

Rank 4 toroidal hypertopes

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Abstract

We classify the regular toroidal hypertopes of rank 4. Their automorphism groups are the quotients of infinite irreducible Coxeter groups of euclidean type with 4 generators. We also prove that there are no toroidal chiral hypertopes of rank 4.

Keywords: Regularity, chirality, toroidal, thin geometries, hypermaps, abstract polytopes.

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1 Introduction

A toroidal polytope is an abstract polytope that can be seen as a tessellation on a torus. By abstract polytope we mean a combinatorial structure resembling a classical polytope described by incidence relationships. Highly symmetric types of these polytopes are well known and understood, in particular the regular and chiral toroidal polytopes have been classified for rank 3 by Coxeter in 1948 [5], see also [6], and for any rank by McMullen and Schulte [10]. Regular toroidal polytopes and also regular toroidal hypertopes, which we define below, are strongly related to a special class of Coxeter groups, the infinite irreducible Coxeter groups of euclidean type which are also known as affine Coxeter groups (see, for example [11, page 73]). The symmetry groups of regular tessellations of euclidean space are precisely the affine Coxeter groups with string diagrams (see [11, Theorem 3B5]).

When we talk about a tessellation we mean, informally, a locally finite collection of polytopes which cover \mathbb{E}^n in a face-to-face manner. A toroidal polytope can then be seen as a "quotient" of a tessellation by linearly independent translations. For a precise definition of a toroidal polytope see [8]. The concept of a hypertope has recently been introduced by Fernandes, Leemans and Weiss (see [7]). A hypertope can be seen as a generalization of

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a polytope. Or, from another perspective, as a generalization of a hypermap. For more information on hypermaps see [4]. In this paper we will classify the rank 4 regular toroidal hypertopes.

Each affine Coxeter group in rank 4 (which are usually denoted by \widetilde{C}_3 , \widetilde{B}_3 and \widetilde{A}_3), as we shall see, can be associated with the group $\widetilde{C}_3 = [4, 3, 4]$, the symmetry group of the cubic tessellation of \mathbb{E}^3 . The *Coxeter Complex*, denoted by \mathcal{C} , of \widetilde{C}_3 can be seen as the simplicial complex obtained by the barycentric subdivision of the cubic tessellation $\{4, 3, 4\}$. The Coxeter complex for the other two rank 4 affine Coxeter groups can be obtained by doubling the rank 3 simplicies for \widetilde{B}_3 and quadrupling them for \widetilde{A}_3 . For details on the construction of \mathcal{C} see [9, Section 6.5] or [11, Section 3B]. We note that \mathcal{C} partitions \mathbb{E}^3 .

A *regular toroidal hypertope* (see Section 2 for a precise definition) can be seen as a quotient \mathcal{C}/Λ_I by a normal subgroup of translations, denoted Λ_I where I represents a generating set identifying the normal subgroup. In particular the quotient induced by a normal subgroup of translations in the string affine Coxeter group \widetilde{C}_3 yields the three families of regular rank 4 toroidal polytopes, while the other two affine Coxeter groups with non-string diagrams do not yield regular polytopes, but as we shall see below, regular hypertopes.

2 C-groups and hypertopes

Details of the concepts we review here are given in [7] and [11]. A *C-group of rank p* is a pair (G, S) such that G is a group and $S = \{r_0, \dots, r_{p-1}\}$ is a generating set of involutions of G that satisfy the following property:

$$\forall I, J \subseteq \{0, \dots, p-1\}, \langle r_i : i \in I \rangle \cap \langle r_j : j \in J \rangle = \langle r_k : k \in I \cap J \rangle.$$

This is known as the *intersection property* which will be referred to later.

A subgroup of G generated by a subset of S is called a *parabolic subgroup*. A parabolic subgroup generated by a single element of S is called *minimal* and a parabolic subgroup generated by all but one element of S is called *maximal*. For $J \subseteq \{0, \dots, p-1\}$, we define $G_J := \langle r_j : j \in J \rangle$ and $G_i := \langle r_j : r_j \in S, r_j \neq r_i \rangle$.

A C-group is a *string C-Group* if $(r_i r_j)^2 = 1_G$ for all i, j with $|i - j| > 1$. A *Coxeter diagram* $C(G, S)$ of a C-group (G, S) is a graph whose vertex set is S and two vertices, r_i and r_j are joined by an edge labelled by $o(r_i r_j)$, the order of $r_i r_j$. We use the convention that if an edge is labeled 2 it is dropped and not labeled if the order of the product of the corresponding generators is 3. Thus the Coxeter diagram of a string C-group is a string.

Affine Coxeter groups are C-groups and those with string diagrams are associated with toroidal polytopes. Hypertopes are generalizations of polytopes and we can, however, find toroidal hypertopes whose automorphism groups are quotients of any affine Coxeter group. We start with the definition of an *incidence system*.

Definition 2.1. An *incidence system* $\Gamma := (X, *, t, I)$ is a 4-tuple such that

- X is a set whose elements are called *elements* of Γ ;
- I is a set whose elements are called *types* of Γ ;
- $t: X \rightarrow I$ is a *type function* that associates to each element $x \in X$ of Γ a type $t(x) \in I$;

- $*$ is a binary relation of X called *incidence*, that is reflexive, symmetric and such that for all $x, y \in X$, if $x * y$ and $t(x) = t(y)$ then $x = y$.

A *flag* is a set of pairwise incident elements of Γ and the *type* of a flag F is $\{t(x) : x \in F\}$. A *chamber* is a flag of type I . An element x is said to be *incident* to a flag F when x is incident to all elements of F and we write $x * F$.

