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Right quadruple convexity*

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Abstract

A set of four points $w, x, y, z \in \mathbb{R}^d$ (always $d \geq 2$) form a rectangular quadruple if their convex hull is a non-degenerate rectangle. The set M is called rq-convex if for every pair of its points we can find another pair in M, such that the four points form a rectangular quadruple. In this paper we start the investigation of rq-convexity in Euclidean spaces.

Keywords: rq-convex sets, parallelotopes, finite sets, Platonic solids.

Math. Subj. Class.: 53C45, 53C22

1 Introduction

Let $\mathcal{F}$ be a family of sets in $\mathbb{R}^d$. A set $M \subset \mathbb{R}^d$ is called $\mathcal{F}$-*convex* if for any pair of distinct points $x, y \in M$ there is a set $F \in \mathcal{F}$ such that $x, y \in F$ and $F \subset M$.

The third author proposed at the 1974 meeting on Convexity in Oberwolfach the investigation of this very general kind of convexity. Usual convexity, affine linearity, arc-wise connectedness, polygonal connectedness, are just some examples of $\mathcal{F}$-convexity (for suitably chosen families $\mathcal{F}$).

Blind, Valette and the third author [1], and also Böröczky Jr [2], investigated rectangular convexity. Magazanik and Perles dealt with staircase connectedness [5]. The third author

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studied right convexity [8]; then the second and the third author generalized the latter type of convexity and investigated the right triple convexity (see [7] and [6]). All these concepts are particular cases of $\mathcal{F}$-convexity. The rectangular convexity is obtained if $\mathcal{F}$ is the family of all non-degenerate rectangles in $\mathbb{R}^d$.

In this paper we present a discretization of rectangular convexity, the right quadruple convexity, which constitutes a generalization of rectangular convexity. As usual, for $M \subset \mathbb{R}^d$, $\mathrm{bd}M$ denotes its boundary, $\mathrm{int}M$ its interior, $\mathrm{diam}M = \sup_{x,y\in M} \|x-y\|$ its diameter, and $\mathrm{conv}M$ its convex hull. A set of four points $w, x, y, z \in \mathbb{R}^d$ (always $d \geq 2$) form a *rectangular quadruple* if $\mathrm{conv}\{w, x, y, z\}$ is a non-degenerate rectangle. Let $\mathcal{R}$ be the family of all rectangular quadruples. Here, we shall choose $\mathcal{F}$ to be this family $\mathcal{R}$.

Let $M \subset \mathbb{R}^d$. A pair of points $x, y \in M$ is said to enjoy the *rq-property in M* if there exists another pair of points $z,\ w \in M$, such that $\{w, x, y, z\}$ is a rectangular quadruple. The set M is called *rq-convex*, if every pair of its points enjoys the rq-property in M. This property is the *right quadruple convexity*.

Let $A \subset \mathbb{R}^d$. We call A^* an *rq-convex completion* of A, if A^* is rq-convex, $A^* \supset A$ and card$(A^*\backslash A)$ is minimal (but possibly infinite). Let $\gamma(A) = \mathrm{card}(A^*\backslash A)$, which is called the *$rq$-convex completion number* of A, in case A is finite. For finite n, let $\gamma(n) = \sup\{\gamma(A) :\ \mathrm{card}A = n\}$.

For distinct $x, y \in \mathbb{R}^d$, let $\overline{xy}$ be the line through x, y, xy the line-segment from x to y, H_{xy} the hyperplane through x orthogonal to $\overline{xy}$, and C_{xy} the hypersphere of diameter xy. For $S_1,\ S_2 \subset \mathbb{R}^d$, let $d(S_1, S_2) = \inf\{d(x, y) \mid x \in S_1,\ y \in S_2\}$ denote the *distance* between S_1 and S_2. The d-dimensional unit ball (centred at $\mathbf{0}$) is denoted by B_d $(d \geq 2)$. Let us remark that every open set in $\mathbb{R}^d$ is rq-convex.

2 Not simply connected rq-convex sets

In $\mathbb{R}^2$, all compact rectangularly convex sets are conjectured to be extremely circular and symmetric. A planar convex set is *extremely circular* if its set of extreme points lies on a circle. Analogously, it is reasonable to conjecture that all compact rq-convex sets have an extremely circular and symmetric convex hull. Consequently, when investigating a compact connected rq-convexset M, we may reasonably start by assuming that $\mathrm{conv}M$ is extremely circular and symmetric. We shall now take $\mathrm{bdconv}M$ to be a circle. If M is simply connected we get the disc. So, assume $(\mathrm{conv}M) \setminus M \neq \emptyset$.

Theorem 2.1. *If* $\mathrm{conv}M$ *is a disc and* $(\mathrm{conv}M) \setminus M$ *lies in a circular disc of radius r at distance at least* $(\sqrt{3}-1)r$ *from* $\mathrm{bdconv}M$*, then M is rq-convex.*

This theorem gives a useful sufficient condition for the rq-convexity of a set M which is not simply connected, regardless the shape of $(\mathrm{conv}M) \setminus M$. Notice that it allows both M and its complement to have arbitrarily many components.

