Fast recognition of direct and strong products[*]

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Abstract

This note describes fast algorithms for computing the prime factors of connected, non-bipartite graphs with respect to the direct product, and of connected graphs with respect to the strong product. The complexities are $O(m \min(n^2, \Delta^3))$ for the direct product, and $O(m\, a(G)\Delta)$ for the strong, where n is the order of the graph G to be factored, m its size, $a(G)$ its arboricity, and Δ its maximum degree. That is, the complexities are linear in m for fixed Δ.

Keywords: Graph products, algorithms.

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1   Introduction

The Cartesian, direct, and strong products of graphs are the only nontrivial, associative products defined on the Cartesian products of the vertex sets having the property that the projections onto the factors are weak homomorphisms [4]. These products enjoy many interesting algebraic properties, such as unique prime factorization. Connected graphs have unique prime factor decompositions with respect to the Cartesian and the strong product in the class Γ of simple graphs [9, 10, 1, 7], and connected nonbipartite graphs have unique prime factor decompositions with respect to the direct product in the class Γ_0 of graphs where loops but not multiple edges are allowed [7].

In the case of the Cartesian product the prime factors can be computed in linear time, that is, in $O(m)$ time, where m is the size of the graph; see [6].

For the other two products the situation is different. For the strong product the paper [2] by Feigenbaum and Schäffer was the first that presented a polynomial algorithm for the unique factorization of connected graphs. This was then extended to the factorization of nonbipartite connected graphs with respect to the direct product by Imrich [5]. In both papers the main aim was to show that the factorization was possible in polynomial time, hence neither of [2, 5] contains an estimate of the complexity of the algorithm presented there. A rough estimate shows that it is $O(n^{4.5})$ in [2] and $O(n^5)$ in [5], where n is the order of the graph.

Both algorithms depend on the factorization of auxiliary graphs with respect to the Cartesian product. The auxiliary graphs are called *Cartesian skeletons* and are defined algorithmically in [2, 5]. The algorithmic definition makes it difficult to work with them.

Later, non-algorithmic definitions were given [3], along with efficient algorithms that compute them. However, it was not known whether it was possible to find the prime factors with respect to the direct and the strong products within the same time complexities needed for computing the skeletons.

The present paper shows that this is indeed the case. We prove that connected, nonbipartite graphs can be factored over the direct product in $O(m \min(n^2, \Delta^3))$ time, and that the prime factors of connected graphs with respect to the strong product can be found in $O(m\, a(G)\Delta)$ time, where n is the order, m the size, Δ the maximum degree, and $a(G)$ the arboricity of the graph G that is to be factored. (The *arboricity* $a(G)$ of a graph G is the minimum number of forests into which $E(G)$ can be partitioned. It is easily seen that $a(G) \leq \Delta$.)

Although both algorithms that are presented here have complexities close to $O(n^4)$ in the worst case, which just slightly better than that of the algorithms of Feigenbaum and Schäffer [2] (for the strong product) and of Imrich [5] (for the direct product), their main advantage is that their complexities for graphs with known bounds on the maximum degree or the arboricity (in the case of the strong product) and can be explicitly stated.

In particular, we wish to point out that, for fixed Δ and growing m, we have $m \leq n\Delta$, hence n also grows, and Δ^3 must eventually become smaller than n^2. Also note that $a(G)\Delta \leq \Delta^2$. Hence the complexities grow linearly in m for fixed Δ. Algorithms with such complexities are called *quasilinear*.

On the way, in Section 2.1, we also prove new bounds on the number of prime factors of a graph with respect to the strong and the direct product.

2   Definitions

We consider finite graphs G which may have loops but not multiple edges, and denote the class of these graphs by Γ_0, while $\Gamma \subset \Gamma_0$ is the class of graphs without loops. An edge joining g to g' is denoted gg'. The open neighborhood of a vertex g is denoted $N(g)$, or $N_G(g)$ when it is necessary to indicate the graph under discussion. The closed neighborhood of g is $N[g] = N(g) \cup \{g\}$. Again, we often write this as $N_G[g]$.

