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Trilateral matroids induced by n_3-configurations

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Abstract

We define a new class of a rank-3 matroid called a trilateral matroid. When defined, the ground set of such a matroid consists of the points of an n_3-configuration, and its bases are the point triples corresponding to non-trilaterals within the configuration. We characterize which n_3-configurations induce trilateral matroids and provide several examples.

Keywords: Configurations, trilaterals, matroids.

Math. Subj. Class.: 05B30, 51E30, 05C38, 05B35

1 Introduction

A *(combinatorial) n_3-configuration* $\mathcal{C}$ is an incidence structure consisting of n distinct points and n distinct blocks for which each point is incident with three blocks, each block is incident with three points, and any two points are incident with at most one common block. If $\mathcal{C}$ may be depicted in the real projective plane using points and having (straight) lines as its blocks, then it is said to be *geometric*. As observed in [6] (pg. 17–18), it is evident that every geometric n_3-configuration is combinatorial, but the converse of this statement does not hold.

A *trilateral* in a configuration is a cyclically ordered set $\{p_0, b_0, p_1, b_1, p_2, b_2\}$ of pairwise distinct points p_i and pairwise distinct blocks b_i such that p_i is incident with b_{i-1} and b_i for each $i \in \mathbb{Z}_3$ [2]. We may without ambiguity shorten this notation by listing only the points of the trilateral as $\{p_0, p_1, p_2\}$, or more simply as $p_0p_1p_2$. A configuration is *trilateral-free* if no trilateral exists within the configuration. Unless stated otherwise, the n_3-configurations we shall examine are point-line configurations, so that the blocks are lines. But we shall investigate an example of a point-plane configuration in Section 3.

Following the terminology of [7], we define a *matroid* M to be an ordered pair $(E, \mathcal{B})$ consisting of a finite ground set E and a nonempty collection $\mathcal{B}$ of subsets of E called *bases* which satisfy the *basis exchange property:*

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Definition 1.1. If $B_1, B_2 \in \mathcal{B}$ and $x \in B_1 - B_2$, then there exists $y \in B_2 - B_1$ such that $B_1 - x \cup y \in \mathcal{B}$.

It is a consequence of this definition that any two bases of M share the same cardinality; this common cardinality is called the *rank* of the matroid. See [7], pg. 16–18 for the details.

It is a standard result that any n_3-configuration $\mathcal{C}$ defines a rank-3 *linear matroid*, or *vector matroid*, $M(\mathcal{C}) = (E, \mathcal{B})$ whose ground set E consists of the points $\{p_1, p_2, \ldots, p_n\}$ of $\mathcal{C}$ and whose set of bases $\mathcal{B}$ consists of the point triples $\{p_a, p_b, p_c\}$ which are *not* collinear in $\mathcal{C}$. Hence the cardinality of $\mathcal{B}$ is $\binom{n}{3} - n$ for the linear matroid $M(\mathcal{C})$ induced by $\mathcal{C}$.

In this work we pose the following associated question: under what conditions do the *trilaterals* of an n_3-configuration $\mathcal{C}$ induce a rank-3 matroid $M_{tri}(\mathcal{C}) = (E, \mathcal{B})$ whose ground set E again consists of the points of $\mathcal{C}$, but now whose bases are the point triples corresponding to non-trilaterals? This question, to our knowledge, has not previously been considered in the literature on configurations and matroids.

Definition 1.2. A *trilateral matroid* $M_{tri}(\mathcal{C}) = (E, \mathcal{B})$, when it exists, is a matroid defined on the set E of points of an n_3-configuration $\mathcal{C}$ whose set of bases $\mathcal{B}$ consists of all of the non-trilaterals of $\mathcal{C}$. When $M_{tri}(\mathcal{C})$ exists, we say that $\mathcal{C}$ *induces* $M_{tri}(\mathcal{C})$.

We shall see that, in contrast to the linear matroid setting, seldom is it the case that an n_3-configuration $\mathcal{C}$ induces a trilateral matroid $M_{tri}(\mathcal{C})$. But thankfully such matroids do exist; for instance, any trilateral-free configuration induces a trilateral matroid, since in this setting *every* point triple forms a base of the matroid. In other words, if $\mathcal{C}$ is a trilateral-free n_3-configuration, then $M_{tri}(\mathcal{C})$ exists and furthermore $M_{tri}(\mathcal{C}) \cong U_{3,n}$, the uniform matroid of rank 3 on n points. Thus our initial motivation to define this new class of matroids stems from the desire to enlarge the class of trilateral-free configurations.

For purposes of instruction, we regard an example of a 15_3-configuration which induces a trilateral matroid on its points. Here is a combinatorial description of this configuration.

l_1	l_2	l_3	l_4	l_5	l_6	l_7	l_8	l_9	l_{10}	l_{11}	l_{12}	l_{13}	l_{14}	l_{15}
1	1	1	2	2	3	3	4	5	5	7	7	9	10	13
2	4	6	4	6	8	11	6	8	9	8	9	11	11	14
3	5	7	14	10	12	13	12	10	13	14	15	12	15	15

This configuration has 10 trilaterals:

t_1	t_2	t_3	t_4	t_5	t_6	t_7	t_8	t_9	t_{10}
1	1	1	2	3	7	9	9	9	11
2	2	4	4	11	14	11	11	13	13
4	6	6	6	12	15	13	15	15	15

In Figure 1 we see both a diagram of this 15_3-configuration and a geometric representation of its trilateral matroid. In the geometric representation, each trilateral (that is, each non-basis element) is collinear.

