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The symmetric genus spectrum of abelian groups

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Abstract

Let $\mathcal{S}$ denote the set of positive integers that appear as the symmetric genus of a finite abelian group and let $\mathcal{S}_0$ denote the set of positive integers that appear as the strong symmetric genus of a finite abelian group. The main theorem of this paper is that $\mathcal{S} = \mathcal{S}_0$. As a result, we obtain a set of necessary and sufficient conditions for an integer g to belong to $\mathcal{S}$. This also shows that $\mathcal{S}$ has an asymptotic density and that it is approximately 0.3284.

Keywords: Symmetric genus, strong symmetric genus, Riemann surface, abelian groups, genus spectrum, density.

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1 Introduction

Let G be a finite group. Among the various genus parameters associated with G, one of the most important is the *symmetric* genus $\sigma(G)$, the minimum genus of any Riemann surface on which G acts faithfully. The origins of this parameter can be traced back over a century to the work of Hurwitz, Poincare, Burnside and others. The modern terminology was introduced in the important article [10].

A natural problem is to determine the positive integers that occur as the symmetric genus of a group (or a particular type of group), that is, to determine the symmetric genus spectrum for the particular type of group. Important results about the symmetric genus spectrum of all finite groups were obtained by Conder and Tucker [1]. They showed that the symmetric genus spectrum of finite groups contains well over 88 percent of all positive integers. In particular, they showed that if g is any non-negative integer such that g is not congruent to 8 or $14 \pmod{18}$, then g is in the spectrum [1, Theorem 1.2]. However, there are no known gaps in the spectrum, and evidence suggests that there are none. Here see [1, Conjecture 1.3].

Our focus here is the symmetric genus spectrum of abelian groups. Let

$$\mathcal{S} = \{g \in \mathbb{N} : g = \sigma(A) \text{ for some abelian group } A\}$$

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denote the symmetric genus spectrum of abelian groups. Henceforth, we will refer to $\mathcal{S}$ simply as the "spectrum."

Closely related to the symmetric genus is the *strong symmetric* genus $\sigma^0(G)$, the minimum genus of any Riemann surface on which G acts faithfully and preserving orientation. Obviously $\sigma(G) \le \sigma^0(G)$ always, but in some (important) cases, the two parameters agree. If the group G does not have a subgroup of index 2, then G cannot act on a surface reversing orientation and thus $\sigma(G) = \sigma^0(G)$. In particular, if G is a group of odd order, then $\sigma(G) = \sigma^0(G)$. Spectrum questions about this parameter have been considered, with some success. The basic problem was settled for the family of all finite groups in [6]: there is a group of strong symmetric genus g, for all $g \in \mathbb{N}$. The strong symmetric genus spectrum of abelian groups was studied in [3]. Let

$$\mathcal{S}_0 = \left\{ g \in \mathbb{N} : g = \sigma^0(A) \text{ for some abelian group } A \right\}.$$

Necessary and sufficient conditions were developed in [3] for an integer g to belong to the spectrum of abelian groups for this parameter; further, this spectrum was shown to have an asymptotic density, approximately equal to 0.3284. In addition, the strong symmetric genus spectrum of nilpotent groups was shown to have lower asymptotic density greater than or equal to $\frac{8}{9}$ in [9].

Our original expectation was that the two spectra $\mathcal{S}$ and $\mathcal{S}_0$ would have a significant intersection, but that there would be integers that are in each spectrum but not the other. Interestingly, this is not the case. Our main result is the following.

Theorem 1.1. $\mathcal{S} = \mathcal{S}_0$.

Our fundamental tool here is the result that determines the symmetric genus $\sigma(A)$ of an abelian group A [5, Theorem 5.7]. An easy but important consequence of this result is that the spectrum $\mathcal{S}$ contains the entire congruence class $g \equiv 1 \pmod 4$.

To establish the containment $\mathcal{S} \subset \mathcal{S}_0$, we show that, given an abelian group A, either $\sigma(A) \equiv 1 \pmod 4$ or $\sigma(A) = \sigma^0(A)$ (or both), unless the Sylow 2-subgroup group of A has rank 2 and is isomorphic to $Z_2 \times Z_{2^\ell}$, for some $\ell \ge 1$. Our approach utilizes the strong constraints on the Sylow 2-subgroup of a group (not necessarily abelian) acting on a surface of even genus or a surface of genus congruent to $3 \pmod 4$; here see [7, Theorem 8] and [8, Theorem 5]. In the exceptional case (in which the Sylow 2-subgroup has a special form), we show that there exists an abelian group A_1 such that $\sigma(A) = \sigma^0(A_1)$.