Given graphs H and K, the Cartesian product $H \,\square\, K$, the direct product $H \times K$ and the strong product $H \boxtimes K$, are defined on the Cartesian product $V(H) \times V(K)$ of the

vertex sets of the factors and have the following edge sets:

$$
\begin{aligned}
E(H \,\square\, K) &= \{(h,k)(h',k') \mid hh' \in E(H), k = k', \text{ or } h = h', kk' \in E(K)\}, \\
E(H \times K) &= \{(h,k)(h',k') \mid hh' \in E(H) \text{ and } kk' \in E(K)\}, \\
E(H \boxtimes K) &= E(H \,\square\, K) \cup E(H \times K).
\end{aligned}
$$

(Examples are shown in Figure 1.) All three products are commutative and associative. Also, the complete graph K_1 on one vertex is a unit for $\square$ and $\boxtimes$, as $K_1 \,\square\, H \cong H$ and $K_1 \boxtimes H \cong H$ for all graphs H. The graph K_1^* consisting of a single vertex with a loop satisfies $K_1^* \times H \cong H$, and is the unit for $\times$.
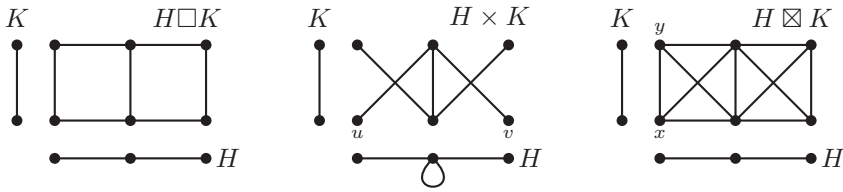

Figure 1: The three standard graph products

Let $a = (a_1, a_2, \ldots, a_k)$ be a vertex of a product $G = G_1 * G_2 * \cdots * G_k$, where $*$ designates any of the symbols $\square$, $\times$, or $\boxtimes$. Then the G_i-*layer* G_i^a of G_i through a is defined as the subgraph of G induced by the set of vertices

$$\{(a_1, a_2, \ldots, a_{i-1}, x_i, a_{i+1} \ldots, a_k) \mid x_i \in V(G_i)\}\,.$$

For the Cartesian and the strong product, the G_i-layers are isomorphic to G_i. For the direct product, $G_i^a \cong G_i$ if all a_j have a loop in G_j for $j \neq i$. Otherwise G_i^a has no edges.

We call the mapping $a \mapsto a_i$ a *projection*. For a subgraph H of G, it restricts to a mapping $p_i : H \to G_i$. For the direct product this is a homomorphism; for the other two products it is a weak homomorphism[1].

If $X \subseteq V(G)$, the subgraph of G induced on X is denoted $\langle X \rangle$, or $\langle X \rangle_G$ if there is a risk of ambiguity. As a consequence of the definitions, if $X_i \subseteq V(G_i)$ for $1 \leq i \leq k$, then

$$\langle X_1 \times X_2 \times \cdots \times X_k \rangle_{G_1 \square G_2 \square \cdots \square G_k} = \langle X_1 \rangle_{G_1} \,\square\, \langle X_2 \rangle_{G_2} \,\square\, \cdots \,\square\, \langle X_k \rangle_{G_k}, \quad (2.1)$$

where the $\times$ indicates the Cartesian product of sets.

A nontrivial graph is called *prime* with respect to a particular product if whenever it is represented as a product of two factors, one of the factors is the unit for the product. As already mentioned, connected graphs have unique prime factor decompositions with respect to the Cartesian and the strong product in Γ, and connected nonbipartite graphs have unique prime factor decompositions with respect to the direct product in Γ_0.

There are two significant equivalence relations R and S on the vertex set of a graph. To motivate this, note that Cartesian products possess a certain degree of rigidity; any automorphism of $H \,\square\, K$ is induced by automorphisms of H and K (or their transposition

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[1]Recall that a homomorphims is an edge-preserving map, whereas a weak homomorphims either preserve edges or maps them into single vertices.
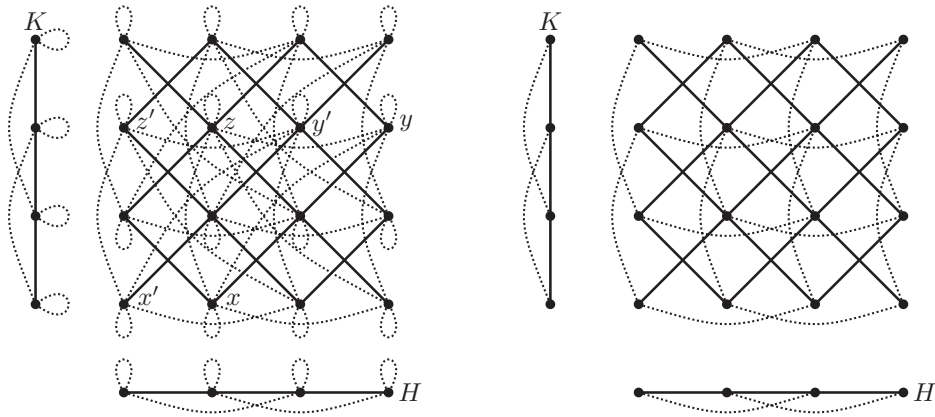
Figure 2: Left: Graphs $H, K, H \times K$ (solid) and $H^s, K^s, (H \times K)^s$ (dotted). Right: Graphs $H, K, H \times K$ (solid) and Cartesian skeletons $S(H), S(K), S(H \times K)$ (dotted).

