

# Efficient enumeration of rooted maps of a given orientable genus by number of faces and vertices

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## Abstract

We simplify the recurrence satisfied by the polynomial part of the generating function that counts rooted maps of positive orientable genus  $g$  by number of vertices and faces. We have written an optimized program in C++ for computing this generating function and constructing tables of numbers of rooted maps, and we describe some of these optimizations here. Using this program we extended the enumeration of rooted maps of orientable genus  $g$  by number of vertices and faces to  $g = 4, 5$  and  $6$  and by number of edges to  $g = 5$  and  $6$  and conjectured a further simplification of the generating function that counts rooted maps by number of edges. Our program is documented and available on request, allowing anyone with a sufficiently powerful computer to carry the calculations even further.

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## 1 Introduction: definitions and history

A *map* is defined topologically as a 2-cell imbedding of a connected graph, loops and multiple edges allowed, in a 2-dimensional surface. The *faces* of a map are the connected components of the complement of the graph in the surface. In this article the surface is

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assumed to be without boundary and orientable, with an orientation already attributed to it (clockwise, say), so that it is completely described by a non-negative integer  $g$ , its *genus*. For short, a map on a surface of genus  $g$  will be called a *genus- $g$  map*. A *planar map* is a genus-0 map (a map on a sphere) and a *toroidal map* is a genus-1 map (a map on a torus or donut). If a map on a surface of genus  $g$  has  $v$  vertices,  $e$  edges and  $f$  faces, then by the Euler-Poincaré formula [7, chap. 9]

$$v - e + f = 2(1 - g). \quad (1.1)$$

Two maps are *equivalent* if there is an orientation-preserving homeomorphism between their imbedding surfaces that takes the vertices, edges and faces of one map into the vertices, edges and faces of the other. A *dart* of a map or graph is a semi-edge. A loop is assumed to be incident twice with the same vertex, so that every edge, whether or not it is a loop, is incident to two darts. The *degree* of a vertex is the number of darts incident to it. The face incident to a dart  $d$  is the face incident to the edge containing  $d$  and on the left of an observer on  $d$  facing away from the vertex incident to  $d$  and the degree of a face is the number of darts incident to it. A *rooted map* is a map with a distinguished dart, its *root*. Two rooted maps are equivalent if there is an orientation-preserving homeomorphism between their imbedding surfaces that takes the vertices, edges, faces and the root of one map into the vertices, edges, faces and the root of the other. A *combinatorial map* is a connected graph with a cyclic order imposed on the darts incident to each vertex, representing the order in which the darts of a (topological) map are encountered during a rotation around the vertex according to the orientation of the imbedding surface. The darts incident to a face are encountered by successive application of the following pair of actions: go from the current dart to the dart on the other end of the same edge and then to the next dart incident to the same vertex according to the cyclic order. In this way the faces of a combinatorial map can be counted, so that its genus can be calculated from (1.1). Two combinatorial maps are equivalent if they are related by a map isomorphism – a graph isomorphism that preserves this cyclic order – with an analogous definition for the equivalence of two rooted combinatorial maps.

It is worth noting that rooted combinatorial maps with  $e$  edges are in one-to-one correspondence with torsion-free subgroups of index  $2e$  in the triangle group  $\Delta(\infty, 2, \infty) = \langle x, y, z | y^2 = xyz = 1 \rangle \cong Z * Z_2$  [12]. Rooted combinatorial maps are indeed a permutation representation of these groups. Details about this correspondence may be found e.g. in [13].

By *enumerating* maps with a given set of properties, whether rooted or not, we mean counting the number of equivalence classes of maps with these properties. It was shown in [12] that each equivalence class of topological maps is uniquely defined by an equivalence class of combinatorial maps; so for the purposes of enumeration, the term “map” can be taken to mean “combinatorial map”.

Let  $m_g(v, f)$  be the number of rooted genus- $g$  maps with  $v$  vertices and  $f$  faces. By face-vertex duality, this number is equal to the number  $m_g(f, v)$  of rooted genus- $g$  maps with  $f$  vertices and  $v$  faces. The generating function that counts rooted genus- $g$  maps is the following formal power series in two variables  $u$  and  $w$ :

$$M_g(w, u) = \sum_{v, f \geq 1} m_g(v, f) w^v u^f. \quad (1.2)$$

Rooted maps were introduced in [15] because they are easier to count than unrooted

maps; this is because only the trivial map automorphism preserves the root [16], so that rooted maps can be counted without considering map automorphisms. In [15], W. T. Tutte found a closed-form formula for the number of rooted planar maps with  $n$  edges. In [16], he found a parametric system of equations defining  $M_0(w, u)$ . In [1] D. Arquès obtained the simpler expression

$$M_0(w, u) = pq(1 - 2p - 2q) \tag{1.3}$$

with the parameters  $p$  and  $q$  defined by

$$w = p(1 - p - 2q) \tag{1.4}$$

and

$$u = q(1 - 2p - q), \tag{1.5}$$

where  $p = q = 0$  when  $w = u = 0$ . In [16], a recursive formula was found for the number of rooted planar maps given the number of vertices, the number of edges, and the degree of the face containing the root; these numbers of maps were then added over all possible degrees of this face and the result expressed in terms of generating functions. In [17], the first author generalized this method to obtain a recursive formula for the number of maps of genus  $g$  with a distinguished dart in each vertex given the number of vertices and the degree of each one; these numbers were then multiplied by the appropriate factor and added over all possible non-increasing sequences of vertex-degrees summing to  $2e$  to obtain the number of rooted maps of genus  $g$  with  $e$  edges and  $v$  vertices. A table of these numbers of maps with up to 14 edges appears in [17] (see [19] for a published account of this work and a table of maps with up to 11 edges) but no attempt was made there to express this result in terms of generating functions. We note here that a similar generalization in which the degrees of all the faces are known but only some of them have a distinguished edge on their boundary, and these faces must be of degree at least 3, appears in [8], where it is attributed to Tutte under the name of Tutte's recursion equations.

