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Saturated 2-plane drawings with few edges

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Abstract

A drawing of a graph is k-plane if every edge contains at most k crossings. A k-plane drawing is saturated if we cannot add any edge so that the drawing remains k-plane. It is well-known that saturated 0-plane drawings, that is, maximal plane graphs, of n vertices have exactly $3n-6$ edges. For $k > 0$, the number of edges of saturated n-vertex k-plane graphs can take many different values. In this note, we establish some bounds on the minimum number of edges of saturated 2-plane graphs under various conditions.

Keywords: Saturated drawing, 2-planar, graphs, discharging.

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1 Preliminaries

In a drawing of a graph in the plane, vertices are represented by points, edges are represented by curves connecting the points, which correspond to adjacent vertices. The points (curves) are also called vertices (edges). We assume that an edge does not go through any vertex, and three edges do not cross at the same point. A graph together with its drawing is a *topological graph*. A drawing or a topological graph is *simple* if any two edges have at most one point in common, that is either a common endpoint or a crossing. In particular, there is no self-crossing. In this paper, we assume the underlying graph has neither loops nor multiple edges.

For any $k \geq 0$, a topological graph is *k-plane* if each edge contains at most k crossings. A graph G is *k-planar* if it has a *k-plane* drawing in the plane.

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There are several versions of these concepts, see e.g. [4]. The most studied one is when we consider only simple drawings. A graph G is *simple k-planar* if it has a *simple k-plane* drawing in the plane.

A simple k-plane drawing is *saturated* if no edge can be added so that the obtained drawing is also simple k-plane. The 0-planar graphs are the well-known planar graphs. A plane graph of n vertices has at most $3n-6$ edges. If it has exactly $3n-6$ edges, then it is a triangulation of the plane. If it has fewer edges, then we can add some edges so that it becomes a triangulation with $3n-6$ edges. That is, saturated plane graphs have $3n-6$ edges.

Pach and Tóth [6] proved the maximum number of edges of an n-vertex (simple) 1-planar graph is $4n-8$. Brandenburg et al. [3] noticed that saturated simple 1-plane graphs can have much fewer edges, namely $\frac{45}{17}n+O(1)\approx 2.647n$. Barát and Tóth [2] proved that a saturated simple 1-plane graph has at least $\frac{20n}{9}-O(1)\approx 2.22n$ edges.

For any k,n, let $s_k(n)$ be the minimum number of edges of a saturated n-vertex simple k-plane drawing. With these notations, $\frac{45n}{17}+O(1)\geq s_1(n)\geq\frac{20}{9}n-O(1)$. For $k>1$, the best bounds known for $s_k(n)$ are shown by Auer et al [1] and by Klute and Parada [5]. Interestingly for $k\geq 5$ the bounds are very close.

In this note, we concentrate on 2-planar graphs on n vertices. Pach and Tóth [6] showed the maximum number of edges of a (simple) 2-planar graph is $5n-10$. Auer et al [1] and Klute and Parada [5] proved that $\frac{4n}{3}+O(1)\geq s_2(n)\geq\frac{n}{2}-O(1)$. We improve the lower bound.

Theorem 1.1. *For any $n>0$, $s_2(n)\geq n-1$.*

A drawing is *l-simple* if any two edges have at most l points in common. By definition a simple drawing is the same as a 1-simple drawing. Let $s_k^l(n)$ be the minimum number of edges of a saturated n-vertex l-simple k-plane drawing. In [5] it is shown that $\frac{4n}{5}+O(1)\geq s_2^2(n)\geq\frac{n}{2}-O(1)$ and $\frac{2n}{3}+O(1)\geq s_2^3(n)\geq\frac{n}{2}-O(1)$. We make the following improvements:

Theorem 1.2. (i) *$s_2^2(3)=3$, and $\lfloor 3n/4\rfloor\geq s_2^2(n)\geq\lfloor 2n/3\rfloor$ for $n\neq 3$,*

(ii) *$s_2^3(3)=3$, and $s_2^3(n)=\lfloor 2n/3\rfloor$ for $n\neq 3$.*

The saturation problem for k-planar graphs has many different settings, we can allow self-crossings, parallel edges, or we can consider non-extendable *abstract* graphs. See [4] for many recent results and a survey.

2 Proofs

Definition 2.1. Let G be a topological graph and u a vertex of degree 1. For short, u is called a *leaf* of G. Let v be the only neighbor of u. The pair (u,uv) is called a *flag*. If there is no crossing on uv, then (u,uv) is an *empty flag*.