Note that the configuration contains two *complete quadrangles*. The first complete quadrangle is determined by the point set $\{1, 2, 4, 6\}$, and the second by $\{9, 11, 13, 15\}$. This means, for example, that no three points in $\{1, 2, 4, 6\}$ are collinear, and each pair of points is incident to a line of the configuration. So all four point triples present within $\{1, 2, 4, 6\}$ give trilaterals, and hence are not bases of the matroid. Thus every 2-element subset of $\{1, 2, 4, 6\}$ is independent, but no 3-element subset of $\{1, 2, 4, 6\}$ is. Therefore

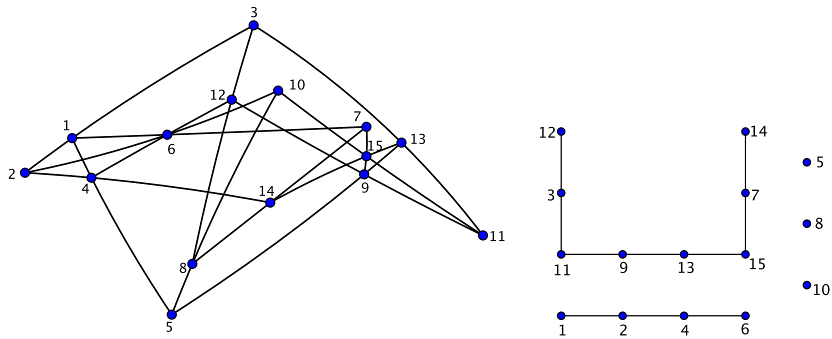


Figure 1: A 15_3-configuration with 10 trilaterals, and a geometric representation of the matroid induced by these trilaterals.

the four-point line that represents the level of dependency of $\{1, 2, 4, 6\}$ in the geometric representation is appropriate. This minor is isomorphic to $U_{2,4}$, which is the unique excluded minor for the class of binary matroids ([7], pg. 501).

We must note that there is a fundamental difference between trilateral matroids and linear matroids. Admittedly a finite set of points and lines in the plane gives a (linear) matroid if and only if any pair of lines meet in at most one point. For suppose there exist two points a and b which are met by two lines, so that points a, b, c are collinear, points a, b, d are collinear, but a, b, c, d are not all on one line. Pick a new point e so that c, d, and e are not collinear, and so that a, b, and e are not collinear. Let $B_1 = abe$ and $B_2 = cde \in \mathcal{B}$; both are bases of the linear matroid. We have $B_1 - B_2 = ab$ and $B_2 - B_1 = cd$. Let $x = e \in B_1 - B_2$, so $B_1 - x = cd$. But if $y \in B_2 - B_1 = ab$, then $B_1 - x \cup y$ equals either abc or abd, neither of which is a base.

Hence a linear matroid cannot have two points common to more than one line. But a trilateral matroid can; if both abc and abd are trilaterals, then the configuration has a chance to induce a trilateral matroid if trilaterals acd and bcd are also present, meaning that points c and d are incident to a particular line of the configuration. In other words, points $\{a, b, c, d\}$ form a *complete quadrangle* within the configuration. We shall explore this necessity further in Theorem 1.7.

Any point of an n_3-configuration is incident to three lines; these three lines are then incident to six points which are distinct from the original point and from each other. Consequently, the maximum number of trilaterals incident to a given point is $\binom{6}{3} - 3 = 12$, since lines are not trilaterals. This maximum is achieved by *every* point of the Fano 7_3-configuration (the smallest n_3-configuration) given in Figure 2.

Proposition 1.3. *Suppose an n_3-configuration $\mathcal{C}$ induces a trilateral matroid $M_{tri}(\mathcal{C}) = (E, \mathcal{B})$. Then each point of the configuration is incident to at most six trilaterals.*

Proof. Let a be a point in $\mathcal{C}$, and let abc, ade, and afg be the lines in $\mathcal{C}$ incident to a. Each of these lines belongs to $\mathcal{B}$, and hence there are at most $\binom{6}{3} - 3 = 12$ trilaterals incident to a, namely

$$abd, abe, abf, abg, acd, ace, acf, acg, adf, adg, aef, \text{ and } aeg.$$

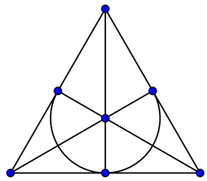


Figure 2: The Fano 7_3-configuration.

Since $B_1 = abc$ and $B_2 = ade$ are bases of $M_{tri}(\mathcal{C})$, the basis exchange property applies to them. This means that if $x \in B_1 - B_2 = bc$, there must exist some $y \in B_2 - B_1 = de$ such that $B_1 - x \cup y \in \mathcal{B}$. Consequently, letting $x = b$, we find at least one of acd and ace must be a base, hence not a trilateral. Likewise, letting $x = c$, it follows that at least one of abd and abe is not a trilateral.

Applying a similar analysis to the pair of bases $B_1 = abc, B_2 = afg$, we find that at least one of acf and acg is not a trilateral, and at least one of abf and abg is not a trilateral. Finally, given $B_1 = ade, B_2 = afg$, we find that at least one of aef and aeg is not a trilateral, and at least one of adf and adg is not a trilateral. Hence at least six of the 12 possible non-collinear triples are not trilaterals, so at most six are trilaterals. □

Corollary 1.4. *Suppose an n_3-configuration $\mathcal{C}$ induces a trilateral matroid $M_{tri}(\mathcal{C}) = (E, \mathcal{B})$. Then $\mathcal{C}$ contains at most $2n$ trilaterals.*

Although Corollary 1.4 admittedly serves as a crude necessary condition for an n_3-configuration to induce a trilateral matroid, it does permit us to eliminate some of the smallest n_3-configurations from consideration, such as the Fano 7_3-configuration (which contains 28 trilaterals) and also the Möbius-Kantor 8_3-configuration (which contains 24 trilaterals). Additionally, two of the three non-isomorphic 9_3-configurations may be dismissed from consideration by this criterion, although the Pappus 9_3-configuration, which contains 18 trilaterals, is still a possibility. We shall soon see, though, that the Pappus configuration does not induce a trilateral matroid on its points.

The upper bound indicated by Proposition 1.3 is sharp, for it turns out that the Desargues 10_3-configuration induces a trilateral matroid. Each of the points of the Desargues configuration is incident to six trilaterals.