if they are isomorphic). This is not so with the direct and strong products. For example, in Figure 1 the vertices u and v of $H \times K$ can be transposed. The same is true for x and y in $H \boxtimes K$. It is easy to see the reason for this. In the case of the direct product, the interchange is possible because $N(u) = N(v)$; it is possible for the strong product because $N[x] = N[y]$. We declare that two vertices x, y are in *relation R* if $N(x) = N(y)$, and they are in *relation S* if $N[x] = N[y]$. Notice that both R and S are equivalence relations.

We say a graph is *R-thin* if every R-equivalence class consists of a single vertex; it is *S-thin* if every S-equivalence class has a single vertex. In discussions of prime factoring over the direct product, it is helpful (at least initially) to assume that all graphs are R-thin. For the strong product, we assume S-thinness.

Another important concept is the Boolean square. The *Boolean square G^s* of a graph G has vertex set $V(G^s) = V(G)$ and edges $E(G^s) = \{xy \mid N_G(x) \cap N_G(y) \neq \emptyset\}$. The left side of Figure 2 shows graphs H, K and $H \times K$ (bold) and their Boolean squares (dotted). It is easy to confirm that

$$(G_1 \times G_2)^s = G_1^s \times G_2^s,$$

and this is indeed reflected in the figure. Furthermore, letting $\mathcal{N}G$ denote the graph G after removal of all loops, we have

$$\mathcal{N}((G_1 \times G_2)^s) = \mathcal{N}G_1^s \boxtimes \mathcal{N}G_2^s. \tag{2.2}$$

The most important concept in this paper is a certain subgraph of G^s called the Cartesian skeleton. It is obtained from G^s by removal of the so-called dispensable edges.

We call an edge xy of G^s *dispensable* if $x = y$ or if there exists some $z \in V(G)$ for which both of the following statements hold.

1. $N_G(x) \cap N_G(y) \subset N_G(x) \cap N_G(z)$ or $N_G(x) \subset N_G(z) \subset N_G(y)$,
2. $N_G(y) \cap N_G(x) \subset N_G(y) \cap N_G(z)$ or $N_G(y) \subset N_G(z) \subset N_G(x)$.

In Figure 2 (left) the edge xy is dispensable because $N_G(x) \cap N_G(y) \subset N_G(x) \cap N_G(z)$ and $N_G(y) \cap N_G(x) \subset N_G(y) \cap N_G(z)$. Also, $x'y'$ is dispensable, as $N_G(x') \subset N_G(z') \subset N_G(y')$ and $N_G(y') \cap N_G(x') \subset N_G(y') \cap N_G(z')$.
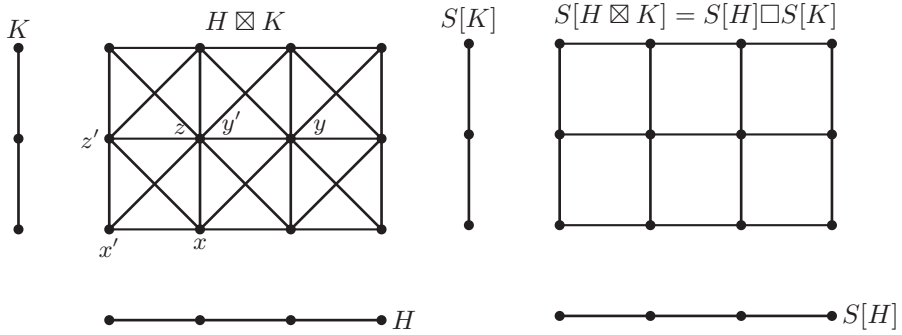
Figure 3: Left: Graphs $H, K, H \boxtimes K$; Right: Closed Cartesian skeletons $S[H], S[K]$, $S[H \boxtimes K]$.

