{4})^{\dagger} *_A \hat{b}^{1\dagger}_{1}$ $\stackrel{03}{[-i]}\stackrel{12}{(-)}\stackrel{56}{(-)}$ $\stackrel{03}{(-i)}\stackrel{12}{(-)}\stackrel{56}{(-)} *_A \stackrel{03}{(+i)}\stackrel{12}{[+]}\stackrel{56}{[+]}$

The opposite way, the algebraic multiplication of any Clifford even "basis vector" ${}^{II}\hat{\mathcal{A}}^{m\dagger}_{f}$ by γ^a leads to the corresponding Clifford odd "basis vector" $\hat{b}^{m\dagger}_{f'}$.
Multiplying the first column of *even* II ${}^{II}\hat{\mathcal{A}}^{m\dagger}_{1}$ in Table 10.5 by γ^0, for example, $\gamma^0 *_A {}^{II}\hat{\mathcal{A}}^{m\dagger}_{1}$, leads to the fourth column of odd I $\hat{b}^{m\dagger}_{1}$, according to Eq. (10.10).
And similarly if multiplying any Clifford even "basis vector" ${}^{I}\hat{\mathcal{A}}^{m\dagger}_{f}$ by $\tilde{\gamma}^a$ leads to the corresponding Clifford odd "basis vector" $\hat{b}^{m\dagger}_{f'}$.

These resemble a kind of supersymmetry: The same number of the Clifford odd and the Clifford even "basis vectors", and the simple relations between fermions and bosons.

It is useful to point out again, that the Clifford odd and the Clifford even "basis vectors" differ essentially in their properties:
i. While the Clifford odd "basis vectors" in even dimensional spaces appear in $2^{\frac{d}{2}-1}$ families, each family having $2^{\frac{d}{2}-1}$ members, and have their Hermitian conjugated partners in a separate group, again with $2^{\frac{d}{2}-1} \times 2^{\frac{d}{2}-1}$ contributions, appear the Clifford even "basis vectors" in even dimensional spaces in two groups, each with $2^{\frac{d}{2}-1} \times 2^{\frac{d}{2}-1}$ members, having the Hermitian conjugated partners within the same group. They have no families.
ii. The Clifford odd "basis vectors" in even dimensional spaces carry the eigenvalues of the Cartan subalgebra members, Eq. (10.7), $\pm\frac{i}{2}$ or $\pm\frac{1}{2}$. The Clifford even "basis vectors" in even dimensional spaces carry the eigenvalues of the Cartan subalgebra members, Eq. (10.7), $(\pm i, 0)$ or $(\pm 1, 0)$.

The two figures, Fig. 10.1 and Fig. 10.2. manifest the differences among the Clifford odd and the Clifford even "basis vectors".

In the case that the group $SO(5,1)$ — manifesting as $SU(3)\times U(1)$ and representing the colour group with quantum numbers (τ^3, τ^8) and the "fermion" group with the quantum number τ' — is embedded into $SO(13,1)$ the triplet would represent quarks (and antiquarks), and the singlet leptons (and antileptons).
The corresponding gauge fields, presented in Table 10.1 and Fig. 10.2, if belonging to the sextet, would transform the triplet of quarks among themselves, changing the colour and leaving the "fermion" quantum number equal to $\frac{1}{6}$.

We can see that ${}^{I}\hat{\mathcal{A}}^{m\dagger}_{3}$ with $(m = 2, 3, 4)$, if applied on the $SU(3)$ singlet $\hat{b}^{1\dagger}_{4}$ with $(\tau' = -\frac{1}{2}, \tau^3 = 0, \tau^8 = 0)$, transforms it to $\hat{b}^{m(=2,3,4)\dagger}_{4}$, respectively, which are members of the $SU(3)$ triplet. All these Clifford even "basis vectors" have τ' equal to $\frac{2}{3}$, changing correspondingly $\tau' = -\frac{1}{2}$ into $\tau' = \frac{1}{6}$ and bringing the needed values of τ^3 and τ^8.

In Table 10.1 we find $(6+4)$ Clifford even "basis vectors" ${}^{I}\hat{\mathcal{A}}^{m\dagger}_{f}$ with $\tau' = 0$. Six of them are Hermitian conjugated to each other — the Hermitian conjugated partners are denoted by the same geometric figure on the third column. Four of them are self-adjoint and correspondingly with $(\tau' = 0, \tau^3 = 0, \tau^8 = 0)$, denoted in the third column of Table 10.1 by ◯. The rest 6 Clifford even "basis vectors" belong to one

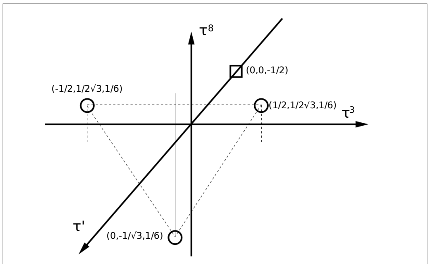

Fig. 10.1: This figure is taken from Ref. [7]. The representations of the subgroups $SU(3)$ and $U(1)$ of the group $SO(5,1)$, the properties of which appear in Tables (10.5, 10.2) for the Clifford odd "basis vectors", are presented. (τ^3, τ^8, τ') can be calculated if using the relations $\tau^3 = \frac{1}{2}(-S^{12} - iS^{03})$, $\tau^8 = \frac{1}{2\sqrt{3}}(S^{12} - iS^{03} - 2S^{56})$, $\tau' = -\frac{1}{3}(S^{12} - iS^{03} + S^{56})$. On the abscissa axis, on the ordinate axis and on the third axis, the eigenvalues of the superposition of the three Cartan subalgebra members, (τ^3, τ^8, τ'), are presented. One notices one triplet,denoted by ○ with the values $\tau' = \frac{1}{6}$, $(\tau^3 = -\frac{1}{2}, \tau^8 = \frac{1}{2\sqrt{3}}, \tau' = \frac{1}{6})$, $(\tau^3 = \frac{1}{2}, \tau^8 = \frac{1}{2\sqrt{3}}, \tau' = \frac{1}{6})$, $\tau^3 = 0, \tau^8 = -\frac{1}{\sqrt{3}}, \tau' = \frac{1}{6})$, respectively, and one singlet denoted by the square. $(\tau^3 = 0, \tau^8 = 0, \tau' = -\frac{1}{2})$. The triplet and the singlet appear in four families, with the family quantum numbers presented in the last three columns of Table 10.2.

.

triplet with $\tau' = \frac{2}{3}$ and (τ^3, τ^8) equal to $[(0, -\frac{1}{\sqrt{3}}), (-\frac{1}{2}, \frac{1}{2\sqrt{3}}), (\frac{1}{2}, \frac{1}{2\sqrt{3}})]$ and one antitriplet with $\tau' = -\frac{2}{3}$ and $((\tau^3, \tau^8)$ equal to $[(-\frac{1}{2}, -\frac{1}{2\sqrt{3}}), (\frac{1}{2}, -\frac{1}{2\sqrt{3}}), (0, \frac{1}{\sqrt{3}})]$.
Each triplet has Hermitian conjugated partners in anti-triplet and opposite. In Table 10.1 the Hermitian conjugated partners of the triplet and antitriplet are denoted by the same signum: $({}^{I}\hat{\mathcal{A}}^{1\dagger}_{1}, {}^{I}\hat{\mathcal{A}}^{4\dagger}_{3})$ by $\star\star$, $({}^{I}\hat{\mathcal{A}}^{1\dagger}_{2}, {}^{I}\hat{\mathcal{A}}^{3\dagger}_{3})$ by $\bullet$, and $({}^{I}\hat{\mathcal{A}}^{2\dagger}_{3}, {}^{I}\hat{\mathcal{A}}^{1\dagger}_{4})$ by $\odot\odot$.
The octet, two triplets and four singlets are presented in Fig. 10.2.

The same Fig. 10.2 represents also the Clifford even ${}^{II}\hat{\mathcal{A}}^{m\dagger}_{f}$, although their expressions with the products of two nilpotents and one projector or three projectors, presented on Table 10.5, differ from ${}^{I}\hat{\mathcal{A}}^{m\dagger}_{f}$.
The way of application of the Clifford even "basis vectors" ${}^{II}\hat{\mathcal{A}}^{m\dagger}_{f}$ on $\hat{b}^{m'\dagger}_{f`}$ follow Eqs. (10.21, 10.22), while the application of ${}^{I}\hat{\mathcal{A}}^{m\dagger}_{f}$ on $\hat{b}^{m'\dagger}_{f`}$ follow Eqs. (10.19, 10.20).

iii. In odd dimensional spaces, $d = 2n+1$, only half of "basis vectors" demonstrate the properties which they demonstrate in even dimensional spaces, the properties which empower the Clifford odd "basis vectors" to describe the internal space of fermion fields and the Clifford even "basis vectors" to describe the internal space of the corresponding gauge fields: This half belongs to $d' = 2n$ and does demonstrate these properties. The other half, obtained from the first half by the application of $S^{0\,2n+1}$, this second half of the Clifford odd "basis vectors", although anticommuting, demonstrate properties of the Clifford even "basis vectors", and the second half of the Clifford even "basis vectors", although commuting, demonstrate properties of the Clifford odd "basis vectors" in even dimensional spaces:

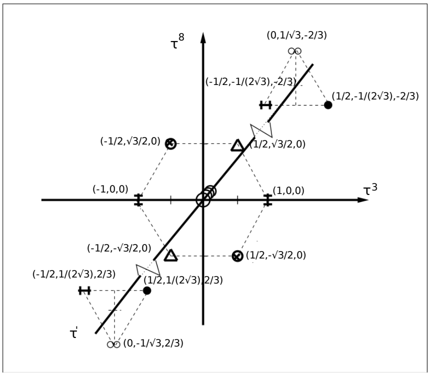

Fig. 10.2: This figure is taken from Ref. [7]. The Clifford even "basis vectors" ${}^{I}\hat{\mathcal{A}}^{m\dagger}_{f}$ in the case that $d = (5+1)$ are presented by the eigenvalues of the commuting operators of the subgroups $SU(3)$ and $U(1)$ of the group $SO(5,1)$: ($\tau^3 = \frac{1}{2}(-\mathcal{S}^{12} - i\mathcal{S}^{03})$, $\tau^8 = \frac{1}{2\sqrt{3}}(\mathcal{S}^{12} - i\mathcal{S}^{03} - 2\mathcal{S}^{56})$, $\tau' = -\frac{1}{3}(\mathcal{S}^{12} - i\mathcal{S}^{03} + \mathcal{S}^{56})$). Their properties appear in Table 10.1. The abscissa axis carries the eigenvalues of τ^3, the ordinate axis carries the eigenvalues of τ^8 and the third axis carries the eigenvalues of τ'. One notices four singlets with ($\tau^3 = 0, \tau^8 = 0, \tau' = 0$), denoted by $\bigcirc$, representing four self adjoint Clifford even "basis vectors" ${}^{I}\hat{\mathcal{A}}^{m\dagger}_{f}$, with $(f = 1, m = 4)$, $(f = 2, m = 3)$, $(f = 3, m = 1)$, $(f = 4, m = 2)$, one sextet of three pairs, Hermitian conjugated to each other, with $\tau' = 0$, denoted by $\triangle$ (${}^{I}\hat{\mathcal{A}}^{2\dagger}_{1}$ with ($\tau' = 0, \tau^3 = -\frac{1}{2}, \tau^8 = -\frac{3}{2\sqrt{3}}$) and ${}^{I}\hat{\mathcal{A}}^{4\dagger}_{4}$ with ($\tau' = 0, \tau^3 = \frac{1}{2}, \tau^8 = \frac{3}{2\sqrt{3}}$)), by $\ddagger$ (${}^{I}\hat{\mathcal{A}}^{3\dagger}_{1}$ with ($\tau' = 0, \tau^3 = -1, \tau^8 = 0$) and ${}^{I}\hat{\mathcal{A}}^{4\dagger}_{2}$ with $\tau' = 0, \tau^3 = 1, \tau^8 = 0$)), and by $\otimes$ (${}^{I}\hat{\mathcal{A}}^{2\dagger}_{2}$ with ($\tau' = 0, \tau^3 = \frac{1}{2}, \tau^8 = -\frac{3}{2\sqrt{3}}$) and ${}^{I}\hat{\mathcal{A}}^{3\dagger}_{4}$ with ($\tau' = 0, \tau^3 = -\frac{1}{2}, \tau^8 = \frac{3}{2\sqrt{3}}$)), and one triplet, denoted by $\star\star$ (${}^{I}\hat{\mathcal{A}}^{4\dagger}_{3}$ with ($\tau' = \frac{2}{3}, \tau^3 = \frac{1}{2}, \tau^8 = \frac{1}{2\sqrt{3}}$)), by $\bullet$ (${}^{I}\hat{\mathcal{A}}^{3\dagger}_{3}$ with ($\tau' = \frac{2}{3}, \tau^3 = -\frac{1}{2}, \tau^8 = \frac{1}{2\sqrt{3}}$)), and by $\odot\odot$ (${}^{I}\hat{\mathcal{A}}^{2\dagger}_{3}$ with ($\tau' = \frac{2}{3}, \tau^3 = 0, \tau^8 = -\frac{1}{\sqrt{3}}$)), as well as one antitriplet, Hermitian conjugated to triplet, denoted by $\star\star$ (${}^{I}\hat{\mathcal{A}}^{1\dagger}_{1}$ with ($\tau' = -\frac{2}{3}, \tau^3 = -\frac{1}{2}, \tau^8 = -\frac{1}{2\sqrt{3}}$)), by $\bullet$ (${}^{I}\hat{\mathcal{A}}^{1\dagger}_{2}$ with ($\tau' = -\frac{2}{3}, \tau^3 = \frac{1}{2}, \tau^8 = -\frac{1}{2\sqrt{3}}$)), and by $\odot\odot$ (${}^{I}\hat{\mathcal{A}}^{4\dagger}_{1}$ with ($\tau' = -\frac{2}{3}, \tau^3 = 0, \tau^8 = \frac{1}{\sqrt{3}}$)).

The still anticommuting Clifford odd "basis vectors" (the Clifford even operators $S^{0\,2n+1}$ do not change either oddness or eveness of the "basis vectors") appear in two separate groups with $2^{\frac{2n}{2}-1} \times 2^{\frac{2n}{2}-1}$, each with their Hermitian conjugated partners within the same group having no families; and still commuting Clifford even "basis vectors" appear in $2^{\frac{2n}{2}-1}$ families, each with $2^{\frac{2n}{2}-1}$ members, having their Hermitian conjugated partners $2^{\frac{2n}{2}-1} \times 2^{\frac{2n}{2}-1}$ in a separate group.

These secoind half of "basis vectors" obviously resemble properties of the internal spaces of the ghost scalar fields, used in the quantum field theory to make contributions of the Feynman diagrams finite, Ref. [8].

10.2.5 Relations between $SO(9,1)$ description of the internal spaces in *string theories* and with the Clifford even and odd "basis vectors"

In this subsection the first step is done to compare the properties of the Clifford odd and the Clifford even "basis vectors" with the properties of the *string theories* in the case of $d = (9+1)$.
Let us first repeat what we learned about the Clifford even and the Clifford odd "basis vectors" in even dimensional spaces.

There are in $d = 2(2n+1)$ and $d = 4n$ dimensional spaces $2^{\frac{d}{2}-1}$ Clifford odd families, each family having $2^{\frac{d}{2}-1}$ members. The Clifford odd "basis vectors" have their Hermitian conjugated partners in a separate group of $2^{\frac{d}{2}-1}$ families with $2^{\frac{d}{2}-1}$ members. In a tensor product with the basis in ordinary space the Clifford odd "basis vectors" together with their Hermitian conjugated partners form creations and annihilation operators, which fulfil on the vacuum state the postulates of the second quantized fermion fields, manifesting all the properties of the fermion fields [7], Eq. (34) and Subsect. 2.4.
There are in even dimensional spaces two times $2^{\frac{d}{2}-1} \times 2^{\frac{d}{2}-1}$ Clifford even basis vectors, with their Hermitian conjugated partners within the same group. In a tensor product with the basis in ordinary space the Clifford even "basis vectors" form creations and annihilation operators, which fulfil the postulates of the second quantized boson fields, manifesting all the properties of the boson fields to the corresponding fermion fields [7], Subsect. 2.4.
Both are represented by the points in the ordinary space.
Let us generalise to any even d what we learned about the description of the Clifford even 'basis vectors" with the algebraic products of the Clifford odd "basis vectors" and their Hermitian conjugated partners in Subsect. 10.2.4 when we treated the case $d = (5+1)$.
Looking at the properties of both kinds of the Clifford even "basis vectors", ${}^{I}\hat{\mathcal{A}}^{m\dagger}_{f}$ and ${}^{II}\hat{\mathcal{A}}^{m\dagger}_{f}$, manifesting momentum in only transverse dimensions (with $\mathcal{S}^{ab}$ not equal $\mathcal{S}^{03}$), we found in Subsect. 10.2.4, in Table 10.3 and in Table 10.4 that to both groups of the Clifford even "basis vectors" all family members m and all families f contribute:
a. To ${}^{I}\hat{\mathcal{A}}^{m\dagger}_{f}$, all the family members m for a particular family f and their Hermitian conjugated partners contribute in $\hat{b}^{m'\dagger}_{f`} *_A (\hat{b}^{m''\dagger}_{f`})^\dagger$, using only half of possibilities ($\frac{1}{2} \times 2^{\frac{d}{2}-1} \times 2^{\frac{d}{2}-1}$), the other half possibilities contribute to $\mathcal{S}^{12} = 0$. Each family f' of $\hat{b}^{m'\dagger}_{f`} *_A (\hat{b}^{m''\dagger}_{f`})^\dagger$ generates the same eight Clifford even ${}^{I}\hat{\mathcal{A}}^{m\dagger}_{f}$ as are the ones presented in Table 10.3 for $f` = 1$.
b. To ${}^{II}\hat{\mathcal{A}}^{m\dagger}_{f}$, all the families $f`$ of a particular member $m'$ and their Hermitian conjugated partners contribute in $(\hat{b}^{m'\dagger}_{f`})^\dagger *_A \hat{b}^{m'\dagger}_{f`'}$, using only half of possibilities, the other half contribute to $\mathcal{S}^{12} = 0$. Each family member $m'$ generates in $(\hat{b}^{m'\dagger}_{f`})^\dagger *_A \hat{b}^{m'\dagger}_{f`'}$ the same eight Clifford even ${}^{II}\hat{\mathcal{A}}^{m\dagger}_{f}$ as are the ones presented in Table 10.4 for $m' = 1$.

Let us now try to relate the description of the internal spaces of bosons described by the Clifford even "basis vectors", as explained in Subsect. 10.2.4 and the way how *string theories* consider the IIA or IIB superstring model for closed RNS strings as presented in Ref. [15] in Subsect. 11.6.3, where the spectrum is described.
We find in $d = (9+1)$, according to what it is discussed in this subsection, in the case that we are interested only on those internal degrees of freedom of the Clifford even basis vectors of each of the two kinds, ${}^{I}\hat{\mathcal{A}}^{m\dagger}_{f}$ and ${}^{II}\hat{\mathcal{A}}^{m\dagger}_{f}$, which manifest momentum in only transverse dimensions (with $\mathcal{S}^{ab}$ not equal $\mathcal{S}^{03}$), $\frac{1}{2} \times 2^{\frac{d=10}{2}-1} \times 2^{\frac{d=10}{2}-1} = 8 \times 16 = 128$ of ${}^{I}\hat{\mathcal{A}}^{m\dagger}_{f}$ and 128 of ${}^{II}\hat{\mathcal{A}}^{m\dagger}_{f}$, together 256 of both kinds of the Clifford even "basis vectors", representing the boson fields. These are also possibilities suggested in Ref. [15] for closed strings in $d = (9+1)$; for the left-right movers or right-left movers forming the closed boson strings of AII and BII kind, manifesting the momentum in only transverse dimensions they found 256 possibilities.

10.3 Conclusions

Let be repeated: In a long series of works [1–8] one of the authors in collaboration with another author and other collaborators, has found the phenomenological success with the model named the *spin-charge-family* theory with the properties: The creation and annihilation operators for fermion and boson fields are described as tensor products of the Clifford odd (for fermions) and the Clifford even (for bosons) "basis vectors" and basis in ordinary space, explaining the second quantization postulates. The theory, assuming in $d \geq (13+1)$ a simple starting action with fermions which interact with gravity only, offers the explanation for all the assumptions of the *standard model* before the electroweak break: For the properties of quarks and leptons and antiquarks and antileptons, for the corresponding gauge fields, for the appearace of the scalar higgs and Yukawa couplings, as well as for several cosmological observations, making several predictions.
Since the *strings theories* are offering the way for explaining so far observed fermion and boson second quantized fields, with gravity included, by offering the renormalizability of the theory by extending the point fermions and bosons into strings and by offering the supersymmetry among fermions and bosons, we expect that in the low energy limit the *string theories* coincide with our predictions provided that we extend points in the ordinary space to strings, hoping that this would help to solve the problem of renormalisability of the *spin-charge-family* theory.
What we present in this contribution is the first step towards relating the ŝpin-charge-family theory with the *string theories*. This first step is promising: We represent the two groups of the Clifford even "basis vectors", ${}^{I}\hat{\mathcal{A}}^{m\dagger}_{f}$ and ${}^{II}\hat{\mathcal{A}}^{m\dagger}_{f}$, as the algebraic products of the Clifford odd "basis vectors" and their Hermitian conjugated partners.
The products $\hat{b}^{m'\dagger}_{f'} *_A (\hat{b}^{m''\dagger}_{f'})^{\dagger}$ represent ${}^{I}\hat{\mathcal{A}}^{m\dagger}_{f}$ and $(\hat{b}^{m'\dagger}_{f'})^{\dagger} *_A \hat{b}^{m'\dagger}_{f'}$ represent ${}^{II}\hat{\mathcal{A}}^{m\dagger}_{f}$. The discussions in Subsect. 10.2.4 and Subsect.10.2.5 demonstrate that in both cases *all the families and family members participate* when describing the Clifford even "basis vectors", either ${}^{I}\hat{\mathcal{A}}^{m\dagger}_{f}$ or ${}^{I}\hat{\mathcal{A}}^{m\dagger}_{f}$, in the case that both boson

fields demonstrate spins in the transverse dimensions only ($\mathcal{S}^{03} = 0$, the rest of Cartan subalgebra members $\mathcal{S}^{ab}$, Eq.(10.7), are in general non zero and can represent $\mathcal{S}^{12} = \pm 1$ as well as charges in adjoint representations). But only half products of $\hat{b}^{m'\dagger}_{f`}$ and $(\hat{b}^{m''\dagger}_{f`'})^\dagger$ are involved, the other half determine the cases when $\mathcal{S}^{03}$ is nonzero.
In Subsect. 10.2.5, we find in the case $d = (9+1)$, and that we are interested only on those internal degrees of freedom of the Clifford even basis vectors of each of the two kinds, ${}^{I}\hat{\mathcal{A}}^{m\dagger}_{f}$ and ${}^{II}\hat{\mathcal{A}}^{m\dagger}_{f}$, which manifest momentum in only transverse dimensions (with $\mathcal{S}^{ab}$ not all equal to $\mathcal{S}^{03} = 0$), $\frac{1}{2} \times 2^{\frac{d=10}{2}-1} \times 2^{\frac{d=10}{2}-1} = 8 \times 16 = 128$ possibilities of ${}^{I}\hat{\mathcal{A}}^{m\dagger}_{f}$ and 128 possibilities of ${}^{II}\hat{\mathcal{A}}^{m\dagger}_{f}$, together 256 members of both kinds of the Clifford even "basis vectors", representing the boson fields. These are also possibilities suggested in Ref. [15] for closed strings in $d = (9+1)$.

The algebraic products $\hat{b}^{m'\dagger}_{f`} *_A (\hat{b}^{m''\dagger}_{f`})^\dagger$ *can be interpreted as left and right movers and* $(\hat{b}^{m'\dagger}_{f`})^\dagger *_A \hat{b}^{m''\dagger}_{f`}$ *as right and left movers of the closed boson strings.*

Let us add that the relation among $SO(n)$ *and all the rest groups is known for a long time [1,16,17], but not in this way as presented in this paper and in [7,8]. This way offers, to our understanding, the new way towards understanding the second quantized fermion and boson fields, with the gravity included.*

Still a hard work is needed to make the next step towards relating the *string theories* and the *spin-charge-family* theory. But since the description of the internal spaces of fermion and boson fields with the Clifford odd and Clifford even "basis vectors", respectively, is simple and well defined, it might bring a new understanding of the theory of our world.

10.4 Useful relations

This appendix, taken from Ref. [7], contains some helpful relations needed for the reader of this paper. For more detailed explanations and for proofs, the reader is kindly asked to read [6] and the references therein.
For fermions, the operator of handedness Γ^d is determined as follows:

$$\Gamma^{(d)} = \prod_a (\sqrt{\eta^{aa}} \gamma^a) \cdot \begin{cases} (i)^{\frac{d}{2}}\,, & \text{for d even}\,, \\ (i)^{\frac{d-1}{2}}\,, & \text{for d odd}\,, \end{cases} \tag{10.23}$$

The vacuum state for the Clifford odd "basis vectors", $|\psi_{oc}>$, is defined as

$$|\psi_{oc}> = \sum_{f=1}^{2^{\frac{d}{2}-1}} \hat{b}^{m}_{f} {}_{*_A} \hat{b}^{m\dagger}_{f} \,|\,1\,> \,. \tag{10.24}$$

Taking into account that the Clifford objects γ^a and $\tilde{\gamma}^a$ fulfil relations of Eq. 10.5, one obtains beside the relations presented in Eq. (10.10) some new ones ones.

$$
\begin{aligned}
&\stackrel{ab}{(k)}\stackrel{ab}{(-k)} = \eta^{aa} \stackrel{ab}{[k]}\,, \quad \stackrel{ab}{(-k)}\stackrel{ab}{(k)}= \eta^{aa} \stackrel{ab}{[-k]}\,, \quad \stackrel{ab}{(k)}\stackrel{ab}{[k]}= 0\,, \quad \stackrel{ab}{(k)}\stackrel{ab}{[-k]}=\stackrel{ab}{(k)}\,,\\
&\stackrel{ab}{(-k)}\stackrel{ab}{[k]} = \stackrel{ab}{(-k)}\,, \quad \stackrel{ab}{[k]}\stackrel{ab}{(k)}=\stackrel{ab}{(k)}\,, \quad \stackrel{ab}{[k]}\stackrel{ab}{(-k)}= 0\,, \quad \stackrel{ab}{[k]}\stackrel{ab}{[-k]}= 0\,,\\
&\stackrel{ab}{(\tilde{k})}\stackrel{ab}{(k)} = 0\,, \quad \stackrel{ab}{(\tilde{k})}\stackrel{ab}{(-k)}= -i\eta^{aa} \stackrel{ab}{[-k]}\,, \quad \stackrel{ab}{\widetilde{(-k)}}\stackrel{ab}{(k)}= -i\eta^{aa} \stackrel{ab}{[k]}\,, \quad \stackrel{ab}{(\tilde{k})}\stackrel{ab}{[k]}= i \stackrel{ab}{(k)}\,,\\
&\stackrel{ab}{(\tilde{k})}\stackrel{ab}{[-k]} = 0\,, \quad \stackrel{ab}{\widetilde{(-k)}}\stackrel{ab}{[k]}= 0\,, \quad \stackrel{ab}{\widetilde{(-k)}}\stackrel{ab}{[-k]}= i \stackrel{ab}{(-k)}\,, \quad \stackrel{ab}{[\tilde{k}]}\stackrel{ab}{(k)}=\stackrel{ab}{(k)}\,,\\
&\stackrel{ab}{[\tilde{k}]}\stackrel{ab}{(-k)} = 0\,, \quad \stackrel{ab}{[\tilde{k}]}\stackrel{ab}{[k]}= 0\,, \quad \stackrel{ab}{\widetilde{[-k]}}\stackrel{ab}{[k]}=\stackrel{ab}{[k]}\,, \quad \stackrel{ab}{[\tilde{k}]}\stackrel{ab}{[-k]}=\stackrel{ab}{[-k]}\,,
\end{aligned}
\tag{10.25}
$$

The algebraic multiplication among $\stackrel{ab}{(\tilde{k})}$ and $\stackrel{ab}{[\tilde{k}]}$ goes as in the case of $\stackrel{ab}{(k)}$ and $\stackrel{ab}{[k]}$

$$
\begin{aligned}
&\stackrel{ab}{(\tilde{k})}\stackrel{ab}{[\tilde{k}]} = 0\,, \quad \stackrel{ab}{[\tilde{k}]}\stackrel{ab}{(\tilde{k})}=\stackrel{ab}{(\tilde{k})}\,, \quad \stackrel{ab}{(\tilde{k})}\stackrel{ab}{\widetilde{[-k]}}=\stackrel{ab}{(\tilde{k})}\,, \quad \stackrel{ab}{[\tilde{k}]}\stackrel{ab}{\widetilde{(-k)}}= 0\,,\\
&\stackrel{ab}{\widetilde{(-k)}}\stackrel{ab}{(\tilde{k})} = \eta^{aa} \stackrel{ab}{[-k]}\,, \quad \stackrel{ab}{\widetilde{(-k)}}\stackrel{ab}{\widetilde{[-k]}}= 0\,,
\end{aligned}
\tag{10.26}
$$

One can further find that

$$
\begin{aligned}
&S^{ac} \stackrel{ab}{(k)}\stackrel{cd}{(k)} = -\frac{i}{2}\eta^{aa}\eta^{cc} \stackrel{ab}{[-k]}\stackrel{cd}{[-k]}\,, \quad S^{ac} \stackrel{ab}{[k]}\stackrel{cd}{[k]}= \frac{i}{2} \stackrel{ab}{(-k)}\stackrel{cd}{(-k)}\,,\\
&S^{ac} \stackrel{ab}{(k)}\stackrel{cd}{[k]} = -\frac{i}{2}\eta^{aa} \stackrel{ab}{[-k]}\stackrel{cd}{(-k)}\,, \quad S^{ac} \stackrel{ab}{[k]}\stackrel{cd}{(k)}= \frac{i}{2}\eta^{cc} \stackrel{ab}{(-k)}\stackrel{cd}{[-k]}\ .
\end{aligned}
\tag{10.27}
$$

10.5 Some properties of the Clifford odd and even "basis vectors" are presented

In this App. the properties of the Clifford odd and Clifford even "basis vectors" for the case that $d = (5+1)$ are presented in the table 10.5, taken from Ref. [7]. Table 10.5 presents the 64 $(= 2^{d=6})$ "eigenvectors" of the Cartan subalgebra members of the Lorentz algebra, S^{ab} and $\mathcal{S}^{ab}$, Eq. (10.7).

The Clifford odd "basis vectors" — they appear in 4 $(= 2^{\frac{d=6}{2}-1})$ families, each family has 4 members — are products of an odd number of nilpotents, either of three or one. They appear in the group named $odd\, I\, \hat{b}^{m\dagger}_{f}$. Their Hermitian conjugated partners appear in the second group named $odd\ II\ \hat{b}^{m}_{f}$. Within each of these two groups the members are mutually orthogonal (which can be checked by using Eq. (10.25)); $\hat{b}^{m\dagger}_{f} *_A \hat{b}^{m'\dagger}_{f'} = 0$ for all (m, m', f, f'). Equivalently, $\hat{b}^{m}_{f} *_A \hat{b}^{m'}_{f'} = 0$ for all (m, m', f, f').
The Clifford even "basis vectors" are products of an even number of nilpotents — of either two or none in this case. They are presented in Table 10.5 in two groups, each with 16 $(= 2^{\frac{d=6}{2}-1} \times 2^{\frac{d=6}{2}-1})$ members, as *even* I $\mathcal{A}^{m\dagger}_{f}$ and *even* II $\mathcal{A}^{m\dagger}_{f}$. One can easily check, using Eq. (10.25), that the algebraic product ${}^{I}\mathcal{A}^{m\dagger}_{f} *_A {}^{II}\mathcal{A}^{m'\dagger}_{f'} = 0 = {}^{II}\mathcal{A}^{m\dagger}_{f} *_A {}^{I}\mathcal{A}^{m'\dagger}_{f'}, \forall\, (m, m', f.f')$, Eq. (10.17). An overview of the Clifford even

Table 10.5: $2^d = 64$ "eigenvectors" of the Cartan subalgebra of the Clifford odd and even algebras — the superposition of odd and even products of γ^a's — in $d = (5+1)$-dimensional space are presented, divided into four groups. The first group, odd I, is chosen to represent "basis vectors", named $\hat{b}^{m\dagger}_f$, appearing in $2^{\frac{d}{2}-1} = 4$ "families" ($f = 1, 2, 3, 4$), each "family" with $2^{\frac{d}{2}-1} = 4$ "family" members ($m = 1, 2, 3, 4$). The second group, odd II, contains Hermitian conjugated partners of the first group for each family separately, $\hat{b}^m_f = (\hat{b}^{m\dagger}_f)^\dagger$. Either odd I or odd II are products of an odd number of nilpotents (one or three) and projectors (two or none). The "family" quantum numbers of $\hat{b}^{m\dagger}_f$, that is the eigenvalues of $(\tilde{S}^{03}, \tilde{S}^{12}, \tilde{S}^{56})$, are for the first *odd I* group appearing above each "family", the quantum numbers of the family members (S^{03}, S^{12}, S^{56}) are written in the last three columns. For the Hermitian conjugated partners of *odd I*, presented in the group *odd II*, the quantum numbers (S^{03}, S^{12}, S^{56}) are presented above each group of the Hermitian conjugated partners, the last three columns tell eigenvalues of $(\tilde{S}^{03}, \tilde{S}^{12}, \tilde{S}^{56})$. The two groups with the even number of γ^a's, *even I* and *even II*, each group has their Hermitian conjugated partners within its group, have the quantum numbers f, that is the eigenvalues of $(\tilde{S}^{03}, \tilde{S}^{12}, \tilde{S}^{56})$, written above column of four members, the quantum numbers of the members, (S^{03}, S^{12}, S^{56}), are written in the last three columns. To find the quantum numbers of $(\mathcal{S}^{03}, \mathcal{S}^{12}, \mathcal{S}^{56})$ one has to take into account that $\mathcal{S}^{ab} = S^{ab} + \tilde{S}^{ab}$.

"basis vectors" $(\tilde{S}^{03}, \tilde{S}^{12}, \tilde{S}^{56})$	m →	f = 1 $(\frac{i}{2}, -\frac{1}{2}, -\frac{1}{2})$	f = 2 $(-\frac{i}{2}, -\frac{1}{2}, \frac{1}{2})$	f = 3 $(-\frac{i}{2}, \frac{1}{2}, -\frac{1}{2})$	f = 4 $(\frac{i}{2}, \frac{1}{2}, \frac{1}{2})$	S^{03}	S^{12}	S^{56}
odd I $\hat{b}^{m\dagger}_f$		03 12 56	03 12 56	03 12 56	03 12 56			
	1	(+i)[+][+]	[+i][+](+)	[+i](+)[+]	(+i)(+)(+)	$\frac{i}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
	2	[−i](−)[+]	(−i)(−)(+)	(−i)[−][+]	[−i][−](+)	$-\frac{i}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$
	3	[−i][+](−)	(−i)[+][−]	(−i)(+)(−)	[−i](+)[−]	$-\frac{i}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$
	4	(+i)(−)(−)	[+i](−)[−]	[+i][−](−)	(+i)[−][−]	$\frac{i}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$
(S^{03}, S^{12}, S^{56})	→	$(-\frac{i}{2}, \frac{1}{2}, \frac{1}{2})$ 03 12 56	$(\frac{i}{2}, \frac{1}{2}, -\frac{1}{2})$ 03 12 56	$(\frac{i}{2}, -\frac{1}{2}, \frac{1}{2})$ 03 12 56	$(-\frac{i}{2}, -\frac{1}{2}, -\frac{1}{2})$ 03 12 56	$\tilde{S}^{03}$	$\tilde{S}^{12}$	$\tilde{S}^{56}$
odd II $\hat{b}^m_f$	1	(−i)[+][+]	[+i][+](−)	[+i](−)[+]	(−i)(−)(−)	$-\frac{i}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$
	2	[−i](+)[+]	(+i)(+)(−)	(+i)[−][+]	[−i][−](−)	$\frac{i}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$
	3	[−i][+](+)	(+i)[+][−]	(+i)(−)(+)	[−i](−)[−]	$\frac{i}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$
	4	(−i)(+)(+)	[+i](+)[−]	[+i][−](+)	(−i)[−][−]	$-\frac{i}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
$(\tilde{S}^{03}, \tilde{S}^{12}, \tilde{S}^{56})$	→	$(-\frac{i}{2}, \frac{1}{2}, \frac{1}{2})$ 03 12 56	$(\frac{i}{2}, -\frac{1}{2}, \frac{1}{2})$ 03 12 56	$(-\frac{i}{2}, -\frac{1}{2}, -\frac{1}{2})$ 03 12 56	$(\frac{i}{2}, \frac{1}{2}, -\frac{1}{2})$ 03 12 56	S^{03}	S^{12}	S^{56}
even I ${}^I\mathcal{A}^m_f$	1	[+i](+)(+)	(+i)[+](+)	[+i][+][+]	(+i)(+)[+]	$\frac{i}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
	2	(−i)[−](+)	[−i](−)(+)	(−i)(−)[+]	[−i][−][+]	$-\frac{i}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$
	3	(−i)(+)[−]	[−i][+][−]	(−i)[+](−)	[−i](+)(−)	$-\frac{i}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$
	4	[+i][−][−]	(+i)(−)[−]	[+i](−)(−)	(+i)[−](−)	$\frac{i}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$
$(\tilde{S}^{03}, \tilde{S}^{12}, \tilde{S}^{56})$	→	$(\frac{i}{2}, \frac{1}{2}, \frac{1}{2})$ 03 12 56	$(-\frac{i}{2}, -\frac{1}{2}, \frac{1}{2})$ 03 12 56	$(\frac{i}{2}, -\frac{1}{2}, -\frac{1}{2})$ 03 12 56	$(-\frac{i}{2}, \frac{1}{2}, -\frac{1}{2})$ 03 12 56	S^{03}	S^{12}	S^{56}
even II ${}^{II}\mathcal{A}^m_f$	1	[−i](+)(+)	(−i)[+](+)	[−i][+][+]	(−i)(+)[+]	$-\frac{i}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
	2	(+i)[−](+)	[+i](−)(+)	(+i)(−)[+]	[+i][−][+]	$\frac{i}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$
	3	(+i)(+)[−]	[+i][+][−]	(+i)[+](−)	[+i](+)(−)	$\frac{i}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$
	4	[−i][−][−]	(−i)(−)[−]	[−i](−)(−)	(−i)[−](−)	$-\frac{i}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$

"basis vectors" and their Hermitian conjugated partners for the case $d = (5+1)$ can be found in Ref. [11].
While the Clifford odd "basis vectors" are (chosen to be) left handed, $\Gamma^{(5+1)} = -1$, their Hermitian conjugated partners have opposite handedness, Eq. 10.23 in App. 10.4 [4].

Acknowledgments

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References

1. N. Mankoč Borštnik, "Spinor and vector representations in four dimensional Grassmann space", *J. of Math. Phys.* **34** (1993) 3731-3745, "Unification of spin and charges in Grassmann space?", hep-th 9408002, IJS.TP.94/22, Mod. Phys. Lett.**A (10)** No.7 (1995) 587-595;
2. A. Borštnik Bračič, N. S. Mankoč Borštnik, "On the origin of families of fermions and their mass matrices", hep-ph/0512062, Phys. Rev. **D 74** 073013-28 (2006).
3. N.S. Mankoč Borštnik, H.B.F. Nielsen, *J. of Math. Phys.* **43**, 5782 (2002) [arXiv:hep-th/0111257]. "How to generate families of spinors", *J. of Math. Phys.* **44** 4817 (2003) [arXiv:hep-th/0303224].
4. N.S. Mankoč Borštnik, D. Lukman, "Vector and scalar gauge fields with respect to $d = (3+1)$ in Kaluza-Klein theories and in the *spin-charge-family theory*", *Eur. Phys. J. C* **77** (2017) 231.
5. N.S. Mankoč Borštnik, H.B.F. Nielsen, "Understanding the second quantization of fermions in Clifford and in Grassmann space", *New way of second quantization of fermions — Part I and Part II*, [arXiv:2007.03517, arXiv:2007.03516].
6. N. S. Mankoč Borštnik, H. B. Nielsen, "How does Clifford algebra show the way to the second quantized fermions with unified spins, charges and families, and with vector and scalar gauge fields beyond the *standard model*", Progress in Particle and Nuclear Physics, http://doi.org/10.1016.j.ppnp.2021.103890 .
7. N. S. Mankoč Borštnik, "How Clifford algebra helps understand second quantized quarks and leptons and corresponding vector and scalar boson fields, opening a new step beyond the standard model", Reference: NUPHB 994 (2023) 116326 , [arXiv: 2210.06256, physics.gen-ph V2].

8. N. S. Mankoč Borštnik, "Clifford odd and even objects in even and odd dimensional spaces", Symmetry 2023,15,818-12-V2 94818, https:doi.org/10.3390/sym15040818, [arxiv.org/abs/2301.04466] , https://www.mdpi.com/2073-8994/15/4/818 Manuscript ID: symmetry-2179313.

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[4] Let us check the hadedness of the chosen representation: $\Gamma^{5+1}\hat{b}^{1\dagger}_{1}(\equiv \stackrel{03}{(+i)}\stackrel{12}{[+]}\stackrel{56}{[+]}) = \sqrt{(-1)^5}\, i^3 (\frac{2}{i})^3 S^{03} S^{12} S^{56} (\stackrel{03}{(+i)}\stackrel{12}{[+]}\stackrel{56}{[+]}) = \frac{i^4 2^3}{i^3} \frac{i}{2}\frac{1}{2}\frac{1}{2} (\stackrel{03}{(+i)}\stackrel{12}{[+]}\stackrel{56}{[+]}) = -1(\stackrel{03}{(+i)}\stackrel{12}{[+]}\stackrel{56}{[+]}))$.

9. N.S. Mankoč Borštnik N S, ”The spin-charge-family theory is explaining the origin of families, of the Higgs and the Yukawa couplings”, *J. of Modern Phys.* **4** (2013) 823 [arXiv:1312.1542].
10. N. S. Mankoč Borštnik, ”New way of second quantized theory of fermions with either Clifford or Grassmann coordinates and *spin-charge-family* theory ” [arXiv:1802.05554v4, arXiv:1902.10628], N. S. Mankoč Borštnik, ”How Clifford algebra can help understand second quantization of fermion and boson fields”, [arXiv: 2210.06256. physics.gen-ph V1] .
11. N. S. Mankoč Borštnik, ”How Clifford algebra can help understand second quantization of fermion and boson fields”, [arXiv: 2210.06256. physics.gen-ph V1] ,
12. N. S. Mankoč Borštnik, ”Clifford odd and even objects offer description of internal space of fermions and bosons, respectively, opening new insight into the second quantization of fields”, The 13^{th} Bienal Conference on Classical and Quantum Relativistic Dynamics of Particles and Fields IARD 2022, Prague, $6-9$ June, [http://arxiv.org/abs/2210.07004] .
13. M. Blagojević, *Gravitation and gauge symmetries*, IoP Publishing, Bristol 2002.
14. H.B. Nielsen, M. Ninomiya, “Novel String Field Theory and Bound State, Projective Line,and sharply 3-transitive group”, [arXiv:2111.05106v1, physics.gen-ph].
15. K. Wray, “An Introduction to String Theory”.
16. E.H.El Kinani, “Between Quantum Virasoro Algebra $\mathcal{L}_\rfloor$ and Generalized Clifford Algebras”, [arXiv:math-ph/0310044].
17. Pavšič, M. *The Landscape of Theoretical Physics: Global View*; van der Merwe, A., Ed.; Kluwer Academic Publishers: New York, NY, USA, 2001.

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11 A Transformation Groupoid and Its Representation — A Theory of Dimensionality

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Abstract. In higher dimensional physics there are usually two ways of dimensional reduction. One is by Kaluza-Klein theory and another by braneworld. In this talk I would like to discuss a third way of dimensional reduction. It is remarkably succinct, integrated by the groupoid and represented by the operation. Additionally, since it has a symmetry, it suggests an unknown conservation law based on Noether's theorem.
This paper is dedicated to the memory of the late Prof. Ichiro Yokota of Shinshu University, who was known for his research of cellular decompositions of classical Lie groups and realizations of exceptional Lie groups.