Definition 2.2. An *incidence geometry* is an incidence system Γ where every flag is contained in a chamber. The *rank* of Γ is the cardinality of I .

Let $\Gamma := (X, *, t, I)$ be an incidence system and F a flag of Γ . The *residue* of F in Γ is the incidence system $\Gamma_F := (X_F, *_F, t_F, I_F)$ where

- $X_F := (x \in F : x * F, x \notin F)$;
- $I_F := I \setminus t(F)$;
- t_F and $*_F$ are the restrictions of t and $*$ to X_F and I_F .

If each residue of rank at least 2 of Γ has a connected incidence graph then Γ is said to be *residually connected*. Γ is *thin* when every residue of rank 1 contains exactly 2 elements.

Furthermore, Γ is *chamber-connected* when for each pair of chambers C and C' , there exists a sequence of chambers $C =: C_0, C_1, \dots, C_n := C'$ such that $|C_i \cap C_{i+1}| = |I| - 1$ (here we say that C_i and C_{i+1} are *adjacent*). An incidence system is *strongly chamber-connected* when all of its residues of rank at least 2 are chamber-connected.

Proposition 2.3 ([7, Proposition 2.1]). *Let Γ be a thin incidence geometry. Then Γ is residually connected if and only if Γ is strongly chamber-connected.*

A *hypertope* is a strongly chamber-connected thin incidence geometry. To reinforce the relationship between polytopes and hypertopes we will sometimes refer to the *elements* of a hypertope Γ as *hyperfaces* of Γ , and *elements of type I* as *hyperfaces of type I* .

Let $\Gamma := (X, *, t, I)$ be an incidence system. An *automorphism* of Γ is a mapping $\alpha : (X, I) \rightarrow (X, I) : (x, t(x)) \mapsto (\alpha(x), t(x))$ where

- α is a bijection on X inducing a bijection on I ;
- for each $x, y \in X$, $x * y$ if and only if $\alpha(x) * \alpha(y)$;
- for each $x, y \in X$, $t(x) = t(y)$ if and only if $t(\alpha(x)) = t(\alpha(y))$.

An automorphism α is *type-preserving* when, for each $x \in X$, $t(\alpha(x)) = t(x)$. We denote by $\text{Aut}(\Gamma)$ the group of automorphisms of Γ and by $\text{Aut}_I(\Gamma)$ is the group of type-preserving automorphisms of Γ .

An incidence system Γ is *flag transitive* if $\text{Aut}_I(\Gamma)$ is transitive on all flags of type J for each $J \subseteq I$. It is *chamber-transitive* if $\text{Aut}_I(\Gamma)$ is transitive on all chambers of Γ . Furthermore, it is *regular* if the action of $\text{Aut}_I(\Gamma)$ is semi-regular and transitive.

Proposition 2.4 ([7, Proposition 2.2]). *Let Γ be an incidence geometry. Γ is chamber-transitive if and only if it is flag-transitive.*

A *regular hypertope* is a flag transitive hypertope. We note that every abstract regular polytope is a regular hypertope. The last concept we introduce here before we construct all rank 4 regular toroidal hypertopes is that of coset geometries.

Proposition 2.5 ([14]). *Let p be a positive integer and $I := \{1, \dots, p\}$ a finite set. Let G be a group together with a family of subgroups $(G_i)_{i \in I}$, X the set consisting of all cosets $G_i g$, $g \in G$, $i \in I$ and $t: X \rightarrow I$ defined by $t(G_i g) = i$. Define an incidence relation $*$ on $X \times X$ by :*

$$G_i g_1 * G_j g_2 \text{ if and only if } G_i g_1 \cap G_j g_2 \text{ is non-empty in } G.$$

*Then the 4-tuple $\Gamma := (X, *, t, I)$ is an incidence system having a chamber. Moreover, the group G acts by right multiplication as a group of type-preserving automorphisms of Γ . Finally, the group G is transitive on the flags of rank less than 3.*

Whenever Γ is constructed as in the above proposition it is written as $\Gamma(G; (G_i)_{i \in I})$ and if it is an incidence geometry it is called a *coset geometry*. If G acts transitively on all chambers of Γ (thus also flags of any type) we say that G is *flag transitive* on Γ or that Γ is *flag transitive*.

Now we note that we can construct a coset geometry $\Gamma(G; (G_i)_{i \in I})$ using a C-group (G, S) or rank p by setting $G_i := \langle r_j : r_j \in S, j \in I \setminus \{i\} \rangle$ for all $i \in I := \{0, \dots, p-1\}$.

We introduce the following proposition which lets us know that constructions we use produce regular hypertopes.

Proposition 2.6 ([7, Theorem 4.6]). *Let $(G, \{r_0, \dots, r_{p-1}\})$ be a C-group of rank p and let $\Gamma := \Gamma(G; (G_i)_{i \in I})$ with $G_i := \langle r_j : r_j \in S, j \in I \setminus \{i\} \rangle$ for all $i \in I := \{0, \dots, p-1\}$. If Γ is flag transitive, then Γ is a regular hypertope.*

Henceforth, we restrict our considerations to rank 4. Let $G = \langle r_0, r_1, r_2, r_3 \rangle$ be an affine Coxeter group where each r_i is a reflection through an associated affine hyperplane, H_i . Let \mathcal{C} be the Coxeter complex of G formed from the hyperplanes H_i 's. Here r_1, r_2 and r_3 will stabilize a point which, without loss of generality, can be assumed to be the origin o in \mathbb{E}^3 . Then the maximal parabolic subgroup G_0 is a finite crystallographic subgroup, which is a group that leaves a central point fixed. For details, see [3, pages 108–109]. The normal vectors to the reflection planes of the generators of G_0 are called the *fundamental roots*. The images of the fundamental roots under G_0 form a *root system* for G_0 .