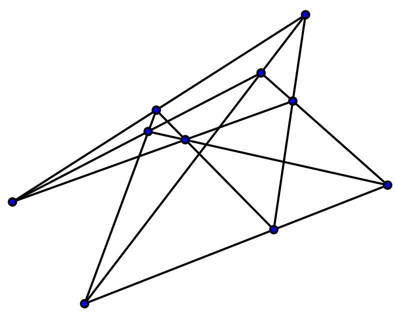


Figure 3: The Desargues 10_3-configuration.

We now establish our main result. This will require the introduction of two types of geometric obstructions (near-complete quadrangles and near-pencils) that, when present within an n_3-configuration $\mathcal{C}$, individually preclude the existence of $M_{tri}(\mathcal{C})$.

Definition 1.5. A *near-complete quadrangle* $[ab:cd]$ consists of four points a, b, c, and d of the configuration, no three of which are collinear, for which five of the six possible lines connecting each pair of points exist within the configuration, except for the pair cd.

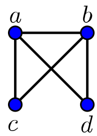


Figure 4: Near-complete quadrangle $[ab:cd]$.

For example, we note the presence of the near-complete quadrangle $[ab : cd]$ in the Pappus configuration in Figure 5.

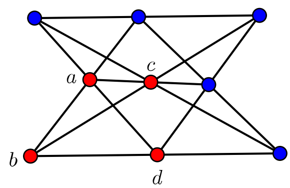


Figure 5: The Pappus 9_3-configuration.

It is important to note that, by our conventions, a complete quadrangle determined by points $\{a, b, c, d\}$ does *not* contain a near-complete quadrangle $[ab: cd]$, since there exists a line in the configuration incident to both c and d. So the Desargues configuration, for example, possesses five complete quadrangles but no near-complete quadrangle.

As we shall witness in greater detail, n_3-configurations which induce trilateral matroids may contain complete quadrangles. Indeed, in a linear matroid, given any two points, at most one line passes between them. But, two trilaterals (call them abc and abd) may share the points a, b provided that acd and bcd are also trilaterals, that is, that line cd is also present within the configuration.

Definition 1.6. A *near-pencil* $[a:bcd]$ consists of four points a, b, c, and d of the configuration, with a incident to each of b, c, and d, and with bcd a line of the configuration.

We regard the near-pencil $[a : bcd]$ in the Möbius-Kantor 8_3-configuration given in Figure 7.

The notations $[ab:cd]$ and $[a:bcd]$ for a near-complete quadrangle and a near-pencil, respectively, are similar in that the points incident to three of the lines which determine the object appear to the left of the colon, and those points incident to two lines appear to the right of the colon.

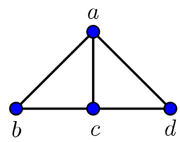


Figure 6: Near-pencil $[a\!:\!bcd]$.

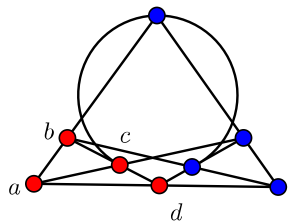
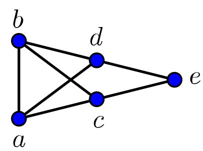


Figure 7: The Möbius-Kantor 8_3-configuration.

Theorem 1.7. *Let $\mathcal{C}$ be an n_3-configuration, and let $\mathcal{B}$ be the set of the non-trilaterals of $\mathcal{C}$. Then $\mathcal{C}$ induces a trilateral matroid $M_{tri}(\mathcal{C})$ if and only if no four points of $\mathcal{C}$ determine either a near-complete quadrangle or a near-pencil.*

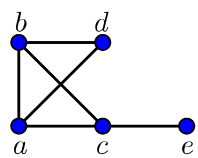
Proof. $(\Rightarrow)$ First suppose that $\mathcal{C}$ contains a near-complete quadrangle $[ab\!:\!cd]$. Let e be the third point on line ace.

Case 1: bde is a line in $\mathcal{C}$. Then the following subfiguration is present inside $\mathcal{C}$.



Let $B_1 = ace$ and $B_2 = bde$; both $B_1, B_2 \in \mathcal{B}$. Then $B_1 - B_2 = ac$ and $B_2 - B_1 = bd$. Let $x = c \in B_1 - B_2$; then $B_1 - x = ae$. But both abe and ade are trilaterals, so $B_1 - x \cup y \notin \mathcal{B}$ for all $y \in B_2 - B_1$. Hence $\mathcal{B}$ cannot be the set of bases of a matroid.

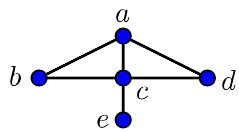
Case 2: bde is not a line in $\mathcal{C}$. Then inside of $\mathcal{C}$ we have



Note that edge be cannot be present, for if so point b would have four lines incident to it, but every point in an n_3-configuration is incident to three lines.

Let $B_1 = abe, B_2 = acd \in \mathcal{B}$. Take $e \in B_1 - B_2$; we have $B_1 - e = ab$. But $B_2 - B_1 = cd$, and both abc and abd are trilaterals. Hence $\mathcal{B}$ cannot be the set of bases of a matroid.

Now suppose $\mathcal{C}$ contains a near-pencil $[a\!:\!bcd]$ as indicated in the diagram. Let e be the third point on line ace.



We have $B_1 = ace, B_2 = bcd \in \mathcal{B}$. Choose $e \in B_1 - B_2$. Then $B_1 - e = ac$. But $B_2 - B_1 = bd$, and both abc and acd are trilaterals. Hence $\mathcal{B}$ cannot be the set of bases of a matroid.

($\Leftarrow$) Suppose that $\mathcal{C}$ does not induce a trilateral matroid $M_{tri}(C)$. Since $\mathcal{B}$ cannot be the set of bases of a matroid, there must exist a pair B_1, B_2 in $\mathcal{B}$ for which the basis exchange property is violated. So there must exist $x \in B_1 - B_2$ such that for all $y \in B_2 - B_1$, $B_1 - x \cup y$ is a trilateral.

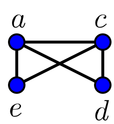
There are several cases to consider, some of which are vacuous.

