Coincident-point rigidity in normed planes

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Abstract

A bar-joint framework (G, p) is the combination of a graph G and a map p assigning positions, in some space, to the vertices of G. The framework is rigid if every edge-length-preserving continuous motion of the vertices arises from an isometry of the space. We will analyse rigidity when the space is a (non-Euclidean) normed plane and two designated vertices are mapped to the same position. This non-genericity assumption leads us to a count matroid first introduced by Jackson, Kaszanitsky and the third author. We show that independence in this matroid is equivalent to independence as a suitably regular bar-joint framework in a non-Euclidean normed plane with two coincident points; this characterises when a regular non-Euclidean normed plane coincident-point framework is rigid and allows us to deduce a delete-contract characterisation. We then apply this result to show that an important construction operation (generalised vertex splitting) preserves the stronger property of global rigidity in non-Euclidean normed planes and use this to construct rich families of globally rigid graphs when the non-Euclidean normed plane is analytic.

Keywords: Bar-joint framework, global rigidity, non-Euclidean framework, count matroid, recursive construction, normed spaces, analytic norm.

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1 Introduction

A bar-joint *framework* (G, p) is the combination of a graph $G = (V, E)$ and a map $p\colon V \to \mathbb{R}^d$ assigning positions to the vertices of G (and hence lengths to the edges). Note that in this article graphs are taken to be finite and simple. Intuitively, the framework is *rigid* if every edge-length-preserving continuous motion of the vertices arises from an isometry of $\mathbb{R}^d$. More strongly, (G, p) is *globally rigid* if every framework in $\mathbb{R}^d$, on the same graph, with the same edge lengths actually has the same distance between every pair of vertices.

The rigidity and global rigidity of bar-joint frameworks in Euclidean spaces has been intensely studied in recent years (e.g. [2, 3, 13, 16, 21, 23]) and has a rich history going as far back as classical work of Euler and Cauchy on Euclidean polyhedra. In the last decade, work on rigidity has been generalised to various non-Euclidean normed spaces (e.g. [6, 9, 10, 19, 20]). All of these results concern characterising the combinatorial nature of the 'generic' behaviour of frameworks. This article extends this to frameworks with two points lying in the same location. The difficulty that already arises in this context shows how necessary the genericity assumption in those papers really was. Frameworks with coincident points have been considered in the Euclidean context [12, 14] and applied to global rigidity there [4], as well as for frameworks on surfaces [17].

Beyond the natural extension towards non-generic frameworks (and thus nearer to being of potential use in applications), we are motivated by the study of global rigidity in non-Euclidean normed planes. The first and third author recently instigated research in this direction [10] proving global rigidity for an infinite class of graphs in non-Euclidean analytic normed planes. In this paper we use our analysis of frameworks with two coincident points to improve this result by creating a substantially richer class of globally rigid graphs.

We now give a short outline of what follows. After introducing the necessary background on the theory of rigid frameworks in normed planes, coincident-point frameworks and the relevant notion of graph sparsity, in Section 2, the majority of the paper is contained in Section 3. Here we provide a detailed geometric analysis of the effect of certain graph operations on the rigidity of a coincident-point framework in a normed plane. In Section 4 we combine these geometric results with combinatorial results of [17] to establish a purely combinatorial characterisation of independence in the 'coincident-point normed plane rigidity matroid' and we deduce from this a delete-contract characterisation of coincident-point rigidity in any strictly convex non-Euclidean normed plane. By delete-contract characterisation we mean that we characterise the coincident-point rigidity of a graph G in terms of the rigidity of two graphs related to G; the graphs obtained from G by deleting the edge between the coincident vertices and the graph obtained by contracting the two coincident vertices. In Section 5 we provide our other main results; these concern global rigidity. We deduce from our delete-contract characterisation that another graph operation preserves global rigidity, and we use this result alongside the results of [10] to establish global rigidity in the special case of non-Euclidean analytic normed planes for a rich family of graphs.

We conclude the introduction with a brief comparison with the more familiar Euclidean case to give context for the reader. Both our characterisation of independence in the coincident-point normed plane rigidity matroid and our delete-contract characterisation are precise analogues of results obtained by Fekete, Jordán and Kaszanitzky for the Euclidean case [12]. Furthermore, in the Euclidean case generic global rigidity in the plane is completely characterised [16]. Our results provide a key step towards establishing an analogue of that result in non-Euclidean analytic normed planes. It is worth noting though that the

results of [12] came later than the Euclidean plane characterisation and, to our knowledge, have not been used to provide an alternative proof of the global rigidity characterisation in the Euclidean plane. The non-Euclidean normed plane case requires both subtly different combinatorics and geometry which motivated our deployment of this technique. We would expect that our application to global rigidity through 'generalised vertex splitting' (defined in Section 5) could be adapted to the Euclidean case.

2 Rigidity and uv-coincident frameworks in normed spaces

2.1 Rigidity in normed spaces

Let X be a real finite-dimensional normed space with norm $\| \cdot \|$. We define a *support functional* of $z \in X$ to be a linear functional $f \colon X \to \mathbb{R}$ such that $f(z) = \|z\|^2$ and $\sup_{\|x\|=1} f(x) = \|z\|$. It follows from the Hahn-Banach theorem that every point has a support functional and every linear functional of X is the support functional of a point in X. A non-zero point in X is said to be *smooth* if it has exactly one support functional, and we shall denote the unique support functional of a smooth point z by φ_z. We say X is *smooth* if every non-zero point in X is smooth, and *strictly convex* if every linear functional of X is the support functional of at most one, and hence exactly one, point in X.[1] We note that for normed planes (2-dimensional normed spaces), strict convexity is equivalent to the property that any two linearly independent smooth points have linearly independent support functionals.

Now let (G, p) be a framework in X; that is the combination of a graph $G = (V, E)$ and a map $p \colon V \to X$ (called a *placement* of G). A *finite flex* of (G, p) is a continuous path $\alpha \colon [0, 1] \to X^V$ where $\alpha(0) = p$ and $\|\alpha_x(t) - \alpha_y(t)\| = \|p_x - p_y\|$ for each edge $xy \in E$ and every $t \in [0, 1]$. If every framework $(G, \alpha(t))$ is congruent to (G, p), i.e. there exists an isometry $f_t \colon X \to X$ so that $\alpha_x(t) = f_t(p_x)$ for every $x \in V$, then we say α is *trivial*. We now define (G, p) to be *(continuously) rigid* if every finite flex of (G, p) is trivial.