To establish the reverse containment $\mathcal{S}_0 \subset \mathcal{S}$, we utilize the characterization of the integers in the spectrum $\mathcal{S}_0$ in [3, Theorem 1]. For each integer g satisfying one of the five conditions in that result, we exhibit an abelian group G such that $g = \sigma(G)$.

2 Background results

Let A be a non-trivial finite abelian group of rank r. Then A has the standard canonical representation

$$A \cong Z_{m_1} \times Z_{m_2} \times \cdots \times Z_{m_r}, \tag{2.1}$$

with invariants $m_1, m_2, \ldots, m_r$ subject to $m_1 > 1$ and $m_i \mid m_{i+1}$ for $1 \le i < r$.

The abelian group A also has another canonical form that is useful in calculating genus parameters of abelian groups. Define the *alternate canonical form* for A as the direct product of three subgroups T, B and D of A. First, the group D is the subgroup of A

generated by the factors Z_{m_s}, where m_s is divisible by 4. Then write $A = D \times \overline{D}$. Now let T be the Sylow 2-subgroup of $\overline{D}$, which is elementary abelian, and B be its direct summand of odd order. Therefore, $A = T \times B \times D$. Define $t = \mathrm{rank}(T)$, $b = \mathrm{rank}(B)$, $d = \mathrm{rank}(D)$. It follows that $r = \mathrm{rank}(A) = d + \max(b, t)$. We point out that this notation differs from that used in [5].

The groups of symmetric genus zero are the classical, well-known groups that act on the Riemann sphere (possibly reversing orientation) [2, §6.3.2]. The abelian group A has symmetric genus zero if and only if A is Z_n, $Z_2 \times Z_{2n}$, or $(Z_2)^3$; see [2, §6.3.2].

The groups of symmetric genus one have also been classified, at least in a sense. These groups act on the torus and fall into 17 classes, corresponding to quotients of the 17 Euclidean space groups [2, §6.3.3]. Each class is characterized by a presentation, typically a partial one. The abelian group A has symmetric genus one if and only if A is $Z_m \times Z_{mn}$ with $m \geq 3$, $Z_2 \times Z_2 \times Z_{2n}$ with $n \geq 2$, or $(Z_2)^4$; see [2, §6.3.3].

Let A be a finite abelian group. The strong symmetric genus of A has been completely determined by Maclachlan [4, Theorem 4], and if A has odd order, then $\sigma(A) = \sigma^0(A)$.

The focus of [5] was the determination of the symmetric genus of an abelian group of even order. The approach was to show that, among the various genus actions of A, there is one induced by an NEC group with a signature of one of three types. We established the following result [5, Theorem 3.10].

Theorem A. *Let A be an abelian group of even order. Among the NEC groups with minimal non-euclidean area that act on A, there is a group Γ whose signature has one of the following forms:*

(I) $(g, +, [\lambda_1, \ldots, \lambda_n], \{\ \})$;

(II) $(0, +, [\lambda_1, \ldots, \lambda_s], \{(\ )^k\})$ *for some* $k \geq 1$;

(III) $(0, +, [\,], \{(\ )^u, (2^v)\})$ *for some* $v \geq 2$.

Furthermore, in cases (I) and (II), λ_i divides λ_{i+1} for $1 \leq i \leq r - 1$.

In (III) the notation (2^v) means, as usual, a period cycle with v link periods equal to 2.

We denote by $\tau(A)$ (here and in [5]) the minimum genus of any action of A induced by an NEC group of Type II. The size of the largest elementary abelian 2-group factor of A determines whether $\sigma(A)$ is given by an action induced by a group of Type I, II or III. The main result of [5] is the following [5, Theorem 5.7].

Theorem B. *Let A be an abelian group of even order with canonical form*

$$A \cong (Z_2)^a \times Z_{m_1} \times Z_{m_2} \times \cdots \times Z_{m_q},$$

where $m_1 > 2$. If the symmetric genus $\sigma(A) \geq 2$, then

(i) $\sigma(A) = 1 + |A| \cdot (a + 3q - 4)/8$, *if* $a \geq q + 2$;

(ii) $\sigma(A) = \tau(A)$, *if* $1 \leq a \leq q + 1$;

(iii) $\sigma(A) = \min\{\sigma^0(A), \tau(A)\}$, *if* $a = 0$.

