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Spectra of signed graphs and related oriented graphs

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Abstract

For every oriented graph G', there exists a bipartite signed graph $\dot{H}$ such that the spectrum of $\dot{H}$ contains the full information about the spectrum of the skew adjacency matrix of G'. This allows us to transfer some problems concerning the skew eigenvalues of oriented graphs to the framework of signed graphs, where the theory of real symmetric matrices can be employed. In this paper, we continue the previous research by relating the characteristic polynomials, eigenspaces and the energy of G' to those of $\dot{H}$. Simultaneously, we address some open problems concerning the skew eigenvalues of oriented graphs.

Keywords: Oriented graph, signed graph, eigenvalues, characteristic polynomial, eigenspaces, energy.

Math. Subj. Class. (2020): 05C20, 05C22, 05C50

1 Introduction

For a finite simple undirected graph $G = (V, E)$, an *oriented graph* G' is a pair (G, σ'), where σ' is the edge *orientation* satisfying $\sigma'(ij) \in \{i, j\}$, for every $ij \in E$. Similarly, a *signed graph* $\dot{G}$ is a pair $(G, \dot{\sigma})$, where $\dot{\sigma}$ is the edge *signature* satisfying $\dot{\sigma}(ij) \in \{+1, -1\}$, for every $ij \in E$. In both cases, G is referred to as the *underlying graph*. The *order* n is the number of vertices of G. The edge set of G' consists of oriented edges, where the edge ij is oriented from i to j if $\sigma'(ij) = j$; this is designated by $i \to j$ (or $j \leftarrow i$). The edge set of $\dot{G}$ consists of positive and negative edges.

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The *skew adjacency matrix* $S_{G'} = (s_{ij})$ of G' is the $n \times n$ matrix such that $s_{ij} = 0$ if ij is not an edge of G', $s_{ij} = 1$ if $i \to j$, and $s_{ij} = -1$ otherwise. This matrix is skew symmetric and differs from the adjacency matrix of G' whose (i, j)-entry is 0 whenever $i \leftarrow j$. The characteristic polynomial, the eigenvalues and the spectrum of $S_{G'}$ are known as the *skew characteristic polynomial*, the *skew eigenvalues*, the *skew spectrum* of G', respectively. To easy language and be consistent with the forthcoming terminology for signed graphs, in this article we omit the prefix 'skew'. The spectrum of G' consists of purely imaginary numbers, the non-zero eigenvalues come as complex conjugates, and thus the rank of $S_{G'}$ is even. Also, $S_{G'}^{\intercal} S_{G'} = -S_{G'}^2$.

The *adjacency matrix* $A_{\dot{G}}$ of $\dot{G}$ is obtained from the standard $(0, 1)$-adjacency matrix of G by reversing the sign of all edges mapped to -1 by $\dot{\sigma}$. By the *characteristic polynomial*, the *eigenvalues* and the *spectrum* of $\dot{G}$ we mean the characteristic polynomial, the eigenvalues and the spectrum of $A_{\dot{G}}$, respectively. Since $A_{\dot{G}}$ is symmetric, its eigenvalues are real.

An *r-cube* or a *hypercube* Q_r is the r-regular graph of order 2^r with vertex set $\{0, 1\}^r$ (all possible binary r-tuples) in which two vertices are adjacent if and only if they differ in exactly one coordinate. Accordingly, an oriented (resp. signed) r-cube is an oriented (signed) graph underlined by Q_r. If Γ is either an oriented graph or a signed graph, then its *energy* $\mathcal{E}(\Gamma)$ is the sum of modulus of its eigenvalues.

It follows from [19] that every oriented graph G' is related to a bipartite signed graph $\dot{H}$ in such a way that the spectrum of $\dot{H}$ contains the full information about the spectrum of G'. All necessary details about this relation are given in the next section. This means that the theory of spectra of oriented graphs is strongly connected to the theory of spectra of signed graphs, and that many problems concerning spectral parameters of oriented graphs can be transferred to the framework of signed graphs, where the entire theory of real symmetric matrices can be employed. In this way, some known results on oriented graphs are proved in an elementary way [19], some open problems are resolved [15] and some known results concerning signed graphs are transferred to the context of oriented graphs [18].

In this article we continue the research by expressing the coefficients of the characteristic polynomial of G' in terms of the coefficients of the characteristic polynomial of $\dot{H}$. We also generate the eigenspaces of G' on the basis of the eigenvectors of $\dot{H}$ and relate the energies of both graphs. The results on characteristic polynomials and energies are of particular interest since they address some open problems posed in literature. The results on eigenspaces are followed by an immediate application in the engineering domain.

Section 2 contains additional terminology and notation, along with a short review of results of [19]. In particular, oriented graphs are related to signed graphs in the forthcoming Theorem 2.1. Some comments and results that arise directly from this theorem are given in Section 3. Characteristic polynomials, eigenspaces and energies are considered in Sections 4–6, respectively. Some notes on particular (oriented or signed) hypercubes are given in Section 7.

2 Preliminaries

We say that an oriented or a signed graph is connected, regular, or bipartite if the same holds for its underlying graph. Similarly, a matching (or a perfect matching) refers to the matching in the underlying graph.

Two oriented (resp. signed) graphs with the same underlying graph are switching equivalent if there is a subset U of the vertex set V, such that one of them is obtained by reversing the orientation (sign) of every edge located between U and $V \setminus U$. In matrix terminology, G_1' and G_2' are switching equivalent if there exists a diagonal matrix S with ± 1 on the main diagonal, such that $S_{G_2'} = S^{-1} S_{G_1'} S$, and similarly for signed graphs. S is referred to as the *switching matrix*. Observe that the spectrum remains unchanged under the switching operation.

We say that an oriented even cycle $C_{2\ell}'$ is *oriented uniformly* if by traversing along the cycle we pass through an odd (resp. even) number of edges oriented in the route direction for ℓ odd (even), where the 'route direction' refers to any of two possible directions: clockwise or counterclockwise. A *canonical orientation* in a bipartite graph G is the orientation which orients all the edges from one colour class to the other. Clearly, in this orientation every cycle is oriented uniformly.