In [5] an improvement on the method of [17] was introduced: to count rooted genus- $g$  maps it is sufficient to know the degree of the first  $g + 1$  vertices and to distinguish a dart of only the first vertex as the root, thus reducing the number of maps that have to be considered. Using doubly-rooted maps, D. Arquès [2] obtained the analogue of (1.3) for toroidal maps:

$$M_1(w, u) = \frac{pq(1 - p - q)}{\left[ (1 - 2p - 2q)^2 - 4pq \right]^2}. \tag{1.6}$$

From this result, he obtained a closed-form formula for the number of rooted toroidal maps with  $e$  edges and another one for the number of rooted toroidal maps with  $v$  vertices and  $f$  faces. In [6] a generating function was obtained for the number of rooted maps of genus 2 and 3 with  $e$  edges.

In [9] the second author generalized (1.6) and obtained a general form for the generating function  $M_g(w, u)$  counting rooted maps of any genus  $g > 0$ :

$$M_g(w, u) = \frac{pq(1 - p - q) P_g(p, q)}{\left[ (1 - 2p - 2q)^2 - 4pq \right]^{5g-3}}, \tag{1.7}$$

where  $P_g(p, q)$  is a symmetric polynomial in  $p$  and  $q$  of total degree bounded by  $6g - 6$  with integral coefficients (in what follows, unless otherwise specified, all the polynomials defined here are polynomials in  $p$  and  $q$ ).

Let us now briefly explain the interest of this result for rooted map counting. In [3, 9] the polynomials  $P_g[p, q]$  were calculated for  $g = 2$  and  $3$ . In this article we extend this calculation to  $g = 4, 5$  and  $6$ . To derive the generating function  $M_g(w, u)$  from  $P_g[p, q]$ , we compute  $p$  and  $q$  as power series in  $w$  and  $u$  iteratively from (1.4) and (1.5) and then we replace  $p$  and  $q$  by their respective power series in (1.7) to obtain  $M_g(w, u)$ . By (1.2) the number of rooted maps of genus  $g$  with  $v$  vertices and  $f$  faces is the coefficient of  $w^v u^f$  in  $M_g(w, u)$ . In this way we enumerate rooted maps of genus  $4, 5$  and  $6$  with  $v$  vertices and  $f$  faces for any  $v$  and  $f$  greater than or equal to  $1$ .

The polynomial  $P_g$  in (1.7) is defined in terms of another polynomial  $T_g$  of degree bounded by  $10g - 8$  by

$$P_g = \frac{T_g}{(1 - p)^{4g-2}}, \tag{1.8}$$

and that polynomial, in turn, is defined in terms of a family of polynomials  $R_g(n_1, \dots, n_r)$  in  $p$  and  $q$  by

$$T_g = R_{g-1}(0, 0) + \sum_{j=1}^{g-1} q(1 - p - q)R_j(0)R_{g-j}(0). \tag{1.9}$$

The degree of the polynomial  $R_g(n_1, \dots, n_r)$  is defined by the equation

$$\text{deg } R_g(n_1, \dots, n_r) = 2(n_1 + \dots + n_r) + 7r + 10g - 12. \tag{1.10}$$

The polynomials  $R_g(n_1, \dots, n_r)$  are defined recursively in terms of several other families of polynomials and a recursively-defined family of rational functions of  $p$  and  $q$ . We have two finite families of polynomials in  $p$  alone defined by the following two sets of equations.

$$\begin{aligned} K_0(p) &= -p; & K_1(p) &= -1 - p; & K_2(p) &= -1; \\ K_m(p) &= 0 \text{ for all } m \geq 3. \end{aligned} \tag{1.11}$$

$$\begin{aligned} L_0(p) &= -p; & L_1(p) &= -1 - 2p; \\ L_2(p) &= -2 - p; & L_3 &= -1; \\ L_k(p) &= 0 \text{ for all } k \geq 4. \end{aligned} \tag{1.12}$$

In what follows, the parameter  $p$  will be omitted, so that these polynomials will be referred to as  $K_m$  and  $L_k$ . We then have two polynomials  $H$  and  $J$  (in  $p$  and  $q$ ) defined by

$$J = q(1 - p - q) \tag{1.13}$$

and

$$H = (1 - 2p - 2q)^2 - 4pq. \tag{1.14}$$

Finally we have an infinite family  $(E_k)_{k \geq 1}$  of rational functions of  $p$  and  $q$ , all but the first two of which are polynomials, defined recursively by

$$\begin{aligned} E_1 &= \frac{1}{2J(1 - p)^2}; \\ E_2 &= \frac{-p - 4q + 2p^2 + 4q^2 + 4pq}{2J(1 - p)^2}; \\ E_3 &= -1; \\ E_k &= -J(1 - p)^2 \sum_{i=2}^{i=k-1} E_i E_{k+1-i} \text{ for all } k \geq 4. \end{aligned} \tag{1.15}$$

To make the recursive definition of the polynomials  $R_g(n_1, \dots, n_r)$  comprehensible, we first explain the abbreviations and conventions we use. For any positive integer  $r$ ,  $[r]$  denotes the sequence  $(2, \dots, r)$  if  $r \geq 2$  and the empty sequence if  $r = 1$ . For any subsequence  $X$  of  $[r]$ ,  $[r] - X$  denotes the subsequence of the elements of  $[r]$  that are not in  $X$ . For any sequence  $(n_2, \dots, n_r)$  of integers,  $N_X$  denotes the sequence of those  $n_i$  such that  $i$  is in  $X$  and  $N_j$  denotes the sequence  $(n_2, \dots, n_{j-1}, n_{j+1}, \dots, n_r)$ . By convention, a sum over an empty domain is equal to zero.

The polynomials  $R_0(n_1)$  are not defined. The anchor of this recursive definition is

$$R_0(0, 0) = (1 - p)^2. \tag{1.16}$$

If  $g = 0$  and  $r = 2$  but  $(n_1, n_2) \neq (0, 0)$ , then we have

$$R_0(n_1, n_2) = (1 - p)^2 (-n_2 H E_{n_1+n_2+2} - (n_2 + 1) E_{n_1+n_2+3}) + 2J(1 - p)^2 \sum_{\substack{i+j+k=n_1+1 \\ i>0, k<n_1}} (-1)^{j+1} H^j E_i R_0(k, n_2). \tag{1.17}$$

We note that (1.16) is a special case of (1.17) where  $n_1 = n_2 = 0$ .