Proof. Let Q be a square circumscribed to the disc D of radius r including $(\mathrm{conv}M) \setminus M$. We may suppose that the origin $\mathbf{0}$ is the centre of D, so $D = rB_2$.

We have $Q \subset \mathrm{conv}M$. Indeed, Q is obviously included in the disc concentric with D of radius $\sqrt{2}r$, which in turn must be included in $\mathrm{conv}M$, since the distance from D to $\mathrm{bdconv}M$ is at least $(\sqrt{3}-1)r > (\sqrt{2}-1)r$.

Let $x, y \in M$. We verify the rq-convexity of M at these two points.

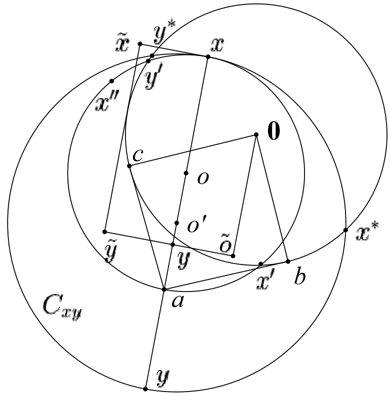

Figure 1: $x \in D, y \notin D$.

Case 1: $x, y \in D$. Choose Q to have a side s parallel to xy. Then x, y and their orthogonal projections on s are vertices of a rectangle. Moreover, these latter vertices lie in $(\mathrm{conv}M) \setminus D$.

Case 2: $x \in D$, $y \notin D$. Let Γ_2, Γ_3 be the two circles concentric with $\mathrm{bd}D$, of radii $r\sqrt{2}$, $r\sqrt{3}$. Let a be the point of $\overline{xy} \cap \Gamma_2$ such that $x \notin ay$. In case $y \in ax$, consider the rectangle $xy\tilde{y}\tilde{x}$, such that $\overline{xy}$ separates $\mathbf{0}$ from $\tilde{x}$, $\tilde{y}$ (if $\mathbf{0} \notin \overline{xy}$) and $\overline{\tilde{x}\tilde{y}}$ is tangent to $\mathrm{bd}D$. See Figure 1. Let $\tilde{o}$ be the orthogonal projection of $\mathbf{0}$ onto $\overline{y\tilde{y}}$. We have

$$\|\tilde{y}\|^2 = \|\tilde{y} - \tilde{o}\|^2 + \|\tilde{o}\|^2 = r^2 + \|\tilde{o}\|^2 \leq r^2 + \|y\|^2 \leq r^2 + \|a\|^2 = 3r^2.$$

Hence, $\tilde{y}$, and of course $\tilde{x}$ too, lie in M, and the rq-property is satisfied in x, y.

We reconsider the case $y = a$. Choose $b, c \in \mathrm{bd}D$ such that $ab\mathbf{0}c$ be a square and $b, \mathbf{0}$ be not separated by $\overline{xy}$. Let $\{x', y'\} = C_{ax} \cap \mathrm{bd}D$, where x' is closer than y' from b. Let x'' be the point of C_{ax} diametrally opposite to x'. For any position of x,

$$\angle \mathbf{0}ax' \leq \angle \mathbf{0}ab.$$

Hence, $\angle \mathbf{0}ax'' \geq \angle \mathbf{0}ac$, whence $x'' \notin D$. (This confirms the rq-property in x, a.) It follows that $\angle x'oy' < \pi$, where o is the centre of C_{ax} and the angle is measured towards $\mathbf{0}$.

In case $a \in xy$, consider the circle C_{xy} with centre o', which cuts $\mathrm{bd}D$ in x^*, y^* (the former being closer than the latter from b). We have

$$\angle xo'x^* < \angle xo'x' < \angle xox'$$

and

$$\angle xo'y^* < \angle xo'y' < \angle xoy',$$

whence

$$\angle x^*o'y^* < \angle x'oy' < \pi,$$

where both angles are taken towards $\mathbf{0}$. Consequently, the points $x^+, y^+ \in C_{xy}$ diametrally opposite to x^*, y^* (respectively) lie outside D. They also lie in different half-circles determined by x, y on C_{xy}. Of these two half-circles, at least one is contained in the disc $\mathrm{conv}M$.

So, either $\{x, y, x^*, x^+\} \subset M$ or $\{x, y, y^*, y^+\} \subset M$, and the rq-property is again satisfied at x, y.

Case 3: $x, y \notin D$. Besides the trivial cases $x, y \in \mathrm{int}M$ and $x, y \in \mathrm{bdconv}M$, we only have the simple situation $x \in \mathrm{int}M$, $y \in \mathrm{bdconv}M$. In that situation, the circle C_{xy} has necessarily two opposite arcs in M starting at x, respectively y. This proves the rq-property at x, y. □

Conjecture 2.2. *Each simply connected rq-convex set in $\mathbb{R}^2$ is convex.*

3 Unbounded rq-convex sets

An infinite family $\mathcal{K}$ of closed convex sets is said to be *uniformly bounded below* if, for some $\lambda > 0$, each of the sets contains a translate of the disc λB_2.