A cycle $\dot{C}$ in a signed graph is *positive* if the product of its edge signs $\dot{\sigma}(\dot{C})$ is 1. Otherwise, it is *negative*. A signed graph is said to be *homogeneous* if all edges have the same sign, e.g., if its edge set is empty. It is *balanced* if it switches to its underlying graph; equivalently, it does not contain negative cycles. The negation $-\dot{G}$ of a signed graph is obtained by reversing the sign of every edge of $\dot{G}$. Observe that if $\dot{G}$ is bipartite, then it is switching equivalent to $-\dot{G}$ and they share the same spectrum.

We proceed with results of [19]. The signature $\dot{\sigma}$ is *associated* with the orientation σ' (and also $\dot{G}$ is *associated* with G') if

$$\dot{\sigma}(ik)\dot{\sigma}(jk) = s_{ik}s_{jk} \text{ holds for every pair of edges } ik \text{ and } jk. \tag{2.1}$$

Being associated is a symmetric relation. We write $\dot{\sigma} \sim \sigma'$ to indicate that $\dot{\sigma}$ and σ' are mutually associated. The following results hold: If $\dot{\sigma} \sim \sigma'$, then $-S_{G'}^2 = A_{\dot{G}}^2$. For a graph G and an orientation σ', there exists a signature $\dot{\sigma}$ associated with σ' if and only if G is bipartite.

The next result gives a crucial relation between the spectrum of an oriented graph and the spectrum of a related signed graph. If G' is an oriented graph, then its *bipartite double* $\mathrm{bd}(G')$ is the oriented graph whose skew adjacency matrix is the Kronecker product $S_{\mathrm{bd}(G')} = A_{K_2} \otimes S_{G'}$, where K_2 is the complete graph with 2 vertices. This definition extends the definition of a bipartite double of an ordinary graph, where $\mathrm{bd}(G)$ has $A_{\mathrm{bd}(G)} = A_{K_2} \otimes A_G$ as the adjacency matrix. Evidently, if G underlies G', then $\mathrm{bd}(G)$ underlies $\mathrm{bd}(G')$. In our notation, exponents denote multiplicities of the eigenvalues.

Theorem 2.1. *Let $G' = (G, \sigma')$ be an oriented graph with $\mathrm{rank}(S_{G'}) = 2k$. The following statements hold true:*

(i) *If G' is bipartite and $\dot{\sigma} \sim \sigma'$, then $\pm i\lambda_1, \pm i\lambda_2, \dots, \pm i\lambda_k, 0^{(n-2k)}$ are the eigenvalues of G' if and only if $\pm\lambda_1, \pm\lambda_2, \dots, \pm\lambda_k, 0^{(n-2k)}$ are the eigenvalues of $\dot{G} = (G, \dot{\sigma})$.*

(ii) *If G' is non-bipartite, $H' = (\mathrm{bd}(G), \sigma'')$ is a bipartite double of G' and $\dot{\sigma} \sim \sigma''$, then $\pm i\lambda_1, \pm i\lambda_2, \dots, \pm i\lambda_k, 0^{(n-2k)}$ are the eigenvalues of G' if and only if $(\pm\lambda_1)^{(2)}, (\pm\lambda_2)^{(2)}, \dots, (\pm\lambda_k)^{(2)}, 0^{(2n-4k)}$ are the eigenvalues of $\dot{H} = (\mathrm{bd}(G), \dot{\sigma})$.*

Therefore, G' is related to a bipartite signed graph whose spectrum gives the full information on the spectrum of G'. If G' is bipartite, a required signed graph is its associate. If G' is non-bipartite, then a required signed graph is associated with $\mathrm{bd}(G')$. One may

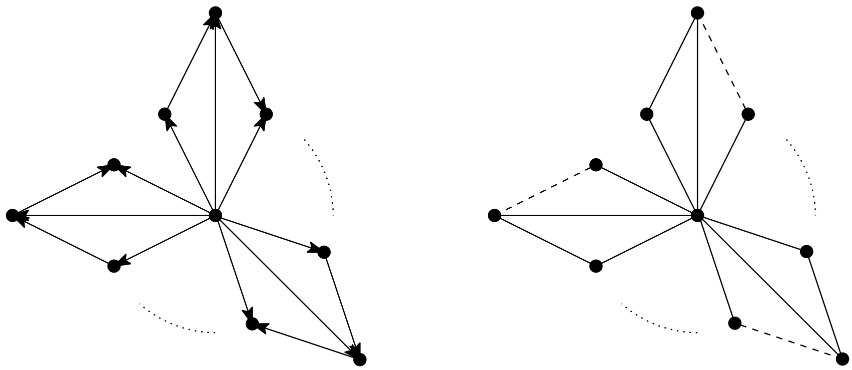

Figure 1: An infinite family of oriented graphs G' and signed graphs $\dot{G}$ such that $-S^2_{G'} = A^2_{\dot{G}}$. Negative edges are dashed.

notice that item (ii) of the previous theorem covers the bipartite case in the sense that, if G' is bipartite, then $\mathrm{bd}(G')$ consists of two copies of G' and $\dot{H}$ also has two identical copies, each associated with G'. However, this would lead to unnecessary complicating, as there is no need to deal with a bipartite double if G' is already bipartite. Thus, the bipartite case is separated in item (i) of the same theorem.

3 Comments to Theorem 2.1

In the bipartite case, G' is associated with $\dot{G}$ if and only if it is associated with $-\dot{G}$. Hence, G' actually has two associates (with the same spectrum). Similarly, $\dot{G}$ is associated with G' if and only if it is associated with the oriented graph obtained by reversing the orientation of every edge of G'. Again, $\dot{G}$ has two associates which share the same spectrum. Henceforth, when we say 'let $\dot{G}$ be a signed graph associated with G'' (or something similar), we always mean that $\dot{G}$ is allowed to take any of the two options (that differ up to negation).