Definition 2.2. Let G be an l-simple 2-plane topological graph. If an edge contains two crossings, then its piece between the two crossings is a *middle segment*. The edges of G divide the plane into cells. A cell C is *special* if it is bounded only by middle segments and isolated vertices. Equivalently, C is *special*, if there is no vertex on its boundary, apart from isolated vertices. An edge that bounds a special cell is also *special*.

Let G be a saturated l-simple 2-plane topological graph, where $1 \leq l \leq 3$. Suppose a cell C contains an isolated vertex v. Since G is saturated, C must be a special cell and there is no other isolated vertex in C. Now suppose C is an empty special cell. Each boundary edge contains two crossings. Therefore, if we put an isolated vertex in C, then the topological graph remains saturated. So if we want to prove a lower bound on the number of edges, we can assume without loss of generality that each special cell contains an isolated vertex.

Claim 2.3. *A special edge can bound at most one special cell.*

Proof. Suppose uv is a special edge and let pq be its middle segment. If uv bounds more than one special cell, then there is a special cell on both sides of pq, C_1 and C_2 say. Let p be a crossing of the edges uv and xy. There is no crossing on xy between p and one of the endpoints, x say. Therefore, one of the cells C_1 and C_2 has x on its boundary, a contradiction. □

Proof of Theorem 1.1. Suppose G is a saturated simple 2-plane topological graph of n vertices and e edges. We assume that each special cell contains an isolated vertex.

Claim 2.4. *All flags are empty in G.*

Proof. Let (u, uv) be a flag. Suppose to the contrary there is at least one crossing on uv. Let p be the crossing on uv closest to u, with edge xy. Since it is a 2-plane drawing, there is no crossing on xy between p and one of the endpoints, x say. In this case, we can connect u to x along up and px. Since the drawing was saturated, u and x are adjacent in G, and $x \neq v$, that contradicts to $d(u) = 1$. □

Remove all empty flags from G. Observe the resulting topological graph G' is also saturated. If we can add an edge to G', then we could have added the same edge to G.

Suppose to the contrary that G' contains a flag (v, vw). Since G' is saturated, the flag is empty by Claim 2.4. In G, vertex v had degree at least 2, so v had some other neighbors, $u_1, \ldots, u_m$ say, in clockwise order. The flags $(u_i, u_i v)$ were all empty. However, u_1 can be connected to w, which is a contradiction. Therefore, there are no flags in G'. On the other hand, the graph G' may contain isolated vertices. Let n' and e' denote the number of vertices and edges of G'. Since $n - n' = e - e'$, it suffices to show that $e' \geq n' - 1$. If there are no isolated vertices in G', then $e' \geq n'$ is immediate.

We assign weight 1 to each edge. If G' has no edge, then it has one vertex and we are done. We discharge the weights to the vertices so that each vertex gets weight at least 1. If uv is not a special edge, then it gives weight $1/2$ to both endpoints u and v. Suppose now that uv is a special edge. It bounds the special cell C containing the isolated vertex x. If $d(u) = 2$, then uv gives weight $1/2$ to u, if $d(u) \geq 3$, then it gives weight $1/3$ to u. We similarly distribute the weight to vertex v. We give the remaining weight of uv to x.

We show that each vertex gets weight at least 1. This holds immediately for all vertices of positive degree. We have to show the statement only for isolated vertices. Let x be an isolated vertex in a special cell C bounded by $e_1, e_2, \ldots, e_m$ in clockwise direction. Let $e_i = u_i v_i$ such that the oriented curve $\overrightarrow{u_i v_i}$ has C on its right. See Figure 1 for $m = 5$. Let p_i be the crossing of e_i and e_{i+1}. Indices are understood modulo m. In general, it may happen that some of the points in $\{\, u_i, v_i \mid i = 1, \ldots, m \,\}$ coincide. For each vertex u_i or v_i of degree at least 3, the corresponding boundary edge of C has a remainder charge