Case 1: $B_1 = B_2$. Then $B_1 - B_2 = \emptyset$, so a violation of the basis exchange property cannot occur in this circumstance.

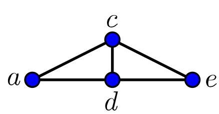
Case 2: $B_1 = abc$, $B_2 = abd$ (distinct letters label distinct points in $\mathcal{C}$.) Then $B_1 - B_2 = c$ and $B_2 - B_1 = d$. For a violation to occur, we require that $B_1 - c \cup d$ be a trilateral. But $B_1 - c \cup d = B_2 \in \mathcal{B}$. Hence no violation can occur in this case as well.

Case 3: $B_1 = abc$, $B_2 = ade$. Then $B_1 - B_2 = bc$ and $B_2 - B_1 = de$. Without loss of generality we assume that $x = b$. For a violation of the basis exchange property to occur, both acd and ace must be trilaterals.

Subcase 3.1: ade is a non-collinear non-trilateral. Then $[ac : de]$ is a near-complete quadrangle.



Subcase 3.2: ade is a line. Then $[c\!:\!ade]$ is a near-pencil.



Case 4: $B_1 = abc$, $B_2 = def$, so $B_1 \cap B_2 = \emptyset$. We may let $x = a$ without loss of generality. So for a violation of the basis exchange property to occur, all three of bcd, bce and bcf must be trilaterals.

Subcase 4.1: Two of d, e, f are collinear with b. Without loss of generality, we assert that bde is a line. Then $[c\!:\!bde]$ is a near-pencil.

Subcase 4.2: No two of d, e, f are collinear with b. Then b must be incident to four lines, a contradiction. □

2 Examples

We have already observed, by Corollary 1.4, that the Fano 7_3-configuration, the Möbius-Kantor 8_3-configuration, and two of the three 9_3-configurations cannot induce trilateral matroids. It is worth noting that the Fano configuration contains no near-complete quadrangle, but many near-pencils; given any line abc of the Fano configuration, and any fourth point d not on this line, then $[d : abc]$ is a near-pencil. Since by Figure 5 we see that the Pappus 9_3-configuration contains a near-complete quadrangle, by Theorem 1.7 it also cannot induce a trilateral matroid.

It is worth noting that there is a matroid associated with the Fano configuration in the sense that no three-element subset of the point set can be an independent set, since every point triple determines a trilateral. But this is really a degenerate case; the matroid is $U_{2,7}$, so every 2-element subset of the point set is independent, but no 3-element subset is. Since $U_{2,7}$ is a rank-2 matroid, and not rank-3, we will not deem it to be a trilateral matroid.

The smallest configuration which does generate a rank-3 trilateral matroid is the Desargues 10_3-configuration provided in Figure 3. There we may readily observe that the configuration contains neither a near-complete quadrangle nor a near-pencil. Since the Desargues configuration contains 20 trilaterals, there are $\binom{10}{3} - 20 = 100$ bases in the associated matroid. Each of the other nine 10_3-configurations contains at least one near-complete quadrangle, and therefore the Desargues configuration is the smallest configuration which induces a trilateral matroid.

Figure 8 depicts a geometric representation of the the trilateral matroid induced by the Desargues configuration in the following fashion. If three points happen to be collinear in the geometric representation, then these points describe a trilateral in the original configuration. Each of the five four-point lines in this representation thus describes four point triples which determine trilaterals; these four points consequently are associated with a complete quadrangle in the Desargues configuration. The Desargues configuration contains five such complete quadrangles, and each point of the configuration is involved in two quadrangles. So we arrive at the star in Figure 8, which is itself a $(10_2, 5_4)$-configuration. This means that there are ten points, with each point incident to two lines, and five lines, with each line incident to four points.

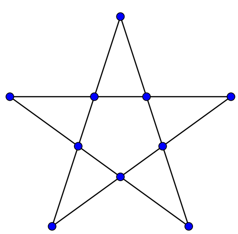


Figure 8: A geometric representation of the trilateral matroid associated with the Desargues configuration.

Interestingly, there is no 11_3-configuration which induces a trilateral matroid. In fact, each of the 31 11_3-configurations contains at least one near-complete quadrangle.

Among the 229 12_3-configurations, there is only one which does not contain a near-complete quadrangle. This configuration also happens not to contain a near-pencil, and hence induces a trilateral matroid on its points. This configuration is the Coxeter 12_3-configuration shown in Figure 9.

12A:

l_1	l_2	l_3	l_4	l_5	l_6	l_7	l_8	l_9	l_{10}	l_{11}	l_{12}
1	1	1	2	2	3	3	4	5	5	6	7
2	4	6	6	8	4	9	8	7	9	9	8
3	5	7	10	11	12	11	10	11	10	12	12

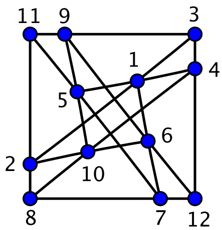


Figure 9: The Coxeter 12_3-configuration.

The automorphism group of this configuration has order 72. This configuration is listed as D88 in Daublebsky von Sterneck's enumeration of the first 228 12_3-configurations in 1895 [4]; the last of the 229 12_3-configurations was found much later in 1990 by Gropp [5]. All 229 12_3-configurations have been recently re-examined in [1], and the provided geometric realization of D88 in Figure 9 stems from this work. Again, by inspection, we see that no near-complete quadrangle is present, as well as no near-pencil.

This configuration contains 12 trilaterals. Each point of the configuration is incident to three of them, with no pair of points belonging to the same trilateral. So these trilaterals are blocks of another 12_3-configuration defined on the same set of points, namely

t_1	t_2	t_3	t_4	t_5	t_6	t_7	t_8	t_9	t_{10}	t_{11}	t_{12}
1	1	1	2	2	3	4	4	5	6	6	7
2	3	5	3	8	9	5	8	9	7	9	8
6	4	7	11	10	12	10	12	11	12	10	11

It is not hard to see that this configuration is isomorphic to the previous one. In fact, this is the first instance of a more general phenomenon.