Since determining whether a framework is rigid is computationally challenging [1], we follow the literature and linearise the problem. First, let (G, p) be a *well-positioned* framework, i.e. the point $p_x - p_y$ is smooth for each edge $xy \in E$. An *infinitesimal flex* of (G, p) is a map $u \colon V \to X$ where $\varphi_{p_x - p_y}(u_x - u_y) = 0$ for all $xy \in E$. An infinitesimal flex is *trivial* if there exists a linear map $T \colon X \to X$ and a point $z_0 \in X$ so that $u_x = T(p_x) + z_0$ for every vertex $x \in V$, and the map T is tangent to the linear isometry group of X at the identity map. Importantly, when X has finitely many linear isometries – for example, when X is a non-Euclidean normed plane [25, page 83] – the only trivial infinitesimal flexes are those that stem from translations, i.e., infinitesimal flexes $u = (u_x)_{x \in V}$ where there exists $z \in X$ such that $u_x = z$ for all $x \in V$. We now say that a well-positioned framework (G, p) is *infinitesimally rigid* if every infinitesimal flex of (G, p) is trivial.

For a d-dimensional normed space X, a well-positioned framework (G, p) in X, and a fixed basis $b_1, \ldots, b_d$ of X, we can define the *rigidity matrix* to be the $|E| \times d|V|$ matrix

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[1]Here we have opted to use a more relevant – but still equivalent – definition for strict convexity. The more conventional definition for the property is as follows: a normed space is said to be strictly convex if $\|tx + (1 - t)y\| < 1$ for all $x, y \in X$ with $\|x\| = \|y\| = 1$ and each $0 < t < 1$. To see the equivalence, note that if $\sup_{\|z\|=1} f(z) = 1$ and $\|x\| = \|y\| = 1$, then $f(x) = f(y) = 1$ if and only if $\|tx + (1-t)y\| = 1$ for all $0 < t < 1$: this latter fact stems from the inequality $tf(x) + (1-t)f(y) \leq f(tx + (1-t)y) \leq \|tx + (1-t)y\| \leq 1$.

$R(G, p)$, where for every $e \in E$, $x \in V$ and $i \in \{1, \ldots, d\}$ we have

$$R(G, p)_{e,(x,i)} = \begin{cases} \varphi_{p_x - p_y}(b_i) & \text{if } e = xy, \\ 0 & \text{otherwise.} \end{cases}$$

The choice of basis used to define $R(G, p)$ can be arbitrary as we are only interested in the sets of linearly independent rows of the matrix. We say a well-positioned framework is *independent* if $\operatorname{rank} R(G, p) = |E|$, *minimally (infinitesimally) rigid* if it is both independent and infinitesimally rigid, and *regular* if $\operatorname{rank} R(G, p) \geq \operatorname{rank} R(G, q)$ for all other well-positioned frameworks (G, q). It is immediate that all independent and/or infinitesimally rigid frameworks are regular. Given k is the dimension of the linear space of trivial infinitesimal flexes of (G, p), it can be shown that so long as the affine span of the set $\{p_x : x \in V\}$ is X, the framework (G, p) will be infinitesimally rigid if and only if $\operatorname{rank} R(G, p) = d|V| - k$; see [6, Proposition 3.13]. Consequently any well-positioned framework where the affine span of the set $\{p_x : x \in V\}$ is X, is minimally rigid if and only if $|E| = \operatorname{rank} R(G, p) = d|V| - k$.

We can link infinitesimal rigidity to rigidity with the following result.

Theorem 2.1. *Let (G, p) be a well-positioned framework in a normed space X.*

  (i) [7, Theorem 3.7] *If (G, p) is infinitesimally rigid, then it is rigid.*

  (ii) [6, Theorem 1.1 & Lemma 4.4] *If (G, p) is regular and rigid, and the set of smooth points in X is open, then (G, p) is infinitesimally rigid.*

We shall make use of the following perturbation result throughout the paper. It will be convenient to refer to properties of placements rather than frameworks. To this end we say that a placement p of G has property P if the framework (G, p) has property P.

Lemma 2.2 ([6, Lemmas 4.1 and 4.4]). *For any graph G and any normed space X, the set of well-positioned placements of G in X is a conull (i.e. the complement of a set with Lebesgue measure zero) subset of X^V, and the set of regular placements of G in X is a non-empty open subset of the set of well-positioned placements.*

We say that a graph is *rigid* (respectively, *independent*, *minimally rigid*) if it has an infinitesimally rigid (respectively, independent, minimally rigid) placement.

Whether a graph $G = (V, E)$ is rigid/independent in a normed plane can be determined by simple sparsity counting conditions. For $\emptyset \neq U \subseteq V$, $i_G(U)$ will denote the number of edges in the subgraph, $G[U]$, of G induced by U. For non-negative integers k, ℓ, we say G is (k, ℓ)-*sparse* if $i_G(U) \leq k|U| - \ell$ for every $\emptyset \neq U \subseteq V$ with $|U| \geq k$; if G is (k, ℓ)-sparse and $|E| = k|V| - \ell$, then we say G is (k, ℓ)-*tight*.

Theorem 2.3 ([5]). *A graph G is minimally rigid in a non-Euclidean normed plane X if and only if G is $(2, 2)$-tight.*

For a family $\mathcal{S} = \{S_1, S_2, \ldots, S_k\}$ of subsets $S_i \subseteq V$, $1 \leq i \leq k$, we say that $\mathcal{S}$ is a *cover* of $F \subseteq E$ if $F \subseteq \{xy : \{x, y\} \subseteq S_i \text{ for some } 1 \leq i \leq k\}$. We can combine Theorem 2.3 with [17, Section 3.1] (which simply applies a classical result of Edmonds [11] on matroids induced by submodular functions) to obtain the following result.

Corollary 2.4. *Let* (G, p) *be a well-positioned framework in a non-Euclidean normed plane* X. *Let* $\mathcal{S}$ *be the set of all covers* $\mathcal{X} := \{X_1, \ldots, X_k\}$. *Given* $s \colon \mathbb{N} \to \{0, 1\}$ *is the map with* $s(x) = 1$ *if* $x = 2$ *and* $s(x) = 0$ *otherwise, we have*

$$\operatorname{rank} R(G, p) \leq \min_{\mathcal{X} \in \mathcal{S}} \sum_{i=1}^{k} \left(2|X_i| - 2 - s(|X_i|) \right),$$

with equality if and only if (G, p) *is regular. Moreover it suffices to minimise over all covers* $\mathcal{Y} := \{Y_1, \ldots, Y_k\}$ *of the edge set* E *where* $|Y_i| \geq 2$ *for each* i *and* $|Y_i \cap Y_j| \leq 1$ *for all* $i \neq j$, *with equality only if* $\min\{|Y_i|, |Y_j|\} = 2$.

