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11 Clifford odd and even objects, offering description of internal space of fermion and boson fields, respectively, open new insight into next step beyond standard model

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Abstract. In a long series of works the author demonstrated, together with collaborators, that the model named the *spin-charge-family* theory offers the explanation for all in the *standard model* assumed properties of fermion and boson fields, with the families of fermions and the Higgs's scalars included. The theory starts with a simple action in $\geq (13+1)$-dimensional space-time with massless fermions which interact with massless gravitational fields only (vielbeins and the two kinds of spin connection fields). The internal spaces of fermion and boson fields are described by the Clifford odd and even objects, respectively. The corresponding odd and even "basis vectors" in a tensor product with the basis in ordinary momentum or coordinate space define the creation and annihilation operators, which explain the second quantization postulates for fermion and boson fields. The break of the starting symmetry leads at low energies to the action for families of quarks and leptons and the corresponding gauge fields, with Higgs's fields included, offering several predictions and several explanations of the observed cosmological phenomena. The properties of the odd dimensional spaces are also discussed.

Povzetek: V dolgem nizu člankov je avtorica, skupaj s sodelavci, pokazala, da ponuja model, ki ga avtorica poimenuje teorija *spinov-nabojev-družin*, razlago za vse v *standardnem modelu* privzete lastnosti fermionskih in bozonskih polj, vključno z družinami fermionov in Higgsovimi skalarji. Teorija predpostavi preprosto akcijo v $\geq (13+1)$-razsežnem prostoru-času, v kateri fermioni nimajo mase, interagirajo pa samo z brezmasnim gravitacijskim poljem (tetradani, ki določajo gravitacijsko polje v običajnem prostoru in dvema vrstama spinskih povezav, ki so umeritvena polja Lorentzovih transformacij v notranjem prostoru fermionov). Notranji prostor fermionov opiše avtorica z "bazičnimi vektorji", ki so lihi objekti Clifordove algebre, notranji prostor bozonov pa s Cliffordovo sodimi objekti. Ustrezni lihi in sodi "bazični vektorji" v tenzorskem produktu z bazo v prostoru gibalnih količin definirajo kreacijske in anihilacijske operatorje antikomutirajočih fermionskih polj in komutirajočih bozonskih polj, kar pojasni postulate za drugo kvantizacijo za fermionska in bozonska polja. Zlomitev začetne simetrije akcije vodi pri nizkih energijah do akcije kot jo predpostavi *standardni model*— za družine kvarkov in leptonov in za ustrezna umeritvena polja ter za Higgsove skalarje. Teorija ponuja števine napovedi in pojasni vzroke za kozmološka opaženja. Predstavi tudi lastnosti Cliffordovih objektov v prostorih z lihim številom dimenzij.

Keywords: Second quantization of fermion and boson fields in Clifford space; beyond the standard model; Kaluza-Klein-like theories in higher dimensional space, explanation of appearance of families of fermions, scalar fields, fourth family, dark matter.

11.1 Introduction

The *standard model* (with massive neutrinos added) has been experimentally confirmed without raising any serious doubts so far on its assumptions, which remain unexplained [1].
The assumptions of the *standard model* has in the literature several explanations, mostly with many new not explained assumptions. The most popular seem to be the grand unifying theories ([1–6].
Among the questions for which the answers are needed are:
i. Where do fermions, quarks and leptons, originate?
ii. Why do family members, quarks and leptons, manifest so different masses if they all start as massless?
iii. Why are charges of quarks and leptons so different and why have the left handed family members so different charges from the right handed ones?
iv. Where do antiquarks and antileptons originate?
v. Where do families of quarks and leptons originate and how many families do exist?
vi. What is the origin of boson fields, of vector fields which are the gauge fields of fermions?
vii. What is the origin of the Higgs's scalars and the Yukawa couplings?
viii. How are scalar fields connected with the origin of families and how many scalar fields determine properties of the so far (and others possibly be) observed fermions and of weak bosons?
ix. Why have the scalar fields half integer weak and hyper charge? Do possibly exist also scalar fields with the colour charges in the fundamental representation ?
ix. Could all boson fields, with the scalar fields included, have a common origin?
x. Where does the *dark matter* originate? Does the *dark matter* consist of fermions?
xi. Where does the "ordinary" matter-antimatter asymmetry originate?
xii. Where does the dark energy originate?
xiii. How can we understand the postulates of the second quantized fermion and boson fields?
xiv. What is the dimension of space? $(3+1)$?, $((d-1)+1)$?, ∞?
xv. Are all the fields indeed second quantized with the gravity included? And consequently are all the systems second quantized (although we can treat them in simplified versions, like it is the first quantization and even the classical treatment), with the black holes included?
xvi. And many others.