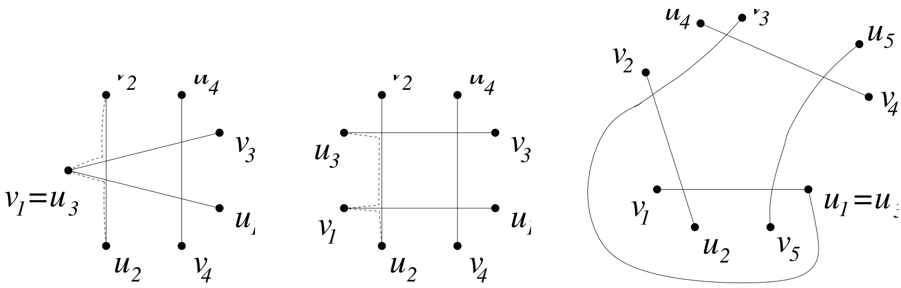

Figure 1: Case 1, $d(v_1) \geq 4$, Case 2, $d(v_1) \geq 3$ and Case 2, $u_1 = u_3$.

at least $1/6$. We have to prove that (with multiplicity) at least 6 of the vertices u_i, v_i have degree at least 3. Consider vertex v_i.

Case 1: $v_i = u_{i+2}$. The vertex $v_i = u_{i+2}$ can be connected to u_{i+1} along the segments $v_i p_i$ and $p_i u_{i+1}$, that are crossing-free segments of the corresponding edges. Similarly, $v_i = u_{i+2}$ can be connected to v_{i+1} along $v_i p_{i+1}$ and $p_{i+1} v_{i+1}$. Since the drawing was simple and saturated, u_i, u_{i+1}, v_{i+1}, v_{i+2} are all different and they are already connected to $v_i = u_{i+2}$, so it has degree at least 4.

Case 2: $v_i \neq u_{i+2}$. The vertex v_i can be connected to u_{i+1} as before, and to u_{i+2} along $v_i p_i$, $p_i p_{i+1}$ and $p_{i+1} u_{i+2}$. Since the drawing was saturated, v_i is already adjacent to u_i, u_{i+1}, u_{i+2}. Unless $u_i = u_{i+2}$, vertex v_i has degree at least 3. Note that $u_{i+1} \neq u_i$ and $u_{i+1} \neq u_{i+2}$, since the drawing was 1-simple.

We can argue analogously for u_i. We conclude that v_i has degree 2 only if $u_i = u_{i+2}$, and u_i has degree 2 only if $v_i = v_{i-2}$.

Recall that m is the number of bounding edges of the special cell C. For $m = 3$, it is impossible that $u_i = u_{i+2}$ or $v_i = v_{i-2}$, therefore, for $i = 1, 2, 3$ all six vertices u_i, v_i have degree at least 3.

Let $m > 3$, and suppose v_1 has degree 2, consequently $u_1 = u_3$. In this case, we prove that u_m, u_1, u_2, u_3, v_m, v_2 all have degree at least 3.

We show it for u_2, the argument is the same for the other vertices. Let γ be the closed curve formed by the segments $u_1 p_1$, $p_1 p_2$ and $p_2 u_3$. (We have $u_1 = u_3$.) Suppose $d(u_2) = 2$. By the previous observations, $v_m = v_2$. However, v_m and v_2 lie on different sides of γ, therefore they cannot coincide. Therefore, there are always at least six vertices u_i, v_i, with multiplicity, which have degree at least 3, so the isolated vertex x gets weight at least 1. This concludes the proof. □

We recall that $s_2^3(n)$ denotes the minimum number of edges of a saturated n-vertex 3-simple 2-plane drawing.

Proof of Theorem 1.2. We start with the upper bounds. Let

$$f(n) = \begin{cases} 3 & \text{if } n = 3 \\ \lfloor 3n/4 \rfloor & \text{otherwise.} \end{cases}$$

First we construct a saturated 2-plane, 2-simple topological graph with n vertices and $f(n)$ edges, for every n. Let $k \geq 3$. A *k-propeller* is isomorphic to a star with k edges as an

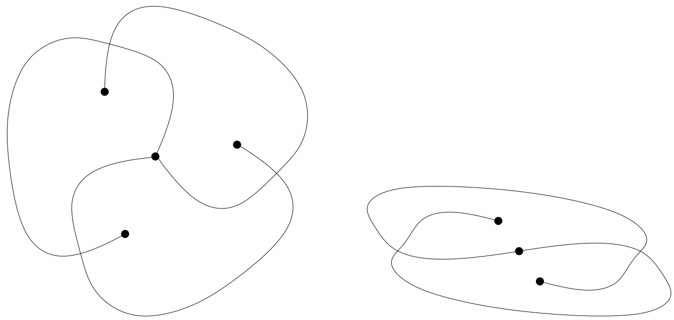

Figure 2: A 3-propeller and a 2-propeller.

abstract graph, drawn as in Figure 2. Clearly it is a saturated 2-plane, 2-simple topological graph with $k+1$ vertices, k edges and the unbounded cell is special.