2.2 uv-coincident rigidity and uv-sparse graphs

Let $G = (V, E)$ be a graph with vertices $u, v \in V$, and let X be a normed space. A framework (G, p) in X is uv-*coincident* if $p_u = p_v$; if the framework $(G - uv, p)$ is well-positioned, then we say that (G, p) is a *well-positioned* uv-*coincident framework*. Since $p_u = p_v$, we consider $G - uv$ so as to maintain smoothness of the support functionals associated with the framework; otherwise, no uv-coincident framework with uv as an edge would be well-positioned.

A well-positioned uv-coincident framework (G, p) is *infinitesimally rigid* if $(G - uv, p)$ is infinitesimally rigid in X. Given the linear space $X^V/uv := \{q \in X^V : q_u = q_v\}$, we say that a well-positioned uv-coincident framework (G, p) is *regular* if $\operatorname{rank} R(G - uv, p) \geq \operatorname{rank} R(G - uv, q)$ for all $q \in X^V/uv$, and *independent* if $uv \notin E$ and (G, p) is independent in X. A well-positioned uv-coincident framework (G, p) is *minimally (infinitesimally) rigid* if it is both infinitesimally rigid and independent. We say a graph G is uv-*rigid* (respectively, uv-*independent*, *minimally* uv-*rigid*) if there exists a uv-coincident framework (G, p) that is infinitesimally rigid (respectively, independent, minimally rigid).

By applying the same methods used to prove Lemma 2.2, we can obtain the natural analogue for uv-coincident frameworks. The two main observations for proving the result are: (i) the set of smooth points of a normed space form a conull subset (i.e. the complement of a set with Lebesgue measure zero) of X and (ii) the map $p \mapsto R(G - uv, p)$ is lower semi-continuous.

Lemma 2.5. *For any graph* G *and any normed space* X, *the set of well-positioned* uv-*coincident placements of* G *in* X *is a conull subset of* X^V/uv, *and the set of regular* uv-*coincident placements of* G *in* X *is a non-empty open subset of the set of well-positioned* uv-*coincident placements.*

As will be shown in Section 4, uv-rigidity in non-Euclidean normed planes is closely related to the following sparsity property of graphs.

Let $G = (V, E)$ be a graph and let u, v be two distinct vertices of G. Let $\mathcal{X} = \{X_1, X_2, ..., X_k\}$ be a family with $X_i \subseteq V$, $1 \leq i \leq k$. We say that $\mathcal{X}$ is uv-*compatible* if $u, v \in X_i$ and $|X_i| \geq 3$ hold for all $1 \leq i \leq k$. We define the *value* of non-empty subsets of V and of uv-compatible families, denoted $\operatorname{val}(\cdot)$, as follows. For $\emptyset \neq U \subseteq V$, we let

$$\operatorname{val}(U) = 2|U| - t_U,$$

where $t_U = 4$ if $U = \{u, v\}$, $t_U = 3$ if $U \neq \{u, v\}$ and $|U| \in \{2, 3\}$, and $t_U = 2$

otherwise. For a uv-compatible family $\mathcal{X} = \{X_1, X_2, \ldots, X_k\}$ we let

$$\text{val}(\mathcal{X}) = \left(\sum_{i=1}^{k} \text{val}(X_i) \right) - 2(k-1) = 2 + \sum_{i=1}^{k} (2|X_i| - t_{X_i} - 2).$$

Note that if $\mathcal{X} = \{U\}$ is a uv-compatible family containing only one set then the two definitions agree, i.e. $\text{val}(\mathcal{X}) = \text{val}(U)$ holds.

We say that G is uv-*sparse* if for all $U \subseteq V$ with $|U| \geq 2$ we have $i_G(U) \leq \text{val}(U)$ and for all uv-compatible families $\mathcal{X}$ we have $i_G(\mathcal{X}) := \left| \bigcup_{i=1}^{k} E(G[X_i]) \right| \leq \text{val}(\mathcal{X})$. A graph G is uv-*tight* if it is uv-sparse and $|E| = 2|V| - 2$. Note that if G is uv-sparse then $uv \notin E$. It was shown in [17] that if $G = (V, E)$ is a graph and $u, v \in V$ are distinct vertices of G then $\mathcal{I} = \{F : F \subseteq E \text{ and } (V, F) \text{ is } uv\text{-sparse}\}$ is the family of independent sets of a matroid $\mathcal{M}_{uv}$ on E.

It is straightforward to construct $(2, 2)$-sparse graphs which are not uv-sparse. Perhaps the simplest way is to notice that the complete bipartite graph $K_{2,3}$, with the part of size two comprising of u and v, is clearly $(2, 2)$-sparse but fails to be uv-sparse. To see this let v_1, v_2, v_3 be the vertices in the part of size three and consider the uv-compatible family $\mathcal{X} = \{X_1, X_2, X_3\}$ where $X_1 = \{u, v, v_1\}$, $X_2 = \{u, v, v_2\}$ and $X_3 = \{u, v, v_3\}$. Then $i_G(\mathcal{X}) = 2 + 2 + 2 = 6$ and $\text{val}(\mathcal{X}) = (2 \cdot 3 - 3) + (2 \cdot 3 - 3) + (2 \cdot 3 - 3) - 2(3 - 1) = 5$.