We have pointed out in the previous section that $-S^2_{G'} = A^2_{\dot{G}}$ holds whenever G' and $\dot{G}$ are associated. However, this identity is not exclusively reserved for associated graphs. For example, Figure 1 illustrates infinite families of graphs that are not mutually associated in the sense of the equality (2.1) (since they are non-bipartite), but satisfy the previous matrix identity. In this case, they share the same underlying graph, but the identity can occur even if they do not. In fact, the identity occurs if and only if for every pair i, j of vertices, $-s^{(2)}_{ij} = a^{(2)}_{ij}$ holds, where an exponent indicates that we deal with the entry of matrix square. This leads to the following result.

Theorem 3.1. *If an oriented graph G' and a signed graph $\dot{G}$ are defined on the same vertex set, then $-S^2_{G'} = A^2_{\dot{G}}$ holds if and only if for every pair of vertices i, j*

$$\big|\{k : (i \to k \wedge j \to k) \vee (i \leftarrow k \wedge j \leftarrow k)\}\big| - \big|\{k : (i \to k \wedge j \leftarrow k) \vee (i \leftarrow k \wedge j \to k)\}\big|$$

in G' is equal to

$$\big|\{k : \dot{\sigma}(ik)\dot{\sigma}(jk) = 1\}\big| - \big|\{k : \dot{\sigma}(ik)\dot{\sigma}(jk) = -1\}\big|$$

in $\dot{G}$.

To demonstrate an application of Theorem 2.1, we deduce a known result. Observe that a bipartite canonically oriented graph is associated with a homogeneous signed graph, necessarily switching equivalent to its underlying graph, and so sharing the spectrum with it. Since bipartite oriented graphs are switching equivalent if and only if associated signed graphs are switching equivalent (where the equivalence is realized by the same switching matrix), we deduce the following result (conjectured in [7], and proved in [3]): If $\pm\mathrm{i}\lambda_1, \pm\mathrm{i}\lambda_2, \dots, \pm\mathrm{i}\lambda_n$ are the skew eigenvalues of a bipartite oriented graph $G' = (G, \sigma')$, then the eigenvalues of G are $\pm\lambda_1, \pm\lambda_2, \dots, \pm\lambda_n$ if and only if G' switches to a canonically oriented graph.

Here are more details on the cycle structure of $\dot{H}$ of Theorem 2.1(ii). It will be used in the forthcoming sections.

Theorem 3.2. *Let G' and $\dot{H}$ be as in* Theorem 2.1(ii). *For every odd cycle C' of G', the signed cycle $\dot{C}$ of $\dot{H}$ associated with $\mathrm{bd}(C')$ is negative.*

Proof. If the vertices of C', labelled in the natural order, are $1, 2, \dots, \ell$, then the vertices of its bipartite double $\mathrm{bd}(C')$ are divided into the colour classes, say $A = \{a_1, a_2, \dots, a_\ell\}$ and $B\{b_1, b_2, \dots, b_\ell\}$, the edges of $\mathrm{bd}(C')$ are $a_1b_2, b_2a_3, a_3b_4, \dots, b_{\ell-1}a_\ell, a_\ell b_1, b_1a_2, \dots, a_{\ell-1}b_\ell, b_\ell a_1$, and these edges inherit the orientation from G' in the sense that $i \to j$ implies $a_i \to b_j$ and $b_i \to a_j$.

Now, if the edges of G' are oriented in the route direction, so are the edges of $\mathrm{bd}(G')$. In this case, the edges of the associated signed cycle $\dot{C}$ alternate in sign, which in particular means that $\dot{C}$ is negative (as it counts exactly ℓ negative edges and ℓ is odd) and the edges a_ib_j and b_ia_j differ in sign (again, since ℓ is odd). If the edges of G' are not oriented in the route direction, then C' is obtained from the previous cycle by reversing the orientation of some edges. The desired conclusion follows since changing the orientation of a single edge ij changes the orientation of both a_ib_j and b_ia_j (in $\mathrm{bd}(C')$) and changes the sign of both a_ib_j and b_ia_j (in $\dot{C}$), so it does not change the signature of $\dot{C}$. □

Corollary 3.3. *Let G' and $\dot{H}$ be as in* Theorem 2.1(ii). *Then $\dot{H}$ is unbalanced and* $\mathrm{bd}(G')$ *contains a cycle that is not oriented uniformly.*

Proof. The first statement follows from the previous theorem. The second one follows from an easy observation that a negative cycle in $\dot{H}$ corresponds to a cycle in $\mathrm{bd}(G')$ that is not oriented uniformly. □

4 Characteristic polynomials

Here are relations between the coefficients of the characteristic polynomials.

Theorem 4.1. *If $\sum_{i=0}^{n} s_i x^{n-i}$ is the characteristic polynomial of an oriented graph G', then $s_i = 0$ for i odd and $s_i \geq 0$ for i even. If G' is bipartite and $\sum_{i=0}^{\lfloor n/2 \rfloor} a_{2i}x^{n-2i}$ is the characteristic polynomial of an associated signed graph, then*

$$a_{2i} = (-1)^i s_{2i}. \tag{4.1}$$

If G' is non-bipartite and $\sum_{i=0}^{n} a_{2i}x^{n-2i}$ is the characteristic polynomial of $\dot{H}$ (where $\dot{H}$ is as in the formulation of Theorem 2.1(ii)*), then*

$$a_{2i} = (-1)^i \sum_{\substack{\ell = -\min\{i, n-i\} \\ i \equiv \ell \ (\mathrm{mod}\ 2)}}^{\min\{i,n-i\}} s_{i-\ell} s_{i+\ell}. \tag{4.2}$$

Proof. Under the assumptions of Theorem 2.1, the characteristic polynomial of the skew adjacency matrix of G' is

$$x^n + s_1 x^{n-1} + \cdots + s_{n-1} x + s_n = x^{n-2k}(x^2 + \lambda_1^2)(x^2 + \lambda_2^2) \cdots (x^2 + \lambda_k^2). \tag{4.3}$$

We immediately obtain $s_i = 0$ for i odd (since s_i is the ith elementary symmetric polynomial in eigenvalues), and $s_i \geq 0$ for i even.