The *Cartesian skeleton* of a graph G is the graph $S(G)$ obtained from G^s by removing all dispensable edges. For example, the dotted lines in Figure 2 (right) are the Cartesian skeletons of H, K and $H \times K$. This figure illuminates a general principle that was proved in [3]: If the product $H \times K$ is R-thin and has no isolated vertices, then

$$S(H \times K) = S(H) \,\square\, S(K). \tag{2.3}$$

For the strong product a similar construction, the *closed Cartesian skeleton* $S[G]$, is useful. It is a subgraph of G obtained by removing all dispensable edges of G, where an edge xy of G is dispensable if for some $z \in V(G)$ both of the following conditions hold:

1 (strong). $N_G[x] \cap N_G[y] \subset N_G[x] \cap N_G[z]$ or $N_G[x] \subset N_G[z] \subset N_G[y]$,
2 (strong). $N_G[y] \cap N_G[x] \subset N_G[y] \cap N_G[z]$ or $N_G[y] \subset N_G[z] \subset N_G[x]$.

Figure 3 shows graphs H, K and $H \boxtimes K$ on the left and their closed Cartesian skeletons $S[H], S[K]$ and $S[H \boxtimes K]$ on the right. Notice the edge xy (for example) is dispensable, because $N_G[x] \cap N_G[y] \subset N_G[x] \cap N_G[z]$ and $N_G[y] \cap N_G[x] \subset N_G[y] \cap N_G[z]$. Also, $x'y'$ is dispensable, as $N_G[x'] \subset N_G[z'] \subset N_G[y']$ and $N_G[y'] \cap N_G[x'] \subset N_G[y'] \cap N_G[z']$.

If $H \boxtimes K$ is S-thin, then, similarly to Equation (2.3), we have

$$S[H \boxtimes K] = S[H] \,\square\, S[K], \tag{2.4}$$

as noted in reference [3], in which it is also shown that the Cartesian skeletons of connected nonbipartite graphs are connected. Moreover, the condition of nonbipartiteness can be dropped in the case of the closed Cartesian skeleton: $S[G]$ is connected if G is.

Note that Equations (2.3) and (2.4) express equality of graphs, not just isomorphism. That is, e.g., graphs $S(H \times K)$ and $S(H) \,\square\, S(K)$ have identical vertex sets and edge sets.

The article [3] also presents algorithms and complexity analysis for computing $S(G)$ and $S[G]$. The skeleton $S(G)$ can be computed in $\min\{O(mn^2), O(m\Delta^3)\}$ time. Its space complexity is determined by the size of the output and is thus between $O(n)$ and $O(n^2)$.

On the other hand, the closed Cartesian skeleton $S[G]$ can be computed in $O(m\,a(G)\Delta)$ time, where $a(G)$ is the *arboricity* of G, that is, the minimum number of forests into which

the $E(G)$ can be partitioned. One can show that $\delta/2 < a(G) \leq \Delta$, where δ is the minimum degree of G.[2]

2.1   Bounds on the number of factors

We will need bounds on the number of prime factors of graphs relative to different products.

A simple computation shows that a product of k nontrivial graphs has at least 2^k vertices. In other words, a graph on n vertices can have at most $\log_2 n$ factors with respect to any product. However, note that the strong product of k copies of K_2 is K_{2^k}, and that every vertex has degree $2^k - 1$. Thus any vertex of a strong product of k nontrivial connected factors has degree at least $2^k - 1$.

For the Cartesian product it is well known that a connected graph has at most δ factors.

Lemma 2.1. *Suppose a graph G has n vertices and minimum degree δ. Then:*

1. *G has at most $\log_2 n$ factors with respect to the Cartesian, direct, or strong product.*

2. *If G is connected, it has at most δ factors with respect to the Cartesian product.*

3. *If G is connected, it has at most $\log_2(\delta+1)$ factors with respect to the strong product.*

Proof. We have already addressed the first two statements. If G is connected, then so are all of its strong product factors. If G has k factors, then $\delta \geq 2^k - 1$, so $\log_2(\delta+1) \geq k$. $\square$

For the direct product we have the following corollary.