Theorem 2.1. *Suppose that an n_3-configuration $\mathcal{C}$ has n trilaterals, with every point incident to three trilaterals and no pair of points incident to more than one trilateral. Let $\mathcal{C}_{tri}$ be the n_3-configuration formed by these n trilaterals. Then $\mathcal{C}_{tri} \cong \mathcal{C}$.*

Proof. It suffices to show that the dual of $\mathcal{C}$ and the dual of $\mathcal{C}_{tri}$ are isomorphic. Regard one of the lines of the respective duals; call this line p. This is a point of each of the original configurations. The local structure is indicated by the diagram in Figure 10.

We associate the line a with the trilateral t_a as follows: of the three trilaterals incident to p, t_a is chosen so that a is *not* involved in determining this trilateral. In a similar manner, line b is identified with trilateral t_b and line c is identified with trilateral t_c. Our hypotheses allow us to carry this correspondence across the respective dual configurations, with the resulting correspondence between the points of $\mathcal{C}$ and of $\mathcal{C}_{tri}$ (the blocks of the duals) the identity map. Therefore $\mathcal{C}_{dual} \cong (\mathcal{C}_{tri})_{dual}$, whence $\mathcal{C} \cong \mathcal{C}_{tri}$. □

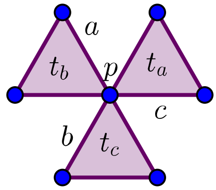


Figure 10: Lines and trilaterals incident to point p.

Next, among the 2036 13_3-configurations, there are four which do not contain a near-complete quadrangle. And among these four, there is only one which does not contain a near-pencil. This is Configuration 13A, given in Figure 11. The automorphism group of

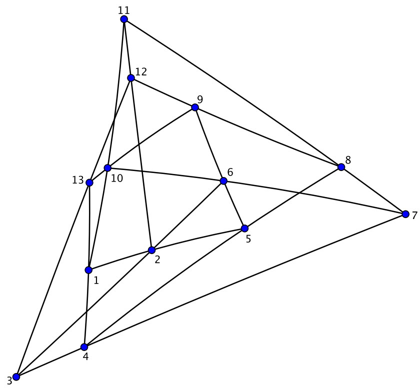


Figure 11: A 13_3-configuration which induces a trilateral matroid.

this configuration has order 39. The configuration contains 13 trilaterals, and each point is incident to three trilaterals, with no pair of points incident to more than one trilateral. So we may derive an associated 13_3-configuration by listing these trilaterals:

t_1	t_2	t_3	t_4	t_5	t_6	t_7	t_8	t_9	t_{10}	t_{11}	t_{12}	t_{13}
1	1	1	2	2	3	4	4	5	6	6	7	9
2	3	5	3	8	11	5	8	10	7	9	8	11
4	7	6	9	10	12	11	13	12	13	11	12	13

By Theorem 2.1 this configuration is isomorphic to Configuration 13A. Configuration 13A is also isomorphic to the cyclic configuration $C_3(13, 1, 4)$, given combinatorially by regarding the lines $\{j, j + 1, j + 4\}$ mod 13 for $0 \leq j \leq 12$:

l_1	l_2	l_3	l_4	l_5	l_6	l_7	l_8	l_9	l_{10}	l_{11}	l_{12}	l_{13}
0	1	2	3	4	5	6	7	8	9	10	11	12
1	2	3	4	5	6	7	8	9	10	11	12	0
4	5	6	7	8	9	10	11	12	0	1	2	3

One may employ these point labels to construct the *Paley graph* of order 13 as follows. Draw an edge between labels a and b if and only if $a - b$ is a perfect square mod 13. This means that $a - b$ can be ± 1, ± 3, or ± 4 mod 13. We thus obtain the following graph where each edge is contained in exactly one triangle, and each triangle in the graph corresponds to a trilateral of the 13_3-configuration.

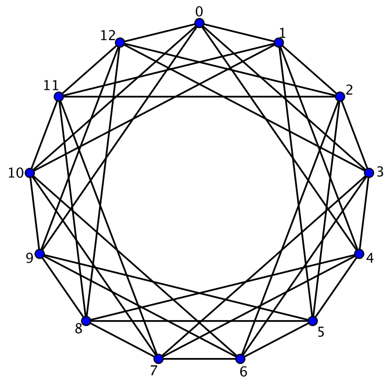


Figure 12: Paley graph associated with Configuration 13A.

More generally, the cyclic n_3-configuration $C_3(n, k, m)$ is given by the lines $\{j, j + k, j + m\}$ mod n for $0 \le j \le n - 1$.

Proposition 2.2. *For $n \ge 13$, the cyclic configuration $C_3(n, 1, 4)$ induces a trilateral matroid on n trilaterals which equals the linear matroid on $C_3(n, 3, 4)$. In other words, $M_{tri}(C_3(n, 1, 4)) = M(C_3(n, 3, 4))$. Moreover, $C_3(n, 1, 4) \cong C_3(n, 3, 4)$.*

Proof. In order to determine the trilaterals of $C_3(n, 1, 4)$, it suffices to ascertain the trilaterals which involve 0, and then extend from this via a cyclic pattern. The trilaterals involving 0 are:

- $0\ 3\ 4$ (using the lines $\{0, 1, 4\}$, $\{3, 4, 7\}$, and $\{n - 1, 0, 3\}$)
- $n-4\, n-3\, 0$ (using the lines $\{n-4, n-3, 0\}$, $\{n-1, 0, 3\}$, and $\{n-5, n-4, n-1\}$)
- $n - 3\ 0\ 1$ (using the lines $\{n - 4, n - 3, 0\}$, $\{n - 3, n - 2, 1\}$, and $\{0, 1, 4\}$)

Since $n \ge 13$, no extra trilateral involving 0 is formed (for example, if $n = 12$, then 0 4 8 would be a trilateral.) Hence we see, after extending cyclically, that the trilaterals of $C_3(n, 1, 4)$ form their own configuration, namely $C_3(n, 3, 4)$, and thus $M_{tri}(C_3(n, 1, 4))$ is the linear matroid corresponding to $C_3(n, 3, 4)$. Finally we may recognize that $C_3(n, 1, 4)$ is isomorphic to $C_3(n, 3, 4)$ either by utilizing Theorem 2.1 or by applying the correspondence $t \to (4 - t) \mod n$. □

It turns out that $C_3(16, 1, 4)$ and $C_3(16, 1, 7)$ are the smallest examples of non-isomorphic cyclic $C_3(n, k, m)$ configurations having n trilaterals each, and hence their corresponding trilateral matroids (which are isomorphic to the linear matroids associated with the respective original configurations) are non-isomorphic to each other as well.