If G' is bipartite, then the characteristic polynomial of an associated signed graph reads $\sum_{i=0}^{\lfloor n/2 \rfloor} a_{2i} x^{n-2i} = x^{n-2k}(x^2 - \lambda_1^2)(x^2 - \lambda_2^2) \cdots (x^2 - \lambda_k^2)$, which yields that even coefficients alternate in sign, and $|a_{2i}| = s_{2i}$. This implies the equality (4.1).

If G' is non-bipartite, then the characteristic polynomial of $\dot{H}$ is

$$\sum_{i=0}^{n} a_{2i} x^{n-2i} = \left(x^{n-2k}(x^2 - \lambda_1^2)(x^2 - \lambda_2^2) \cdots (x^2 - \lambda_k^2)\right)^2.$$

Comparing it with (4.3), we arrive at

$$a_{2i} = (-1)^i \sum_{\ell = -\min\{i,n-i\}}^{\min\{i,n-i\}} s_{i-\ell} s_{i+\ell},$$

but we know that odd coefficients under the sum are zero, so the summation reduces to even coefficients, which leads to (4.2). □

For $2i \leq n$, we note that the formula (4.2) is simplified to

$$a_{2i} = (-1)^i \sum_{\ell=0}^{i} s_{2\ell} s_{2(i-\ell)}.$$

We visualize this in the following example.

Example 4.2. Let G' be the oriented graph with 10 vertices illustrated in Figure 1. The coefficients of its characteristic polynomial are $(s_0, s_2, \ldots, s_{10}) = (1, 15, 60, 92, 48, 0)$. The coefficients of the characteristic polynomial of $\dot{H}$ of Theorem 2.1(ii) are

$$(a_0, a_2, \ldots, a_{20}) = (1, -30, 345, -1984, 6456, -12480, 14224, -8832, 2304, 0, 0).$$

Say, $345 = a_4 = s_0 s_4 + s_2^2 + s_4 s_0 = 60 + 225 + 60$ or $-8832 = a_{14} = (-1)^7(s_{10}s_4 + s_8 s_6 + s_6 s_8 + s_4 s_{10}) = 0 - 2 \cdot 92 \cdot 48 + 0$, as in Theorem 4.1.

We recall that a basic figure in a graph is a disjoint union of edges and cycles. If $\sum_{i=0}^{n} a_i x^i$ is the characteristic polynomial of the adjacency matrix of a signed graph $\dot{G} = (G, \dot{\sigma})$, then from [4]

$$a_i = \sum_{B \in \mathcal{B}_i} (-1)^{p(B)} \dot{\sigma}(B) 2^{|c(B)|},$$

where $\mathcal{B}_i$ is the set of basic figures on i vertices in G, $p(B)$ is the number of components of B, $c(B)$ is the set of cycles in B and $\dot{\sigma}(B)=\prod_{\dot{C}\in c(B)}\dot{\sigma}(\dot{C})$. This result can be extended to oriented graphs on the basis of Theorems 2.1 and 4.1. However, this has already been performed in [5, pages 4516–4517] and [9, Theorem 2.3], and so we just refer the reader to these references. It is worth mentioning that these results address the research problem of [1, Section 6], asking for an interpretation of the coefficients s_i in terms of G'.

We conclude this section by the following observation. If the characteristic polynomials of G (the underlying graph), $\dot{G}=(G,\dot{\sigma})$ (a signed graph) and $G'=(G,\sigma')$ (an oriented graph) are $\sum_{i=0}^{n}a_i(G)x^{n-i}$, $\sum_{i=0}^{n}a_i(\dot{G})x^{n-i}$ and $\sum_{i=0}^{n}s_i(G')x^{n-i}$, respectively, then $a_i(G)=a_i(\dot{G})=s_i(G')\pmod 2$, for $1\leq i\leq n$, as $A_G=A_{\dot{G}}=S_{G'}\pmod 2$. Since $s_i(G')=0$ for i odd, this in particular means that $a_i(G)$ and $a_i(\dot{G})$ are even, whenever i is odd. Moreover, we have the following consequence.

Theorem 4.3. *For a graph G, let $\dot{\mathcal{G}}$ (resp. $\mathcal{G}'$) consist of all signed graphs (oriented graphs) having G as the underlying graph. If there is at least one $\Gamma\in\{G\}\cup\dot{\mathcal{G}}\cup\mathcal{G}'$, such that the determinant of Γ is odd, then G, all signed graphs of $\dot{\mathcal{G}}$ and all oriented graphs of $\mathcal{G}'$ are non-singular (i.e. their determinant is non-zero).*

Proof. This result follows from the previous observation applied to $a_0(G), a_0(\dot{G})$ and $s_0(G')$. Indeed, if for example $a_0(\dot{G})$ is odd, then $a_0(\dot{G})=a_0(G)=1\pmod 2$, and the same holds for the elements of $\dot{\mathcal{G}}\cup\mathcal{G}'$ in the role of G. □

5 Eigenspaces

Let $\mathcal{E}(\lambda)$ and $\mathcal{E}(-\lambda)$ be the eigenspaces of eigenvalues λ ($\neq 0$) and $-\lambda$ of a bipartite signed graph $\dot{G}$, and $\mathcal{E}(\mathrm{i}\lambda)$, $\mathcal{E}(-\mathrm{i}\lambda)$ the eigenspaces of $\mathrm{i}\lambda$ and $-\mathrm{i}\lambda$ belonging to the spectrum of an associated oriented graph G'. Then $\mathcal{E}(\mathrm{i}\lambda)\cup\mathcal{E}(-\mathrm{i}\lambda)$ is spanned (in $\mathbb{C}^n$) by the union of bases of $\mathcal{E}(\lambda)$ and $\mathcal{E}(-\lambda)$. Indeed, since $-S_{G'}^2=A_{\dot{G}}^2$, the eigenspace of λ^2 for $A_{\dot{G}}^2$ coincides with the eigenspace of $-\lambda^2$ for S^2, and thus the former eigenspace is spanned by the union of eigenbases of λ and $-\lambda$ (for $A_{\dot{G}}$), and the latter one is spanned by the union of eigenbases of $\mathrm{i}\lambda$ and $-\mathrm{i}\lambda$ (for $S_{G'}$). By the same reasoning, if 0 is an eigenvalue of $\dot{G}$, then it has the same eigenspace for $\dot{G}$ and G'. However, we can say more.