Thus Theorem B gives the symmetric genus of an abelian group A in terms of the invariants of A and the numbers $\sigma^0(A)$ and $\tau(A)$.

The main result in [3] is the characterization [3, Theorem 1] of the integers in the spectrum $\mathcal{S}_0$, and this will be important here.

Theorem C. *Let $g \geq 2$. Then $g \in \mathcal{S}_0$ if and only if g satisfies one of the following conditions:*

(i) $g \equiv 1 \pmod 4$ *or* $g \equiv 55 \pmod{81}$*;*

(ii) $g - 1$ *is divisible by* p^4 *for some odd prime* p*;*

(iii) $g - 1$ *is divisible by* a^2 *for some odd integer* a *with* $(a-1) \mid g$*;*

(iv) $g - 1$ *is divisible by* $b^2a^2(a-1)$ *for some odd integers* $a, b > 1$*, with* $a \equiv 3 \pmod 4$*.*

3 $\mathcal{S}_0 \subset \mathcal{S}$

To establish the containment $\mathcal{S}_0 \subset \mathcal{S}$, we use the characterization of the integers in the spectrum $\mathcal{S}_0$ in Theorem C. For each integer g satisfying one of the five conditions in that result, we exhibit an abelian group G such that $g = \sigma(G)$. This is quite easy, as we shall see.

In this section, we will assume that A is always written in alternate canonical form, $A = T \times B \times D$.

We begin by noting some consequences of Theorem B. Directly from part (i) we have the following; this formula was also pointed out in [7, p. 4094].

Proposition 3.1. $\sigma(Z_2^3 \times Z_{2m}) = 1 + 4m$ *for any integer* $m \geq 2$*.*

Since $\sigma((Z_2)^4) = 1$ and $\sigma((Z_2)^5) = 5$ [7] (the general genus formula is $\sigma((Z_2)^n) = 1 + 2^{n-3}(n-4)$ [5, Corollary 5.4]), it follows that the spectrum $\mathcal{S}$ contains the entire congruence class $g \equiv 1 \pmod 4$. These odd integers are also in $\mathcal{S}_0$ [3, p. 342].

A special case of Theorem B [5, p. 423] will be important here.

Theorem 3.2. *Let the abelian group A have alternate canonical form $A = T \times B \times D$. If T is trivial, then $\sigma(A) = \sigma^0(A)$.*

Let A be a finite abelian group. Then $\sigma(A) = \sigma^0(A)$ in another important case.

Lemma 3.3. *Let A be an abelian group of rank three or more. If the Sylow 2-subgroup S_2 of A is cyclic, then $\sigma(A) = \sigma^0(A)$.*

Proof. By Theorem 3.2, we may assume that T is non-trivial. If S_2 is cyclic, then we must have $S_2 = Z_2$ and $D = 1$. Now write $A \cong Z_2 \times B \cong Z_2 \times Z_{\beta_1} \times Z_{\beta_2} \times \cdots \times Z_{\beta_b}$ for $b \geq 3$, where each β_i is odd. In this case, $t = 1$, $d = 0$ and $b \geq 3$. Since $t \leq d + 1$, [5, p. 416] gives

$$\tau(A) = 1 + \frac{1}{2}|A| \left(-1 + \sum_{i=1}^{b} \left(1 - \frac{1}{\beta_i} \right) \right)$$

(see also (4.1) in Section 4).

The group A has canonical form $A = Z_{\beta_1} \times Z_{\beta_2} \times \cdots \times Z_{2\beta_b}$, and is the image of a Fuchsian group Γ with signature $(0, +, [\beta_1, \ldots, 2\beta_b, 2\beta_b], \{\ \})$. When we calculate the genus arising from the action of this Fuchsian group on A, we get that it is equal to $\tau(A)$. Now applying Maclachlan's formula shows that $\sigma^0(A) \leq \tau(A)$ and hence $\sigma(A) = \sigma^0(A)$ by Theorem B. □

Theorem 3.4. $\mathcal{S}_0 \subset \mathcal{S}$.

Proof. First suppose that $g \equiv 1 \pmod 4$. Then $g \in \mathcal{S}$ by Proposition 3.1 (and the comments after its statement).