Povzetek: Teorijski fiziki osnovnih delcev in polj, ki razlagajo lastnosti opazljivih pojavov ob predpostavki, da ima prostor-čas več kot le štiri razsežnosti, morajo razložiti, zakaj višjih dimenzij ne opazimo. Teorije Kaluza-Kleinovega tipa privzamejo, da so ostale dimenzije ujete v neopazljivo majhne razsežnosti. Teorije "brainworld" predpostavijo, da je naše vesolje tridimenzionalna membrana več razsežnega prostora. Avtor predlaga nov navčin redukcije števila dimenzij z grupoidi, predstavljenimi z operacijo. Ker vsebuje simetrijo, določa ta način nov (še neznan) ohranitveni zakon. Avtor posveča prispevek prof. Ichiri Yokotu z univerze Shinshu, ki je znan po svojih raziskavah celičnih dekompozicij klasičnih Liejevih grup in po realizaciji izjemnih Liejevih grup.

11.1 Prologue: The Motive and Background

Since the dawn of modern cosmology (early 20th century), the 3-dimensional sphere or hyperboloid has been the model for the universe we live in. The Poincaré Conjecture was not proven at that time.
In 1921-26, T. Kaluza and O. Klein proposed that gravity and electromagnetic forces can be unified by adding one extra dimension of space to the 4-dimensional space plus time called 5-dimensional space-time (later shown to be insufficient). The extra 1-dimensional space, which we cannot perceive, is confined within the subatomic particles as an extremely small closed space. This is generally known as the Kaluza-Klein theory (KK theory), which is one of the essential theories in string theory. Although more complicated Calabi-Yau manifolds are also used today, the essential idea is the same as the KK theory. Moreover, 10^{500} different universes are produced from this theory (multiverses). Experiments have been

conducted to search for this compactified extra dimension, but they have yet to be verified. None of the particle physicists have been able to answer why space of more than three dimensions is compactly wound up in the first place. Additionally, such an embedding from higher dimensional space to lower dimensional space should be diffeomorphism. However, are they really diffeomorphism?
An idea called D-brain (Dirichlet membrane: oscillations of string particles due to Dirichlet boundary conditions) and braneworld are becoming mainstream in string theory. Those ideas are that the extra dimensions are not compactified, but that our universe is a 3-dimensional space (4-dimensional space-time) floating within higher dimensional space. If this is the case, it is not surprising that there are subatomic particles as well that are eternally moving in a two-dimensional plane within 'our three-dimensional space'. No such strange subatomic particles have yet been discovered.
So, could we not discuss this in a simpler way or model?

11.2 Concrete Insights into History of Spatial Dimensions

What is the spatial dimension? Let us delve deeper into this matter.
The history of spatial dimensions is as follows:
Euclid: Definition of points, lines, planes.
Aristotle: The three-dimensional volume is 'perfect' and there are no dimensions beyond the third dimension (in his celestial theory).
R. Descartes, P. Fermat: Co-ordinate geometry.
B. Riemann: In Riemannian geometry he introduced the line element (an extension of the Pythagorean theorem), which made it possible for the first time to mathematically discuss spaces of four or more dimensions.
D. Hilbert: Based on the orthogonality of vectors, n-dimensional space is a space in which any number of base vectors are orthogonal to each other and the norm is defined. It is called a Hilbert space.
These ideas assume that lower dimensional space is a subspace of higher (or same) dimensional space. However, is this assumption correct? That will be obviously doubtful, especially when we look at the strangeness of extra-dimensional (higher dimensional) space in modern physics.
Now let us consider again the spatial dimension from the low-dimensional case. Consider the relationship between co-ordinate geometry and its degrees of freedom (or the number of variables).

11.3 Degree of Freedom (Lower Dimension to Higher Dimension)

The simplest explanation is that of the direction from point A to point B in a two-dimensional space.
We empirically think that point B is only one arbitrary point (as in Cartesian co-ordinates).

However, is this true? This is a special case because the degree of freedom of the point means moving from A to B. The direction towards point B is completely guaranteed. Suppose then that the point from A to B has no degree of freedom: indecisive to one direction. How would that point move? A point that is not given a 'degree of freedom' cannot determine a single direction. In other words, the point will move in 'whole event' directions (the entire 360°).

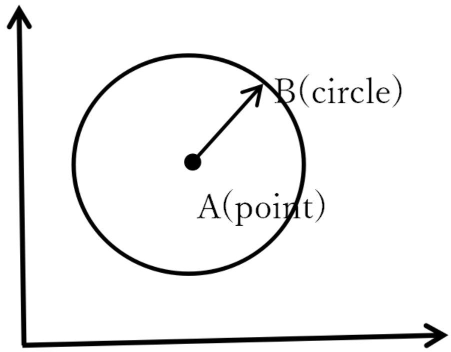


Fig. 1

This appears to be a wave (pulse wave) propagating in space with no medium. What does this mean? Here are some things to keep in mind. The original 1-dimensional space is not a subspace of the 2-dimensional plane. A subspace is only a part of the 2-dimensional space. If there is a 1-dimensional straight line as a subspace, the degrees of freedom have 'already' been determined (e.g., from point A to point B).

Now consider the degree of freedom (direction). If a point has 2 degrees of freedom in the 2-dimensional space, it is a point in the 2-dimensional space. On the other hand, what does it mean if a point has only 1 degree of freedom in the 2-dimensional space?

Conclusion: What does it mean that a point is in the 2-dimensional space with only 1 degree of freedom? A point has only 1 degree of freedom in the 2-dimensional space. If the point originally existed in the 1-dimensional space with only 1 degree of freedom, and it is moved to the 2-dimensional space, then the point keeps having only 1 degree of freedom in the 2-dimensional space.

Now consider the degree of freedom as a stochastic event. The fact that a point moving in the 2-dimensional space has 2 degrees of freedom means that the point can go in any direction in 360°. On the other hand, if a point has only 1 degree of freedom in the 2-dimensional space, it cannot decide in which direction to go. Therefore, the point can only move forward in 'whole event' directions. Thus, the point can only move in the 2-dimensional space like a pulse wave.

Let us consider the same thing in the 2-dimensional Cartesian co-ordinate system. In ordinary Cartesian co-ordinates, there are two degrees of freedom (x, y). If there is only one degree of freedom, then there is only x (or y). If x has a value, it can be shown for example as $x = 2$. This is a 'point' located at 2 in the 1-dimensional space (the number line), but if we consider this in the 2-dimensional space, of course we do not consider it to be (2, 0). Note that by considering $y = 0$, we have

given y a degree of freedom. Strictly speaking, we consider this $x = 2$ to be a 'straight line' parallel to the y-axis and passing through 2 on the x-axis.
If we consider this in the context mentioned above, a point moved from the 1-dimensional space to the 2-dimensional space has no degree of freedom on the y-axis, so it occupies all points on the y-axis (i.e., a 'whole event' in probability theory). Therefore, $x = 2$ can be interpreted as meaning that $x = 2$ is a straight line. It makes sense also in conventional mathematics.
Furthermore, consider $x = 2$ in the 3-dimensional space. Since there is only one degree of freedom in the 3-dimensional space, it can be regarded as a plane parallel to the $y - z$ plane by the same consideration as shown in Fig. 2.

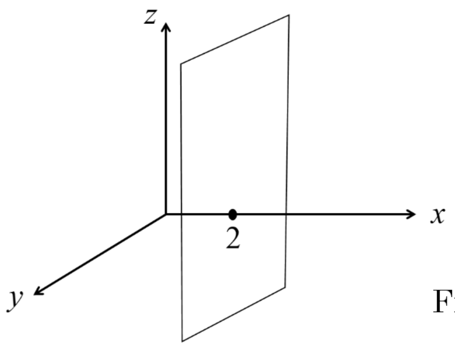


Fig. 2

Now consider a point with one degree of freedom that is transferred from the 1-dimensional space to higher dimensional space and then the first example which spreads out in concentric circles as described earlier. This is a case in the polar co-ordinates. Therefore, if the point $x = 2$ is transferred to the 2-dimensional space, it is a circle of radius 2 as shown in Fig. 3. If the point is transferred to the 3-dimensional space, it is a sphere of radius 2 as shown in Fig. 4.

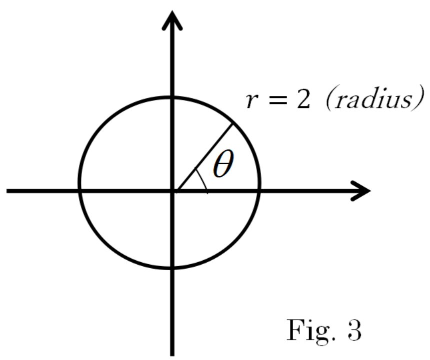


Fig. 3

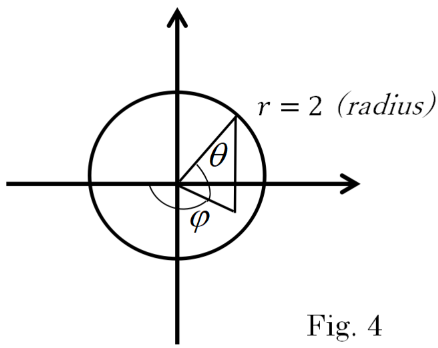


Fig. 4

Degree of Freedom (Higher Dimension to Lower Dimension)

So far, we have considered points that are transferred from lower dimensional space to higher dimensional space. How does the other way around go? The best example of this is the relationship between the equations of a circle and a curved surface in the homogeneous co-ordinates. For example, the equation of a paraboloid $x^2 + y^2 = z^2$ is, by algebraic manipulation, the equation of a circle $\left(\frac{x}{y}\right)^2 + \left(\frac{y}{z}\right)^2 = X^2 + Y^2 = 1$, as shown in Fig. 5 and Fig. 6.

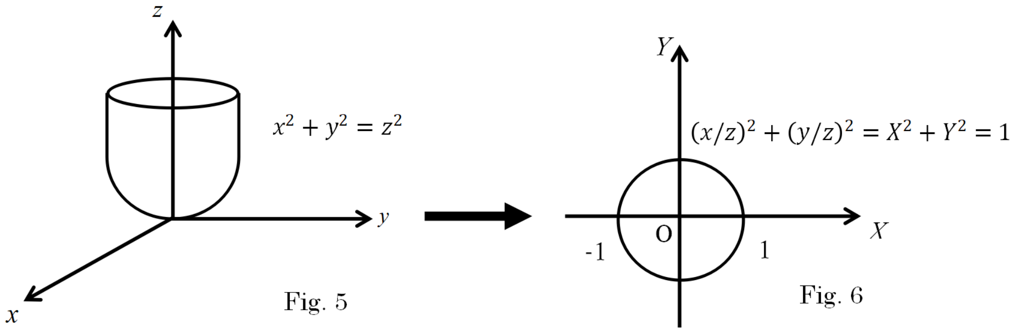


Fig. 5 Fig. 6

This can be said to be a projection of a curved surface in the 3-dimensional space on to a circle in the 2-dimensional space. In other words, it is a transfer or mapping from higher dimensional space to lower dimensional space.
From now on, following the term of the homogenous co-ordinates, the movement from m-dimensional space to n-dimensional space is referred to as 'projection'.
Finally for this chapter, let us end the discussion above as follows:
Lower dimensional space is not a subspace of higher dimensional space. They are disjoint to each other.
Any point in the n-dimensional space has n-degrees of freedom: it has n-variables or co-ordinates as $x,\ y,\ z$ of $(x,\ y,\ z)$.
If a point is transferred to a different dimensional space, the number of variables or the degree of freedom never changes.

11.4 Description with Matrices

Let us introduce a specific matrix operator to project a point between mutual dimensions. This matrix is different from a conventional one. It includes a special operator needing a temporary variable for operation, because the number of variables of a point before and after this operation is different. One can demonstrate it with attention to this fact, for example, operating by an operator E_{12} to project a point A_1 in the 1-dimensional space into the 2-dimensional space (A_2). The equation is $A_2 = E_{12}A_1$,
$\therefore \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & D \end{pmatrix} \begin{pmatrix} a \\ T \end{pmatrix} = \begin{pmatrix} a \\ DT \end{pmatrix}, \ldots$ (4.1)
where D is a matrix element making the dimension higher and T is a temporary variable to correspond to the 2-dimensional space after the operation. Therefore, DT denotes all the real numbers of y at once. This process of Eq. 4.1 is shown in Fig. 7.

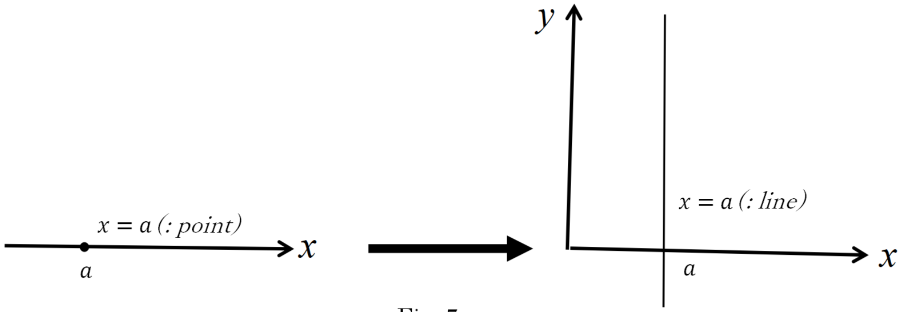


Fig. 7

.
Operating another case from the 1-dimensional space to the 3-dimensional space, then:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & D & 0 \\ 0 & 0 & D \end{pmatrix} \begin{pmatrix} a \\ T \\ T \end{pmatrix} = \begin{pmatrix} a \\ DT \\ DT \end{pmatrix} \dots (4.2)$$

Similarly, the 3-dimensional space to the 2-dimensional space is operated as:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & D^{-1} \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} a \\ b \\ D^{-1}c \end{pmatrix} \dots (4.3)$$

D^{-1} denotes an element making the dimension lower and the inverse of D. Eq. 4.3 is shown in Fig. 8. Then, if returning the point projected from the 3-dimensional space into the 2-dimensional space by Eq. 4.3 to the original dimensional space, the operation is as follows:

$$\begin{aligned} \begin{pmatrix} x \\ y \\ z \end{pmatrix} &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & D \end{pmatrix} \begin{pmatrix} a \\ b \\ D^{-1}c \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & D^{-1} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & D \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} \\ &= \begin{pmatrix} a \\ b \\ c \end{pmatrix} \cdots (4.4) \end{aligned}$$

$$\therefore E_{32}E_{23}(=(E_{23})^{-1}E_{23}) = E_{33} \equiv 1 \cdots (4.5)$$

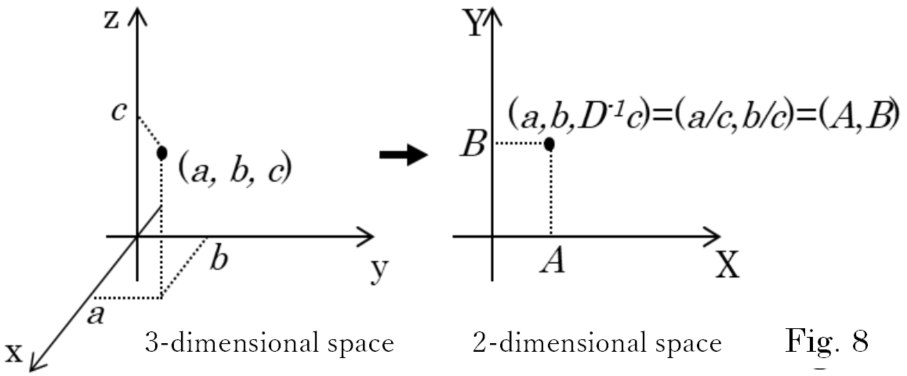


Fig. 8

The general operator which is a dimensional unit matrix E_{lm} is:

If $l < m, E_{lm} = diag(\overbrace{1,1,1,\ldots,1,1,1}^{l},\overbrace{D,D,D,\ldots,D,D,D}^{m-l})$.

If $m < l, E_{lm} = (E_{ml})^{-1} = diag(\overbrace{1,1,1,\ldots,1,1,1}^{m},\overbrace{D^{-1},D^{-1},D^{-1},\ldots,D^{-1},D^{-1},D^{-1}}^{l-m})$.

$\therefore E_{jk}E_{kj} = E_{jj} \equiv 1 \equiv E_{kj}E_{jk} = E_{kk}. \ldots$ (4.6)

Furthermore, $E_{0n} = diag(\overbrace{D,D,D,\ldots,D,D,D}^{n})$,

$$E_{n0} = (E_{0n})^{-1} = diag(\overbrace{D^{-1},D^{-1},D^{-1},\ldots,D^{-1},D^{-1},D^{-1}}^{n}).$$

Note: This is not the best example though, the case below is in a series of projections from higher dimensional space to lower dimensional space:

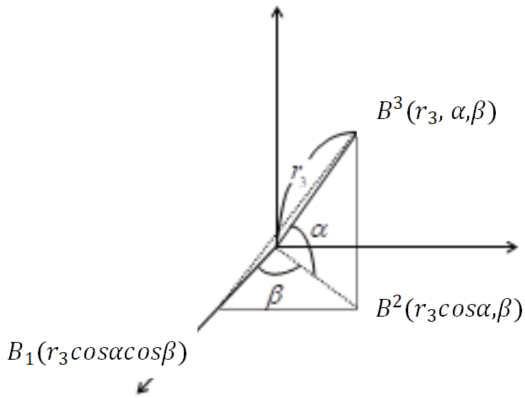
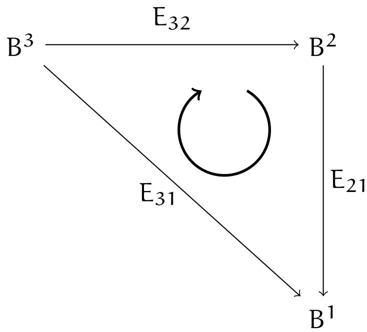


Fig. 9



11.5 The Groupoid

Proposition. In the former chapter, matrices in a series of operations (partially functional) make the group action. They are indicated by equations as follows:
i. $E_{lm}E_{mn} = E_{ln}$ (automorphism besides closure , proven in Chap. 4), … (5.1)
ii. $(E_{kl}E_{lm})E_{mn} = E_{kl}(E_{lm}E_{mn})$ (associative), … (5.2)
iii. $E_{jk}E_{kj} = E_{jj} \equiv 1 \equiv E_{kj}E_{jk} = E_{kk}$ (inverse), … (5.3)
or $E_{lm}E_{ml} = E_{lm}\,(E_{lm})^{-1} = (E_{ml})^{-1}\,E_{ml} \equiv 1$, (identity), … (5.4)
iv. $E_{kl}E_{lm}E_{ml} = E_{kl}E_{lm}\,(E_{lm})^{-1} = E_{kl}$ (right identity), … (5.5)
and $E_{lk}E_{kl}E_{lm} = (E_{kl})^{-1}\,E_{kl}E_{lm} = E_{lm}$ (left identity), … (5.6)
v. $(E_{lm}E_{mn})^{-1} = (E_{mn})^{-1}\,(E_{lm})^{-1}$, … (5.7),
vi. $E_{jj} \equiv 1$ (identity equivalent to the scalar value). … (5.8)
(i) and (vi) are peculiar to the groupoid.
The operators in chapter 2 explicitly show the groupoid mentioned above. However, we have never calculated such matrices. Therefore, we need to check and verify that they really work.
Proof. At first, of the formula (i) is as follows:
a1) If $0 < l < m < n$ (projecting into higher dimensions),

$$
\begin{aligned}
x' = E_{lm}x &= diag(\overbrace{1,1,\dots,1}^{l},\overbrace{D,D,\dots,D}^{m-l})(x_1,x_2,\dots,x_l,\overbrace{T,T,\dots,T}^{m-l})^T \\
&= (x_1,x_2,\dots,x_l,\overbrace{DT,DT,\dots,DT}^{m-l})^T \\
\therefore E_{mn}x' &= diag(\overbrace{1,1,\dots,1}^{m},\overbrace{D,D,\dots,D}^{n-m})(x1,x2,\dots,xl,\overbrace{DT,DT,\dots,DT}^{m-l},\overbrace{T,T,\dots,T}^{n-m})^T \\
&= (x_1,x_2,\dots,x_l,\overbrace{DT,DT,\dots,DT}^{n-l})^T \\
&= diag(\overbrace{1,1,\dots,1}^{l},\overbrace{D,D,\dots,D}^{n-l})(x_1,x_2,\dots,x_l,\overbrace{T,T,\dots,T}^{n-l})^T = E_{ln}x.
\end{aligned}
$$

a2) If $0 < l < n < m$ (projecting into higher dimensions),

$$
\begin{aligned}
x' = E_{lm}x &= diag(\overbrace{1,1,\dots,1}^{l},\overbrace{D,D,\dots,D}^{m-l})(x_1,x_2,\dots,x_l,\overbrace{T,T,\dots,T}^{m-l})^T \\
&= (x_1,x_2,\dots,x_l,\overbrace{DT,DT,\dots,DT}^{m-l})^T \\
\therefore\ E_{mn}x' &= (E_{nm})^{-1}x' \\
&= diag(\overbrace{1,1,\dots,1}^{m},\overbrace{D^{-1},D^{-1},\dots,D^{-1}}^{m-n})\times \\
&\times (x_1,x_2,\dots,x_l,\overbrace{DT,DT,\dots,DT}^{m-l},\overbrace{T,T,\dots,T}^{m-n})^T \\
&= (x_1,x_2,\dots,x_l,\overbrace{DT,DT,\dots,DT}^{n-l},\overbrace{T,T,\dots,T}^{m-n})^T.
\end{aligned}
$$

Remark. Since T is temporary, they are gone. For example,

$$
\begin{aligned}
\begin{pmatrix}1 & 0 & 0\\ 0 & D^{-1} & 0\\ 0 & 0 & D^{-1}\end{pmatrix}\begin{pmatrix}a\\ DT\\ DT\end{pmatrix} &= \begin{pmatrix}1 & 0 & 0\\ 0 & D^{-1} & 0\\ 0 & 0 & D^{-1}\end{pmatrix}\begin{pmatrix}1 & 0 & 0\\ 0 & D & 0\\ 0 & 0 & D\end{pmatrix}\begin{pmatrix}a\\ T\\ T\end{pmatrix} \\
&= \begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}a\\ T\\ T\end{pmatrix} \\
&= \begin{pmatrix}a\\ T\\ T\end{pmatrix} = a.
\end{aligned}
$$

See also Eq. 4.1 , 4.2 , 4.3

$$\therefore = (x_1, x_2, \dots, x_l, \overbrace{DT, DT, \dots, DT}^{n-l})^T$$
$$= diag(\overbrace{1, 1, \dots, 1}^{l}, \overbrace{D, D, \dots, D}^{n-l})(x_1, x_2, \dots, x_l, \overbrace{T, T, \dots, T}^{n-l})^T = E_{ln}x.$$

a3) If $0 < m < l < n$ (projecting into higher dimensions),

$$x' = E_{lm}x = (E_{ml})^{-1}x = diag(\overbrace{1, 1, \dots, 1}^{m}, \overbrace{D^{-1}, D^{-1}, \dots, D^{-1}}^{l-m})(x_1, x_2, \dots, x_l)^T$$
$$= (x_1, x_2, \dots, x_l,\ D^{-1}x_{m+1}, D^{-1}x_{m+2}, \dots, D^{-1}x_l)^T.$$

$$\therefore\ E_{mn}x' = diag(\overbrace{1, 1, \dots, 1}^{m}, \overbrace{D, D, \dots, D}^{m-n})\times$$
$$\times (x_1, x_2, \dots, x_l,\ \overbrace{D^{-1}x_{m+1}, D^{-1}x_{m+2}, \dots, D^{-1}x_l}^{l-m}\ , \overbrace{T, T, \dots, T}^{n-l})^T$$
$$= (x_1, x_2, \dots, x_l, \overbrace{DT, DT, \dots, DT}^{n-l})^T$$
$$= diag(\overbrace{1, 1, \dots, 1}^{l}, \overbrace{D, D, \dots, D}^{n-l})(x_1, x_2, \dots, x_l,\ \overbrace{T, T, \dots, T}^{n-l})^T = E_{ln}x.$$

b1) If $0 < n < m < l$ (projecting into lower dimensions),

$$x' = E_{lm}x = (E_{ml})^{-1}\,x = diag(\overbrace{1, 1, \dots, 1}^{m}, \overbrace{D^{-1}, D^{-1}, \dots, D^{-1}}^{l-m})(x_1, x_2, \dots, x_l)^T$$
$$= (x_1, x_2, \dots, x_l, D^{-1}x_{m+1}, D^{-1}x_{m+2}, \dots, D^{-1}x_l)^T.$$

$$\therefore\ E_{mn}x' = (E_{nm})^{-1}\,x'$$
$$= diag(\overbrace{1, 1, \dots, 1}^{n}, \overbrace{D^{-1}D^{-1}, \dots, D^{-1}}^{m-n}, \overbrace{1, 1, \dots, 1}^{l-m})\times$$
$$\times (x_1, x_2, \dots, x_m, D^{-1}x_{m+1}, D^{-1}x_{m+2}, \dots, D^{-1}x_l)^T$$
$$= (x_1, x_2, \dots, x_n, D^{-1}x_{n+1}, D^{-1}x_{n+2}, \dots, D^{-1}x_l)^T$$
$$= diag(\overbrace{1, 1, \dots, 1}^{n}, \overbrace{D^{-1}, D^{-1}, \dots, D^{-1}}^{l-n})(x_1, x_2, \dots, x_l)^T = (E_{nl})^{-1}\,x = E_{ln}x.$$

b2) If $0 < n < l < m$ (projecting into lower dimensions),

$$x' = E_{lm}x = diag(\overbrace{1,1,\ldots,1}^{l},\overbrace{D,D,\ldots,D}^{m-l})(x_1,x_2,\ldots,x_l,\overbrace{T,T,\ldots,T}^{m-l})^T$$
$$= (x_1,x_2,\ldots,x_m,\ \overbrace{DT,DT,\ldots,DT}^{m-l})^T.$$

$$\therefore\ E_{mn}x' = (E_{nm})^{-1}x'$$
$$= diag(\overbrace{1,1,\ldots,1}^{n},\overbrace{D^{-1},D^{-1},\ldots,D^{-1}}^{m-n})\ (x_1,x_2,\ldots,x_m,\overbrace{DT,DT,\ldots,DT}^{m-l})^T$$
$$= (x_1,x_2,\ldots,x_n,D^{-1}x_{n+1},D^{-1}x_{n+2},\ldots,D^{-1}x_l,\overbrace{T,T,\ldots,T}^{m-l})^T$$
$$= (x_1,x_2,\ldots,x_n,D^{-1}x_{n+1},D^{-1}x_{n+2},\ldots,D^{-1}x_l)^T$$
$$= diag(\overbrace{1,1,\ldots,1}^{n},\overbrace{D^{-1},D^{-1},\ldots,D^{-1}}^{l-n})(x_1,x_2,\ldots,x_l)^T = (E_{nl})^{-1}x = E_{ln}x.$$

b3) If $0 < m < n < l$ (projecting into lower dimensions),

$$x' = E_{lm}x = (E_{ml})^{-1}x = diag(\overbrace{1,1,\ldots,1}^{m},\overbrace{D^{-1},D^{-1},\ldots,D^{-1}}^{l-m})(x_1,x_2,\ldots,x_l)^T$$
$$= (x_1,x_2,\ldots,x_m,\ D^{-1}x_{m+1},D^{-1}x_{m+2},\ldots,D^{-1}x_l)^T.$$

$$\therefore\ E_{mn}x' = diag(\overbrace{1,1,\ldots,1}^{n},\overbrace{D,D,\ldots,D}^{n-m},\overbrace{1,1,\ldots,1}^{n-l})\times$$
$$\times\ (x_1,x_2,\ldots,x_m,D^{-1}x_{m+1},D^{-1}x_{m+2},\ldots,D^{-1}x_l)^T$$
$$= (x_1,x_2,\ldots,x_n,D^{-1}x_{n+1},D^{-1}x_{n+2},\ldots,D^{-1}x_l)^T$$
$$= diag(\overbrace{1,1,\ldots,1}^{n},\overbrace{D^{-1},D^{-1},\ldots,D^{-1}}^{l-n})(x_1,x_2,\ldots,x_l)^T = (E_{nl})^{-1}x = E_{ln}x.$$

Proof of (ii) of associative law is as follow: Since $(E_{kl}E_{lm})E_{mn} = E_{km}E_{mn} = E_{kn}$, and
$E_{kl}(E_{lm}E_{mn}) = E_{kl}E_{ln} = E_{kn}$ from (i), $(E_{kl}E_{lm})E_{mn} = E_{kl}(E_{lm}E_{mn})$.
Proof of (iii) follows the rule of Eq. 4.6. Another proof for Eq. 5.4 is from the formula (i), $f\left(AA^{-1}\right) = I = f(A)f(A^{-1})$. $\therefore\ f(A^{-1}) = f(A)^{-1}$, where $f\left(A^{-1}\right) = E_{ml}$ and $f(A)^{-1} = (E_{lm})^{-1}$.
Proof of (iv) is trivial from (ii).
Proof of (v), $E_{lm}E_{mn}\,(E_{lm}E_{mn})^{-1} = E_{lm}E_{mn}\,(E_{mn})^{-1}\,(E_{lm})^{-1} = E_{ll} \equiv 1$.
At last, proof of (vi) is as follows. For $G = E_{lm}$, the scalar multiplication by 1 in field k holds as

$s : 1 \times G = G \times 1 \to G$. It is compatible with the matrix multiplications in G. Then, $E_{mn} = I_m E_{mn} \equiv 1E_{mn} = E_{mn}1 \equiv E_{mn}I_n = E_{mn}$. Since it is 'mapping to itself' in the narrow sense of the word of our discussion, it is equivalent to the conventional unit matrices.

Since this groupoid is homomorphism from (i), it can be considered as a representation of groupoid. Strictly speaking, it is automorphism. This will be proven later.

From Another Viewpoint

The groupoid mentioned above is partially functional, not for any two elements arbitrarily taken from G. We will not therefore consider it as a group. However, even though that is a groupoid, we must take notice that group axioms do not claim that such a whole process (binary operation) should be done. To confirm this, let us try to give five conditions as group axioms as follows.

(1). We randomly take any two elements in a set G.

(2). For any two elements taken from G, the operation is closed in G, s. t. for any a, b, c in G, $ab = c$.

(3). For any a, b, c in G, $(ab)c = a(bc)$: associative law holds.

(4). There exists unique identity e.

(5). For each a in G, its inverse b exists s. t. $ab = ba = e$.

What we must pay attention to is whether the first condition should be included in axioms of groups. If accepting it, we should introduce a concept of axiom in probability theory. That is, in group theory, we assume the whole event for any two elements arbitrarily taken from G in the manner of probability theory, then define operation such for any elements taken from G at random. In other words, we should consider so-called sample space or measure theory for probability in group theory. Group axioms naturally do not claim such a process and another axiomatic system in probability theory.

Remark. In conventional algebra, the number of combinations is at most $\mathbb{N} \times \mathbb{N}$ (Descartes product). However, the number of the operators' combinations is $\mathbb{N} \times \mathbb{N} \times \mathbb{N} \times \mathbb{N}$. Even in the case of $E_{lm}E_{mn}$, the number is $\mathbb{N} \times \mathbb{N} \times \mathbb{N}$. So, we must contemplate and rethink these issues.

11.6 The Groupoid Representation

Claim. Equation of the groupoid $E_{lm}E_{mn} = E_{ln}$ is automorphism.

Proof. At first the automorphism is proven as follows. Let $f(A)$ be E_{lm}, $f(B)$ be E_{mn}.

a1) If $0 < l < m < n$ (projecting into higher dimensions),

For $f(A)(B)$,

$$(f(A))(x) = x' = E_{lm}x = diag(\overbrace{1, 1, \ldots, 1}^{l}, \overbrace{D, D, \ldots, D}^{m-l})(x_1, x_2, \ldots, x_l, \overbrace{T, T, \ldots, T}^{m-l})^T$$
$$= (x1, x2, \ldots, x_l, \overbrace{DT, DT, \ldots, DT}^{m-l})^T.$$

Then, $(f(B))\,(x') = E_{mn}x'$

$$= diag(\overbrace{1,1,\ldots,1}^{m},\overbrace{D,D,\ldots,D}^{n-m})(x_1,x_2,\ldots,x_l,\overbrace{DT,DT,\ldots,DT}^{m-l},\overbrace{T,T,\ldots,T}^{n-m})^T$$

$$= (x_1,x_2,\ldots,x_l,\overbrace{DT,DT,\ldots,DT}^{n-l})^T$$

$$= diag(\overbrace{1,1,\ldots,1}^{l},\overbrace{D,D,\ldots,D}^{n-l})(x_1,x_2,\ldots,x_l,\overbrace{T,T,\ldots,T}^{n-l})^T = E_{ln}x.$$

For$f(AB)$, $(f(AB))\,(x) = E_{lm}E_{mn}x$

$$= diag(\overbrace{1,1,\ldots,1}^{l},\overbrace{D,D,\ldots,D}^{m-l},\overbrace{1,1,\ldots,1}^{n-m})diag(\overbrace{1,1,\ldots,1}^{m},\overbrace{D,D,\ldots,D}^{n-m})\times$$

$$\times\,(x_1,x_2,\ldots,x_l,\overbrace{T,T,\ldots,T}^{n-l})^T$$

$$= (x_1,x_2,\ldots,x_l,\overbrace{DT,DT,\ldots,DT}^{n-l})^T$$

$$= diag(\overbrace{1,1,\ldots,1}^{l},\overbrace{D,D,\ldots,D}^{n-l})(x_1,x_2,\ldots,x_l,\overbrace{T,T,\ldots,T}^{n-l})^T = E_{ln}x.$$

$\therefore f(AB) = f(A)(B)$.

a2) If $0 < l < n < m$ (projecting into higher dimensions),

For$f(A)f(B)$, $(f(A))(x) = x' = E_{lm}x$

$$= diag(\overbrace{1,1,\ldots,1}^{l},\overbrace{D,D,\ldots,D}^{m-l})(x_1,x_2,\ldots,x_l,\overbrace{T,T,\ldots,T}^{m-l})^T$$

$$= (x_1,x_2,\ldots,x_l,\overbrace{DT,DT,\ldots,DT}^{m-l})^T$$

Then, $(f(B))\,(x') = E_{mn}x'$

$$= (E_{nm})^{-1}\,x'$$

$$= diag(\overbrace{1,1,\ldots,1}^{m},\overbrace{D^{-1},D^{-1},\ldots,D^{-1}}^{m-n})(x_1,x_2,\ldots,x_l,\overbrace{DT,DT,\ldots,DT}^{m-l})^T$$

$$= (x_1,x_2,\ldots,x_l,\overbrace{DT,DT,\ldots,DT}^{n-l},\overbrace{T,T,\ldots,T}^{m-n})^T$$

$$= (x_1,x_2,\ldots,x_l,\overbrace{DT,DT,\ldots,DT}^{n-l})^T$$

$$= diag(\overbrace{1,1,\ldots,1}^{l},\overbrace{D,D,\ldots,D}^{n-l})(x_1,x_2,\ldots,x_l,\overbrace{T,T,\ldots,T}^{n-l})^T = E_{ln}x.$$

$$\text{For} f(AB), (f(AB))(x) = (E_{lm}E_{mn})x$$
$$= diag(\overbrace{1,1,\ldots,1}^{l},\overbrace{D,D,\ldots,D}^{m-l})diag(\overbrace{1,1,\ldots,1}^{m},\overbrace{D,D,\ldots,D}^{m-n})\times$$
$$\times\,(x_1,x_2,\ldots,x_l,\overbrace{DT,DT,\ldots,DT}^{m-l},\overbrace{T,T,\ldots,T}^{m-n})^T$$
$$= (x_1,x_2,\ldots,x_l,\overbrace{DT,DT,\ldots,DT}^{n-l},\overbrace{T,T,\ldots,T}^{m-n})^T$$
$$= (x_1,x_2,\ldots,x_l,\overbrace{DT,DT,\ldots,DT}^{n-l})^T$$
$$= diag(\overbrace{1,1,\ldots,1}^{l},\overbrace{D,D,\ldots,D}^{n-l})(x_1,x_2,\ldots,x_l,\overbrace{T,T,\ldots,T}^{n-l})^T$$
$$= E_{ln}x. \therefore f(AB) = f(A)(B).$$

a3) If $0 < m < l < n$ (projecting into higher dimensions),

$$\text{For} f(A)f(B), (f(A))(x) = x' = E_{lm}x = (E_{ml})^{-1}\,x$$
$$= diag(\overbrace{1,1,\ldots,1}^{m},\overbrace{D^{-1},D^{-1},\ldots,D^{-1}}^{l-m})(x_1,x_2,\ldots,x_l)^T$$
$$= (x_1,x_2,\ldots,x_m,\ D^{-1}x_{m+1},D^{-1}x_{m+2},\ldots,D^{-1}x_l)^T.$$

$$\text{Then}, (f(B))(x^{'}) = E_{mn}x^{'}$$
$$= diag(\overbrace{1,1,\ldots,1}^{m},\overbrace{D,D,\ldots,D}^{n-m})\times$$
$$\times\,(x_1,x_2,\ldots,x_m,\ \overbrace{D^{-1}x_{m+1},D^{-1}x_{m+2},\ldots,D^{-1}x_l}^{l-m}\ ,\overbrace{T,T,\ldots,T}^{n-l})^T$$
$$= (x_1,x_2,\ldots,x_l,\overbrace{DT,DT,\ldots,DT}^{n-l})^T$$
$$= diag(\overbrace{1,1,\ldots,1}^{l},\overbrace{D,D,\ldots,D}^{n-l})(x_1,x_2,\ldots,x_l,\ \overbrace{T,T,\ldots,T}^{n-l})^T = E_{ln}x.$$

$$\text{For} f(AB), (f(AB))(x) = (E_{lm}E_{mn})x$$
$$= diag(\overbrace{1,1,\ldots,1}^{m},\overbrace{D^{-1},D^{-1},\ldots,D^{-1}}^{l-m},\overbrace{1,1,\ldots,1}^{n-l})diag(\overbrace{1,1,\ldots,1}^{m},\overbrace{D,D,\ldots,D}^{n-m})\times$$
$$\times\ (x_1,x_2,\ldots,x_l,\overbrace{T,T,\ldots,T}^{n-l})^T$$
$$= (x_1,x_2,\ldots,x_l,\overbrace{DT,DT,\ldots,DT}^{n-l})^T$$
$$= diag(\overbrace{1,1,\ldots,1}^{l},\overbrace{D,D,\ldots,D}^{n-l})(x_1,x_2,\ldots,x_l,\overbrace{T,T,\ldots,T}^{n-l})^T = E_{\text{ln}}x.$$
$$\therefore f(AB) = f(A)(B).$$

b1) If $0 < n < m < l$ (projecting into lower dimensions),

$$\begin{aligned}\text{For} f(A)f(B), (f(A))(x) &= x'z\\ &= E_{lm}x\\ &= (E_{ml})^{-1}\,x\\ &= diag(\overbrace{1,1,\ldots,1}^{m},\overbrace{D^{-1},D^{-1},\ldots,D^{-1}}^{l-m})(x_1,x_2,\ldots,x_l)^T\\ &= (x_1,x_2,\ldots,x_l,\ D^{-1}x_{m+1},D^{-1}x_{m+2},\ldots,D^{-1}x_l)^T.\end{aligned}$$

$$\text{Then}, (f(B))(x) = E_{mn}x' = (E_{nm})^{-1}\,x'$$
$$= diag(\overbrace{1,1,\ldots,1}^{n},\overbrace{D^{-1}D^{-1},\ldots,D^{-1}}^{m-n},\overbrace{1,1,\ldots,1}^{l-m})\times$$
$$\times\ (x_1,x_2,\ldots,x_m,D^{-1}x_{m+1},D^{-1}x_{m+2},\ldots,D^{-1}x_l)^T$$
$$= (x_1,x_2,\ldots,x_n,D^{-1}x_{n+1},D^{-1}x_{n+2},\ldots,D^{-1}x_l)^T$$
$$= diag(\overbrace{1,1,\ldots,1}^{n},\overbrace{D^{-1},D^{-1},\ldots,D^{-1}}^{l-n})(x_1,x_2,\ldots,x_l)^T = (E_{nl})^{-1}\,x = E_{\text{ln}}x.$$

For $f(AB), (f(AB))(x) = (E_{lm}E_{mn})x$

$$= diag(\overbrace{1,1,\ldots,1}^{m},\overbrace{D^{-1},D^{-1},\ldots,D^{-1}}^{l-m})\ diag(\overbrace{1,1,\ldots,1}^{n},\overbrace{D^{-1},D^{-1},\ldots,D^{-1}}^{m-n}\overbrace{1,1,\ldots,1}^{l-m})\times$$

$$\times\,(x_1,x_2,\ldots,x_l)^T$$

$$= (x_1,x_2,\ldots,x_n,\ D^{-1}x_{n+1},D^{-1}x_{n+2},\ldots,D^{-1}x_l)^T$$

$$= diag(\overbrace{1,1,\ldots,1}^{n},\overbrace{D^{-1},D^{-1},\ldots,D^{-1}}^{l-n})(x_1,x_2,\ldots,x_l)^T$$

$$= (E_{nl})^{-1}x = E_{ln}x. \therefore f(AB) = f(A)(B).$$

b2) If $0 < n < l < m$ (projecting into lower dimensions),

For $f(A)f(B), (f(A))(x) = x' = E_{lm}x$

$$= diag(\overbrace{1,1,\ldots,1}^{l},\overbrace{D,D,\ldots,D}^{m-l})(x_1,x_2,\ldots,x_l,\overbrace{T,T,\ldots,T}^{m-l})^T$$

$$= (x_1,x_2,\ldots,x_l,\ \overbrace{DT,DT,\ldots,DT}^{m-l})^T.$$

Then, $(f(B))(x') = E_{mn}x' = (E_{nm})^{-1}x'$

$$= diag(\overbrace{1,1,\ldots,1}^{n},\overbrace{D^{-1},D^{-1},\ldots,D^{-1}}^{m-n})(x_1,x_2,\ldots,x_l,\overbrace{DT,DT,\ldots,DT}^{m-l})^T$$

$$= (x_1,x_2,\ldots,x_n,D^{-1}x_{n+1},D^{-1}x_{n+2},\ldots,D^{-1}x_l,\overbrace{T,T,\ldots,T}^{m-l})^T$$

$$= (x_1,x_2,\ldots,x_n,D^{-1}x_{n+1},D^{-1}x_{n+2},\ldots,D^{-1}x_l)^T$$

$$= diag(\overbrace{1,1,\ldots,1}^{n},\overbrace{D^{-1},D^{-1},\ldots,D^{-1}}^{l-n})(x_1,x_2,\ldots,x_l)^T$$

$$= (E_{nl})^{-1}x = E_{ln}x.$$

$\text{For} f(AB), (f(AB))(x) = (E_{lm}E_{mn})x$

$= \text{diag}(\overbrace{1, 1, \ldots, 1}^{l}, \overbrace{D, D, \ldots, D}^{m-l})\text{diag}(\overbrace{1, 1, \ldots, 1}^{n}, \overbrace{D^{-1}, D^{-1}, \ldots, D^{-1}}^{m-n})\times$

$\times (x_1, x_2, \ldots, x_l, \overbrace{T, T, \ldots, T}^{m-l})^T$

$= (x_1, x_2, \ldots, x_n, D^{-1}x_{n+1}, D^{-1}x_{n+2}, \ldots, D^{-1}x_l, \overbrace{T, T, \ldots, T}^{m-l})^T$

$= (x_1, x_2, \ldots, x_n, D^{-1}x_{n+1}, D^{-1}x_{n+2}, \ldots, D^{-1}x_l)^T$

$= \text{diag}(\overbrace{1, 1, \ldots, 1}^{n}, \overbrace{D^{-1}, D^{-1}, \ldots, D^{-1}}^{l-n})(x_1, x_2, \ldots, x_l)^T$

$= (E_{nl})^{-1}x = E_{ln}x. \therefore f(AB) = f(A)(B).$

b3) If $0 < m < n < l$ (projecting into lower dimensions),

$\text{For} f(A)f(B), (f(A))(x) = x' = E_{lm}x = (E_{ml})^{-1}x$

$= \text{diag}(\overbrace{1, 1, \ldots, 1}^{m}, \overbrace{D^{-1}, D^{-1}, \ldots, D^{-1}}^{l-m})(x_1, x_2, \ldots, x_l)^T$

$= (x_1, x_2, \ldots, x_m, D^{-1}x_{m+1}, D^{-1}x_{m+2}, \ldots, D^{-1}x_l)^T.$

Then, $(f(B))(x') = E_{mn}x'$

$= \text{diag}(\overbrace{1, 1, \ldots, 1}^{n}, \overbrace{D, D, \ldots, D}^{n-m}, \overbrace{1, 1, \ldots, 1}^{n-l})\times$

$\times (x_1, x_2, \ldots, x_m, D^{-1}x_{m+1}, D^{-1}x_{m+2}, \ldots, D^{-1}x_l)^T$

$= (x_1, x_2, \ldots, x_n, D^{-1}x_{n+1}, D^{-1}x_{n+2}, \ldots, D^{-1}x_l)^T$

$= \text{diag}(\overbrace{1, 1, \ldots, 1}^{n}, \overbrace{D^{-1}, D^{-1}, \ldots, D^{-1}}^{l-n})(x_1, x_2, \ldots, x_l)^T = (E_{nl})^{-1}x = E_{ln}x.$

$\text{For} f(AB), (f(AB))(x) = (E_{lm}E_{mn})x$

$= \text{diag}(\overbrace{1, 1, \ldots, 1}^{m}, \overbrace{D^{-1}, D^{-1}, \ldots, D^{-1}}^{l-m})\text{diag}(\overbrace{1, 1, \ldots, 1}^{m}, \overbrace{D, D, \ldots, D}^{n-m}, \overbrace{1, 1, \ldots, 1}^{n-l})\times$

$\times (x_1, x_2, \ldots, x_l)^T$

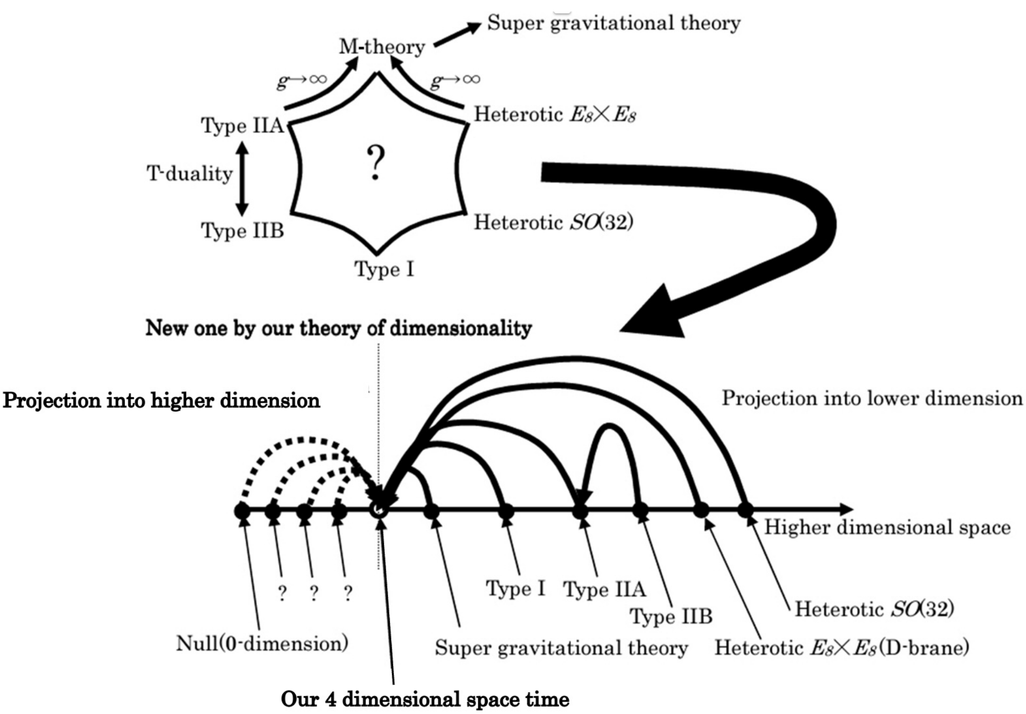
$= (x_1, x_2, \ldots, x_n, D^{-1}x_{n+1}, D^{-1}x_{n+2}, \ldots, D^{-1}x_l)^T$

$= \text{diag}(\overbrace{1, 1, \ldots, 1}^{n}, \overbrace{D^{-1}, D^{-1}, \ldots, D^{-1}}^{l-n})(x_1, x_2, \ldots, x_l)^T$

$= (E_{nl})^{-1}x = E_{ln}x. \therefore f(AB) = f(A)(B).$

11.7 Epilogue

The Invariant and Symmetry — Towards Noether's Theorem
In theoretical physics, an invariant means a physical system unchanged under mathematical operation. It is also called symmetry. Noether's Theorem states that every differential symmetry of the action of a physical system with conservative forces has a corresponding conservation law. Noether's theorem holds not only differential symmetries but also discrete symmetries. Examples of the discrete symmetries are parity and selection rule in quantum theory. So, what is the invariant in the groupoid that we have discussed? The invariant is conservation of the degree of freedom. Wherever a point is projected, its degree of freedom is conserved. It suggests that if higher dimensional physics were described by the groupoid, we might find an unknown physical conservation law.
What is a preferable unification of string theories?
The following diagrams present two methods of the unification of string theory. The upper model is by M-Theory. The lower one is by our theory of dimensionality.