Corollary 2.2. *A connected, nonbipartite graph has at most $\log_2(\delta\Delta + 1)$ factors with respect to the direct product.*

Proof. Let $G = G_1 \times G_2 \times \cdots \times G_k$. Then $G^s = G_1^s \times G_2^s \times \cdots \times G_k^s$, so any bound on the number of factors of G^s also bounds the number of factors of G. Equation (2.2) implies

$$\mathcal{N}G^s = \mathcal{N}G_1^s \boxtimes \mathcal{N}G_2^s \boxtimes \cdots \boxtimes \mathcal{N}G_k^s.$$

Hence k is also bounded by the number of factors of $\mathcal{N}G^s$ over the strong product. As $\log_2(\delta_{\mathcal{N}G^s} + 1) \geq k$, the assertion will follow as soon as we establish $\delta_G \Delta_G \geq \delta_{\mathcal{N}G^s}$.

Indeed, for any vertex x of $\mathcal{N}G^s$ the definition of G^s yields $\deg_G(x)\Delta_G \geq \deg_{\mathcal{N}G^s}(x)$. If x has minimum degree in G, this is $\delta_G \Delta_G \geq \deg_{\mathcal{N}G^s}(x)$; thus $\delta_G \Delta_G \geq \delta_{\mathcal{N}G^s}$. $\square$

3   Thin graphs and the direct product

Suppose we want to compute the prime factorization $G = G_1 \times G_2 \times \cdots \times G_k$ of an R-thin, connected, nonbipartite graph G. The compatibility of the Cartesian skeleton with the direct product as expressed in Equation (2.3) implies

$$S(G) = S(G_1) \square S(G_2) \square \cdots \square S(G_k). \tag{3.1}$$

Because $S(G)$ is connected, there is a Cartesian prime factorization

$$S(G) = \mathop{\square}_{i \in I} H_i \tag{3.2}$$

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[2]These relations are not hard to show. The right side is routine. For the left side one invokes a theorem of Nash-Williams [8], see [4, Exercise 20.2] and the hint thereto.

that is unique up to the order and isomorphisms of the factors. By [4, p. 296], this equals the presentation in (3.1) or a refinement of it. Thus the number of Cartesian factors of $S(G)$ can be larger than the number of direct factors of G. Since we can compute $S(G)$ and its prime factorization over $\square$, our task is to find a partition $J_1, J_2, \ldots, J_k$ of I such that

$$S(G_i) = \underset{j \in J_i}{\square} H_j$$

for all $i, 1 \leq i \leq k$.

Section 24.3 of [4] shows that each J_i is a minimal subset of I for which the H_{J_i}-layers of $H_{J_i} \square H_{I \setminus J_i}$ (where $H_{J_i} = \square_{j \in J_i} H_j$ and $H_{I \setminus J_i} = \square_{j \in I \setminus J_i} H_j$) correspond to the layers of a factor of G with respect to the direct product. In other words, each J_i is a minimal subset of I for which $G = p_{J_i}(G) \times p_{I \setminus J_i}(G)$, where p_{J_i} is the projection of $V(G) = V(S(G))$ onto $V(H_{J_i})$, and $p_{J_i}(G)$ is the smallest graph for which this projection is a homomorphism. (The graph $p_{I \setminus J_i}(G)$ is defined similarly.)

It is also shown there that the complexity of checking whether a subset $J \subseteq I$ induces a factoring $G = p_J(G) \times p_{I \setminus J}(G)$ is $O(m|I|)$. As I has $2^{|I|}$ subsets, the complexity of checking them all is $O(m \, 2^{|I|} |I|)$.

Now, the Cartesian skeleton $S(G)$ has the same number of vertices as G. By Lemma 2.1, we infer that $|I| \leq \log_2 n$. Hence the prime factors of G with respect to the direct product can be computed from the Cartesian skeleton in $O(m2^{\log_2 n} \log_2 n) = O(mn \log n)$ time.

Recall that the Cartesian skeleton can be computed in $\min\{O(mn^2), O(m\Delta^3)\}$ time. Clearly $O(mn \log n) \leq O(mn^2)$, so if $n \log_2 n \leq \Delta^3$, then the prime factorization of G over the direct product (from $S(G)$) does not cost more than the computation of $S(G)$. But, if $\Delta^3 < n \log_2 n$, then the computation of the Cartesian skeleton is cheaper than the computation of the prime factorization as presented above.

If we could somehow reduce the number of subsets of I that must be investigated, then we might be able to retain the time complexity $O(m\Delta^3)$ for the prime factorization of G even when $\Delta^3 < n \log_2 n$. This is indeed possible.