The lattice, Λ , generated by the root system is called the *root lattice*, and the fundamental roots form the integral basis for Λ . The region enclosed by the fundamental roots is called the *fundamental region*. This lattice gives us (and can be identified with) the translation subgroup $T \leq G$ generated by the root lattice of G_0 , note that $G = G_0 \rtimes T$ [3]. For convenience we identify the translations with its vectors in addition a lattice also corresponds with its generating translation.

If I is a set of linearly independent translations in T , and let $T_I \leq T$ be the subgroup generated by I . Then the sublattice $\Lambda_I \leq \Lambda$ is the lattice induced by oT_I , the orbit of the origin under T_I . We note that \mathcal{C} is a regular hypertope and each simplex in \mathcal{C} represents a chamber where each vertex of the simplex is an element of a different type. In rank 4, when the quotient \mathcal{C}/Λ_I is a regular hypertope, we say it is a *regular toroidal hypertope* of rank 4. \mathcal{C}/Λ_I is a regular hypertope (and thus a regular toroidal hypertope) when Λ_I is large enough to ensure the corresponding group satisfies the intersection condition and when Λ_I invariant under G_0 , i.e. $r_i \Lambda_I r_i = \Lambda_I$ for $i = 1, 2, 3$. It is important to note that, in addition to denoting a lattice, Λ_I is also denotes a set of vectors as well as a translation subgroup of G along those vectors. If I consists of all permutations and changes in sign of the coordinates of some vector s then we will write Λ_s .

3 Toroidal polytopes constructed from the group $\tilde{C}_3 = [4, 3, 4]$

We begin, necessarily, with generating regular toroidal hypertopes (or, in this case, polytopes) whose automorphism groups are quotients of the group \tilde{C}_3 , the affine Coxeter Group $[4, 3, 4]$. As generators of $[4, 3, 4]$ we take ρ_1, ρ_2, ρ_3 to be reflections in the hyperplanes with normal vectors $(1, -1, 0)$, $(0, 1, -1)$, $(0, 0, 1)$ respectively, and ρ_0 the reflection in the plane through $(1/2, 0, 0)$ with normal vector $(1, 0, 0)$ (see Figure 1). Then,

$$\begin{aligned} (x, y, z)\rho_0 &= (1 - x, y, z), \\ (x, y, z)\rho_1 &= (y, x, z), \\ (x, y, z)\rho_2 &= (x, z, y), \\ (x, y, z)\rho_3 &= (x, y, -z). \end{aligned} \tag{3.1}$$

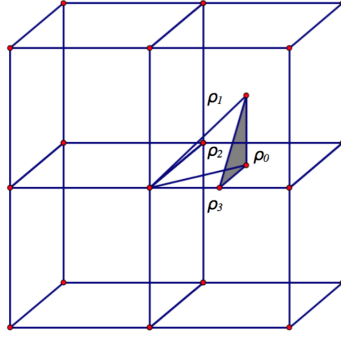


Figure 1: Fundamental simplex of $[4, 3, 4]$.

In this case, the construction described in Section 2 will yield the regular polytopes since $[4, 3, 4]$ is a string group. We denote by τ the corresponding tessellation $\{4, 3, 4\}$ of the Euclidean plane by cubes and by T it's full translation subgroup, where T is generated by the usual basis vectors, $T = \langle (1, 0, 0), (0, 1, 0), (0, 0, 1) \rangle$.

Let H_i be the planes fixed by ρ_i . The simplex bounded by the reflection planes H_i is a fundamental simplex of $[4, 3, 4]$ and is denoted ε , it is a simplex in the Coxeter complex of \tilde{C}_3 . Let F_i be the vertex of the fundamental simplex not on H_i then $\{F_0, F_1, F_2, F_3\}$ represents a flag of τ , and the set of all j -faces of $\tau = \{4, 3, 4\}$ is represented by the orbit of F_j under \tilde{C}_3 .

The regular polytope which results from factoring the regular tessellation $\tau = \{4, 3, 4\}$ by a subgroup Λ of T which is normal in $[4, 3, 4]$, is denoted by τ/Λ (as above).

We let Λ_s be the translation subgroup (or lattice) generated by the vector s and its images under the stabilization of the origin in $[4, 3, 4]$ and hence under permutations and changes of sign of its coordinates. The regular polytope τ/Λ_s is denoted by $\{4, 3, 4\}_s := \{4, 3, 4\}/\Lambda_s$ and the corresponding group $[4, 3, 4]/\Lambda_s$ is written as $[4, 3, 4]_s$. The following Lemma lists all possible such subgroups of T .

Lemma 3.1. *Let Λ be a subgroup of T , and if for every $a \in \Lambda$, the image of a under all changes of sign and permutations of coordinates (which is conjugation of a by the*

stabilization of the origin in $[4, 3, 4]$ is also in Λ , then $\Lambda = \langle (x, 0, 0), (0, x, 0), (0, 0, x) \rangle, \langle (x, x, 0), (-x, x, 0), (0, -x, x) \rangle$ or $\langle (x, x, x), (2x, 0, 0), (0, 2x, 0) \rangle$.

Proof. As adapted from page 165 from Abstract Regular Polytopes [11].