Theorem 3.1. *Let $\mathcal{K}$ be a family of pairwise disjoint closed convex sets in $\mathbb{R}^d$. If $\mathcal{K}$ is finite or uniformly bounded below, then the closure of the complement of $\bigcup \mathcal{K}$ is rq-convex.*

Proof. We may assume that all sets in $\mathcal{K}$ possess interior points, because the case of empty interior is irrelevant. Let M be the closure of $\mathbb{R}^d \setminus \bigcup \mathcal{K}$, and choose $x, y \in M$. Clearly, the only interesting case is when $x, y \in \mathrm{bd}M$.

The condition of uniform boundedness below for infinite $\mathcal{K}$ guarantees that $x \in \mathrm{bd}M$ only if x is a boundary point of some member of $\mathcal{K}$.

Let M' be the intersection of M with an arbitrary 2-dimensional plane $\Pi \ni x, y$. For some $K_x, K_y \in \mathcal{K}$, $x \in \mathrm{bd}K_x$, $y \in \mathrm{bd}K_y$. Consider the supporting hyperplane H_x of K_x at x, the line $H'_x = H_x \cap \Pi$, and analogously H_y and H'_y. If H'_x, H'_y are not orthogonal to xy, there are six different situations in the neighbourhood of x and y, depicted in Figure 2 (subfigures (a)-(f)). (In the figure only the generic case is depicted, when $K_x \cap \Pi$ and $K_y \cap \Pi$ are not degenerate; but the proof works in all cases.)

In the situations of Figure 2 (subfigures (a),(c) and (e)), the circle C_{xy} has two opposite arcs inside M, so M has the rq-property at x, y. In the cases of Figure 2 (subfigures (b),(d) and (f)), a thin rectangle with xy as a side has its short sides in M, so the rq-property is again verified.

The only remaining case is that of at least one of the lines H'_x, H'_y, say the first, being orthogonal to xy. In this case, there is a short line-segment $x(x + v) \subset M$ in any direction v orthogonal to xy. Now, if $y + v \in M$, we found the right quadruple $\{x, y, y + v, x + v\}$. If $y + v \notin M$, i.e. $y + v \in \mathrm{int}K_y$, then $y - v \notin K_y$. Thus, $\{x, y, y - v, x - v\}$ is a suitable rectangular quadruple. □

We can drop the convexity condition if the considered sets are bounded.

Theorem 3.2. *The complement of any bounded set in $\mathbb{R}^d$ is rq-convex.*

The easy proof is left to the reader.

A *plane tiling* $\mathcal{T}$ is a countable family $\{T_1, T_2, \ldots\}$ of closed sets with non-empty interiors, which cover the plane without gaps or overlaps. Every closed set $T_i \in \mathcal{T}$ is called a *tile of* $\mathcal{T}$. We consider the special case in which each tile is a polygon. If the corners and sides of a polygon coincide with the vertices and edges of the tiling, we call the tiling

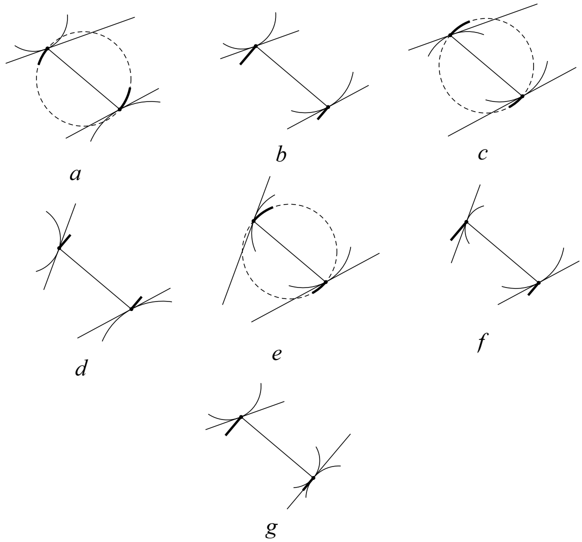

Figure 2: Illustration for the proof of Theorem 3.1.

edge-to-edge. A so-called *type* describes the neighbourhood of any vertex of the tiling. If, for example, in some cyclic order around a vertex there are a triangle, then another triangle, then a square, next a third triangle, and last another square, then its type is $(3^2.4.3.4)$. We consider plane edge-to-edge tilings in which all tiles are regular polygons, and all vertices are of the same type. Thus, the vertex-type defines our tiling up to similarity.

There exist precisely eleven such tilings [3]. These are (3^6), $(3^4.6)$, $(3^3.4^2)$, $(3^2.4.3.4)$, $(3.4.6.4)$, $(3.6.3.6)$, (3.12^2), (4^4), $(4.6.12)$, (4.8^2), and (6^3). They are called *Archimedean tilings.*

Theorem 3.3. *The Archimedean tilings* (4^4)*,* (3^6)*,* (6^3)*,* $(3.6.3.6)$*,* $(3^4.6)$*,* $(3.3.4.3.4)$*,* $(4.8.8)$ *have* rq*-convex vertex sets.*

Theorem 3.4. *The vertex sets of the Archimedean tilings* $(3.3.3.4.4)$*,* $(3.4.6.4)$*,* $(4.6.12)$*,* $(3.12.12)$ *are not* rq*-convex.*

The proofs of Theorems 3.3 and 3.4 are also left to the reader.