But before stating it in the next proposition, we first recall an elemental fact about vertex neighborhoods in direct products, namely if $G = G_1 \times G_2 \times \cdots \times G_k$, then

$$N_G((a_1, a_2, \ldots, a_k)) = N_{G_1}(a_1) \times N_{G_2}(a_2) \times \cdots \times N_{G_k}(a_k), \qquad (3.3)$$

where $\times$ is the Cartesian product of sets. Taking induced subgraphs, this becomes

$$\langle N_G((a_1, a_2, \ldots, a_k)) \rangle_G = \langle N_{G_1}(a_1) \rangle_{G_1} \times \langle N_{G_2}(a_2) \rangle_{G_2} \times \cdots \times \langle N_{G_k}(a_k) \rangle_{G_k}, \qquad (3.4)$$

where $\times$ is the direct product of graphs. A subgraph of a product that is a product of subgraphs of the factors is called a *box* relative to the product. The above equation implies that the subgraph induced on any open neighborhood of a graph is a box relative to any direct product factorization of the graph.

Proposition 3.1. *If G is a connected, nonbipartite R-thin graph of order n and size m, then its direct product prime factorization can be found in $\min\{O(mn^2), O(m\Delta^3)\}$ time.*

Proof. By the above arguments it suffices to treat the case $\Delta^3 < n^2$; see also [4, p. 296].

Let us have a closer look at the Cartesian skeleton of a graph G with prime factorization $G_1 \times G_2 \times \cdots \times G_k$. By our previous discussion we have:

$$G \;=\; G_1 \times G_2 \times \cdots \times G_k, \tag{3.5}$$
$$S(G) \;=\; S(G_1) \,\square\, S(G_2) \,\square\, \cdots \,\square\, S(G_k), \tag{3.6}$$
$$G^s \;=\; G_1^s \times G_2^s \times \cdots \times G_k^s. \tag{3.7}$$

Furthermore, given $a \in V(G)$, the layers G_i^a, $S(G_i)^a$ and $(G_i^s)^a$ have the same vertex sets. Moreover, the layers $(G_i^s)^a$ are all isomorphic to G_i^s. Given $v \in V(G^s)$, Equation (3.4) applied to (3.7) gives

$$\langle N_{G^s}(v)\rangle_{G^s} = \bigtimes_{i=1}^{k} \langle N_{G_i^s}(v_i)\rangle_{G_i^s}. \tag{3.8}$$

Notice that all $\langle N_{G_i^s}(v_i)\rangle_{G_i^s}$ are connected, but that they need not be prime (with respect to the direct product), and that the G_i^s may not be R-thin.

Equation (3.3) applied to (3.7) gives $N_{G^s}(v) = N_{G_1^s}(v_1) \times N_{G_2^s}(v_2) \times \cdots \times N_{G_k^s}(v_k)$. Consider the subgraph of $S(G) = S(G_1) \,\square\, S(G_2) \,\square\, \cdots \,\square\, S(G_k)$ induced on this vertex set. Using Equation (2.1), we get

$$\langle N_{G^s}(v)\rangle_{S(G)} = \bigsquare_{i=1}^{k} \langle N_{G_i^s}(v_i)\rangle_{S(G_i)}. \tag{3.9}$$

Call this box $S(G)(v)$.

Comparing Equations (3.9) and (3.6), we see that $S(G)(v)$ contains all edges of $S(G_i)^v$ that are incident with v for any i. Thus, the connected component of $S(G)(v)$ through v contains edges from all $S(G_i)$. Clearly it is a box, and we denote it by B. By Equation (3.9) we have

$$
\begin{array}{ccccccccc}
S(G) & = & S(G_1) & \square & S(G_2) & \square & \cdots & \square & S(G_k) \\
\cup\vert & & \cup\vert & & \cup\vert & & & & \cup\vert \\
B & = & B_1 & \square & B_2 & \square & \cdots & \square & B_k .
\end{array}
\tag{3.10}
$$

Observe that the number of vertices of $S(G)(v)$ is the same as the number of vertices of $N_{G^s}(v)$, which is at most Δ^2, a bound for the maximum degree of G^s. But then the number of Cartesian factors of $S(G)(v)$ is at most $2\log_2 \Delta$, and this also bounds the number of Cartesian factors of B. Thus the factoring of B in Equation (3.10) has a refinement $B = B_1' \,\square\, B_2' \,\square\, \cdots \,\square\, B_\ell'$, where $\ell \leq 2\log_2 \Delta$, and each B_s' is a prime factor of some B_i. (Recall that – as mentioned in the introduction – computing the prime factorization of $S(G)$ is linear in m, and hence this is also the case for B.)