Let s be the smallest positive integer from all coordinates of vectors in Λ , then we can assume that $(s, s_2, s_3) \in \Lambda$. Then $(-s, s_2, s_3) \in \Lambda$ and thus $2se_1 \in \Lambda$ and so too are each $2se_i$. By adding and subtracting multiples of these we can find a vector all of whose coordinates are values between $-s$ and $+s$. It then follows that Λ is generated by the all permutations of $(s^k, 0^{3-k})$ with all changes of sign for some $k \in \{1, 2, 3\}$. (Note that in rank n , k can be only 1, 2 or n . Since otherwise $(s^k, 0^{n-k}) - (0, s^k, 0^{n-k-1}) \in \Lambda$ and so $(s, s, 0^{n-2}) \in \Lambda$ if k is odd or $(s, 0^{n-1})$ is if k is even. Though $n = 3$ in rank 4.)

If $k = 1$ then we have the first basis mentioned in the Lemma, the second if $k = 2$ and the third when $k = 3$. \square

It follows that $\Lambda_s = s\Lambda_{(1^k, 0^{n-k})}$, and thus, as can be seen in [11, Theorem 6D1], we have the following theorem.

Theorem 3.2. *The only regular toroidal polytopes constructed from $[4, 3, 4]$ are $\{4, 3, 4\}_s$ where $s = (s, 0, 0), (s, s, 0)$ or (s, s, s) and $s \geq 2$.*

Proof. Since conjugation of vectors in Λ by ρ_1, ρ_2 and ρ_3 are precisely all permutations of coordinates and changes of sign, this theorem follows directly from Lemma 3.1. \square

The following theorem also appears in [11] along with its proof. This theorem describes the group of each toroid. To arrive at the following result (and subsequent related results in sections 4 and 5) we note that the mirror of reflection ρ_0 is $x = 1/2$ while the mirrors for ρ_1, ρ_2 are $x = y$ and $y = z$ respectively and the mirror for ρ_3 is $z = 0$.

Theorem 3.3 ([11, Theorem 6D4]). *Let $s = (s^k, 0^{3-k})$, with $s \geq 2$ and $k = 1, 2, 3$. Then the group $[4, 3, 4]_s$ is the Coxeter group $[4, 3, 4] = \langle \rho_0, \rho_1, \rho_2, \rho_3 \rangle$, where the generators are specified in (3.1), factored out by the single extra relation which is*

$$\begin{aligned}(\rho_0\rho_1\rho_2\rho_3\rho_2\rho_1)^s &= id, \text{ if } k = 1, \\(\rho_0\rho_1\rho_2\rho_3\rho_2)^{2s} &= id, \text{ if } k = 2, \\(\rho_0\rho_1\rho_2\rho_3)^{3s} &= id, \text{ if } k = 3.\end{aligned}$$

As explained in [11], a geometric argument can be used to verify the intersection property for these groups when $s \geq 2$. However, note that $[4, 3, 4]_s$ does not satisfy the intersection condition when $s \leq 1$ and thus is not a C-Group. We show the breakdown of the intersection condition for $s = 1$ by way of example for $k = 1$ where cases for $k = 2, 3$ follow similar arguments.

When $s = 1$, the identity $\rho_0\rho_1\rho_2\rho_3\rho_2\rho_1 = id$ tells us that $\rho_0 \in \langle \rho_1, \rho_2, \rho_3 \rangle$ so G does not satisfy the intersection property.

4 Toroidal hypertopes whose automorphism group is $\tilde{B}_3 (= S_n)$

Let $\{\rho_0, \rho_1, \rho_2, \rho_3\}$ be the set of generators of $[4, 3, 4]$ as in the previous section and ε the corresponding fundamental simplex. We can double this fundamental simplex by replacing

the generator ρ_0 with $\tilde{\rho}_0 = \rho_0\rho_1\rho_0$. Then $\tilde{\rho}_0$ is a reflection through the hyperplane through the point $(1, 0, 0)$ with normal vector $(1, 1, 0)$. The transformation of a general vector by $\tilde{\rho}_0$ is

$$(x, y, z)\tilde{\rho}_0 = (1 - y, 1 - x, z). \quad (4.1)$$

Then $\{\tilde{\rho}_0, \rho_1, \rho_2, \rho_3\}$ generates \tilde{B}_3 , a subgroup of $[4, 3, 4]$ of index 2. The Coxeter diagram for this group is the non-linear diagram in Figure 2. In this section we let $G(=\tilde{B}_3) := \langle \tilde{\rho}_0, \rho_1, \rho_2, \rho_3 \rangle$ and let $\mathcal{C}(\tilde{B}_3)$ be the Coxeter complex of $G = \tilde{B}_3$.

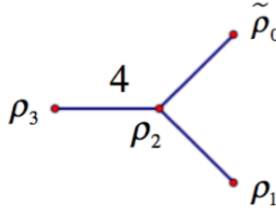


Figure 2: Coxeter diagram for \tilde{B}_3 .

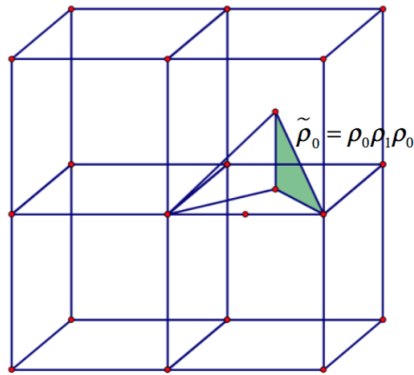
The fundamental simplex of \tilde{B}_3 is the simplex in Figure 3 bounded by the planes H_1, H_2, H_3 (fixed by ρ_1, ρ_2, ρ_3 respectively) and H_0 (now fixed by $\tilde{\rho}_0$). Let, as above, F_i be the vertices of the fundamental simplex opposite to H_i . The orbit of each vertex, F_j of the fundamental simplex of \tilde{B}_3 represents the set of hyperfaces of type j . Since this fundamental simplex shares vertices F_0, F_2 and F_3 with the fundamental simplex of $\langle \rho_0, \rho_1, \rho_2, \rho_3 \rangle$ we will use the same names for hyperfaces as the names in Section 3, namely, vertices, faces and facets. Though the orbit of F_1 (which is isomorphic to the orbit of F_0 since the maximal parabolic subgroups generated by excluding ρ_1 or $\tilde{\rho}_0$ are isomorphic) will be called hyperedges.