3   Recursive operations

Let $G = (V, E)$ be a graph. The 0-*extension* operation (on a pair of distinct vertices $a, b \in V$) adds a new vertex z and two edges za, zb to G. The 1-*extension* operation (on edge $ab \in E$ and vertex $c \in V \setminus \{a, b\}$) deletes the edge ab, adds a new vertex z and edges za, zb, zc. The *vertex-to-H move* adds a copy of a $(2, 2)$-tight graph H with $V(H) \cap V = \{w\}$, along with an arbitrary replacement of each edge xw by an edge of the form xy with $y \in V(H)$. A *vertex-to-4-cycle move* takes a vertex w with neighbours $v_1, v_2, \ldots, v_k$ for any $k \geq 2$, splits w into two new vertices w, w' with $w' \notin V$, adds edges $wv_1, w'v_1, wv_2, w'v_2$ and then arbitrarily replaces edges xw with edges of the form xy where $x \in \{v_3, \ldots, v_k\}$ and $y \in \{w, w'\}$. All $(2, 2)$-tight graphs can be constructed from a single vertex by a sequence of 0- and 1-extensions, vertex-to-4-cycle and vertex-to-K_4 operations; see [22, Theorem 3.1] for more details. The operations we use are illustrated in Figures 1 and 2.
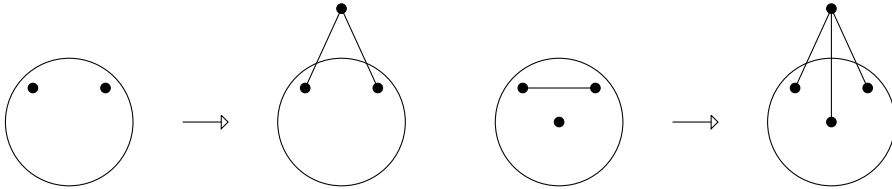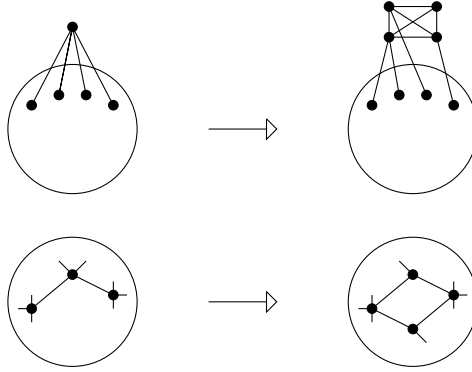

Figure 1: 0-extension and 1-extension.

Figure 2: The vertex-to-H (with H being the complete graph on 4 vertices) and vertex-to-4-cycle operations.

We shall need the following specialized versions. First, suppose that $|V \cap \{u, v\}| = 1$. The *0-extension that adds u* (respectively, *0-extension that adds v*) operation is a 0-extension where $z = u$ and $v \in V \setminus \{a, b\}$ (respectively, with $z = v$ and $u \in V \setminus \{a, b\}$). The *vertex-to-4-cycle move that adds u* (respectively, *vertex-to-4-cycle move that adds v*) is a vertex-to-4-cycle move where $w = v$ and $\{w, w'\} = \{u, v\}$ (respectively, $w = u$ and $\{w, w'\} = \{u, v\}$). The *vertex-to-H move that adds u* (respectively, *vertex-to-H move that adds v*) is a vertex-to-H move where $w = v$ and $u \in V(H) \setminus V$ (respectively, $w = u$ and $v \in V(H) \setminus V$), and the graph H is uv-tight.

Now suppose $u, v \in V$ are two distinct vertices. The *uv-0-extension* operation is a 0-extension on a pair a, b with $\{a, b\} \neq \{u, v\}$. The *uv-1-extension* operation is a 1-extension on some edge ab and vertex c for which $\{u, v\}$ is not a subset of $\{a, b, c\}$. The *uv-vertex-to-4-cycle* and *uv-vertex-to-H* moves are simply any vertex-to-4-cycle and vertex-to-H moves applied to a graph containing both u and v.

We can immediately obtain the following result using the proof technique of [5, Lemmas 5.1 and 5.2]. In particular, since the uv-0- and uv-1-extensions are local operations that relate to at most one of u and v, their coincidence does not have any effect on the proofs presented in [5].

Lemma 3.1. *Let G be a graph that contains both u and v, and let G' be formed from G by either a uv-0-extension or a uv-1-extension. If G is uv-independent in a normed plane X, then G' is uv-independent in X.*

The next lemma shows 0-extensions that add either u or v preserve independence. It should be noted that our proof technique requires strict convexity.

Lemma 3.2. *Let $G = (V, E)$ be a graph that contains u but not v, and let X be a strictly convex normed plane. Suppose G' is formed from G by a 0-extension that adds v. Then G' is uv-independent if and only if G is independent.*

Proof. We note that as G' contains G as a subgraph, if G' is uv-independent then G will be independent. Suppose there is an independent placement p of G in X. By applying translations, we may suppose that $p_u = 0$. Let v_1, v_2 be the two neighbours of v in G'. We may also assume that p_{v_1} and p_{v_2} are linearly independent and smooth; indeed if this

was not true, we could apply Lemma 2.2 to (G, p) to find a placement of G where it is true. Define p' to be the well-positioned placement of G' with $p'_x = p_x$ for all $x \in V$ and $p'_v = p_u$. From our choice of placement of G', we see that

$$R(G', p') = \left[\begin{array}{c|c} R(G, p) & \mathbf{0}_{|E| \times 2} \\ \hline A & -\varphi_{p_{v_1}} \\ B & -\varphi_{p_{v_2}} \end{array} \right]$$

for some $1 \times 2|V|$ matrices A and B. Hence (G', p') is independent if and only if $\varphi_{p_{v_1}}, \varphi_{p_{v_2}}$ are linearly independent. Since p_{v_1}, p_{v_2} are linearly independent and X is strictly convex, the pair $\varphi_{p_{v_1}}, \varphi_{p_{v_2}}$ are linearly independent as required. $\square$

For the vertex-to-4-cycle move we will use the technique of [17, Lemma 11] to show that a vertex-to-4-cycle move which creates two coincident vertices preserves independence. Similarly to the previous result, we will require that the normed plane in question is strictly convex.