From (3.10), we see that each Cartesian prime factor B_s' of B is a prime factor of some B_i, and hence i is the only index $1 \leq i \leq k$ for which $S(G_i)$ has prime factors H_j with $E(B_s'^v) \cap E(H_j^v) \neq \varnothing$. Form a new partition $J_1', J_2', \ldots, J_\ell'$ of I, where J_s' consists of indices $j \in I$ for which $E(B_s'^v) \cap E(H_j^v) \neq \varnothing$. This new partition need not be unique, for an $E(H_j^v)$ may meet layers of several prime factors of B. To be definite, we define the J_s' inductively by first declaring $J_s' = \{j \in I \mid E(B_1'^v) \cap E(H_j^v) \neq \emptyset\}$ and thereafter $J_s' = \{j \in I - \bigcup_{\lambda=1}^{s-1} J'_\lambda \mid E(B_s'^v) \cap E(H_j^v) \neq \emptyset\}$.

Letting the ℓ graphs

$$H_s' = \bigsquare_{j \in J_s'} H_j$$

play the role of the H_i we thus see that we can find the prime factors of G from $S(G)$ in

$$O(m2^{2\log\Delta}2\log\Delta) = O(m\Delta^2\log\Delta) = O(m\Delta^3)$$

time, which is the same time complexity as that for computing $S(G)$ if $\Delta^3 \le n^2$. □

4 Thin graphs and the strong product

Let G be a connected S-thin graph. It is uniquely factorable into prime graphs with respect to the strong product, say as

$$G = G_1 \boxtimes G_2 \boxtimes \cdots \boxtimes G_k.$$

By [3], the strong Cartesian skeleton of G is the Cartesian product of the strong Cartesian skeletons of the G_i, in symbols,

$$S[G] = S[G_1] \square S[G_2] \square \cdots \square S[G_k]. \tag{4.1}$$

Similarly to the case of the direct product, there is a unique prime factorization

$$S[G] = \bigsquare_{i \in I} H_i.$$

It is equal to the the presentation in Equation (4.1) or to a refinement of it. Again our task is to find a partition $J_1, J_2, \ldots, J_k$ of I such that

$$S[G_i] = \bigsquare_{j \in J_i} H_j$$

for all $i, 1 \le i \le k$. As before we have to check all minimal subsets J_i of I. We have to ensure that the H_{J_i}-layers of $H_{J_i} \square H_{I \setminus J_i}$ (where $H_{J_i} = \bigsquare_{j \in J_i} H_j$ and $H_{I \setminus J_i} = \bigsquare_{j \in I \setminus J_i} H_j$) correspond to the layers of a factor of G with respect to the strong product. (Again, see Section 24.3 of [4].)

Here too the complexity of doing this is $O(m2^{\log_2 |I|} \log_2 |I|)$. Since the bound $\log_2 n$ for $|I|$ yields the bound $O(mn\log n)$, which will usually be larger than $O(m\,a(G)\Delta)$, we follow a similar approach as before.

Before beginning, we note that in the case of the strong product, Formulas (3.3) and (3.4) play out as follows. If $G = G_1 \boxtimes G_2 \boxtimes \cdots \boxtimes G_k$, then

$$N_G[(a_1, a_2, \ldots, a_k)] = N_{G_1}[a_1] \times N_{G_2}[a_2] \times \cdots \times N_{G_k}[a_k], \tag{4.2}$$

where $\times$ is the Cartesian product of sets.

Proposition 4.1. *Let G be a connected S-thin graph of order n and size m. Then its prime factors with respect to the strong product can be computed in $O(m\,a(G)\Delta)$ time.*

Proof. The proof parallels that of Proposition 3.1. Choose a vertex v of minimal degree in G and consider its closed neighborhood

$$N_G[v] = N_{G_1}[v_1] \times N_{G_2}[v_2] \times \cdots \times N_{G_k}[v_k].$$

By Equation (2.1), the subgraph of $S[G] = S[G_1] \square S[G_2] \square \cdots \square S[G_k]$ induced on this vertex set is the box

$$\langle N_G[v] \rangle_{S[G]} = \langle N_{G_1}[v_1] \rangle_{S[G_1]} \square \langle N_{G_2}[v_2] \rangle_{S[G_2]} \square \cdots \square \langle N_{G_k}[v_k] \rangle_{S[G_k]}.$$

