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Products of subgroups, subnormality, and relative orders of elements

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Abstract

Let G be a group. We give an explicit description of the set of elements $x \in G$ such that $x^{|G:H|} \in H$ for every subgroup of finite index $H \leqslant G$. This is related to the following problem: given two subgroups H and K, with H of finite index, when does $|HK : H|$ divide $|G : H|$?

Keywords: Relative order, product of subgroups, subnormal subgroup.

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1 Introduction

Let G be an arbitrary group, and let us write $H \leqslant_f G$ to say that H is a subgroup of G of finite index. Let $x \in G$ and $H \leqslant_f G$. If H is a normal subgroup of G, then it is easy to see that $x^{|G:H|} \in H$. The same is not true in general: fixed $H \leqslant_f G$, the set $\{x \in G : x^{|G:H|} \in H\}$ may not even be closed under multiplication (take $G = \mathrm{Sym}(3)$ and $H = \langle (1\ 2) \rangle$). The goal of this paper is to understand this phenomenom and its implications. As far as we can see, this has not been dealt with before in the literature.

Definition 1.1. Let $x \in G$ and $H \leqslant G$. The *relative order* of x with respect to H is

$$o_H(x) := |\langle x \rangle : \langle x \rangle \cap H|.$$

The following result is proved in Section 2.

Lemma 1.2. *Let $n \geq 1$. Then $x^n \in H$ if and only if $o_H(x)$ is finite and divides n.*

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Given $H, K \leqslant G$, $|HK : H|$ is the cardinality of the set of all cosets of H which are intersected by K (we refer to Section 2 for more details). Since $o_H(x) = |H\langle x \rangle : H|$, we obtain

Corollary 1.3. *$x^{|G:H|} \in H$ if and only if $|H\langle x \rangle : H|$ divides $|G : H|$.*

If $H, K \leqslant_f G$, then $|HK : H|$ divides $|G : H|$ if and only if $|HK : K|$ divides $|G : K|$. If G is finite, both are equivalent to $|HK|$ dividing $|G|$. In Section 3, we prove the following two results:

Proposition 1.4. *Let $H \lhd \lhd G$. Then $|HK : K|$ divides $|G : K|$ for every $K \leqslant_f G$.*

Theorem 1.5. *Let $H \leqslant_f G$. Then $H \lhd \lhd G$ if and only if $|HK : H|$ divides $|G : H|$ for every $K \leqslant G$.*

The converse of Proposition 1.4 is not true in general (see Example 5.11). In particular, some attention is needed with subgroups of infinite index. During the preparation of this manuscript, the author found out that the finite version of Theorem 1.5 already appeared in [5, Theorem 2]. In Section 4, we study the following class of subgroups

Definition 1.6. A subgroup $H \leqslant_f G$ is *exponential* if $x^{|G:H|} \in H$ for every $x \in G$.

This is a generalization of subnormality, and we prove that it is equivalent to normality in some cases, namely for the Hall subgroups of a finite group and for the maximal subgroups of a solvable group. From the dual point of view, in Section 5 we study the set

$$S(G) := \{x \in G : x^{|G:H|} \in H \text{ for every } H \leqslant_f G\}.$$

At first glance $S(G)$ is quite elusive, and indeed working directly with the definition is not easy. Using the results of Section 3, we give an elementary proof of the next theorem. Given $N \lhd G$, let $F_N(G)$ be the preimage of $F(G/N)$, where $F(G)$ denotes the Fitting subgroup of G.

Theorem 1.7. *If G is any group, then $S(G) = \cap_{N \lhd_f G} F_N(G)$.*

In particular, $S(G) = F(G)$ when G is finite (Proposition 5.1). Of course, Theorem 1.7 implies that $S(G)$ is closed under multiplication, a fact which is not immediately clear from the definition.

2 Preliminaries

We start with the proof of the key Lemma 1.2.