For $n = 1, 2, 3$, a complete graph of n vertices satisfies the statement. For $n \geq 4$, $n \equiv 0 \mod 4$, consider $n/4$ disjoint 3-propellers such that each of them is in the unbounded cell of the others. For $n \geq 4$, $n \equiv 1, 2, 3 \mod 4$, replace one of the propellers by an isolated vertex, a K_2, and a 4-propeller, respectively. This implies the upper bound in (i), that is, $s_2^2(n) \leq f(n)$.

Now we construct a saturated 2-plane, 3-simple topological graph with n vertices and $\lfloor 2n/3 \rfloor$ edges, for every n. A 2-*propeller* is isomorphic to a path of 2 edges as an abstract graph, drawn as in Figure 2. Clearly it is a saturated 2-plane, 3-simple topological graph with 3 vertices, 2 edges and the unbounded cell is special.

For $n \equiv 0 \mod 3$, take $n/3$ disjoint 2-propellers such that each of them is in the unbounded cell of the others. For $n \equiv 1, 2 \mod 3$, add an isolated vertex or an independent edge. This implies the upper bound in (ii), $s_2^3(n) \leq \lfloor 2n/3 \rfloor$.

We prove by induction on n that $s_2^2(n) \geq \lfloor 2n/3 \rfloor$ and $s_2^3(n) \geq \lfloor 2n/3 \rfloor$. It is trivial for $n \leq 4$. Let $n > 4$ and assume that $s_2^2(m), s_2^3(m) \geq \lfloor 2m/3 \rfloor$ for every $m < n$. Let G be a saturated 2-plane, 2-simple or 3-simple drawing with n vertices and e edges. We may assume again that every special cell contains an isolated vertex.

Suppose that (u, uv) is an empty flag. We remove u from G. Analogous to the proof of Theorem 1.1, the obtained topological graph is saturated, it has $n-1$ vertices and $e-1$ edges. By the induction hypothesis, $e - 1 \geq \lfloor 2(n-1)/3 \rfloor$, which implies that $e \geq \lfloor 2n/3 \rfloor$. Therefore, we assume for the rest of the proof that G does not contain empty flags.

Claim 2.5. *If (u, uv) is a flag, then either $d(v) \geq 3$ or u and v are included in a 2-propeller.*

Proof. Since G does not contain empty flags, there is a crossing on uv. Let p be the crossing on uv closest to u, with edge xy. There is no crossing on xy between p and one of the endpoints, x say, and $x \neq u$ by the assumptions. We can connect u to x along the segments up and px. Since the drawing was saturated, u and x are adjacent in G. Since u has degree 1, $x = v$. This implies $d(v) \geq 2$. We exclude parallel edges, so $y \neq u$.

Suppose $d(v) = 2$. There is a crossing on the segment py of vy, otherwise we could connect u to y along the segments up and py contradicting the degree assumption on u. Let q be the crossing of vy and ab. There is no crossing on ab between q and one of the endpoints, a say. If a and u are on the same side of edge vy (that is, the directed edges $\overrightarrow{ab}$

and $\overrightarrow{ab}$ cross the directed edge $\overrightarrow{vy}$ from the same side), then we can connect u to a along the segments up, pq, qa. Therefore $a = v$, so either $d(v) \geq 3$, or $b = u$, and edges uv and vy form a 2-propeller. Note that this case is possible only if G is 3-simple.

So we may assume that a is on the other side. If $a = v$, then $d(v) \geq 3$, so we also assume that $a \neq v$. Consider now the edge uv. If there was no crossing on the segment pv of uv, then we can connect u to a along up, the segment pv of yv, the segment vp of uv, pq, and qa. Therefore, there is a crossing on the segment pv of uv. Let r be this crossing of uv with edge cd, and we can assume there is no crossing on the segment cr. (Here, c or d might coincide with a.) If c and y are on the same side of uv (that is, the directed edges $\overrightarrow{vy}$ and $\overrightarrow{dc}$ cross the directed edge $\overrightarrow{vu}$ from the same side), then we can connect u to c along up, px, xr, rc, which means that $c = v$, so $d(v) \geq 3$. If c and y are on opposite sides of uv, then we can connect c to v, so they are already connected. Therefore, $c = y$. However, we assumed that $\overrightarrow{vy}$ and $\overrightarrow{dc}$ cross the directed edge $\overrightarrow{vu}$ from the opposite sides, so there is another crossing of uv and vy. If G is 2-simple, this is impossible and we are done. If G is 3-simple, then this crossing can only be r, so $c = y$ and $d = x$. Now the edges uv and vy form a 2-propeller. $\square$

In a graph G, a connected component with at least two vertices is an *essential component*. If G has only one essential component, then G is *essentially connected*.