Lemma 3.3. *Let $G = (V, E)$ and $G' = (V', E')$ be graphs and let X be a strictly convex normed plane.*

(i) *If G is independent in X and G' is formed from G by a vertex-to-4-cycle move that adds either u or v, then G' is uv-independent in X.*

(ii) *If G is uv-independent in X and G' is formed from G by a uv-vertex-to-4-cycle move, then G' is uv-independent in X.*

Proof. Suppose that G is uv-independent (respectively, independent). Using Lemma 2.5 (respectively, Lemma 2.2), choose a uv-independent (respectively, independent) placement p of G in X so that p_w, p_{v_1}, p_{v_2} are not collinear. By applying translations to p, we shall assume that $p_w = 0$. Now define p' to be the placement of G' with $p'_x = p_x$ for all $x \in V$ and $p'_{w'} = p_w$. The pair (G', p') form a well-positioned uv-coincident framework due to our choice of p'. Since X is strictly convex, the pair $\varphi_{p_{v_1}}, \varphi_{p_{v_2}}$ are linearly independent. Define G'' to be the graph formed from G' by replacing each edge $w'v_i$ for $3 \leq i \leq k$ with the edge wv_i. Then

$$R(G'', p') = \left[\begin{array}{c|c} R(G, p) & \mathbf{0}_{|E| \times 2} \\ \hline A & \varphi_{p'_{w'} - p'_{v_1}} \\ B & \varphi_{p'_{w'} - p'_{v_2}} \end{array} \right] = \left[\begin{array}{c|c} R(G, p) & \mathbf{0}_{|E| \times 2} \\ \hline A & -\varphi_{p_{v_1}} \\ B & -\varphi_{p_{v_2}} \end{array} \right],$$

for some $1 \times 2|V|$ matrices A and B. Since p_{v_1}, p_{v_2} are linearly independent and X is strictly convex, the pair $\varphi_{p_{v_1}}, \varphi_{p_{v_2}}$ are linearly independent. Hence $R(G'', p')$ has linearly independent rows. To prove that G' is uv-independent in X we will describe a series of rank-preserving row operations that will form $R(G', p')$ from $R(G'', p')$.

As $\varphi_{p_{v_1}}$ and $\varphi_{p_{v_2}}$ are linearly independent, there exist for each $3 \leq i \leq k$ a unique pair of values α_i and β_i such that

$$\alpha_i \varphi_{p_{v_1}} + \beta_i \varphi_{p_{v_2}} = \varphi_{p_{v_i}} = \varphi_{p'_{v_i} - p'_z},$$

where $z \in \{w, w'\}$ is chosen so that $v_i z \in E'$. For $1 \leq i \leq k$, let (wv_i) denote the row of $R(G'', p')$ corresponding to the edge wv_i, and similarly let $(w'v_1)$ and $(w'v_2)$ denote the

rows of $R(G'', p')$ corresponding to edges $w'v_1$ and $w'v_2$ respectively. For $v_i \in N_{G'}(w')$, let $[w'v_i]$ denote the row of $R(G', p')$ corresponding to the edge $w'v_i$. Now, for all $v_i \in N_{G'}(w')\backslash\{v_1, v_2\}$, we have

$$[w'v_i] = (wv_i) - \alpha_i(wv_1) - \beta_i(wv_2) + \alpha_i(w'v_1) + \beta_i(w'v_2).$$

These row operations, when applied $R(G'', p')$, preserve linear independence and form the matrix $R(G', p')$. Therefore the rows of $R(G', p')$ are linearly independent. $\qquad\square$

We now prove that vertex-to-H operations that add either u or v and uv-vertex-to-H operations will preserve uv-independence.

Lemma 3.4. *Let $G = (V, E)$ and $G' = (V', E')$ be graphs and let X be any non-Euclidean normed plane.*

(i) *Suppose G is independent in X and G' is formed from G by a vertex-to-H move that adds either u or v. If H is minimally uv-rigid in X, then G' is uv-independent in X.*

(ii) *Suppose G is uv-independent in X and G' is formed from G by a uv-vertex-to-H move. If H is minimally rigid in X, then G' is uv-independent in X.*

Proof. If (i) holds, let (H, q) be a minimally rigid uv-coincident framework in X and (G, p) be an independent framework in X, while if (ii) holds, let (H, q) be a minimally rigid framework in X and (G, p) be an independent uv-coincident framework in X. By applying translations we may assume $q_w = p_w = 0$. For any matrix A with columns corresponding to a vertex subset of $V \cup V(H)$, define A_w to be the matrix where we delete all columns corresponding to the vertex w. Given the fixed basis $b_1, b_2 \in X$ used to define our rigidity matrices in X, we define the matrix

$$M := \left[\begin{array}{c|c} R(H, q)_w & \mathbf{0}_{|E(H)|\times(2|V|-2)} \\ \hline A & R(G, p)_w \end{array}\right]$$

where A is the $|E| \times (2|V(H)| - 2)$ matrix with entries

$$A_{e,(y,i)} = \begin{cases} \varphi_{p_y - p_w}(b_i) & \text{if } e = xw, \\ 0 & \text{otherwise.} \end{cases}$$

By our choices of p and q, the matrix M has linearly independent rows.

For each $n \in \mathbb{N}$, choose a well-positioned uv-coincident framework (G', p^n) where $p_x^n = q_x/n$ for each $x \in V(H)$ and $\|p_x^n - p_x\| < 1/n$ for each $x \in V$ (this framework can be seen to exist from Lemma 2.5). Define M_n to be the matrix formed from multiplying each row of $R(G', p^n)_w$ corresponding to an edge of H by n. As the map $x \mapsto \varphi_x$ is continuous on the set of smooth points of X (see [24, Theorem 25.5]), the sequence of matrices $(M_n)_{n\in\mathbb{N}}$ will converge to M. Hence for sufficiently large $N \in \mathbb{N}$, the matrix M_{n_0} (and hence $R(G', p^{n_0})_w$) will have linearly independent rows. By setting $p' = p^{n_0}$, we obtain our desired independent uv-coincident framework (G', p'). $\qquad\square$

4 Characterising coincident-point independence

With the geometric results of the previous section in hand, we can use the combinatorics of [17] to prove the difficult sufficiency direction of our main result on coincident frameworks. We begin with the following result which can be extracted from the proof of [17, Theorem 4].

Proposition 4.1 ([17])**.** *Any uv-tight graph on at least five vertices can be constructed from either a $(2,2)$-tight graph with at least four vertices that contains exactly one of u and v, or from the graph consisting of two copies of K_4 intersecting in a single vertex $x \notin \{u,v\}$ where u and v are in different copies of K_4 (see* Figure 3*), by a sequence of 0-extensions that add u or v, vertex-to-4-cycle and vertex-to-H moves that add u or v, uv-0- and uv-1-extensions, and uv-vertex-to-4-cycle and uv-vertex-to-H moves.*

Sketch of proof. Since the proof of [17, Theorem 4] is long and technical we provide a sketch of the proposition to orient the interested reader into how it can be extracted from that theorem. It is easy to see that every graph generated as described in the statement is uv-tight. For the converse, firstly [17, Theorem 4] is stated for independence in $\mathcal{M}_{uv}$, i.e. for uv-sparse graphs, but we can extend to a base E of $\mathcal{M}_{uv}$ which induces a graph $G = (V, E)$ that, since $|V| \geq 5$, necessarily has $2|V| - 2$ edges and hence is uv-tight.