11.8 Acknowledgements

I appreciate Prof. Susan Hansen, who voluntarily and patiently listened to explanations of my research; proofread all the slides to correct grammatical errors; and offered me significant suggestions and opinions for English expression. I also appreciate Dr. Astri Kleppe and Rei Takaba to their supports for this paper's TeX format.

References

[1] Felix Klein (1872), Vergleichende Betrachtungen über neuere geometrische Forschungen. Mathematische Annalen Vol. 43, 1893, pp. 63-100. (Also: Gesammelte Abh., Springer 1921, Vol. 1, pp. 460-497). Also known as Erlangen Programme, about English translation: https://arxiv.org/abs/0807.3161

[2] Euich Miztani, Projections and Dimensions, Communications in Applied Geometry, Research India Publications, ISSN 2249-4986, 2011, Vol. 1, pp. 7-18. https://dx.doi.org/10.37622/CAG/1.1.2011.7-16

[3] Euich Miztani, A Transportation Group and Its Subgroups, Advances in Algebra, Research India Publications, ISSN 0973-6964, 2015, Vol. 8, no. 1, pp. 25-31. https://dx.doi.org/10.37622/AA/8.1.2015.25-31

[4] Euich Miztani, A Group Representation — Notations for "Projections and Dimensions", Communications in Applied Geometry, Research India Publications, ISSN 2249-4286, 2016, Vol. 3, no. 1, pp. 21-27. https://dx.doi.org/10.37622/CAG/3.1.2016.21-27

For a quick grasp of this research, see also:
http://dx.doi.org/10.13140/RG.2.2.11950.38720
http://dx.doi.org/10.13140/RG.2.2.21097.93285

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12 Our Dark Matter Stopping in Earth

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Abstract. We have long worked on a model for dark matter, in which the dark matter consists of small bubbles of a new speculated type of vacuum, which are pumped up by some ordinary matter such as diamond, so as to resist the pressure of the domain wall separating the two vacua. Here we put forward thoughts on, how such macroscopic pearls would be cleaned off passing through the atmosphere and the Earth, and what would their distribution be as a function of the depth of their stopping point and the distribution of the radiation emitted from them, if, as we take it in our model, they radiate 3.5 keV electron and photon radiation after having been excited during the passage into the Earth. The purpose of such an estimation of the radiation distribution is to explain the truly mysterious fact that among all the underground experiments seeking dark matter colliding with the Earth material only the DAMA-LIBRA experiment has seen any evidence of dark matter. This is an experiment based on the *solid NaI* and being rather deep 1400 m, and it is our point that we can arrange/fit the main radiation to appear in the relatively deep DAMA-LIBRA site, and explain that the dark matter pearls cannot stop in a fluid, such as xenon in the xenon based experiments.

Povzetek: Avtorja predstavita model za temno snov, ki ga dopolnjujeta že vrsto let. Njuno temno snov tvorijo mehurčki v njunem vakuumu, iz katerega se dvigujejo pri trku z nekaterimi snovmi, kot je denimo diamant, in vzdržujejo pritisk domenske stene med dvema vakuumova. Tokrat ocenjujeta gostota mehurčkov pri prehodu skozi ozračje in skozi plasti, ki sestavljajo površino naše Zemlje. Privzameta, da pri tem sprožajo 3,5 keV elektronskega in fotonskega sevanja. Njun študij ima namen pojasniti, zakaj od vseh eksperimentov samo DAMA/LIBRA izmeri letno modulacijo temne snovi. Njun zaključek, ko prilagodita parametre modela, je da NaI v merilni aparaturi DAMA/LIBRA ustavlja mehurčke, tekoči Xe pa jih ne.

12.1 Introduction

It is still a great *mystery* of what the *dark matter*, of which one mostly has seen the effect of its gravitation, *consists*. An exceptionally great mystery in this connection is that among the experiments looking for dark matter hitting the Earth and being detected deep underground - to avoid the cosmic radiation background - there is only one experiment, DAMA-LIBRA [1] which has seen any evidence for dark matter. The experiments based on the fluid scintillator, fluid xenon, typically claim direct disagreement with DAMA-LIBRA, by obtaining so low upper limits on the

** Speaker at the Workshop "What comes Beyond the Standard Models" in Bled, 2023 cross section for the dark matter - using a WIMP model - that the observations by DAMA could not avoid having been seen in e.g. LUX. Our model has dark matter, that is not as weakly interacting as the usual WIMP model assumes. Rather our dark matter pearls consist of bubbles of a new (speculated) type of vacuum, containing ordinary matter and compressed to an outrageously high density. So, although still being per kg much less interacting than ordinary matter, our dark matter is much more strongly interacting per kg than the WIMPs usually considered. Thus our pearls of dark matter should not be called WIMPs but rather only IMPs (Interacting Massive Particles). The essential point for the present paper is, what happens to our model dark matter when hitting the Earth. It is not so much that they consist of a new type of vacuum etc. but rather that they are macroscopic objects causing a much bigger interaction that matters. However they still must be so massive compared to their cross section, that they do not just function like ordinary matter. But that does not prevent them getting stopped in the Earth, although with an appreciably longer stopping length than ordinary matter.

It is rather easy in our model to adjust parameters, so as to obtain whatever stopping length one might want for our dark matter pearls. At least we can easily believe that we could fit them to have a stopping length of the order of the depth of the DAMA-LIBRA experiment. Then if they were arranged to emit most of their excitation energy where they get stopped, they would be appreciably more visible to experiments in the depth of DAMA-LIBRA than in other depths.

If really the stopping of dark matter is needed for their easy observation, then the experiments with a fluid scintillator would be severely disfavoured because a dark matter pearl cannot really fully stop in a fluid. A little piece of fluid around the pearl will at least by gravity follow the pearl as it falls down, and it would spend much less time in a liquid xenon experiment than in a NaI(Tl) one. Even other NaI(Tl) experiments, if having a lower depth under the Earth surface than DAMA, might see only a little of the dark matter, because it passes through such experiments too fast and too little of the radiation from its excitation would be detected at such higher up experiments.

So our model of dark matter consists of small but still macroscopic pearls, in the sense of each consisting of many atoms, then made effectively "darker" by having these atoms concentrated by the domain wall between the new vacuum and the ordinary vacuum, which we assume to be degenerate (i.e. with the same energy density). We shall first briefly mention the evidence from dwarf galaxies for the interaction of dark matter with itself.

12.2 Our dark matter, stopping in the Earth, etc.

The two most crucial properties of our model [2–14] are as follows.

- Our dark matter is not so dark as WIMPs. Our dark matter pearls interact so much that they get stopped in the Earth.
  We actually speculate that our dark matter pearls get essentially stopped in the Earth, down from their $\sim 300\text{km}/\text{s}$ galactic velocity, at a depth around that

of DAMA-LIBRA. Velocity dependent fits of their “inverse darkness” $\frac{\sigma}{M}$ by Correa [15] from studying dwarf Galaxies is seen in Figure 12.1, which was motivated by the failure of the model that dark matter has only gravitational interactions as seen in Figure 12.2.

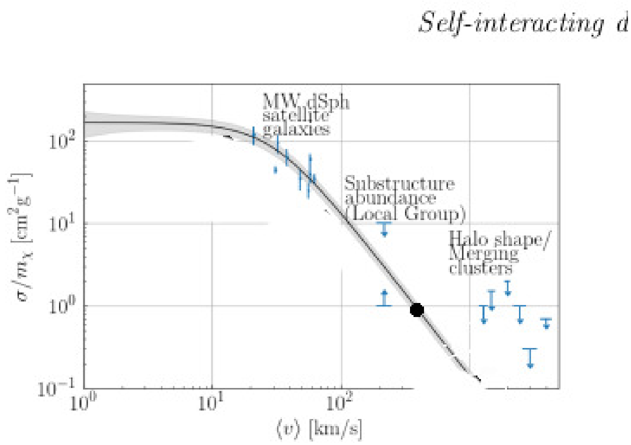

Figure 7. Same as Fig. 6, but extended to cover the range of MW- (~200 km/s) and cluster-size (~1000 – 5000 km/s) haloes' velocities. The figure shows upper and lower limits for σ/m_χ taken for substructure abundance studies (e.g. Volgelsberger et al. 2012 and Zavala et al. 2013), as well as based on halo shape/ellipticity studies and cluster lensing surveys (see text).

Fig. 12.1: Self-interaction cross section over mass of the particles (pearls in our model) obtained by fitting velocities of (of course) the ordinary matter in dwarf galaxies around our Milky Way.

From Figure 2 we obtain the “inverse darkness” values:

$$\frac{\sigma}{M} \approx 150\mathrm{cm}^2/\mathrm{g} = 15\mathrm{m}^2/\mathrm{kg}\,(\text{low velocity}) \tag{12.1}$$

$$\frac{\sigma}{M} \approx 1.5\mathrm{cm}^2/\mathrm{g} = 0.15\mathrm{m}^2/\mathrm{kg}\,(200\mathrm{km/s}) \tag{12.2}$$

But here we must admit that with these “inverse darknesses” the pearls would not go so deep as we would like to make them stop in the depth of the DAMA-LIBRA experiment. So we have to help our model a little bit by declaring that the pearls have caught up dust around them so as to get less dark, but that in the atmosphere or the upper shielding of the Earth this dust is washed off. Washing off the dust makes the cross section smaller presumably without losing much weight, so that after such a cleaning the inverse darkness is decreased and thus the stopping length is increased.

- The dark matter particles can be excited to radiate 3.5 keV X rays or presumably also electrons of the same energy per particle.

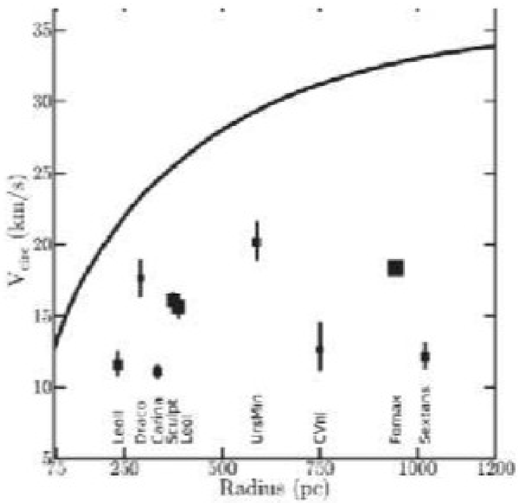

Fig. 12.2: This figure should illustrate, that when Correa simulated the dark matter distribution, under the assumption of a purely gravity interaction, and regained a prediction for the star velocities it was *not* successful. So there is a phenomenological call for e.g. interaction of the dark matter with itself.

12.3 A hard problem: How can ANAIS disagree with DAMA-LIBRA?

The two underground experiments, ANAIS [16] and DAMA-LIBRA [1], and also a third one COSINE-100 [17], which has not yet reached sufficient accuracy to properly disagree with either of the first two, looking for dark matter are *very similar*:

- Both use NaI(Tl= Thallium),
- Both look for seasonal variation,
- Depth is 1400 m ~ 4200 m.w.e. for DAMA-LIBRA, while ANAIS is at 2450m.w.e. (COSINE-100 has a similar depth to ANAIS)

But ANAIS "sees" no dark matter yet and claims to deviate by 3 σ from DAMA-LIBRA, which sees 0.0103 cpd/kg/keV. Our idea to resolve this problem is that: *Dark matter passes quickly by ANAIS, but slows down or stops at DAMA-LIBRA,* see Figure 12.3. If the DAMA-LIBRA signal is due to radiating dark matter pearls then this could solve the problem.

The COSINE-100 group searched for an annual modulation amplitude in their data taken over 3 years in the 1-6 keV region with the phase fixed to the DAMA value of 152.5 days. They subtracted their calculated time dependent background and obtained a positive amplitude of 0.0067 ± 0.0042 cpd/kg/keV, which is to be compared with the DAMA-LIBRA amplitude of 0.0103 cpd/kg/keV. The large error on the COSINE-100 amplitude means that it is consistent with both the DAMA-LIBRA result and no seasonal variation at all. But then they analysed their own COSINE-100 data in the "same way as DAMA" [18] by subtracting a constant background taken to be the average rate over one year, and generated what they consider to be a spurious seasonal variation. In fact they found a modulation amplitude of the opposite phase -0.0441 ± 0.0057 cpd/kg/keV (the wrong season has over abundance). Interestingly the WIMP model could not obtain such a result, but it is possible with our dark matter pearls.

In our model an experiment at some given depth, would have just those dark matter particles stop in the instrument, which have just the right velocity.

12.3.1 How slowdown can help and a huge day/night effect

In Figure 12.3 we illustrate how one must think about dark matter particles coming in and at first having high speed but then slowing down due to the interaction with the earth or stone in the shielding. When the speed gets low even the Earth's gravity can make so much extra acceleration that the trajectories of the dark matter pearls become curved.

But now we have to call attention to the fact that our model would - if we do not as we shall do in a moment find a way of avoiding it - predict a huge variation of the counting rate between day and night. The point is that the solar system moves with about 200 km/s relative to the dark matter center of mass system and, although the spread in the dark matter velocity in the radial direction to and from the center of the galaxy is presumably large, the spread in the direction around the galaxy in which the solar system moves will be much smaller. It might be as small as say ± 90 km/s in which case the average of the dark matter velocity seen from the Earth or the Sun would be say 2 times as large as the spread. On the other hand with stopping dark matter, as in our model, the dark matter can only come into the Earth and to an underground experiment from the side of the Earth on which the experiment is located. It can only come from the direction conceived as from above as seen from the experiment. This means that there will be a huge difference in the rate of visible impacts depending on whether the experiment in on the forward or the backward side of the Earth relative to the motion of the Earth seen from the dark matter average rest system.

When the experiment in question is on the front side of the Earth in its motion relative to the dark matter rest system, most of the dark matter can be overtaken

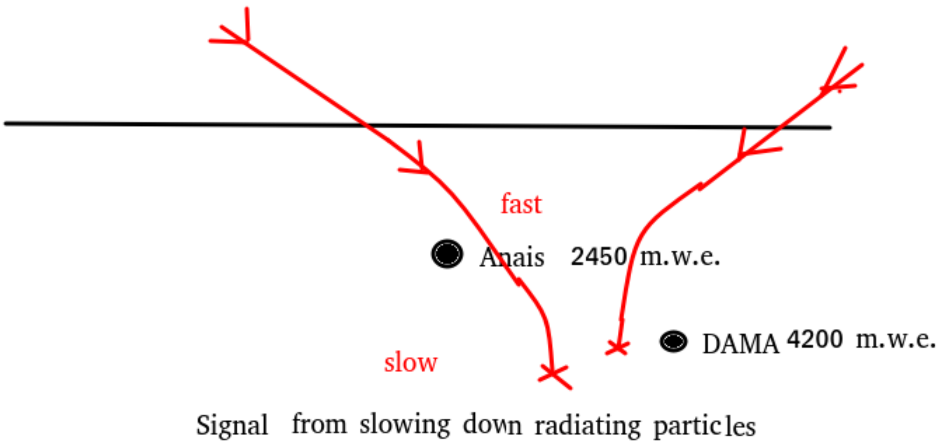

Fig. 12.3: Under the assumption, that dark matter after excitation under the stopping radiates say electrons, and that it is these electrons, that have the best chance to be observed, the fast moving dark matter pearls in the region closer to the surface of the Earth leave only weak signals, while a much stronger signal will be left in the counters of DAMA-LIBRA in a depth, where the dark matter pearls have slowed down.(m. w. e. = meter water equivalent). Notice that we have drawn the trajectories as a bit curved, especially where the dark matter pearls have almost stopped, so much as to move slower. Then namely the effect of the Earth's gravity becomes relatively larger.

by the Earth and fall into the experiment from the sky side, and one should get a very large rate of impact in this situation. Twelve hours later the experiment will be on the back side and now only very few dark matter particles have such a high individual velocity compared to the average velocity of the dark matter relative to the Earth, that they can run in the opposite direction to the majority of the dark matter. So in this case only relatively few dark matter particles can be observed.
For an underground experiment only particles running down towards its site can hit it, except that the Earth can overtake some slow ones. See Figure 12.4.
In Figure 12.5 we see the situation of the Earth running relative to the center of mass for dark matter. With the experiment in front of a huge part of the dark matter, all except the white tip would hit the Earth, while in the case of the experiment being on the backside only a small tip, the green one, will hit the Earth.
In Figure 12.6 we have added some small displacement lines to illustrate how the borders between the amounts of dark matter hitting and not hitting the Earth is shifted by the relatively small velocity changes due to the season. When the experiment is in front in the motion towards the dark matter center of mass motion the hitting rate increases in the summer when the relative velocity is larger. However, when we are in the time of day when the experiment is on the backside the larger summer relative velocity will mean that the little tip of dark matter hitting the Earth gets even smaller. So in the time of day-night in which

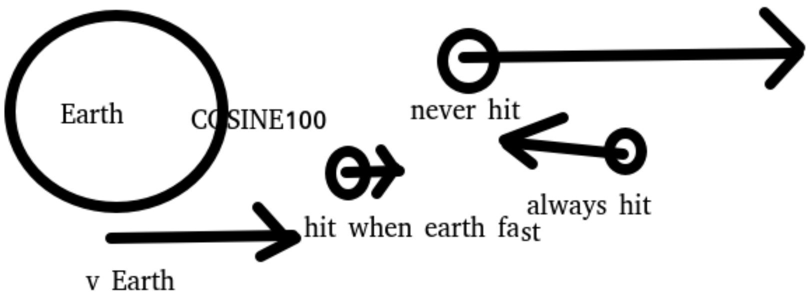

Fig. 12.4: If the stopping length is so short as we speculate in our model, it is only dark matter that penetrates into the Earth from the side of the experiment, e.g. COSINE-100, that has any chance of being detected, while in WIMP-models dark matter particles coming through the earth-interior before interacting is not at all excluded. This figure concentrates on three examples of the velocity of particles having managed to come in on the front side of the Earth say, on which side the experiments lies. Whether dark matter particles now hit the earth, and how fast, depends on the relative velocity.

the dark matter mainly comes in from the side of the Earth opposite to that of the experiment, the seasonal effect is actually opposite to that when the dark matter comes in on the same side as the experiment, i.e there are most counts in the winter and fewer in the summer.

This is definitely a very interesting prediction of ours, if the data on the detection of the dark matter is sufficiently detailed that one can distinguish day and night so to say. But now in detail our own model has it that the dark matter pearls are counted by means of the radiation of say electrons which are sent out delayed compared to the time at which the pearls were excited. If the decay lifetime is several days, then the very big day night oscillation gets washed out in our model and would not be seen. Indeed if the day night effect was as strong as we predict - percentwise stronger than the seasonal one - at first, i.e. if we did not have the washout by the pearls being excited and decaying with a several day lifetime scale, then it should have been easy for DAMA-LIBRA to have observed the day-night variation! (So it means that a model of our type, with only a limited penetration into the Earth, would not work unless we also have a washout due to the decay lifetime being of order of at least days.)

When our dark matter pearls slow down, they either stop completely or continue falling slowly under gravity.

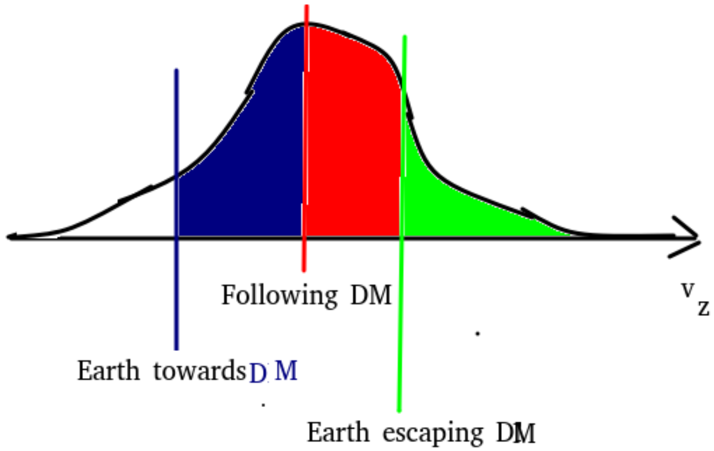

Fig. 12.5: On this figure the Gaussian distribution is the distribution of the dark matter particle velocities in the rest frame for the average velocity of the dark matter, the center of mass. Let us imagine the Earth at first coming in from the right side of the figure with the site of the experiment, COSINE-100 say, on the place having the left direction in zenith. Then, if the velocity towards the dark matter of the Earth - moving to the left - is very high the dark matter in the velocity bands denoted on the figure as dark blue, red and green, i.e. all the ranges except the white one, would hit the Earth on the side of this experiment. But now if the Earth moved slower compared to the dark matter, say it followed the center of mass for the dark matter, then only the red and the green amounts of dark matter would hit the Earth. And if the Earth "moved away" in the sense that the experiment was on the backside of the Earth compared to its velocity in the dark matter center of mass frame, then say only the green part of the dark matter would hit the Earth in the region of the experiment. In this case, where the experiment is on the backside relative to the motion only the very fastest part, the green band say, would be able to overtake the Earth and hit the experiment.

- If they stop completely, we should find a lot of stationary dark matter to be dug out as heavy pearls (much like if it were gold dust and one should wash it out).
- If they sink slowly, they of course go deeper inside the Earth.

12.4 Crude estimate of stopping distance and mass of a dark matter pearl

For simplicity we shall only consider the motion in one direction, down, and ignore the rest. The depth, into which a dark matter particle will penetrate before (effectively) stopping, will be a smooth function of its velocity relative to the Earth. So "topologically" the *distribution of stopping-depths* will reflect the *initial velocity spectrum* in the downward direction, as illustrated in Figure 12.7.

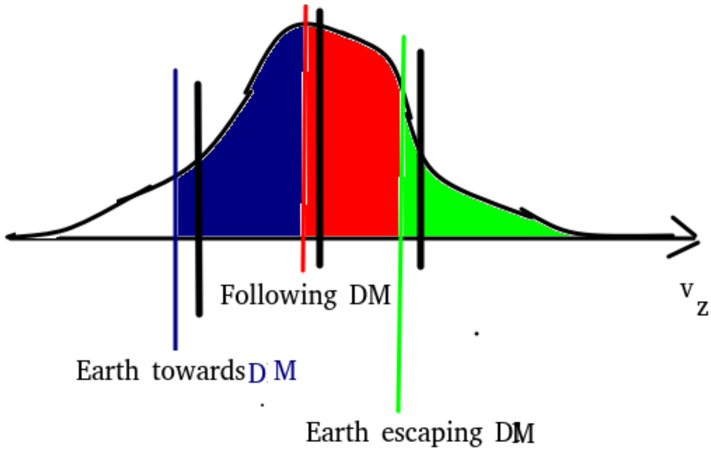

Fig. 12.6: Like in figure 12.5 we consider here a Gaussian dark matter velocity distribution, and look for how with different velocities of the Earth (with the experiment on the left side) moving in the dark matter sees more dark matter in the experiment the faster this Earth moves forward. E.g. if the Earth moves with the experiment on the left side to the left, only the white part moves too fast for the Earth to catch up. But now we are interested in how small variations in the Earth's velocity, exemplified by the vertical lines drawn to the right of the color separation places on the figure. They symbolize a little bit slower Earth. The interesting point is that the effect of such a small velocity lowering is not the same independent of how fast the Earth moves already compared to the dark matter average. Even relatively the extra velocity gives different extra contributions, different even taken relative to the original one.

We note that DAMA-LIBRA is about twice as deep down as ANAIS and COSINE-100, see Figure 12.8. This figure illustrates that in our model ANAIS and COSINE-100 could, in principle, have annual modulation amplitudes with the opposite phase to DAMA-LIBRA, but of course it is not necessary.
In order for our dark matter pearls to stop just at DAMA-LIBRA crudely, we need a penetration depth of the order of magnitude:

$$\text{Penetration depth } L = \frac{M}{\sigma} * 30/\rho_{stone} \approx 1\text{km} \tag{12.3}$$

$$\Rightarrow \frac{M}{\sigma} \approx 10^5 \text{kg/m}^2 \tag{12.4}$$

$$\Rightarrow \frac{\sigma}{M} \approx 10^{-5} \text{m}^2/\text{kg} \tag{12.5}$$

The number 30 is a crude estimate involving a logarithm and factors of order unity roughly - see equation (12.42).
We may compare this value of $\frac{\sigma}{M}$ for reaching just the DAMA wanted order of magnitude with the numbers given in equations (12.1, 12.2). This means that from the low velocity value (12.1) we need a decrease by a factor $1.5 * 10^6$ and from

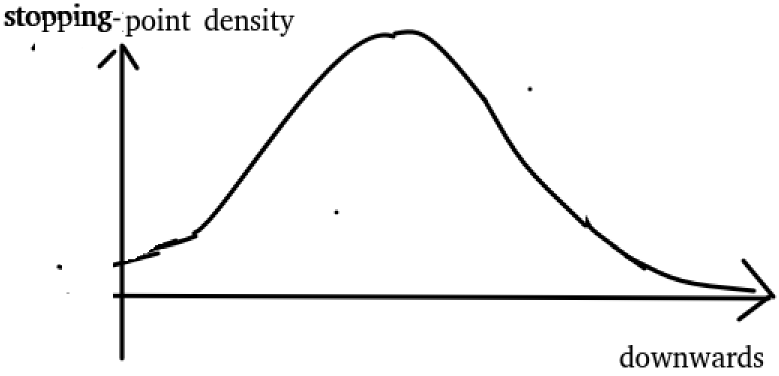

Fig. 12.7: If one has a definite formula for the drag force stopping the pearls as a function of the velocity, then this formula will give the effective stopping depth - or rather the penetration distance - as a function of the initial impact velocity (we consider only the vertical direction as an approximation). And then there will be simply a transformation from the impact velocity in the vertical direction to the depth at the point of stopping (only "effectively", if the particle really continues with a much lower fixed velocity, which is then neglected here). Here we have drawn symbolically the image under this transformation of the initial vertical velocity distribution. It still bears some similarity to the supposed Gaussian velocity distribution.

the $200 km/s$ value (12.2) a decrease by $1.5 * 10^4$. Thus we must take it that so much dust around the dark matter pearl is washed off during the impact, that the cross section goes down by a factor of the order of respectively a million and 10000. Taking it that the relevant velocity is of the order of 200 km/s, the factor ten thousand is enough.

12.4.1 Size of the bubble

The need for assuming the dirt to be washed off the pearl means that the inverse darkness $\frac{\sigma}{M} = 10^{-5} m^2/kg$, identified with the one needed for the penetration depth just reaching DAMA-LIBRA, must be that of the supposedly hard bubble making up the main and most heavy part of the dark matter pearls. Now we used to think that we could estimate the density of this bubble filled with ordinary matter under high pressure by using a dimensional argument [8] - of course rather uncertain - to obtain the Fermi-momentum from the HOMO-LUMO gap E_H identified with the $E_H = 3.5\,keV$ X-ray photon energy line likely to be associated with the dark matter:

$$E_H = \sqrt{2}\left(\frac{\alpha}{c}\right)^{3/2} E_f. \quad (12.6)$$

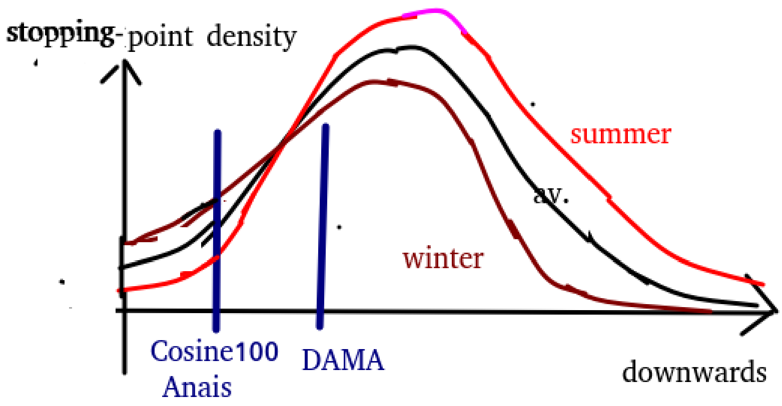

Fig. 12.8: This is again the vertical density distribution transformed into the depth spectrum (ignoring the horizontal components of the velocity), but now we have shown three variants corresponding to the seasons in which the velocity of the dark matter relative to the Earth is different. In summer this relative velocity is higher and thus the stopping spectrum is for summer on average a bit deeper. On the contrary the winter spectrum lies a bit less deep. An average spectrum is also drawn, which we could consider to be that for autumn and spring. Rather than to be realistic, the drawing has been made with the purpose of illustrating that indeed there is the possibility of the relatively close to the surface lying experiments ANAIS and COSINE-100 getting *less counts in the summer than in winter* opposite to the DAMA-LIBRA-experiment, which is about twice as deep (1400 m) and should see most events in the summer.

Here E_f is the Fermi energy in the compressed ordinary matter inside the new vacuum bubble, and α is the fine structure constant, which for our a bit special dimensional argument has been taken to be of dimension velocity, so that it is α/c that is the usual fine structure constant $1/137.036....$ We actually even calculated the $\sqrt{2}$, but that would clearly be far outside the expected accuracy. From this dimensional argumentation we then got the density of the material inside the new vacuum bubble:

$$\rho_B = 5 * 10^{11} kg/m^3. \tag{12.7}$$

Of course the radius of the bubble with the compressed ordinary matter is given as

$$R = \sqrt[3]{\frac{3M}{\rho_B * 4\pi}} = \sqrt[3]{M} * 8 * 10^{-4} kg^{-1/3} m. \tag{12.8}$$

and cross section

$$\sigma = \pi R^2 = \pi^{1/3} 3^{2/3} 4^{-2/3} M^{2/3} \rho_B^{-2/3} \quad (12.9)$$
$$= 1.21 * (M/\rho_b)^{2/3} = 1.9 * 10^{-8} kg^{-2/3} m^2 * M^{2/3}. \quad (12.10)$$

So we obtain

$$\frac{\sigma}{M} = 1.9 * 10^{-8} kg^{-2/3} m^2 (\sqrt[3]{M})^{-1} \quad (12.11)$$

To achieve the value of $\sigma/M = 10^{-5} m^2/kg$ to just reach DAMA we need

$$10^{-5} m^2/kg = 1.9 * 10^{-8} kg^{-2/3} m^2 M^{-1/3} \quad (12.12)$$
$$\text{giving } \sqrt[3]{M} = 1.9 * 10^{-3} kg^{1/3} \quad (12.13)$$
$$\text{i.e. } M = 7 * 10^{-9} kg. \quad (12.14)$$

In a moment we shall see below, by considering the energy of impact of the dark matter, that we would for the purpose of what DAMA sees have preferred the value $9.4 * 10^{-17} kg$.
The density of dark matter in the region of the solar system is

$$D_{sun} = 0.3\, GeV/cm^3. \quad (12.15)$$

The rate of impact energy is then

$$\text{"}Rate\text{"} = v * D_{sun} \quad (12.16)$$
$$= 300 km/s * 0.3 * 1.79 * 10^{-27} kg/(10^{-6} m^3) \quad (12.17)$$
$$= 1.6 * 10^{-16} kg/m^2/s \quad (12.18)$$
$$= 1.4 * 10^{-11} kg/m^2/day \quad (12.19)$$

Now let us call the mass of the individual pearls M and then we can compute the number density of pearls of dark matter:

$$\text{number density falling} = 1.4 * 10^{-11} kg/m^2/day/M \quad (12.20)$$

Spreading over a kilometer we obtain:

$$\text{number density } = 1.4 * 10^{-14} kg/m^3/day/M \quad (12.21)$$

For an earth density $= 3000 kg/m^3$ we then have:

$$\text{number per kg of earth} = 4.7 * 10^{-18} /M per\ day \quad (12.22)$$
$$\text{DAMA saw } S_M = 0.01 cpd/kg/keV \quad (12.23)$$
$$\text{Over say 5 keV:} \quad 5 keV S_m = 0.05 cpd/kg \quad (12.24)$$

If there is only one count per particle,we get

$$4.7 * 10^{-18} kg/M cpd = 0.05 cpd \quad (12.25)$$
$$\text{giving } M = 9.4 * 10^{-17} kg \quad (12.26)$$
$$= 5.3 * 10^{10} GeV \quad (12.27)$$

12.4.2 Discussion of mass value

The agreement of the mass $M = 9.4 * 10^{-17}kg$ needed to get the DAMA number observations with just one electron emission from each dark matter pearl and the number crudely estimated from the dimensional argument and the wish for the appropriate stopping length $7 * 10^{-9}kg$ is far from perfect. But we shall have in mind that the mass came in via a third root, so that a miscalculation of say ρ_B, the density of the compressed ordinary matter by just one order of magnitude would lead to a factor 1000 in the mass. This density ρ_B again depends on the third power of the Fermi energy E_f, so that indeed the mass we need to achieve the wanted inverse darkness $\frac{\sigma}{M} = 10^{-5}m^2/kg$ will go as the ninth power, if there is a mistake in our E_f estimate. So our 8 orders of magnitude deviation in the mass M corresponds to E_f only being about one order of magnitude wrong.
But our model is tensioned in the direction that the density of the bubbles should go up and we should preferably get several counts out of the same dark matter pearl. It should sit in the NaI(Tl) and radiate for say several days. Then we could tolerate somewhat heavier pearls and still get the required number of counts.
So in reality we can consider our model to be successful, because the deviations were not more than about 8 orders of magnitude in the mass, corresponding to one order of magnitude in the Fermi energy.

12.5 Model

We now give a short review of our dark matter pearl model:

- Our "new physics": There are several different phases of the vacuum, but all with same energy density (*Multiple Point Criticality Principle* = *MPP* [19–25]). But even this "several vacuum phases" hypothesis is not truly new physics, if we believe the speculation that the quark masses have been adjusted so that two phases of the QCD-theory can be degenerate [13]: One phase with chiral symmetry spontaneously broken, and another one where the breaking should rather be said to be due to the quark masses.
- Our dark matter particles are macroscopic objects consisting of bubbles of a second vacuum filled with some ordinary matter e.g. carbon.
- There is a skin/wall separating the two vacuum phases with a tension of the order of

$$S^{1/3} \sim 10\,MeV. \tag{12.28}$$

- The nucleons have a lower potential in the inside-the-pearl-vacuum than in the outside vacuum, wherein we live, by a difference

$$\Delta V \approx 3\,MeV. \tag{12.29}$$

  The value $\Delta V \approx 3\,MeV$ was fitted to the inside material having a gap - a homolumo gap - between the empty and the filled electron-states arranged to let the dark matter preferably emit X-rays with the observed energy of 3.5 keV.

- The dark matter pearls are so macroscopic and of such a size compared to mass that they get stopped in the Earth at a distance of the order of the depth of the DAMA-LIBRA experiment (1400 m stone).
- The pearls are excitable so as to radiate X-ray photons of just the energy needed to give the "mysterious 3.5 keV line".

12.5.1 The 3.5 keV line

The observation of a mysterious new X-ray emission line at 3.5 keV was reported [26, 27] in 2014. It was detected in the Andromeda galaxy, the Perseus cluster and different combinations of other galaxy clusters. Later the line was detected in the Milky Way Center [28]. This line has been suggested to originate from dark matter and our model has been adjusted to fit the observations. However this interpretation and even the very existence of the line is controversial. There are indeed several details in the observation of this 3.5 keV X-ray, which a priori does not look supportive for the hypothesis that this line really comes from dark matter: In fact Jeltema et al. [29] have seen it in the Kepler Supernova Remnant where there is far too little dark matter for them to be able to see it, unless dark matter interacts not only with dark matter but also with ordinary matter. Also the distribution seen from the Center of the Milky Way does not a priori look so much like dark matter produced X-rays. Furthermore there are some problems with the details of fitting the 3.5 keV radiation from the outskirts of the Perseus galaxy cluster. Assuming that the dark matter can be brought to radiate the X-rays by interaction not only with other dark matter particles, but also with ordinary matter, we believe we can improve the understanding of these mentioned three problems in fitting the 3.5 keV radiation observations with dark matter pearls.
We should point out that the observations of the 3.5 keV line are only seen as very small deviations of the spectra observed from what is understood from the various ions in the sources. They are on the borderline of being statistically significant (see Figure 12.9 for the Perseus Cluster and Figure 12.10 for Andromeda taken from from Iakubovskyi [30]):
We identify the emission of some particle or another as seen in DAMA-LIBRA and potentially in COSINE-100 etc. with the emission from the mysterious 3.5 keV line. So it becomes of course crucial that the energy seen in the events of the seasonal varying type in DAMA-LIBRA is indeed just 3.5 keV. In Figure 12.11 we show a fit (red) to the DAMA-LIBRA low-energy spectrum [31] consisting of a background model (grey/dashed) and a Gaussian distribution function (green). The parameters of the Gaussian are shown in the figure and the energy is given as 3.15 keV in remarkably good agreement!

12.6 Distributions

Really to get an idea of how the radiation from excited dark matter pearls get distributed in the Earth, we must take into account that the distribution of their incoming direction is presumably very smoothly distributed over the sky. We shall now make a couple of somewhat special approximations and derive the

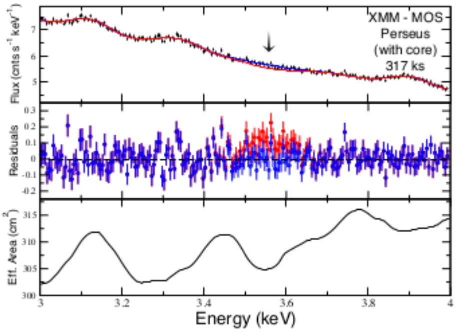

Fig. 1: The combined MOS spectrum of Perseus cluster scaled to 3-4 keV energy range. On top of their best-fit model, the series of the single-bin residuals corresponding to the extra emission line at 3.57 keV is shown in red. (Adapted from Figure 7 in [70]).

Fig. 12.9: Spectrum of X-rays from the galaxy cluster Perseus. The curve is fitted to the expected X-ray spectrum from known ions. The little deviation at 3.5 keV could be from dark matter?

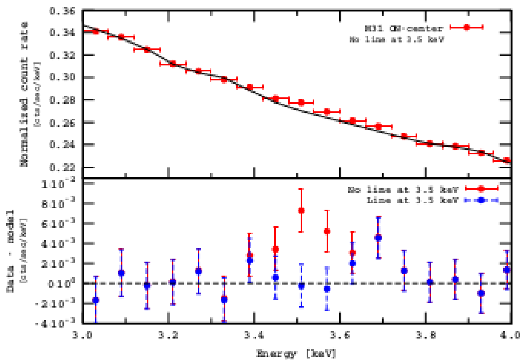

Fig. 2: The same as in the previous Figure 1 but for the combined spectrum of Andromeda galaxy. (Adapted from Figure 1 in [71]).

Fig. 12.10: X-ray spectrum from Andromeda.