Proof of Lemma 1.2. Let $ord_H(x) := \min\{n \geq 1 : x^n \in H\}$. We first notice that $o_H(x) = ord_H(x)$. Indeed, from the definitions we have $o_H(x) = o_{H\cap\langle x \rangle}(x)$ and $ord_H(x) = ord_{H\cap\langle x \rangle}(x)$. The fact that $o_{H\cap\langle x \rangle}(x) = ord_{H\cap\langle x \rangle}(x)$ is a simple exercise. Now, the "if" part of the statement is trivial. On the other hand, if $x^n \in H$ for some $n \geq 1$, then clearly $ord_H(x) < \infty$. Let $n = q \cdot ord_H(x) + r$ with $r, q \geq 0$ and $r < ord_H(x)$. Since H is a subgroup, the fact that $x^n = x^{q \cdot ord_H(x)} x^r \in H$ implies that $x^r \in H$, which in turn means $r = 0$. $\square$

The bulk of this paper is about finite groups. We summarize here the basic tools and notation that are used with regard to general non-finite groups. Let G be an arbitrary group and $H, K \leqslant G$. If H and K have finite index, then so has $H \cap K$, and $|G : H \cap K| = |G : H||H : H \cap K|$. As we have said in the introduction, we write $|HK : H|$ for the cardinality of the set of all cosets of H which are intersected by K. This is not accidental, because the product set $HK = \{hk : h \in H, k \in K\}$ is a union of cosets of H. It is not relevant to distinguish between left-cosets and right-cosets, since $k \in Hx$ if and only if $k^{-1} \in x^{-1}H$. We also observe that $|HK : H| = |K : H \cap K| = |KH : H|$.
The *finite residual* $R(G)$ is the intersection of the subgroups of G of finite index. If $R(G) = 1$, then G is said to be *residually finite*. It is easy to check that $G/R(G)$ is always residually finite. Finally, the *Fitting subgroup* $F(G)$ is defined as the subgroup generated by the nilpotent normal subgroups, and coincides with the set of the elements $x \in G$ such that the normal closure $\langle x\rangle^G$ is nilpotent [1]. In general, this is a stronger condition than $\langle x\rangle$ being subnormal in G. If G is finite, then $F(G)$ itself is nilpotent, i.e. it is the largest nilpotent normal subgroup.

3 Products of subgroups

The proof of Proposition 1.4 follows immediately from the following

Lemma 3.1. *Let $H \lhd M \leqslant G$, and let $K \leqslant_f G$. Then $|HK : K|$ divides $|MK : K|$.*

Proof. We have to prove that the ratio

$$\frac{|MK : K|}{|HK : K|} = \frac{|M : M \cap K|}{|H : H \cap K|}$$

is an integer. Now $H \lhd M$ implies that $H(M \cap K)$ is a subgroup of M, and so we can write

$$\begin{aligned} |M : M \cap K| &= |M : H(M \cap K)||H(M \cap K) : M \cap K| \\ &= |M : H(M \cap K)||H : H \cap K|. \end{aligned}$$

In particular, the original ratio equals $|M : H(M \cap K)|$. □

We continue with the easiest direction of Theorem 1.5.

Lemma 3.2. *Let $H \leqslant_f M \leqslant_f G$, and let $K \leqslant G$. Then $\frac{|HK:H|}{|MK:M|} = |M \cap K : H \cap K|$.*

Proof. We have

$$\begin{aligned} \frac{|HK : H|}{|MK : M|} &= \frac{|K : H \cap K|}{|K : M \cap K|} \\ &= \frac{|K : M \cap K||M \cap K : H \cap K|}{|K : M \cap K|} \\ &= |M \cap K : H \cap K|. \end{aligned}$$

□

We prove the claim of Theorem 1.5 by induction on the subnormal defect of H, so let $H \lhd_f M \lhd\lhd_f G$, and $K \leqslant G$. Using Lemma 3.2, we have

$$\frac{|G : H|}{|HK : H|} = \frac{|G : M||M : H|}{|MK : M||M \cap K : H \cap K|}.$$

By induction, it is sufficient to prove that $\frac{|M:H|}{|M\cap K:H\cap K|}$ is an integer. Now $H \lhd M$ implies that $H(M \cap K)$ is a subgroup of M, and so we can write

$$\begin{aligned} |M : H| &= |M : H(M \cap K)||H(M \cap K) : H| \\ &= |M : H(M \cap K)||M \cap K : H \cap K|. \end{aligned}$$

This concludes the proof of the "only if" part.