Suppose G has a vertex, w, of degree 2. If $w \in \{u,v\}$ then $G - w$ is $(2,2)$-tight. If $w \notin \{u,v\}$ then an easy argument shows that $G - w$ is uv-tight.

Therefore we may assume that the minimum degree is exactly 3, however it is much harder to reduce degree 3 vertices. [17, Theorem 4] deals firstly with three straightforward special cases. Firstly, if there is a 4-cycle in G containing u and v then uv is not an edge of this 4-cycle and we see that G is obtained from a $(2,2)$-tight graph by a vertex-to-4-cycle operation that split u into u and v. Secondly, if G contains a uv-tight subgraph H such that $V(H) \subsetneq V$ then we may assume H is a maximal such subgraph (that is there is no vertex in $V \setminus V(H)$ with more than one neighbour in $V(H)$). Then G/H (the graph obtained from G by contracting all vertices of H to a single vertex) is $(2,2)$-tight and G is obtained from G/H by a vertex-to-H move that expands u into a subgraph H that contains u and v. Thirdly, if G contains a degree 3 vertex contained in a subgraph of G isomorphic to K_4 and there is a vertex $x \in V \setminus V(H)$ such that $|V(H) \cap N(x)| = 2$ (and since we may assume the second special case does not occur $\{u,v\} \not\subset V(H) \cup \{x\}$), then we may apply a uv-vertex-to-H move to a uv-tight graph $G/(H \cup \{x\})$ to obtain G.

The proof is then completed by applying the arguments in Cases 5 and 6 of [17, Theorem 4], which use the fact we do not have the special structures we just dealt with, to analyse all possibilities for reducing a vertex of degree 3. Note that it is still not true that 1-extensions and uv-1-extensions suffice, however it is true that using precisely the operations listed in the proposition is sufficient. $\square$

We will also require the following lemmas.

Lemma 4.2. *Let $G = (V, E)$ be a graph with at most 4 vertices that contains both u and v, and let X be a strictly convex non-Euclidean normed plane. Then G is uv-sparse if and only if it is uv-independent in X.*

Proof. The only graphs on 4 or fewer vertices that are not uv-sparse are those which contain the edge uv, and if G contains the edge uv then it is not uv-independent. Suppose $uv \notin E$. We note that G must be a subgraph of $K_4 - uv$, so it is sufficient to consider the
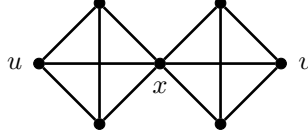
Figure 3: A uv-tight graph that is one of the base graphs of the construction described in Proposition 4.1.

case $G = K_4 - uv$. As G can be formed from $G - u$ by a 0-extension that adds u, G is uv-independent by Theorem 2.3 and Lemma 3.2. □

Lemma 4.3. *Let $G = (V, E)$ be the graph consisting of two copies of K_4 intersecting in a single vertex $x \notin \{u, v\}$, where u and v are in different copies of K_4 (see* Figure 3*). Then G is minimally uv-rigid in any non-Euclidean normed plane X.*

Proof. Let $V_u = \{x, u, a_u, b_u\}$ and $V_v = \{x, v, a_v, b_v\}$ be the vertex sets of the two copies of K_4 in G. As can be seen in Figure 3, $V_u \cap V_v = \{x\}$. By Theorem 2.3, there exists a placement $p^u \colon V_u \to X$ so that the framework (K_{V_u}, p^u), where K_{V_u} is the complete graph with vertex set V_u, is minimally rigid in X. Define the placement $p \colon V \to X$ by setting $p_{a_v} = p^u_{a_u}$, $p_{b_v} = p^u_{b_u}$, $p_v = p^u_u$, and $p_y = p^u_y$ for all $y \in V_u$. We now note that (G, p) is a minimally rigid uv-coincident framework; this follows from the fact that joining two minimally rigid frameworks in a non-Euclidean normed plane produces a minimally rigid framework, since the trivial infinitesimal flexes of a non-Euclidean normed plane correspond only to translations. Hence G is minimally uv-rigid as required. □

Theorem 4.4. *A graph is uv-independent in a strictly convex non-Euclidean normed plane X if and only if it is uv-sparse.*

Proof. First suppose G is uv-independent in X. Let G/uv denote the graph obtained from G by contracting the vertex pair u, v into a new vertex which we denote as z[2]. Let (G, p) be a regular (and hence independent) uv-coincident framework in X. We obtain a framework $(G/uv, p^{uv})$ in X by putting $p^{uv}_z = p_u = p_v$ and $p^{uv}_x = p_x$ for all $x \in V \setminus \{u, v\}$. For any $U \subseteq V$, the (possibly uv-coincident) induced subframework $(G[U], p|_U)$ is independent. Hence, if $\{u, v\} \not\subseteq U$, then $i_G(U) \leq \mathrm{val}(U)$ by Theorem 2.3. Since the case when $U = \{u, v\}$ is trivial, it now remains to show that $i_G(\mathcal{X}) \leq \mathrm{val}(\mathcal{X})$ for all uv-compatible families $\mathcal{X}$ in G. (Note that the case when $U \subseteq V$ and $\{u, v\} \subseteq U$ will be included by taking $\mathcal{X} = \{U\}$.)

Let $\mathcal{X} = \{X_1, \ldots, X_k\}$ be a uv-compatible family and consider the subgraph $H = (U, F)$ of G, where $U = \bigcup_{i=1}^k X_i$ and $F = \bigcup_{i=1}^k E(G[X_i])$. By contracting the vertex pair u, v in H, we obtain the graph H/uv. Define q to be the restriction of p to the vertex set U and q^{uv} to be the restriction of p^{uv} to the vertex set $U - \{u, v\} + z$. We have $\mathcal{X}_{uv} = \{X_1/uv, \ldots, X_k/uv\}$ is a cover of $E(H/uv)$ where X_i/uv denotes the set that we

---

[2]For us, a contraction will always be the more general vertex-contraction (which does not require u and v be adjacent) not the stricter edge-contraction (which does require u and v be adjacent).