4 rq-convex skeleta of parallelotopes

As already remarked in [1], for $d \geq 3$, there is not even any conjectured characterization of rectangularly convex sets in $\mathbb{R}^d$. Among the sets mentioned in [1] as rectangularly convex we find the cylinder $K \times [0,1]$ with a $(d-1)$-dimensional compact convex set K as basis. In particular, any *right parallelotope*, i.e. the cartesian product of d pairwise orthogonal line-segments, is rectangularly convex and, a fortiori, rq-convex.

Theorem 4.1. *The* 1*-skeleton of any right parallelotope is* rq*-convex.*

Proof. Let $P = I_1 \times I_2 \times \ldots \times I_d$ be our parallelotope, where $I_i = [\mathbf{0}, a_i]$ $(i = 1, \ldots, d)$. We verify the rq-property at the points x, y belonging to the 1-skeleton of P.

Case 1: x, y belong to parallel edges of P. We have without loss of generality

$$x = (x_1, 0, \dots, 0),$$
$$y = (y_1, a_1, \dots, a_i, 0, \dots, 0).$$

Then we choose as third and fourth point

$$u = (y_1, 0, \dots, 0),$$
$$v = (x_1, a_1, \dots, a_i, 0, \dots, 0).$$

Indeed, $xuvy$ is a rectangle.

Case 2: x, y belong to two edges of P having a common endpoint. Say without loss of generality that

$$x = (x_1, 0, \dots, 0),$$
$$y = (0, y_2, 0, \dots, 0).$$

Then take

$$u = (x_1, 0, a_3, 0, \dots, 0),$$
$$v = (0, y_2, a_3, 0, \dots, 0),$$

completing the vertex set of a rectangle $xuvy$.

Case 3: x, y belong to two non-parallel disjoint edges of P. If

$$x = (x_1, 0, \dots, 0),$$
$$y = (0, \dots, 0, y_i, a_{i+1}, \dots, a_d),$$

then we choose

$$u = (x_1, 0, \dots, 0, a_{i+1}, \dots, a_d),$$
$$v = (0, \dots, 0, y_i, 0, \dots, 0),$$

and again we get the vertices of a rectangle.

Case 4: x, y belong to the same edge of P. This is immediate. □

Contrary to the case of an arbitrary cylinder, the following is true.

Theorem 4.2. *The boundary of any right parallelotope is rq-convex.*

Proof. Take x, y on the boundary of the parallelotope P. We show that they have the rq-property.

If x, y belong to the same facet F, choose their orthogonal projections onto the opposite facet F'; the four points are vertices of a rectangle.

If $x \in F$, $y \in F'$, choose the projection x' of x onto F' and the projection y' of y onto F; we get the rectangle $xx'yy'$.

If x, y belong to two adjacent facets F, F^*, respectively, take the orthogonal projections x^* and y^* of x and y (respectively) onto $F \cap F^*$. We complete the rectangles $xx^*y^*\tilde{y}$ and $yy^*x^*\tilde{x}$. Then, clearly, $\{x, \tilde{x}, y, \tilde{y}\} \subset \mathrm{bd}P$ is a rectangular quadruple. □

Theorem 4.3. *Not every convex cylinder has an rq-convex boundary.*

Proof. Take a cylinder $Z = E \times [0,1] \subset \mathbb{R}^3$, where $E \subset \mathbb{R}^2$ is convex and $\mathrm{bd}E$ is a long ellipse. Choose x on the long axis of $\mathrm{bd}E \times \{1\}$, close to one of its endpoints $\{e\} = \{e_p\} \times \{1\}$, and let $\{e'\} = \{e'_p\} \times \{1\}$ be the other endpoint. Let $\{y_\varepsilon\} = \{e'_p\} \times \{\varepsilon\}$, where $\varepsilon \geq 0$. See Figure 3.

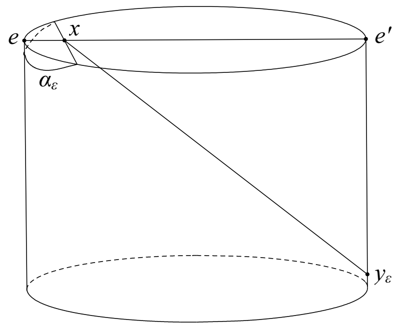

Figure 3: A convex cylinder without rq-convex boundary.

The plane H_{xy_ε} cuts $(\mathrm{bd}E) \times [0,1]$ along an arc α_ε of an ellipse. Let $f(\varepsilon) = d(\{x\}, \alpha_\varepsilon)$. The function f is increasing, and $f(0) > 0$.

The plane $H_{y_\varepsilon x}$ cuts $\mathrm{bd}Z$ along a closed curve (reduced to a single point if $\varepsilon = 0$), of diameter $g(\varepsilon)$. This function g is also increasing, and $g(0) = 0$.