Next suppose that $g \equiv 55 \pmod{81}$. Write $g = 55+81j$. Let $G = Z_3 \times Z_3 \times Z_3 \times Z_{3n}$, where $n = j+1$. Then by Maclachlan's formula, $g = \sigma^0(G)$; here also see [3, p. 344]. But since the Sylow 2-subgroup of G is clearly cyclic, we also have $\sigma(G) = \sigma^0(G) = g$ by Lemma 3.3.

Now assume that $g = 1 + mp^4$ for an odd prime p. First let $p \geq 5$ and set $G = Z_p \times Z_p \times Z_p \times Z_{mp}$. Then $g = \sigma^0(G)$; see [3, p. 344]. Again the Sylow 2-subgroup of G is cyclic, and $\sigma(G) = \sigma^0(G) = g$. For the prime $p = 3$, see the calculations in [3, p. 344] and use Lemma 3.3.

Suppose that g satisfies condition (iii) of Theorem C for some odd a. Then, as shown in the proof of Proposition 5 of [3, p. 343], g is the strong symmetric genus of the group $G = Z_a \times Z_a \times Z_{an}$ for some n. Once again, $\sigma(G) = \sigma^0(G) = g$ by Lemma 3.3.

Finally, assume that g satisfies condition (iv) of Theorem C for some odd $a, b > 1$, with $a \equiv 3 \pmod 4$. Then $g = \sigma^0(G)$, where G is a group of the form $Z_a \times Z_{ab} \times Z_{abn}$ [3, p. 343], a group with a cyclic Sylow 2-subgroup, so that $\sigma(G) = \sigma^0(G) = g$ by Lemma 3.3. □

4 The Type II genus

The Type II genus $\tau(A)$ was considered in [5, §4]. The genus $\tau(A)$ is the minimum genus of any action of A induced by an NEC group with signature $(0, +, [\lambda_1, \ldots, \lambda_s], \{(\,)^k\})$ for some $k \geq 1$ (the integer k is the number of empty period cycles). Among the signatures of the NEC groups that induce the Type II genus $\tau(A)$, Lemmas 4.2 and 4.3 of [5] identify one value of k that can be used to calculate $\tau(A)$. These two lemmas are correct. Unfortunately, there is a mistake in [5, Formula (4.5)], which is used in the final determination of $\tau(A)$. We correct that here.

Let $A = T \times B \times D$ be the alternate canonical form for A. Remember that $t = \operatorname{rank}(T)$, $b = \operatorname{rank}(B)$, $d = \operatorname{rank}(D)$ and so $r = \operatorname{rank}(A) = d + \max(b,t)$. The odd order group B is generated by elements with orders $\beta_1, \ldots, \beta_b$, where β_i divides β_j for $i < j$.

In the case in which $t \leq d+1$, the formula for $\tau(A)$ [5, p. 416] is correct. Note that $k = t$ gives minimal area by [5, Lemma 4.2]. In the new notation, this formula is

$$\tau(A) = 1 + \frac{1}{2}|A| \left(k - 2 + \sum_{i=1}^{b} \left(1 - \frac{1}{\beta_i}\right) + \sum_{i=1}^{d+1-t} \left(1 - \frac{1}{\delta_i}\right) \right), \tag{4.1}$$

where the group D is generated by elements with orders $\delta_1, \ldots, \delta_d$ satisfying δ_i divides δ_j for $i < j$.

Next, we consider the case $t > d+1$.

Theorem 4.1. *Let A be an abelian group in alternate canonical form. Suppose that $t > d+1$ and k is given by [5, Lemma 4.3]. There are two cases and in each case, define $\nu = b + d - k + 1$.*

(i) Suppose that $t+d$ is odd. Then $k = (t+d+1)/2$;

(a) If $b \leq (k-1) - d$, then $\tau(A) = 1 + \frac{1}{2}|A|(k-2)$;

(b) If $b > (k-1) - d$*, then* $\nu \geq 1$ *and*

$$\tau(A) = 1 + \frac{1}{2}|A|\left(k - 2 + \sum_{i=1}^{\nu}\left(1 - \frac{1}{\beta_i}\right)\right);$$

(ii) Suppose that $t + d$ *is even. Then* $k = (t+d)/2$*;*

(a) If $b \leq (k-1) - d$*, then* $\tau(A) = 1 + \frac{1}{2}|A|(k - \frac{3}{2})$*;*

(b) If $b > (k-1) - d$*, then* $\nu \geq 1$ *and*

$$\tau(A) = 1 + \frac{1}{2}|A|\left(k - 2 + \sum_{i=1}^{\nu-1}\left(1 - \frac{1}{\beta_i}\right) + \left(1 - \frac{1}{2\beta_\nu}\right)\right).$$