Claim 2.6. *We can assume without loss of generality that G is essentially connected.*

Proof. Suppose to the contrary G has at least two essential components. We define a partial order on the essential components of G: $G_i \prec G_j$ if and only if G_i lies in a bounded cell of G_j. Let G_1 be a minimal element with respect to $\prec$ and let G_2 be the union of all other essential components. There is a cell C of G, which is bounded by both G_1 and G_2. Let C correspond to cell C_1 of G_1 and cell C_2 of G_2. By the definition of G_1, C_1 is the unbounded cell of G_1. Since G is saturated, at least one of C_1 or C_2 is a special cell, otherwise G_1 and G_2 can be connected.

For $i = 1, 2$, let H_i be the topological graph G_i together with an isolated vertex in every special cell. Let n_i denote the number of vertices and e_i the number of edges in H_i. We notice $e = e_1 + e_2$ and $n = n_1 + n_2 - 1$ if exactly one of C_1 and C_2 is a special cell. Also $n = n_1 + n_2 - 1$ if both of them are special cells, since we can add 1 isolated vertex instead of 2. By the induction hypothesis, we have $e_i \geq \lfloor 2n_i/3 \rfloor$, so $e \geq \lfloor 2n_1/3 \rfloor + \lfloor 2n_2/3 \rfloor$, and it is easy to check, that for any $n_1, n_2 \geq 2$, $\lfloor 2n_1/3 \rfloor + \lfloor 2n_2/3 \rfloor \geq \lfloor 2(n_1 + n_2 - 1)/3 \rfloor$. Therefore, $e \geq \lfloor 2n_1/3 \rfloor + \lfloor 2n_2/3 \rfloor \geq \lfloor 2(n_1 + n_2 - 1)/3 \rfloor = \lfloor 2n/3 \rfloor$. So, if G is not essentially connected, then we reduce the problem and proceed by induction. $\square$

Assume the 3-simple 2-plane drawing G has a flag (u, uv). If $d(v) = 1$, then G is isomorphic to K_2 and the theorem holds. If $d(v) = 2$, then G contains a 2-propeller u, v, w by Claim 2.5. Since G is essentially connected, but there is an isolated vertex in every special cell, there is an isolated vertex x in the special cell of the 2-propeller. Therefore, if $d(v) = 2$ and $d(w) = 1$, then G is isomorphic to a 2-propeller plus an isolated vertex and we are done. If $d(v) = 2$ and $d(w) > 1$, then remove vertices u, v, x. We removed 3 vertices and 2 edges, so we can use induction.

In the rest of the proof, we assume that every leaf of G is adjacent to a vertex of degree at least 3, and there is no 2-propeller subgraph in G. We give weight $3/2$ to every edge. We discharge the weights to the vertices and show that either every vertex gets weight at least 1, or we can prove the lower bound on the number of edges by induction.

Let uv be an edge. Vertex u gets $1/d(u)$ weight and v gets $1/d(v)$ weight from uv. Every edge has a non-negative remaining charge.

If uv is a special edge, then it is easy to verify that uv bounds only one special cell, and the special cell contains an isolated vertex by the assumption, just like in the proof of Claim 2.3. In this case, edge uv gives the remaining charge to this isolated vertex. After the discharging step, any vertex x with $d(x) > 0$ gets charge at least 1.

Now let x be an isolated vertex, its special cell being C. We distinguish several cases.

Case 1: The special cell C has two sides. Let u_1v_1 and u_2v_2 be the bounding edges. They cross twice, in p and q say, so there are no further crossings on u_1v_1 and u_2v_2. The four endpoints are either distinct, or two of them u_1 and u_2 might coincide, if G was 3-simple. Suppose the order of crossings on the edges is u_ipqv_i, for $i = 1, 2$. If the vertices u_1 and u_2 are distinct, then they can be connected along u_1p and pu_2. Therefore, u_1 and u_2 are either adjacent or coincide in G. Similarly, v_1 and v_2 are also adjacent. Therefore, all four endpoints have degree at least 2, and both u_1v_1 and u_2v_2 give at most charge $1/2$ to its endpoints. Their remaining charges are at least $1/2$, so x gets at least charge 1.