We first note the following, probably known, result. If $\mathbf{x}$ is an eigenvector associated with the eigenvalue λ ($\neq 0$) of a bipartite signed graph $\dot{G}$, then by negating the entries on one colour class, we get an eigenvector associated with $-\lambda$. Indeed, with an appropriate vertex labelling, the adjacency matrix has the form

$$A_{\dot{G}}=\begin{pmatrix} O & B\\ B^{\intercal} & O\end{pmatrix} \tag{5.1}$$

If $\mathbf{x}=\begin{pmatrix}\mathbf{x}_1\\ \mathbf{x}_2\end{pmatrix}$, then $A_{\dot{G}}\begin{pmatrix}-\mathbf{x}_1\\ \mathbf{x}_2\end{pmatrix}=\begin{pmatrix}B\mathbf{x}_2\\ -B^{\intercal}\mathbf{x}_1\end{pmatrix}=-\lambda\begin{pmatrix}-\mathbf{x}_1\\ \mathbf{x}_2\end{pmatrix}$, as desired. If $A_{\dot{G}}$ has no the previous form, the eigenvectors are permutationally equal, and the result follows. Henceforth, we assume that the adjacency matrix of a bipartite signed graph is as in (5.1).

Theorem 5.1. *Let $\mathbf{x}_j=\begin{pmatrix}\mathbf{x}_1\\ \mathbf{x}_2\end{pmatrix}$ and $\mathbf{x}_{-j}=\begin{pmatrix}-\mathbf{x}_1\\ \mathbf{x}_2\end{pmatrix}$ be eigenvectors associated with distinct eigenvalues λ_j and $-\lambda_j$ of a bipartite signed graph $\dot{G}$, and let, with a consistent vertex labelling, G' be an associated oriented graph. Then $\mathbf{x}_j+\mathrm{i}\mathbf{x}_{-j}$ and $\mathbf{x}_j-\mathrm{i}\mathbf{x}_{-j}$ are eigenvectors associated with $\mathrm{i}\lambda_j$ and $-\mathrm{i}\lambda_j$.*

Proof. If the adjacency matrix of $\dot{G}$ has the form (5.1), then the skew adjacency matrix of G' is $S_{G'} = \begin{pmatrix} O & B \\ -B^\intercal & O \end{pmatrix}$. Indeed, in the adjacency matrix of the underlying graph G, B has no negative entries, and reversing the sign of an edge uv changes the sign of both a_{uv}, a_{vu} in $A_{\dot{G}}$ and both s_{uv}, s_{vu} in $S_{G'}$.

We compute

$$\begin{aligned} S_{G'}(\mathbf{x}_j \pm \mathrm{i}\mathbf{x}_{-j}) &= \begin{pmatrix} B\mathbf{x}_2 \\ -B^\intercal \mathbf{x}_1 \end{pmatrix} \pm \mathrm{i} \begin{pmatrix} B\mathbf{x}_2 \\ B^\intercal \mathbf{x}_1 \end{pmatrix} = \begin{pmatrix} \lambda_j \mathbf{x}_1 \\ -\lambda_j \mathbf{x}_2 \end{pmatrix} \pm \mathrm{i} \begin{pmatrix} \lambda_j \mathbf{x}_1 \\ \lambda_j \mathbf{x}_2 \end{pmatrix} \\ &= \pm \mathrm{i}\lambda_j \left(\begin{pmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{pmatrix} \pm \mathrm{i} \begin{pmatrix} -\mathbf{x}_1 \\ \mathbf{x}_2 \end{pmatrix} \right) = \pm \mathrm{i}\lambda_j (\mathbf{x}_j \pm \mathrm{i}\mathbf{x}_{-j}), \end{aligned}$$

as desired. □

The previous theorem tells us that, in the bipartite case, the eigenspaces of $\mathrm{i}\lambda_j$ and $-\mathrm{i}\lambda_j$ of G' contain vectors of the form

$$\mathbf{y}_{\mathrm{i}j} = \begin{pmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{pmatrix} + \mathrm{i} \begin{pmatrix} -\mathbf{x}_1 \\ \mathbf{x}_2 \end{pmatrix} \text{ and } \mathbf{y}_{-\mathrm{i}j} = \overline{\mathbf{y}}_{\mathrm{i}j} = \begin{pmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{pmatrix} - \mathrm{i} \begin{pmatrix} -\mathbf{x}_1 \\ \mathbf{x}_2 \end{pmatrix},$$

respectively. Conversely, if these eigenvectors are given, then the eigenvectors associated with λ_j and $-\lambda_j$ of $\dot{G}$ are easily computed. We proceed with the non-bipartite case.