It is possible, however, for a non-cyclic n_3-configuration to induce a trilateral matroid on its n trilaterals, with the trilaterals capable of determining an n_3-configuration in their own right, without the original configuration needing to be cyclic. We have already seen

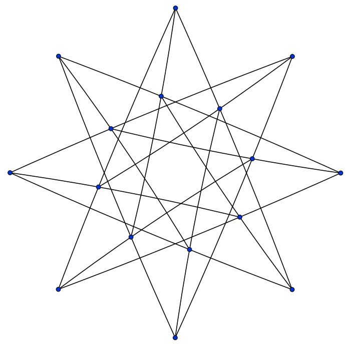


Figure 13: A non-cyclic 16_3-configuration whose trilateral matroid is isomorphic to the linear matroid associated with the configuration.

one example of this with the Coxeter 12_3-configuration given in Figure 9. Another example is the 16_3-configuration provided in Figure 13 whose automorphism group has order 32.

It is additionally possible for an n_3-configuration possessing n trilaterals to induce a trilateral matroid that is not isomorphic to the linear matroid associated with the original configuration. Figure 14 gives a diagram of such a configuration, a 20_3-configuration containing 20 trilaterals. It contains two points which are involved in six trilaterals and four points involved in four trilaterals. A geometric representation of the matroid is also provided.

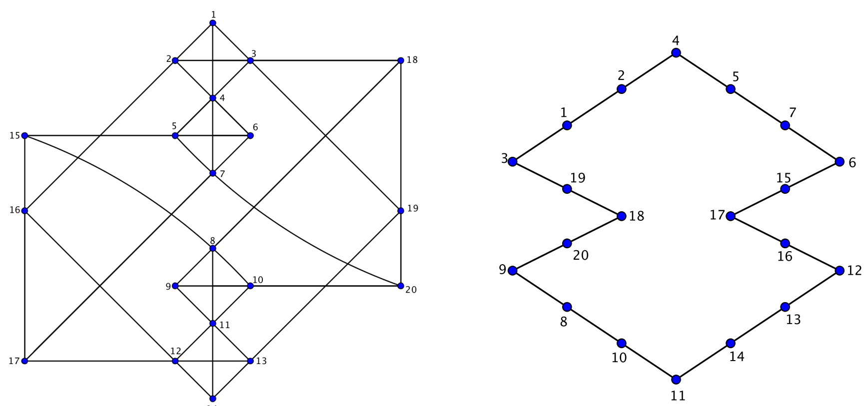


Figure 14: A 20_3-configuration with 20 trilaterals whose trilateral matroid is not isomorphic to the linear matroid of the configuration, and a geometric representation of its trilateral matroid.

We next offer an example of of an 18_3-configuration possessing 20 trilaterals which induces a trilateral matroid. In Figure 15 we provide a picture of this configuration (with several pseudolines) and the accompanying geometric representation of its trilateral matroid. This example presents another instance, in addition to the Desargues 10_3-configuration, of an n_3-configuration containing more than n trilaterals which induces a trilateral matroid.

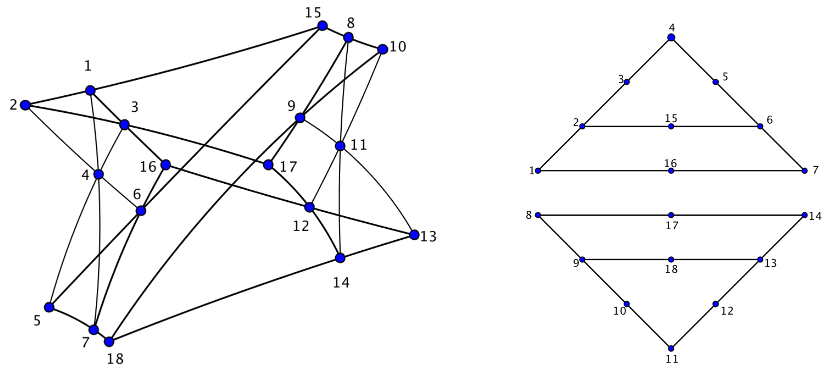


Figure 15: An 18_3-configuration with 20 trilaterals, and a geometric representation of its trilateral matroid.

Note that this configuration contains four complete quadrangles.

We now return to the enumeration of the smallest n_3-configurations which induce trilateral matroids. There are four 14_3-configurations which do so. We label these configurations as 14A, 14B, 14C and 14D, and provide combinatorial depictions of them.