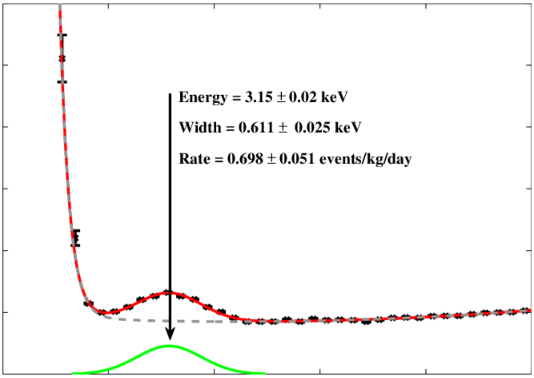

Fig. 12.11: Here the seasonally varying part of the DAMA-LIBRA data is fitted to a Gaussian with the average energy 3.15 keV. This is remarkably close to the line observed from Perseus Cluster and Andromeda and other galaxy clusters etc.

distribution at the depth of the point under the surface of the Earth at which one detects the radiation.
We shall make 4 ideal calculations based on the following assumptions:

- What is counted is the emissions from the dark matter particles of some radiation (electrons and/or X-rays, actually with energy 3.5 keV), which is assumed to go on with a constant rate for a long time (we may think of weeks) after the impact of the pearl. This means that in order to obtain a distribution with respect to depth, we need to calculate the length of time during which a dark matter particle is in each little interval of depth.
- At first we assume that gravity is so weak that we can ignore it and assume that the pearls stop completely. Then in the second calculation we suppose that the pearls continue to fall directly downwards under gravity with a constant very low velocity. After the set in of this "terminal velocity" the pearls contribute the same amount of radiation in all depth layers below the point where the "terminal velocity" has set in. This means that we consider that there are two different stages in the motion of a dark matter particle:
  - A fast stage from when the dark matter particle enters the Earth through its surface until the particle has slowed down and we assume it moves so fast that it has no time to emit significant amounts of radiation. So during this stage we can neglect the radiation and the particle is effectively invisible.
  - In the next stage we first consider the case where the dark matter particle stops and stays sitting in the Earth until its amount of excitation has burned out (after a long time, say a week, but taken at least as an average as a fixed constant). This case is illustrated by the *black* rectangle in Figure 12.12.
- Further, as a simplifying assumption, we assume that all the particles have the same stopping length (in the earth) and have the same incoming velocity. So all the dark matter particles hitting the Earth's surface at a certain point end up and stop (or in the second case considered below go into the next stage with a terminal velocity) at a point lying on a half sphere around the point of impact with the Earth, having a radius equal to the stopping length.

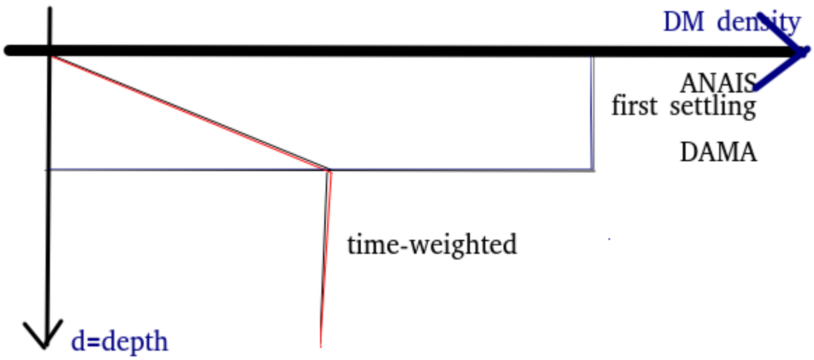

Fig. 12.12: We consider a model in which the incoming direction of the dark matter pearls has a constant distribution over the sky. At first we assume they stop completely after passing through a ”stopping length” of earth and only then radiate significantly. Then we get a flat distribution of radiation delivery as a function of the depth down to a deepest depth equal to the ”stopping length”, beyond which there is no more probability for finding any radiation. This corresponds to the black rectangle on the figure. However if the pearls do not stop completely, but begin to fall with a constant velocity determined by a balance between the friction and the gravitational attraction by the Earth after reaching the ”stopping length”, then the distribution of the radiation delivered is given by the red curve as a function of the depth.

- Again for simplicity, we assume that the distribution of the directions from which the dark matter particles come is evenly distributed over the sky. But it is of course in our model with a stopping length much shorter than the radius of the Earth, only the half sphere where the particles going downwards can come from that gets populated. Particles seeking to come upwards are stopped on the other side of the Earth.
- We also assume that the radiation (of electrons or X-rays of energy 3.5 keV) emitted during the first stage when the pearls pass quickly through the medium (essentially the earth, or the experimental apparatus scintillator) is negligible compared to the radiation emitted at the stopping point or, in the second case, when the dark matter sinks (before it runs out of excitation energy).

Let us now illustrate the rather trivial calculation of the radiation intensity observed:
First notice that if the dark matter particle comes in with a direction making an angle θ with the downward vertical, then the depth to which it has reached when

the particle has stopped is

$$\text{stopping depth } = \text{“}\mathit{stopping\ length}\text{''} * \cos(\theta) \quad (12.30)$$

Now the area on the half sphere - which by the assumption of an even distribution on the sky is proportional to the fraction of the dark matter particles ending up there - corresponding to the depth z being in the infinitesimal interval dz is given as

$$dz = \text{“}\mathit{stopping\ length}\text{''} * d\cos\theta \quad (12.31)$$

$$= (-)\text{“}\mathit{stopping\ length}\text{''} * \sin\theta * d\theta \quad (12.32)$$

while

$$d\text{“}\mathit{area}\text{''} = 2\pi * (\text{“}\mathit{stopping\ length}\text{''})^2 * d\theta * \sin\theta. \quad (12.33)$$

so that

$$dz \propto d\text{“}\mathit{area}\text{''} \propto d\text{“}\mathit{probability}\text{''} \text{ up to } z = \text{“}\mathit{stopping\ depth}\text{''}. \quad (12.34)$$

This means that the radiation is present only down to the depth just equal to the stopping length, and in the interval with lower depth than that the radiation rate is quite constant. This is the black rectangular distribution given in Figure 12.12, which is namely the case in which the dark matter particles sit still on the surface of the half sphere and radiate out.

In the second case the dark matter particles almost stop and then start moving straight downwards with a constant terminal velocity, which is so slow that they contribute significant amounts of radiation at lower depths. Hence one gets the radiation rate for a depth z to be given by the integral of the rectangular distribution over all the higher depths:

$$\text{“}\mathit{Rate}\text{''}(z) \propto \int_0^z \text{“}\mathit{rate}\text{''}_{\mathit{rectangular}}(z')dz', \quad (12.35)$$

It is easy to see that the resulting radiation rate “Rate''(z) has a non-zero slope in the uppermost stopping length, while it becomes constant under this domain.

Next we want to repeat the above two calculations including the effect of gravitation under the fast stage, which we have neglected so far. We perform this improvement very crudely assuming:

- We can ignore the variation as a function of the angle θ of the time the stopping in the fast stage takes. I.e. whatever the angle and thereby the depth in which the particle essentially stops it takes the same time, counted from the passage through the Earth's surface.
- Much like in an Einstein elevator the particle falls compared to the motion ignoring gravity by the amount that a free fall in the same time would have caused an object to fall.

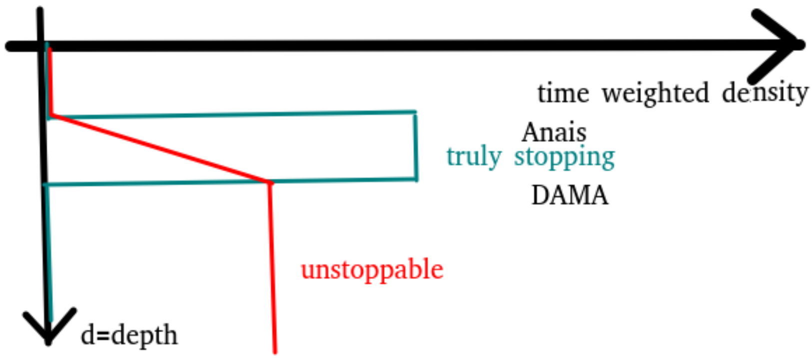

Fig. 12.13: Here we take into account that the Earth's gravity acts on the dark matter pearl even while it is in the fast stage of its motion. Then even a pearl coming in almost horizontally will sink a bit before it stops or falls with its terminal velocity So in fact the black rectangle from figure 12.12 will essentially be lowered by a constant amount in depth. Also the distribution for the delivery of the radiation in the case of the pearl having a terminal velocity gets a corresponding lowering of the red curve from figure 12.12.

With these assumptions the effect of introducing gravity on the curves of Figure 12.12 is simply to move them downward by the free fall distance. We here mean the free fall distance of a body falling for as long as the stopping time. The uppermost range of depth corresponding to this falling distance, becomes completely free of radiation in our approximations. So an experiment so close to the surface should see no dark matter at all.
These moved down curves are drawn in Figure 12.13.
If we could arrange say that ANAIS should lie so close to the surface as the falling distance during the stopping time then, under the rather simplified assumptions just presented, we could explain the lack of observation of dark matter.

12.7 Size

The dragging force stopping the dark matter pearl is

$$F_D = C_D \frac{Av^2 \rho_{material\ outside\ pearl}}{2} \tag{12.36}$$

where v is its velocity and $\rho_{material\ outside\ pearl}$ the density of the fluid or material through which the pearl falls. C_D is the drag coefficient and is of order unity at high speed. $A \approx \sigma$ is the area shown to the motion.

The equation of motion (Newton's second law) becomes:

$$M\dot{v} = -F_D = -\frac{C_D \sigma v^2 \rho_{material\ outside\ pearl}}{2} \tag{12.37}$$

which can be rewritten and integrated to give the stopping length $L_{stopping}$ as follows:

$$\frac{\dot{v}}{v^2} = -\frac{\sigma}{M} * \frac{C_D}{2}\rho_{material\ outside\ pearl} \tag{12.38}$$

$$-\frac{1}{v} = -t * \frac{\sigma}{M} * \frac{C_D}{2}\rho_{material\ outside\ pearl} + const. \tag{12.39}$$

$$v = \frac{1}{t * \frac{\sigma}{M} * \frac{C_D}{2}\rho_{material\ outside\ pearl} - const.} \tag{12.40}$$

$$L_{stopping} = \int v dt = \int_{v=300km/s}^{v\ small} \frac{dt}{t * \frac{\sigma}{M} * \frac{C_D}{2}\rho_{material\ outside\ pearl} - \cdots} \tag{12.41}$$

$$\approx -(\frac{\sigma}{M} * \frac{C_D}{2}\rho_{material\ outside\ pearl})^{-1} \ln(\frac{\text{“small”}}{300km})$$

Rewriting the estimate of the stopping length we have

$$L_{stopping} \approx \frac{1}{\frac{\sigma}{M} * \frac{C_D}{2}\rho_{material\ outside\ pearl} \ln(\frac{300km/s}{\text{“small”}})}$$

$$L_{stopping}\rho_{material\ outside\ pearl}\sigma \approx M\frac{2}{C_D}\ln(\frac{300km/s}{\text{“small”}}) \tag{12.42}$$

Now $L_{stopping}\sigma * \rho_{material\ outside\ pearl}$ is the amount of material pressed away by the passage of the pearl through the earth. So noticing that $\frac{2}{C_D} * \ln\frac{300km/s}{\text{“small”}}$ is just of order unity, say 30, we see that the mass of the material pressed away by the the pearl is only an order of unity times bigger than the mass M of the pearl itself.

If the pearl essentially stops at the DAMA depth with 4200m w.e. then we have

$$4200m * 1000kg/m^3 = \frac{M}{\sigma}\frac{2}{C_D} * \ln\frac{300km/s}{\text{“small”}} \tag{12.43}$$

$$\text{giving } \frac{\sigma}{M} = \frac{1}{4.2 * 10^6 kg/m^2 * 30} \tag{12.44}$$

$$= 10^{-7} m^2/kg. \tag{12.45}$$

This should be compared with the inverse darkness obtained by Correa for velocities around 300 km/s from her analysis of dwarf galaxies [15]

$$\frac{\sigma}{M} = 1cm^2/g = 0.1m^2/kg. \tag{12.46}$$

This means we need that so much dust washed off the pearls coming into the Earth that their cross section is diminished by a factor 1 million. So in area, they should have lost a factor 10^6 and in linear scale a factor 10^3.

12.8 Physics of hoped for phase transition

By fitting to our dark matter model we found the order of magnitude of the tension S of the domain wall between the two phases should lie in the range

$$S = (\text{few MeV})^3 \text{ to say } (30\,\text{MeV})^3 \tag{12.47}$$

This indicates that the physics involved in making this two vacua, if it is right, should be in an energy range which is *not at all new*.
The number $(30MeV)^3$ is what letting the domain walls replace the cosmological constant (the dark energy) would require [14].
We obtained the relationship between the mass M and surface tension S for our pearls mainly by adjusting the density of the strongly compressed material of ordinary matter inside the new type of vacuum to have a HOMOLUMO gap suitable for emitting a 3.5 keV X-ray line. We further assumed that the pearls are not far from collapsing and spitting out the nuclei inside, in spite of a potential keeping them in of $\Delta V \approx 3MeV$. This relationship is illustrated here by giving examples of the mass M for a few values of the cubic root of the surface tension S.

$$S^{1/3} = 1\,\text{GeV} \Rightarrow M = 24\,\text{ton} \tag{12.48}$$

$$\text{while} \quad S^{1/3} = 100\,\text{MeV} \Rightarrow M = 24\,\text{mg} \tag{12.49}$$

$$\text{and} \quad S^{1/3} = 10\,\text{MeV} \Rightarrow M = 2.4 * 10^{-14}\,\text{kg} = 1.4 * 10^{13}\,\text{GeV}. \tag{12.50}$$

If we take the tension to be say $(10MeV)^3$, then we should look for physics at this scale to find out what could be used to cause the phase transition making the two vacua.

12.9 Conclusion

We have briefly reviewed our model for the dark matter being pieces of a new vacuum phase - which should though be understandable in terms of Standard Model physics, namely the Nambu JonaLasinio chiral symmetry breaking and QCD - filled under high pressure with ordinary matter.
But mainly we have searched for a way to resolve the seemingly very hard mystery that, while the DAMA-LIBRA experiment has collected more and more evidence for dark matter other experiments have not yet seen any. In particular the ANAIS and COSINE-100 experiments are very similar in that they also use NaI(Tl) scintillator but do not confirm the seasonal variation in their data which DAMA-LIBRA use to single out the dark matter. AMAIS and COSINE-100 claim a 3 σ disagreement with DAMA-LIBRA but COSINE-100 has less data. Even more, all the xenon scintillator based experiments having searched for dark matter have seen nothing. The solution we have proposed to this disagreement is that dark matter is macroscopic and interacts more strongly than assumed in other models such as WIMP theories, but still sufficiently dark to fit observations. WIMPS are not significantly stopped by the earth shielding. However our dark matter pearls are stopped in

a depth of order of that of the DAMA-LIBRA experiment 1400m, after passing through the Earth's surface with a high galactic velocity of the order of 200 km/s. In our model the dark matter can be excited and then emit electrons or photons with the preferred energy of 3.5 keV (corresponding to the controversial X-ray line associated with dark matter). So we can claim that the main effect seen by DAMA-LIBRA is this 3.5 keV radiation emitted by stopped or very much slowed down dark matter particles. But the dark matter particles pass through the experiments like ANAIS and COSINE-100 so fast that there is not sufficient time for them to give an observable signal. If really we take the dark matter to only be effectively observable when stopped and having time to emit its excitation energy, then the *fluid* xenon used by the majority of the underground experiments looking for dark matter cannot keep the dark matter stationary and thus, under such assumptions, have no chance to "see" the dark matter. At least if the Earth's gravitation is sufficient to drive the dark matter particles through the liquid xenon the liquid xenon experiments cannot observe them.

Very crucial for our speculation is that the signal DAMA-LIBRA sees is indeed 3.5 keV radiation emitted by stopped dark matter particles. It is a *remarkable fact* that the DAMA-LIBRA spectrum of the seasonally varying component is fitted by an essentially Gaussian distribution in energy with an average energy $3.15\ \mathrm{keV}$ *close to the* $3.5\ \mathrm{keV}$ *line*.

We remark that in our model we can even obtain a negative seasonal variation (i.e. with more events in winter than in summer). Interestingly COSINE-100 generated such an effect, which they consider as spurious, by using the DAMA-LIBRA background subtraction procedure on their data.

But if now dark matter is indeed, as we suggest, stopped in a depth under the Earth's surface of the order of the depth of DAMA, 1400 m, then one should - of course - seek to *dig dark matter out at this depth.* In water equivalent depth it would accidentally be very close to the bottom of the oceans, 5 km of water. The dark matter should be easily distinguished by having an abnormally high specific weight.

So finally we believe that the DAMA-LIBRA results, contrary to other experiments, point to the need for a macroscopic model of dark matter and that the mean free path in earth for galactic velocity dark matter must be of the order of 1400 m.

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References

1. R. Bernabei et al, Prog. Part. Nucl. Phys. **114**, 103810 (2020).
2. C. D. Froggatt and H. B. Nielsen, Phys. Rev. Lett. **95** 231301 (2005) [arXiv:astro-ph/0508513].
3. C.D. Froggatt and H.B. Nielsen, Proceedings of Conference: C05-07-19.3 (Bled 2005); arXiv:astro-ph/0512454.

4. C. D. Froggatt and H. B. Nielsen, Int. J. Mod. Phys. A **30** no.13, 1550066 (2015) [arXiv:1403.7177].
5. C. D. Froggatt and H. B. Nielsen, Mod. Phys. Lett. **A30** no.36, 1550195 (2015) [arXiv:1503.01089].
6. H.B. Nielsen, C.D. Froggatt and D. Jurman, PoS(CORFU2017)075.
7. H.B. Nielsen and C.D. Froggatt, PoS(CORFU2019)049.
8. C. D. Froggatt, H. B. Nielsen, "The 3.5 keV line from non-perturbative Standard Model dark matter balls", arXiv:2003.05018.
9. H. B. Nielsen (speaker) and C.D. Froggatt, "Dark Matter Macroscopic Pearls, 3.5 keV -ray Line, How Big?", 23rd Bled Workshop on What comes beyond the Standard Models (2020), arXiv:2012.00445.
10. C. D. Froggatt and H.B.Nielsen, "Atomic Size Dark Matter Pearls, Electron Signal", 24th Bled Workshop on What comes beyond the Standard Models (2021), arXiv:2111.10879.
11. C. D. Froggatt and H.B. Nielsen, "Atomic Size Pearls being Dark Matter giving Electron Signal", arXiv:2203.02779.
12. H. B. Nielsen (speaker) and C.D. Froggatt, "Dusty Dark Matter Pearls Developed", 24th Bled Workshop on What comes beyond the Standard Models (2022), arXiv:2303.06061.
13. C.D. Froggatt and H.B. Nielsen PoS(CORFU2022)003, arXiv:2305.12291
14. C.D. Froggatt and H.B. Nielsen PoS(CORFU2022)205, arXiv:2305.18645
15. C. A. Correa, MNRAS **503** 920 (2021) [arXiv:2007.02958].
16. J. Amaré et al, Phys. Rev. **D103**, 102005 (2021) [arXiv:2103.01175]
17. G. Adhikari et al. Phys. Rev. **D106**, 052005 (2022)
18. Adhikari, G. et al., Sci. Rep. **13**, 4676 (2023).
19. D. L. Bennett, C. D. Froggatt and H. B. Nielsen, NBI-HE-94-44, GUTPA-94-09-3, Presented at Conference: C94-07-20 (ICHEP 1994), p.0557-560.
20. D. L. Bennett, C. D. Froggatt and H. B. Nielsen, NBI-95-15, GUTPA-95-04-1, Presented at Conference: C94-09-13 (Adriatic Meeting 1994), p.0255-279 [arXiv:hep-ph/9504294].
21. D. L. Bennett and H. B. Nielsen, Int. J. Mod. Phys. **A9** 5155 (1994).
22. D. L. Bennett, C. D. Froggatt and H. B. Nielsen, NBI-HE-95-07, Presented at Conference: C94-08-30 (Wendisch-Rietz) p.394-412.
23. C. D. Froggatt and H. B. Nielsen, Phys. Lett. **B368** 96 (1996) [arXiv:hep-ph/9511371].
24. H.B. Nielsen (Speaker) and C.D. Froggatt, Presented at Conference: C95-09-03.1 (Corfu 1995); arXiv:hep-ph/9607375.
25. Gia Dvali, arXiv:1107.0956.
26. E. Bulbul, M. Markevitch, A. Foster et al., ApJ. **789**, 13 (2014) [arXiv:1402.2301];
E. Bulbul, M. Markevitch, A. Foster et al., ApJ. **831**, 55 (2016) [arXiv:1605.02034].
27. A. Boyarsky, O. Ruchayskiy, D. Iakubovskyi and J. Franse, Phys. Rev. Lett. **113**, 251301 (2014) [arXiv:1402.4119].
28. A. Boyarsky, J. Franse, D. Iakubovskyi and O. Ruchayskiy, Phys. Rev. Lett. **115**, 161301 (2015) [arXiv:1408.2503].
29. T. Jeltema and S. Profumo, MNRAS **450**, 2143 (2015) [arXiv:1408.1699].
30. D.A. Iakubovskyi, Advances in Astronomy and Space Physics **6**, 3-15 (2016) [arXiv:1510.00358]
31. Z. Ahmed et al, Phys. Rev. **D81**, 042002 (2010) [arXiv:0907.1438]

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13 Deriving Locality, Gravity as Spontaneous Breaking of Diffeomorphism Symmetry

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Abstract. Looking for ideas for a fundamental physics that could almost work by itself, not making but mild assumptions, we are inspired by the earlier work by Astri Kleppe and myself. We showed that assuming diffeomorphism symmetry in addition to only "mild assumptions" we could obtain obtain effectively a local Lagrangian density,**without assuming that the action were at all local!** However, we point out in the present article, that unless there is a spontaneous breakdown of the full diffeomorphism invariance by gravitational like fields having non-zero vacuum expectation values, there would be no propagation possible, in other words the resulting effective local theory would be superlocal, so no communication between the different space-time points. We suggest that having a projective space or some similar space with some more structure, it could be in practice enough to get the effective locality, so the full diffeomorphism symmetry might not not be really needed. Finally we found a phenomenological feature of the CMB (= Cosmic Microwave Background Radiation) low l spherical harmonic expansion of the CMB fluctuations which we interpret as a sign of space-time in the background in which space-time is embedded is a projective space, rather than just a manifold.

Povzetek: Avtor želi definirati naravne zakone s čim manj privzetki in zahtevami. Prve korake sta naredila z Astri Kleppe v skupnem članku: Predpostavila sta maksimalno simetrijo prostora-časa, ki sta jo omejila z zelo blagimi zahtevami. Privedlo ju je do skoraj lokalne Lagrangeove gostote, ne da bi zahtevala lokalnost. Pokazala pa sta, da taka Lagrangova gostota nima dinamike, če te popolne simetrije ne zlomi gravitacija ali gravitaciji podobno polje z od nič različno vakuumsko pričakovano vrednostjo. Avtor tokrat predlaga projektni prostor, ali prostor z nekaj več strukture, ki bi prinesel učinkovito lokalnost. Tako blago zlomitev simetrije prinaša že CMB (kozmično mikrovalovno sevanje ozadja).

13.1 Introduction

In the search for a theory beyond the physics we know today we have long attempted "Random Dynamics", which consists of asking:
Are there some laws of nature that **can be derived** from other ones in some (e.g. low energy-) limit ? If so we should leave the derivable law out of the fundamental model, and assume the ones needed in derivation as **a step more fundamental**.

The example today: **Locality** $\Leftarrow$ **Diffeomorphism symmetry** (this derivation Astri Kleppe and myself [2] already claim to have performed/ proved)

** Speaker at the Workshop "What comes Beyond the Standard Models" in Bled, 1923

So: Assume that **diffeomorphism symmetry** or something similar - e.g. projective geometry symmetry, or simplectomorphic symmetry - is a very fundamental principle, while **we do not assume locality as fundamental** at the same level.

In the future we shall of course attempt **also to derive diffeomorphism symmetry** from something else, and that might e.g. be by assuming some "large amount" of symmetry acting on a space. Actually Masao Ninomiya and myself have stressed, that an infinite space acted upon in an sharply 3-transitive (see these concepts "sharply 3-transitive" below in section 13.6) way is already close to be the projective line [13].

13.1.1 Philosophic Speculative Introduction, Plan

Let us argue a bit looking at the present work as seeking a theory for gravity, behind or beyond gravity:

- Introduction (we do not know what is behind gravity)
- Argument for a geometry with local scale and projective symmetry:
  - Astri Kleppe and I could derive **locality** of the effectively resulting action. We take it that since we can get locality out from starting from a symmetry postulate that has the wole symmetry of the manifold, **without putting in explicitly locality**, then it means that we can avoid the extra assumption, i.e. we have a simplification, if we assume such a symmetry like the whole diffeomorphism symmetry.
  - Now we Only get gravity after a background field is assumed, like $g^{\mu\nu}(x)$. Actually unless we have such a field non-zero in vacuum, we get no propagation, i.e. the different space time points do not get connected. So you make say: We need as a very abstract need for physical theory, that there is connection between the different space time events. Then we must have some field like $g^{\mu\nu}(x)$ or some vierbeins or the like that can do the same job of introducing propagations of the field through space time. This is not really a derivation of gravity, but it is a little bit in that direction in as far as this argument present a **need** for gravity. In usual theories like genrel relativity one could say that we just have introduced gravity because of phenomenological need, one simply has known for long effects of gravity, and thus we need it. Here I seek to say: we need something less specific, propagation of particles, and then because we have for other reasons(the beauty of deriving locality) assumed too much symmetry, we have come to need some fields, which of the most obvious type looks just like gravity fields. So we came to assume too much symmetry so that gravity could not be avoided for **other than simply phenomelogical reasons of seeing gravitational forces**.
    In this point of view the gravity fields represent the fields giving the certain needed spontaneous break down of the too much assumed symmetry.
    This might make us think that it is less fundamental, but of course it of something from which it can be constructed has to be in the theory. Psychologically this way of looking at it might make or a bit less keen on

looking at the effects of the fundamental theory, we look for, expected from gravity, because after all the gravity was just the presumably composite field breaking the too much assumed symmetry.
That is to say we could use this way of thinking as an encouragement not to look for that we should have to bother with all the terrible topologically complicated space times, which are almost unavoidable in quantum gravity. They are in so huge amounts that they would be pretty hard to treat mathematically.

- Projective geometry (a possibility beyond gravity)
Rather we might take the here put forward suggestion of looking at gravity as being embedded into the manifold. The manifold is there at the most fundamental level, at first we could say. But then we may get the hope that the terrible topological forms of manifolds might be avoided by modifying a bit the theory by replacing the manifold by a topologically less terrible structure with sufficiently similar symmetries that we could at least approximately still derive the locality principle. A proposal of a very nice structure behaving similarly to the manifold but with much simpler topological form is the **projective spaces.** So the speculative model which we might propose here is that we live in back ground space time which is a projective space. The structure of such a space time is so simple that seen form small scale it is like a flat space, and thus has the promising feature af providing a kind of explanation for how flat space time is in practice.
- A further hope brought by the projective space idea, is that there is hope to characterize the projective space by defining it by means of its symmetry properties. Indeed Masao Ninomiya and myself recently brought attention to that the projective line (= the projective pace of dimension 1) were close to come out by assuming a space to be sharply 3-transitively transformed under a group. Just assuming such 3-transitivity you begin to see structures like the field (the real field say) without putting in the concept of a field directly. Remember that the manifold is defined by means of coordinates, which are differentiable with respect to each other, and thus you define manifolds mathematically in a way already using the field-concept as wellknown. If the dream could come through of defining the background space time (now assumed to be a projective space time) without explicitly putting in our real numbers, but rather getting it out somewhat similar to studies of 3-transitive actions of groups, then it would be very nice and suggestive on us being on the right track for the fundamental theory, because we could then claim: we did not even put in our real numbers, no they came out instead!
Even though this requirement 3-transitivity only leads to ‘almost fields” and not completely to the real fields as one is accustomed to use in the physical geometry, it is till much better than to have to put the fields in completely from outside.
- A phenomenological support of projective geometry.
Then what could in the long run if it works out really support the idea of a background projective space would be if what we shall present below in section 13.9 is really working out and we have a phenomenological observational indication that we indeed live in a world embedded in a projective space time.

- Conclusion:
  A very optimistic Random Dynamics dream, might look like this: Almost whatever a very complicated mathematical structure would be like, it would if it is very big unavoidably have some similarity of some parts with some other parts. Such a similarity - presumably approximate only - would naturally be expressed by some approximate **symmetry**, which of course in the mathematical language means that there is some group G acting on the compicated structure being the world, say X, just as talked about in section 13.6. Then it is needed to find out which properties of such a system a group acting on a set/structure is most likely to be the type relevant for such an attempt of a theory of everything.
  With some empirical support I and Don Bennet found what characterizes the Standard Model group $S(U(2)\times U(3))$ (which is gauge group in O'Raifeartaigh.s sense [10] (Group Structure of Gauge theories,University Press Cambridge (1986)), namely that the representation of the acting gauge group could be found to be essentially in volume the smallest possible compared to a volume constructed for the group itself [6–9]. Taking this to say " The group shall be so large compared to the set7structure it acts on as possible (in Nature of fundamental physics)", and noting that that the n in the (sharp) n-transitivity of a group action roughly means that the group is the nth cross product power of the set X on which it acts. In fact you can say that the group has been brought in correspondance to the cross product $X \times X \times \cdots \times X$ (with n factors X=). Really a manifold or a projective space are spaces with relatively many of symmetries, so there is good hope to find that such a big group compared to the object acted upon could favor just embedding spaces, we suggest in this article. But then since gravity is needed for getting propagation and the same big group compared to the representation or almost the same the space acted upon, then the Standard model gauge group could also come from such principle. Taking it that the fermion representations and the Higgs representation in the Standard Model are indeed among the **smallest** faithful ones, there is not much in our present knowledge about the physical laws, which would not be almost unavoidable the here sugested system.

13.1.2 Listing of Sections

In the following section 13.2 we shall present and discuss the mentioned theorem of Astri Kleppes and mine, in which we get locality without putting it in, while the detailed proof of this theorem will be postponed to section 13.4 . Next in section 13.3 we put forward the problem, that we cannot get a world with propagation of fields/particles without a spontaneous breakdown by having a non-zero $g^{\mu\nu}$-field (with upper indices) in vacuum. In the subsection 13.3.1 we give a new way of making dimensional reduction in the spirit of the world being embedded into the manifold (this is the manifold always used in gerel Relativity) or some projective space which is similar to the manifold. In the subsection13.3.2 we again review the point of the gravity being needed for propagation.
In section 13.4 we give the real proof of our derivation of the principle of locality together of course with the statement of the "mild" assumptions, such as the

analyticity - as a function**al** - of the otherwise so general action, that it is **not** by assumption local.
Now it is often the most interesting about Random Dynamics derivations, that they do not succeed completely and also the derivation of locality is only partly succesful. In section 13.5 we thus tell that one of the results from this not quite succesfulness of the locality derivation is that we obtain an idea to derive (with in addition only very "mild" assumptions) an old postulate of ours called " Multiple Point Criticality Principle" (=MPP).
In the next section 13.6 we review the mathematical concept of a group action acting n-transitively.
In section 13.7 we suggest that one should exercise finding the effective action - the now local one - which obeys the symmetries in our model left over, so that we can give at least an idea about have one by a bit more work might see that essentially the usual einstein general relativity comes out of the model, in this article we have mainly left this for the reader, but we hope to come through in another article.
The next section 13.8 is then mainly a review of projective geometry, which we consider a promising candidate for a space to replace the full manifold. It may not lead to quite as perfect locality as the manifold, but after all phenomenological quantum field theories, as we know them and use them, have deviations from locality at very short distancies, so if we get that in our model, it might be an advantage. Indeed in section 13.11 we speculate that such a slight breaking of locality might give us the hope of obtaining an ultraviolet cut off theory in spite of the these manifold or projective space-times are a priori having infinitely small distances on the same footing as the large ones.
In secton 13.12 we put forward the remark that our model of the embedding into a projective space time might give some hope for being able to explain the appearance of the huge almost flat space time volumes, we find phenomenologically; and in section 13.13 we deliver the speculation of characterising a projective space as a set/space on which a group acts in an especially strong way. To say sharply n-transitively is connected with problems in as far as there are no true more than 3-transitive, but anyway... infinite spaces.
In section 13.14 we conclude and resume the article.

13.2 Astri Kleppes and mine Theorem:

Even taking an action $S[\phi]$ depending on many fields defined over a space-time manifold **not to be a priori local at all** but only to obey

- 1. $S[\phi]$ is Taylor-expandable as a **functional**,
- 2. It is **symmetric** under the **diffeomorphism symmetry**,
- 3. We observe it only with so long wave lengths that only products of fields up to some limited dimension is observed,

then the effectively observed theory will have a weak form of locality, in the sense, that the action will be observed as one of the form

$$S[\phi] = F(\int \mathcal{L}_1(\phi(x))d^dx, ..., \int \mathcal{L}_n(\phi(x))d^dx), \tag{13.1}$$

meaning the action functional $S[\phi]$ would be a **function** of a series of usual local action integrals $\int \mathcal{L}_i(\phi(x))d^d x$, but presumably not itself of this form.

The field $\phi(x)$ is a common name and can stand for fields with many different transformation properties under the diffeomorphism symmetry

The field $\phi(x)$ is just short hand for any of the many fields we know (or do not even know) like

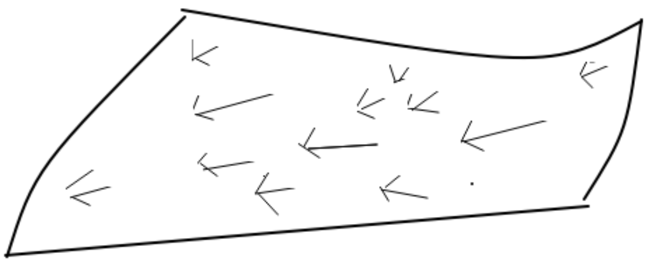
$$\phi(x) = g^{\mu\nu}(x), V_a^\mu(x), \varphi(x), A_\mu(x), \sqrt{g}, ... \tag{13.2}$$

or even combinations(products) of them with their various transformation properties under diffeomorphism symmetry.

With diffeomorphisms we think of bijective maps of the manifold on which the theory is defined onto itself having the property of being many times continuous differentiable.

They are smooth deformations of the manifold M say.

A Diffeomorphism transforms bijectively the Manifold and is continuos differentiable (some number of times)



We can likely (almost) do with less than full diffeomorphism symmetry

We believe the argument for locality (which I still owe you) could go through with similar symmetry such as:

- A projective space time with as symmetry the projective maps of this space onto itself.(We shall give the reader a reminder about projective spaces in section 13.8.)
- The symplectimorphisms of a non-commutative space-time with a symplectic structure on it.(The symplectomorphisms are maps preserving the/a symplectic structure defined on the space. [4])
- Of course a true manifold with its diffeomorphisms is o.k.

But a Minkowski space time **with a distance** between two points that cannot be varied by the symmetry of the goemetry would **not** be suitable for our derivation of locality.

Essential is that you by the symmetry can move one point around even keeping another point fixed, so that the **only** kept information on the relative position of a pair of points is if they coincide or not.
We did NOT get full/true Locality out: Only $S[\phi]$ **of the Form** $F(\int \mathcal{L}_1(x)d^dx, ..., \int \mathcal{L}_n(x)d^dx)$**.**
With Random Dynamics derivations you are often **not quite** successful as here:

- **True locality:** $S[\phi] = \int \mathcal{L}(x)d^dx$.
- **But Only got:** $S[\phi] = F(\int \mathcal{L}_1(x)dx, ..., \int \mathcal{L}_n(x)dx)$

But that is precisely **interesting** because then the suggestion is that nature may only have the **not quite** successful form of e.g. locality:
In fact it is suggested: **The coupling constants such as the fine structure constant or the Higgs mass or the cosmological constant "knows" about what goes on far away.**
The Coupling Constants "knowing" about Remote Happenings is an Advantage
That the effective coupling constants or the cosmological constant etc. depend on integrals like $\int \mathcal{L}_i(x)d^dx$ gives at least **hope for solving finetuning problems** such as:
Why is the cosmological constant so fantastically small - in say Planck units -?
Now we can at least hope it is so small to make universe big or flat...**But if the coupling constant (say cosmological constant) did not "know" about the remote, it could not adjust to it.**
Another fine tuning problem is: Why is the Higgs mass and thereby the weak scale so small compared to Planck scale or unification scale (if there were unification)?

13.3 The Propagation Problem, Need for Gravity

Another Trouble for it being a full Success: No Propagation without a spontaneous breaking $g^{\mu\nu}$**. I.e.** $< g^{\mu\nu} > \neq 0$
If there were no spontaneous breakdown, so that in vacuum all fields ϕ had zero expectation value, then there would be **too much locality**, superlocality:
There would not be place for useful derivatives in the Lagrangian density $\mathcal{L}(x)$, because ∂_μ could only be contracted to the $dx^\mu \wedge dx^\mu \wedge \cdots dx^\rho$, but that gives only an action which is a boundary integral only (integral of total derivative).
So a gravity field with vacuum expectation value is needed (smells like deriving gravity as needed at least).
Viewing Gravity from reparametrization Invariant Fundamental Theory
Gravity or Physics with propagation, needs a breaking of scale invarince since the equation of motion

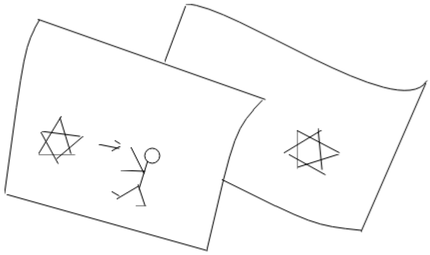
$$g^{\mu\nu}\partial_\mu\partial_\nu\phi = 0 \tag{13.3}$$

needs an upper index $g^{\mu\nu}$ for being reparametrization invariant.
A non-zero $g^{\mu\nu}$ represent a spontaneous break down of a symmetry involving say scaling or reparametrizations. Similar ideas by [1].
The theorem: Spontaneous breaking of Reparametrization Symmetry Needed

Theorem: *If a theory with reparametrization invariance is not spontaneously broken - meaning the vacuum is totally reparametrization invariant - then propagation in this world is impossible.*
The speculative suggestion: If **reparametrization** transformations constitute a **fundamental symmetry**, there would be no waves going from one point to another; so only to the extend that vacuum has some **breaking** of the symmetry of **reparametrization** we can get propagation. So not much interesting physics could go on without this spontaneous breaking. **Gravity-like fields** - basically $g^{\mu\nu}$ - non-zero in vacuum are **needed**.

13.3.1 New Way of Reducing Dimensionality, use Degenerate $g^{\mu\nu}$

Side remark on Dimensional Reduction in the Spontaneous Breakdown
Even if dimension d of the fundamental space were high, we could have that the rank of the $g^{\mu\nu}$ tensorfield in vacuum was lower. In that case the world in which we could propagate would be of the lower dimension.
But say spinorfields would - you might think - anyway have to have numbers of components matching the fundamental dimension, but alas: There is no spinor representation of the general linear group which the transformation group of the tangent space for the symmetries, we care for in this talk! So can a model with reparametriztion invariance as fundamental symmetry have spinors at all?
Well, even in ordinary general relativity, we know that the spinor fields are indeed w.r.t. curved space indices scalars - they only have the so called flat indices, which are really only enumerations of various vierbein fields - so they are indeed **scalars**. This means that without the vierbeins the fermion fields could not propagate. So realistically we have to think of the breaking of the too large symmetry comes by means of vierbein fields rather than by a true $g^{\mu\nu}(x)$ alone. In fact the most likely model is presumably that there is bound state combination of the vierbeins making up the $g^{\mu\nu}(x)$.



If in vacuum the rank of the upper-index metric tensor/matrix is lower than the dimensionality of the manifold, then there appears surfaces on which signals can propagate, but from surface to surface it cannot.
We embed Gravity and the rest into Spacetime WITHOUT metric, as Just Manifold or Projective Space or some Noncommutative phase space, ...
The idea of embedding the gravitational manifold from general relativity into an embedding space is an old one, see e.g. [5]. But Sheikin and Paston embed

the general relativity space into a flat metric space time. In the present work we are interested in **embedding into a geometrical space**, which have **no metric** but rather has such symmetries that locally it is part of the symmetry that a small neighborhood can be deformed and scaled up or down in size, so that a metric would be forbidden by the symmetry and at best be allowed as spontaneous symmetry breaking. The space in which to embed in our present work is rather thus thought upon as either a **pure manifold** with no further structure **or a projective space-time**.

In spaces with local linear deformation like: Reparametrization invariant or Projective space, No Signature

It is the $g^{\mu\nu}(x)$ that has the signature - in physical world 3+1 -, so without $g^{\mu\nu}$ Minkowski and Euclidean spaces are the same.

Our Point of View: Start with a locally Scaling and deformation Containing Symmetry.

Having in mind our work with Astri Kleppe [2,3] of **deriving locality of the theory from a reparametrization** invariant theory - though allowed not to be invariant under variations of the measure - for an extremely general action, we suggest to assume a symmetry involving - at least locally - such a reparametrization invariance to be assumed before e.g. getting gravity.

That is to say: We want to assume either reparametrization invariance or something like that, and **after that** hopefully derive or understand gravity and locality.

Geometries with (local) scale and deformation symmetry.

Examples of how you can have local deformation symmetry:

- **Full Reparametrization Symmetry** This is the symmetry of coordinate shifts in the General relativity.
- **Projective geometry space** The symmetry of the projective space is a smaller group than that of general relativity. (I am personally especially attached to projective geometry, because I made my living from teaching it for 6 years.). M. Ninomiya and me [13].
- **Symplectomorphic invariant space.**

By local scale symmetry we mean that there are symmetries, so that in the tangent space to any point we have symmetry under scaling up by any (real) factor this tangent space.

In the present work we really want to have **locally not only scale invariance but invariance under any real linear transformation** $GL(d, R)$.

Starting Point: Locally General Linear Transformation Symmetry

Our starting assumption - in this work - is that there are such symmetries assumed that for every point on the manifold you must have symmetry under general linear transformations in the tangent space:

$$(dx^1, dx^2, ..., dx^d) \to ((Adx)^1, (Adx)^2, ..., (Adx)^d) \tag{13.4}$$

$$= (A^1_\mu dx^\mu, A^2_\mu dx^\mu, ..., A^d_\mu dx^\mu) \tag{13.5}$$

$$\text{for any real matrix} A^\mu_\nu \in M_{d\times d} (\text{in the curled indices}). \tag{13.6}$$

Having such a symmetry will be enough for guaranteeing that we for a general functionally analytic action $S[\mathtt{fields}]$ shall get locality [2,3].

13.3.2 Resume propagation

Propagation Requires Breaking of the Locally General Linear Symmetry
Usually the propagation of particles in say free approximation is given by a D'alembertian equation of motion

$$(\Box + m^2)\phi = 0 \tag{13.7}$$

but to have local general linear transformation invariance:

$$(g^{\mu\nu}\partial_\mu\partial_\nu + m^2)\phi = 0 \tag{13.8}$$

we need the $g^{\mu\nu}$!

If such a $g^{\mu\nu}$ is non-zero in vacuum, we have a **spontaneous break down** of the symmetry, because the $g^{\mu\nu}$ field transform non-trivially under the local general linear transformations.
So we only can propagate (normally) any particles, provided we break (spontaneously) this locally general linear symmetry!

13.4 Locality Derivation Argument

Reviewing our " Derivation of Locality"
Let us review Astri Kleppes and HBN's [2,3] "derivation" of locality under the assumptions of **diffeomorphism symmetry** (invariance under reparamterizations) for a very general action S not being a priori local but rather only having the diffeomorphism symmetry and being **Taylor expandable** in **local fields**:

$$S[\phi] = \sum_n \frac{1}{n!} \sum \text{(or integral)} \frac{\partial}{\partial\phi(x_1)} \cdots \frac{\partial}{\partial\phi(x_n)} S[0]\phi(x_1)\cdots\phi(x_n),$$

where here ϕ stands for very general fields, possibly with many indices, and S **is an a priori non-local action.**
("Mild assumptions": Taylorexapndability, Finite order Lagrange term only observed (low energy))
Setting for the Derivation of Locality of the Action $S[\phi]$
The field $\phi(x)$ can be so general, that it can stand for all the fields, we know, or do not know yet

$$\phi(x) = A_\mu(x), g^{\mu\nu}(x), \psi_\alpha(x), ... \tag{13.9}$$

x is a coordinate point, but in the spaces like manifold, or projective space, there is always a coordinate choice needed.
The action $S[\phi]$ is a functional of the fields ϕ and is assumed

- Taylor expandable (functional Taylor expansion)
- But **not assumed local**, since it is the point to derive/prove **locality**

The Taylor Expansion for Functional in Integral form:

The **functional Taylor expansion** in the more functional notation (i.e. without imagining a lattice cut off say):

$$S[\phi] = \sum_n \frac{1}{n!} \int \frac{\delta}{\delta\phi(x_1)} \cdots \frac{\delta}{\delta\phi(x_n)} S[0]_{\mu\cdots\nu} \phi(x_1) \cdots \phi(x_n) dx_1^\mu \cdots dx_n^\mu.$$

Here the $\frac{\delta}{\delta\phi(x)}$ means functional derivative

The Crucial Point: All Points can by Symmetry (Diffeomorphism symmetry) be brought into Any Other one, Transitivity

When there is no distance a priori in the just manifold with diffeomorphism symmetry or the projective space, you at least, if you do not go to higher order interaction with several fields multiplied, **field at one point will interact the same way with fields at any other point, except the very point itself.** Thus you either interactions between all points, or interaction of the fields at the same point, i.e. locality.