For the rest of the proof, suppose C is bounded by $e_1, e_2, \ldots, e_m$ in clockwise direction, $e_i = u_iv_i$ such that $\overrightarrow{u_iv_i}$ has C on its right.

Case 2: $m = 3$. If none of the bounding edges is a flag, then we are done since each of those edges give weight at least $1/2$ to x. Suppose that u_1 is a leaf. We can connect u_1 to v_2 along segments of the edges u_1v_1 and u_2v_2. Since u_1 is a leaf and the drawing was saturated, u_1 and v_2 are adjacent, consequently $v_1 = v_2$. Similarly, we can connect u_1 to v_3, so $v_1 = v_2 = v_3$.

If u_2 is not a leaf, then u_1v_1 and u_3v_3 both give at least $1/6$ to x, and u_2v_2 gives at least $2/3$, so we have charge at least 1 for x. The same applies if u_3 is not a leaf. So assume u_1, u_2 and u_3 are all leaves. If there are no other edges in G, then we can see from the crossing pattern that G is a 3-propeller and an isolated vertex. That is, $n = 5$ and $e = 3$ and the required inequality holds.

Suppose there are further edges. By Claim 2.6, G is essentially connected. Since u_1, u_2, u_3 are leaves, v_1 is a cut vertex. Let $H_1 = G \setminus \{x, u_1, u_2, u_3\}$. The induced subgraph H_1 has $n - 4$ vertices and $e - 3$ edges, and it is saturated. Therefore, by the induction hypothesis, $e - 3 \geq f(n - 4)$. Notice that $f(n) \leq f(n - 4) + 3$, consequently $e \geq f(n)$.

Case 3: $m > 3$. Each edge gives at least $1/6$ charge to x by Claim 2.5. If an edge is not a flag, then it gives at least $1/2$ charge to x. If there is at least one non-flag bounding edge, we are done. Suppose that each edge u_iv_i is a flag (that is, $d(u_i)$ or $d(v_i)$ is 1). We may also assume that u_1 is a leaf. Now, as in the previous case, we can argue that $v_3 = v_2 = v_1$. It implies u_2 and u_3 are leaves, and by the same argument, $v_5 = v_4 = v_3 = v_2 = v_1$. We can continue and finally we obtain that all v_i are identical and all u_i are leaves. So the vertices u_i, v_i $1 \leq i \leq m$ form a star, and they have the same crossing pattern as an m-propeller. Therefore, u_i, v_i $1 \leq i \leq m$ span an m-propeller. We can finish this case exactly as Case 2. If there are no further edges in G, then the graph is an m-propeller and an isolated vertex. That is, $n = m + 2$ and $e = m$ and the inequality holds. If there are further edges, then v_1 is a cut vertex, and we can apply induction. This concludes the proof of Theorem 1.2. □

Remarks

- We have established lower and upper bounds on the number of edges of a saturated, k-simple, 2-plane drawing of a graph. As we mentioned in the introduction, this problem has many modifications, generalizations. Probably the most natural modification is that instead of graphs already drawn, we consider saturated *abstract* graphs. A graph G is saturated l-simple k-planar, if it has an l-simple k-planar drawing but adding any edge, the resulting graph does not have such a drawing. Let $t_k^l(n)$ be the minimum number of edges of a saturated l-simple k-planar graph of n vertices. By definition, $s_k^l(n) \leq t_k^l(n)$. We are not aware of any case when the best lower bound on $t_k^l(n)$ is better than for $s_k^l(n)$. On the other hand, it seems to be much harder to establish an upper bound construction for $t_k^l(n)$ than for $s_k^l(n)$. In fact, we know nontrivial upper bounds only in two cases, $t_1^1(n) \leq 2.64n + O(1)$ [3] and $t_2^1(n) \leq 2.63n + O(1)$ [1], the latter without a full proof.

  It is known that a k-planar graph has at most $c\sqrt{k}n$ edges [6], so $t_k^l(n) \leq c\sqrt{k}n$, for some $c > 0$.

  Problem 1. Prove that for every $c > 0$, $t_k^l(n) \leq c\sqrt{k}n$ if k, l, n are large enough.

- For any n and k, the best known upper and lower bounds on s_k^l decrease or stay the same as we increase l. This would suggest that $s_k^l \leq s_k^{l-1}$ for any n, k, l, or at least if n is large enough, however, we cannot prove it.

  Problem 2. Is it true, that for any k and l, and n large enough, $s_k^l \leq s_k^{l-1}$?

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