3.1 The Kegel-Wielandt-Kleidman theorem, revisited

Definition 3.3. Let G be a finite group, $H \leqslant G$, and let p be a prime. Then H is *p-subnormal* in G if $H \cap P$ is a p-Sylow of H for every p-Sylow P of G.

We characterize p-subnormality with the following

Lemma 3.4. *A subgroup H is p-subnormal if and only if $|HP|$ divides $|G|$ for every p-Sylow $P \leqslant G$.*

Proof. We have that $H \cap P$ is a p-Sylow of H if and only if $|H : H \cap P| = |HP : P|$ is not divisible by p. Since $|H : H \cap P|$ is a divisor of $|G|$, the last condition is equivalent to $|HP : P|$ dividing $|G : P|$, i.e. $|HP| \mid |G|$. □

The famous Kegel-Wielandt conjecture [3, 7], proved by Kleidman [4] using the classification of the finite simple groups, says that $H \lhd \lhd G$ whenever H is p-subnormal for every p.

Theorem 3.5 (Kegel-Wielandt conjecture)**.** *If $|HP|$ divides $|G|$ for every Sylow subgroup $P \leqslant G$, then $H \lhd \lhd G$.*

See [2] for some consequences of p-subnormality for a single p. The "if" part of Theorem 1.5 follows easily. Let $H \leqslant_f G$, and assume that $|HK : H|$ divides $|G : H|$ for every $K \leqslant G$. Let $N \lhd_f G$ be the normal core of H, and let $N \leqslant K \leqslant G$ be any intermediate subgroup. Working with G/N and K/N, Theorem 3.5 gives $H/N \lhd\lhd G/N$, i.e. $H \lhd\lhd G$.

We point out that Kegel [3] did not use the classification to prove Theorem 3.5 when H is solvable. We give a very short proof in the case where H is nilpotent, which is enough for the characterization of $S(G)$ we will present in Section 5.

Lemma 3.6 (Kegel-Wielandt for nilpotent subgroups)**.** *Let $H \leqslant G$ be a nilpotent subgroup of the finite group G. If $|HP|$ divides $|G|$ for every Sylow subgroup $P \leqslant G$, then $H \lhd\lhd G$.*

Proof. Suppose that H is not subnormal, and in particular $H \nleqslant F(G)$. So there exists a p-element x such that $x \in H \setminus F(G)$. Since $x \notin O_p(G)$, there exists a p-Sylow P of G such that $x \notin P$. By hypothesis $H \cap P$ is a p-Sylow of H and, since H is nilpotent, $H \cap P$ contains all the p-elements of H. This contradicts the fact that $x \notin P$. □

Levy [5] proves the same result when H is a p-subgroup of G. Another consequence of Theorem 1.5 is that p-subnormality for every p implies that $|HK|$ divides $|G|$ for every $K \leqslant G$. We provide an elementary proof of this fact.

Lemma 3.7. *Let G be a finite group and $H \leqslant G$. If $|HP|$ divides $|G|$ for every Sylow $P \leqslant G$, then $|HK|$ divides $|G|$ for every $K \leqslant G$.*

Proof. Let $K \leqslant G$. We have to show that $|HK : K| = |H : H \cap K|$ divides $|G : K|$. Let p^α be a prime power that divides $|H : H \cap K|$. Since p^α is arbitrary, it is sufficient to prove that $p^\alpha \mid |G : K|$. Let $P_0 \leqslant K$ be a p-Sylow of K, and let $P \leqslant G$ be a p-Sylow of G such that $P \cap K = P_0$. Of course, $p^\alpha \mid |H : H \cap P_0|$. By hypothesis $|H : H \cap P| = |HP : P|$ divides $|G : P|$, and so is not divisible by p. Therefore, $p^\alpha \mid |H \cap P : H \cap P_0|$. Now $|H \cap P : H \cap P_0| = |(H \cap P)P_0 : P_0|$, and this divides $|P : P_0|$ because P is a p-group. So $p^\alpha \mid |P : P_0|$, and then of course $p^\alpha \mid |G : P_0|$. Since $p \nmid |K : P_0|$, we obtain $p^\alpha \mid |G : K|$ as desired. □

4 Exponential subgroups

We write $H \leqslant_{exp} G$ if $x^{|G:H|} \in H$ for all $x \in G$. We observe immediately that exponentiality is preserved by quotients.