If  $(g, r) \neq (0, 2)$ , then

$$R_g(n_1, \dots, n_r) = \text{term}_1 + \text{term}_2 + \text{term}_3 + \text{term}_4, \tag{1.18}$$

where

$$\text{term}_1 = 2J(1 - p)^2 \sum_{\substack{i+j+k=n_1+1 \\ i>0, k<n_1}} (-1)^{j+1} H^j E_i R_g(k, n_2, \dots, n_r) \tag{1.19}$$

(we note that the second line of (1.17) is a special case of (1.19) where  $g = 0$  and  $r = 2$ ),

$$\text{term}_2 = J \sum_{\substack{k+l+m=n_1+1 \\ 0 \leq j \leq g \\ X \subseteq [r] \\ (j, X) \neq (0, \emptyset) \\ (j, X) \neq (g, [r])}} K_m H^m R_j(k, N_X) R_{g-j}(l, N_{[r]-X}), \tag{1.20}$$

$$\text{term}_3 = \sum_{i+j+m=n_1+1} K_m H^m R_{g-1}(i, j, N_{[r]}) \tag{1.21}$$

and

$$\text{term}_4 = \sum_{j=2}^r \left( \begin{array}{l} n_j \sum_{k+l=n_1+n_j+2} L_k H^{k+1} R_g(l, N_j) \\ + (n_j + 1) \sum_{k+l=n_1+n_j+3} L_k H^k R_g(l, N_j) \end{array} \right). \tag{1.22}$$

It was shown in [9] that each polynomial  $R_g(n_1, \dots, n_r)$  is symmetric in all its variables. This was made possible by distinguishing a dart incident to each of the vertices whose degree is considered, which increases the size of the coefficients but does not increase the number of polynomials that have to be calculated.

We note here that in the account of these results published in [3] formula (1.17) and the sum in (1.15) are missing; the formulas are presented correctly in [9]. At that time the second author, programming in Maple, calculated the polynomial  $P_g$  and the generating

function  $M_g(w, u)$  for  $g = 2$  and  $g = 3$  (these results are published in [3]) and also computed the generating function that counts rooted maps of genus 4 by number of edges. This result was recently included in [13], where it was used to count both rooted and unrooted maps of genus 4 by number of edges.

Recently, the second author extended his enumeration results to genus 5. The first author, programming mainly in C, optimized the calculation of the polynomials  $R_g(n_1, \dots, n_r)$  and thus extended the enumeration by number of vertices and faces, as well as by number of edges, to genus 6. Although each author used a different algorithm and a different programming language, we both obtained the same answers, and the numbers of rooted maps we calculated agree with the tables in [17], providing evidence of the correctness of our results. An account of these extensions is given in Sections 2 and 3. Tables of rooted map numbers are given in Appendix A. A discussion of the enumeration of rooted genus- $g$  maps by number of edges appears in Section 4 and the polynomial part of each of the corresponding generating functions appears in Appendix B. Finally, a discussion of some open problems appears in Section 5.

## 2 Results from the Maple program

A first version of the Maple code written in 1998 implemented recurrence relations between the rational functions introduced in [4] for the computation of the generating functions  $M_g$ . It was not designed for efficiency but for validating formulas from [4]. That code has also been used for validating the formulas from (1.2) through (1.22) for the first values of  $g$ ,  $r$  and  $n_1, \dots, n_r$  (these formulas were first obtained from a long computation that was done by hand and is thus error-prone). When executed in 1998 with Maple V for computing  $M_4(w, u)$  that code ran into a fundamental limitation (wired into the Maple kernel) of a maximal number of 65,535 terms in any polynomial.

That old code has been recently replaced by a simpler code implementing directly the recursion between polynomials described by the formulas from (1.2) through (1.22). The code is short (less than 400 lines) and resembles the mathematical formulas as much as possible in order to detect errors. All the results obtained by this new code match known results in rooted map enumeration. For all these reasons, it can be considered as a reference for the debugging of optimized implementations.

With a personal computer running under Windows XP with an Intel Core 2 Duo CPU at 2.19 GHz and 3.5 Gb of memory, and a Maple 14 release supporting larger objects, the next two generating functions  $M_4(w, u)$  and  $M_5(w, u)$  were successfully computed in 4 minutes and 5 hours, respectively. It was, however, not sensible to continue using this inefficient prototype for computing the next generating functions. A better idea was to write an independent implementation optimizing memory space and execution time.

## 3 Optimizations and the C program

Aside from the advantage in execution speed that C has over Maple, the first author optimized the calculation of the polynomials  $R_g(n_1, \dots, n_r)$ . One of these optimizations was made possible by the following observation.

**Proposition 3.1.** *For any  $(g, r) \neq (0, 1)$  and any sequence  $n_1, \dots, n_r$ , the polynomial  $R_g(n_1, \dots, n_r)$  is divisible by  $(1 - p)^2$ .*

*Proof.* We use generalized induction on the degree of a polynomial of the form  $R_g(n_1, \dots,$

$n_r$ ), which we call an *R-polynomial*.

**Basic step (degree 2).** The only *R-polynomial* of degree 2 is  $R_0(0, 0) = (1 - p)^2$ : see (1.16).

**Induction step.** Suppose that the degree  $d$  of a given *R-polynomial*  $R_g(n_1, \dots, n_r)$ , as defined by (1.10), is greater than 2 and that every *R-polynomial* of degree  $< d$  is divisible by  $(1 - p)^2$ . We show that  $R_g(n_1, \dots, n_r)$  is also divisible by  $(1 - p)^2$ . Since every *R-polynomial* on the right side of equations (1.17), (1.19), (1.20), (1.21) and (1.22) is of degree  $< d$ , it follows from the induction hypothesis that each such polynomial is divisible by  $(1 - p)^2$ . We examine each of these equations in turn.