Spelling a bit out the Functional Taylor Expansion:

The **functional Taylor expansion** in the more functional notation:

$$S[\phi] = \sum_n \frac{1}{n!} \int \frac{\delta}{\delta\phi(x_1)} \cdots \frac{\delta}{\delta\phi(x_n)} S[0]_{\mu\cdots\nu} \phi(x_1) \cdots \phi(x_n) dx_1^\mu \cdots dx_n^\mu.$$

The symbol $\delta/\delta\phi(x)$ means functional derivative.

- If we did not allow δ-functions so that we could get no contribution from cases where two of space-variables x_i and x_j say coincide, and if there were a symmetry like a translation $x_i \rightarrow x_i + a$ for all vectors a the terms in the functional Taylor expansion could only depend on $\phi(x_j)$ via the integral $\int \phi(x_j)dx_j$. So the whole taylor expanded action would be a function of such integrals $\int \phi(x)dx$. (for the type of symmetries we want to assume one could not even construct such integrals, except if $\phi(x)$ transform as a density- like $\sqrt{g}$)

In the **functional Taylor expansion:**

$$S[\phi] = \sum_n \frac{1}{n!} \int \frac{\delta}{\delta\phi(x_1)} \cdots \frac{\delta}{\delta\phi(x_n)} S[0]_{\mu\cdots\nu} \phi(x_1) \cdots \phi(x_n) dx_1^\mu \cdots dx_n^\mu.$$

we can **without delta functions** $\delta(..)$ only get the Taylor expanded (action) $S[\phi]$ to depend on integrals of the type

$$\int \phi(x)dx^\mu \quad \text{, which are invariant under the symmetries} \tag{13.10}$$

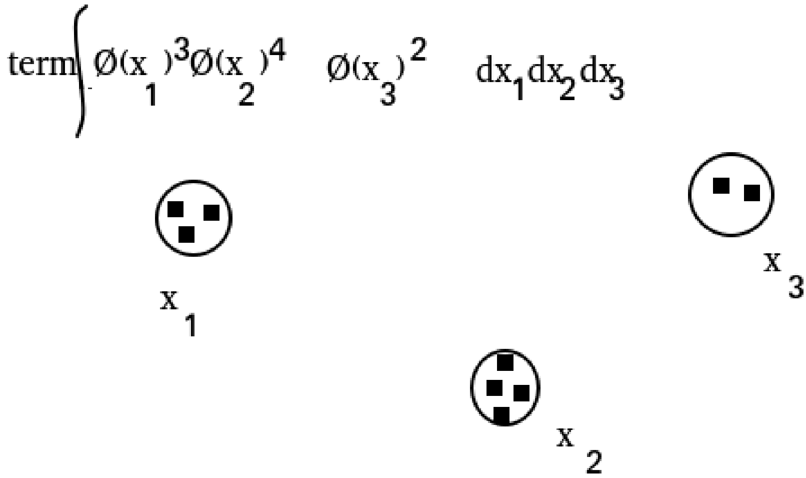
$$\text{(of a manifold say)} \tag{13.11}$$

- But if we allow δ-functions - in the functional derivatives of the functional to be exapnded $S[\phi]$ then one can get integrals allowed involving products of fields $\phi(x^\mu)$ taken at the **same** point.

What can occur, if we allow the δ-functions ?
If we allow the $\delta(...)$ functions and require the symmetry of the diffeomorphism of the manifold, the integrals on which the being Taylor-expanded quantity $S[\phi]$ can depend, must be integrals **symmetric under the prescribed (diffeomorphism) symmetry** of **the form**

$$S_i[\phi] = \int L_{i\ \mu\nu\rho\kappa}(x)dx^\mu \wedge dx^\nu \wedge \cdots \wedge dx^\kappa \tag{13.12}$$

where now the $L_i(\phi(x), ..., \phi(x))_{\mu\nu\rho\sigma}$ is a product of several of the $\phi(x)$ fields at the **same** point $x = x^\mu$. If these arguments for the $\phi(x)$ are not at the same point, then you can by the diffeomorphism symmetry transform them around separately and the integral would be required to **factorize** into integrals each only having fields from **one point** (kind of locality).



The $S_i[\phi] = \int L_{i\ \mu\nu\rho\kappa}(x)dx^\mu \wedge dx^\nu \wedge \cdots \wedge dx^\kappa$ **are ordinary local Actions, but...**
We did **not** derive that the action functional we discussed $S[\phi]$ was of the form $S_i[\phi] = \int \mathcal{L}_{i\ \mu\nu\rho\kappa}(x)dx^\mu \wedge dx^\nu \wedge \cdots \wedge dx^\kappa$, but **only that it could only depend on the fields via such integrals**. So rather only the expanded action is a **function** of such integrals.
We derived a form of the Diffeomorphism Invariant Taylor Expandable Action $S[\phi]$ **as a Function** $F(S_1[\phi], S_2[\phi], ..., S_n[\phi])$ **of usual *local* action integral.**
Indeed the terms in the functional Taylor expansion will be of the form that groups of factors are at the same point inside the groups, and that then these point are integrated over all the space(manifold). Denoting the possible integral over local field combinations

$$S_i[\phi] = \int \mathcal{L}_{i\ \mu\nu\rho\kappa}(x)dx^\mu \wedge dx^\nu \wedge \cdots \wedge dx^\kappa \tag{13.13}$$

we get the **form**

$$S[\phi] = F(S_1[\phi], ..., S_n[\phi]). \tag{13.14}$$

This form was studied by my student Stillits [12]
But this was not Really Locality!
The form, which we derived from the diffeomorphism invariance

$$S[\phi] = F(S_1[\phi], ..., S_n[\phi]) \tag{13.15}$$

is not truly local; we should have had a linear combination

$$S[\phi] \overset{=}{wanted}\; a_1 S_1[\phi] + ... + a_n S_n[\phi]. \tag{13.16}$$

However, if we construct the **equation of motion** by putting the functional derivative of (13.17) to zero, we get the wanted form (13.16).
Equations of Motion got Already Local, but...
The equations of motion for an action of the form - derived from the diffeomorphism symmetry

$$S[\phi] = F(S_1[\phi], ..., S_n[\phi]) \tag{13.17}$$

becomes

$$0 = \frac{\delta S[\phi]}{\delta \phi(x)} \tag{13.18}$$

$$= \sum_i \frac{dF}{dS_i}|_\phi * \frac{\delta S_i[\phi]}{\delta \phi} \tag{13.19}$$

$$= \frac{\delta}{\delta \phi} \sum_i \frac{dF}{dS_i}|_\phi * S_i[\phi] \text{ considering } \tfrac{dF}{dS_i}|_\phi = a_i \text{ constants}$$

The coefficients $\frac{dF}{dS_i}|_\phi$ do depend on the fields ϕ, but integrated over all time and all space.
Effectively these coefficients

$$a_i = \frac{dF}{dS_i}|_\phi \tag{13.20}$$

to the various possible local actions $S_i[\phi]$ **do depend on the fields** ϕ but since they depend via integrals over all space time, we can in pracsis take them as constants. Indeed they are the **coupling constants** which we just fit to experiments. But it means that our lack of completing the derivation of locality means: **The coupling constants - say fine structure constant etc. - depends huge integrals over space time**, although composed in a way which depends on the fundamental non-local action, which we do not know (yet?).
We did not get full locality! Coupling constants depend on all space-time
We got, that **the Lagrangian density only depends on the fields in the point you write this Lagrangian density**, and that is practical locality, **but we did not get**

locality for the coupling constants in the sense that they with our derivation "know about" what goes on all over space and time, including even future.
. This suggests that the question of what the **initial conditions** should be at least in principle needs an extra discussion.
In principle there is a backreation for any choice of initial condition because it influences the couplings depending on the development of it also to far future.

13.5 MPP

Prediction of Severel Degenerate Vacua ("Mulitple Point Criticality Principle")
This great importance of integrals over all space time of the fields could very easily lead to limitations for such overall space time integrals.
Such specification of a non-local (even in time) is analogous to **extensive quantities** in thermodynamics:
If you specify them you risk to put your system into a phase transition point.
If you specify several of them you easily end up with several phases in a ballance.
Here the analogues of the **intensive quantities** like temperature and chemical potentials are the coupling constants, which with our incomplete locality derivation depend on what goes on or will go on or has gone on in the universe.
When Ice and Water in equilibrium Temperature 0^0
When one has the situation of slush - that there is both water and ice - then ones knows the temperature must be zero. cold but not extremely cold.

In the analogous way we have what we called **The Multiple Point Criticality Principle** when in space time one has several vacua in balance taken to mean that they have the same energy density. To find a good argument for this suggested principle we speculated that the some integrals of the type $S_i[\phi]$ got specified values analogously to fixing extensive quantities in chemistry. Then it could easily

be that the specified quantities could only be realized when there were indeed several (vauum) phases analogous to the some specified combinations of a number of mols water, total energy and volume could enforce there to be slush and maybe even water wapor and the temperature and pressure could be enforced to be at the triple point.
With the Action derived from our locality derivation involving strongly the many integrals over all space time, one could easily imagine that by some way of getting a selfconsistent solution it could turn out that several of these integrals get so restricted, that one has such a situation similar to the slush one, that it was needed to have several phases of vacuum in space-time.And if they should be in equilibrium by having the same energy densities say, then coupling constants might end up in some critical point where the phases could coexist.
At least we can say, that since such integrals appear in the form we argued for, seeking consistent solutions for the equations of motions - and that is not quite trivial in our still not quite local action, since the timedevelopment for some set of couplings should avoid making the integrals over all space time into values giving a different set of couplings- might give restrictions reminiscent of the ones you get by fixing the over-all space time integrals.
So it is not unlikely that our not quite local action would lead to our earlier proposed **multiple point criticality principle.**
This would be a success if we could get the Multiple point criticality principle out as extra premium from the attempt to derive locality, because Colin Froggatt and claim to have PREdicted the mass of the Higgs boson before the Higgs boson were found experimentally, by means of the multiple point criticality principle.
We PREdicted the Higgs Mass by Several Balancing Vacua (MPP) before the Higgs was found

The painting of me together with the Danish finance minister - whom I only met many years later - were painted in the beginning of 90's much before the Higgs was observed in LHC (=Large Hadron Collider) with 3 σ in 2012 and finally estblished in 2014. Neverthe less you can see the mass of the Higgs particle written as $135GeV \pm 10GeV$ (only the 1 is hidden behind Mogens Lukketofts head), but

in our article in Phys. Rev. we have the 135 ± 10 GeV. (The measured mass turned out 125 GeV).
This we like to take as a support for the multiple point criticality principle, and thus if this could be a consequence of the incompletely local action form for that even this form is being a little supported.
Actually it would be rather impossible to see how such phases with same energy density could come about in a world with complete locality.
If namely one vacuum did not appear before after some time in the universe development - and that must be so because there were so hot in the beginning, that there were no vacuum proper anywhere - then how could any coupling constant or the Higgs mass adjust to make such a vacuum obtain a special value for its energy density say, when the vacuum had yet never existed ? At least it looks that some "non-locality" of this type must exist: Higgs mass or other parameters in the theory such as coupling constants and the cosmological constant must have been informed from the beginning about e.g. vacuum properties of vacua first existing long after.
This type of lacking locality is precisely the one we did not mannage to derive.

13.6 Transitivity

More on Transitivity
When a group G acts on a space X

$$\alpha : G \times X \to X \tag{13.21}$$
$$\text{denoting } \alpha(g, x) = gx \tag{13.22}$$
$$\text{so if } gx_1 = x_2, \tag{13.23}$$

it means the group element g brings the element $x_1 \in X$ into $x_2 \in X$, then we say G **acts n-transitively provided there for any two ordered sets of n different points in X, $(x_1, x_2, ..., x_n)$ and $(y_1, y_2, ..., y_n)$ exist a group element $g \in G$ such that**

$$gx_i = y_i \text{ for all } i. \tag{13.24}$$

We say it is sharply n-transitive, when this g is unique.
d-dimensional projective space has a symmetry group acting almost(!) $(d+2)$-transitively
Examples:

- Under the action of diffeomorphisms on a manifold the action is infinitely transitive.
- In d-dimensional projective space $PS(d, R)$ the symetry group acts essentially $(d+2)$-transitively, but not truly so, because the image of points say on a line remains on a line. Only the projective line $PS(1, R)$ is truly 3-transitive.
- Euclidean spaces are only (1-)transitive under their symmetry. It is the translation group that acts transitively. When the group conserves the length of line there can be no even 2-transitivity.

13.7 Repeat

Repeating Argument using Action
Instead of looking at the equation of motion we could ask, if we could make an action

$$S[\mathtt{fields}] = \int \mathcal{L}_{\mu\nu\rho\sigma}(x) dx^\mu \wedge dx^\nu \wedge dx^\rho \wedge dx^\sigma, \tag{13.25}$$

which is invariant under our symmetry having locally generallinear symmetry and at the same time can describe a propagation of some fields.
If a field ϕ shall not be determined locally by the other fields, but appear in equation(s) with derivatives, there must be a derivative acting on ϕ i.e. say $\partial_\mu\phi$ occurring in the Lagrangian density $\mathcal{L}_{\mu\nu\rho\sigma}$; but with what to contract the lower index μ on $\partial_\mu\phi$? To some field with an upper curled index like a vierbein V_a^μ or a $g^{\mu\nu}$? Yes but if we work in vacuum and there were no spontaneous break down of the symmetry these fields would be zero.
Continuing repeating Derivation of Need for Spontaneous Breaking of Locally General linear symmetry
Looking for making

$$S[\mathtt{fields}] = \int \mathcal{L}_{\mu\nu\rho\sigma}(x) dx^\mu \wedge dx^\nu \wedge dx^\rho \wedge dx^\sigma, \tag{13.26}$$

invariant under the symmetry, but still with fields propagating even with vacuum not breaking the symmetry. (We shall show you cannot find such an action.)
Can it help to let the $\partial_\mu\phi$ combination be contracted with a dx^μ to give it a chance to propagate?
In fact

$$dx^\mu * \frac{\partial\phi(x)}{x^\mu} = d\phi(x) \tag{13.27}$$

is a total derivative. If you now wanted to make the term second order in the ∂_μ, you would use yet another of the factors in the measure

$$d^d x = dx^1 \wedge dx^2 \wedge \cdots \lambda dx^{n-1} \lambda dx^n, \tag{13.28}$$

and the second order term would be like

$$\frac{\partial\phi}{\partial x^\mu} dx^\mu \wedge \frac{\partial\phi}{\partial x^\nu} dx^\nu = 0 \text{for same } \phi \text{ in the two factors.} \tag{13.29}$$

or different ϕ's, ϕ_a and ϕ_b

$$\frac{\partial\phi_a}{\partial x^\mu} dx^\mu \wedge \frac{\partial\phi_b}{\partial x^\nu} dx^\nu = d\phi_a \wedge d\phi_b \text{ a toal derivative.} \tag{13.30}$$

Couple it directly to the dx^μ's?
Seems not to give a propagating equation of motion usual type.
Let us here also remark that to get such an integral over all space time as we discuss, one also has to include a quantity that transforms as the usually known

$\sqrt{g}$ the determinant of the metric tensor with lower indices. It is e.g.

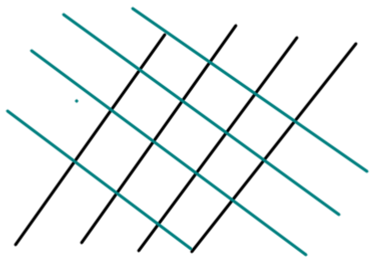
$$\int \sqrt{g} d^d x = \text{“}4 - volume\text{''} \tag{13.31}$$

$$\int \frac{\partial \phi}{\partial x^\mu} * \frac{\partial \phi}{\partial x^\nu} g^{\mu\nu} \sqrt{g} d^d x \quad , \tag{13.32}$$

which are meaningful reparametrization invariant space time integrals, while there is no meaningful 4-volume of the manifold nor of the projective space time **without the $\sqrt{g}$ or something replacing it.** The projective space has no size in itself. Also for that you need a spontaneous breakdown.
In reality is what we need for being able to get a propagation for a scalar ϕ the combined quantity $g\mu\nu\sqrt{g}$ where then g is determinant of the metric with lower indices being the inverse of the one of the upper index.

13.8 Projective

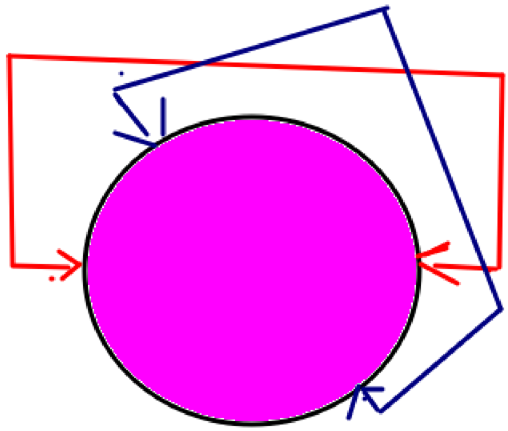
In Plane Projective Geometry All (different) Lines Cross in a Point



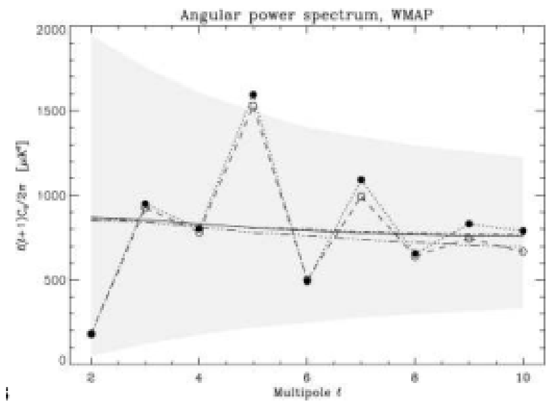
Bundles of parallel lines are identified with points on the line at infinity. So parallel lines cross there.
Really one makes a description of the d-dimesnional projective space by taking a $d+1$ dimensional vector space and identifuing the rays (i.e. sets of vectors proportinal to each other. A class of such non-zero to each other proportional vectors is called a ray) with the points in the d-dimensional projective space. The lines in the projective space are then identified with the two dimensional subspaces of the vector space, and the projective plans with the three-dimensional subspaces, and so on.
It is thus possible to make projective spaces corresponding to different fields in as far as onecan have vector spaces with diffeerent field. In this article we are interested in using the real field only.

13.9 Phenomenological Evidence for World being a Projective Space-time

Infinite far out points in opposite direction identified in projective geometry



We here stress, that since in projective geometry a bundle of Parallel lines is considered **only one point on the line or plane or etc. at infinity** there is no distinction between the point at infinity in one direction along the bundle of parallel lines and the point on the infinite plane or whatever. There is **only** one point at infinity for each bunch of lines.
This is illustrated on the figure by the arrows pointing to identified directions so to speak.
If somehow our universe really were a projective space, then you might see the same object in the two opposite directions. That would give of course a correlation of e.g. the radiation coming from two opposite almost infinities. They would fluctuate in similar way because of being the same point on the infinite plane (in three dimensions)
Lowest l WMAP fluctuations



The analysis of the microwave background radiation is typically done by resolving the fluctuation of the temperature as function of the point on the sky into a

description expanded on spherical harmonics. Thus one presents e.g. the size of such fluctuation connected to the various spherical harmonics, which are marked by $(l.m)$. We shall have in mind that the even l spherical harmonics have the same value in exact opposite directions, while the odd l ones have just opposite values in opposite directions.
On the figure we have the experimentally found fluctuations as function of l (averaged over m) for the first few lowest l's.
Remarkable:**Even l fluctuations are relatively low, while the odd l ones are relatively high**
We know that microwave backgound radiation comes from 13.7 millird light years away, so if the universe should really be a projective space, the infinite plane or infinite thre-space if we think of the fourdimensional space time as projective, sould be not much further away than 13.7 millard light years if we should be able toobserve it.
If Projective Space "seen" in CMB-fluctuations, then Universe Not Much bigger than Visible Universe
The just shown:

- Y_{lm}-proportinal modes in temperature variation over sky with **even** l have **lower** fluctuation.
- Y_{lm}-proportional modes in temperature variation over sky with **odd** l have **higher** fluctuation.

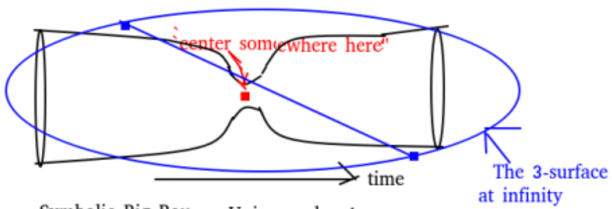
if taken seriously implies that **the visible universe edge is not very far from where there is the identification of the diametrically opposite points** (on say the infinite line). So Universe would not be so huge as the very accurate flatness would indicate!

13.9.1 Correcting for my mistake

Post Talk Slide: Naively you expect the opposite!
I got very confused and chocked, when I persented the foregoing slides, because naivly thinking of a three-dimensional Projective space:
Since the points in opposite directions are **related, in fact the same region,** the **even l** spherical harmonics, should show a **big** fluctuation, because they add together by continuity across the infinite plane related regions, and thus counts really the same fluctuation in two ends as statistically independent and thus over estimate the fluctuation. Oppositely weighting with an **odd l** spherical harmonic you add with a relative minus sign two close by (across the infinite plane) region contributions and should get approximately zero. Thus the fluctuations for **odd l should be small.**
Thus my chock: The "evidence" I had believed, had the wrong sign!
We Forgot the Time Direction...
We should not have looked for opposite points in the purely 3-dimensional space, Rather than looking **for the opposite point on the infinite 3-space we should look for opposite points also in the time direction i.e. in the 4-dimensional space-time.**

If we take that what should be Big Bang is rather just the narrowest point, in some sort of bouncing universe, than we can at least speculate to have a mirror symmetry in this the narrow universe region. Let us consider this time usually taken as big bang to be rather a center / origo of the space time, in the sense that we consider the opposite ends of lines extending from this big-bang like region and claim that because of a three-space at infinity having the fluctuations are approximately the same in the two ends of a line through this big-bang region .

Imagine embedding a Bouncing Universe Space-time into a 4-dimensional Projective space, filling it out



Symbolic Big Bounce Universe development
Two opposite directions will have one before and one after the bounce.`

This drawing is very symbolic, but of course one can see that the two **directions** corresponding to the same point on the infinite 3-space, now have opposite times too. That is to say that one of them are in the prebigbang time (which probably do not exist, but the projective space has no singular start so it goes extremely badly with the big bang theory; so to have any chance with projective space-time we better give up big-bang).

In Projective Space No Singularities like Big Bang, so better Bounce

In the Projective space(s) all points are symmetric with each other (transitivity of the group action) and thus no point-singularities, so that a **big bang start would not fit well into the projective spacetime**.

Therefore we rather imagine here a **big bounce model** in which there is contracting universe before it reexpands, although such a model has rather many problems with second law of thermodynamics, and with how an about to crunch universe can get its contraction turned to an expansion. But an **empty De Sitter space can bounce**, so we might postulate that in the middle of times the universe is an empty De Sitter space.

So two opposite time directions in respectively the expanding and the crunching halfs of the space time.

Inflaton Field goes up to top of Potential to Stand there as Long as possible to get preferably slowest roll

Let us imagine as our model to get at all a reasonable embedding into the projective space, and a preferably as long as achievable inflation, locally the inflaton field in crunch-part of time in the anti-inflaton period runs up the potential hill and stops very close to the top of the potential. Then it falls down again, first extremely slowly and then unavoidably faster. (we have in mind that it goes down the opposite side of the hill as it came up, but it may not matter so much for the model, but we think it this way crossing the peak and describe as if so.)

Opposite Point Identified by the 3-space at infinity, are opposite in both space and time.
If an event at the recombination era at 370000 years after the big bounce (we call the the very time when the bouncing occur for time =0) is supposed to be sufficiently far out to be close to a point on the 3-space at infinity, then this approximating point at infinity is identified with a point at infinity in the opposite direction in space, meaning on the sky, but it shall **also be opposite in time.**
The latter presumably means it shall be on the crunching time sector if we have looked at a search the point opposite to a point in the expanding time sector.
Speculate biggest fluctuation in the Time of reaching the Potential Peak
Let us further assume that the largest fluctuation in the inflaton field comes from the **time** at which the inflaton just reaches the peak of the potential varies randomly from region to region in space. (One should have in mind, that in a bouncing universe model there is an inflation period during which the inflaton field is only tinily varying and stands almost close to the top of the potential peak. This period extends both into the time after time=0, and into the time beforee time=.0. Here we imagine that the inflaton crawls up from one side of the peak/hill in the "crunching era (before time=0)" and rolls down by developping further in the same direction. It rolls down on the opposite side of the hill so to say.)
If we have points that are relatively far from each other such fluctuation of the times of peaking should be strongly fluctuating.
Now notice: If it happens early from the point of view of the expanding universe then it happens also early by the point of view of the crunching universe, but that has for this compared our expanding universe the opposite effect, because of the time reversal.
Prediction of the Sign-inversion by the time reflection not detail dependent
So whether late or early gives larger or smaller CMB radiation, then the time reflected point will always give the opposite to the not timereversed one. So we will get the odd l spherical harmonics get the biggest fluctuation, and the even ones the smallest!
Conclusion on CMB-fluctuation Prediction
Assuming:

- Bouncing Universe
- **Time** of Inflaton field Reaching the Peak (for the inflaton effective potential) being the most important quantity causing the fluctuations in the CMB radiation.
- The Crunching Universe (the development before most narrow universe in the bouncing) behave Time reversal invariant to the expanding one. (I.e. such a model needs that you have an oppositely going second law of thermodynamics, so it is indeed a picture with very great problems making the story nor so trustable)

we get: **The odd l shperical harmonic modes shall have the largest fluctuations**, while the **even l one the smallest,** contrary to the intuition forgetting the time to be also reflected. Via timereveral and the assumption that the main fluctuation is in the very moment at which the inflaton field is on the top of the effective potential we get that big or small temperature at a given time is opposite for the "crunching world before time =0" and " the our world after the time at the peak" becomes opposite. This phnomenon can then give the opposite sign for the CMb fluctuations predicted from our picture relative to what one would see forgetting also to reflect in time.

13.9.2 Statistics of the CMB deviation for small l

We must admit there are only ca. 2 standard deviations from also the low l fluctuation observations agree with the statistical model, so there is only two standard deviations to build the story about the projective space on. So it is very weakly supported.
Only 2 s.d. from statistically understood low l modes In spite of the statistical significance of the observed even-odd asymmetry we used to support Projective geometry is only $\sim 2s.d.$ theorists sought to explain these low l fluctuations, e.g. R. Mayukh et al, [14] by Superstring excitations, and by "Punctuated inflation" [15].

13.10 Inermediate Conclusion

We have reviewed an older work by Astri Kleppe and myself deriving the locality form

$$S_{eff}[\phi, ...] = \int \mathcal{L}(\phi(x), \partial_\mu \phi(x), ...g^{\mu\nu}(x), ...)d^d x \tag{13.33}$$

for the effective action, **even though the a priori action** $S[\phi, ..]$ **is not assumed local.** under the "not mild assumption" of diffeomorphism symmetry, which though approximately is suggested to be replaceable by a projective geometry or other space which at least locally as symmetry can be deformed, spread out or in in different directions.
Our main interest in the present article was to point out, that in order that such a derivation be useful for phenomenological physics a gravity-like field, say $g^{\mu\nu}(x)$ or better the combination $g^{\mu\nu}\sqrt{g}$ or some corresponding vierbeins have to take non-zero expectation values in vacuum. Thus gravity gets a status as a spontaneous breaking set of fields needed to have at all the rather abstract property of the physics model of there being propagation, or say interaction between different points/events even in an indirect way. In other words, were it not for the gravity spontaneous breaking, then different events in space time would have no interaction with each other. We also review that the fact, that the derivation is after all not complete, but only leads to an effective locality in as far as the effective coupling constants and mass parameters in the quantum field theory resulting **depend actually on integrals over all space and all time**, so that they are formally not at all local. Only when we by a formal kind of swindle take these on all over the space time extended integrals depend "coupling constants" as constants we obtain the true local theory. (but of course they are constants as function of time.)

This slight lack of full locality we suggested to be useful for solving some finetuning problems, such as the smallness problem of the cosmological constant, and it could also lead to the by us since long beloved speculation of "Multiple Point Criticality Principle" saying that there are several types of vacuum, and that they have the same energy density(= cosmological constant).
More like an outlook for future work we hoped for a specification of a manifold, or taken as very similar a projective space, by an assumption of there being so much symmetry, that the set of geometrical points/events in the space-time should be transformed under the sharp n-transitive action of a group G on a system/set of events X. Higher than n equal 3 is actually not possible, but one may hope to come over this problem by defining a concept of the action of a group G being only **almost** n-transitive. In fact here are namely classifications that an infinite set cannot have a sharp n-transitive action for n more than 3.
Finally we represented an argument, that the small only about 2 standard deviations of the microwave background radition fluctuation data from the prediction of the Standard Cosmological Model, could be understood as a reflection of the embedding of our space time into a projective space-time. It turned out that it being a space-time, and not only the space, is crucial for the sign of the effect, the sign of the deviation from the Standard Cosmological Model.
Thus the model, that the space-time is embedded into presumably a 4-dimensional space-time, which is a projective space, has some tiny phenomenological advance for it.

13.11 Cut off

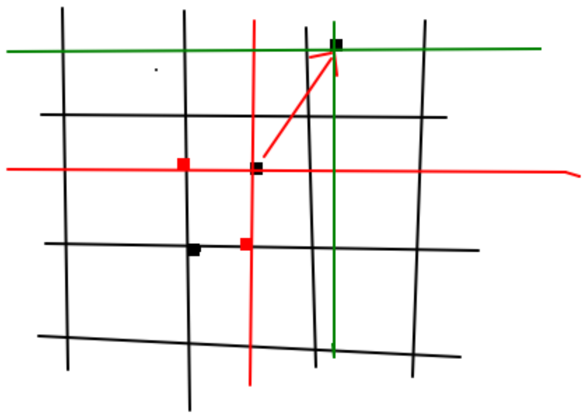
A speculative Attempt to Cut off Gravity
Encourraged by the even phenomenological support for an embedding of the space-time in a projective space, and by the derivation of effective locality, even if you do not impose it, we propose to replace the **manifold** with its diffeomorphism symmetry, by a projective space, with a little less but very similar symmetry, namely of the group of projective transformations.
Then the derivation of **locality will not be as perfect as with the diffeomorphism symmetric manifold**, and rather there will not be some possibility for **non-local terms in the final action being expanded**.
But this one could consider an advantage, because it could function as a cut off, having a chance of allowing a theory with less renormalization problems.

Dream of Dynamical Lattice Embedded in Projective Space
If some non-local interaction - allowed the projective symmetry - led to a dynamical lattice embedded in the projective space, we could have a dynamical lattice spontaneously breaking the projective symmetry by giving it (essentially) a metric. It could function as a cut-off (=regularized) gravity, and other fields might be restricted to only be able to **propagate** along the lattice, because it could be that there would be no truly continuous gravity field excepthe fields on the lattice points.

It might be not over optimistic to calculate possibilites for such a lattice of points embedded in the projective space-time.

A Lattice proposal Embedded in a d-dimensional Projective space-time, with some special points
Since the non-local interaction allowed by the Projective symmetry will be by among $d + 3$ points, it will be the best chance to get a suitable lattice, having $d + 1$ points in the projective space being especially marked.
We could let these "marked points" to be at first the origin and d points on the surface at infinity. The lowest number of points for which a (non-local) action dependence would be $d + 3$, so by fixing as marked $d + 1$ marked, we would have a favourite lattice relation when the two points further needed for nonlocal action term being allowed could be considered neighbors in a more conventional lattice.

What determines the Lattice ?
We must imagine that the lattice giving the analogue of a vacuum must come from minimizing something - the energy or some action - and if so, then it would be simplest that each time you have the $d + 3$ points needed for having a nonlocal interaction, then the minimization of whatever puts such $d + 3$ points in a certain configuration having meaning under the projective symmetry.
So we should only a few favourite relative positions of such $d + 3$ points realized again and again in the lattice.



With a Couple (for d=2) of favvourite 5 point combinations we get a lattice two dimensional integer coordinat lattice (though in log coordinate)

It turns out that each step of going from one point to the next by a favourite configuration, you step the same step each time in logartithmic coordinates.

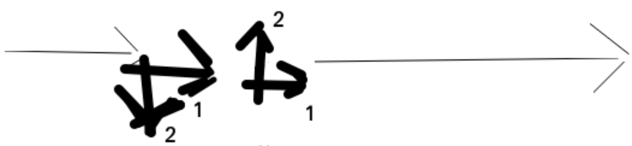
But it gives an apriori **flat world** which has a dynamical gravity so it can be curved, but first crude prediction is flat.

13.12 Hugeness of the Universe

The Hugeness of the Universe ?
Dirac wondered about the huge numbers of order 10^{20}, that e.g. the age of the universe is of the order of $(10^{20})^3$ time the Planck time.
Assuming a **projective space** background for our space time could in an a priori unexpected way enforce the existence of very - infinitely - extended space time region(s)!
Argument goes:

- The projective space of even dimension is non-orientable.
- That enforces a hyper-surface, on which the $g^{\mu\nu}$ is is of rank one less - say for normal rank 4 it has 3 there.
- But then there $g_{\mu\nu} = \infty$.
- Appraoching this degeneracy surface the volume relative to the coordinates grow so much that an infinite universe in space and time pops out.

Non-orientability of Even Dimensional Projective space
Most easily seen in the even dimension $d = 2$.

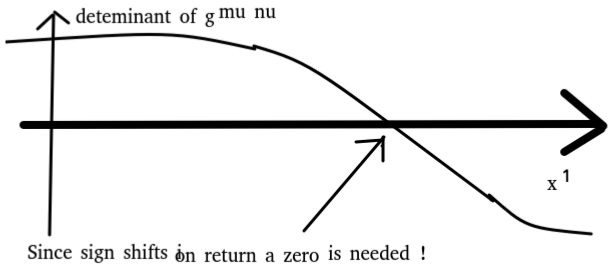


Move a little coordinate system around to the line at infinity and back from opposite side.

The Needed $g^{\mu\nu}(x)$ must be degenerate along a 3-surface
The determinant $\det(g^{\mu\nu})$ cannot avoid a zero surface of dimension 3 in a 4 dimensional projective space. The sign of this determinant namely represents an orientation.
Write it the coordinates chosen locally x^1, x^2, x^3, x^4 and in a certain order say 1,2,3,4. Then

$$\begin{aligned} &\text{If } \det(g^{\mu\nu}) > 0, \text{ orientation is that of ordered coordinates} \\ &\text{If } \det(g^{\mu\nu}) < 0, \text{orientation is opposite coordinates in their order} \end{aligned} \quad (13.34)$$

Think of the determinant $\det(g^{\mu\nu})$ followed around to infinite line and back the other way



We really needed upper index $g^{\mu\nu}$, so it must be "fundamentally" an effective (?) field
But the lower index ones $g_{\mu\nu}$ could just a definiton of an inverse.
I.e. the $g_{\mu\nu}$ with lower indices would just be the defined as the inverse

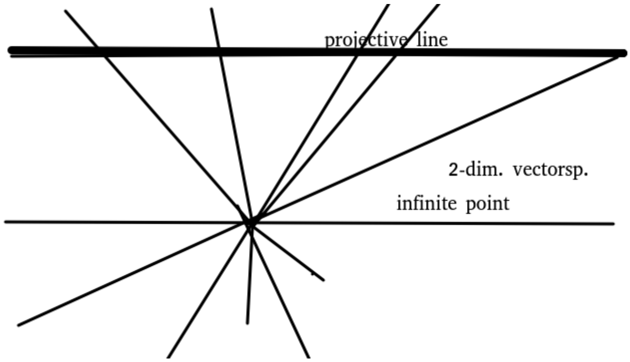
$$g_{\mu\nu} = (-1)^{\mu+\nu} \frac{\det g^{\cdot\cdot}|_{\text{left out}_{\mu\nu}}}{\det(g^{\cdot\cdot})} \tag{13.35}$$

So when $\det(g^{\mu\nu}) = 0$ (generically) all matrix elements of $g_{\mu\nu}$ go to infinity. And so near by all distances between the point in the projective space become huge.

13.13 Characterization

Characterization of Projective line as 3-transitive
In last years Bled talk I presented a work with Masao Ninomiya [13], in which we showed that requiring for a group acting on space X in sharply 3-transitive way, essentially led you to the projective line (= a one dimensional projective space.
Hope to somehow characterize projective spaces by some form of n-transitivity (maybe next years talk?)

Projective line
The projective line is the real axis extended with one point at infinity.
Projective space of d dimension as set of Rays in Vector space of $d + 1$



13.14 Conclusion

Conclusion

- We centered the present talk about: That Astri Kleppe and I could derive : **Principle of locality, that the action effectively is an integral over a local Lagrange density - only depending on fields defined at a single space-time point -.**
  From: **Diffeomorphism symmetry i.e. a manifold** and some milder assumptions, Taylor expandability, keeping only low dimensional terms.

13.15 Concluison (continued)

The locality-derivation were not quite successful:

- A couple of (not quite) successes turned out promising:
  - Only getting a function form $S[\phi] = F(S_1[\phi], ..., S_n[\phi])$, where the $S_i[\phi]$ are truly local actions, meant: couplings could depend on what goes on all over at all times, we could likely get our several vacua with same energy density.
  - Would get superlocality and thus no propagation, unless we have some spontaneous breaking of diffeomorphism symmetry down to a metric space time, by **gravity fields**
- A phenomenological argument - based on only 2 s.d. though - for embedding of world in a **projective space-time**.

13.16 Conclusion (a bit old)

- Screwed logic: If we want to use Astris and my way to derive locality, then space-time must be diffeomorphism invariant, or it might still go with less symmetry, such as the **Projective space time**.
- With such diffeomorphism or projective space symmetry, there would be no propagations of signals no waves, if there does not appear by spontaneous break down a $g^{\mu\nu}$ (with upper indices) being non-zero.
- So gravity is activity due to a needed for propagation, non-zero field.

13.17 Speculative Conclusion

Fundamentally we have for some reason a projective space or a manifold with diffeomorphism invariance - in any case a space-time with symmetry group acting in a practically n-transitive way with a high n - but then either a field $g^{\mu\nu}(x)$ or some corresponding vierbein fields $V_a^{\mu}(x)$ (also with upper curved indices) get non-zero in the vacuum. This makes possible propagation of waves/particles along the direction of the subspace of the tangent space spanned by this $g^{\mu\nu}(x)$ or these vierbeins $V_a^{\mu}(x)$. So at the end the end **the Einstein general relativity four-space is embedded into the more fundamental general manifold or projective space.**

13.18 Acknowledgements:

I thank Danai Roumelioti for calling my attention to the works by R. Percacci [1], and her and George Zoupanos for discussions. Also I thank the Niels Bohr Institute for staying there as emeritus and for support to the Bled-Workshop, where I also talked about this subject. Most of the work presented here was a review of my work with Astri Kleppe, with whom I had very many discussions.

References

1. Roberto Percacci, “spontaneous Soldering”, Physics Letters vol. **144** number 1.2, 23. august 1984.
   R. Percacci, “The Higgs phenomenon in Quantum Gravity”, arXiv: 0712.3545v2 [hep-th] 8. jan. 2008.
2. H. B. Nielsen and A. Kleppe (The Niels Bohr Institute, Copenhagen, Denmark and SACT, Oslo, Norway)
   “Towards a Derivation of Space” arXiv: 1403.1410v1 [hep-th] 6 Mar 2014.
3. H.B. Nielsen and A. Kleppe BLED WORKSHOPS IN PHYSICS VOL. 20, NO. 2 Proceedings to the 22nd Workshop What Comes Beyond . . . (p. 135) Bled, Slovenia, July 6–14, 2019
   “ Deriving Locality From Diffeomorphism Symmetry in a Fiber Bundle Formalism ”
4. C. Godbillon, ”Géométrie différentielle et mécanique analytique” , Hermann (1969).
5. A.A. Sheykin and S.A. Paston The approach to gravity as a theory of embedded surface AIP Conference Proceedings 1606, 400–406 (2014)
6. Holger Bech Nielsen, Niels Bohr Institutet, Blegdamsvej 15 -21 DK 2100Copenhagen “Small Representation Explaing, Why Standard Model Group”, PoS(CORFU2014)045 (Corfu 2014).
7. H. B. Nielsen, “Dimension Four Wins the Same Game as the Standard Model Group,” arXiv:1304.6051 [hep-ph].
8. Don Bennett and H. B. Nielsen, “Seeking...”, Contribution to the workshop “Beyond the Standard Models”, Bled 2011
9. H. B. Nielsen, “Small Representation Principle”, Contribution to the workshop “Beyond the Standard Models”, Bled 2013.
10. O’Raifeartaigh, Group Structure of Gauge theories,University Press Cambridge (1986)

11. P. Bielewicz, K. M. Górski, A. J. Banday "Low-order multipole maps of cosmic microwave background anisotropy derived from WMAP", Monthly Notices of the Royal Astronomical Society, Volume 355, Issue 4, December 2004, Pages 1283–1302, https://doi.org/10.1111/j.1365-2966.2004.08405.x
12. Stillits, Thesis for Cand.Scient.degree at The Niels Bohr Insitute.
13. H.B. Nielsen, Masao Ninomiy "String Field Theory and Bound State, Projective Line, and sharply 3-transitive group" Proceedings contribution to the 24th Bled-Workshop "What comes beyond the Standard Models" Bled, July 3-11, 2021, Slovenia; arXiv:2111.05106 [physics.gen-ph] (or arXiv:2111.05106v1 [physics.gen-ph] for this version)
    https://doi.org/10.48550/arXiv.2111.05106
14. Mayukh R. Gangopadhyaya, Grant J. Mathews, Kiyotomo Ichiki, Toshitaka "Explaining low ' Anomalies in the CMB Power Spectrum with Resonant Superstring Excitations during Inflation" Eur. Phys.J., arXiv:1701.00577v5[astro-ph.CO] 17 sep. 2018
15. Mussadiq H. Qureshi, Asif Iqbal, Manzoor A. Malik, and Tarun Souradeep "Low-l power suppression in punctuated inflation" Published 7 April 2017; 2017 IOP Publishing Ltd and Sissa Medialab srl Journal of Cosmology and Astroparticle Physics, Volume 2017, April 2017 Citation Mussadiq H. Qureshi et al JCAP04(2017)013, DOI 10.1088/1475-7516/2017/04/013

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14 The gauge coupling unification in Grand Unified Theories based on the group E_8

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Abstract. We consider a theory with the gauge group E_8, assuming that the gauge symmetry breaking pattern is $E_8 \to E_7 \times U_1 \to E_6 \times U_1 \to SO_{10} \times U_1 \to SU_5 \times U_1 \to SU_3 \times SU_2 \times U_1$ and vacuum expectation values are acquired only by components of the representations 248. It is demonstrated that, in this case, there are several options for the relations between the gauge couplings of the resulting theory, but only one of them gives $\alpha_3 = \alpha_2$ and $\sin^2\theta_W = 3/8$. Also, it is the only option for which the resulting theory can include all MSSM superfields.

Povzetek: Avtor študira zlomitve simetrije E_8 za umeritvena polja ob predpostavki, da se simetrija zlomi najprej v $E_7 \times U_1$, ta pa naprej preko $E_6 \times U_1$ in preko $SO_{10} \times U_1$ v $SU_5 \times U_1$ in končno v $SU_3 \times SU_2 \times U_1$, s privzetkom, da imajo neničelno pričakovano vrednost samo upodibitve 248. Poišče razmerja med sklopitvenimi konstantami za tako izbrane zlomitve simetrij. Izkaě se, da le ena med njimi ponudi $\alpha_3 = \alpha_2$ in $\sin^2\theta_W = 3/8$. Izkaže se tudi, da le v tem primeru vključuje nastala teorija vsa superpolja MSSM.