[1] This introduction is similar to the one appearing in the arxiv:2210.07004. Also most of sections and subsections are similar. There are, however, some new parts added.

In a long series of works ([1–3,5,23,25,27–29,31,32] and the references therein), the author has succeeded, together with collaborators, to find the answer to many of the above, and also to other open questions of the *standard model*, as well as to several open cosmological questions, with the model named the *spin-charge-family* theory. The more work is put into the theory the more answers the theory offers. The theory assumes that the space has more than $(3+1)$ dimensions, it must have $d \geq (13+1)$, so that the subgroups of the $SO(13,1)$ group, describing the internal space of fermions by the superposition of odd products of the Clifford objects γ^a's, manifest from the point of view of $d = (3+1)$-dimensional space the spins, handedness and charges assumed for massless fermions in the *standard model*. Correspondingly each irreducible representation of the $SO(13,1)$ group carrying the quantum numbers of quarks and leptons and antiquarks and antileptons, represents one of families of fermions, the quantum numbers of which are determined by the second kind of the Clifford objects, by $\tilde{\gamma}^a$ (by $\tilde{S}^{ab}$ $(= \frac{i}{4}\{\tilde{\gamma}^a, \tilde{\gamma}^b\}_-)$.
Fermions interact in $d = (13+1)$ with gravity only, with vielbeins (the gauge fields of momenta) and the two kinds of the spin connection fields, the gauge fields of the two kinds of the Lorentz transformations in the internal space of fermions, of $S^{ab} (= \frac{i}{4}\{\gamma^a, \gamma^b\}_-)$ and of $\tilde{S}^{ab}$ $(= \frac{i}{4}\{\tilde{\gamma}^a, \tilde{\gamma}^b\}_-)$.
The theory assumes a simple starting action ([5] and the references therein) for the second quantized massless fermion and antifermion fields, and the corresponding massless boson fields in $d = 2(2n+1)$-dimensional space

$$\begin{aligned}
\mathcal{A} &= \int d^dx \, E \, \frac{1}{2} \, (\bar{\psi} \, \gamma^a p_{0a} \psi) + h.c. + \\
&\quad \int d^dx \, E \, (\alpha \, R + \tilde{\alpha} \, \tilde{R}) \, , \\
p_{0a} &= f^{\alpha}{}_a p_{0\alpha} + \frac{1}{2E} \{p_\alpha, E f^{\alpha}{}_a\}_- \, , \\
p_{0\alpha} &= p_\alpha - \frac{1}{2} S^{ab} \omega_{ab\alpha} - \frac{1}{2} \tilde{S}^{ab} \tilde{\omega}_{ab\alpha} \, , \\
R &= \frac{1}{2} \{f^{\alpha[a} f^{\beta b]} \; (\omega_{ab\alpha,\beta} - \omega_{ca\alpha} \, \omega^{c}{}_{b\beta})\} + h.c. \, , \\
\tilde{R} &= \frac{1}{2} \{f^{\alpha[a} f^{\beta b]} \; (\tilde{\omega}_{ab\alpha,\beta} - \tilde{\omega}_{ca\alpha} \, \tilde{\omega}^{c}{}_{b\beta})\} + h.c. \, .
\end{aligned} \tag{11.1}$$