Therefore, for $\varepsilon > 0$ small enough,

$$g(\varepsilon) < f(0) < f(\varepsilon).$$

Choose $y = y_\varepsilon$. The above inequalities show that there is no rectangle $xyy'x'$ with $x', y' \in \mathrm{bd}Z$.

Consider now the sphere C_{xy}. The set $C_{xy} \cap Z$ has four components: a component Z_1 containing x, another one Z_2 containing y, a third Z_3 containing the point e^* diametrically opposite to e' in C_{xy}, and a fourth, $\{e'\}$. It is easily seen that the only pairs of diametrically opposite points in $Z_1 \cup Z_2 \cup Z_3 \cup \{e'\}$ are (x, y) and (e', e^*). But $e^* \in \mathrm{int}Z$, so $\mathrm{bd}Z$ is not rq-convex. □

5 rq-convexity of finite sets

In these last two sections, we shall use the following notation. For $x, y \in \mathbb{R}^d$, we set $W_{xy} = H_{xy} \cup H_{yx} \cup C_{xy}$. Let $\mathcal{A}$ be the family of all finite point sets in $\mathbb{R}^2$.

Theorem 5.1. *For any set $A \in \mathcal{A}$ with $\mathrm{card}A = n \geq 3$, we have $\gamma(A) \leq n^2 - 2n$.*

Proof. If A is included in a line L, consider a line L' parallel to L and the orthogonal projection A' of A onto L'. Then obviously $A \cup A'$ is rq-convex and $\mathrm{card}A' = n \leq n^2 - 2n$, since $n \geq 3$.

If A is not included in any line, let $A = \{a_1, a_2, \ldots, a_n\}$, and assume that a_1a_2 is a side of the polygon $\mathrm{conv}A$. Obviously there are at most $n-2$ lines $L_1, \ldots, L_{n-2}$ passing through the remaining points of A and parallel to $\overline{a_1a_2}$. Also, there are at most n lines $L'_1, \ldots, L'_n$ passing through the points of A orthogonally onto $L_0 = \overline{a_1a_2}$.

The set
$$A' = \bigcup_{0 \le i \le n-2; 1 \le j \le n} (L_i \cap L'_j)$$
is obviously rq-convex and has at most $n(n-1)$ points, whence $\gamma(A) \le n^2 - 2n$. $\square$

Thus, for any $n \ge 3$, $\gamma(n) \le n^2 - 2n$. In particular, $\gamma(3) = 3$.

Theorem 5.2. *There are precisely two kinds of* 6*-point* rq*-convex sets in* $\mathcal{A}$*, shown in Figure 4.*

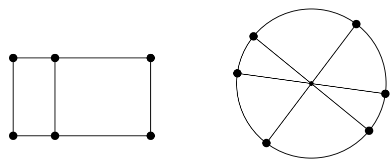

Figure 4: 6-point rq-convex sets.

Proof. Let $F = \{a, b, c, d, e, f\}$ be a 6-point rq-convex set. We assume without loss of generality that $\{a, b, c, d\} \in \mathcal{R}$, where $\angle abc = \frac{\pi}{2}$. By the definition of rq-convexity, e, f must meet one of the following seven conditions.

C_1. $\{\{e, f, a, b\}, \{e, f, c, d\}\} \subset \mathcal{R}$;

C_2. $\{\{e, f, a, c\}, \{e, f, b, d\}\} \subset \mathcal{R}$;

C_3. $\{\{e, f, a, d\}, \{e, f, b, c\}\} \subset \mathcal{R}$;

C_4. $\{\{e, f, a, b\}, \{e, f, a, c\}, \{e, f, a, d\}\} \subset \mathcal{R}$;

C_5. $\{\{e, f, b, a\}, \{e, f, b, c\}, \{e, f, b, d\}\} \subset \mathcal{R}$;

C_6. $\{\{e, f, c, a\}, \{e, f, c, b\}, \{e, f, c, d\}\} \subset \mathcal{R}$;

C_7. $\{\{e, f, d, a\}, \{e, f, d, b\}, \{e, f, d, c\}\} \subset \mathcal{R}$.

Clearly, C_1 and C_3 generate the same kind of set F, and so do C_4, C_5, C_6 and C_7.

Case 1: e, f satisfy C_1. By the definition of rq-convexity, $e, f \in W_{ab} \cap W_{cd}$, and so $e, f \in (H_{ab} \cup H_{ba}) \setminus \{a, b, c, d\}$ and ef, ab are parallel. Without loss of generality, we may suppose $e \in H_{ab}$, $f \in H_{ba}$, which leads to the three solutions depicted in Figure 5, all of them providing a 6-point set of the first type.

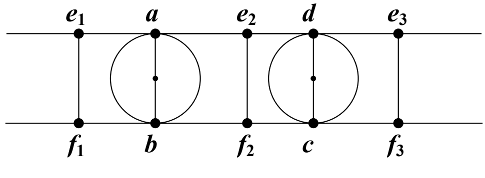

Figure 5: e, f satisfy C_1.