Theorem 5.2. *Let G' be a non-bipartite oriented graph and $\mathbf{y}$ and $\overline{\mathbf{y}}$ eigenvectors associated with the eigenvalues $\mathrm{i}\lambda_j$ and $-\mathrm{i}\lambda_j$, respectively. Then the pair $\begin{pmatrix} \mathbf{y} \\ \mathbf{y} \end{pmatrix}, \begin{pmatrix} \mathbf{y} \\ -\mathbf{y} \end{pmatrix}$ (resp. $\begin{pmatrix} \overline{\mathbf{y}} \\ \overline{\mathbf{y}} \end{pmatrix}, \begin{pmatrix} \overline{\mathbf{y}} \\ -\overline{\mathbf{y}} \end{pmatrix}$) is associated with $\mathrm{i}\lambda_j$ (resp. $-\mathrm{i}\lambda_j$) in the bipartite double $\mathrm{bd}(G')$, where $S_{\mathrm{bd}(G')} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes S_{G'}$. Let $\dot{H}$ be a signed graph associated with $\mathrm{bd}(G')$, and k the multiplicity of λ_j in the spectrum of $\dot{H}$. The first (resp. second) pair is spanned by $\mathbf{x}_{j\ell} + \mathrm{i}\mathbf{x}_{-j\ell}$ (resp. $\mathbf{x}_{j\ell} - \mathrm{i}\mathbf{x}_{-j\ell}$), for $1 \leq \ell \leq k$, where $\mathbf{x}_{j\ell}$ and $\mathbf{x}_{-j\ell}$ span the eigenspaces of λ_j and $-\lambda_j$, respectively.*

Proof. Since $S_{\mathrm{bd}(G')}$ is the tensor product, the eiegnvectors of $\mathrm{i}\lambda_j$ for $\mathrm{bd}(G')$ are $(1,1)^\intercal \otimes \mathbf{y}$ and $(1,-1)^\intercal \otimes \mathbf{y}$, and similarly for $-\mathrm{i}\lambda_j$. This establishes the proof of the first part of the statement.

By Theorem 5.1, the eigenvectors of $\mathrm{i}\lambda_j$ for $\mathrm{bd}(G')$ are $\mathbf{x}_{j\ell} + \mathrm{i}\mathbf{x}_{-j\ell}$, for $1 \leq \ell \leq k$. They are obviously linearly independent (because $\mathbf{x}_{j\ell}$ and $\mathbf{x}_{-j\ell}$ are). Since the multiplicity of $\mathrm{i}\lambda_j$ in the spectrum of $\mathrm{bd}(G')$ is k (cf. Theorem 2.1(ii)), its geometric multiplicity is at most k and the desired conclusion follows. □

We proceed with an application in control theory. The following differential equation is the standard model of multi-agent single-input linear control systems:

$$\frac{d\mathbf{x}}{dt} = M\mathbf{x} + \mathbf{b}u, \tag{5.2}$$

where the scalar $u = u(t)$ is the control input, $M \in \mathbb{R}^n \times \mathbb{R}^n$ and $\mathbf{x}, \mathbf{b} \in \mathbb{R}^n$. The system is controllable if for any $\mathbf{x}^\star$ and time $t^\star$, there exists a control function $u(t)$, $0 \leq t \leq t^\star$, such that the solution of the differential equation gives $\mathbf{x}^\star = \mathbf{x}(t^\star)$ irrespective of $\mathbf{x}(0)$. There are many controllable criteria, one of which is the Popov-Belevitch-Hautus Test to be found in any of [6, 11]. Accordingly, the system is controllable if and only if there is no $\mathbf{z} \in \mathbb{C}^n \setminus \{\mathbf{0}\}$ such that $\mathbf{z}^*M = \lambda\mathbf{z}^*$ and $\mathbf{z}^*\mathbf{b} = 0$, where $*$ denotes the complex conjugate.

Theorem 5.3. *Let A be the adjacency matrix of a bipartite signed graph and S the skew adjacency matrix of an associated oriented graph. If the system (5.2) with $M = A$ is controllable, then it is controllable with $M = S$.*

Proof. Since the system with $M = A$ is controllable, we have $\mathbf{x}^\intercal\mathbf{b} \neq 0$ for every eigenvector of A. Moreover A has no repeated eigenvalues. Indeed, if $\mathbf{x}_1$ and $\mathbf{x}_2$ are linearly independent eigenvectors associated with the same eigenvalue, with $\mathbf{x}_1^\intercal\mathbf{b} = b_1 \neq 0$ and $\mathbf{x}_2^\intercal\mathbf{b} = b_2 \neq 0$, then $(b_2\mathbf{x}_1 - b_1\mathbf{x}_2)^\intercal\mathbf{b} = 0$.

As $\mathbf{z}^*S = \lambda\mathbf{z}^*$ is equivalent to $S\mathbf{z} = \lambda\mathbf{z}$, and $\mathbf{z}$ is an eigenvector for S if and only if $\mathbf{z}^*$ is an eigenvector for the same matrix, we have to show that for every eigenvector $\mathbf{y}$ of S, $\mathbf{y}^*\mathbf{b} = 0$ holds.

Since A and S share the same eigenspace for 0, the claim holds for eigenvectors associated with 0 (if any). Every eigenvector associated with $\mathrm{i}\lambda_j$ has the form $z(\mathbf{x}_j + \mathrm{i}\mathbf{x}_{-j})$, where $z = z_1 + \mathrm{i}z_2 \neq 0$, and $\mathbf{x}_j$ and $\mathbf{x}_{-j}$ are as in the formulation of Theorem 5.1. As before, we may take $\mathbf{x}_j^\intercal\mathbf{b} = b_1 \neq 0$ and $\mathbf{x}_{-j}^\intercal\mathbf{b} = b_2 \neq 0$. Then

$$\begin{aligned}(z(\mathbf{x}_j + \mathrm{i}\mathbf{x}_{-j}))^*\mathbf{b} &= \big(z_1\mathbf{x}_j - z_2\mathbf{x}_{-j} + \mathrm{i}(z_1\mathbf{x}_{-j} + z_2\mathbf{x}_j)\big)^*\mathbf{b} \\ &= (z_1\mathbf{x}_j - z_2\mathbf{x}_{-j})^\intercal\mathbf{b} - \mathrm{i}(z_1\mathbf{x}_{-j} + z_2\mathbf{x}_j)^\intercal\mathbf{b} \\ &= z1b_1 - z_2b_2 + \mathrm{i}(z_1b_2 + z_2b_1).\end{aligned}$$

Equating with 0, we obtain $z_1 = \frac{b_2}{b_1}z_2$ and $z_2 = -\frac{b_2}{b_1}z_1$, which leads to the impossible scenario $z_1 = z_2 = 0$. Hence, the system with $M = S$ is controllable. □

6 Energy

We start with the following result concerning signed graphs. There is a similar result for oriented graphs reported in [1].