Here [2] $f^{\alpha[a} f^{\beta b]} = f^{\alpha a} f^{\beta b} - f^{\alpha b} f^{\beta a}$. f^a_α, and the two kinds of the spin connection fields, $\omega_{ab\alpha}$ (the gauge fields of S^{ab}) and $\tilde{\omega}_{ab\alpha}$ (the gauge fields of $\tilde{S}^{ab}$), manifest in $d = (3+1)$ as the known vector gauge fields and the scalar gauge fields taking

[2] $f^{\alpha}{}_a$ are inverted vielbeins to $e^a{}_\alpha$ with the properties $e^a{}_\alpha f^\alpha{}_b = \delta^a{}_b$, $e^a{}_\alpha f^\beta{}_a = \delta^\beta_\alpha$, $E = \det(e^a{}_\alpha)$. Latin indices $a, b, .., m, n, .., s, t, ..$ denote a tangent space (a flat index), while Greek indices $\alpha, \beta, .., \mu, \nu, ..\sigma, \tau, ..$ denote an Einstein index (a curved index). Letters from the beginning of both the alphabets indicate a general index ($a, b, c, ..$ and $\alpha, \beta, \gamma, ..$), from the middle of both the alphabets the observed dimensions $0, 1, 2, 3$ ($m, n, ..$ and $\mu, \nu, ..$), indexes from the bottom of the alphabets indicate the compactified dimensions ($s, t, ..$ and $\sigma, \tau, ..$). We assume the signature $\eta^{ab} = diag\{1, -1, -1, \cdots, -1\}$.

care of masses of quarks and leptons and antiquarks and antileptons and the weak boson fields [27] [3]

While in any even dimensional space the superposition of odd products of γ^a's, forming the Clifford odd "basis vectors", offer the description of the internal space of fermions with the half integer spins, (manifesting in $d = (3+1)$ properties of quarks and leptons and antiquarks and antileptons, with the families included if $d = (13+1)$, the superposition of even products of γ^a's, forming the Clifford even "basis vectors", offer the description of the internal space of boson fields with integer spins, manifesting as gauge fields of the corresponding Clifford odd "basis vectors".

From the point of view of $d = (3+1)$ one family of the Clifford odd "basis vectors" with $2^{\frac{d=14}{2}-1}$ members manifest spins, handedness and charges of quarks and leptons and antiquarks and antileptons appearing in $2^{\frac{d=14}{2}-1}$ families, while their Hermitian conjugated partners appear in another group of $2^{\frac{d}{2}-1}$ members in $2^{\frac{d}{2}-1}$ families [4].

The Clifford even "basis vectors" appear in two groups, each with $2^{\frac{d}{2}-1} \times 2^{\frac{d}{2}-1}$ members, with the Hermitian conjugated partners within the same group and have correspondingly no families. The Clifford even "basis vectors" manifest from the point of view of $d = (3+1)$ all the properties of the vector gauge fields before the electroweak break and for the scalar fields causing the electroweak break (as assumed by the *standard model*).

Tensor products of the Clifford odd and Clifford even "basis vectors" (describing the internal space of fermions and bosons, respectively) with the basis in ordinary space form the creation operators to which the "basis vectors" transfer either anticommutativity or commutativity. The Clifford odd "basis vectors" transfer their anticommutativity to creation operators and to their Hermitian conjugated partners annihilation operators for fermions. The Clifford even "basis vectors" transfer their commutativity to creation operators and annihilation operators for bosons. Correspondingly the anticommutation properties of creation and annihilation operators of fermions explain the second quantization postulates of Dirac for fermion fields, while the commutation properties of creation and annihilation operators for bosons explain the corresponding second quantization postulates for boson fields [5].