As before, let B be the component of this box containing v. Then B is a box with no more than $\delta + 1$ vertices, and it factors as

$$
\begin{array}{ccccccccc}
S[G] & = & S[G_1] & \square & S[G_2] & \square & \cdots & \square & S[G_k] \\
\cup| & & \cup| & & \cup| & & & & \cup| \\
B & = & B_1 & \square & B_2 & \square & \cdots & \square & B_k \, .
\end{array}
\tag{4.3}
$$

This factoring of B has a refinement $B = B'_1 \square B'_2 \square \cdots \square B'_\ell$, where $\ell \leq \log_2(\delta + 1)$, and each B'_s is a prime factor of some B_i. As in the proof of Proposition 3.1, we can form a new partition $J'_1, J'_2, \ldots, J'_\ell$ of I, where J'_s consists of indices $j \in I$ for which $E({B'_s}^v) \cap E(H^v_j) \neq \varnothing$. Arguing as before, we see that we can find the prime factors of G from $S[G]$ in

$$
O\big(m 2^{\log \delta} \log(\delta + 1)\big) = O(m\, \delta \log \delta)
$$

time. From $\delta < 2a(G)$ we infer that $O(m\, \delta \log \delta) = O(m\, a(G)\Delta)$. The observation that the prime factors of B can be computed in $O(\delta\Delta)$ time completes the proof. $\qquad\square$

We wish to point out that we have no guarantee that $N_G[v]$ is thin, which excludes it from our present factorization methods.

5   Factoring graphs that are not thin

Up to here we have factored only thin graphs. The reason is that we made strong use of properties of the Cartesian skeleton that do not hold if the graphs are not thin. In order to factor a graph G that is not thin we first compute the quotient graphs G/R or G/S. These are formed from G by contracting all R-classes (respectively all S-classes) of G to single vertices. The resulting graphs are thin, and factored by the methods just described. (See Section 8.2 of [4, 24.4].) Notice that one can compute G/R, respectively G/S, in $O(m)$ time.

Once G/R, respectively G/S, has been factored, the question is whether this leads to factorization of G. While every factorization of G induces a factorization of G/R, respectively G/S, this is not true in the other direction. For example, consider the graph G consisting of a triangle C_3 on the vertices $v_0 v_1 v_2$ and two pendant edges $v_0 a$ and $v_0 b$. It is not R-thin, because the vertices a and b have the same neighborhoods. As G has 5 vertices, it is prime. Contracting the R-class $\{a, b\}$ to a single vertex, we obtain the thin graph G/R, which is a triangle with one pendant edge. It is easily seen that this graph is the direct product of two copies of an edge with a loop. Similar examples are possible for the strong product.

Hence, we observe that G may have fewer prime factors than G/R, respectively G/S. We now use [4, 24.4] to compute them. First the direct product.

By [4, 24.4], connected, nonbipartite graphs G of size m and order n can be factored with respect to the direct product in $O(m 2^k k)$ time, where k is a bound on the number of direct factors of G. With the general bound $k \leq \log_2 n$ this yields the bound $O(mn^2)$ for the factorization of G.

If $\Delta^3 < n^2$ we use the bound $k \leq \log_2(\delta\Delta)$ from Corollary 2.2. It yields the estimate

$$
m 2^k k \leq m 2^{\log_2(\delta\Delta)} \log_2(\delta\Delta) \leq m\delta\Delta(\log_2 \delta + \log_2 \Delta) \leq 2m\Delta^3 \, .
$$

Theorem 5.1. *The prime factors (over the direct product) of a connected, nonbipartite graph of size m, order n, and maximum degree Δ can be found in $O(m \min(n^2, \Delta^3))$ time.*

For the strong product we have the same complexity $O(m2^k k)$, but here k is the number of strong factors of G. Using the bound $\log_2(\delta + 1)$ for k we thus obtain the following.

$$O(m2^k k) = O\big(m2^{\log_2(\delta+1)} \log_2(\delta + 1)\big) = O\big(m(\delta + 1)\delta\big) = O(m\, a(G)\Delta)\,.$$

Theorem 5.2. *The prime factors (over the strong product) of a connected graph of size m, arboricity $a(G)$, and maximum degree Δ can be computed in $O(m\, a(G)\Delta)$ time.*

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