Case 2: e, f satisfy C_2. By the definition of rq-convexity, we have $e, f \in W_{ac} \cap W_{bd}$, so e, f are antipodal points of C_{ac}; see Figure 6.

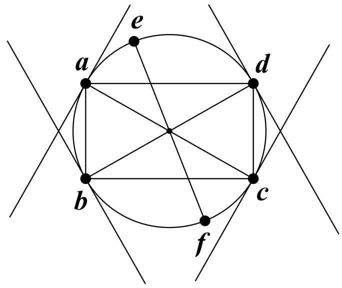

Figure 6: e, f satisfy C_2.

Case 3: e, f satisfy C_4. By the definition of rq-convexity, $(e,\ f) \in W_{ab} \cap W_{ac} \cap W_{ad}$. But $W_{ab} \cap W_{ac} \cap W_{ad} = \{a, b, d\}$, so we obtain no solution in this case. See Figure 7. □

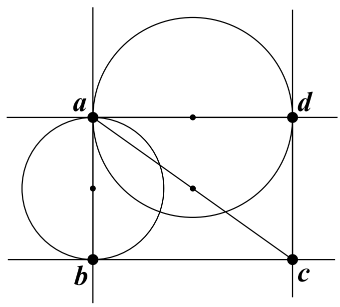

Figure 7: e, f satisfy C_3.

Theorem 5.3. *There are precisely three kinds of* 8*-point* rq*-convex sets, shown in Figure 8.*

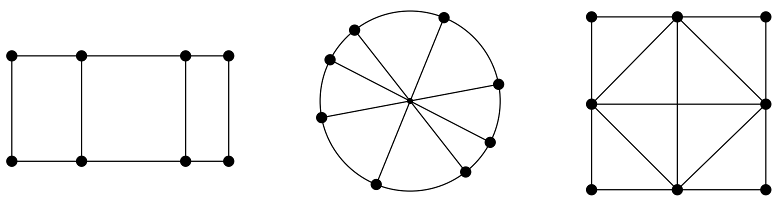

Figure 8: 8-point rq-convex sets.

The proof is ten pages long, so we decided not to include it into the paper. It is a case-by-case examination, treating separately those sets which contain a 6-point rq-convex subset and those which do not. It can be read in [4].

Theorem 5.4. *The smallest odd cardinality of an* rq*-convex set in* $\mathbb{R}^2$ *is* 9.

Proof. It is obvious that every rq-convex set contains a rectangular quadruple and quickly seen that no fifth point can be added to a rectangular quadruple keeping rq-convexity. Similarly, knowing what a 6-point rq-convex set looks like (Theorem 5.2), it is an easy exercise to establish that there is no 7-point rq-convex set containing a 6-point rq-convex set.

Next, we will consider the case that a 7-point rq-convex set does not contain any 6-point rq-convex subset. Let $F = \{a, b, c, d, e, f, g\}$ be such a set. Suppose a, c realise the diameter of F. Since F is rq-convex, there is another pair of antipodal points of C_{ac} in F, say $\{b, d\}$. Hence $\{a, b, c, d\} \in \mathcal{R}$, and put $\mathrm{conv}\{a, b, c, d\} = R$.

For the set of points $\{x, y\} \subset \{e, f, g\}$, if there exist $z, w \in \{a, b, c, d\}$ such that $\{w, x, y, z\} \in \mathcal{R}$, then we say that $\{x, y\}$ is *rq-good*. Next we will prove that for any two points $x, y \in \{e, f, g\}$, $\{x, y\}$ is not rq-good. Suppose $\{e, f\}$ is rq-good. Then there exist $z, w \in \{a, b, c, d\}$, such that $\{e, f, z, w\} \in \mathcal{R}$.

Case 1: zw is a diagonal of R. Without loss of generality, we assume $zw = ac$, so $e, f \in W_{ac}$. Since $\{a, c\}$ realise the diameter of F, e, f are antipodal on C_{ac}. But so we obtain a 6-point rq-convex subset of F, which contradicts our assumption about F.

Case 2: zw is an edge of R. We assume without loss of generality that $zw = ab$. Then $e, f \in W_{ab}$. Clearly, e, f must be antipodal points of C_{ab}. Take a diameter a_0b_0 of C_{ab} orthogonal to $\overline{ab}$, such that $\overline{ab}$ separates a_0 from cd. We may suppose that e belongs to the (smaller) arc of C_{ab} from a to a_0; see Figure 9. If $\|a-b\| \le \|b-c\|$, then $\|c-e\| > \|c-a\|$,

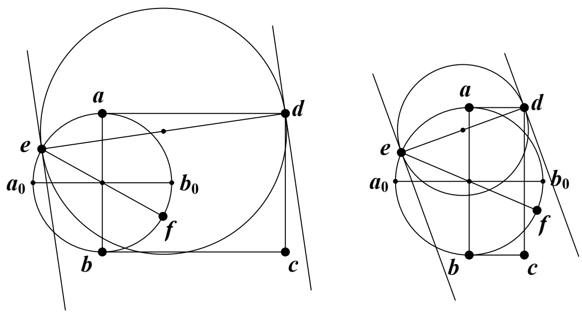

Figure 9: zw is an edge of R.

which is impossible. So, $\|a - b\| > \|b - c\|$.