Theorem 6.1. *Let $\mathcal{E}(\dot{G})$ be the energy of a signed graph $\dot{G}$ having n vertices, m edges, average vertex degree $\overline{d} = \frac{2m}{n}$ and eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$. Then*

$$\sqrt{2\left(m + \binom{n}{2}\Big(\prod_{i=1}^{n}|\lambda_i|\Big)^{\frac{2}{n}}\right)} \leq \mathcal{E}(\dot{G}) \leq n\sqrt{\overline{d}}. \tag{6.1}$$

Both equalities hold if and only if $\dot{G}$ is either edgeless or has exactly two distinct eigenvalues and these eigenvalues are equal in absolute value.

Proof. The proof of inequalities of (6.1) is an imitation of the proof of [1, Theorem 2.5] (concerning oriented graphs). Namely, both follow from the next chain of inequalities and equalities:

$$2\Big(m+\binom{n}{2}\Big(\prod_{i=1}^{n}|\lambda_i|\Big)^{\frac{2}{n}}\Big)\leq\sum_{i=1}^{n}|\lambda_i|^2+2\sum_{1\leq i<j\leq n}|\lambda_i||\lambda_j|=\Big(\sum_{i=1}^{n}|\lambda_i|\Big)^2$$
$$\leq n\sum_{i=1}^{n}|\lambda_i|^2=2mn=n^2\overline{d}.$$

To clarify, the first inequality in above chain follows from the inequality between the geometric mean and the arithmetic mean, in our case:

$$\Big(\prod_{i=1}^{n}|\lambda_i|\Big)^{\frac{2}{n}}=\sqrt[n(n-1)]{\prod_{i=1}^{n}|\lambda_i|^{2(n-1)}}=\sqrt[n(n-1)]{\prod_{1\leq i<j\leq n}(|\lambda_i||\lambda_j|)^2}$$
$$\leq\frac{2\sum_{1\leq i<j\leq n}|\lambda_i||\lambda_j|}{n(n-1)}.$$

The second inequality in the chain is a consequence of the Cauchy-Schwarz inequality.

Consider now the equality cases. The first equality in (6.1) holds if and only if $|\lambda_i||\lambda_j|$ is a constant for every $i\neq j$. This occurs if either $\lambda_i=0$, for all i, or $\lambda_i=\pm\lambda_j\neq 0$, for all $i\neq j$. The second equality holds if and only if $(|\lambda_1|,|\lambda_2|,\dots,|\lambda_n|)$ is a constant vector (possibly zero). Hence, both inequalities hold if and only if $\dot{G}$ is as in the statement formulation. □

We observe that, in the particular case of graphs, the equalities in (6.1) are attained if and only if G a disjoint union of either isolated vertices or isolated edges. In the 'signed' case, there are many other examples. All of them are regular with an even number of vertices, say $2n$, and a symmetric spectrum of the form $[\lambda^n,(-\lambda)^n]$. These signed graphs belong to the class of strongly regular signed graphs in the sense of definition of [13]; we note in passing that this definition generalizes the concept of strongly regular graphs. Considering the minimal polynomial, we deduce that the common vertex degree is equal to λ^2. Therefore, λ is the square root of an integer. Disconnected examples are not of interest, since each of them is a disjoint union of connected ones. We know from [14] that a signed r-cube without positive quadrangles has the spectrum $\big[\sqrt{r}^{2^{r-1}},(-\sqrt{r})^{2^{r-1}}\big]$. There are no other examples for $\lambda\leq\sqrt{3}$. Signed graphs with spectrum $[2^n,(-2)^n]$ are completely determined in [10, 16] (see also [12]). For those with $[\sqrt{5}^n,(-\sqrt{5})^n]$, see [17]. Some other constructions can be found in [8]. We also remark that all signed line graphs with the required spectrum are constructed in [16].

Observe next that the upper bound (6.1) does not depend on the eigenvalues of $\dot{G}$. This, in particular, means that it simultaneously holds for a signed graph $\dot{G}$ and its underlying graph G. According to the previous discussion, in the connected case, this bound is simultaneously attained for $\dot{G}$ and G if and only if $G\cong K_1$ or $G\cong K_2$; so, in exactly two simple cases, and in both $\dot{G}$ switches to G. For the remaining connected signed graphs that attain this bound, their underlying graphs do not attain it. This leads to an assumption that the energy of a signed graph could often be larger than the energy of its underlying graph. In this context we have experimented with a large number of connected graphs having small order, or small number of edges (i.e., obtained by inserting few edges to a tree), or large number of edges (i.e., obtained by deleting few edges of a complete graph). Our conclusions are

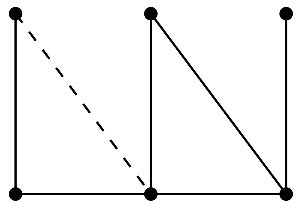

Figure 2: The signed graph $\dot{G}$ with $7.814 \approx \mathcal{E}(\dot{G}) < \mathcal{E}(G) \approx 7.996$.

summarized as follows: $\mathcal{E}(G) = \mathcal{E}(\dot{G})$ holds if $\dot{G}$ switches to G, or G is non-bipartite and $\dot{G}$ switches to $-G$. (For example, the latter occurs for every unicyclic graph with an odd cycle). We have encountered hundreds of examples in which $\mathcal{E}(G) < \mathcal{E}(\dot{G})$. An example in which $\mathcal{E}(G) > \mathcal{E}(\dot{G})$ is illustrated in Figure 2. Motivated by these experiments, we formulate the following problems.

Problem 6.2. Determine graphs G such that $\mathcal{E}(G) \leq \mathcal{E}((G, \dot{\sigma}))$ holds for every signature $\dot{\sigma}$ defined on the edge set of G.