Now the translation subgroup of G is different from the one translation subgroup in the previous section since the set of vertices of $\{4, 3, 4\}$ now represent vertices and hyperedges (hyperfaces of type 0 and 1 respectively). The translation subgroup associated with this fundamental simplex is $T = \langle (1, 1, 0), (-1, 1, 0), (0, -1, 1) \rangle$.

We then note that the translation by vector $(1, 1, 0)$ is the transformation (by right multiplication) $w_1 = \tilde{\rho}_0\rho_2\rho_3\rho_2\rho_1\rho_2\rho_3\rho_2$, $(-1, 1, 0)$ is $w_2 = \rho_1\rho_2\rho_3\rho_2\tilde{\rho}_0\rho_2\rho_3\rho_2$ and $(0, -1, 1)$ is $w_3 = \rho_2\rho_3\rho_2\rho_1\rho_2\rho_1\tilde{\rho}_0\rho_2\rho_3\rho_1$.

Now, to form a root lattice Λ we have the freedom to choose the crystallographic subgroup G_0 by fixing either a vertex or a hyperedge (see [3, pages 108–109]). We choose to leave out $\tilde{\rho}_0$ since this reflection does not fix F_0 . Doing so leaves $[3, 4]$ as the subgroup we are conjugating with, which is the same as was for $[4, 3, 4]$. We also note that if we chose ρ_1 rather than $\tilde{\rho}_0$ then the result is functionally the same since we are still conjugating by $[3, 4] = \langle \tilde{\rho}_0, \rho_2, \rho_3 \rangle$ and this corresponds to forming a torus with its corners at hyper-edges.

We now note that although the same conditions are satisfied as in Lemma 3.1, T is now a different set. So instead we have the following lemma.

Figure 3: Fundamental simplex of \tilde{B}_3 .

Lemma 4.1. *If $T = \langle (1, 1, 0), (-1, 1, 0), (0, -1, 1) \rangle$, $\Lambda \leq T$ a subgroup and if for every $\mathbf{a} \in \Lambda$, the image of \mathbf{a} under changes of sign and permutations of coordinates is also in Λ , then $\Lambda = \langle (2x, 0, 0), (0, 2x, 0), (0, 0, 2x) \rangle$, $\langle (x, x, 0), (-x, x, 0), (0, -x, x) \rangle$ or $\langle (2x, 2x, 2x), (4x, 0, 0), (0, 4x, 0) \rangle$.*

Proof. We will only modify the proof of Lemma 3.1. In that proof we arrive at a generating set $(s^k, 0^{3-k})$ for each $k \in \{1, 2, 3\}$, given that T is different than the translation subgroup of Section 3.

Similar arguments to the ones used in the proof to Lemma 3.1 can now be used to show that for $k = 1$ or $k = 3$, s is even. For $k = 2$, Λ is generated by permutations and changes of sign of $(s, s, 0)$. This needs no further examination since it is clearly in T . \square

As in the previous section, we describe the groups that will be used to construct each of the toroids. We denote by G_s the quotient \tilde{B}_3/Λ_s . Earlier we noted w_1 as the translation $(1, 1, 0)$ while $(\tilde{\rho}_0 \rho_2 \rho_3 \rho_2 \rho_1)^2$ is the translation $(2, 0, 0)$ and $(\tilde{\rho}_0 \rho_2 \rho_3 \rho_1 \rho_2 \rho_3)^3$ is the translation $(2, 2, 2)$. And now that the mirror for $\tilde{\rho}_0$ is $y = 1 - x$.

Theorem 4.2. *Let $\mathbf{s} = (2s, 0, 0)$, $(s, s, 0)$ with $s \geq 2$ or $(2s, 2s, 2s)$ with $s \geq 1$. Then the group $G_s = \tilde{B}_3/\Lambda_s$ is the Coxeter group $\tilde{B}_3 = \langle \tilde{\rho}_0, \rho_1, \rho_2, \rho_3 \rangle$ with Coxeter diagram in Figure 2, factored out by the single extra relation which is*

$$\begin{aligned} (\tilde{\rho}_0 \rho_2 \rho_3 \rho_2 \rho_1)^{2s} &= id \text{ if } \mathbf{s} = (2s, 0, 0), \\ (\tilde{\rho}_0 \rho_2 \rho_3 \rho_2 \rho_1 \rho_2 \rho_3 \rho_2)^s &= id \text{ if } \mathbf{s} = (s, s, 0), \\ (\tilde{\rho}_0 \rho_2 \rho_3 \rho_1 \rho_2 \rho_3)^{3s} &= id \text{ if } \mathbf{s} = (2s, 2s, 2s). \end{aligned}$$

Here, as in Section 3, we have that G_s fails the intersection property for small enough s . However, because the fundamental simplex is doubled, this time when $\mathbf{s} = (2s, 2s, 2s)$, G_s satisfies the intersection condition for $s \geq 1$ while $s \geq 2$ is still necessary for the other two vectors. Verifying that G_s fails the intersection condition for $s = 1$ when $\mathbf{s} = (2s, 0, 0)$ and $(s, s, 0)$ follows similar calculations as those done in Section 3. Namely, when $s = 1$ for the

first vector, we arrive at the identity $\tilde{\rho}_0 \rho_2 \rho_3 \rho_2 \tilde{\rho}_0 = \rho_1 \rho_2 \rho_3 \rho_2 \rho_1$ and for the second vector we have the identity $\tilde{\rho}_0 \rho_2 \rho_3 \rho_2 = \rho_2 \rho_3 \rho_2 \rho_1$. Which violates the intersection condition.