Equation (1.17). The first line contains a factor  $(1 - p)^2$ . The term  $E_{n_1+n_2+3}$  is a polynomial for any non-negative  $n_1$  and  $n_2$ . The term  $E_{n_1+n_2+2}$  is a polynomial unless  $n_1 = n_2 = 0$ , but in this case  $E_{n_1+n_2+2}$  is multiplied by  $n_2 = 0$ ; so the first line of (1.17) is divisible by  $(1 - p)^2$ . In the second line, each term of the sum contains a polynomial  $R_0(k, n_2)$ , which, by the induction hypothesis, is divisible by  $(1 - p)^2$ . This factor of  $(1 - p)^2$  could be cancelled by  $E_1$  or  $E_2$ , but the sum is nevertheless a polynomial, and the factor  $(1 - p)^2$  by which the sum is multiplied ensures that the second line of (1.17) too is divisible by  $(1 - p)^2$ ; so the right side of (1.17) is divisible by  $(1 - p)^2$ .

Equation (1.19). By an argument similar to the one used for the second line of (1.17), the right side of (1.19) is divisible by  $(1 - p)^2$ .

Equations (1.20)-(1.22). Each term in the sum contains at least one *R-polynomial* that is divisible by  $(1 - p)^2$ ; so the right side of each of these equations is divisible by  $(1 - p)^2$ . It follows from (1.18) that  $R_g(n_1, \dots, n_r)$  is divisible by  $(1 - p)^2$ , which completes the proof. □

We now modify equations (1.8)-(1.10), and (1.16)-(1.22) in the light of Proposition 3.1. We introduce a new family of polynomials (which we call *S-polynomials*) defined by

$$S_g(n_1, \dots, n_r) = R_g(n_1, \dots, n_r)/(1 - p)^2 \tag{3.1}$$

and we also let

$$U_g = T_g/(1 - p)^2. \tag{3.2}$$

Then  $U_g$  is a polynomial of degree  $10(g - 1)$  and (1.8)-(1.10) become (3.3)-(3.5), respectively.

$$P_g = \frac{U_g}{(1 - p)^{4g-4}}, \tag{3.3}$$

$$U_g = S_{g-1}(0, 0) + q(1 - p - q) (1 - p)^2 \sum_{j=1}^{g-1} S_j(0)S_{g-j}(0). \tag{3.4}$$

$$\text{deg } S_g(n_1, \dots, n_r) = 2(n_1 + \dots + n_r) + 7(r - 2) + 10g. \tag{3.5}$$

Also, (1.16)-(1.22) become (3.6)-(3.12), respectively.

$$S_0(0, 0) = 1, \tag{3.6}$$

$$\begin{aligned}
 S_0(n_1, n_2) &= (-n_2 H E_{n_1+n_2+2} - (n_2 + 1) E_{n_1+n_2+3}) \\
 &\quad + 2J(1-p)^2 \sum_{\substack{i+j+k=n_1+1 \\ i>0, k<n_1}} (-1)^{j+1} H^j E_i S_0(k, n_2). \tag{3.7}
 \end{aligned}$$

If  $(g, r) \neq (0, 2)$ , then

$$S_g(n_1, \dots, n_r) = \text{term}_5 + \text{term}_6 + \text{term}_7 + \text{term}_8, \tag{3.8}$$

where

$$\text{term}_5 = 2J(1-p)^2 \sum_{\substack{i+j+k=n_1+1 \\ i>0, k<n_1}} (-1)^{j+1} H^j E_i S_g(k, n_2, \dots, n_r), \tag{3.9}$$

$$\begin{aligned}
 \text{term}_6 &= J(1-p)^2 \sum_{\substack{k+l+m=n_1+1 \\ 0 \leq j \leq g \\ X \subseteq [r] \\ (j, X) \neq (0, \emptyset) \\ (j, X) \neq (g, [r])}} K_m H^m S_j(k, N_X) S_{g-j}(l, N_{[r]-X}), \tag{3.10}
 \end{aligned}$$

$$\text{term}_7 = \sum_{i+j+m=n_1+1} K_m H^m S_{g-1}(i, j, N_{[r]}), \tag{3.11}$$

and

$$\text{term}_8 = \sum_{j=2}^r \left( \begin{array}{l} n_j \sum_{k+l=n_1+n_j+2} L_k H^{k+1} S_g(l, N_j) \\ + (n_j + 1) \sum_{k+l=n_1+n_j+3} L_k H^k S_g(l, N_j) \end{array} \right). \tag{3.12}$$

Since  $R_g(n_1, \dots, n_r)$  is symmetric in all its variables, so is  $S_g(n_1, \dots, n_r)$ ; so only those polynomials  $S_g(n_1, \dots, n_r)$  with  $n_1 \leq \dots \leq n_r$  are treated. In all the  $S$ -polynomials on the right side of each of the equations (3.9)-(3.12), only the first two variables can violate these inequalities; so they are inserted into their proper slots among the remaining variables to preserve the inequalities. Also, equation (3.4) is symmetric in  $j$  and  $g - j$ , equation (3.10) is symmetric in  $k$  and  $l$  and equation (3.11) is symmetric in  $i$  and  $j$ ; so the calculations there can be cut almost in half. In equation (3.12), each polynomial  $S_g(l, N_j)$  is calculated only once and then used twice. The following easily proved observations can be used to avoid calculating a polynomial that is identically 0:  $\text{term}_5 = 0$  if  $n_1 = 0$ ,  $\text{term}_6 = 0$  if  $g + r \leq 2$ ,  $\text{term}_7 = 0$  if  $g = 0$ ,  $\text{term}_8 = 0$  if  $r = 1$  or  $(g, r) = (0, 2)$ . From these observations, it follows that the only term that could possibly contribute to  $S_0(n_1)$  is  $\text{term}_5$ . From (3.9) it follows by generalized induction on  $n_1$  that  $S_0(n_1) = 0$  for all  $n_1 \geq 0$ ; so these polynomials do not have to be defined.