In Sect. 11.2 the Grassmann and the Clifford algebra are explained and creation and annihilation operators described as a tensor products of the "basis vectors"

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[3] Since the multiplication with either γ^a's or $\tilde{\gamma}^a$'s changes the Clifford odd "basis vectors" into the Clifford even objects, and even "basis vectors" commute, the action for fermions can not include an odd numbers of γ^a's or $\tilde{\gamma}^a$'s, what the simple starting action of Eq. (19.1) does not. In the starting action γ^a's and $\tilde{\gamma}^a$'s appear as $\gamma^0\gamma^a\hat{p}_a$ or as $\gamma^0\gamma^c\, S^{ab}\omega_{abc}$ and as $\gamma^0\gamma^c\, \tilde{S}^{ab}\tilde{\omega}_{abc}$.

[4] The appearance of the condensate of two right handed neutrinos causes that the number of the observed families reduces to two at low energies decoupled groups of four groups.

[5] The creation and annihilation operators for either fermion or boson fields with the momenta zero, have no dynamics, and consequently no influence on clusters of fermion and boson fields.

offering explanation of the internal spaces of fermion (by the Clifford odd algebra) and boson (by the Clifford even algebra) fields and the basis in ordinary space.
In Subsect. 11.2.1 the "basis vectors" are introduced and their properties presented.
In Subsect. 11.2.2 the properties of the Clifford odd and even "basis vectors" are demonstrated in the toy model in $d = (5+1)$. The simplest cases with $d = (1+1)$ and $d = (3+1)$ are also added.
In Subsect. 11.2.3 the properties of the creation and annihilation operators for the second quantized fields are described.
In Sect. 11.3 a short overview of the achievements and predictions so far of the *spin-charge-family* theory is presented,
Sect. 11.4 presents what the reader could learn from the main contribution of this talk.
In Sect. 11.5 the properties of Clifford odd and Clifford even "basis vectors" in odd dimensional spaces are presented, demonstrating how much properties of "basis vectors" in odd dimensional spaces differ from the properties in even dimensional spaces.

11.2 Creation and annihilation operators for fermions and bosons

The second quantization postulates for fermions [16–18] require that the creation operators and their Hermitian conjugated partners annihilation operators, depending on a finite dimensional basis in internal space, that is on the space of half integer spins and on charges described by the fundamental representations of the appropriate groups, and on continuously infinite number of momenta (or coordinates) ([5], Subsect. 3.3.1), fulfil anticommutation relations.
The second quantization postulates for bosons [16–18] require that the creation and annihilation operators, depending on finite dimensional basis in internal space, that is on the space of integer spins and on charges described by the adjoint representations of the same groups, and on continuously infinite number of momenta (or coordinates) ([5], Subsect. 3.3.1), fulfil commutation relation.
I demonstrate in this talk that using the Clifford algebra to describe the internal space of fermions and bosons, the creation and annihilation operators which are tensor products of the internal basis and the momentum/coordinate basis, not only fulfil the appropriate anticommutation relations (for fermions) or commutation relations (for bosons) but also have the required properties for either fermion fields (if the internal space is described with the Clifford odd products of γ^a's) or for boson fields (if the internal space is described with the Clifford even products of γ^a's). The Clifford odd and Clifford even "basic vectors" correspondingly offer the explanation for the second quantization postulates for fermions and bosons, respectively.
There are two Clifford subalgebras which can be used to describe the internal space of fermions and of bosons, each with 2^d members. In each of the two subalgebras there are $2 \times 2^{\frac{d}{2}-1} \times 2^{\frac{d}{2}-1}$ Clifford odd and $2 \times 2^{\frac{d}{2}-1} \times 2^{\frac{d}{2}-1}$ Clifford even "basic vectors" which can be used to describe the internal space of fermion fields, the