Proof. Among the NEC groups that induce the Type II genus $\tau(A)$, let Γ be one with signature $(0; +; [\lambda_1, \ldots; \lambda_s]; (\,)^k)$ in which the number k of empty period cycles is given by $k = [(t + d + 1)/2]$ [5, Lemma 4.3, p. 414]. It follows that $k \geq d + 1$. The group Γ has generators $x_1, \ldots, x_s$, $e_1, \ldots, e_k$, and involutions $c_1, \ldots, c_k$. The defining relations for Γ consist of $x_1 \cdots x_s e_1 \cdots e_k = 1$, conditions on the order of the elements x_i and certain elements commuting. Clearly, one generator is redundant, and, since $\mu(\Gamma)$ is minimal, we may assume that e_k is that generator. Now let $a_1, \ldots, a_r$ be a generating set for A in canonical form so that the orders of these elements satisfy the standard divisibility condition. With $\mu(\Gamma)$ minimal, the elements $e_1, \ldots, e_{k-1}$ are mapped onto the subgroup generated by the $k-1$ elements $a_{r-k+2}, \ldots, a_r$ of highest order. In particular, since $k-1 \geq d$, the subgroup D of A is contained in the image of $\langle e_1, \ldots, e_{k-1}\rangle$.

If $t + d$ is odd, then $2k - 1 = t + d$. Since $t + d$ is the rank of the Sylow 2-subgroup S_2 of A, we have that S_2 is contained in the image of $\langle c_1, \ldots, c_k, e_1, \ldots, e_{k-1}\rangle$. It follows that $\langle T, D\rangle$ is contained in the image of $\langle c_1, \ldots, c_k, e_1, \ldots, e_{k-1}\rangle$. If $t + d$ is even, then $2k = t + d$. In this case, there is an additional generator x_ℓ so that $\langle T, D\rangle$ is contained in the image of $\langle x_\ell, c_1, \ldots, c_k, e_1, \ldots, e_{k-1}\rangle$.

If $b \leq (k-1) - d$, then the images of the generators e_i which are not mapped into D generate all of B. Therefore, if $t + d$ is odd, then A is the image of the NEC group with signature $(0; +; [\,]; \{(\,)^k\})$ and $\tau(A) = 1 + \frac{1}{2}|A|(k - 2)$. If $t + d$ is even, then we need a generator x_1 in order to map onto A, and with $\mu(\Gamma)$ minimal, $|x_1| = 2$. Therefore, if $t + d$ is even, then A is the image of the NEC group with signature $(0; +; [2]; \{(\,)^k\})$ and $\tau(A) = 1 + \frac{1}{2}|A|(k - \frac{3}{2})$.

The last case is when $b > (k-1) - d$. Let E be the subgroup of A generated by the images of $e_1, \ldots, e_{k-1}$. Then the subgroup B can be decomposed as $B = B_1 \times B_2$, where $B_2 = B \cap E$. Let $\nu = b + d - k + 1$ so that ν is the rank of B_1 and $\nu \geq 1$. We need generators $x_1, \ldots, x_\nu$ to map onto B_1. If $t + d$ is odd, then A is the image of the NEC group with signature $(0; +; [\beta_1, \ldots, \beta_\nu]; \{(\,)^k\})$ and

$$\tau(A) = 1 + \frac{1}{2}|A|\left(k - 2 + \sum_{i=1}^{\nu}\left(1 - \frac{1}{\beta_i}\right)\right).$$

Suppose that $t + d$ is even. As in the previous case, we need generators $x_1, \ldots, x_\nu$ to map onto B_1. However, the generator x_ν must map onto an element of order $2\beta_\nu$ for the NEC group to map onto A. This is because there is the extra involution not in the image of $\langle c_1, \dots, c_k\rangle$. If $t + d$ is even, then A is the image of the NEC group with signature $(0; +; [\beta_1, \dots, \beta_{\nu-1}, 2\beta_\nu]; \{(\ )^k\})$ and

$$\tau(A) = 1 + \frac{1}{2}|A| \left(k - 2 + \sum_{i=1}^{\nu-1} \left(1 - \frac{1}{\beta_i}\right) + \left(1 - \frac{1}{2\beta_\nu}\right)\right). \qquad \square$$

5 General results

Again in this section, we assume that A is written in alternate canonical form, $A = T \times B \times D$.