get from X_i by identifying u and v. By Corollary 2.4, we have

$$
\begin{aligned}
\operatorname{rank} R(H/uv, q^{uv}) &\leq \sum_{i=1}^{k} (2|X_i/uv| - 2 - s(|X_i/uv|)) \\
&= \sum_{i=1}^{k} (2|X_i| - 2 - t_{X_i}) \\
&= \operatorname{val}(\mathcal{X}) - 2.
\end{aligned}
$$

Every vector μ^{uv} in the kernel of $R(H/uv, q^{uv})$ determines a unique vector μ in the kernel of $R(H, q)$ with $\mu_u = \mu_v = \mu_z^{uv}$ and $\mu_x = \mu_x^{uv}$ for all for all $x \in U \setminus \{u, v\}$. Hence $\dim \ker R(H, q) \geq \dim \ker R(H/uv, q^{uv})$. The rigidity matrix $R(H, q)$ has linearly independent rows since $R(G, p)$ has linearly independent rows, hence we have

$$
i_G(\mathcal{X}) = \operatorname{rank} R(H, q) \leq \operatorname{rank} R(H/uv, q^{uv}) + 2 \leq \operatorname{val}(\mathcal{X}).
$$

Thus G is uv-sparse.

We prove the sufficiency by induction on $|V|$. Suppose that G is uv-sparse. If $|V| \leq 4$, then G is uv-independent in X by Lemma 4.2. So we may suppose that $|V| \geq 5$. By adding additional edges, if necessary, we may assume G is uv-tight[3]. By Proposition 4.1, G can be constructed from either a $(2, 2)$-tight graph containing exactly one of u and v, or the graph pictured in Figure 3, by the operations defined in Section 3. Furthermore, as X is strictly convex, the corresponding geometric operations preserve minimal rigidity in X (see Section 3). The result now follows from Theorem 2.3 (i.e., every $(2, 2)$-tight graph is independent in X) and Lemma 4.3. $\qquad\square$

We next use this result to prove the following delete-contract characterisation of uv-rigidity.

Theorem 4.5. *Let G be a graph with distinct vertices u, v, and let X be a strictly convex non-Euclidean normed plane. Then G is uv-rigid in X if and only if $G - uv$ and G/uv are both rigid in X.*

Proof. Suppose that G is uv-rigid. It is immediate from the definition that $G - uv$ must be rigid. Choose a regular uv-coincident placement p of G, and define p^{uv} to be the placement of G/uv where $p_x^{uv} = p_x$ for all $x \in V - \{u, v\}$ and (given that z is the vertex obtained from u and v during the contraction) $p_z^{uv} = p_u = p_v$. Given an infinitesimal flex μ^{uv} of $(G/uv, p^{uv})$, we can form an infinitesimal flex μ of (G, p) by setting $\mu_x = \mu_x^{uv}$ for all $x \in V - \{u, v\}$ and $\mu_u = \mu_v = \mu_z^{uv}$. Since (G, p) is infinitesimally rigid as a uv-coincident framework, we must have that $\mu = (\lambda)_{x \in V}$ (and hence $\mu^{uv} = (\lambda)_{x \in V(G/uv)}$) for some vector $\lambda \in X$. Thus $(G/uv, p^{uv})$ is infinitesimally rigid and G/uv is rigid.

The converse follows from Theorem 4.4 as in the proof of [17, Theorem 1]. $\qquad\square$

We conjecture that the last two results apply in arbitrary non-Euclidean normed planes.

Conjecture 4.6. *Let $G = (V, E)$ be a graph and let $u, v \in V$ be distinct vertices. Then G is uv-independent in a non-Euclidean normed plane X if and only if G is uv-sparse.*

---

[3]Recall that uv-sparse graphs are the independent sets of a matroid and, when $|V| \geq 5$, the bases of this matroid have rank $2|V| - 2$.

Indeed extending the proof of Theorem 4.4 to the non-convex case requires only improvements to Lemmas 3.2 and 3.3. For the first of these, the issue is that 0-extensions that add v require us to precisely place v on top of the placement of u. However in the not strictly convex case, the proof of [5, Lemmas 5.1] requires one to choose the position of v carefully so that the support functionals of the edges incident to v guarantee linear independence. For the latter case, both the vertex-to-4-cycle move that adds v and the uv-vertex-to-4-cycle move have similar complications that would need to be resolved.

5   Global rigidity in analytic normed planes

A framework (G, p) in a normed space X is said to be *globally rigid* if every other framework (G, q) in X with $\|p_v - p_w\| = \|q_v - q_w\|$ for every edge $vw \in E$ is congruent to (G, p). A graph is then said to be *globally rigid* in X if the set

$$\big\{p \in X^V : (G, p) \text{ is globally rigid}\big\}$$

has a non-empty interior as a subset of the linear space X^V with the product topology inherited from X. It can be quickly seen that any globally rigid framework/graph will also be rigid.

Although much is known about global rigidity in Euclidean spaces, very little is known about the property for non-Euclidean normed spaces. The results that are known are only for *analytic normed spaces*, i.e., normed spaces where the norm restricted to the non-zero points is a real analytic function. As well as being strictly convex ([10, Lemma 3.1]), analytic normed spaces have many useful properties, including the following.

Lemma 5.1. *Let G be a graph with distinct vertices u, v and let X be a non-Euclidean analytic normed space.*

  (i) *The set of all $p \in X^V$ where (G, p) is a regular framework is an open conull subset of X^V.*

  (ii) *The set of all $p \in X^V/uv$ where (G, p) is a regular uv-coincident framework is an open conull subset of X^V/uv.*

Proof. If $\dim X = 1$ then the result follows immediately from noticing that all well-positioned frameworks and uv-coincident frameworks are regular. Suppose $\dim X \geq 2$. It was shown in [10, Proposition 3.2] that the set of well-positioned but non-regular placements of G are exactly the zero set of a non-constant analytic function defined on the connected open conull set of well-positioned placements. This gives (i). For (ii) we can use the same technique to show that the set of well-positioned but non-regular uv-coincident placements of G are exactly the zero set of a non-constant analytic function defined on the connected open conull set of well-positioned uv-coincident placements. The result now holds as the zero set of a non-constant analytic function with connected domain is always a closed null subset (see [10, Proposition 2.3]). □

Importantly, we can define a large class of globally rigid graphs in any non-Euclidean analytic normed plane.