14.1 Introduction

Supersymmetric extensions of the Standard Model are one of the best candidates for describing the physics beyond it [1]. The Minimal Supersymmetric Standard Model (MSSM) is the simplest of these extensions. It is a gauge theory with the group $SU_3 \times SU_2 \times U_1$ and softly broken supersymmetry. Consequently, MSSM contains 3 gauge coupling constants e_3, e_2, and e_1. (The number of gauge coupling constants is equal to the number of factors in the gauge group.) Quarks, leptons, and Higgs fields are components of the chiral matter superfields which are collected in Table 14.1. In this table we also present their quantum numbers with respect to SU_3, SU_2, and U_1 subgroups (the representations for SU_3 and SU_2 and the hypercharge for U_1). Also we indicate that there are 3 generations of the chiral superfields which include quarks and leptons.
Note that for the superfields which include left quarks and leptons we use the brief notations

$$Q = \begin{pmatrix} \widetilde{U} \\ \widetilde{D} \end{pmatrix}; \qquad L = \begin{pmatrix} \widetilde{N} \\ \widetilde{E} \end{pmatrix}. \tag{14.1}$$

Superfield	SU_3	SU_2	U_1 (Y)	Superfield	SU_3	SU_2	U_1 (Y)
$3\times Q$	3	2	$-1/6$	$3\times N$	1	1	0
$3\times U$	3	1	$2/3$	$3\times E$	1	1	-1
$3\times D$	3	1	$-1/3$	H_d	1	2	$1/2$
$3\times L$	1	2	$1/2$	H_u	1	2	$-1/2$

Table 14.1: Quantum numbers of the various MSSM chiral matter superfields.

It is important that the quantum numbers of various MSSM superfields are not accidental. In particular, they satisfy the anomaly cancellation conditions, see ,e.g., [2,3],

$$\mathrm{tr}\Big(T^A\{T^B,T^C\}\Big)=0, \tag{14.2}$$

where T^A are the generators of the representation in which the chiral matter superfields lie. This equation is needed for the renormalizability of the theory. In MSSM this condition should be verified for all (10) possible ways of placing the gauge fileds on the external lines in the triangle diagram. The nontrivial relations needed for anomaly cancellation appear for
1. two SU_3 gauge fields and 1 U_1 gauge field;
2. two SU_2 gauge fields and 1 U_1 gauge field;
3. three U_1 gauge fields.
The corresponding relations needed for the anomaly cancellation are written in the form

$$\begin{aligned}&Y_U+Y_D+2\,Y_Q=0; \qquad 3\,Y_Q+Y_L=0;\\ &3\,Y_U^3+3\,Y_D^3+Y_E^3+6\,Y_Q^3+2\,Y_L^3=0.\end{aligned} \tag{14.3}$$

Therefore, there is a question if the quantum numbers of MSSM superfields are accidental, and how they appear.
The origin of the quantum numbers of various (super)fields can presumably be explained with the help of the Grand Unification idea. Similarly to the nonsupersymmetric case first considered in [4], the MSSM superfields of a single generation (including the right neutrino) can be accommodated in 3 irreducible representations of the group SU_5

$$1+5+10 \tag{14.4}$$

in such a way that

$$1\sim N;\quad 5_i\sim\begin{pmatrix}D_1\\ D_2\\ D_3\\ \widetilde{E}\\ -\widetilde{N}\end{pmatrix};\quad 10^{\,ij}\sim\begin{pmatrix}0 & U_3 & -U_2 & \widetilde{U}^1 & \widetilde{D}^1\\ -U_3 & 0 & U_1 & \widetilde{U}^2 & \widetilde{D}^2\\ U_2 & -U_1 & 0 & \widetilde{U}^3 & \widetilde{D}^3\\ -\widetilde{U}^1 & -\widetilde{U}^2 & -\widetilde{U}_3 & 0 & E\\ -\widetilde{D}^1 & -\widetilde{D}^2 & -\widetilde{D}^3 & -E & 0\end{pmatrix}. \tag{14.5}$$

In this case after the symmetry breaking $SU_5 \to SU_3 \times SU_2 \times U_1$ all fields of the low-energy theory will have correct quantum numbers.
The SU_5 symmetry can be broken down to the subgroup $SU_3 \times SU_2 \times U_1$ with the elements

$$\omega_5 = \begin{pmatrix} \omega_3 e^{-i\beta_Y/3} & 0 \\ 0 & \omega_2^* e^{i\beta_Y/2} \end{pmatrix} \in SU_3 \times SU_2 \times U_1 \subset SU_5. \tag{14.6}$$

by a vacuum expectation value (vev) of the Higgs field in the adjoint representation 24. Then, from the SU_5 tensor transformations

$$N \to N; \qquad 5_i \to (\omega_5)_i{}^j 5_j; \qquad 10^{\,ij} \to (\omega_5^*)^i{}_k (\omega_5^*)^j{}_l\, 10^{\,kl} \tag{14.7}$$

one obtains that with respect to the subgroup $SU_3 \times SU_2 \times U_1$ all chiral superfields have the same quantum numbers as the MSSM superfields. The further symmetry breaking $SU_3 \times SU_2 \times U_1 \to SU_3 \times U_1^{em}$ is usually made by vacuum expectation values of the Higgs superfields in the representations 5 and 5 . However, in this case the doublet-triplet splitting requires fine tuning [5,6].
The anomaly cancellation in this model occurs due to the relation

$$\mathrm{tr}\big(T^A\{T^B, T^C\}\big)\Big|_5 + \mathrm{tr}\big(T^A\{T^B, T^C\}\big)\Big|_{\overline{10}} = 0. \tag{14.8}$$

Because the group SU_5 is simple, there is the only gauge coupling constant e_5 in the SU_5 Grand Unified Theory (GUT). This implies that in the low-energy $SU_3 \times SU_3 \times U_1$ theory 3 coupling constants should be related to each other. This relation is written as

$$\alpha_2 = \alpha_3; \qquad \sin^2\theta_W = 3/8, \tag{14.9}$$

where $\mathrm{tg}\,\theta_W \equiv e_1/e_2$. If we introduce the notation $\alpha_1 \equiv \frac{5}{3}\cdot\frac{e_1^2}{4\pi}$, then the gauge coupling unification condition takes the simplest form $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_5$. This condition is in a good agreement with the well-known renormalization group behaviour of the running gauge couplings in MSSM [7–9].
The field content of the SU_5 GUT indicates on a possibility of the existence of a theory with a wider SO_{10} symmetry [10,11] because the superfields of a single generation can arise from a single irreducible (spinor) SO_{10} representation

$$16\Big|_{SO_{10}} = 1(5) + 5(-3) + \;10\,(1)\Big|_{SU_5\times U_1}. \tag{14.10}$$

However, the symmetry breaking pattern

$$SO_{10} \to SU_5 \to SU_3 \times SU_2 \times U_1 \tag{14.11}$$

has some drawbacks. In particular, for the symmetry breaking one needs (super)fields in sufficiently large representations (no less than 45 of SO_{10} and 24 of SU_5). Moreover, the simplest (supersymmetric) SU_5 model is excluded by the modern experimental limits on the proton lifetime [12]. A more convenient symmetry breaking pattern is

$$SO_{10} \to SU_5 \times U_1 \to SU_3 \times SU_2 \times U_1. \qquad (14.12)$$

It corresponds to the flipped SU_5 model [13–16]. In this case the chiral matter superfields containing quarks and leptons are situated in the representations $3 \times \Big(\overline{10}(1) + 5(-3) + 1(5)\Big)$ in a different way,

$$1 \sim E; \quad 5_i \sim \begin{pmatrix} U_1 \\ U_2 \\ U_3 \\ \widetilde{E} \\ -\widetilde{N} \end{pmatrix}; \quad \overline{10}^{ij} \sim \begin{pmatrix} 0 & D_3 & -D_2 & \widetilde{U}^1 & \widetilde{D}^1 \\ -D_3 & 0 & D_1 & \widetilde{U}^2 & \widetilde{D}^2 \\ D_2 & -D_1 & 0 & \widetilde{U}^3 & \widetilde{D}^3 \\ -\widetilde{U}^1 & -\widetilde{U}^2 & -\widetilde{U}^3 & 0 & N \\ -\widetilde{D}^1 & -\widetilde{D}^2 & -\widetilde{D}^3 & -N & 0 \end{pmatrix}, \qquad (14.13)$$

so that the superfields corresponding to the right up and down quarks and leptons are swapped. (That is why this model is called "flipped".)

The $SU_5 \times U_1$ symmetry is broken down to $SU_3 \times SU_2 \times U_1^{(Y)}$ by vevs of Higgses in the representations $10(-1)$ and $10\,(1)$, and the group $U_1^{(Y)}$ appears as a superposition of the $SU(5)$ transformations with

$$\omega_5 = \exp\left\{ \frac{i\alpha_Y}{30} \begin{pmatrix} 2 \cdot 1_3 & 0 \\ 0 & -3 \cdot 1_2 \end{pmatrix} \right\} \qquad (14.14)$$

and the U_1 transformations with $\omega_1 = \exp(-iQ\alpha_Y/5)$, where Q is the U_1 charge normalized by Eq. (14.10). This model

1. allows to naturally split Higgs douplet and triplet [14, 17–19];
2. does not require higher representations for the breaking of the SU_5 symmetry;
3. satisfies present limits on the proton lifetime [20] (see also [21–23]).

The flipped SU_5 model has 2 coupling constants e_5 and e_1. However, if it is considered as a remnant of the SO_{10} theory, then they are related to each other as

$$e_5 = \frac{e_{10}}{\sqrt{2}}; \qquad e_1 = \frac{e_5}{2\sqrt{10}} = \frac{e_{10}}{4\sqrt{5}}. \qquad (14.15)$$

Then for the residual $SU_3 \times SU_2 \times U_1$ theory we obtain the standard relations (14.9).

Also for constructing various GUTs it is possible to consider larger groups, for example, the exceptional group E_6 [24] (see [25] for a review) or even E_8 [26–31]. Using of the exceptional groups E_7 and E_8 is complicated by the fact that they have only real representation and the corresponding theories are not chiral [32]. However, it is known that the Lie algebras used for constructing various GUTs can be considered as a part of the E-series if we also include in it some classical Lie algebras, see Fig. 14.1.

Here (following [33]) we will investigate a possibility of constructing GUT based on the group E_8, which is the largest of them, assuming that the symmetry breaking pattern is

$$E_8 \to E_7 \times U_1 \to E_6 \times U_1 \to SO_{10} \times U_1 \to SU_5 \times U_1 \to SU_3 \times SU_2 \times U_1, \quad (14.16)$$

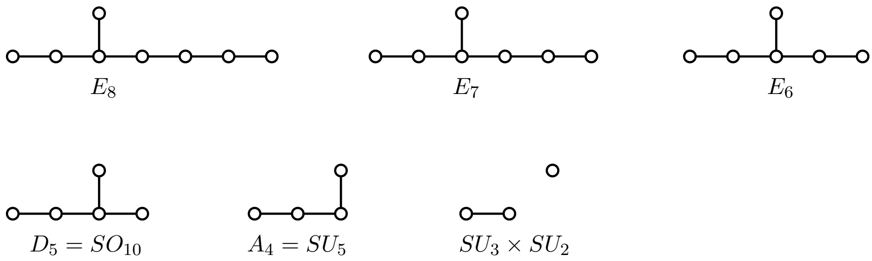

Fig. 14.1: Dynkin diagrams for the Lie algebras of the E-series with some classical algebras included

and vevs responsible for the various symmetry breakings are acquired only by certain parts of the fundamental representation of the group E_8 (of the dimension 248). Namely, we will investigate the unification of the gauge couplings and study a possibility of obtaining the conditions (14.9).
The paper is organized as follows. In section 14.2 we present explicit commutation relations for the exceptional Lie algebras of the E-series. In the next section 14.3 we discusss the symmetry breaking by vevs of scalar fields in various parts of the representation 248 of E_8. The relations between the coupling constants of the non-Abelian groups are derived in section 14.4. The similar relations of the U_1 coupling constants for various options of the symmetry breaking are discussed in section 14.5. Conclusion contains a brief summary of the results.

14.2 Explicit commutation relations for the exceptional Lie algebras of the E-series

For investigating the symmetry breaking pattern (14.16) we will use the explicit construction of all Lie algebras entering it. In particular, we will need the explicit commutation relations for the exceptional algebras E_8, E_7, and E_6, which will be presented in this section.

14.2.1 The Γ-matrices in diverse dimensions

For constructing the commutation relations for the exceptional Lie algebras in the explicit form we will need the Γ-matrices in the space of a dimension D and the Euclidean signature. By definition, they satisfy the condition

$$\{\Gamma_i, \Gamma_j\} = 2\delta_{ij} \cdot 1. \tag{14.17}$$

It is known that for an even D they have the size $2^{D/2} \times 2^{D/2}$, and for an odd D their size is $2^{(D-1)/2} \times 2^{(D-1)/2}$. In particular, for $D = 2, 3$ as the Γ-matrixes one can choose the Pauli matrices,

$$\Gamma_1^{(2)} = \sigma_1; \quad \Gamma_2^{(2)} = \sigma_2; \qquad \Gamma_1^{(3)} = \sigma_1; \quad \Gamma_2^{(3)} = \sigma_2; \quad \Gamma_3^{(3)} = \sigma_3. \qquad (14.18)$$

For larger values of D the Γ-matrices are constructed with the help of mathematical induction. Suppose that we have constructed them in an odd dimension D. Then in the next (even) dimension $D + 1$ the Γ-matrices have the size in two times larger and can be chosen in the form

$$\Gamma_i^{(D+1)} = \begin{pmatrix} 0 & \Gamma_i^{(D)} \\ \Gamma_i^{(D)} & 0 \end{pmatrix}, \quad i = 1, \dots, D; \qquad \Gamma_{D+1}^{(D+1)} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}. \qquad (14.19)$$

In this case it is also possible to construct the matrix

$$\Gamma_{D+2}^{(D+1)} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad (14.20)$$

which satisfies the conditions

$$\left\{\Gamma_{D+2}^{(D+1)}, \Gamma_i^{(D+1)}\right\} = 0, \quad i = 1, \dots D+1; \qquad \left(\Gamma_{D+2}^{(D+1)}\right)^2 = 1. \qquad (14.21)$$

This implies that in the next (odd) dimension $D + 2$ as Γ-matrices one can take the Γ-matrices from the previous (even) dimension supplemented by the matrix $\Gamma_{D+2}^{(D+1)}$,

$$\Gamma_i^{(D+2)} \equiv \Gamma_i^{(D+1)}, \quad \text{where} \quad i = 1, \dots D+2. \qquad (14.22)$$

Thus, the induction step is completed.
The above constructed Γ-matrices are Hermitian, $(\Gamma_i)^+ = \Gamma_i$. Therefore, for odd i they are symmetric, while for even i they are antisymmetric. In an even dimension D the charge conjugation matrix is defined as

$$C \equiv \Gamma_1 \Gamma_3 \dots \Gamma_{D-1} \qquad (14.23)$$

and satisfies the conditions

$$C\,\Gamma_i\,C^{-1} = -(-1)^{D/2}\,(\Gamma_i)^T; \qquad C^{-1} = C^+ = C^T = (-1)^{D(D-2)/8}\,C. \qquad (14.24)$$

Consequently, for the antisymmetrized products of Γ-matrixes the identities

$$\begin{aligned}
&(\Gamma_{i_1 i_2 \dots i_k} C)^T = (-1)^{(D-2k)(D-2k-2)/8}\,\Gamma_{i_1 i_2 \dots i_k} C; \\
&(\Gamma_{i_1 i_2 \dots i_k} \Gamma_{D+1} C)^T = (-1)^{(D-2k)(D-2k+2)/8}\,\Gamma_{i_1 i_2 \dots i_k} \Gamma_{D+1} C
\end{aligned} \qquad (14.25)$$

are valid in the case of even D.

14.2.2 Notations

We will denote the generators of the fundamental representation by t_A, where $A = 1, \dots, \dim G$. They are normalized by the condition

$$\mathrm{tr}(t_A t_B) = \frac{1}{2} g_{AB}, \tag{14.26}$$

where g_{AB} is a (symmetric) metric, and g^{AB} is the corresponding inverse matrix. For an arbitrary representation R the generators T_A satisfy the equations

$$\mathrm{tr}(T_A T_B) = T(R) g_{AB}; \qquad [T_A, T_B] = i f_{AB}{}^C T_C, \tag{14.27}$$

where $f_{AB}{}^C$ are the structure constants. The expression $f_{ABC} \equiv g_{CD} f_{AB}{}^D$ is totally antisymmetric, and

$$(T_{Adj\,A})^C{}_B = i f_{AB}{}^C; \qquad C_2\, g_{AB} \equiv -f_{AC}{}^D f_{BD}{}^C. \tag{14.28}$$

In particular, from these equations we obtain $g^{AB}[T_A, [T_B, T_C]] = C_2 T_C$. Also we note that for irreducible representations

$$C(R)_i{}^j \equiv g^{AB}(T_A T_B)_i{}^j = C(R)\, \delta_i{}^j, \quad \text{where} \quad C(R) = T(R) \cdot \frac{\dim G}{\dim R}. \tag{14.29}$$

14.2.3 The group E_8

According to [32], the fundamental representation of the group E_8 coincides with the adjoint representation and has the dimension 248. The group E_8 has the maximal subgroup $SO_{16} \subset E_8$, with respect to which

$$248\Big|_{E_8} = 120 + 128\Big|_{SO_{16}}, \tag{14.30}$$

where 120 is the adjoint representation of SO_{16}, and 128 is its representation by Majorana-Weyl (right, for the definiteness) spinors. Therefore, the E_8 generators can be written as the set [34]

$$t_A = \big\{ t_a,\ t_{ij} \big\}, \tag{14.31}$$

where $i, j = 1, \dots, 16$ and $a = 1, \dots, 128$. The commutation relations of the group E_8 can be presented in the form

$$\begin{aligned}
&[t_{ij}, t_{kl}] = \frac{i}{\sqrt{120}}\Big(\delta_{il} t_{jk} - \delta_{jl} t_{ik} - \delta_{ik} t_{jl} + \delta_{jk} t_{il}\Big);\\
&[t_{ij}, t_a] = -\frac{i}{\sqrt{480}}(\Gamma^{(16)}_{ij})_a{}^b t_b;\\
&[t_a, t_b] = -\frac{i}{2\sqrt{480}}(\Gamma^{(16)}_{ij} C^{(16)})_{ab} t_{ij}.
\end{aligned} \tag{14.32}$$

where charge conjugation matrix in $D = 16$ denoted by $C^{(16)}$ is symmetric, and the matrices $\Gamma^{(16)}_{ij} C^{(16)}$ are antisymmetric. The corresponding metric has the form

$$g^{\mathbf{AB}} \to \begin{pmatrix} \frac{1}{4}\left(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}\right) & 0 \\ 0 & (C^{(16)})^{ab} \end{pmatrix}. \tag{14.33}$$

In particular, it is easy to verify the identity

$$g^{\mathbf{AB}} t_{\mathbf{A}} t_{\mathbf{B}} = \frac{1}{2} t_{ij} t_{ij} + (C^{(16)})^{ab} t_a t_b = \frac{1}{2}. \tag{14.34}$$

14.2.4 The group E_7

To describe the group E_7, it is convenient to use its subgroup $SO_{12} \times SO_3$. Then [32]

$$\begin{aligned} 56\Big|_{E_7} &= [12, 2] + [32, 1]\Big|_{SO_{12}\times SO_3}; \\ 133\Big|_{E_7} &= [1, 3] + [32', 2] + [66, 1]\Big|_{SO_{12}\times SO_3}, \end{aligned} \tag{14.35}$$

where 32 and $32'$ are right and left spinor representations of SO_{12}. The indices of the left SO_{12} spinors are denoted by dots. Therefore, the E_7 generators can be written as the set

$$t_{\mathbf{A}} = \left\{ t_{ij}, t_\alpha, t_{a\dot{A}} \right\}, \tag{14.36}$$

where $a, b = 1, 2$; $\alpha, \beta = 1, \ldots, 3$; $i, j = 1, \ldots, 12$. Their commutation relations take the form [33]

$$\begin{aligned} &[t_\alpha, t_\beta] = \frac{i}{\sqrt{12}} \varepsilon_{\alpha\beta\gamma} t_\gamma; \qquad [t_\alpha, t_{ij}] = 0; \\ &[t_\alpha, t_{a\dot{A}}] = -\frac{1}{2\sqrt{12}} (\sigma_\alpha)_a{}^b t_{b\dot{A}}; \qquad [t_{ij}, t_{a\dot{A}}] = -\frac{i}{2\sqrt{24}} (\Gamma^{(12)}_{ij})_{\dot{A}}{}^{\dot{B}} t_{a\dot{B}}; \\ &[t_{ij}, t_{kl}] = \frac{i}{\sqrt{24}} \Big(\delta_{il} t_{jk} - \delta_{jl} t_{ik} - \delta_{ik} t_{jl} + \delta_{jk} t_{il} \Big); \\ &[t_{a\dot{A}}, t_{b\dot{B}}] = \frac{i}{4\sqrt{24}} (\sigma_2)_{ab} (\Gamma^{(12)}_{ij} C^{(12)})_{\dot{A}\dot{B}} t_{ij} + \frac{1}{2\sqrt{12}} (C^{(12)})_{\dot{A}\dot{B}} (\sigma_\alpha \sigma_2)_{ab} t_\alpha. \end{aligned} \tag{14.37}$$

and the corresponding metric is

$$g^{\mathbf{AB}} \to \begin{pmatrix} \frac{1}{4}\left(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}\right) & 0 & 0 \\ 0 & \delta_{\alpha\beta} & 0 \\ 0 & 0 & (\sigma_2)^{ab} (C^{(12)})^{\dot{A}\dot{B}} \end{pmatrix}. \tag{14.38}$$

Note that the matrices $\sigma_2 = i\sigma_1\sigma_3$ and $C^{(12)}$ are antisymmetric, while the matrices $\sigma_\alpha \sigma_2$ and $\Gamma^{(12)}_{ij} C^{(12)}$ are symmetric, so that this metric is really symmetric.

In the explicit form the generators of the fundamental representation 56 are written as

$$t_{ij} = \frac{i}{\sqrt{24}} \begin{pmatrix} (\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk})\delta_a^b & 0 \\ 0 & \frac{1}{2}(\Gamma_{ij}^{(12)})_A{}^B \end{pmatrix};$$
$$t_\alpha = \frac{1}{2\sqrt{12}} \begin{pmatrix} \delta_{kl}(\sigma_\alpha)_a{}^b & 0 \\ 0 & 0 \end{pmatrix};$$
$$t_{d\dot{D}} = \frac{i}{2\sqrt{12}} \begin{pmatrix} 0 & (\sigma_2)_{da}(\Gamma_k^{(12)})_{\dot{D}}{}^B \\ (\Gamma_l^{(12)}C^{(12)})_{A\dot{D}}\delta_d^b & 0 \end{pmatrix}. \quad (14.39)$$

As a check, one can verify that

$$C(56) = \frac{1}{2}t_{ij}t_{ij} + t_\alpha t_\alpha + (\sigma_2)^{ab}(C^{(12)})^{\dot{A}\dot{B}}t_{a\dot{A}}t_{b\dot{B}} = \frac{19}{16} = \frac{1}{2}\cdot\frac{133}{56}. \quad (14.40)$$

14.2.5 The group E_6

For describing the group E_6 we will use its maximal subgroup $SO_{10} \times U_1$ with respect to that

$$27\Big|_{E_6} = 1(4) + 10(-2) + 16(1)\Big|_{SO_{10}\times U_1};$$
$$27\Big|_{E_6} = 1(-4) + 10(2) + 16(-1)\Big|_{SO_{10}\times U_1};$$
$$78\Big|_{E_6} = 1(0) + 16(-3) + 16(3) + 45(0)\Big|_{SO_{10}\times U_1}, \quad (14.41)$$

where 16 and 16 are the right and left spinor representations of SO_{10}. However, now we will use a single spinor index $a = 1, \ldots, 32$, so that the E_6 generators can presented as the set

$$t_{\mathbf{A}} = \{t_{ij}, t_a, t\}. \quad (14.42)$$

In this case their commutation relations are written in the form [33]

$$[t_{ij}, t_{kl}] = \frac{i}{\sqrt{12}}\Big(\delta_{il}t_{jk} - \delta_{jl}t_{ik} - \delta_{ik}t_{jl} + \delta_{jk}t_{il}\Big);$$
$$[t_{ij}, t] = 0; \qquad [t, t_a] = \frac{1}{4}(\Gamma_{11}^{(10)})_a{}^b t_b; \qquad [t_{ij}, t_a] = -\frac{i}{2\sqrt{12}}(\Gamma_{ij}^{(10)})_a{}^b t_b;$$
$$[t_a, t_b] = -\frac{i}{4\sqrt{12}}(\Gamma_{ij}^{(10)}C^{(10)})_{ab}t_{ij} + \frac{1}{4}(\Gamma_{11}^{(10)}C^{(10)})_{ab}t, \quad (14.43)$$

and the corresponding metric is

$$g^{AB} \to \begin{pmatrix} \frac{1}{4}\big(\delta_{ik}\delta_{jl}-\delta_{il}\delta_{jk}\big) & 0 & 0\\ 0 & (C^{(10)})^{ab} & 0\\ 0 & 0 & 1 \end{pmatrix}. \qquad (14.44)$$

Note that in $D = 10$ the matrix $C^{(10)}$ is symmetric and coincides with its inverse, while the matrices $\Gamma^{(10)}_{ij}C^{(10)}$ and $\Gamma^{(10)}_{11}C^{(10)}$ are antisymmetric.
In the explicit form the generators of the fundamental representation 27 are written as

$$t_{ij} = \frac{i}{\sqrt{12}}\begin{pmatrix} 0 & 0 & 0\\ 0 & \delta_{ik}\delta_{jl}-\delta_{il}\delta_{jk} & 0\\ 0 & 0 & \frac{1}{4}\left[\Gamma^{(10)}_{ij}(1+\Gamma^{(10)}_{11})\right]_a{}^b \end{pmatrix};$$

$$t = \frac{1}{12}\begin{pmatrix} 4 & 0 & 0\\ 0 & -2\delta_{kl} & 0\\ 0 & 0 & \frac{1}{2}(1+\Gamma^{(10)}_{11})_a{}^b \end{pmatrix}; \qquad (14.45)$$

$$t_d = \frac{1}{\sqrt{96}}\begin{pmatrix} 0 & 0 & \sqrt{2}\big(1+\Gamma^{(10)}_{11}\big)_d{}^b\\ 0 & 0 & \left[\Gamma^{(10)}_k(1+\Gamma^{(10)}_{11})\right]_d{}^b\\ \sqrt{2}\left[(1+\Gamma^{(10)}_{11})C^{(10)}\right]_{ad} & \left[(1+\Gamma^{(10)}_{11})\Gamma^{(10)}_l C^{(10)}\right]_{ad} & 0 \end{pmatrix}$$

Again, as a check, one can verify that

$$C(27) = \frac{1}{2}t_{ij}t_{ij} + (C^{(10)})^{ab}t_a t_b + t^2 = \frac{13}{9} = \frac{1}{2}\cdot\frac{78}{27}. \qquad (14.46)$$

14.3 The symmetry breaking

In this section we will analyze the representations which can be used for realizing the symmetry breaking pattern (14.16) and find some relations between the coupling constants which appear at various stages of this symmetry breaking.

14.3.1 The symmetry breaking $E_8 \to E_7 \times U_1$

Let us investigate if it is possible to break the E_8 symmetry by vev of the representation 248. For this purpose we consider the embedding

$$E_8 \supset SO_{16} \supset \underbrace{SO_{12}\times SO_3}_{\subset E_7} \times SO_3, \qquad (14.47)$$

for which

$$
248\Big|_{E_8} = 120 + 128\Big|_{SO_{16}} = \underbrace{[1,1,3]}_{[1,3]} + \underbrace{[1,3,1] + [66,1,1] + [32',2,1]}_{+[133,1]}
$$
$$
+ \underbrace{[12,2,2] + [32,1,2]}_{+[56,2]\big|_{E_7\times SO_3}}\Big|_{SO_{12}\times SO_3\times SO_3}. \tag{14.48}
$$

Let us assume that a scalar field in the representation 248 is responsible for the symmetry breaking and present it in the form

$$
\varphi = \varphi_{\mathbf{A}}\, g^{\mathbf{AB}} t_{\mathbf{B}} = \frac{1}{2}\varphi_{ij} t_{ij} + \varphi_a (C^{(16)})^{ab} t_b \tag{14.49}
$$

supposing that

$$
(\varphi_{13,14})_0 = (\varphi_{15,16})_0 = v_8. \tag{14.50}
$$

Next, we construct the corresponding little group under which, by definition, a vacuum expectation remains invariant. This implies that it is necessary to find all E_8 generators which commute with $\varphi_0 = v_8(t_{13,14} + t_{15,16})$. Evidently, this vev commutes with all t_{ij} with $i, j = 1, \ldots, 12$, which form the subgroup SO_{12}. Also the little group includes the generators

$$
\widetilde{t}_1 \equiv \frac{1}{\sqrt{2}}(t_{13,16} - t_{14,15}); \qquad \widetilde{t}_2 \equiv \frac{1}{\sqrt{2}}(-t_{13,15} - t_{14,16});
$$
$$
\widetilde{t}_3 \equiv \frac{1}{\sqrt{2}}(t_{13,14} - t_{15,16}); \qquad \widetilde{t}'_3 \equiv \frac{1}{\sqrt{2}}(-t_{13,14} - t_{15,16}).
$$

They form the subgroup $SO_3 \times U_1$ of the little group,

$$
[\widetilde{t}'_3, \widetilde{t}_\alpha] = 0; \qquad [\widetilde{t}_\alpha, \widetilde{t}_\beta] = \frac{i}{\sqrt{60}}\varepsilon_{\alpha\beta\gamma}\widetilde{t}_\gamma. \tag{14.51}
$$

However, the little group is wider than $SO_{12} \times SO_3 \times U_1$ because some generators t_a also commute with the vacuum expectation value. Really,

$$
[\varphi_0, t_a] = v_8\left[t_{13,14} + t_{15,16}, t_a\right] = -\frac{i v_8}{2\sqrt{120}}\left(\Gamma^{(16)}_{13,14} + \Gamma^{(16)}_{15,16}\right)_a{}^b\, t_b. \tag{14.52}
$$

where

$$
-\frac{i}{2}\left(\Gamma^{(16)}_{13,14} + \Gamma^{(16)}_{15,16}\right) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \cdot \frac{1}{2}\left(1 + \Gamma^{(12)}_{13}\right). \tag{14.53}
$$

Therefore, the generators t_a which belong to the little group form two left 32 component SO_{12} spinors, which transform under the spinor representation 2 of the group SO_3. Thus, we obtain that the little group is $E_7 \times U_1$ because

$$133\Big|_{E_7} = [1,3] + [32',2] + [66,1]\Big|_{SO_{12}\times SO_3}. \qquad (14.54)$$

Therefore, by the vev (14.50) the symmetry is broken as

$$E_8 \to E_7 \times U_1. \qquad (14.55)$$

Next, let us relate 2 coupling constants of the resulting theory with the original coupling constant e_8. Comparing the commutation relations of the generators t_{ij} for the groups E_8 and E_7 we see that

$$t_{ij}\Big|_{E_8} = \frac{1}{\sqrt{5}} t_{ij}\Big|_{E_7}. \qquad (14.56)$$

Because $A_\mu = ieA_\mu^A t_A$ and the generators t_{ij} are normalized by the same condition

$$\mathrm{tr}(t_{ij}t_{kl}) = \frac{1}{2}\big(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}\big), \qquad (14.57)$$

the coupling constants for the groups E_7 and E_8 are related as

$$e_7 = \frac{e_8}{\sqrt{5}}. \qquad (14.58)$$

The coupling constant $e_1^{(7)}$ corresponding to the subgroup U_1 depends on the normalization of the U_1 charge. Let us choose the SO_3 generators in the subgroup $E_7 \times SO_3 \subset E_8$ in such a way that

$$[t'_\alpha, t'_\beta] = 2i\varepsilon_{\alpha\beta\gamma} t'_\gamma. \qquad (14.59)$$

and take t'_3 as a generator of the U_1 component of the little group. For this normalization condition

$$248\Big|_{E_8} = 1(0) + 1(2) + 1(-2) + 133(0) + 56(1) + 56(-1)\Big|_{E_7\times U_1}. \qquad (14.60)$$

From the other side, the generators of the SO_3 subgroup in $E_7 \times SO_3 \subset E_8$ normalized in the same way as all E_8 generators satisfy the commutation relation

$$[\widetilde{t}'_\alpha, \widetilde{t}'_\beta] = \frac{i}{\sqrt{60}} \varepsilon_{\alpha\beta\gamma} \widetilde{t}'_\gamma. \qquad (14.61)$$

Comparing it with the commutation relation for t'_α we see that

$$\widetilde{t}'_3 = \frac{1}{4\sqrt{15}} t'_3. \qquad (14.62)$$

This implies that the corresponding couplings are related by the equation

$$e_1^{(7)} = \frac{e_8}{4\sqrt{15}} = \frac{e_7}{4\sqrt{3}}. \qquad (14.63)$$

14.3.2 The symmetry breaking $E_7 \times U_1 \to E_6 \times U_1$

The group E_7 contains the maximal subgroup $E_6 \times U_1$, with respect to which

$$\begin{aligned} 56\Big|_{E_7} &= 27(1) + 27\,(-1) + 1(3) + 1(-3)\Big|_{E_6\times U_1}; \\ 133\Big|_{E_7} &= 1(0) + 27(-2) + 27\,(2) + 78(0)\Big|_{E_6\times U_1}. \end{aligned} \tag{14.64}$$

In particular, the representation 56 contains two E_6 singlets with nontrivial U_1 charges. If one of them acquires a vacuum expectation value, then the little group will contain the factor E_6. Let the vacuum expectation value ν_7 is acquired by the representation $56(1)$ of the group $E_7 \times U_1$, and the corresponding scalar field lies in the representation $1(3)$ of the group $E_6 \times U_1 \subset E_7$. Under the $U_1 \times U_1$ transformations in

$$E_7 \times \underbrace{U_1}_{\beta_1^{(7)}} \supset (E_6 \times \underbrace{U_1}_{\beta_2^{(7)}}) \times \underbrace{U_1}_{\beta_1^{(7)}}. \tag{14.65}$$

the vacuum expectation value changes as $\nu_7 \to \exp\left(i\beta_1^{(7)} + 3i\beta_2^{(7)}\right)\nu_7$. Therefore, it is invariant under the transformations with $\beta_1^{(7)} + 3\beta_2^{(7)} = 0$. Evidently, they constitute the group $U_1 \subset U_1 \times U_1$. Therefore, the little group in this case is $E_6 \times U_1$.

Next, we compare the coupling constants in the original $E_7 \times U_1$ theory and in its $E_6 \times U_1$ remnant. As earlier, comparing the commutation relations for the generators t_{ij} we find the relation between the couplings for the groups E_7 and E_6,

$$t_{ij}\Big|_{E_7} = \frac{1}{\sqrt{2}} t_{ij}\Big|_{E_6} \quad \to \quad e_6 = \frac{e_7}{\sqrt{2}}, \tag{14.66}$$

because

$$A_\mu\Big|_{E_7} = ie_7 A_\mu^A t_A\Big|_{E_7} \quad \to \quad A_\mu\Big|_{E_6} = ie_6 A_\mu^A t_A\Big|_{E_6}. \tag{14.67}$$

For obtaining the coupling constant $e_1^{(6)}$ we write the branching rule for the representation $56(1)$ with respect to the subgroup $E_6 \times U_1 \times U_1$,

$$56(1)\Big|_{E_7\times U_1} = 27(1,1) + 27\,(1,-1) + 1(1,3) + 1(1,-3)\Big|_{E_6\times U_1\times U_1}, \tag{14.68}$$

and choose the charge with respect to the little group in the form

$$Q_1^{(6)} = \frac{1}{2}\Big(-3Q_1^{(7)} + Q_2^{(7)}\Big). \tag{14.69}$$

From Eq. (14.63) we see that the charge $Q_1^{(7)}$ is an eigenvalue of the operator $4\sqrt{3}\,t_1^{(7)}$, where $t_1^{(7)}$ is the generator of the U_1 factor in $E_7 \times U_1$ which is normalized in the same way as the generators of the group E_7.

Let $t\Big|_{U_1\subset E_7}$ be the generator of the U_1 factor in the subgroup $E_6 \times U_1 \subset E_7$ normalized in the same way as all E_7 generators. Then

$$t\Big|_{U_1\subset E_7} = \frac{1}{12}\begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & -3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \quad \text{acting on} \quad \begin{pmatrix} 1 \\ 1 \\ 27 \\ 27 \end{pmatrix}, \tag{14.70}$$

because

$$\mathrm{tr}\left(\left(t\Big|_{U_1\subset E_7}\right)^2\right) = \frac{1}{144}\Big(1\cdot 3^2 + 1\cdot(-3)^2 + 27\cdot 1^2 + 27\cdot 1^2\Big) = \frac{1}{2}. \tag{14.71}$$

Comparing Eq. (14.70) with the first branching rule in (14.64) we see that the charge $Q_2^{(7)}$ is an eigenvalue of the operator $12\,t\Big|_{U_1\subset E_7}$. Therefore, the little group charge (14.69) corresponds to the operator

$$\frac{1}{2}\Big(-3\cdot 4\sqrt{3}\,t_1^{(7)} + 12\,t\Big|_{U_1\subset E_7}\Big) = 12\Big(-\frac{\sqrt{3}}{2}t_1^{(7)} + \frac{1}{2}t\Big|_{U_1\subset E_7}\Big). \tag{14.72}$$

In the right hand side the operator in the brackets is normalized in the same way as the generators of the group E_7. Therefore, the coefficient 12 is equal to the ratio of the couplings e_7 and $e_1^{(6)}$, so that

$$e_1^{(6)} = \frac{e_7}{12}. \tag{14.73}$$

Next, we construct the branching rule of 248 with respect to the subgroup $E_6 \times \underbrace{U_1}_{\beta_1^{(7)}} \times \underbrace{U_1}_{\beta_2^{(7)}} \subset E_7 \times \underbrace{U_1}_{\beta_1^{(7)}}$,

$$\begin{aligned} 248\Big|_{E_8} &= \Big[1(0,0) + 1(2,0) + 1(-2,0)\Big] + \Big[1(0,0) + 27(0,-2) + 27\,(0,2) \\ &+78(0,0)\Big] + \Big[27(1,1) + 27\,(1,-1) + 1(1,3) + 1(1,-3)\Big] + \Big[27(-1,1) \\ &+ 27\,(-1,-1) + 1(-1,3) + 1(-1,-3)\Big]\Big|_{E_6\times U_1\times U_1} \end{aligned} \tag{14.74}$$

and calculate the charge with respect to the little group for each term. As the result we obtain the decomposition

$$\begin{aligned} 248\Big|_{E_8} &= 4\times 1(0) + 2\times 1(3) + 2\times 1(-3) + 2\times 27(-1) + 2\times 27\,(1) \\ &+27(2) + 27\,(-2) + 78(0)\Big|_{E_6\times U_1}. \end{aligned} \tag{14.75}$$

14.3.3 The representations for the further symmetry breaking

The further investigation of the symmetry breaking is made similarly. Vacuum expectation values are acquired by the representations which are present in the branching rules of 248 and contain singlets with respect to the non-Abelian components of the little group with nontrivial U_1 charges, namely, for $E_6 \times U_1 \to SO_{10} \times U_1$

$$27\Big|_{E_6} = 1(4) + 10(-2) + 16(1)\Big|_{SO_{10}\times U_1}, \tag{14.76}$$

for $SO_{10} \times U_1 \to SU_5 \times U_1$

$$16\Big|_{SO_{10}} = 1(-5) + \bar{5}\,(3) + 10(-1)\Big|_{SU_5\times U_1}, \tag{14.77}$$

and for $SU_5 \times U_1 \to SU_3 \times SU_2 \times U_1$

$$10\Big|_{SU_5} = [1,1](6) + [\bar{3}\,,1](-4) + [3,2](1)\Big|_{SU_3\times SU_2\times U_1}. \tag{14.78}$$

However, the further symmetry breaking can be made in a different ways because the U_1 charges of these representations can be different. For example, according to Eq. (14.75), the symmetry breaking $E_6 \times U_1 \to SO_{10} \times U_1$ can be made either by vev of $27(-1)$ or by vev of $27(2)$ (and/or the corresponding conjugated representations). The various options which can appear in the considered symmetry breaking pattern will be analysed in section 14.5.

14.4 Remaining relations between the coupling constants of the non-Abelian groups

The remaining relations between the coupling constants for the non-Abelian groups can also be obtained by comparing the commutation relations for the corresponding generators using the explicit form of the embeddings. For instance, the SO_{10} generators $(t_{ij})_{kl} = i\,(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk})\,/2$ normalized with the metric

$$g_{\mathbf{AB}} \to \delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}; \qquad g^{\mathbf{AB}} \to \frac{1}{4}\Big(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}\Big) \tag{14.79}$$

satisfy the commutation relations

$$[t_{ij}, t_{kl}] = \frac{i}{2}\Big(\delta_{il}t_{jk} - \delta_{jl}t_{ik} - \delta_{ik}t_{jl} + \delta_{jk}t_{il}\Big). \tag{14.80}$$

Comparing this equation with the corresponding relation for E_6 we conclude that

$$t_{ij}\Big|_{E_6} = \frac{1}{\sqrt{3}}t_{ij}\Big|_{SO_{10}} \quad \to \quad e_{10} = \frac{e_6}{\sqrt{3}}, \tag{14.81}$$

because in this case

$$A_\mu\Big|_{E_6} = ie_6 A_\mu^{\mathbf{A}} t_{\mathbf{A}}\Big|_{E_6} \to A_\mu\Big|_{SO_{10}} = ie_{10} A_\mu^{\mathbf{A}} t_{\mathbf{A}}\Big|_{SO_{10}} = \frac{i}{2} e_{10} (A_\mu)_{ij} t_{ij}\Big|_{SO_{10}}. \tag{14.82}$$

For constructing the embedding $U_5 \subset SO_{10}$ we consider a complex 5-component column $z = x + iy$ such that

$$z \equiv x + iy \to \Omega_5 z = (B + iC)(x + iy) = (Bx - Cy) + i(By + Cx), \tag{14.83}$$

where B and iC are the real and purely imaginary parts of the 5×5 matrix $\Omega_5 \in U_5$. The condition $\Omega_5^+ \Omega_5 = 1$ leads to the constraints

$$B^T B + C^T C = 1; \qquad B^T C = C^T B. \tag{14.84}$$

The above transformation of z can equivalently be presented as the transformation of the real 10-component column

$$\begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} B & -C \\ C & B \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}. \tag{14.85}$$

Due to the constraints (14.84) the matrix belongs to the group SO_{10}. (Its determinant is equal to 1 because the U_5 group manifold is connected.) Therefore, it is possible to write the properly normalized generators of SO_{10} corresponding to the subgroup SU_5 in the form

$$t_A\Big|_{SU_5 \subset SO_{10}} = \begin{cases} \dfrac{1}{\sqrt{2}} \begin{pmatrix} t_{A,5} & 0 \\ 0 & t_{A,5} \end{pmatrix} = \dfrac{1}{\sqrt{2}} T(t_{A,5}), \\ \qquad\qquad \text{if } t_{A,5} \text{ is purely imaginary;} \\ \dfrac{i}{\sqrt{2}} \begin{pmatrix} 0 & t_{A,5} \\ -t_{A,5} & 0 \end{pmatrix} = \dfrac{1}{\sqrt{2}} T(t_{A,5}), \\ \qquad\qquad \text{if } t_{A,5} \text{ is real,} \end{cases} \tag{14.86}$$

where the generators of the SU_5 fundamental representation $t_{A,5}$ (with $A = 1, \ldots, 24$) are normalized by the condition

$$\mathrm{tr}\big(t_{A,5} t_{B,5}\big) = \frac{1}{2} \delta_{AB}. \tag{14.87}$$

The properly normalized generator of the U_1 subgroup of SO_{10} in this case takes the form

$$t\Big|_{U_1 \subset SO_{10}} = -\frac{i}{\sqrt{20}} \begin{pmatrix} 0 & -1_5 \\ 1_5 & 0 \end{pmatrix}. \tag{14.88}$$

Due to the factor $1/\sqrt{2}$ in Eq. (14.86) the coupling constants for the groups SO_{10} and SU_5 are related by the equation

$$e_5 = \frac{e_{10}}{\sqrt{2}}. \tag{14.89}$$

Similarly, the last embedding $SU_3 \times SU_2 \times U_1 \subset SU_5$

$$\omega_5 = \begin{pmatrix} e^{-2i\beta_2^{(5)}}\omega_3 & 0 \\ 0 & e^{3i\beta_2^{(5)}}\omega_2 \end{pmatrix} \tag{14.90}$$

gives the well-known equation $e_2 = e_3 = e_5$.