Let A be a finite abelian group so that the integer $g = \sigma(A)$ is in the spectrum $\mathcal{S}$. We want to show that g is in $\mathcal{S}_0$ as well. This is clearly the case if A has rank one or two or A has a trivial factor T in its alternate canonical form. Thus we may assume that A has rank at least 3 and T is not trivial. In particular, A has even order.

Our approach utilizes the Sylow 2-subgroup S_2 of A.

Lemma 5.1. *Let A be an abelian group of rank three or more with $\sigma(A) \geq 2$. If the Sylow 2-subgroup of A is isomorphic to $(Z_2)^3$, then $\tau(A) \equiv 1 \pmod 4$.*

Proof. Since $S_2 \cong (Z_2)^3$, we have $T = S_2$, $D = 1$ and $A = T \times B$. Now $t = 3$, $d = 0$, and $k = 2$. Since $b = 1$ would imply $\sigma(A) = 1$, $b \geq 2$. We have $t > d + 1$ with $t + d$ odd. The Type II genus is $\tau(A) = 1 + |A| \cdot M/2$, where $M = (k - 2 + \sum_{i=1}^{\nu}(1 - \frac{1}{\beta_i}))$ by Theorem 4.1(i)(b). Since $M \cdot |B|$ is an integer and $|A| = 8|B|$, we clearly have $\tau(A) \equiv 1 \pmod 4$. $\square$

Lemma 5.2. *Let A be an abelian group of rank three or more with $\sigma(A) \geq 2$. If the Sylow 2-subgroup of A is isomorphic to $Z_2 \times Z_2 \times Z_{2^\ell}$ for some $\ell \geq 2$, then $\tau(A) \equiv 1 \pmod 4$.*

Proof. In this case we have $T = Z_2 \times Z_2$ and D is cyclic with order divisible by 2^ℓ. Therefore, $t = 2$ and $d = 1$. This implies that $k = t = 2$ by [5, Lemma 4.2]. Since $b = 1$ would imply $\sigma(A) = 1$, $b \geq 2$. We have $t = d + 1$ and by (4.1), the Type II genus $\tau(A) = 1 + |A| \cdot M/2$ where $M = (k - 2 + \sum_{i=1}^{b}(1 - \frac{1}{\beta_i}))$. Since $M \cdot |B|$ is an integer and $|A| = 4 \cdot |D| \cdot |B|$ with $|D|$ divisible by 2^ℓ, again $\tau(A) \equiv 1 \pmod 4$. $\square$

Lemma 5.3. *Let A be an abelian group. If the Sylow 2-subgroup of A is isomorphic to $(Z_2)^4$, then $\tau(A) \equiv 1 \pmod 4$.*

Proof. Since $S_2 \cong (Z_2)^4$, we have $T = S_2$, $D = 1$ and $A = T \times B$. Now $t = 4$, $d = 0$, $k = 2$ and $b \geq 1$. We have $t > d+1$ with $t+d$ even. We apply Theorem 4.1(ii) and get that $k = 2$. If $b = 1$, then $M = 1/2$ by Theorem 4.1(ii)(a). If $b \geq 2$, then by Theorem 4.1(ii)(b), $M \cdot 2|B|$ is an integer. In either case, $2M \cdot |B|$ is an integer. Since $|A| = 16|B|$, we again have $\tau(A) \equiv 1 \pmod 4$. $\square$

Theorem 5.4. *Let A be an abelian group of rank three or more with $\sigma(A) \geq 2$. Then either $\sigma(A) \equiv 1 \pmod 4$ or $\sigma(A) = \sigma^0(A)$ (or both), unless the Sylow 2-subgroup of A has rank 2 and is isomorphic to $Z_2 \times Z_{2^\ell}$, for some $\ell \geq 1$.*

Proof. First, if A has odd order, then $\sigma(A) = \sigma^0(A)$. Assume, then, that A has even order so that $\sigma(A)$ is given by Theorem B. Let A have canonical form

$$A \cong (Z_2)^a \times Z_{m_1} \times Z_{m_2} \times \cdots \times Z_{m_q},$$

as in Theorem B. It is easy to see in case (i), we always have that $\sigma(A) \equiv 1 \pmod 4$.

Suppose $a \leq q+1$. By Theorem B, $\sigma(A)$ is either equal to $\sigma^0(A)$ or $\tau(A)$. Let A act on a surface X of genus $g = \sigma(A) \geq 2$, and write $|A| = 2^n \cdot m$, where m is odd.