All the  $S$ -polynomials are stored in a single one-dimensional array  $s$ . A preliminary recursion does not calculate any of these polynomials. Instead, it calculates all the quadruples  $(d, g, r, c)$  of parameters of the  $S$ -polynomials that will later be calculated, where  $d = \text{deg } S_g(n_1, \dots, n_r)$  and  $c$  is an integer coding the sequence  $(n_1, \dots, n_r)$ , and stores the list of quadruples in four parallel arrays, one array for each of the four parameters  $d, g, r, c$  and one element of all four arrays for each quadruple  $(d, g, r, c)$ . The program then sorts the four parallel arrays by degree  $d$  using bucket sort, computes the number of  $S$ -polynomials that have to be calculated and the total number of terms in these polynomials and stores in two arrays the index in  $s$  and the one in the four parallel arrays of the first term for each degree  $d$ . Then the  $S$ -polynomials are calculated in increasing order of their

degree and stored in  $s$ . This can be done non-recursively because all the  $S$ -polynomials that need to be used will have already been stored and need only be found by searching the four parallel arrays, starting with the first index for the appropriate degree  $d$ , for the appropriate parameters, and adding  $(d+1)(d+2)/2$  to the index in  $s$  each time the index in the four parallel arrays is increased by 1. Once the last polynomial  $S_{g-1}(0, 0)$  has been calculated, first (3.4) is used to calculate  $U_g$  and then (3.3) is used to calculate  $P_g$  and its coefficients are stored in a text file, which is available from the first author on request.

The number of  $S$ -polynomials that have to be calculated is roughly the total number of partitions of all the positive integers up to  $10(g-1)$ . For each of these polynomials, the most expensive calculation is  $\text{term}_6$ , because the sum there runs over all the partitions of the sequence  $[r] = (2, \dots, r)$ , where  $r$  can be as great as  $g+1$ , and involves multiplying two  $S$ -polynomials. The time-complexity of calculating  $P_g$  is therefore exponential in  $g$ , but the optimizations made here nevertheless made it possible to calculate  $P_g$  for a greater value of  $g$  than was possible previously. Another program computes a table of numbers of rooted genus- $g$  maps counted by number of vertices and faces by reading this file and using (1.2) if  $g \geq 1$  or (1.3) if  $g = 0$ . Tables of numbers of rooted genus- $g$  maps for any  $g \leq 6$  and with up to any reasonable number of edges are available from the first author on request.

The programs were written mainly in C. The one that computes the polynomials is about 2000 lines long and the one that computes the tables is about 300 lines long. They both use the C++ library CLN to do arithmetic on big integers because CLN reads arithmetic expressions in C that use only addition, multiplication and subtraction; only statements involving quotients, remainders, input/output of big integers and file management had to be modified. Since CLN requires a GNU compiler, XCODE was downloaded and installed by Jerome Tremblay, a computer technician at UQAM, who also downloaded and installed CLN and wrote sample C++ statements for input/output of big numbers and file management.

The programs were executed on a 2004 Macintosh GR4 computer. The time taken to compute the polynomial  $P_g$  varied from run to run. In Table 1 we show, for each  $g$  from 1 to 6, the number of  $S$ -polynomials that were calculated, the total number of terms in all these  $S$ -polynomials, and a typical execution time. Once the  $S$ -polynomials had been computed and stored, it took the computer only 48 seconds to make a table of numbers of genus-6 maps with up to 42 edges counted by number of vertices and faces.

Source codes for both programs are included in release 0.3.2 and higher of the MAP project [11]. The polynomials  $P_2$  and  $P_3$  appear in [3]. The polynomials  $P_4$ ,  $P_5$  and  $P_6$  are too large to be reproduced here. The coefficients of the polynomials  $P_g[p, q]$  for  $1 \leq g \leq 6$  are included in the MAP project and available from the first author on request. For  $1 \leq g \leq 6$  tables of numbers of rooted genus- $g$  maps up to 20 edges are given in Appendix A. More numbers of rooted genus- $g$  maps for any  $g \leq 6$  and with up to any reasonable number of edges can be obtained from code included in the MAP project and are available from the first author on request.

## 4 Counting by number of edges

To compute the generating function  $M_g(z) = z^{2g-2}M_g(z, z)$  that counts rooted genus- $g$  maps by number of edges alone, we use the substitution obtained in [13], which is a more compact form of the one obtained in [9] and published in [3]. Let

$$p = q = m, \tag{4.1}$$

$g$	number of $S$ -polynomials	total number of terms	execution time
1	1	1	instantaneous
2	16	507	1 second
3	67	7407	10 seconds
4	205	49796	2 minutes
5	543	235410	20 minutes
6	1314	900114	3.5 hours

Table 1: Evaluation of the computation cost

where

$$z = m(1 - 3m) \text{ and } m = 0 \text{ when } z = 0. \tag{4.2}$$

By substituting from (4.1) into (1.4) and (1.5) to express  $w$  and  $u$  in terms of  $m$  and then substituting into (1.7), we obtain the following equation for  $g \geq 1$ :

$$M_g(z) = \frac{m^{2g}(1 - 3m)^{2g-2}P_g(m, m)}{(1 - 6m)^{5g-3}(1 - 2m)^{5g-4}}. \tag{4.3}$$

For  $g = 0$ , we substitute into (1.3) instead of (1.7) and obtain

$$M_0(z) = (1 - 3m)^{-2}(1 - 4m). \tag{4.4}$$

The first author computed  $M_g(z)$  from the computed values of  $M_g(w, u)$  for  $g \leq 6$ . The program divides the polynomial  $P_g(m, m)$  by  $1 - 2m$  as often as possible. The program then divides the resulting polynomial by 2 and by 3 as often as possible, extracts the appropriate constant factor and then stores the resulting generating function in another text file, also available from the first author. The second author computed  $M_g(z)$  directly for  $g \leq 6$ . We then compared our formulas and verified that they agree. The formulas for  $P_g(m, m)$  for  $1 \leq g \leq 6$  appear in Appendix B. Now  $P_g(m, m)$  is of degree  $6g - 6$ , but we found experimentally in 2009 that for  $1 \leq g \leq 6$ ,  $P_g(m, m)$  is divisible by  $(1 - 2m)^{2g-2}$ , so that the quotient is only of degree  $4g - 4$ , and we conjectured that this is the case for any positive integer  $g$ . In 2010, the second author proved that conjecture [10].

### 5 Some interesting open problems

The recurrences satisfied by the  $R$ - and  $S$ -polynomials both result from proofs by induction. After the right conjecture has been guessed by observing the first computed terms, these proofs are not difficult to find, but they are tedious and error-prone due to the length of the expressions involved. Thus they are good candidates for automation. We plan to develop a suitable formal framework for assisting this kind of proofs with a computer algebra system. The challenge is to shorten the chain of conjectures and proofs about the general pattern of generating functions for counting rooted maps.