Clifford odd "basic vectors", and of boson fields, the Clifford even "basic vectors" in any even d. $d = (13+1)$ offers the explanation for all the properties of fermion fields, with families included, and of boson fields which are the gauge fields of fermion fields.
In any even d, $d = 2(2n+1)$ or $d = 4n$, any of the two Clifford subalgebras offers twice $2^{\frac{d}{2}-1}$ irreducible representations, each with $2^{\frac{d}{2}-1}$ members, which can represent "basis vectors" and their Hermitian conjugated partners. Each irreducible representation offers in $d = (13+1)$ the description of the quarks and the antiquarks and the leptons and the antileptons (with the right handed neutrinos and left handed antineutrinos included in addition to what is) assumed by the *standard model*.
There are obviously only one kind of fermion fields and correspondingly also of their gauge fields observed. There is correspondingly no need for two Clifford subalgebras.
The reduction of the two subalgebras to only one with the postulate in Eq. (19.6), (Ref. [5], Eq. (38)) solves this problem. At the same time the reduction offers the quantum numbers for each of the irreducible representations of the Clifford subalgbebra left, γ^a's, when fermions are concerned ([5] Subsect. 3.2).
Boson fields have no families as it will be demonstrated.

Grassmann and Clifford algebras

The internal space of anticommuting or commuting second quantized fields can be described by using either the Grassmann or the Clifford algebras [1–3,31]. What follows is a short overview of Subsect.3.2 of Ref. [5] and of references cited in [5].
In Grassmann d-dimensional space there are d anticommuting (operators) θ^a, and d anticommuting operators which are derivatives with respect to θ^a, $\frac{\partial}{\partial\theta_a}$,

$$\{\theta^a, \theta^b\}_+ = 0\,, \qquad \{\frac{\partial}{\partial\theta_a}, \frac{\partial}{\partial\theta_b}\}_+ = 0\,,$$
$$\{\theta_a, \frac{\partial}{\partial\theta_b}\}_+ = \delta_{ab}\,, (a,b) = (0,1,2,3,5,\cdots,d)\,. \qquad (11.2)$$

Defining [32]

$$(\theta^a)^\dagger = \eta^{aa}\frac{\partial}{\partial\theta_a}\,, \quad \text{leads to} \quad (\frac{\partial}{\partial\theta_a})^\dagger = \eta^{aa}\theta^a\,, \qquad (11.3)$$

with $\eta^{ab} = diag\{1,-1,-1,\cdots,-1\}$.
θ^a and $\frac{\partial}{\partial\theta_a}$ are, up to the sign, Hermitian conjugated to each other. The identity is the self adjoint member of the algebra. The choice for the following complex properties of θ^a and correspondingly of $\frac{\partial}{\partial\theta_a}$ are made

$$\{\theta^a\}^* = (\theta^0, \theta^1, -\theta^2, \theta^3, -\theta^5, \theta^6, ..., -\theta^{d-1}, \theta^d)\,,$$
$$\{\frac{\partial}{\partial\theta_a}\}^* = (\frac{\partial}{\partial\theta_0}, \frac{\partial}{\partial\theta_1}, -\frac{\partial}{\partial\theta_2}, \frac{\partial}{\partial\theta_3}, -\frac{\partial}{\partial\theta_5}, \frac{\partial}{\partial\theta_6}, ..., -\frac{\partial}{\partial\theta_{d-1}}, \frac{\partial}{\partial\theta_d})\,. \qquad (11.4)$$

The are 2^d superposition of products of θ^a, the Hermitian conjugated partners of which are the corresponding superposition of products of $\frac{\partial}{\partial\theta_a}$.

There exist two kinds of the Clifford algebra elements (operators), γ^a and $\tilde{\gamma}^a$, expressible with θ^a's and their conjugate momenta $p^{\theta a} = i\,\frac{\partial}{\partial \theta_a}$ [2], Eqs. (11.2, 11.3),

$$\gamma^a = (\theta^a + \frac{\partial}{\partial \theta_a})\,, \quad \tilde{\gamma}^a = i\,(\theta^a - \frac{\partial}{\partial \theta_a})\,,$$
$$\theta^a = \frac{1}{2}\,(\gamma^a - i\tilde{\gamma}^a)\,, \quad \frac{\partial}{\partial \theta_a} = \frac{1}{2}\,(\gamma^a + i\tilde{\gamma}^a)\,, \tag{11.5}$$