As F is rq-convex, $\{e, d\}$ enjoys the rq-property and there exist $p, q \in \{a, b, c, f, g\}$ such that $p, q \in W_{ed}$. Clearly, $a, b, c \notin H_{de}$. Since $d \notin C_{ab}$, we have $f \notin H_{de}$. Further, we easily verify that $a, b, c, f \notin H_{ed} \cup H_{de}$. It follows that p, q are two antipodal points of C_{ed}. Since $\angle dae > \frac{\pi}{2}$, $\angle dce < \frac{\pi}{2}$, we get $a, c \notin C_{ed}$. Hence, $p = b \in C_{ed}$ or $p = f \in C_{ed}$, and $q = g$.

(i) $p = b \in C_{ed}$. We only can choose g such that b, g are antipodal points of C_{ed}. As $\angle ebf = \frac{\pi}{2}$, b, f, d are collinear. But then we get a 6-point rq-convex set $\{e, b, a, f, g, d\} \subset F$, contradicting our choice of F; see Figure 10.

(ii) $p = f \in C_{ed}$. Now, f, g are antipodal points of C_{ed}. Hence, $efdg$ is a rectangle. The points g and b are separated by $\overline{ad}$, or $a \in dg$ if $e = a_0$. Also,

$$\|d - g\| = \|e - f\| = \|a - b\| > \|b - c\| = \|a - d\|.$$

It follows that $\|c - g\| > \|c - a\|$, and a contradiction is obtained. See Figure 11. Hence,

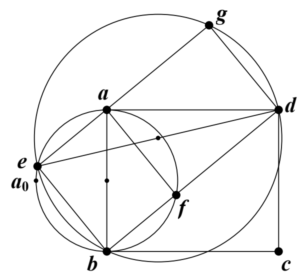

Figure 10: $p = b \in C_{ed}$.

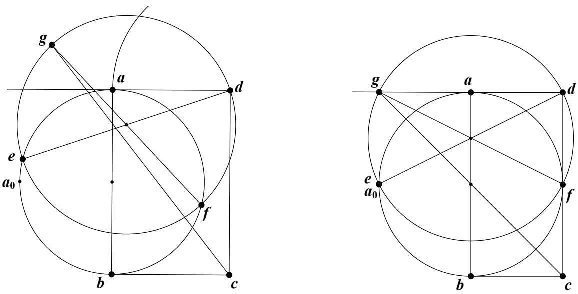

Figure 11: $p = f \in C_{ed}$.

$\{e, f\}$ is not rq-good.

Since F is rq-convex, we must have $\{e, f, g, a\}$, $\{e, f, g, b\}$, $\{e, f, g, c\}$, $\{e, f, g, d\} \in \mathcal{R}$, which is true only for $a = b = c = d$, which is impossible. Thus, there is no 7-point rq-convex set.

On the other hand, a 9-point rq-convex set is easily produced: just take the intersection $L \cap L'$, where L is the union of three horizontal lines and L' the union of three vertical lines. $\square$

Consider now the square lattice $\mathbb{Z}^2$, and the usual norm

$$\|(x, y)\|_m = \max\{|x|, |y|\},$$

defining in $\mathbb{Z}^2$ the discs of radius $n \in \mathbb{Z}$

$$Q(n) = \{(x, y) : \|(x, y)\|_m \leq n\},$$

centred at the origin $\mathbf{0}$. The subset $Q(n) \setminus Q(n-1)$ will be called *boundary of* $Q(n)$, and $Q(n-1)$ its *interior*. Obviously, $Q(n)$ is rq-convex, for any $n \geq 1$, and so is its boundary too. Does $Q(n)$ remain rq-convex if one deletes parts of its interior (but not all of it)?

Theorem 5.5. *The set* $Q(n) \setminus \{\mathbf{0}\}$ *is* rq*-convex.*

Proof. For any pair of points $x = (x_1, x_2)$, $y = (y_1, y_2)$ in $Q(n) \setminus \{\mathbf{0}\}$, consider the points $(x_1, y_2), (y_1, x_2) \in \mathbb{Z}^2$. If none of them is $\mathbf{0}$, the rq-propertyis verified at (x, y), as $(x_1, y_2), (y_1, x_2) \in Q(n)$.

Otherwise, assume without loss of generality that $x_1 = y_2 = 0$. We can also assume that both x_2, y_1 are positive, the other cases being symmetrical. Consider the points $x' = (-x_2, x_2 - y_1)$ and $y' = (y_1 - x_2, -y_1)$. Then $x', y' \in Q(n) \setminus \{\mathbf{0}\}$ and x, y, y', x' are the vertices of a square. □

Perhaps removing several layers of the boundary, thereby giving a set $Q(n) \setminus Q(m)$ for $m < n - 1$, will provide an rq-convex set?