Lemma 4.1. *Let $N \lhd G$, and $N \leqslant H \leqslant G$. Then $H \leqslant_{exp} G$ if and only if $H/N \leqslant_{exp} G/N$.*

Proof. Let $x \in G$ and $H \leqslant_{exp} G$. Then $(Nx)^{|G/N:H/N|} = Nx^{|G:H|} \in H/N$ and so $H/N \leqslant_{exp} G/N$. If $H/N \leqslant_{exp} G/N$, then $Nx^{|G:H|} = (Nx)^{|G/N:H/N|} \in H$, and so $x^{|G:H|} \in H$. □

Since exponential subgroups have finite index, we can apply Lemma 4.1 with the normal core, and work with a finite group. Let G be a finite group and $H \leqslant G$. From Corollary 1.3 and Theorem 1.5, we have

- $H \lhd \lhd G$ if and only if $|HK|$ divides $|G|$ for every $K \leqslant G$;
- $H \leqslant_{exp} G$ if and only if $|HC|$ divides $|G|$ for every cyclic $C \leqslant G$.

We stress that $H \leqslant_{exp} G$ whenever $|G : H|$ is a multiple of the exponent $\exp(G)$.

Remark 4.2. Every finite group of order other than a prime has a non-trivial exponential subgroup: if $\exp(G) < |G|$, then it is sufficient to take any subgroup whose order divides $|G|/\exp(G)$. Otherwise, all the Sylow subgroups of G are cyclic, and it is well known that G is solvable. In particular, G has a non-trivial normal subgroup, which is certainly exponential.
We notice a difference with the stronger condition that HK is a subgroup for every K i.e. H is a *permutable* subgroup. Indeed, it is easy to prove that if HC is a subgroup for every cyclic $C \leqslant G$, then HK is a subgroup for every $K \leqslant G$.

For every $n \geq 1$, let $G^n := \langle \{x^n : x \in G\} \rangle$. The exponential subgroups of G of index n are in correspondence with the subgroups of G/G^n of index n. Since G^n is characteristic, the property of being exponential is preserved by automorphisms. Moreover, we have the following

Lemma 4.3. *Let $H \leqslant G$ have a trivial characteristic core. Then $H \leqslant_{exp} G$ if and only if $|G : H|$ is a multiple of the exponent of G.*

Proof. Let $n = |G : H|$. By the exponentiality of H we have $G^n \leqslant H$. Since G^n is a characteristic subgroup of G contained in H, we obtain $G^n = 1$. But this means exactly that n is a multiple of $\exp(G)$. The converse is trivial. □

In general, there exist non-subnormal exponential subgroups whose index is not a multiple of the exponent. A simple example is $G = C_4 \times \mathrm{Sym}(3)$ and $H \cong C_2 \times C_2$. The following corollaries of Lemma 4.3 are obtained with the same strategy.

Corollary 4.4. *Let $H \leqslant G$ be a Hall subgroup. If $H \leqslant_{exp} G$, then $H \lhd G$.*

Proof. Suppose that H is not normal, and let $N \lhd G$ be the normal core of H. Since H/N is a Hall subgroup of G/N, by induction and Lemma 4.1, we can assume that H is core-free. Now $\exp(G)$ captures every prime dividing $|G|$, and so the contradiction is given by Lemma 4.3. □

Corollary 4.5. *Let $M \leqslant G$ be a maximal subgroup of the solvable group G. If $M \leqslant_{exp} G$, then $M \lhd G$.*

Proof. Suppose that M is not normal, and let $N \lhd G$ be the normal core of M. Since M/N is a maximal subgroup of G/N, by induction and Lemma 4.1, we can assume that M is core-free. Now $|G : M| = q^\alpha$ for some prime power q^α. If G is a q-group we are done. Otherwise, the contradiction is given by Lemma 4.3. □

We cannot drop the hypothesis of solvability in Corollary 4.5: the alternating group $G = \mathrm{Alt}(10)$ has a conjugacy class of maximal subgroups M of size 720. Since $\exp(G) = 2520 = |G : M|$, it appears that M is an exponential maximal subgroup which is not normal.
We conclude this section with the hereditary properties of exponential subgroups.