Once the numbers of rooted maps of genus up to  $g$  are known, the number of unrooted maps up to genus  $g$  can be calculated using the methods presented in [14]. As was mentioned above, the second author collaborated with A. Mednykh to count rooted and unrooted maps of genus 4 by number of edges [13]. It would be interesting to count unrooted genus- $g$  maps by number of vertices and faces for as many values of  $g$  as possible (see [18] for an account of the progress made on this problem).

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### Appendix A

Number  $m_g(v, f)$  of rooted maps of genus  $g$  with  $e$  edges and  $v$  vertices, for  $1 \leq g \leq 6$  and  $2g \leq e \leq 20$ . Lines whose column of  $v$  is empty give the total number  $m_g(e)$  of rooted maps of genus  $g$  with  $e$  edges.

$e$	$v$	$g = 1$	$g = 2$	$g = 3$
2	1	1		
2		1		
3	1	10		
3	2	10		
3		20		
4	1	70	21	
4	2	167		
4	3	70		
4		307	21	
5	1	420	483	
5	2	1720	483	
5	3	1720		
5	4	420		
5		4280	966	
6	1	2310	6468	1485
6	2	14065	15018	
6	3	24164	6468	
6	4	14065		
6	5	2310		
6		56914	27954	1485
7	1	12012	66066	56628
7	2	100156	258972	56628
7	3	256116	258972	
7	4	256116	66066	
7	5	100156		
7	6	12012		
7		736568	650076	113256
8	1	60060	570570	1169740
8	2	649950	3288327	2668750
8	3	2278660	5554188	1169740
8	4	3392843	3288327	
8	5	2278660	570570	
8	6	649950		
8	7	60060		
8		9370183	13271982	5008230
9	1	291720	4390386	17454580
9	2	3944928	34374186	66449432
9	3	17970784	85421118	66449432
9	4	36703824	85421118	17454580
9	5	36703824	34374186	
9	6	17970784	4390386	
9	7	3944928		
9	8	291720		
9		117822512	248371380	167808024
10	1	1385670	31039008	211083730

$e$	$v$	$g = 1$	$g = 2$	$g = 3$
10	2	22764165	313530000	1171704435
10	3	129726760	1059255456	1955808460
10	4	344468530	1558792200	1171704435
10	5	472592916	1059255456	211083730
10	6	344468530	313530000	
10	7	129726760	31039008	
10	8	22764165		
10	9	1385670		
10		1469283166	4366441128	4721384790
11	1	6466460	205633428	2198596400
11	2	126264820	2583699888	16476937840
11	3	875029804	11270290416	40121261136
11	4	2908358552	22555934280	40121261136
11	5	5188948072	22555934280	16476937840
11	6	5188948072	11270290416	2198596400
11	7	2908358552	2583699888	
11	8	875029804	205633428	
11	9	126264820		
11	10	6466460		
11		18210135416	73231116024	117593590752
12	1	29745716	1293938646	20465052608
12	2	678405090	19678611645	196924458720
12	3	5593305476	106853266632	647739636160
12	4	22620890127	276221817810	945068384880
12	5	50534154408	375708427812	647739636160
12	6	65723863196	276221817810	196924458720
12	7	50534154408	106853266632	20465052608
12	8	22620890127	19678611645	
12	9	5593305476	1293938646	
12	10	678405090		
12	11	29745716		
12		224636864830	1183803697278	2675326679856
13	1	135207800	7808250450	174437377400
13	2	3550829360	140725699686	2079913241120
13	3	34225196720	925572602058	8789123742880
13	4	164767964504	2979641557620	17326957790896
13	5	448035881592	5235847653036	17326957790896
13	6	729734918432	5235847653036	8789123742880
13	7	729734918432	2979641557620	2079913241120
13	8	448035881592	925572602058	174437377400
13	9	164767964504	140725699686	
13	10	34225196720	7808250450	
13	11	3550829360		
13	12	135207800		
13		2760899996816	18579191525700	56740864304592
14	1	608435100	45510945480	1384928666550
14	2	18182708362	955708437684	19925913354061
14	3	201976335288	7454157823560	104395235785256
14	4	1137369687454	29079129795702	264477214235234
14	5	3682811916980	63648856688592	357391270819604
14	6	7302676928666	82234427131416	264477214235234
14	7	9145847808784	63648856688592	104395235785256
14	8	7302676928666	29079129795702	19925913354061
14	9	3682811916980	7454157823560	1384928666550
14	10	1137369687454	955708437684	
14	11	201976335288	45510945480	
14	12	18182708362		
14	13	608435100		
14		33833099832484	284601154513452	1137757854901806
15	1	2714556600	257611421340	10369994005800