Theorem 5.6. *The set $Q(n) \setminus Q(n-2)$ is not rq-convex, for any $n \geq 3$.*

Proof. Assume first that $n > 3$. Consider the points $x = (n, 2-n)$ and $y = (n-3, n-1)$. The point $(n-3, 2-n)$ does not belong to $Q(n) \setminus Q(n-2)$. The line H_{yx} meets $Q(n)$ again at $y' = (-n, n-4)$. But the fourth vertex $x' = (3-n, -n-1)$ of the square $xyy'x'$ lies outside $Q(n)$, whence $Q(n) \setminus Q(n-2)$ is not rq-convex.

Now, consider the set $Q(3) \setminus Q(1)$. In this case take the points $x = (3, -1)$, $y = (-1, 2)$. As $(-1, -1) \notin Q(3) \setminus Q(1)$ and $H_{yx} \cap Q(3) \setminus Q(1) = \emptyset$, the result is proven. □

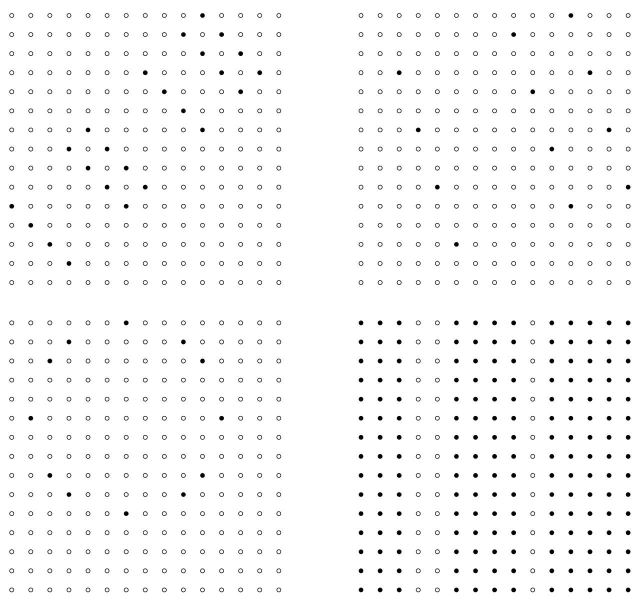

Figure 12: rq-convex proper subsets of $Q(n)$.

It seems that $Q(n) \setminus \{\mathbf{0}\}$ is the only rq-convex proper subset of $Q(n)$ containing the boundary of $Q(n)$ but different from it. However, this is not proven. Other proper subsets of $Q(n)$ which are rq-convex abound. For some examples, see Figure 12, where the solid black dots form rq-convex proper subsets of $Q(n)$.

6 rq-convexity of the vertex sets of Platonic solids

Due to their symmetry, the vertex sets of the cube, regular octahedron, regular dodecahedron, and regular icosahedron are all rq-convex. Among the Platonic solids, only the regular tetrahedron lacks this property. But what is the rq-convex completion number of the vertex set of the regular tetrahedron?

Theorem 6.1. *The rq-convex completion number of the vertex set of the regular tetrahedron is* 3.

Proof. Let $T = \{a, b, c, d\}$ denote the vertex set of a regular tetrahedron in $\mathbb{R}^3$. Obviously, for any $x,\ y \in T$, we have $T \cap W_{xy} = \{x, y\}$. Also, it is easily seen that there is no 5-point rq-convex set containing T. Suppose there is a 6-point rq-convex set $\{a, b, c, d, x, y\}$. The only suitable pair of points $x,\ y \in W_{ab} \cap W_{cd}$ is obtained when $\{x, y\} = (H_{ab} \cap C_{cd}) \cup (H_{ba} \cap C_{cd})$. But then b, c do not enjoy the rq-property in $\{a, b, c, d, x, y\}$. Hence $\gamma(T) \geq 3$.

Next, we only need to prove $\gamma(T) \leq 3$. Let $a_1,\ b_1,\ c_1$ denote the midpoints of $ad,\ bd,\ cd$, respectively. See Figure 13. The line $L_a \ni a_1$ parallel to $\overline{bc}$ and the anal-

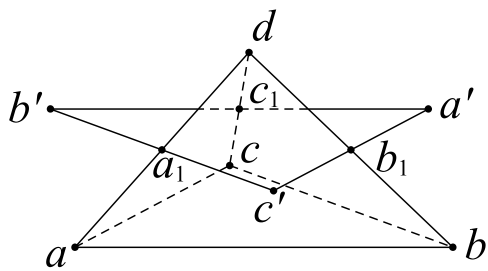

Figure 13: A 7-point rq-convex set containing the vertices of a regular tetrahedron.

ogous lines L_b and L_c are coplanar. Put

$$\{a'\} = L_b \cap L_c, \quad \{b'\} = L_c \cap L_a, \quad \{c'\} = L_a \cap L_b.$$

Obviously, $ab'dc', bc'da', ca'db'$ are squares, while $a'b'ab, b'c'bc, c'a'ca$ are rectangles. Thus, $\{a, b, c, d, a', b', c'\}$ is a 7-point rq-convex set, and $\gamma(T) = 3$. $\square$

Theorem 6.1 reveals the existence of 7-point rq-convex sets in $\mathbb{R}^3$, in contrast with the inexistence of such sets in $\mathbb{R}^2$. What happens in higher dimensions?

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