	l_1	l_2	l_3	l_4	l_5	l_6	l_7	l_8	l_9	l_{10}	l_{11}	l_{12}	l_{13}	l_{14}
	1	1	1	2	2	3	3	4	5	5	6	7	8	9
14A:	2	4	6	4	8	7	8	11	6	12	9	10	13	11
	3	5	7	9	10	12	11	12	13	14	10	14	14	13

	l_1	l_2	l_3	l_4	l_5	l_6	l_7	l_8	l_9	l_{10}	l_{11}	l_{12}	l_{13}	l_{14}
	1	1	1	2	2	3	3	4	5	5	6	6	7	8
14B:	2	4	6	4	9	7	10	11	10	12	8	9	9	11
	3	5	7	8	12	11	12	13	14	13	10	13	14	14

	l_1	l_2	l_3	l_4	l_5	l_6	l_7	l_8	l_9	l_{10}	l_{11}	l_{12}	l_{13}	l_{14}
	1	1	1	2	2	3	3	4	5	5	6	7	7	10
14C:	2	4	6	4	8	6	13	11	8	12	8	9	10	11
	3	5	7	9	10	11	14	12	13	14	9	14	12	13

	l_1	l_2	l_3	l_4	l_5	l_6	l_7	l_8	l_9	l_{10}	l_{11}	l_{12}	l_{13}	l_{14}
	1	1	1	2	2	3	3	4	5	5	6	6	7	7
14D:	2	4	6	4	10	8	12	11	8	10	8	10	9	11
	3	5	7	9	13	11	14	12	13	14	9	12	14	13

These configurations contain 14, 10, 10, and 6 trilaterals, respectively. Also, their automorphism groups have orders 14, 1, 4, and 8, respectively.

Figure 16 gives a realization of Configuration 14A, which is isomorphic to the cyclic configuration $C_3(14, 1, 4)$. Hence we know its trilateral matroid is isomorphic to its linear matroid by Proposition 2.2.

Configurations 14B and 14C both contain 10 trilaterals, so it is conceivable that their associated trilateral matroids could be isomorphic. But they are not, for 14B has three points which are each incident to three trilaterals and one point which is incident to only one trilateral, whereas Configuration 14C has two points each incident to three trilaterals

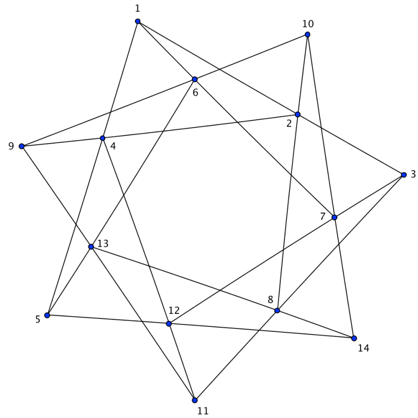


Figure 16: Configuration 14A.

and no point incident to only one trilateral. Figure 17 gives geometric representations of the trilateral matroids associated with Configurations 14B and 14C, respectively.

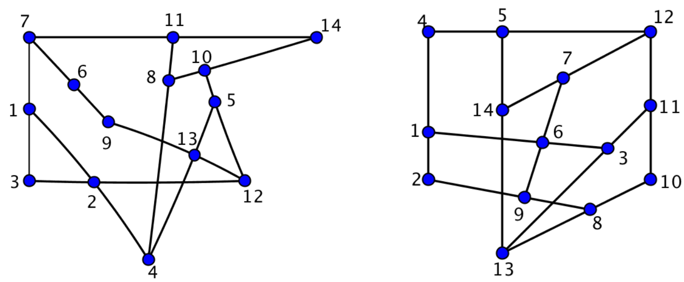


Figure 17: Geometric representations for trilateral matroids for Configurations 14B and 14C.

Figure 18 is a rendering for Configuration 14D with several pseudolines, along with a geometric representation of its associated trilateral matroid.

Proceeding to the $n = 15$ setting, we encounter a substantial increase, to 220, of the number of 15_3-configurations which induce trilateral matroids. One such example is the Cremona-Richmond configuration provided in Figure 19. It is the smallest example of a trilateral-free n_3-configuration. As it is trilateral-free, the trilateral matroid it induces is the uniform matroid on 15 points $U_{3,15}$.

Another example is the cyclic configuration $C_3(15, 1, 4)$, whose induced trilateral matroid (with 15 trilaterals) is isomorphic to the linear matroid on $C_3(15, 1, 4)$ by Proposition 2.2. Its automorphism group has order 30. Each of the other 15_3-configurations which induces a trilateral matroid contains k trilaterals, where $k \in \{4, 6, 7, 8, 9, 10, 11, 12, 13, 14\}$.

It is clearly not the case that for all n, there exists a one-to-one correspondence between the trilateral matroids themselves and the n_3-configurations which induce them. We know this because there are four non-isomorphic trilateral-free 18_3-configurations [3], so each consequently must induce the same uniform matroid on 18 points.

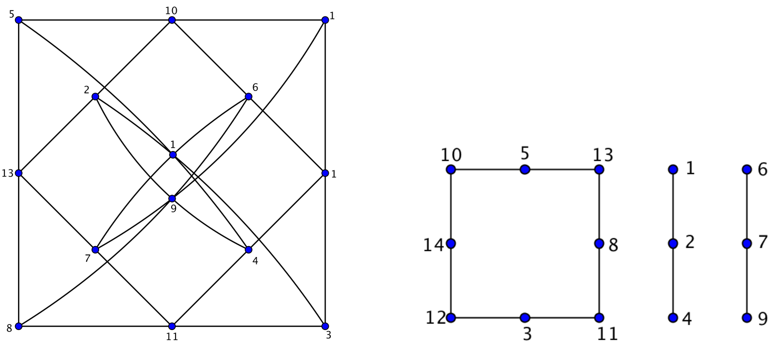


Figure 18: Configuration 14D and its trilateral matroid.

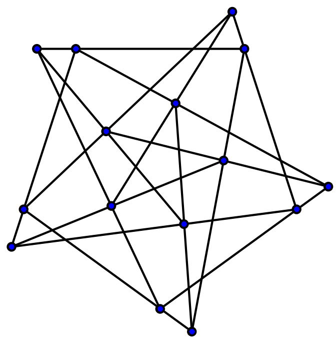


Figure 19: The Cremona-Richmond 15_3-configuration.