MAGMA [1] can be used to verify that G_s satisfies the intersection condition when $\mathbf{s} = (4, 0, 0) = (2s, 0, 0)$, $(2, 2, 0) = (s, s, 0)$ or $(2, 2, 2)$. To see that it also satisfies the intersection condition for greater values of s can be seen with a geometric argument.

The orbit of a base chamber of each parabolic subgroup of G_s can be seen as a collection of chambers which are duplicated at each of the 8 corners of the boundaries of Λ_s . For instance, the subgroup $\langle \rho_1, \rho_2, \rho_3 \rangle$ consists of chambers forming octahedra centred around corner vertex.

Given the collection of chambers in two such subgroups, there will always be some intersection between the collections occurring at the same corner (sometimes it's just the base chamber itself). However, If G_s fails the intersection condition, then there will be an intersection with the chambers of one subgroup centred around one corner that intersect with the chambers of the other subgroup on another corner.

So, given a particular s where G_s satisfies the intersection condition, by increasing s , the corners of Λ_s get further and further apart. So if there are no such intersections for some s , then for larger s there will not be either.

Adopting a similar notation as in the previous section and using Λ_s as defined in Section 2, we now have the following theorem.

Theorem 4.3. *The regular toroidal hypertopes of rank 4 constructed from $G(= \tilde{B}_3) = \langle \tilde{\rho}_0, \rho_1, \rho_2, \rho_3 \rangle$, where the generators are specified in (3.1) and (4.1), are $\mathcal{C}(\tilde{B}_3)/\Lambda_s$ where $\mathcal{C}(\tilde{B}_3)$ is the Coxeter complex of \tilde{B}_3 and $\mathbf{s} = (2s, 0, 0)$, $(s, s, 0)$ with $s \geq 2$ or $(2s, 2s, 2s)$ with $s \geq 1$.*

Proof. To begin we need to find an \mathbf{s} and corresponding Λ_s that is invariant under conjugation by a subgroup of G which is the symmetry group of "vertex"-figure (by vertex we mean, the element that the translations begin from). In this case our subgroup ends up being $[3, 4]$ as was described before Lemma 4.1.

Now, since we are conjugating by $[3, 4] = \langle \rho_1, \rho_2, \rho_3 \rangle$, Λ_s must contain all permutations and changes of sign of any vector in Λ_s (which we discovered in the proof of Theorem 3.2 which is also on page 165 of [11]). Thus, by Lemma 4.1, $\mathbf{s} = (2s, 0, 0)$, $(s, s, 0)$ or $(2s, 2s, 2s)$. However, we still do not know if this construction yields a regular hypertope. To do this, we start by noting that the Coxeter complex $\mathcal{C}(\tilde{B}_3)$ formed from G is precisely the hypertope $\Gamma(G; (G_i)_{i \in I})$ (the construction of which follows from [7]).

So we need to show that $\mathcal{C}(\tilde{B}_3)$ is flag transitive (or, equivalently, chamber transitive). To do so we will note the rank 3 residue $\Gamma_{\tilde{0}} := \Gamma(G_{\tilde{0}}; (G_{\{\tilde{0}, i\}})_{i \in \{1, 2, 3\}})$. This is isomorphic to the cube, a regular polyhedron, which is flag transitive.

So we pick to chambers in $\Gamma(G; (G_i)_{i \in I}) = \mathcal{C}(\tilde{B}_3)$ which can be written as $C_1 = \{G_{\tilde{0}}g_0, G_1g_1, G_2g_2, G_3g_3\}$ and $C_2 = \{G_{\tilde{0}}h_0, G_1h_1, G_2h_2, G_3h_3\}$ for some $g_i, h_i \in G$. Then, since $G = G_{\tilde{0}} \rtimes T$ and T acts transitively on elements of type $\tilde{0}$ there is a translation $t \in G$ such that $C_1t = \{G_{\tilde{0}}h_0, X, Y, Z\}$ which is some chamber that shares the same element of type $\tilde{0}$ as C_2 . Then the chambers C_1t and C_2 are both in some rank 3 residue which is isomorphic to $\Gamma_{\tilde{0}}$. Since this residue is flag transitive, there is some element, $g \in G$ such that $C_1tg = C_2$. Thus $\mathcal{C}(\tilde{B}_3)$ is chamber transitive and thus flag transitive. So, by Proposition 4.6 from [7] this is a regular hypertope.

So now we want to know if $\Gamma(G'; (G'_i)_{i \in I})$ is a regular hypertope where G' is the

group G/Λ_s where $s \geq 2$ (since otherwise G' fails the intersection condition and the resulting construction fails to be thin). Just as before, we take two chambers Φ and Ψ from $\Gamma(G'; (G'_i)_{i \in I})$. Then to each of these chambers we can associate a family of chambers Φ' and Ψ' in $\mathcal{C}(\widetilde{B}_3)$. Since $\mathcal{C}(\widetilde{B}_3)$ is chamber transitive for each $\Phi_j \in \Phi'$ and $\Psi_k \in \Psi'$ there exists $\varphi_{jk} \in G$ where $\Phi_j \varphi_{jk} = \Psi_k$. In particular there exist chambers $\Phi_1 \in \Phi'$ and $\Psi_1 \in \Psi'$ in $\mathcal{C}(\widetilde{B}_3)$ where, since Λ_s is invariant under G , $\Phi_1 \psi = \Psi_1$ and $\psi \in G'$. We can see this by noting that Φ_1 and Ψ_1 are the members of their respective families which lie inside the fundamental region of Λ_s .