Proposition 5.2 ([10])**.** *Let X be a non-Euclidean analytic normed plane. Then the graphs $K_5 - e$ and H, depicted in* Figure 4*, are globally rigid in X. Moreover any graph obtained from either of these by a sequence of degree 3 vertex additions (i.e., add a vertex and join it to three other vertices) and edge additions is globally rigid.*
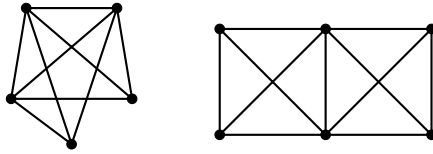
Figure 4: The graphs K_5^- (left) and H (right).

We next increase this class of graphs with the following construction operation introduced in [18]. A *generalised vertex split*, is defined as follows. Choose $z \in V$ and a partition N_u, N_v of the neighbours of z. Next, delete z from G and add two new vertices u, v joined to N_u, N_v, respectively. Finally add two new edges uv, uw for some $w \in V \setminus N_u$. See Figure 5 for an illustration of the operation.
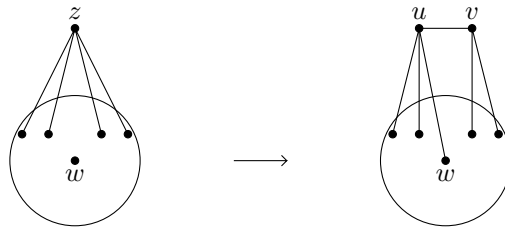

Figure 5: Generalised vertex split.

As the name suggests, this operation generalises the usual vertex splitting operation, see [26], which is the special case when w is chosen to be a neighbour of v. Note also that the special case when u has degree 3 (and $v = z$) is the well known 1-extension operation. Previously it was not known whether the 1-extension operation or a suitably restricted version of the vertex splitting operation preserves global rigidity in any non-Euclidean normed plane X.

As an application of our main result we will deduce that global rigidity can, under certain conditions, be preserved for generalised vertex splits. We will first need the following result which can be seen to follow from adapting the methods in [10, Section 3.2] to allow frameworks with zero-length edges[4].

Lemma 5.3. *Let (G, p) be a uv-coincident framework in a smooth normed space X with finitely many linear isometries. If (G, p) is globally rigid and infinitesimally rigid, then there exists an open neighbourhood $U \subset X^V$ of p where for each $q \in U$ the framework (G, q) is globally rigid.*

Theorem 5.4. *Let G be a globally rigid graph in a non-Euclidean analytic normed plane X. Let G' be a generalised vertex split of G at the vertex z with new vertices u, v and suppose that $G' - uv$ is rigid in X. Then G' is globally rigid in X.*

---

[4]Although it is a prerequisite in [10, Section 3.2] that the frameworks are well-positioned, the proof technique only requires that the squared edge-length map is differentiable. Since the map $x \mapsto \|x\|^2$ is always differentiable at the point 0, we can refine the result so that it holds for frameworks with zero-length edges.

Proof. Since $G'/uv = G$ is globally rigid in X it is also rigid in X by Theorem 2.1. As $G' - uv$ is also rigid in X, Theorem 4.5 implies that G' is uv-rigid in X. Hence by Lemma 5.1, we may choose an infinitesimally and globally rigid framework (G, p) so that if we define (G', p') to be the uv-coincident framework with $p'_x = p_x$ for all $x \in V$ and $p'_u = p'_v = p_z$, then (G', p') will be infinitesimally rigid also. Furthermore, (G', p') will also be globally rigid as (G, p) is globally rigid. We can now use Lemma 5.3 to deduce that (G', q) is globally rigid in X for all q sufficiently close to p'. Hence G' is globally rigid in X also. □

We can now improve upon Proposition 5.2. Here a graph $G = (V, E)$ is *redundantly rigid* in X if $G - e$ is rigid in X for any edge $e \in E$.

Corollary 5.5. *Let G be a graph obtained from K_5^- or H by a sequence of generalised vertex splits that preserve redundant rigidity, edge additions and degree at least 3 vertex additions. Then G is globally rigid in any non-Euclidean analytic normed plane.*

Proof. Follows immediately from Proposition 5.2 and Theorem 5.4. □

Since minimally rigid graphs in X have $2|V| - 2$ edges by Theorem 2.3, it is natural to expect that if $G = (V, E)$ is globally rigid then $|E| \geq 2|V| - 1$. The graphs K_5^- and H both achieve equality, but the inequality is strict for every graph in the infinite family obtained from these as in Proposition 5.2. To illustrate the power of Corollary 5.5 we note that we now have infinitely many globally rigid graphs for which equality holds and that this still holds if we restrict generalised vertex splitting to just one of vertex splitting or 1-extension. Two examples are depicted in Figure 6. The graph on the left is obtained from H by a vertex split and the graph on the right is obtained from H by a 1-extension. Both are globally rigid in X by Corollary 5.5.
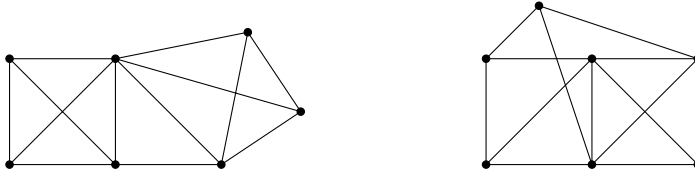

Figure 6: Examples of globally rigid graphs.

6 Concluding remarks

1. Following submission of this article we were able to improve upon Corollary 5.5. Specifically in [8], using the results of this article in a crucial way, we obtained a complete combinatorial description of graphs that are globally rigid in any non-Euclidean analytic normed plane. It turns out that we needed just 1 additional operation to those used in Corollary 5.5: this operation deletes an edge xy and adds two new vertices z, w and 5 new edges xz, xw, yz, yw, zw. In different language, the characterisation of [8] shows that a graph is globally rigid in any non-Euclidean analytic normed plane if and only if it is 2-connected and redundantly rigid (which means that it is still rigid after deleting any edge).

2. Theorem 4.4 and Theorem 4.5 provide a detailed combinatorial understanding of coincident point rigidity for frameworks in strictly convex non-Euclidean normed planes. As noted in the introduction, similar results exist for the Euclidean plane [12] and for frameworks supported on a cylinder in $\mathbb{R}^3$ [17]. Given the applicability of coincident point rigidity to analysing global rigidity (e.g. [4]) it would be interesting to develop analogues of Theorem 4.4 and Theorem 4.5 in other natural settings in rigidity theory. It may also be interesting to explore rigidity for frameworks with larger (or multiple) sets of coincident points. This line of investigation has begun in the case of the Euclidean plane [15].

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