It is of interest to contemplate whether smaller non-isomorphic n_3-configurations exist that induce isomorphic trilateral matroids, and indeed this turns out to be true. In fact, this property is satisfied by the following pair of non-isomorphic 15_3-configurations given in Figure 20. Each contains 8 trilaterals and has a symmetry group of order 48. The

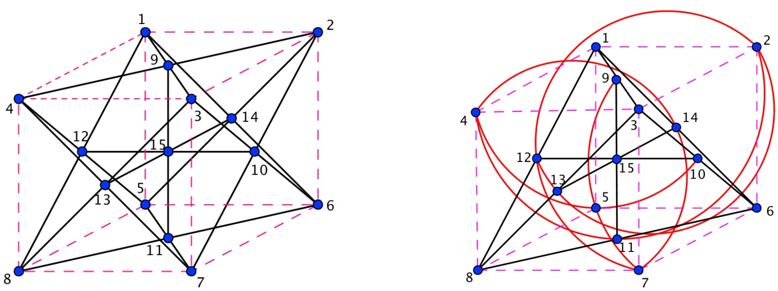


Figure 20: Non-isomorphic 15_3-configurations which induce the same trilateral matroid on 15 points.

set of points for both configurations consists of the eight vertices of a cube, the centers of the six faces of the cube, and the center of the cube itself. In the former configuration the diagonally-opposing points in each face of the cube are incident via a line which passes through the center of the same face, whereas in the latter configuration one pair of diagonally-opposing points in each face are incident via a "line" which passes through the center of the opposite face. The eight trilaterals involved in these respective configurations are identical, and thus their corresponding trilateral matroids are the same. Figure 21 gives this matroid, which is isomorphic to $U_{2,4} \oplus U_{2,4} \oplus U_{3,7}$. Hence the number of

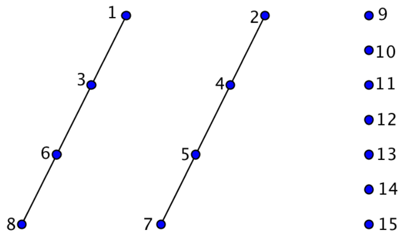


Figure 21: The common trilateral matroid.

trilateral matroids that are induced from 15_3-configurations is smaller than the number of 15_3-configurations which induce trilateral matroids. Our calculations indicate that there are 214 non-isomorphic trilateral matroids that may be found from the 220 15_3-configurations which induce trilateral matroids.

We conclude this section with a table which summarizes the current state of affairs. Here $\#_c(n)$ denotes the number of non-isomorphic n_3-configurations, $\#_{tri}(n)$ denotes the number of these configurations which induce trilateral matroids, and $\#_{mat}(n)$ denotes the number of non-isomorphic trilateral matroids which arise from these configurations.

n	$\#_c(n)$	$\#_{tri}(n)$	$\#_{mat}(n)$
7	1	0	0
8	1	0	0
9	3	0	0
10	10	1	1
11	31	0	0
12	229	1	1
13	2036	1	1
14	21399	4	4
15	245342	220	214

3 A point-plane configuration

A *point-plane n_3-configuration* is an incidence structure consisting of n distinct points and n distinct planes for which each point is incident with three planes, each plane is incident with three points, and any two points are incident with at most one common plane. In such a configuration, we deem a trilateral to be a cyclically ordered set $\{p_0, \pi_0, p_1, \pi_1, p_2, \pi_2\}$ of pairwise distinct points p_i and pairwise distinct planes π_i such that p_i is incident with π_{i-1} and π_i for each $i \in \mathbb{Z}_3$. Once more we may without ambiguity shorten this notation by listing only the points of the trilateral as $\{p_0, p_1, p_2\}$, or more simply as $p_0p_1p_2$.

In Figure 22 we offer an example of a point-plane 12_3-configuration which induces a trilateral matroid on its points. The 12 points are selected from the 20 vertices of the regular dodecahedron so that each of the twelve pentagonal faces contains three points; note that each of the 12 points is the intersection of three faces, so a point-plane 12_3-configuration is achieved. We observe that each of the eight unlabeled red points in the

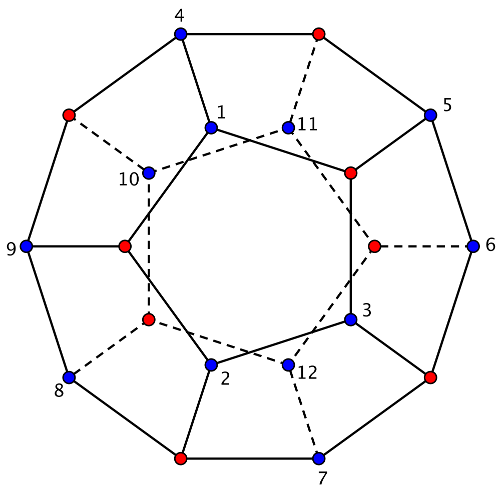


Figure 22: A 12_3 point-plane configuration which induces a trilateral matroid.

diagram corresponds to a trilateral, and that this trilateral may be specified uniquely by cycling through the configuration points that are immediately adjacent to the red point. For example, the triple $\{1, 3, 5\}$ defines a trilateral. We start at 1, then pass through the plane containing both 1 and 3 to 3. We then pass through the plane containing both 3 and 5 to 5, and then finally pass through the plane containing both 5 and 1 back to 1 to complete the cycle. Here are the eight trilaterals.

t_1	t_2	t_3	t_4	t_5	t_6	t_7	t_8
1	1	2	3	4	4	6	8
2	3	7	6	5	9	11	10
9	5	8	7	11	10	12	12

Figure 23 gives a geometric representation of the trilateral matroid.

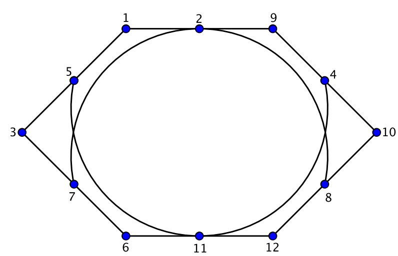


Figure 23: The trilateral matroid of the 12_3 point-plane configuration.

After identifying each trilateral with its corresponding red point in Figure 22, we recognize that the trilateral matroid may also be represented as a point-plane configuration, namely an $(8_3, 12_2)$-configuration. This means the configuration has eight points, with three planes incident to each point, and twelve planes, with two points incident to each plane.

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