$e$	$v$	$g = 1$	$g = 2$	$g = 3$
15	2	91392185080	6216591472728	176357530955320
15	3	1156128848680	56532447160536	1115525500250760
15	4	7506901051000	261637840342860	3505018618003600
15	5	28442316247080	694146691745820	6087558311398000
15	6	67173739068760	1117259292848016	6087558311398000
15	7	102432266545800	1117259292848016	3505018618003600
15	8	102432266545800	694146691745820	1115525500250760
15	9	67173739068760	261637840342860	176357530955320
15	10	28442316247080	56532447160536	10369994005800
15	11	7506901051000	6216591472728	
15	12	1156128848680	257611421340	
15	13	91392185080		
15	14	2714556600		
15		413610917006000	4272100949982600	21789659909226960
16	1	12021607800	1422156202740	73920866362200
16	2	452077562620	38985279745230	1461629029629340
16	3	6447533938280	407653880116680	10933959720960760
16	4	47700234551918	2200626948631386	41491242915292306
16	5	208462422428152	6928413234959820	89390908732820144
16	6	576218752277476	13518984452463630	114899070275212424
16	7	1046677747672360	16842445235560944	89390908732820144
16	8	1274461449989715	13518984452463630	41491242915292306
16	9	1046677747672360	6928413234959820	10933959720960760
16	10	576218752277476	2200626948631386	1461629029629340
16	11	208462422428152	407653880116680	73920866362200
16	12	47700234551918	38985279745230	
16	13	6447533938280	1422156202740	
16	14	452077562620		
16	15	12021607800		
16		5046403030066927	63034617139799916	401602392805341924
17	1	52895074320	7683009544980	505297829133240
17	2	2205359390592	236923660397172	11460411934448048
17	3	35155923872640	2815913391715452	99727841192820016
17	4	293370096957504	17486142956133684	447708887118504600
17	5	1461307573813824	64232028100704156	1165172136542282424
17	6	4660202610532480	148755268498286436	1857975645023518752
17	7	9908748651241088	224686278407291148	1857975645023518752
17	8	14373136466094880	224686278407291148	1165172136542282424
17	9	14373136466094880	148755268498286436	447708887118504600
17	10	9908748651241088	64232028100704156	99727841192820016
17	11	4660202610532480	17486142956133684	11460411934448048
17	12	1461307573813824	2815913391715452	505297829133240
17	13	293370096957504	236923660397172	
17	14	35155923872640	7683009544980	
17	15	2205359390592		
17	16	52895074320		
17		61468359153954656	916440476048146056	7165100439281414160
18	1	231415950150	40729207226400	3331309741059300
18	2	10627956019245	1401097546161936	85694099173907510
18	3	187959014565840	18743188498056288	855779329367736840
18	4	1753945289216484	132344695964811720	4470547991985864322
18	5	9857665477085832	559373367462490656	13767319160210071404
18	6	35839052357422132	1511718920778951024	26522236056202555206
18	7	87930943305742512	2710382626755160416	32904419378927915376
18	8	149314477245194262	3286157560248860532	26522236056202555206
18	9	177882700353757460	2710382626755160416	13767319160210071404
18	10	149314477245194262	1511718920778951024	4470547991985864322
18	11	87930943305742512	559373367462490656	855779329367736840
18	12	35839052357422132	132344695964811720	85694099173907510
18	13	9857665477085832	18743188498056288	3331309741059300
18	14	1753945289216484	1401097546161936	
18	15	187959014565840	40729207226400	
18	16	10627956019245		
18	17	231415950150		

$e$	$v$	$g = 1$	$g = 2$	$g = 3$
18		747672504476150374	13154166812674577412	124314235272290304540
19	1	1007340018300	212347275857640	21280393666593600
19	2	50668344988068	8089830217844928	614960028331370816
19	3	987658610225052	120789163612555200	6968569097113244096
19	4	10229201477344752	960323177351524512	41790549086980226368
19	5	64309102366765200	4616545437250956192	149789855223187292608
19	6	263868150558327376	14358354462488121408	341505418008822731328
19	7	738178726378902064	30044423965980553536	511895831411154922176
19	8	1446563778096423816	43241609165618454096	511895831411154922176
19	9	2017523504473479992	43241609165618454096	341505418008822731328
19	10	2017523504473479992	30044423965980553536	149789855223187292608
19	11	1446563778096423816	14358354462488121408	41790549086980226368
19	12	738178726378902064	4616545437250956192	6968569097113244096
19	13	263868150558327376	960323177351524512	614960028331370816
19	14	64309102366765200	120789163612555200	21280393666593600
19	15	10229201477344752	8089830217844928	
19	16	987658610225052	212347275857640	
19	17	50668344988068		
19	18	1007340018300		
19		9083423595292949240	186700695099591735024	2105172926498512761984
20	1	4365140079300	1090848505817070	132216351453357600
20	2	239250231713210	45732525474843801	4257157940494918160
20	3	5110652802256260	756589971284883792	54217755730994858080
20	4	58364244137596695	6716133365837116980	369061676845849000520
20	5	407372683115470800	36362952155187558600	1518921342035154605600
20	6	1870153808268516280	128656798319026864068	4031165546220945277040
20	7	5905479331377981200	309859885439753598768	7151648337964982801760
20	8	13196809961724011350	520516978029736518606	8640883781524178188980
20	9	21241931655650633720	617910462111714896820	7151648337964982801760
20	10	24868664942648145372	520516978029736518606	4031165546220945277040
20	11	21241931655650633720	309859885439753598768	1518921342035154605600
20	12	13196809961724011350	128656798319026864068	369061676845849000520
20	13	5905479331377981200	36362952155187558600	54217755730994858080
20	14	1870153808268516280	6716133365837116980	4257157940494918160
20	15	407372683115470800	756589971284883792	132216351453357600
20	16	58364244137596695	45732525474843801	
20	17	5110652802256260	1090848505817070	
20	18	239250231713210		
20	19	4365140079300		
20		110239596847544663002	2623742783421329300190	34899691847703927826500

$e$	$v$	$g = 4$	$g = 5$	$g = 6$
8	1	225225		
8		225225		
9	1	12317877		
9	2	12317877		
9		24635754		
10	1	351683046	59520825	
10	2	792534015		
10	3	351683046		
10		1495900107	59520825	
11	1	7034538511	4304016990	
11	2	26225260226	4304016990	
11	3	26225260226		
11	4	7034538511		
11		66519597474	8608033980	
12	1	111159740692	158959754226	24325703325
12	2	600398249550	354949166565	
12	3	993494827480	158959754226	
12	4	600398249550		
12	5	111159740692		
12		2416610807964	672868675017	24325703325
13	1	1480593013900	4034735959800	2208143028375
13	2	10743797911132	14805457339920	2208143028375
13	3	25766235457300	14805457339920	
13	4	25766235457300	4034735959800	
13	5	10743797911132		
13	6	1480593013900		
13		75981252764664	37680386599440	4416286056750
14	1	17302190625720	79553497760100	100940771124360
14	2	160576594766588	420797306522502	223790013148500
14	3	517592962672296	691650582088536	100940771124360
14	4	750260619502310	420797306522502	
14	5	517592962672296	79553497760100	
14	6	160576594766588		
14	7	17302190625720		
14		2141204115631518	1692352190653740	425671555397220
15	1	182231849209410	1302772718028600	3130208769783780
15	2	2089035241981688	9220982517965400	11344028448443832
15	3	8615949311310872	21853758736216200	11344028448443832
15	4	16789118602155860	21853758736216200	3130208769783780
15	5	16789118602155860	9220982517965400	
15	6	8615949311310872	1302772718028600	
15	7	2089035241981688		
15	8	182231849209410		
15		55352670009315660	64755027944420400	28948474436455224
16	1	1763184571730010	18475997006212200	74520697707149580
16	2	24325590127655531	166713517116449940	387689146050297186
16	3	123981042854132536	528887751025584600	633832536898519848
16	4	309197871098871838	762684674663536626	387689146050297186
16	5	415691294404230748	528887751025584600	74520697707149580
16	6	309197871098871838	166713517116449940	
16	7	123981042854132536	18475997006212200	
16	8	24325590127655531		
16	9	1763184571730010		