Thus $\Gamma(G'; (G'_i)_{i \in I})$ is chamber transitive and thus face transitive, so is also a regular hypertope by Proposition 2.6.

For the other two possibilities of Λ , we need only change the added relations, but because the relations were chosen specifically, they will also generate regular hypertopes. \square

5 Toroidal hypertopes whose automorphism group is $\widetilde{A}_3 (= P_n)$

We can show that this group is, yet again a subgroup of $[4, 3, 4]$ by doubling the fundamental simplex a second time (this can be seen geometrically in Figure 5) and now defining $\widetilde{\rho}_3 = \rho_3 \rho_2 \rho_3$ which is a reflection in the plane through $(1, 1, -1)$ with normal vector $(0, 1, 1)$. Transformation of a general vector by $\widetilde{\rho}_3$ is

$$(x, y, z) \widetilde{\rho}_3 = (x, -z, -y). \quad (5.1)$$

Now we let $G(= \widetilde{A}_3) := \langle \widetilde{\rho}_0, \rho_1, \rho_2, \widetilde{\rho}_3 \rangle$ and $\mathcal{C}(\widetilde{A}_3)$ be the Coxeter complex of G . The defining relations for G are implicit in the Coxeter diagram in Figure 4.

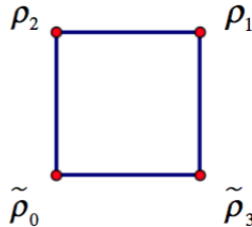
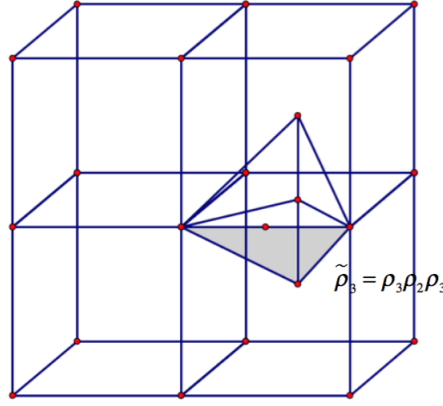


Figure 4: Coxeter diagram for \widetilde{A}_3 .

Here the fundamental simplex of \widetilde{A}_3 is a tetrahedron bounded by the planes H_i (fixed by ρ_i). This fundamental simplex shares the planes fixed by $\widetilde{\rho}_0, \rho_1, \rho_2$ with the fundamental simplex of \widetilde{B}_3 as well as the corresponding vertices. The stabilizers of each vertex of the fundamental simplex are also isomorphic since all maximal parabolic subgroups of \widetilde{A}_3 are pairwise isomorphic. This implies that the set of hyperfaces of types i and j are isomorphic for each $i, j \in \{0, 1, 2, 3\}$.

This fundamental simplex gives us the same translation subgroup as we had in the previous Section. Though now we must use the new generators to find the translations. We define $w_1 = \widetilde{\rho}_0 \rho_2 \rho_1 \widetilde{\rho}_3 \rho_1 \rho_2 = (1, 1, 0)$, $w_2 = \rho_1 \rho_2 \widetilde{\rho}_3 \widetilde{\rho}_0 \widetilde{\rho}_3 \rho_2 = (-1, 1, 0)$ and $w_3 = \rho_2 \rho_1 \widetilde{\rho}_0 \widetilde{\rho}_3 \widetilde{\rho}_0 \rho_1 = (0, -1, 1)$.

Figure 5: Fundamental simplex of \widetilde{A}_3 .

Using these translations, for a translation $(a, b, c) \in \Lambda$, we have that $\rho_1(a, b, c)\rho_1 = (b, a, c)$. In a similar way, conjugating by ρ_2 yields (a, c, b) and conjugating by $\tilde{\rho}_3$ yields $(a, -c, -b)$. So if we conjugate by $\rho_1\rho_2\rho_1$ then we get (c, b, a) and so Λ must have all permutations. Now, from the previous we know Λ must also contain $(a, -b, -c)$ and adding this to (a, b, c) gives $(2a, 0, 0)$, which then subtracted from (a, b, c) is $(-a, b, c)$ and so with all permutations means that Λ must have all permutations and changes of sign.

With this group, we leave out $\tilde{\rho}_0$ to form the crystallographic subgroup G_0 . Though a curiosity of this group is that we use any generator of B_3 to form a crystallographic subgroup and still finish with the same objects. With each choice simply changing where we draw the boundary of the torus. This leaves ρ_1, ρ_2 and $\tilde{\rho}_3$ with which to conjugate Λ . As in the regular case, ρ_1 and ρ_2 show us that Λ must consist of all permutations of the coordinates of vectors.

If (a, b, c) is a general vector in Λ then $\tilde{\rho}_3$ tells us that $(-c, b, -a)$ must also be in Λ and then so also must $(-a, b, -c)$. Adding that to our original general vector tells us that $(0, 2b, 0)$ is also included. So, subtracting that from the general vector finally gives us $(a, -b, c)$. Note that this can just as easily be done with either a or c with some simple permutations.