Assume first that g is even. Then by [7, Theorem 9], A is not a 2-group so that $m \neq 1$. We consider the possibilities for the Sylow 2-subgroup S_2 of A. If S_2 is cyclic, then by $\sigma(A) = \sigma^0(A)$ by Lemma 3.3. Assume then that S_2 is not cyclic. If A contains an element of order 2^{n-1} with $|S_2| = 2^n$, then S_2 is isomorphic to $Z_2 \times Z_{2^{n-1}}$, the exceptional case.

Assume then that A has no elements of order 2^{n-1}, and apply [7, Theorem 8]. Since A and S_2 are abelian, the only possibility is that S_2 is isomorphic to $(Z_2)^3$. But in this case $\tau(A) \equiv 1 \pmod 4$ by Lemma 5.1.

Therefore, by Theorem B, if g is even, then either $\sigma(A) \equiv 1 \pmod 4$ or $\sigma(A) = \sigma^0(A)$, unless $S_2 \cong Z_2 \times Z_{2^\ell}$, for some $\ell \geq 1$.

Now suppose $g \equiv 3 \pmod 4$, and use [8, Theorem 5]. The Sylow 2-subgroup S_2 also acts of the surface X of genus $g \geq 2$. By [8, Theorem 5], S_2 contains an element of order 2^{n-3} or larger. Further, if $\mathrm{Exp}(S_2) = 2^{n-3}$, then S_2 contains a dihedral subgroup of index 4. We consider the possibilities for $\mathrm{Exp}(S_2)$.

If S_2 is cyclic, then by $\sigma(A) = \sigma^0(A)$ by Lemma 3.3.

If S_2 is not cyclic and contains an element of order 2^{n-1}, then S_2 is isomorphic to $Z_2 \times Z_{2^{n-1}}$, the exceptional case.

Suppose $\mathrm{Exp}(S_2) = 2^{n-2}$. Then S_2 is isomorphic to either $Z_2 \times Z_2 \times Z_{2^{n-2}}$ or $Z_4 \times Z_{2^{n-2}}$. If $S_2 \cong Z_2 \times Z_2 \times Z_{2^{n-2}}$, then by Lemma 5.2, $\tau(A) \equiv 1 \pmod 4$. If on the other hand, $S_2 \cong Z_4 \times Z_{2^{n-2}}$, then by Theorem 3.2, $\sigma(A) = \sigma^0(A)$.

Suppose $\mathrm{Exp}(S_2) = 2^{n-3}$ and S_2 has a dihedral subgroup of index 4. Since S_2 is abelian, this forces $n = 4$ and $S_2 \cong (Z_2)^4$. In this case, $\tau(A) \equiv 1 \pmod 4$ by Lemma 5.3.

Therefore, by Theorem B, if $g \equiv 3 \pmod 4$, then either $\sigma(A) = \tau(A) \equiv 1 \pmod 4$ or $\sigma(A) = \sigma^0(A)$, unless $S_2 \cong Z_2 \times Z_{2^\ell}$, for some $\ell \geq 1$. □

A consequence of the proof is perhaps worth noting, in connection with the well-known conjecture that "almost all" groups are 2-groups.

Theorem 5.5. *If A be an abelian 2-group of positive symmetric genus, then*

$$\sigma(A) \equiv 1 \pmod 4.$$

Proof. Assume A is an abelian 2-group with $\sigma(A) \geq 2$. Then $\sigma(A)$ is not even by [7, Theorem 9]. The proof of Theorem 5.4 shows that $\sigma(A)$ cannot be congruent to $3 \pmod 4$ either, since $A = S_2$ and A is not an abelian group of genus zero or one. □

Next we handle the exceptional case in Theorem 5.4.

Theorem 5.6. *Let A be an abelian group of rank three or more with $\sigma(A) \geq 2$. If the Sylow 2-subgroup of A is isomorphic to $Z_2 \times Z_{2^\ell}$, for some $\ell \geq 1$, then there exists an abelian group A_1 such that $\sigma(A) = \sigma^0(A_1)$.*

Proof. Let A have alternate canonical form $A = T \times B \times D$. We consider two cases, $\ell = 1$ and $\ell \geq 2$.