$e$	$v$	$g = 4$	$g = 5$	$g = 6$
16		1334226671709010578	2190839204960030106	1558252224413413380
17	1	15894791312284170	233454817237201560	1457897216520222060
17	2	258634264294653390	2595050050431235488	10115530661997850556
17	3	1587135819804394530	10499075716384241952	23746474580826741940
17	4	4892650539994184868	20269771718252599536	23746474580826741940
17	5	8419549939292302908	20269771718252599536	10115530661997850556
17	6	8419549939292302908	10499075716384241952	1457897216520222060
17	7	4892650539994184868	2595050050431235488	
17	8	1587135819804394530	233454817237201560	
17	9	258634264294653390		
17	10	15894791312284170		
17		30347730709395639732	67194704604610557072	70639804918689629112
18	1	134951136993773100	2682208751185413450	24464684545968004800
18	2	2548272396065512974	35801820369640556595	215810538282954699872
18	3	18451302662846918700	178505550201444784920	675870370096399329024
18	4	68503375296263488977	439591872915483185214	970559177597162956688
18	5	145737674581607574840	588564117958709029644	675870370096399329024
18	6	186553519919803261860	439591872915483185214	215810538282954699872
18	7	145737674581607574840	178505550201444784920	24464684545968004800
18	8	68503375296263488977	35801820369640556595	
18	9	18451302662846918700	2682208751185413450	
18	10	2548272396065512974		
18	11	134951136993773100		
18		657304672067357799042	1901727022434216910002	2802850363447807024080
19	1	1088243826731751690	28449551653853229900	362610922310040035940
19	2	23532893106071038404	447016944351510642564	3931547761898967889520
19	3	197822824662547694148	2677324515710001081372	15658846910857993085360
19	4	866831237081712285138	8127109896970086044280	30002691954232352277608
19	5	2221381417843144801098	13881153040572190501512	30002691954232352277608
19	6	3515647035511186627416	13881153040572190501512	15658846910857993085360
19	7	3515647035511186627416	8127109896970086044280	3931547761898967889520
19	8	2221381417843144801098	2677324515710001081372	362610922310040035940
19	9	866831237081712285138	447016944351510642564	
19	10	197822824662547694148	28449551653853229900	
19	11	23532893106071038404		
19	12	1088243826731751690		
19		13652607304062788395788	50322107898515282999256	99911395098598706576856
20	1	8391311316938069520	281858111998039476900	4848655679592076350570
20	2	205518653220527665304	5131008990500486096250	63004600211616713352227
20	3	1979281881126113225376	36188783145801243558900	308528931105126354302392
20	4	10071757699155275906824	131989618396827099239715	751928550048308520251566
20	5	30468100266480917147760	277921666244135490925320	1003326321163364778495004
20	6	58089920897558352891672	354556747218700475500140	751928550048308520251566
20	7	71823371612912533887168	277921666244135490925320	308528931105126354302392
20	8	58089920897558352891672	131989618396827099239715	63004600211616713352227
20	9	30468100266480917147760	36188783145801243558900	4848655679592076350570
20	10	10071757699155275906824	5131008990500486096250	
20	11	1979281881126113225376	281858111998039476900	
20	12	205518653220527665304		
20	13	8391311316938069520		
20		273469313030628783700080	1257582616997225194094310	3259947795252652107008514

### Appendix B

Polynomial  $P_g(m, m)/(1 - 2m)^{(2g-2)}$  in the generating function  $M_g(z)$ .

$g$	$P_g(m, m)/(1 - 2m)^{(2g-2)}$
1	1
2	$3(7 - 70m + 295m^2 - 636m^3 + 588m^4)$
3	$3^2 \left( \begin{array}{l} 165 - 2596m + 19835m^2 - 102138m^3 + 397742m^4 \\ -1162744m^5 + 2360496m^6 - 2918016m^7 + 1642656m^8 \end{array} \right)$
4	$3^2 \left( \begin{array}{l} 25025 - 465894m + 4245462m^2 - 28633200m^3 \\ +178608786m^4 - 1025233956m^5 + 4855070265m^6 \\ -17709582732m^7 + 48202134300m^8 - 95026128096m^9 \\ +128766120048m^{10} - 107657028288m^{11} + 41956066368m^{12} \end{array} \right)$
5	$3^2 \left( \begin{array}{l} 6613425 - 128153480m + 1123286598m^2 - 7641539820m^3 \\ +68489369190m^4 - 681945904584m^5 + 5453799804351m^6 \\ -33175983024306m^7 + 157025924018370m^8 \\ -590662433458296m^9 + 1778501684246544m^{10} \\ -4258112783048352m^{11} + 7946769062433024m^{12} \\ -11156448512891520m^{13} + 11087677481748480m^{14} \\ -6955529138076672m^{15} + 2071316467035648m^{16} \end{array} \right)$
6	$3^3 \left( \begin{array}{l} 900951975 - 16624244750m + 105922471285m^2 \\ -402327939748m^3 + 9014122899102m^4 \\ -183050473605084m^5 + 2152106046117936m^6 \\ -17716916701552824m^7 + 113738504396139378m^8 \\ -602051461456822740m^9 + 2694620167659984726m^{10} \\ -10264333975933057272m^{11} + 33144207748349404248m^{12} \\ -89851078246171110912m^{13} + 201700042332545251008m^{14} \\ -368052722019205320960m^{15} + 531966143515513800960m^{16} \\ -586003188281237388288m^{17} + 462270648384927677952m^{18} \\ -232608604432295245824m^{19} + 56102738197832792064m^{20} \end{array} \right)$