As in the previous section, we describe the groups of each of the toroids. Earlier we noted w_1 as the translation $(1, 1, 0)$ while $(\tilde{\rho}_0\rho_2\tilde{\rho}_3\rho_1)^2$ is the translation $(2, 0, 0)$ and $(\tilde{\rho}_0\rho_2\rho_1\tilde{\rho}_3)^{3s}$ is the translation $(2, 2, 2)$. And now that the the mirror for $\tilde{\rho}_0$ is $y = 1 - x$ while the mirror for $\tilde{\rho}_3$ is $y = -z$.

Theorem 5.1. *Let $s = (2s, 0, 0), (s, s, 0)$ with $s \geq 2$ or $(2s, 2s, 2s)$ with $s \geq 1$. Then the group $G_s = \widetilde{A}_3/\Lambda_s$ is the Coxeter group $\widetilde{A}_3 = \langle \tilde{\rho}_0, \rho_1, \rho_2, \tilde{\rho}_3 \rangle$ (with Coxeter group specified in Figure 4), factored out by the single extra relation which is*

$$\begin{aligned} (\tilde{\rho}_0\rho_2\tilde{\rho}_3\rho_1)^{2s} &= id \text{ if } s = (2s, 0, 0), \\ (\tilde{\rho}_0\rho_2\rho_1\tilde{\rho}_3\rho_1\rho_2)^s &= id \text{ if } s = (s, s, 0), \\ (\tilde{\rho}_0\rho_2\rho_1\tilde{\rho}_3)^{3s} &= id \text{ if } s = (2s, 2s, 2s). \end{aligned}$$

For the same reasons as in Section 4, the intersection condition is satisfied for $\mathbf{s} = (2s, 2s, 2s)$ when $s \geq 1$.

Theorem 5.2. *The regular toroidal hypertopes of rank 4 induced by $G(= \widetilde{A}_3) = \langle \tilde{\rho}_0, \rho_1, \rho_2, \tilde{\rho}_3 \rangle$ (where the generators are specified in (3.1), (4.1) and (5.1)) are $\mathcal{C}(\widetilde{A}_3)/\Lambda_s$ where $\mathcal{C}(\widetilde{A}_3)$ is the Coxeter complex of \widetilde{A}_3 and $\mathbf{s} = (2s, 0, 0), (s, s, 0)$ for $s \geq 2$ or $(2s, 2s, 2s)$ with $s \geq 1$.*

Proof. We first show that $\mathcal{C}(\widetilde{A}_3)$ is a regular hypertope, which requires showing that it is flag transitive. In the same manner as the proof of Theorem 4.3 we need only show each rank 3 residue is flag transitive, since all rank 3 residues are regular tetrahedra $\mathcal{C}(\widetilde{A}_3)$ is flag transitive. The translation subgroup is the same as in the previous Section and conjugating Λ by $\rho_1, \rho_2, \tilde{\rho}_3$ gives all permutations and changes in sign of a general vector in Λ , the same arguments for Lemma 4.1 and Theorem 4.3 will prove this theorem. \square

6 Non-existence of rank 4 chiral hypertopes

Here we recall that for an abstract polytope to be chiral its automorphism group must have two orbits when acting on flags and that adjacent flags are in different orbits. Chiral polytopes of any rank are examined in depth in [13]. The existence of these objects in any rank was proved in [12]. There is also a notion of chirality in hypermaps as well, see for example, [2]. Similarly we say for a hypertope to be chiral if its automorphism group action has two chamber orbits and adjacent chambers are in different orbits [7].

As in Section 2, given an affine Coxeter group G and associated Coxeter complex \mathcal{C} , we define a subgroup $G_0 \leq G$ as the maximal parabolic subgroup fixing the origin. Then, given a set I of linearly independent translations in G and T_I , the translation subgroup generated by I then we call the lattice Λ_I the lattice induced by the orbit of the origin under T_I . When Λ_I is invariant under the rotation subgroup G_0^+ but there is no automorphism group of G that interchanges adjacent chambers, then in rank 4 we say that the quotient \mathcal{C}/Λ_I is a *chiral toroidal hypertope* of rank 4.

The proof that there are no chiral toroids of rank 4 for the group $[4, 3, 4]$ comes from page 178 from [11] and the same proof can be adapted for the other two rank 4 affine Coxeter groups. The basic idea for the proof is that since \mathcal{C}/Λ is chiral, Λ is invariant under the rotation group $[3, 4]^+$, so Λ contains vectors that are compositions of an even number of permutations with an even number of sign changes or all compositions of an odd number of permutations with an odd number of sign changes. It then goes to show that if $(a, b, c) \in \Lambda$ then $(b, a, c) \in \Lambda$, which is the image of (a, b, c) under an odd permutation, which is a contradiction. Therefore no such Λ can exist.

We will use the same method to show the same is true for the other two groups.

Theorem 6.1. *There are no rank 4 chiral toroidal hypertopes.*

Proof. In [11] it was shown that there are no rank 4 hypertopes constructed from $[4, 3, 4]$, so we show for constructions from \widetilde{B}_3 and \widetilde{A}_3 . In previous sections we found that if Λ is a subgroup of the translations that is invariant under conjugation by the stabilizer of the origin in \widetilde{B}_3 and \widetilde{A}_3 with $(a, b, c) \in \Lambda$, then Λ contains all permutations and changes of sign of (a, b, c) , just as it did with the stabilizer in $[4, 3, 4]$.

Thus conjugation of Λ by the stabilizer of the rotation subgroup of each of these groups is all compositions of even permutations with an even number of sign changes or all compositions of odd permutations with an odd number of sign changes, just as for $[4, 3, 4]$.

So the same arguments and calculations from page 178 in [11] still hold and show that $(b, a, c) \in \Lambda$ and we develop the same contradiction. \square

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