Problem 6.3. Determine (or, at least, characterize) signed graphs $\dot{G}$ with $\mathcal{E}(\dot{G}) < \mathcal{E}(G)$.

We proceed with oriented graphs. If G' is associated with $\dot{G}$, then $\mathcal{E}(G') = \mathcal{E}(\dot{G})$, by Theorem 2.1(i). If $\dot{H}$ is as in Theorem 2.1(ii), then $\mathcal{E}(G') = \frac{\mathcal{E}(\dot{H})}{2}$. Accordingly, if the underlying graph G is regular of degree r, then $\mathcal{E}(G') = n\sqrt{r}$ (the upper bound (6.1)) if and only if G' has exactly two eigenvalues (complex conjugates); this is established in [1], as well. Some examples are constructed in the mentioned reference. Here we note that the search on regular oriented graphs attaining the upper bound (6.1) is reduced to the search on signed graphs with the same spectral property: An associate of a bipartite signed graph with $\mathcal{E}(\dot{G}) = n\sqrt{r}$ has the required spectral property, and if it figures as a bipartite double, then a corresponding constituent has the same property. In this context it is worth mentioning that, on the basis of the results of [10, 16], all oriented graphs with n vertices and spectrum $[\mathrm{i}2^{n/2}, (-\mathrm{i}2)^{n/2}]$ are determined in [15] (their energy attains $2n$); they include infinite families of both bipartite and non-bipartite oriented graphs. More examples can be extracted from known signed graphs with two eigenvalues obtained in the foregoing references.

We also observe that an oriented graph shares the energy with its underlying graph G whenever it is associated with a balanced signed graph. In this context, we point out that, due to Corollary 3.3, $\dot{H}$ of Theorem 2.1(ii) is never balanced. It also shares the energy with G whenever it is associated with a signed graph that switches to $-G$.

This section is concluded with the following result. A conference matrix A is an $n \times n$ matrix with diagonal entries 0 and off-diagonal entries ± 1, satisfying $A^{\intercal}A = (n-1)I$.

Theorem 6.4. *An oriented graph G' (resp. a signed graph $\dot{G}$) with n vertices attains $\mathcal{E}(G') = n\sqrt{n-1}$ ($\mathcal{E}(\dot{G}) = n\sqrt{n-1}$) if and only if its adjacency matrix is the skew-symmetric conference matrix (symmetric conference matrix).*

Proof. If $\mathcal{E}(G') = n\sqrt{n-1}$, by Theorems 2.1(ii) and 6.1, G' is regular of degree $n-1$ and its spectrum is $\left[\mathrm{i}\sqrt{n-1}^{n/2}, -(\mathrm{i}\sqrt{n-1})^{n/2}\right]$. Considering the minimal polynomial of $A_{G'}$, we obtain $A_{G'}^2 = (n-1)I$, which means that $A_{G'}$ is a skew-symmetric conference matrix. Conversely, if $A_{G'}$ is a skew-symmetric conference matrix then, by definition, we have $A_{G'}^2 = (n-1)I$, which gives $\mathcal{E}(G') = n\sqrt{n-1}$.

The proof for $\dot{G}$ is analogous. □

The previous result addresses the open problem (6) of [1, Section 6] related to the existence of skew-symmetric conference matrices that do not give the maximum energy of the corresponding oriented graph. It also partially addresses the problem (2) of the same reference related to determination of oriented graphs with maximum energy, as it gives their characterization via the matrix structure. However, their determination remains a difficult research problem, since it is equivalent to the complete determination of skew-symmetric conference matrices.

7 Notes on hypercubes

Due to [22, Theorem 2.1(iv)] (see also [21, Theorem 2]) a signed graph is balanced if its cycle basis has all positive cycles. Since the induced quadrangles of a signed r-cube contain a cycle basis, we arrive at the following result.

Lemma 7.1 (cf. [22, Theorem 2.1(iv)]). *For $r \geq 2$, a signed r-cube $\dot{Q}_r$ is balanced if and only if it has no negative quadrangles.*

In [2, 3, 20], the authors gave several algorithms which iteratively construct an oriented r-cube such that its energy is either equal to the energy of the underlying graph or attains the upper bound (6.1). (This upper bound reduces to $n\sqrt{r}$.) These algorithms are useful since they give explicit orientations for which the previous settings occur. In this context we offer the following contribution (see also the text below the theorem).

Theorem 7.2. *The following statements hold true:*

(i) *A signed r-cube without negative quadrangle is associated with an oriented r-cube whose energy is equal to the energy of its underlying r-cube.*

(ii) *A signed r-cube is associated with an oriented r-cube whose energy attains the upper bound of (6.1) if and only if it has no positive quadrangle.*

Proof. (i): If every quadrangle (if any) in $\dot{Q}_r$ is positive then $\dot{Q}_r$ is balanced, by Lemma 7.1, and therefore it switches to the underlying graph Q_r. Consequently, $\dot{Q}_r$ and Q_r have the same energy. By Theorem 2.1, $\dot{Q}_r$ shares the energy with an associated oriented cube, and the result follows.

(ii): Since a signed graph and an associated oriented graph share the same energy, it is sufficient to show that the energy $n\sqrt{r}$ is exclusive to a signed cube without positive quadrangles. First, such a cube has this energy (as mentioned in the previous section). Conversely, assume that, for $r \geq 3$, a signed r-cube with the adjacency matrix A has a positive quadrangle, say $uvwz$ (where the vertices are in the natural order). It follows that the (u, w)-entry of A^2 is 2, and so A^2 is not a multiple of the identity matrix, but then the spectrum of A deviates the equality condition of Theorem 6.1, as follows by considering the minimal polynomial. □

In other words, to construct an oriented r-cube whose energy is $\mathcal{E}(Q_r)$ (resp. $n\sqrt{r}$), it is sufficient to take any signed r-cube without negative quadrangle (without positive quadrangle), and construct an oriented associate.

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