Thus, for the coupling constants corresponding to the non-Abelian groups we obtain the relations

$$e_2 = e_3 = e_5 = \frac{e_{10}}{\sqrt{2}} = \frac{e_6}{\sqrt{6}} = \frac{e_7}{\sqrt{12}} = \frac{e_8}{\sqrt{60}}. \tag{14.91}$$

14.5 The coupling constants for the U_1 groups and options for the further symmetry breaking

Next, we need to calculate all coupling constants corresponding to all U_1 groups present in the symmetry breaking pattern. For the symmetry breaking $G \times U_1 \to H \times U_1$ this can be done according to the following algorithm:

1. First, it is necessary to construct the decomposition of the representation which acquires vev with respect to the subgroup $H \times U_1 \times U_1 \subset G \times U_1$.
2. Next, one should find the expression for the little group charge. At all steps except for the last one it is chosen in such a way that this charge takes minimal possible integer values. At the last step the charge normalization is chosen so that the maximal number of MSSM representations has correct U_1 hypercharges.
3. After that, we construct the generators of the group $U_1 \times U_1$ normalized in the same way as the generators of the group G.
4. Next, we construct the generator of the little group and extract from it the operator normalized in the same way as the generators of the group G. The coefficient before it gives the ratio $e_G/e_1^{(H)}$.

With the help of this algorithm for each option of the symmetry breaking we obtain a sequence of the U_1 charges. For each of them finally we calculate $\mathrm{tg}\,\theta_W = e_1^{(Y)}/e_2$.

All options for the symmetry breaking and the corresponding values of $\sin^2\theta_W$ obtained according to the procedure described above are presented in Table 14.2. The various options appear because the scalar fields in the representations 27 of E_6, 16 of SO_{10}, and 10 of SU_5 can have different U_1 charges. However, among these options there is the only one (denoted by B-1-1-1) which gives the correct value of the Weinberg angle. Moreover, this is the only option that contains all representations needed for the accommodation of all chiral MSSM superfields, because in this case the branching rule for the representation 248 of E_8 with respect to the MSSM gauge group $SU_3 \times SU_2 \times U_1$ is [33]

$$248\Big|_{E_8} = 25 \times [1,1](0) + 5 \times [1,1](1) + 5 \times [1,1](-1) + [1,3](0)$$

$$+10 \times [1,2](1/2) + 10 \times [1,2](-1/2) + 10 \times [3,1](-1/3)$$

Option	$E_6 \times U_1$	$SO_{10} \times U_1$	$SU_5 \times U_1$	$\sin^2\theta_W$
B-1-1-1	$27(-1)\Big\vert_{E_6\times U_1}$	$16(-1)\Big\vert_{SO_{10}\times U_1}$	$10(-1)\Big\vert_{SU_5\times U_1}$	**3/8**
B-1-1-2	$27(-1)\Big\vert_{E_6\times U_1}$	$16(-1)\Big\vert_{SO_{10}\times U_1}$	$10(4)\Big\vert_{SU_5\times U_1}$	3/5
B-1-2-1	$27(-1)\Big\vert_{E_6\times U_1}$	$16(3)\Big\vert_{SO_{10}\times U_1}$	$10(-2)\Big\vert_{SU_5\times U_1}$	3/5
B-1-2-2	$27(-1)\Big\vert_{E_6\times U_1}$	$16(3)\Big\vert_{SO_{10}\times U_1}$	$10(3)\Big\vert_{SU_5\times U_1}$	3/4
B-2-1-1	$27(2)\Big\vert_{E_6\times U_1}$	$16(1)\Big\vert_{SO_{10}\times U_1}$	$10(-2)\Big\vert_{SU_5\times U_1}$	3/5
B-2-1-2	$27(2)\Big\vert_{E_6\times U_1}$	$16(1)\Big\vert_{SO_{10}\times U_1}$	$10(3)\Big\vert_{SU_5\times U_1}$	3/4

Table 14.2: Various options for the symmetry breaking $E_6 \times U_1 \to SO_{10} \times U_1 \to SU_5 \times U_1 \to SU_3 \times SU_2 \times U_1$

$$+10 \times [3,1](1/3) + 5 \times [\mathbf{3},\mathbf{1}](\mathbf{2/3}) + 5 \times [3,1](-2/3) + [3,2](-5/6)$$
$$+[3,2](5/6) + 5 \times [3,2](1/6) + 5 \times [\mathbf{3},\mathbf{2}](\mathbf{-1/6}) + [8,1](0)\Big\vert_{SU_3\times SU_2\times U_1}, \tag{14.92}$$

where the representations needed for accommodating the MSSM superfields are maked by the bold font. Also it is interesting to note that B-1-1-1 corresponds to the minimal possible absolute values of the U_1 charges of the above mentioned representations 27, 16, and 10. The values of coupling constants for all steps of symmetry breaking for the option B-1-1-1 are presented in Table 14.3.

Group	Vev	e_G	$e_1^{(G)}$
E_8	248	e_8	—
$E_7 \times U_1$	$56(\pm 1)$	$e_7 = e_8/\sqrt{5}$	$e_1^{(7)} = e_7/4\sqrt{3}$
$E_6 \times U_1$	$27(-1);\ 27(1)$	$e_6 = e_7/\sqrt{2}$	$e_1^{(6)} = e_6/6\sqrt{2}$
$SO_{10} \times U_1$	$16(-1);\ 16(1)$	$e_{10} = e_6/\sqrt{3}$	$e_1^{(10)} = e_{10}/4\sqrt{3}$
$SU_5 \times U_1$	$10(-1);\ 10(1)$	$e_5 = e_{10}/\sqrt{2}$	$e_1^{(5)} = e_5/2\sqrt{10}$
$SU_3 \times SU_2 \times U_1$	$[1,2](\pm 1/2)$	$e_3 = e_2 = e_5$	$e_1^{(Y)} = e_5\sqrt{3/5}$

Table 14.3: Values of the coupling constants for the option B-1-1-1

The options B-1-1-2, B-1-2-1, and B-2-1-1 lead to the same value of the Weinberg angle $\sin^2\theta_W = 3/5$ and to the same branching rule of the representation 248 with respect to $SU_3 \times SU_2 \times U_1$,

$$248\Big|_{E_8} = 19 \times [\mathbf{1},\mathbf{1}](\mathbf{0}) + 8 \times [1,1](1/2) + 8 \times [1,1](-1/2) + [1,3](0)$$
$$+12 \times [1,2](0) + 4 \times [\mathbf{1},\mathbf{2}](\mathbf{1/2}) + 4 \times [\mathbf{1},\mathbf{2}](\mathbf{-1/2}) + 8 \times [3,1](1/6)$$
$$+8 \times [\bar{3},1](-1/6) + 6 \times [\mathbf{3},\mathbf{1}](\mathbf{-1/3}) + 6 \times [\bar{3},1](1/3) + [\mathbf{3},\mathbf{1}](\mathbf{2/3})$$
$$+[\bar{3},1](-2/3) + 2 \times [3,2](-1/3) + 2 \times [\bar{3},2](1/3) + 4 \times [3,2](1/6)$$
$$+4 \times [\bar{\mathbf{3}},\mathbf{2}](\mathbf{-1/6}) + [8,1](0)\Big|_{SU_3\times SU_2\times U_1}. \quad (14.93)$$

However, from this equation we see that the representation $[1,1](-1)$ needed for the superfields corresponding to the right charged leptons is absent in this case. Therefore, these options are not acceptable for phenomenology.
The options B-1-2-2 and B-2-1-2 also lead to the same value of the Weinberg angle $\sin^2\theta_W = 3/4$ and to the same branching rule for the representation 248 with respect to $SU_3 \times SU_2 \times U_1$, which is written as

$$248\Big|_{E_8} = 17 \times [\mathbf{1},\mathbf{1}](\mathbf{0}) + 9 \times [1,1](1/3) + 9 \times [1,1](-1/3) + [1,3](0)$$
$$+[\mathbf{1},\mathbf{2}](\mathbf{1/2}) + [\mathbf{1},\mathbf{2}](\mathbf{-1/2}) + 9 \times [1,2](1/6) + 9 \times [1,2](-1/6)$$
$$+9 \times [3,1](0) + 9 \times [\bar{3},1](0) + 3 \times [3,1](1/3) + 3 \times [\bar{3},1](-1/3)$$
$$+3 \times [\mathbf{3},\mathbf{1}](\mathbf{-1/3}) + 3 \times [\bar{3},1](1/3) + 3 \times [3,2](-1/6) + 3 \times [\bar{3},2](1/6)$$
$$+3 \times [3,2](1/6) + 3 \times [\bar{\mathbf{3}},\mathbf{2}](\mathbf{-1/6}) + [8,1](0)\Big|_{SU_3\times SU_2\times U_1}. \quad (14.94)$$

In this case there are no representations $[1,1](-1)$ needed for the superfields corresponding to the right charged leptons and no representations $[3,1](2/3)$ corresponding to the right upper quarks. Therefore, all these options are also not acceptable for phenomenology.

14.6 Conclusion

Using the group theory we analyzed a possibility of the symmetry breaking pattern

$$E_8 \to E_7 \times U_1 \to E_6 \times U_1 \to SO_{10} \times U_1 \to SU_5 \times U_1 \to SU_3 \times SU_2 \times U_1 \quad (14.95)$$

provided that only parts of the representation 248 can acquire vacuum expectation values. Also we assume that all U_1 groups in the considered chain are different.

We have found that in this case there are 6 different options for the symmetry breaking, and the only one of them leads to the correct value of the Weinberg angle and produces all representations needed for the MSSM chiral matter superfields. It is interesting that this option corresponds to the minimal absolute values of all U_1 charges of the fields responsible for the symmetry breaking.
The representation 248 of the group E_8 is fundamental and adjoint simultaneously, and, evidently, more than one 248 representations are needed for realizing the symmetry breaking considered in this paper. Therefore, there is an interesting possibility [31] to use for the Grand Unification a finite theory obtained from $\mathcal{N} = 4$ SYM with the group E_8 by adding some terms which break extended supersymmetry and do not break the finiteness (proved in [35–39]). The existence of such terms was demonstrated in [40–42]. However, here we did not study dynamics of the considered symmetry breaking pattern and made the investigation only using the group theory methods. This can be the prospect of future research. Making it, one should also take into account some other similar symmetry breaking patterns like

$$E_8 \to E_7 \times U_1 \to E_6 \times U_1 \to SO_{10} \to SU_5 \times U_1 \to SU_3 \times SU_2 \times U_1, \quad (14.96)$$

etc. Although at present it is not clear if one can construct a phenomenologically acceptable theory based on the E_8 group, this possibility is worth considering.

References

1. R.N. Mohapatra: *Unification and Supersymmetry. The Frontiers of Quark - Lepton Physics: The Frontiers of Quark-Lepton Physics*, Springer, 2002.
2. J.A. Minahan, P. Ramond, R.C. Warner: A Comment on Anomaly Cancellation in the Standard Model, Phys. Rev. D **41** (1990), 715–716.
3. A. Bilal: Lectures on Anomalies: arXiv:0802.0634 [hep-th].
4. H. Georgi, S.L.Glashow: Unity of All Elementary Particle Forces, Phys. Rev. Lett. **32** (1974), 438.
5. S. Dimopoulos, H.Georgi: Softly Broken Supersymmetry and SU(5), Nucl. Phys. B **193** (1981), 150–162.
6. N. Sakai: Naturalness in Supersymmetric Guts, Z. Phys. C **11** (1981), 153–157.
7. J.R. Ellis, S. Kelley, D.V. Nanopoulos: Probing the desert using gauge coupling unification, Phys. Lett. B **260** (1991), 131–137.
8. U. Amaldi, W.de Boer, H. Furstenau: Comparison of grand unified theories with electroweak and strong coupling constants measured at LEP, Phys. Lett. B **260** (1991), 447-455.
9. P. Langacker, M.X. Luo: Implications of precision electroweak experiments for M_t, ρ_0, $\sin^2\theta_W$ and grand unification, Phys. Rev. D **44** (1991), 817–822.
10. H. Fritzsch, P.Minkowski: Unified Interactions of Leptons and Hadrons, Annals Phys. **93** (1975), 193-266.
11. H. Georgi: The State of the Art—Gauge Theories, AIP Conf. Proc. **23** (1975), 575–582.
12. R.L. Workman *et al.* [Particle Data Group]: Review of Particle Physics, PTEP **2022** (2022), 083C01.
13. S.M. Barr: A New Symmetry Breaking Pattern for SO(10) and Proton Decay, Phys. Lett. B **112** (1982), 219–222.

14. I. Antoniadis, J.R. Ellis, J.S. Hagelin, D.V. Nanopoulos: Supersymmetric Flipped SU(5) Revitalized, Phys. Lett. B **194** (1987), 231–235.
15. B.A. Campbell, J.R. Ellis, J.S. Hagelin, D.V. Nanopoulos: K.A.Olive, Supercosmology revitalized, Phys. Lett. B **197** (1987), 355–362.
16. J.R. Ellis, J.S. Hagelin, S. Kelley, D.V. Nanopoulos: Aspects of the Flipped Unification of Strong, Weak and Electromagnetic Interactions, Nucl. Phys. B **311** (1988), 1–34.
17. A. Masiero, D.V. Nanopoulos, K. Tamvakis, T.Yanagida: Naturally Massless Higgs Doublets in Supersymmetric SU(5), Phys. Lett. B **115** (1982), 380–384.
18. B. Grinstein: A Supersymmetric SU(5) Gauge Theory with No Gauge Hierarchy Problem, Nucl. Phys. B **206** (1982), 387–396.
19. J. Hisano, T. Moroi, K. Tobe, T.Yanagida: Suppression of proton decay in the missing partner model for supersymmetric SU(5) GUT, Phys. Lett. B **342** (1995), 138–144.
20. J. Ellis, M.A.G. Garcia, N. Nagata, D.V. Nanopoulos, K.A. Olive: Proton Decay: Flipped vs Unflipped SU(5), JHEP **05** (2020), 021.
21. M. Mehmood, M.U. Rehman, Q.Shafi: Observable proton decay in flipped SU(5), JHEP **02** (2021), 181.
22. N. Haba, T. Yamada: Moderately suppressed dimension-five proton decay in a flipped SU(5) model, JHEP **01** (2022), 061.
23. J. Ellis, J.L. Evans, N.Nagata, D.V.Nanopoulos, K.A. Olive: Flipped SU(5) GUT phenomenology: proton decay and $g_\mu - 2$, Eur. Phys. J. C **81** (2021) 1109.
24. F. Gursey, P. Ramond, P.Sikivie: A Universal Gauge Theory Model Based on E6, Phys. Lett. B **60** (1976), 177–180.
25. S.F. King, S. Moretti, R.Nevzorov: A Review of the Exceptional Supersymmetric Standard Model, Symmetry **12** (2020) no.4, 557.
26. S.E. Konshtein, E.S. Fradkin: Asymptotically supersymmetric model of unified interaction based on E8 (in Russian), Pisma Zh. Eksp. Teor. Fiz. **32** (1980), 575–578.
27. N.S. Baaklini: Supergrand unification in E8, Phys. Lett. B **91** (1980), 376–378.
28. N.S. Baaklini: Supersymmetric exceptional gauge unification, Phys. Rev. D **22** (1980), 3118–3127.
29. I.Bars, M.Gunaydin: Grand Unification With the Exceptional Group E8, Phys. Rev. Lett. **45** (1980), 859–862.
30. M. Koca: On tumbling E8, Phys. Lett. B **107** (1981), 73–76.
31. S. Thomas: Softly broken N=4 and E8, J. Phys. A **19** (1986), 1141–1149.
32. R. Slansky: Group Theory for Unified Model Building, Phys. Rept. **79** (1981), 1–128.
33. K. Stepanyantz: The gauge coupling unification in the flipped E_8 GUT, arXiv:2305.01295 [hep-ph].
34. M.B. Green, J.H. Schwarz, E. Witten: *Superstring Theory. Vol. 1: Introduction*, Cambridge University Press, Cambridge, 1988.
35. M.F. Sohnius, P.C. West: Conformal Invariance in N=4 Supersymmetric Yang-Mills Theory, Phys. Lett. B **100** (1981), 245.
36. M.T. Grisaru, W.Siegel: Supergraphity. 2. Manifestly Covariant Rules and Higher Loop Finiteness, Nucl. Phys. B **201** (1982), 292 [erratum: Nucl. Phys. B **206** (1982), 496].
37. P.S. Howe, K.S. Stelle, P.K. Townsend: Miraculous Ultraviolet Cancellations in Supersymmetry Made Manifest, Nucl. Phys. B **236** (1984), 125–166.
38. S. Mandelstam: Light Cone Superspace and the Ultraviolet Finiteness of the N=4 Model, Nucl. Phys. B **213** (1983), 149–168.
39. L. Brink, O. Lindgren, B.E.W. Nilsson: N=4 Yang-Mills Theory on the Light Cone, Nucl. Phys. B **212** (1983), 401-412.
40. A.J. Parkes, P.C. West: $N = 1$ Supersymmetric Mass Terms in the $N = 4$ Supersymmetric Yang-Mills Theory, Phys. Lett. B **122** (1983), 365–367.

41. A. Parkes, P.C. West: Finiteness and Explicit Supersymmetry Breaking of the $N = 4$ Supersymmetric Yang-Mills Theory, Nucl. Phys. B **222** (1983), 269–284.
42. A. Parkes, P.C. West: Explicit Supersymmetry Breaking Can Preserve Finiteness in Rigid $N = 2$ Supersymmetric Theories, Phys. Lett. B **127** (1983), 353–359.

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15 Clean energy from the dark Universe?

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Abstract. Dark matter (DM) came 1933 from unexpected cosmological observations by F. Zwicky. Interestingly, within our solar system, still diverse observations defy conventional explanation. For example, questions arise on the main physical process(es) underlying, e.g., the heating of the different solar atmospheric layers, the triggering of solar flares, the quasi ubiquitous11 years solar cycle, but also the annual heating-up of the stratosphere in January and the mysterious behaviour of our ionosphere (1937-) in December. Streaming DM with its widely assumed velocities of 0.001 c (c = speed of light) offers a viable common scenario following gravitational focusing by the solar system bodies. This fits-in the underlying process behind the solar cycle, which was the first suspected signature of planetary dependency. Since 1859 the challenge is to find an explanation for the underlying physics-mechanism of the suspected remote planetary impact. However, the only known remote planetary force is the extremely feeble tidal force. Therefore, we stress the possible involvement of an external impact by some overlooked "streaming invisible matter", which reconciles all local mysterious observations mimicking a not extant remote planetary force. Unexpected planetary relationships exist for both the dynamic Sun and Earth, reflecting multiple signatures for streaming DM or DM clusters. Thus, applying the reasoning à la Zwicky also locally, it is suggestive for a plethora of observations including puzzling biomedical phenomena. Favorite DM candidates are anti-quark-nuggets, magnetic monopoles, dark photons, or other constituents like the composite "pearls" and other suggestions. Then, anomalies within the solar system can be the manifestation of the dark Universe. The tentative streaming DM scenario enhances spatiotemporally the DM flux favoring conditions for direct DM detection. This can be decisive in extracting also clean energy from the not-so-invisible as anticipated dark sector.

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Povzetek: Da mora obstojati v vesolju temna snov je iz kosmoloških opazovanj prvi naznanil F. Zwicky že 1933 leta. Kako pa se obnaša temna snov v našem osončju in kaj je, na primer, vzrok segrevanju različnih plasti sonč ne atmosfere, sprožanju sončnih izbruhov, navidezno vseprisotnemu 11-letnemu sončnemu ciklu, pa tudi letnemu segrevanju stratosfere v januarju in skrivnostnemu obnašanju naše ionosfere decembra, ostaja odprto vprašanje. Gibanje temne snovi s hitrostmi 0,001 c ponuja možnost za skupno obravnavo vseh teles sonňega sistema.
Od leta 1859 poskušajo fiziki najti razlago za medsebojni vpliv med oddaljenimi člani osončja. Zdi se, da bi meritve lahko pojasnili s prisotnostjo temne snovi. Najbolj popularni kandidati za temno snov so gruče anti-kvarkov, magnetni monopoli, temni fotoni ali drugi predlogi. Temna snov pa morda ponudi možnost za pridobivanje energije brez polucije.

Keywords: planetary relationship, dark sector, invisible matter, gravitational focusing

15.1 INTRODUCTION

The discovery of "Dunkle Materie" (DM) by Fritz Zwicky in 1933 came from unexpected cosmological observations. Today we know that our Universe should be dominated by the mysterious DM, whose composition remains elusive. In between, its name became synonymous with something that does not emit, absorb, or reflect electromagnetic radiation, thus making it difficult to detect. However, following the reasoning of this work, this definition can be misleading, as several counter-examples might be caused by streaming DM, while, at first sight, contradicting the initial definition of DM. Our working hypotheses involve planetary (and solar) gravitational effects on non-relativistic "invisible massive particles" which can then be focused on solar and planetary atmospheres (see Fig. 1). These particles might also interact "strongly", i.e., they can have a large cross-section, with normal matter and radiation. Such DM constituents interact already in the outer atmosphere. However, the screening by the upper atmospheric region can be significant, thus strongly suppressing possible signals in DM searches.
With time, a planetary alignment with an incident invisible stream will repeat provided the stream lasts much longer than the corresponding planetary periodicity, which is a reasonable assumption for our solar system. Often, an observed periodicity of some solar system observable reflects either a single planetary orbital periodicity or a synod of two or more planets, resulting in a repeating signal enhancement. For example, the 11-year solar cycle coincides with the well-established synod of Jupiter-Earth-Venus. This is probably not a random coincidence, and it was suggestive for the streaming DM scenario as it was proposed in [1] and underpinned by several follow-up signatures of solar and terrestrial observations along with a long series of medical data on diagnosed melanomata (a type of skin cancer) [2–7]. A planetary correlation of any observable is then the novel signature from the dark sector, even though there is not a remote planetary force beyond the extremely feeble and smooth over an orbital period tidal force. Fortunately, for the streaming DM scenario, the gravitational deflection of an invisible stream depends inversely proportional to its velocity squared [8]. This favors enormously

non-relativistic speeds like the ones widely assumed for the constituents of the dark Universe ($v \sim 0.001c$, with c being the speed of light).
Additionally, this scenario makes any exo-solar planetary system of potential interest, since they also consist of orbiting gravitational lenses being probably appropriate to gravitationally focus constituents from the dark sector. As an example, even the Moon can focus DM particles on the Earth's location with velocities up to about 400 km/s thus covering a large fraction of the DM phase space [3,9,10]. Notice, throughout this work we often refer to "invisible matter", to distinguish it from the celebrated DM candidates like axions and WIMPs.
The planetary gravitational lensing effects within the solar system become enormous only if invisible matter consists, at least partly, in the form of streams or clusters. Interestingly, recent cosmological considerations [11] arrived at a very large number of "fine-grained" DM streams in our Galaxy (up to $\sim 10^{14}$). Thus, to explain unusual or anomalous observations in our vicinity, we also converge on the existence of streaming "invisible matter" (see e.g. [1,3]). An invisible streaming scenario is suggested also by independent cosmological considerations [11,12], which are founded on a different reasoning. A posteriori we conclude that both findings, namely the anomalous observations within the solar system, or the cosmological "fine-grained" DM axion streams, while they are based on a different logic, they both converge towards streaming DM, or invisible matter which eventually can also include other, theoretically not yet conceived, candidates from the dark sector.

15.2 SOME STRIKING SIGNATURES FROM SPACE

The underlying idea behind this work goes along with the reasoning used by Zwicky that has led to the discovery of DM on cosmological scales. Namely, in the last ~ 160 years, with the quasi-ubiquitous 11-year solar cycle, several unexpected energetic observations within the solar system defy conventional explanation (see e.g. [3]). This could be due to the dark Universe [1], whose manifestation at small scales has been overlooked. Driven by observations, we converge on a class of "invisible" particle candidates from the dark sector, which could interact with a large cross-section with normal matter and are different than the parameter phase space for axions and WIMPs following failed direct searches for a few decades. We conclude therefore that although axions or WIMPs do not fit in, they remain inspiring.
The striking mysterious heating of the solar corona with its unnatural step-like temperature inversion, the unpredictable solar activity, the dynamic Earth's atmosphere, and other observations might all arise from otherwise invisible streams giving rise to spatiotemporally strong flux enhancements due to gravitational lensing effects, within the solar system, by one or more solar system bodies, including the Sun [3]. The underlying dark constituents can be of a diverse nature, being eventually theoretically not yet introduced, which makes their identification even more challenging (see below).
Examples of unpredictable energetic solar observations include: the flaring Sun, its irradiance, and more generally its dynamical behavior [3,13] as it is manifested

by the widely accepted proxy of the solar radio line (F10.7) at 10.7 cm wavelength. The most energetic planetary relationship is observed for the Sun's size variation during one solar cycle [13] by about 1 km, with the relative size variation being at the level of $\sim 10^{-6}$. Of note, the required energy to lift a 1 km thin layer of the photosphere ($\rho \approx 10^{-7}$ gram/cm^3) by 1 km, is enormous ($\sim 10^{30}$ ergs). In addition, a remarkable planetary dependence of the Sun's elemental composition is observed [3,14], which makes it more of a riddle within known physics; this resembles the planetary relationship observed for solar EUV irradiation above ~ 20 eV photon energy, which is another manifestation of the otherwise still mysterious solar corona heating [15]. Similarly, the elemental magnetic bright points on the solar surface also show planetary relationships [3]. WOLF (1859) who observed sunspots (see [3] and ref. 7 therein) was the first to suspect a planetary cause behind the workings of the Sun [1], though the underlying process remained elusive within known physics.

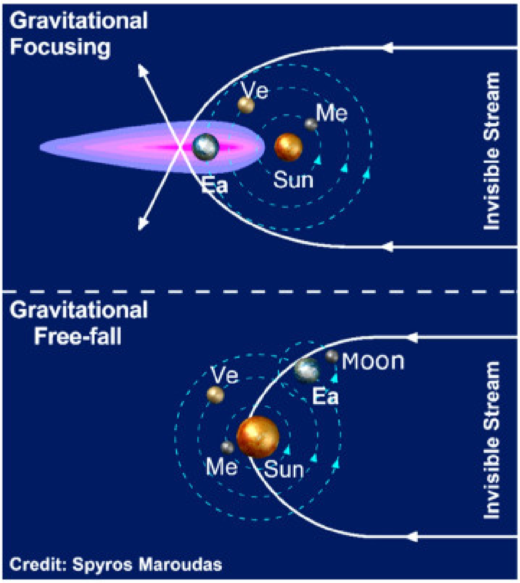

Fig. 1: Schematic representation of the gravitational focusing effects on a DM stream by the solar system. **Top**: Gravitational focusing by the inner solar system towards the Earth location. **Bottom**: the free-fall effect of incident low-speed streams may occasionally dominate planetary gravitational focusing towards the Sun since the flux enhancement increases with $(v_{incident}/v_{escape})^{-2}$, with v_{escape} being the escape velocity from the Sun and $v_{incident}$ the initial particle velocity far away from the Sun. The flux towards the Earth can also be gravitationally modulated by intervening planets. The Moon focuses particles towards the Earth with an incident velocity near the Moon up to ~400 km/s [3, 8–10].

In addition to the aforementioned unexpected planetary relationships of various solar observables, several nearby terrestrial anomalous phenomena occur in our upper atmosphere, which have been known since the 1930s. For example, the mechanism behind the ionosphere's dynamical behavior that shows also unexpected planetary relationship [2] remains elusive. More specifically, there is an annual anomaly, known since 1937 [16], showing about 25% more atmospheric ionization around December than six months apart around June. We recall two extraordinary facts about the ionosphere:

(A) The ionosphere is the most outer terrestrial region that is directly exposed to outer space. Then, any "invisible" constituents from the dark Universe may appear first up there, if they interact with normal matter with a large cross-section (see e.g. ref. [17]). Interestingly, this is still possible for DM constituents following recent publications [17]. Therefore, this requirement is not used here just to support the assumed scenario of this work. In contrast, we recall that the deep underground direct DM searches address extremely feebly interacting DM particles due to the screening of "strongly" interacting dark constituents by the overhead Earth's layers including its atmosphere.

(B) Some cross-disciplinary observations of societal relevance, while the ionosphere is occasionally also involved:

1) The not randomly appearing Earthquakes [14,18], probably happen by accumulating energy deposition inside the Earth, triggering finally an Earthquake, occurring somehow like the aforementioned solar radius variation over relatively long-time intervals. Apparently, it is not necessary for the invisible stream(s) or clusters (see e.g., [19,20]) to provide spatiotemporally confined the entire energy released during an Earthquake. However, it can be the external trigger for an Earthquake to occur. Remarkably, during the largest Earthquakes, the ionosphere's plasma state changes over long distances as has been observed by the orbiting GPS satellites that continuously register the ionospheric plasma for self-calibration purposes [18].

2) Melanoma diagnoses [4–6] show planetary relationships following Mercury's 88-day orbital period. The observed periodic modulation of daily diagnosed melanoma cases strikingly coincides with the lunar geocentric sidereal periodicity of 27.32 days [6]; both periodicities point at a cause of exo-solar origin, which fit-in the tentatively predicted streaming invisible matter scenario [1]. Of note, a plethora of observations have a common feature. Namely, they all show an otherwise unexpected planetary relationship, while there is not some remote planetary force to cause any of them. With time, more and more results emerge following this out-of-the-box approach. This might allow us to finally corner the microscopic nature of the suspected stream(s), being not as "invisible" as they were widely thought to be [3]. In addition, they may point at new windows of opportunity to tap converted energy from the dark sector.

Moreover, following the reasoning underlying this work, it is interesting to find out whether similar behavior is encountered in exo-solar planetary systems [21]. With near-Earth exo-planetary systems, one might be able to establish similar 'exo-planetary' relationships, or even also a cross-correlation with the dynamical behavior of our solar system. Such observations have the potential to expand our horizon within our Galaxy as well as towards the dark Universe, validating the actual working hypotheses behind the present suggestion. In this work, we pinpoint a simple feature as the common signature behind such observations within the solar system. The widely discussed dark sector constituents with a velocity around ~300 km/s, while being in the form of streams, can be efficiently gravitationally focused or deflected within the solar system [1,2,12,19,20] thus causing at the focal region an unexpected planetary relationship.

15.3 DARK MATTER AS CLEAN ENERGY SOURCE

The observations made with a long series of data have established also socially relevant results [4–6]. Recently P. Sikivie proposed DM axions as a potential source of clean energy [22]. However, using the present constraints on axion interaction strength with normal matter as determined by the CAST experiment [23], the expected profit is quite small. On the other hand, following the diverse peaking planetary relationships for several observables [3,13,14], this situation can change, since DM dominates over normal matter in the Universe.
Noticeably, we consider here generic not yet identified DM constituents as being eventually more appropriate than axions for a clean energy source. Among the already multiply established planetary dependencies with solar and terrestrial observables [3], some might lead us to spatiotemporally optimum conditions allowing us to extract efficiently considerable clean energy. Earth-related time windows of opportunity might be fixed annually, while other planetary peaking relationships can be spread during a year and be thus more profitable.
We recall that occasional streaming DM flux enhancements due to gravity effects by the solar system bodies can be several orders of magnitude, i.e., flux amplification factors of up to about 10^5 to 10^8 seem realistic [8,11,24] for reaching a significant converted energy density. For example, in January in the northern hemisphere, an annually peaking stratospheric temperature has been observed live [25,26]. Also, probably more planetary relationships may be discovered, which can be of practical use for energy exploitation.
Interestingly, NASA has developed scientific balloons (see [27–30]) that can stay for months in the upper stratosphere with a payload of up to a few tons. This is encouraging also for the present reasoning since planetary relationships have already been observed for the upper stratosphere's temperature [7] and the ionosphere's degree of ionization [2]. For the stratosphere, a strong peaking planetary relationship has been observed using the orbital positions of Mercury and Venus. Combining Venus and Mercury's orbital positions, a clear peaking relationship for stratosphere's temperature variation in early January in the northern hemisphere [7], might pave the way to:
a) perform DM searches in the upper atmosphere [7,25] contrary but complementary to the widely preferred underground searches, and
b) investigations proposed here aim to establish the optimum conditions to extract energy from occasionally much more invisible matter in the Earth's upper atmosphere.

The possible use of the upper stratosphere in January as a possible converter of DM to energy is just one example. Future atmospheric investigations could provide additional places and time windows of potential interest, to search directly for DM, but also to extract energy from the dark Universe. Thus, planetary lensing or Earth's gravitational self-focusing by the inner Earth mass distribution have the potential to enhance temporally the local DM flux by up to several orders of magnitude (up to $\sim 10^8$x) thus providing new perspectives for DM detection and possibly also an alternative clean energy source.

15.4 SUMMARY

Observationally driven, a planetary relationship can be a key signature pointing on its own at exo-solar impact for a certain observable. So far, the only viable common explanation we have for several otherwise unexpectedly observed behavior due to gravitational focusing of streaming "invisible" matter from one solar system body to another, including the Sun and the Moon. We tentatively identify the assumed streams with constituents from the dark Universe, interacting eventually also with a large cross-section with ordinary matter. However, with the data at hand, we can only speculate about the possible particle candidates.

Implications in ongoing or future DM experiments are obvious. Therefore, we propose that all experiments searching for direct DM signatures, should perform a statistical re-analysis following the reasoning underlying this approach or modify their data acquisition procedure accordingly for future measurements (see ref. [3]]). If a planetary dependency is found also in direct DM searches, this will strengthen the concept of "invisible streams" within our solar system.

With this work, we are aiming to highlight the appearance of such new signatures which are probably still hidden also to other observations. One day one might decipher the properties of the invisible stream(s). Along these lines of reasoning, medical observations made with long data series of cancer diagnoses (=melanoma) have emerged [4–6]. Surprisingly, the main two planetary signatures appeared so far in medicine are:

1) The 88-day orbital periodicity of Mercury using mean monthly data from the northern hemisphere (USA) [4], which has also been independently confirmed [5]. However, the author did not give the appropriate attention to his analysis, which confirmed our previous results, and even for most cancer types, and

2) The sidereal geocentric lunar periodicity (=27.32 days) using daily melanoma diagnoses data from the southern hemisphere (Australia) [6]. Interestingly, following the planetary scenario and the possible signatures that already have been observed [1–3,13,14,18], the underlying stream(s) can only be exo-solar in origin if the periodicity is sidereal since it refers to a reference frame fixed to remote stars. This is of no minor importance for direct DM searches, or for indirect ones following astrophysical/cosmic observations. In short, a wide diversity of signatures showing planetary relationships may allow us to identify the otherwise "invisible" components of the dark Universe. Finally, some favored "invisible candidates" following the observations made so far, are:

a) Anti-Quark Nuggets (AQNs) which have been invented in 2003 by Ariel Zhitnitsky [31] (see also [32-35]). These peculiar objects are inspiring for this work being based on many investigations spanning from the origin of the solar corona heating mystery to the direct detection of fast axions.
b) Magnetic monopoles whose interaction with the ubiquitous magnetic fields make different energy deposition scenarios of potential interest.
c) Dark photons, which can even resonantly convert to real photons if the local plasma density fits in the rest mass of the hidden photon. Contrary to axions or axion-like particles, the kinetic mixing between real photons with hidden sector photons does not require a magnetic field as a catalyst, and this makes them attractive.
d) Pearls [36,37]. A quantitative investigation as it has been undertaken for the AQNs would clarify whether such composite particles also fit in, at least some of the observations made so far, starting with the mysterious solar corona heating, the unpredictable solar flares, and the entire dynamic and mysterious Sun.
e) Some composites of a new quark family with long lifetimes [38].
f) Other invisible constituents, yet to be invented, remain always an option.

15.5 CONCLUSION

The expected DM flux amplification by several orders of magnitude due to gravitational focusing effects by the solar system bodies including the self-focusing effects by the inner Earth [9,10,24] might bring a long-awaited breakthrough not only for the direct DM detection but also for the possible explanation of long-standing solar and terrestrial anomalies. Additionally, the interaction strength of "invisible streaming matter" with normal matter could also be large [17] opening the way for a substantial and clean energy source. The most inspiring particle constituents fitting in several observations are AQNs, magnetic monopoles, and dark photons. However, more emerging candidates, like "pearls", quark composites, etc. should be investigated whether they fit the scenario of this work. Thus, insisting anomalies/mysteries within the solar system might be the unnoticed manifestation of the dark Universe, and they deserve further attention aiming to identify their elemental composition and properties. This can help to tap more efficiently to a source of clean energy from the dark Universe.

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REFERENCES

1. Zioutas, K., Tsagri, M., Semertzidis, Y.K., Papaevangelou, T., Hoffmann, D.H.H., and Anastassopoulos, V., Mod. Phys. Lett. A, 2014, vol. 29, p. 1440008.
2. Bertolucci, S., Zioutas, K., Hofmann, S., and Maroudas, M., Phys. Dark Univ., 2017, vol. 17, p. 13.
3. Zioutas, K., Anastassopoulos, V., Argiriou, A., Cantatore, G., Cetin, S.A., Gardikiotis, A., Hoffmann, D.H.H., Hofmann, S., Karuza, M., Kryemadhi, A., Maroudas,

M., et al., Phys. Sci. Forum, 2021, vol. 2, p. 10.
4. Zioutas, K. and Valachovic, E., Biophys. Rev. Lett., 2020, vol. 13, p. 75.
5. Zioutas, K., Valachovic, E., Maroudas, M., Biophys. Rev. Lett., 2019, vol. 14, p. 11.
6. Zioutas, K., Maroudas, M., Hofmann, S., Kryemadhi, A., and Matteson, E., Biophys. Rev. Lett., 2020, vol. 15, p. 275.
7. Zioutas, K., Argiriou, A., Fischer, H., Hofmann, S., Maroudas, M., Pappa, A., and Semertzidis, Y.K., Phys. Dark Univ., 2020, vol. 28, p. 100497.
8. Hoffmann, D.H.H., Jacoby, J., and Zioutas, K., Astropart. Phys., 2003, vol. 20, p. 73.
9. Sofue, Y., Galaxies, 2020, vol. 8, issue 2, p. 42.
10. Prezeau, G., ApJ, 2015, vol. 814, 122.
11. Kryemadhi, A., Maroudas, M., Mastronikolis, A., and Zioutas, K., arXiv:2210.07367 [astro-ph.IM], 2022.
12. Vogelsberger, M. and White, S.D.M., Mon. Not. R. Astron. Soc., 2011, vol. 413, p. 1419.
13. Zioutas, K., Maroudas, M., and Kosovichev, A., Symmetry, 2022, vol. 14, p. 325.
14. Maroudas, M., Ph.D. thesis, Signals for invisible matter from solar-terrestrial observations, University of Patras / Greece, (2022).
15. Murawski, K., Musielak, Z.E., Poedts, S., Srivastava, A.K., and Kadowaki, L., Accepted Astrophys. Space Sci., 2022, https://doi.org/10.48550/arXiv.2211.12289.
16. Appleton, E.V., Proc. Roy. Soc. London A, 1937, vol. 162, p. 451.
17. Emken, T., Essig, R., Kouvaris, C., and Sholapurkar, M., J. Cosmol. Astropart. Phys., 2019, vol. 9, p. 70.
18. Zioutas, K., Anastassopoulos, V., Argiriou, A., Cantatore, G., Cetin, S., Fischer, H., Gardikiotis, A., Haralambous, H., Hoffmann, D.H.H., Hofmann, S., Karuza, M., Kryemadhi, A., Maroudas, M., Mastronikolis, A., Oikonomou, C., Ozbozduman, K., and Semertzidis, Y.K., Proceedings 15th Intern. Conf. on Meteorology, Climatology and Atm. Phys. COMECAP, 2021, Ioannina, Greece, p. 728, https://www.conferre.tv/comecap2020.
19. Tkachev, I.I., Phys. Lett., 1991, vol. 261, p. 289.
20. Kolb, E.W. and Tkachev, I.I., Phys. Rev. Lett., 1993, vol. 71, p. 3051.
21. Perryman, M., and Zioutas, K., arXiv:2106.15408 [astro-ph.GA], 2021.
22. Sikivie, P., arXiv:2107.14300 [hep-ph], 2021.
23. CAST Collaboration, New CAST limit on the axion–photon interaction, Nature Phys., 2017, vol. 13, p. 584, and Search for DM Axions with CAST-CAPP, Nature Commun 13 (2022) 6180; https://doi.org/10.1038/s41467-022-33913-6.
24. Arza, A., Kryemadhi, A., and Zioutas, K., Phys. Rev. D 108 (2023) 8, 083001, arXiv:2212.10905 [hep-ph].
25. Cantatore, G., Fischer, H., Funk, W., Karuza, M., Kryemadhi, A., Maroudas, M., Semertzidis, Y.K., and Zioutas, K., arXiv:2012.03353 [hep-ex], 2022.
26. Fischer, H., Liang, X., Zhitnitsky, A., Semertzidis, Y.K., and Zioutas, K., Phys. Rev. D, 2018, vol. 98, 043013.
27. NASA, scientific Balloons, Types, 2022, https://www.nasa.gov/scientific-balloons/types-of-balloons.
28. NASA, Scientific Balloons, overview, 2021, https://www.nasa.gov/scientific-

balloons/overview.
29. Gorham, P., Anderson, J., Bernasconi, P., Chakrabarti, S., Guzik, T.G., Jones, W., Kierans, C., Millan, R., Vieregg, A., Walker, C., and Young, E., A Roadmap For Scientific Ballooning 2020-2030, 2022.
30. Columbia Scientific Balloon Facility Homepage; http://www.csbf.nasa.gov/.
31. Zhitnitsky, A., JCAP, 2003, vol. 2003, no. 10, p. 10.
32. Zhitnitsky, A., Phys. Dark Univ., 2018, vol. 22, p. 1.
33. Raza, N., van Waerbeke, L., and Zhitnitsky, A., Phys. Rev. D, 2018, vol. 98, p. 10352
34. Budker, D., Flammbaum, V.V., and Zhitnitsky, A., Symmetry, 2022, vol. 14, p. 459.
35. Zhitnitsky, A.R., Mod. Phys. Lett. A, 2021, vol. 36, p. 2130017.
36. Nielsen, H.B. and Froggatt, C.D., arXiv:2205.08871 [astro-ph.HE], 2022.
37. Nielsen, H.B. and Froggatt. C.D., arXiv:2012.00445 [hep-ph],2020.
38. Borštnik, B.G., Mankoč, N.S., Does dark matter consist of baryons of new stable family quarks?, Phys. Rev. D 80 (2009) 083534.1; https://doi.org/10.1103, PhysRevD.80.083534

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16 Abstracts of talks presented at the Workshop and in the Cosmovia forum

http://bsm.fmf.uni-lj.si/bled2023bsm/presentations.html
https://bit.ly/bled2023bsm

Not all the talks come as articles in this year's Proceedings, but all the talks can be found on the official website of the Workshop and on the Cosmovia forum:
https://bit.ly/bled2023bsm.
Here are the abstracts of the contributors who did not submit an article.

16.1 Albino Hernández-Galeana

Quark masses and mixing from a SU(3) gauge family symmetry

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Abstract: Within the gauged SU(3) family symmetry model I have been working on the scales of the SSB of this family symmetry. The corresponding SU(3) gauge boson masses have to be enough heavy, in the range of a few 1000 TeV's or heavier, in order to account for tree level exchange FCNC's contributions in the u and d quarks and charged leptons.
At present I am exploring the possibility of lowering the gauge boson masses constraints up to a few 100 TeV's. In this direction we need to find out solutions for the quarks and leptons with maximal mixing suppression flavor changing couplings.
I already have separated solutions for u and d quarks and charged leptons with SU(2), subgroup of SU(3), gauge boson masses in the range of 10 - 200 TeV's, with the proper suppression of $Do - \bar{D}o$, $Ko - \bar{K}o$, and $\mu \to eee$ respectively.
A simultaneous global analysis for all quarks and leptons, including sterile neutrinos, is in progress.