Lemma 4.6. *The following are true:*

- *If $H \leqslant_{exp} M \leqslant_{exp} G$, then $H \leqslant_{exp} G$;*
- *The intersection of exponential subgroups is exponential.*

Proof. Let $x \in G$. Since $M \leqslant_{exp} G$, we have $m = x^{|G:M|} \in M$. Then $x^{|G:H|} = m^{|M:H|} \in H$. To prove the second statement, it is sufficient to notice that $|G : H \cap K|$ is a multiple of both $|G : H|$ and $|G : K|$. □

Other important properties of the lattice of the subnormal subgroups are not true for exponential subgroups, and the dihedral group $G = D_{12}$ is a good source of counterexamples. Every subgroup of G whose order is 2 is exponential in G, since $\exp(G) = 6$. Let H be any non-central subgroup of order 2. Now

- The subgroup $H_1 = \langle H, Z(G)\rangle \cong C_2 \times C_2$ provides a counterexample to the statement that two exponential subgroups generate an exponential subgroup: choosing any involution $x \in G \setminus H_1$ we get $x^{|G:H_1|} = x \notin H_1$.
- The subgroup H_2 which satisfies $H < H_2 \cong \mathrm{Sym}(3)$ provides a counterexample to the statement that the intersection of an exponential subgroup of G with any subgroup of G is exponential in that subgroup: choosing any involution $x \in H_2 \setminus H$, we get that H is not exponential in H_2 although it is exponential in G.

5 The set $S(G)$

Let us recall the definition of $S(G)$ given in the introduction:

$$S(G) := \{x \in G : x^{|G:H|} \in H \text{ for every } H \leqslant_f G\}.$$

From Corollary 1.3, we have

$$S(G) = \{x \in G : |H\langle x\rangle : H| \text{ divides } |G : H| \text{ for every } H \leqslant_f G\}.$$

The results of Section 3 allow to settle the finite case easily:

Proposition 5.1. *If G is finite, then $S(G) = F(G)$.*

Proof. Let $x \in G$. Then $x \in S(G)$ if and only if $|H\langle x\rangle|$ divides $|G|$ for every $H \leqslant G$. From Proposition 1.4 and Lemma 3.6, this is equivalent to $\langle x\rangle \lhd\lhd G$, i.e. $x \in F(G)$. □

5.1 A top-down approach

Let G be an arbitrary group and let $R(G) = \cap_{H \leqslant_f G} H$ be its finite residual. The condition in the definition of $S(G)$ is empty on $R(G)$, and so $R(G) \subseteq S(G)$. In fact, $S(G)$ is the preimage of $S(G/R(G))$ under the projection $G \twoheadrightarrow G/R(G)$.

Lemma 5.2. *Let $N \lhd G$. Then $S(G/N) = \{Nx : x^{|G:H|} \in H$ for every $N \leqslant H \leqslant_f G\}$. In particular, $S(G/R(G)) = S(G)/R(G)$.*

Proof. Let $x \in G$ and $N \leqslant H \leqslant_f G$. The equality $(Nx)^{|G:H|} = Nx^{|G:H|}$ implies that $Nx \in H/N$ if and only if $x^{|G:H|} \in H$, and the first part follows because H is arbitrary. The second part follows because $R(G)$ contains all the subgroups of G of finite index. □

As a consequence of Lemma 5.2, we can assume that G is residually finite. Given $N \lhd G$, let $F_N(G)$ be the preimage of $F(G/N)$.

Proof of Theorem 1.7. We have to prove that $S(G) = \cap_{N \lhd_f G} F_N(G)$. Let $x \in S(G)$ and $N \lhd_f G$. From Lemma 5.2 and Proposition 5.1 we have $Nx \in S(G/N) = F(G/N)$, i.e. $x \in F_N(G)$.
On the other hand, let $x \in \cap_{N \lhd_f G} F_N(G)$ and $H \leqslant_f G$. If $N \lhd_f G$ is the normal core of H, then in particular $x \in F_N(G)$. From Proposition 5.1 we have

$$Nx \in \frac{F_N(G)}{N} = F(G/N) = S(G/N),$$

and so Lemma 5.2 provides $x^{|G:H|} \in H$. The proof follows because H is arbitrary. □

The following observation deletes a bunch of terms from $\cap_{N \lhd_f G} F_N(G)$.