First assume that $S_2 \cong (Z_2)^2$. Now $T = S_2$, $D = 1$ and $A = T \times B$, with $t = 2$, $d = 0$, and $k = 1$. Write $A \cong Z_2 \times Z_2 \times B \cong Z_2 \times Z_2 \times Z_{\beta_1} \times Z_{\beta_2} \times \cdots \times Z_{\beta_b}$ for $b \geq 3$, where each β_i is odd. By Theorem 4.1(ii)(b), the Type II genus is

$$\tau(A) = 1 + \frac{1}{2}|A|\left(-1 + \sum_{i=1}^{b-1}\left(1 - \frac{1}{\beta_i}\right) + \left(1 - \frac{1}{2\beta_b}\right)\right).$$

By Theorem B, $\sigma(A) = \min\{\sigma^0(A), \tau(A)\}$. If $\sigma(A) = \sigma^0(A)$, then set $A_1 = A$ and we are done. So we assume that $\tau(A) < \sigma^0(A)$. Maclachlan's formula uses the non-euclidean areas of the groups Δ_0 with signature $(0, +, [\beta_1, \dots, 2\beta_{b-1}, 2\beta_b, 2\beta_b], \{\ \})$ and Γ_p with signatures $(p, +, [\beta_1, \dots, \beta_{b-2p}, \beta_{b-2p}], \{\ \})$ for $p \geq 1$ to obtain $\sigma^0(A)$.

Let $A_1 = Z_{\beta_1} \times \cdots \times Z_{\beta_{b-1}} \times Z_{4\beta_b}$ be an abelian group. Maclachlan's formula for $\sigma^0(A_1)$ uses the minimum non-euclidean areas of the Fuchsian groups Γ_0 with signature $(0, +, [\beta_1, \dots, \beta_{b-1}, 4\beta_b, 4\beta_b], \{\ \})$ and Γ_p for $p \geq 1$ as in the calculation of $\sigma^0(A)$. By assumption the Fuchsian groups Γ_p for $p \geq 1$ give genus larger than $\tau(A)$. The Fuchsian group Γ_0 gives the same genus as $\tau(A)$. Therefore, $\tau(A) = \sigma^0(A_1)$ and so $\sigma(A) = \sigma^0(A_1)$.

Now assume that $S_2 \cong Z_2 \times Z_{2^\ell}$, for some $\ell \geq 2$. Now $T = Z_2$ and D is isomorphic to Z_{m2^ℓ}, where m is odd. We have alternate canonical form $A = Z_2 \times B \times Z_{m2^\ell} \cong Z_2 \times Z_{\beta_1} \times Z_{\beta_2} \times \cdots \times Z_{\beta_{b-1}} \times Z_{m2^\ell}$, where each β_i is odd, β_i divides β_j for $i < j$ and m is divisible by β_{b-1}. It follows that $t = 1$, $b \geq 3$, $d = 1$, and $k = 1$ by [5, Lemma 4.2]. By (4.1) the Type II genus is

$$\tau(A) = 1 + \frac{1}{2}|A|\left(-1 + \sum_{i=1}^{b}\left(1 - \frac{1}{\beta_i}\right) + \left(1 - \frac{1}{m2^\ell}\right)\right).$$

By Theorem B, $\sigma(A) = \min\{\sigma^0(A), \tau(A)\}$. If $\sigma(A) = \sigma^0(A)$, then let $A_1 = A$ and we are done. So we will assume that $\tau(A) < \sigma^0(A)$. Now let $A_1 = Z_{\beta_1} \times \cdots \times Z_{\beta_b} \times Z_{m2^{\ell+1}}$. The Fuchsian group with signature $(0, +, [\beta_1, \dots, \beta_b, m2^{\ell+1}, m2^{\ell+1}], \{\ \})$ has genus action equal to $\tau(A)$. As in the previous case, $\tau(A) = \sigma^0(A_1)$ and so $\sigma(A) = \sigma^0(A_1)$. □

Combining Theorems 5.4 and 5.6 yields the following.

Theorem 5.7. $\mathcal{S} \subset \mathcal{S}_0$.

Of course, Theorems 5.7 and 3.4 provide the proof of Theorem 1.1. Theorem 1.1 and the results in [3] give some interesting results about the symmetric genus spectrum $\mathcal{S}$ of abelian groups. Now Theorem C gives a necessary and sufficient condition for a positive integer to be in the spectrum $\mathcal{S}$. Two other consequences are perhaps worth stating.

Corollary 5.8. *The spectrum $\mathcal{S}$ has an asymptotic density $\delta(\mathcal{S}) \approx 0.3284$.*

Corollary 5.9. *If $g - 1$ is a square-free integer, then $g \notin \mathcal{S}$.*

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