Povzetek:
Avtor privzame za opis pojava družin kvarkov in leptonov grupo $SU3$. Da zagotovi, da so prispevki prehodov na drevesnem nivoju med kvarki in leptoni iste vrste v skladu z meritvami, privzame, da so mase bozonov $SU(3)$ nekaj 1000 TeV ali več.

Zdaj raziskuje pogoje, ki dovolijo, da imajo bozoni $SU(3)$ manjšo maso, do nekaj 100 TeV, nevtralni tokovi med kvarki in med leptoni, ki tečejo med družinami, pa bodo š vedno skladni z meritvami.
Našel je pogoje za kvarke, za leptone pa samo v podgrupi $SU(2)$ grupe SU(3).
Dobro pa kaže tudi za družinsko grupo $SU(3)$ brez omejitev, tudi ko vklju/v ci sterilne nevtrine.

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17 Virtual Institute of Astroparticle physics as the online support for studies of BSM physics and cosmology

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Keywords: astroparticle physics, physics beyond the Standard model, e-learning, e-science, MOOC

17.1 Introduction

Studies in astroparticle physics link astrophysics, cosmology, particle and nuclear physics and involve hundreds of scientific groups linked by regional networks (like ASPERA/ApPEC [1,2]) and national centers. The exciting progress in these studies will have impact on the knowledge on the structure of microworld and Universe in their fundamental relationship and on the basic, still unknown, physical laws of Nature (see e.g. [3,4] for review). The progress of precision cosmology and experimental probes of the new physics at the LHC and in nonaccelerator experiments, as well as the extension of various indirect studies of physics beyond the Standard model involve with necessity their nontrivial links. Virtual Institute of Astroparticle Physics (VIA) [5] was organized with the aim to play the role of an unifying and coordinating platform for such studies.
Starting from the January of 2008 the activity of the Institute took place on its website [6] in a form of regular weekly videoconferences with VIA lectures, covering all the theoretical and experimental activities in astroparticle physics and related topics. The library of records of these lectures, talks and their presentations was accomplished by multi-lingual Forum. Since 2008 there were **220 VIA online lectures**, VIA has supported distant presentations of **192 speakers at 32 Conferences** and provided transmission of talks at **78 APC Colloquiums**.
In 2008 VIA complex was effectively used for the first time for participation at distance in XI Bled Workshop [7]. Since then VIA videoconferences became a natural part of Bled Workshops' programs, opening the virtual room of discussions to the world-wide audience. Its progress was presented in [8–21].
Here the current state-of-art of VIA complex is presented in order to clarify the way in which discussion of open questions beyond the standard models of both particle physics and cosmology were supported by the platform of VIA facility at the hybrid XXVI Bled Workshop. Even without pandemia, there appear other

obstacles, preventing many participants to attend offline meetings and in this situation VIA videoconferencing supported in 2023 traditions of open discussions at Bled meetings combining streams of the offline presentations and support of distant talks and involving distant participants in these discussions.

17.2 VIA structure and activity

17.2.1 The problem of VIA site

The structure of the VIA site was initially based on Flash and is virtually ruined now in the lack of Flash support. The original structure is illustrated by the Fig. 17.1. The home page, presented on this figure, contained the information on the coming and records of the latest VIA events. The upper line of menu included links to directories (from left to right): with general information on VIA (About VIA); entrance to VIA virtual rooms (Rooms); the library of records and presentations (Previous), which contained records of VIA Lectures (Previous → Lectures), records of online transmissions of Conferences (Previous → Conferences), APC Colloquiums (Previous → APC Colloquiums), APC Seminars (Previous → APC Seminars) and Events (Previous → Events); Calendar of the past and future VIA events (All events) and VIA Forum (Forum). In the upper right angle there were links to Google search engine (Search in site) and to contact information (Contacts). The announcement of the next VIA lecture and VIA online transmission of APC Colloquium occupied the main part of the homepage with the record of the most recent VIA events below. In the announced time of the event (VIA lecture or transmitted APC Colloquium) it was sufficient to click on "to participate" on the announcement and to Enter as Guest (printing your name) in the corresponding Virtual room. The Calendar showed the program of future VIA lectures and events. The right column on the VIA homepage listed the announcements of the regularly up-dated hot news of Astroparticle physics and related areas.
In the lack of Flash support this system of links is ruined, but fortunately, they continue to operate separately and it makes possible to use VIA Forum, by direct link to it, as well as direct inks to virtual Zoom room for regular Laboratory and Seminar meetings (see Fig 17.2). The necessity to restore all the links within VIA complex is a very important task to revive the full scale of VIA activity.

17.2.2 VIA activity

In 2010 special COSMOVIA tours were undertaken in Switzerland (Geneva), Belgium (Brussels, Liege) and Italy (Turin, Pisa, Bari, Lecce) in order to test stability of VIA online transmissions from different parts of Europe. Positive results of these tests have proved the stability of VIA system and stimulated this practice at XIII Bled Workshop. The records of the videoconferences at the XIII Bled Workshop were put on VIA site [22].
Since 2011 VIA facility was used for the tasks of the Paris Center of Cosmological Physics (PCCP), chaired by G. Smoot, for the public program "The two infinities" conveyed by J.L.Robert and for effective support a participation at distance at

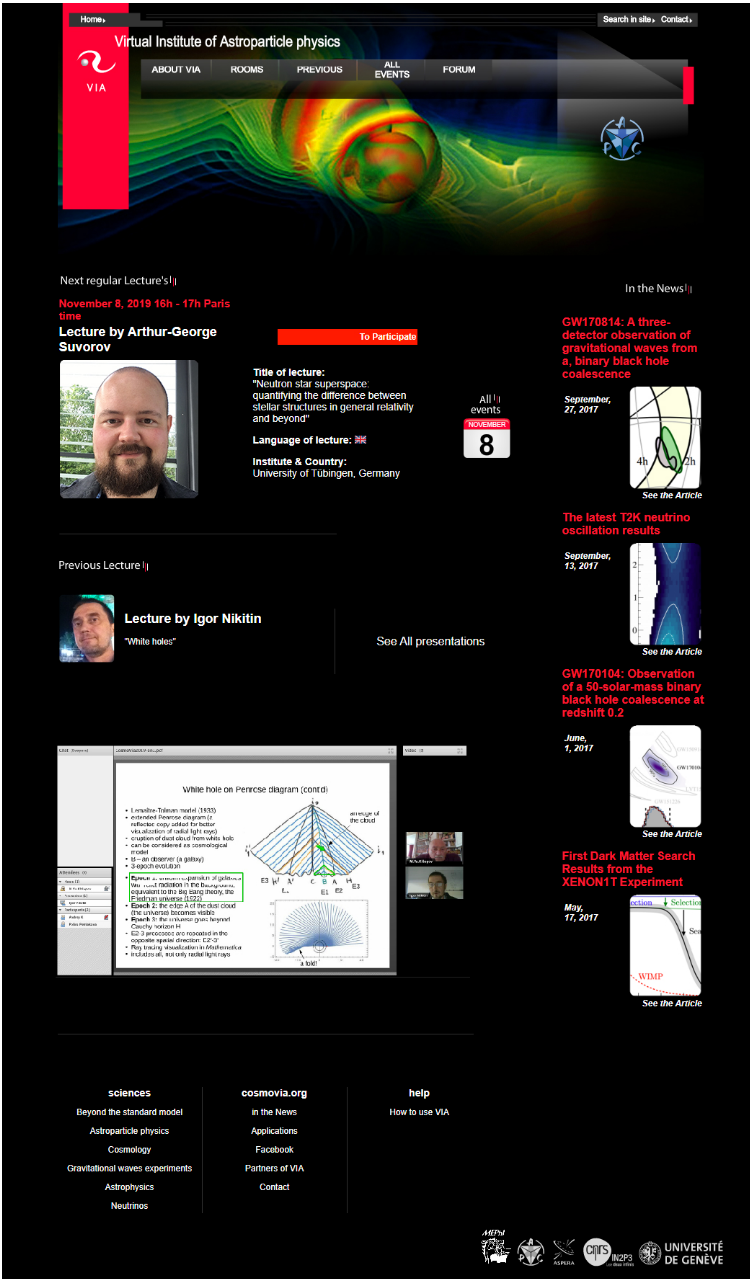

Fig. 17.1: The original home page of VIA site

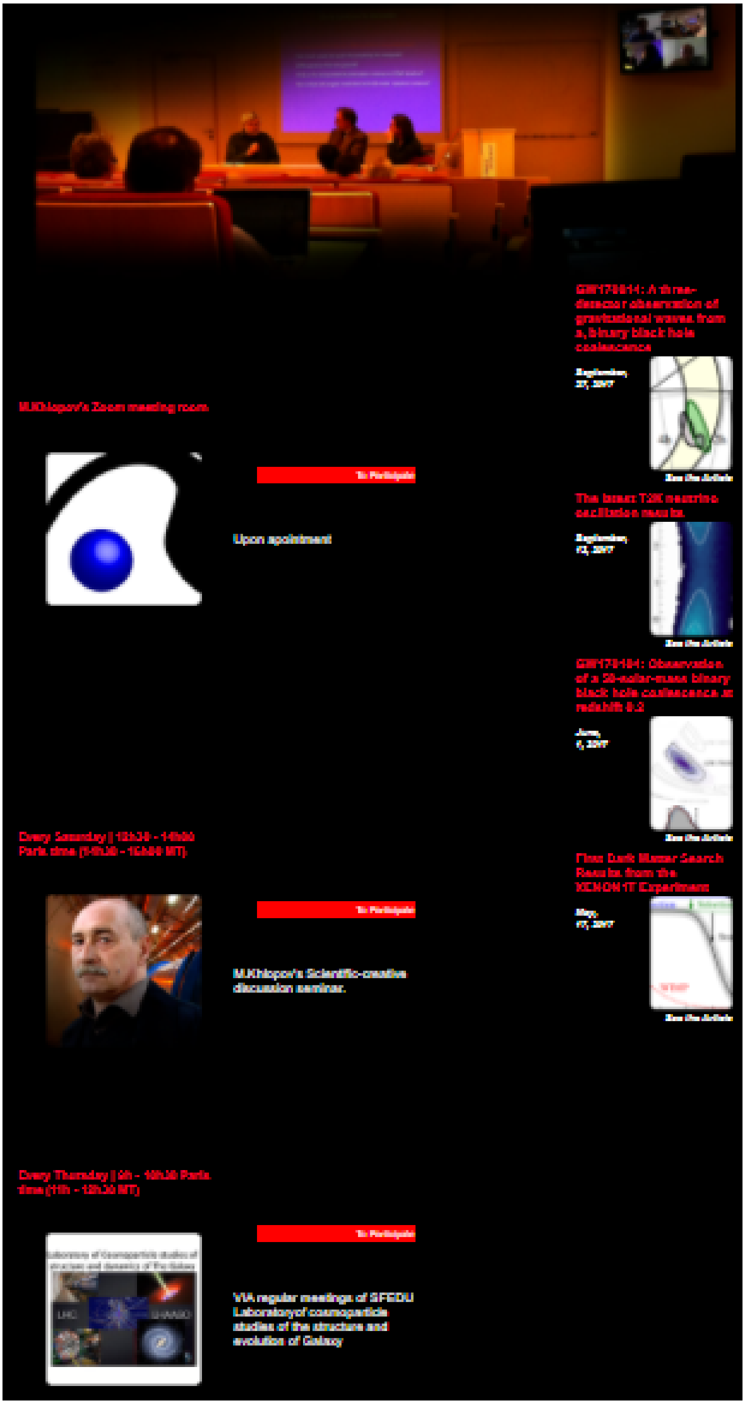

Fig. 17.2: The current home page of VIA site

meetings of the Double Chooz collaboration. In the latter case, the experimentalists, being at shift, took part in the collaboration meeting in such a virtual way.
The simplicity of VIA facility for ordinary users was demonstrated at XIV Bled Workshop in 2011. Videoconferences at this Workshop had no special technical support except for WiFi Internet connection and ordinary laptops with their internal webcams and microphones. This test has proved the ability to use VIA facility at any place with at least decent Internet connection. Of course the quality of records is not as good in this case as with the use of special equipment, but still it is sufficient to support fruitful scientific discussion as can be illustrated by the record of VIA presentation ”New physics and its experimental probes” given by John Ellis from his office in CERN (see the records in [23]).
In 2012 VIA facility, regularly used for programs of VIA lectures and transmission of APC Colloquiums, has extended its applications to support M.Khlopov’s talk at distance at Astrophysics seminar in Moscow, videoconference in PCCP, participation at distance in APC-Hamburg-Oxford network meeting as well as to provide online transmissions from the lectures at Science Festival 2012 in University Paris7. VIA communication has effectively resolved the problem of referee’s attendance at the defence of PhD thesis by Mariana Vargas in APC. The referees made their reports and participated in discussion in the regime of VIA videoconference. In 2012 VIA facility was first used for online transmissions from the Science Festival in the University Paris 7. This tradition was continued in 2013, when the transmissions of meetings at Journées nationales du Développement Logiciel (JDEV2013) at Ecole Politechnique (Paris) were organized [25].
In 2013 VIA lecture by Prof. Martin Pohl was one of the first places at which the first hand information on the first results of AMS02 experiment was presented [24].
In 2014 the 100th anniversary of one of the foundators of Cosmoparticle physics, Ya. B. Zeldovich, was celebrated. With the use of VIA M.Khlopov could contribute the programme of the ”Subatomic particles, Nucleons, Atoms, Universe: Processes and Structure International conference in honor of Ya. B. Zeldovich 100th Anniversary” (Minsk, Belarus) by his talk ”Cosmoparticle physics: the Universe as a laboratory of elementary particles” [26] and the programme of ”Conference YaB-100, dedicated to 100 Anniversary of Yakov Borisovich Zeldovich” (Moscow, Russia) by his talk ”Cosmology and particle physics”.
In 2015 VIA facility supported the talk at distance at All Moscow Astrophysical seminar ”Cosmoparticle physics of dark matter and structures in the Universe” by Maxim Yu. Khlopov and the work of the Section ”Dark matter” of the International Conference on Particle Physics and Astrophysics (Moscow, 5-10 October 2015). Though the conference room was situated in Milan Hotel in Moscow all the presentations at this Section were given at distance (by Rita Bernabei from Rome, Italy; by Juan Jose Gomez-Cadenas, Paterna, University of Valencia, Spain and by Dmitri Semikoz, Martin Bucher and Maxim Khlopov from Paris) and its proceeding was chaired by M.Khlopov from Paris. In the end of 2015 M. Khlopov gave his distant talk ”Dark atoms of dark matter” at the Conference ”Progress of Russian Astronomy in 2015”, held in Sternberg Astronomical Institute of Moscow State University.

In 2016 distant online talks at St. Petersburg Workshop ”Dark Ages and White Nights (Spectroscopy of the CMB)” by Khatri Rishi (TIFR, India) ”The information hidden in the CMB spectral distortions in Planck data and beyond”, E. Kholupenko (Ioffe Institute, Russia) ”On recombination dynamics of hydrogen and helium”, Jens Chluba (Jodrell Bank Centre for Astrophysics, UK) ”Primordial recombination lines of hydrogen and helium”, M. Yu. Khlopov (APC and MEPHI, France and Russia)”Nonstandard cosmological scenarios” and P. de Bernardis (La Sapiensa University, Italy) ”Balloon techniques for CMB spectrum research” were given with the use of VIA system. At the defense of PhD thesis by F. Gregis VIA facility made possible for his referee in California not only to attend at distance at the presentation of the thesis but also to take part in its successive jury evaluation.
Since 2018 VIA facility is used for collaborative work on studies of various forms of dark matter in the framework of the project of Russian Science Foundation based on Southern Federal University (Rostov on Don). In September 2018 VIA supported online transmission of **17 presentations** at the Commemoration day for Patrick Fleury, held in APC.
The discussion of questions that were put forward in the interactive VIA events is continued and extended on VIA Forum. Presently activated in English,French and Russian with trivial extension to other languages, the Forum represents a first step on the way to multi-lingual character of VIA complex and its activity. Discussions in English on Forum are arranged along the following directions: beyond the standard model, astroparticle physics, cosmology, gravitational wave experiments, astrophysics, neutrinos. After each VIA lecture its pdf presentation together with link to its record and information on the discussion during it are put in the corresponding post, which offers a platform to continue discussion in replies to this post.

17.2.3 VIA e-learning, OOC and MOOC

One of the interesting forms of VIA activity is the educational work at distance. For the last eleven years M.Khlopov’s course ”Introduction to cosmoparticle physics” is given in the form of VIA videoconferences and the records of these lectures and their ppt presentations are put in the corresponding directory of the Forum [27]. Having attended the VIA course of lectures in order to be admitted to exam students should put on Forum a post with their small thesis. In this thesis students are proposed to chose some BSM model and to study the cosmological scenario based on this chosen model. The list of possible topics for such thesis is proposed to students, but they are also invited to chose themselves any topic of their own on possible links between cosmology and particle physics. Professor’s comments and proposed corrections are put in a Post reply so that students should continuously present on Forum improved versions of work until it is accepted as admission for student to pass exam. The record of videoconference with the oral exam is also put in the corresponding directory of Forum. Such procedure provides completely transparent way of evaluation of students’ knowledge at distance.
In 2018 the test has started for possible application of VIA facility to remote supervision of student’s scientific practice. The formulation of task and discussion

of progress on work are recorded and put in the corresponding directory on Forum together with the versions of student's report on the work progress.
Since 2014 the second semester of the course on Cosmoparticle physics is given in English and converted in an Open Online Course. It was aimed to develop VIA system as a possible accomplishment for Massive Online Open Courses (MOOC) activity [28]. In 2016 not only students from Moscow, but also from France and Sri Lanka attended this course. In 2017 students from Moscow were accompanied by participants from France, Italy, Sri Lanka and India [29]. The students pretending to evaluation of their knowledge must write their small thesis, present it and, being admitted to exam, pass it in English. The restricted number of online connections to videoconferences with VIA lectures is compensated by the wide-world access to their records on VIA Forum and in the context of MOOC VIA Forum and videoconferencing system can be used for individual online work with advanced participants. Indeed Google Analytics shows that since 2008 VIA site was visited by more than **250 thousand** visitors from **155** countries, covering all the continents by its geography (Fig. 17.3). According to this statistics more than half of these visitors continued to enter VIA site after the first visit. Still the form of individual

Fig. 17.3: Geography of VIA site visits according to Google Analytics

educational work makes VIA facility most appropriate for PhD courses and it could be involved in the International PhD program on Fundamental Physics, which was planned to be started on the basis of Russian-French collaborative agreement. In 2017 the test for the ability of VIA to support fully distant education and evaluation of students (as well as for work on PhD thesis and its distant defense) was undertaken. Steve Branchu from France, who attended the Open Online Course and presented on Forum his small thesis has passed exam at distance. The whole procedure, starting from a stochastic choice of number of examination ticket, answers to ticket questions, discussion by professors in the absence of student and announcement of result of exam to him was recorded and put on VIA Forum [30].

In 2019 in addition to individual supervisory work with students the regular scientific and creative VIA seminar is in operation aimed to discuss the progress and strategy of students scientific work in the field of cosmoparticle physics.
In 2020 the regular course now for M2 students continued, but the problems of adobe Connect, related with the lack of its support for Flash in 2021 made necessary to use the platform of Zoom, This platform is rather easy to use and provides records, as well as whiteboard tools for discussions online can be solved by accomplishments of laptops by graphic tabloids. In 2022 the Open Online Course for M2 students was accompanied by special course "Cosmoparticle physics", given in English for English speaking M1 students. In 2023 the practice of Open Online Course for M2 students was continued.

17.2.4 Organisation of VIA events and meetings

First tests of VIA system, described in [5,7–9], involved various systems of videoconferencing. They included skype, VRVS, EVO, WEBEX, marratech and adobe Connect. In the result of these tests the adobe Connect system was chosen and properly acquired. Its advantages were: relatively easy use for participants, a possibility to make presentation in a video contact between presenter and audience, a possibility to make high quality records, to use a whiteboard tools for discussions, the option to open desktop and to work online with texts in any format. The lack of support for Flash, on which VIA site was originally based, made necessary to use Zoom, which shares all the above mentioned advantages.
Regular activity of VIA as a part of APC included online transmissions of all the APC Colloquiums and of some topical APC Seminars, which may be of interest for a wide audience. Online transmissions were arranged in the manner, most convenient for presenters, prepared to give their talk in the conference room in a normal way, projecting slides from their laptop on the screen. Having uploaded in advance these slides in the VIA system, VIA operator, sitting in the conference room, changed them following presenter, directing simultaneously webcam on the presenter and the audience. If the advanced uploading was not possible, VIA streaming was used - external webcam and microphone are directed to presenter and screen and support online streaming. This experience has found proper place in the current weakening of the pandemic conditions and regular meetings in real can be streamed. Moreover, such streaming can be made without involvement of VIA operator, by direction of webcam towards the conference screen and speaker.

17.2.5 VIA activity in the conditions of pandemia

The lack of usual offline connections and meetings in the conditions of pandemia made the use of VIA facility especially timely and important. This facility supports regular weekly meetings of the Laboratory of cosmoparticle studies of the structure and dynamics of Galaxy in Institute of Physics of Southern Federal University (Rostov on Don, Russia) and M.Khlopov's scientific - creative seminar and their announcements occupied their permanent position on VIA homepage (Fig. 17.2), while their records were put in respective place of VIA forum, like [32] for Laboratory meetings.

The platform of VIA facility was used for regular Khlopov's course "Introduction to Cosmoparticle physics" for M2 students of MEPHI (in Russian) and in 2020 supported regular seminars of Theory group of APC.
The programme of VIA lectures continued to present hot news of astroparticle physics and cosmology, like talk by Zhen Cao from China on the progress of LHAASO experiment or lecture by Sunny Vagnozzi from UK on the problem of consistency of different measurements of the Hubble constant.
The results of this activity inspired the decision to hold in 2020 XXIII Bled Workshop online on the platform of VIA [19].
The conditions of pandemia continued in 2021 and VIA facility was successfully used to provide the platform for various online meetings. 2021 was announced by UNESCO as A.D.Sakharov year in the occasion of his 100th anniversary VIA offered its platform for various events commemorating A.D.Sakharov's legacy in cosmoparticle physics. In the framework of 1 Electronic Conference on Universe ECU2021), organized by the MDPI journal "Universe" VIA provided the platform for online satellite Workshop "Developing A.D.Sakharov legacy in cosmoparticle physics" [33].

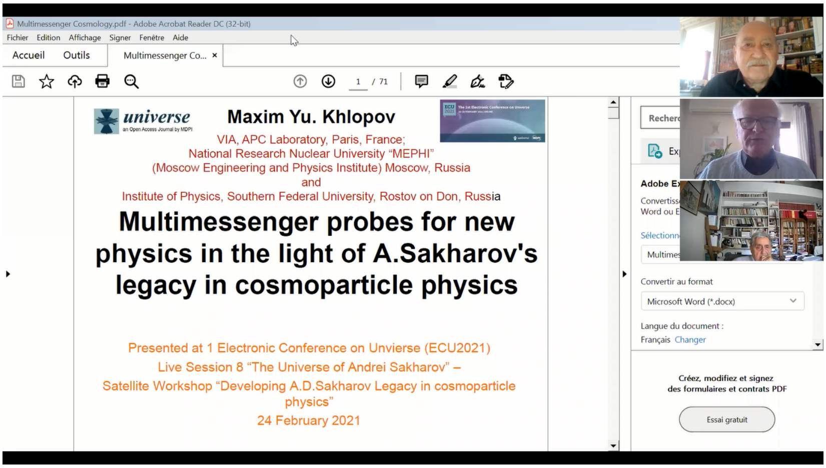

Fig. 17.4: M.Khlopov's talk "Multimessenger probes for new physics in the light of A.D.Sakharov legacy in cosmoparticle physics" at the satellite Workshop "Developing A.D.Sakharov legacy in cosmoparticle physics" of ECU2021.

17.3 VIA platform at the Hybrid XXVI Bled Workshop

VIA sessions at Bled Workshops continued the tradition coming back to the first experience at XI Bled Workshop [7] and developed at XII, XIII, XIV, XV, XVI, XVII, XVIII, XIX, XX, XXI and XXII Bled Workshops [8–18]. They became a regular but supplementary part of the Bled Workshop's program. In the conditions of

pandemia it became the only form of Workshop activity in 2020 [19] and in 2021 [20], as well as substantial part of the hybrid Memorial XXV Bled Workshop in 2022 [21].
During the XXVI Bled Workshop the announcement of VIA sessions was put on VIA home page, giving an open access to the videoconferences at the Workshop sessions. The preliminary program as well as the corrected program for each day were continuously put on Forum with the slides and records of all the talks and discussions [34].
VIA facility tried to preserve the creative atmosphere of Bled discussions. The program of XXVI Bled Workshop combined talks presented in Plemelj House in Bled, which were streamed by VIA facility, as the talk ”How far can we understand nature with the spin-charge-family theory, describing the internal spaces of fermions and bosons with the Clifford algebra” by Norma Mankoc-Borstnik (Fig. 17.5) with talks given in the format videoconferences ”Recent efforts in the DAMA project” by R.Bernabei, (Fig. 17.6), from Rome University, Italy (see records in [34]).

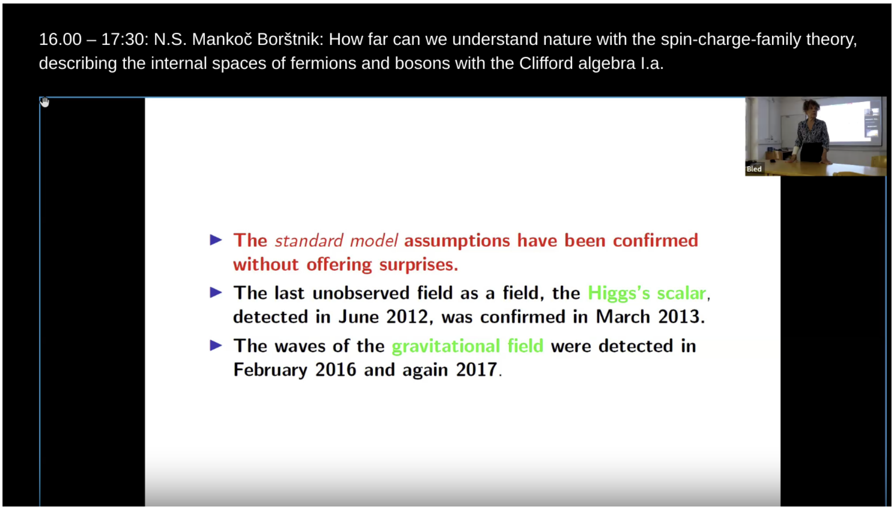

Fig. 17.5: VIA stream of the talk ”How far can we understand nature with the spin-charge-family theory, describing the internal spaces of fermions and bosons with the Clifford algebra” by Norma Mankoc-Borstnik at XXVI Bled Workshop

During the Workshop the VIA virtual room was open, inviting distant participants to join the discussion and extending the creative atmosphere of these discussions to the world-wide audience. The participants joined these discussions from different parts of world. The talk ”Quantum gravity in lab” was given by Andrea Addazi from China (Fig. 17.7), by A. Hernandez-Galeana from Mexico, by S. Roy Chowdhury - from India, by O.M.Lecian from Italy, as well as M.Y. Khlopov gave his talk ”Recent advances of Beyond the Standard model cosmology” (Fig. 17.9)

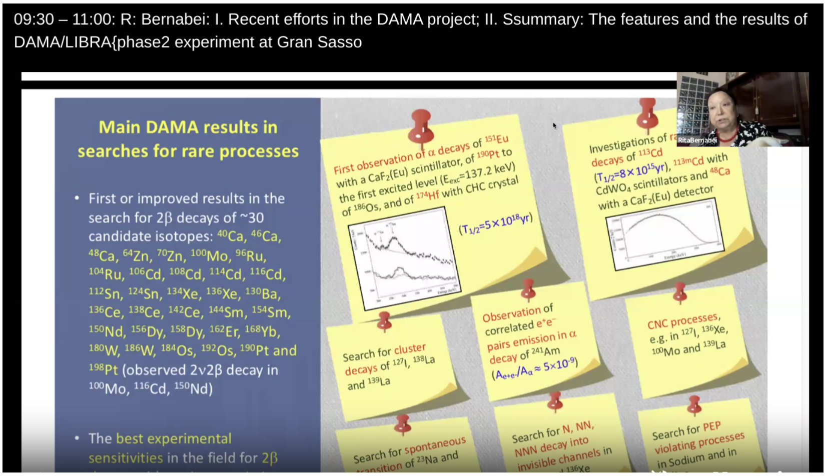

Fig. 17.6: VIA talk ”Recent efforts in the DAMA project” by R.Bernabei from Rome at XXVI Bled Workshop

from Japan. The online talks were combined with presentations in Bled such as ”Dusty dark matter pearls developed” by H.B. Nielsen (Fig. 17.8) .

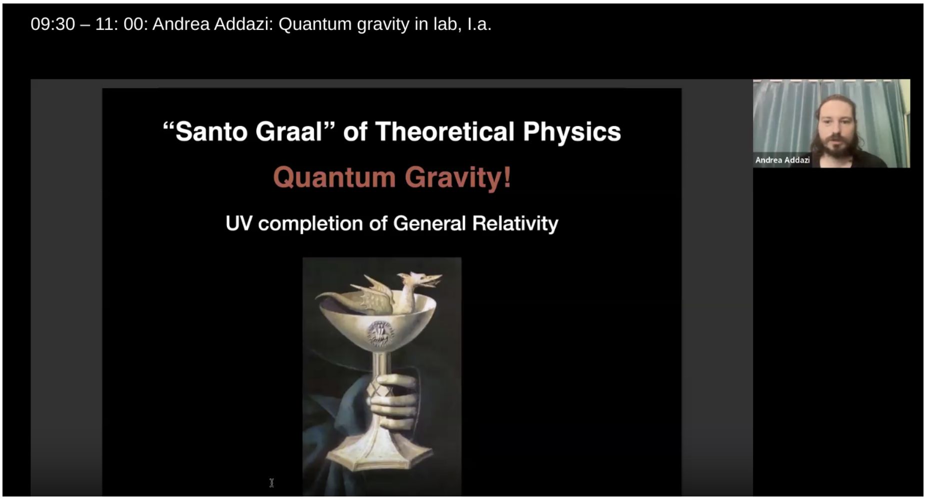

Fig. 17.7: VIA talk ”Quantum gravity in lab” by Andrea Addazi at XXVI Bled Workshop

The distant VIA talks highly enriched the Workshop program and streaming of talks from Bled involved distant participants in fruitful discussions. The use of VIA facility has provided remote presentation of students’ scientific debuts in BSM

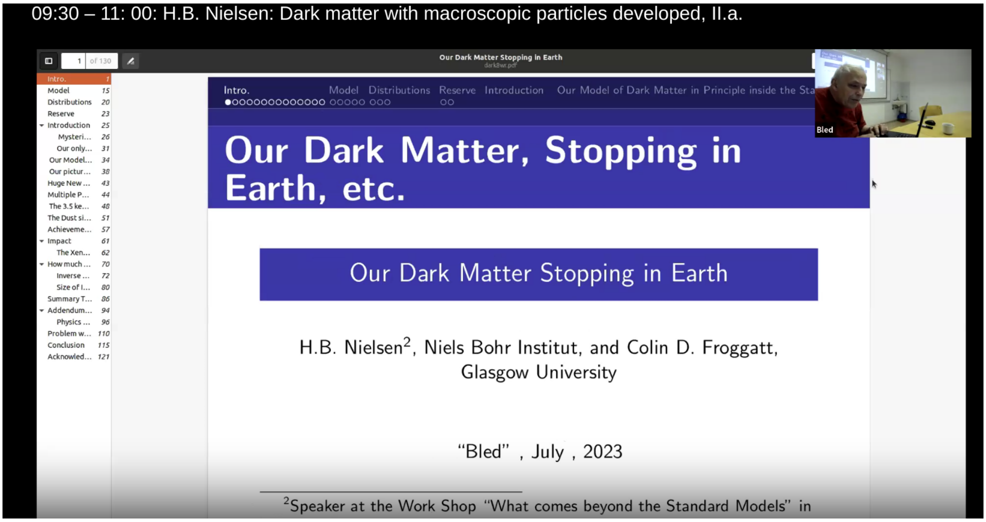

Fig. 17.8: VIA stream of talk "Dark matter with macroscopic particles developed" by Holger Bech Nielsen at XXVI Bled Workshop

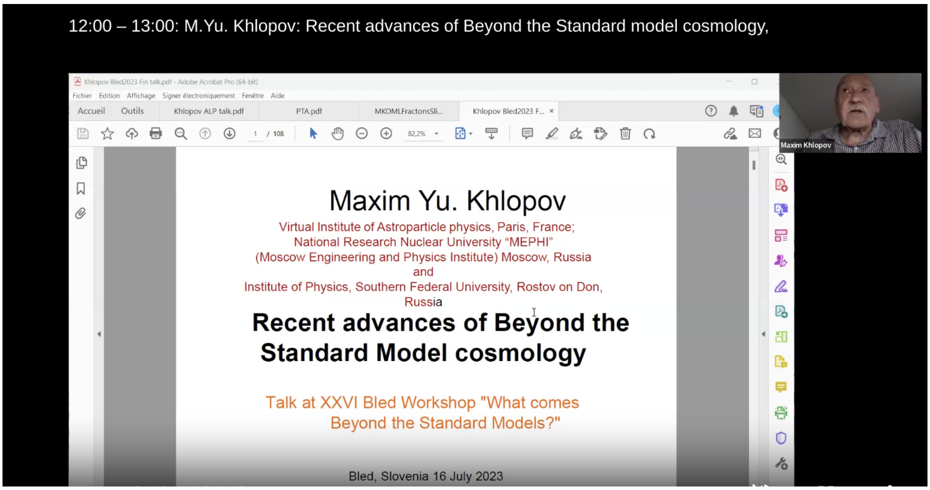

Fig. 17.9: VIA talk "Cosmological reflection of the BSM physics" by Maxim Yu. Khlopov at XXVI Bled Workshop

physics and cosmology. The records of all the talks and discussions can be found on VIA Forum [34].
VIA facility has managed to join scientists from Mexico, USA, France, Italy, Russia, Slovenia, India, China and many other countries in discussion of open problems of physics and cosmology beyond the Standard models. In the current situation, hindering visits of Russian scientists, to Europe it made possible Russian students to present their results and participate in these discussions

17.4 Conclusions

The Scientific-Educational complex of Virtual Institute of Astroparticle physics provides regular communication between different groups and scientists, working in different scientific fields and parts of the world, the first-hand information on the newest scientific results, as well as support for various educational programs at distance. This activity would easily allow finding mutual interest and organizing task forces for different scientific topics of cosmology, particle physics, astroparticle physics and related topics. It can help in the elaboration of strategy of experimental particle, nuclear, astrophysical and cosmological studies as well as in proper analysis of experimental data. It can provide young talented people from all over the world to get the highest level education, come in direct interactive contact with the world known scientists and to find their place in the fundamental research. These educational aspects of VIA activity can evolve in a specific tool for International PhD program for Fundamental physics. Involvement of young scientists in creative discussions was an important aspect of VIA activity at XXVI Bled Workshop. VIA applications can go far beyond the particular tasks of astroparticle physics and give rise to an interactive system of mass media communications.
VIA sessions, which became a natural part of a program of Bled Workshops, maintained in 2023 the platform for online discussions of physics beyond the Standard Model involving distant participants from all the world in the fruitful atmosphere of Bled offline meeting. This discussion can continue in posts and post replies on VIA Forum. The experience of VIA applications at Bled Workshops plays important role in the development of VIA facility as an effective tool of e-science and e-learning.
One can summarize the advantages and flaws of online format of Bled Workshop. It makes possible to involve in the discussions scientists from all the world (young scientists, especially) free of the expenses related with meetings in real (voyage, accommodation, ...), but loses the advantage of nonformal discussions at walks along the beautiful surrounding of the Bled lake and other places of interest. The improvement of VIA technical support by involvement of Zoom provided better platform for nonformal online discussions, but in no case can be the substitute for offline Bled meetings and its creative atmosphere in real, which has revived at the offline XXVI Bled Workshop. One can summarize that VIA facility provides important online addition of the offline Bled Workshop, involving world-wide participants in its creative and open discussions of BSM physics and cosmology.

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References

1. http://www.aspera-eu.org/
2. http://www.appec.org/
3. M.Yu. Khlopov: *Cosmoparticle physics*, World Scientific, New York -London-Hong Kong - Singapore, 1999.
4. M.Yu. Khlopov: *Fundamentals of Cosmic Particle Physics*, CISP-Springer, Cambridge, 2012.
5. M. Y. Khlopov, Project of Virtual Institute of Astroparticle Physics, arXiv:0801.0376 [astro-ph].
6. http://viavca.in2p3.fr/site.html
7. M. Y. Khlopov, Scientific-educational complex - virtual institute of astroparticle physics, 981–862008.
8. M. Y. Khlopov, Virtual Institute of Astroparticle Physics at Bled Workshop, 10177–1812009.
9. M. Y. Khlopov, VIA Presentation, 11225–2322010.
10. M. Y. Khlopov, VIA Discussions at XIV Bled Workshop, 12233–2392011.
11. M. Y. .Khlopov, Virtual Institute of astroparticle physics: Science and education online, 13183–1892012.
12. M. Y. .Khlopov, Virtual Institute of Astroparticle physics in online discussion of physics beyond the Standard model, 14223–2312013.
13. M. Y. .Khlopov, Virtual Institute of Astroparticle physics and "What comes beyond the Standard model?" in Bled, 15285-2932014.
14. M. Y. .Khlopov, Virtual Institute of Astroparticle physics and discussions at XVIII Bled Workshop, 16177-1882015.
15. M. Y. .Khlopov, Virtual Institute of Astroparticle Physics — Scientific-Educational Platform for Physics Beyond the Standard Model, 17221-2312016.
16. M. Y. .Khlopov: Scientific-Educational Platform of Virtual Institute of Astroparticle Physics and Studies of Physics Beyond the Standard Model, 18273-2832017.

17. M. Y. .Khlopov: The platform of Virtual Institute of Astroparticle physics in studies of physics beyond the Standard model, 19383-3942018.
18. M. Y. .Khlopov: The Platform of Virtual Institute of Astroparticle Physics for Studies of BSM Physics and Cosmology, Journal20249-2612019.
19. M. Y. .Khlopov: Virtual Institute of Astroparticle Physics as the Online Platform for Studies of BSM Physics and Cosmology, Journal21249-2632020.
20. M. Y. .Khlopov: Challenging BSM physics and cosmology on the online platform of Virtual Institute of Astroparticle physics, Journal22160-1752021.
21. M. Y. .Khlopov: Virtual Institute of Astroparticle physics as the online platform for studies of BSM physics and cosmology, Journal23334-3472022.
22. $http://viavca.in2p3.fr/what_comes_beyond_the_standard_models_xiii.html$
23. $http://viavca.in2p3.fr/what_comes_beyond_the_standard_models_xiv.html$
24. $http://viavca.in2p3.fr/pohl_martin.html$
25. In http://viavca.in2p3.fr/ Previous - Events - JDEV 2013
26. $http://viavca.in2p3.fr/zeldovich_100_meeting.html$
27. In http://bsm.fmf.uni-lj.si/bled2023bsm/ Cosmovia - Forum- Discussion in Russian - Courses on Cosmoparticle physics
28. In http://bsm.fmf.uni-lj.si/bled2023bsm/ Cosmovia - Forum - Education - From VIA to MOOC
29. In http://bsm.fmf.uni-lj.si/bled2023bsm/ Cosmovia - Forum - Education - Lectures of Open Online VIA Course 2017
30. In http://bsm.fmf.uni-lj.si/bled2023bsm/ Cosmovia - Forum - Education - Small thesis and exam of Steve Branchu
31. $http://viavca.in2p3.fr/john_ellis.html$
32. In http://bsm.fmf.uni-lj.si/bled2023bsm/ Cosmovia - Forum - LABORATORY OF COSMOPARTICLE STUDIES OF STRUCTURE AND EVOLUTION OF GALAXY
33. In http://bsm.fmf.uni-lj.si/bled2023bsm/ Cosmovia - Forum - CONFERENCES - CONFERENCES ASTROPARTICLE PHYSICS - The Universe of A.D. Sakharov at ECU2021
34. In http://bsm.fmf.uni-lj.si/bled2023bsm/ Cosmovia - Forum - CONFERENCES BEYOND THE STANDARD MODEL - XXVI Bled Workshop "What comes beyond the Standard model?"

18 The Equation, a story

For Oskar Sonne, mathematics was the most beautiful of arts. He was captivated by the elegant movement of a calculation, the web of mathematical threads that are drawn from one point to another. But the most remarkable was that a mathematical equation could reveal things that nobody in their wildest dreams could have imagined, like when antimatter was discovered thanks to an equation. Mathematics is wiser than us, Oskar thought, it encompasses everything, and it never lies.

This day he was high up in the air, on his way to a physicist meeting in Norway. He stretched out his legs as best he could in the cramped airplane seat, and looked at the blank sheet of paper lying on the small table in front of him. A flight attendant came by with coffee and hot rolls, and Oskar, who loved coffee, gratefully accepted the cup, and let his gaze wander to the blue infinity outside the window while he carefully took a sip of the warm liquid.

We're just a tiny particle hurtling across the gravitational field, he thought, the rolls and the coffee and the fuselage and the pilots and all of us sitting here – nothing but a single tiny particle en route in time and space. And all the time he heard the woman next to him munching her roll, while intently reading the book on her lap. Throughout his studies Oskar had pondered what the world really *is*, and the wonderful fact we can actually predict how a ball will move through the air, or where a lunar rocket will land, as if each moment is hooked into the next, in a single, limitless now. At times he was sleepless because he could not let go of these thoughts: the world was a spreadsheet, no, thousands upon thousands of spreadsheets.

He glanced at the paper on the table in front of him, thinking about the talk he was going to give. Perhaps he could start by talking about equations in general, that made sense, since there would be many students in the audience. Yes, that was a good move. He took out his pen and wrote: *Introduction*.

The stewardess came by and served more coffee, but he left it on his table, because it was far too hot. He had just finished his roll when he suddenly felt irresistibly tired, and in the next moment he had fallen asleep.

He dreamt that he was standing by a gigantic blackboard where he slowly wrote down the world equation, showing that it had two different solutions. As he tried to explain that the world only uses one of the solutions, it suddenly occurred gto him that the wrong choice had been made. Yes, the world was based on the wrong solution, and because of this all the defects, all the evil, had arisen, and now it was up to him to use the other solution instead.

He wrote obsessively, and the board was soon filled with calculations. Then he suddenly felt a searing pain on his hand, and he sat up, wide awake. The hot coffee had poured out over his hand, as the plane was precariously shaking.

He stared at the sheet of paper on the table, and saw that he had written something in his sleep, it looked like an equation. He always used to write with a fountain pen, and even if the paper was stained by coffee, he could see what was written. His heart began to gallop, because it seemed to him that he had actually written down the other solution from the dream, the solution that the world had not chosen. That must be the reason why the plane was shaking so badly, people screamed and the woman next to him cried, while she quickly produced a small pocket mirror and began to wipe away the mascara that was running down her face. The plane pitched like a boat in rough seas, and Oskar clung to his seat, but in the next moment the rest of the coffee spilled over his entire paper and turned the writing into an illegible blur of blue and brown. The plane immediately righted itself, and people cried with relief, some started lamely clapping their hands.

He stared at the stained sheet of paper. At the top of the page he could still see the word *Introduction*.

When the smiling stewardess walked by and asked if they had any trash they wanted to get rid of, he made a ball out of the paper sheet and let it slide into the black plastic bag that she held out. The woman next to him straightened her hair and gave him warm smile, as if realizing that he had just saved her life.

- Ugh, it's so uncomfortable with that kind of turbulence, isn't it? she said, closing her book before she pushed it into the pocket in front of her. He nodded, and she laid her head back and closed her eyes. A metallic stewardess voice announced that they would be landing in twenty minutes, but the lady next to him had apparently fallen asleep. Before straightening his table, Oskar curiously took out her book and quickly glanced at the title page: *Eichmann in Jerusalem – a report on the banality of evil*. He sat for a while and looked at the book in his hand.

If my dream were true, he thought, it would imply that all evil would been gone if the other solution had been realized. He smiled to himself and shook his head, putting back the book into the seat pocket. Then he leaned his head back onto the seat, closed his eyes, and continued to dream.

A. Kleppe

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Edited by Norma Susana Mankoč Borštnik, Holger Bech Nielsen, Maxim Yu. Khlopov in Astri Kleppe

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