Lemma 5.3. *Let G be a finite group and $N \lhd G$. Then $F(G) \leqslant F_N(G)$.*

Proof. We have that $NF(G)/N \cong F(G)/(N \cap F(G))$ is a nilpotent normal subgroup of G/N. Then $NF(G)/N \leqslant F(G/N) = F_N(G)/N$, and so $NF(G) \leqslant F_N(G)$. □

Corollary 5.4. *If $N, K \lhd_f G$ and $K \leqslant N$, then $F_K(G) \leqslant F_N(G)$.*

As a particular case of Theorem 1.7, we have

Proposition 5.5. *Let G be a group. The following are equivalent:*

(A) $G = S(G)$*;*

(B) *every subgroup of finite index of G is exponential;*

(C) *every finite quotient of G is nilpotent;*

(D) *every subgroup of finite index of G is subnormal.*

Proof. This follows easily from Theorem 1.7. □

We say that a group G is S-free if $S(G) = 1$.

Lemma 5.6. *Let G be a group which is residually S-free. Then $S(G) = 1$.*

Proof. Let $1 \neq x \in G$. By definition, there exists $N \lhd G$ such that $x \notin N$ and $S(G/N) = 1$. In particular $Nx \notin S(G/N)$, and so from Lemma 5.2 we obtain $x \notin S(G)$. Since x is arbitrary, it follows that $S(G) = 1$. □

Corollary 5.7. *If F is a finitely generated free group, then $S(F) = 1$.*

5.2 Baer groups and S-groups

Following a different approach, now we study $S(G)$ starting from the subgroups of G. This will provide a counterexample to the converse of Proposition 1.4.
Let $B(G) := \{x \in G : \langle x \rangle \lhd \lhd G\}$ be the *Baer radical* of G. It is clear that $B(G)$ is a characteristic subgroup. Moreover, $B(G)$ coincides with $F(G)$ if G is finite, but it can be much larger in general (see [1, Example 85]). A group which equals its Baer radical is called a *Baer group*. The same argument in the proof of Proposition 5.1 shows that $B(G) \subseteq S(G)$. We say that a group is an *S-group* if it satisfies the equivalent conditions of Proposition 5.5. It is easy to see that the class of S-groups is closed by subgroups of finite index and quotients. Of course, every Baer group is an S-group.

Proposition 5.8 (Theorem 73 in [1])**.** *A group is a Baer group if and only if every its finitely generated subgroup is subnormal and nilpotent. In particular, every finitely generated Baer group is nilpotent.*

By Propositions 5.5 and 5.8, every finitely generated non-nilpotent p-group is an S-group which is not Baer. The next theorem of Wilson [8] provides many groups with *trivial* Baer radical. We recall that an infinite group is just-infinite if every its proper quotient is finite.

Theorem 5.9 (Theorem 2 in [8])**.** *Let G be a just-infinite group. If $B(G) \neq 1$, then $B(G)$ is a free abelian group of finite rank, which coincides with its own centralizer in G.*

Lemma 5.10. *Let G be a just-infinite p-group. Then $S(G) = G$, but $B(G) = 1$.*

Proof. The fact that $G = S(G)$ follows from Proposition 5.5 and the fact that finite p-groups are nilpotent. If $B(G) \neq 1$, then $B(G)$ is a free abelian group by Theorem 5.9, which contraddicts that G is a p-group. □

Example 5.11 (No converse to Proposition 1.4)**.** Let G be a just-infinite p-group, and let $K \leqslant G$ be any nilpotent subgroup. Since every subgroup of finite index of G is subnormal, from Theorem 1.5 we have that $|HK : H|$ divides $|G : H|$ for every $H \leqslant_f G$. On the other hand, K is not subnormal in G, because $B(G) = 1$.

Finally, it is worth to mention the following theorem of Robinson [6]. Given a group property $\mathcal{P}$, a group is hyper-$\mathcal{P}$ if every its non-trivial homomorphic image has some non-trivial normal subgroup with the property $\mathcal{P}$.

Theorem 5.12 (Theorem 1 in [6])**.** *Let G be a finitely generated hyperabelian or hyperfinite group. If G is an S-group, then G is nilpotent.*

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