offering together $2\cdot 2^d$ operators: 2^d are superposition of products of γ^a and 2^d of $\tilde{\gamma}^a$. It is easy to prove, if taking into account Eqs. (11.3, 11.5), that they form two anticommuting Clifford subalgebras, $\{\gamma^a, \tilde{\gamma}^b\}_+ = 0$, Refs. ([5] and references therein)

$$\{\gamma^a, \gamma^b\}_+ = 2\eta^{ab} = \{\tilde{\gamma}^a, \tilde{\gamma}^b\}_+\,,$$
$$\{\gamma^a, \tilde{\gamma}^b\}_+ = 0\,, \quad (a, b) = (0, 1, 2, 3, 5, \cdots, d)\,,$$
$$(\gamma^a)^\dagger = \eta^{aa}\,\gamma^a\,, \quad (\tilde{\gamma}^a)^\dagger = \eta^{aa}\,\tilde{\gamma}^a\,. \tag{11.6}$$

While the Grassmann algebra offers the description of the "anticommuting integer spin second quantized fields" and of the "commuting integer spin second quantized fields" [5,35], the Clifford algebras which are superposition of odd products of either γ^a's or $\tilde{\gamma}^a$'s offer the description of the second quantized half integer spin fermion fields, which from the point of the subgroups of the $SO(d-1,1)$ group manifest spins and charges of fermions and antifermions in the fundamental representations of the group and subgroups.
The superposition of even products of either γ^a's or $\tilde{\gamma}^a$'s offer the description of the commuting second quantized boson fields with integer spins (as we can see in [9] and shall see in this contribution) which from the point of the subgroups of the $SO(d-1,1)$ group manifest spins and charges in the adjoint representations of the group and subgroups.
The following *postulate*, which determines how does $\tilde{\gamma}^a$'s operate on γ^a's, reduces the two Clifford subalgebras, γ^a's and $\tilde{\gamma}^a$'s, to one, to the one described by γ^a's [2, 14,29,31,32]

$$\{\tilde{\gamma}^a B = (-)^B\, i\, B\gamma^a\}|\psi_{oc}>\,, \tag{11.7}$$

with $(-)^B = -1$, if B is (a function of) an odd products of γ^a's, otherwise $(-)^B = 1$ [14], $|\psi_{oc}>$ is defined in Eq. (19.8) of Subsect. 11.2.1.

After the postulate of Eq. (19.6) it follows:
a. The Clifford subalgebra described by $\tilde{\gamma}^a$'s looses its meaning for the description of the internal space of quantum fields.
b. The "basis vectors" which are superposition of an odd or an even products of γ^a's obey the postulates for the second quantization fields for fermions or bosons, respectively, Sect.11.2.1.
c. It can be proven that the relations presented in Eq. (19.3) remain valid also after

the postulate of Eq. (19.6). The proof is presented in Ref. ([5], App. I, Statement 3a.
d. Each irreducible representation of the Clifford odd "basis vectors" described by γ^a's are equipped by the quantum numbers of the Cartan subalgebra members of $\tilde{S}^{ab}$, chosen in Eq. (19.4), as follows

$$\begin{aligned}
&\mathcal{S}^{03}, \mathcal{S}^{12}, \mathcal{S}^{56}, \cdots, \mathcal{S}^{d-1\; d}, \\
&S^{03}, S^{12}, S^{56}, \cdots, S^{d-1\; d}, \\
&\tilde{S}^{03}, \tilde{S}^{12}, \tilde{S}^{56}, \cdots, \tilde{S}^{d-1\; d}, \\
&\mathcal{S}^{ab} = S^{ab} + \tilde{S}^{ab} = i\,(\theta^a \frac{\partial}{\partial \theta_b} - \theta^b \frac{\partial}{\partial \theta_a})\,. \qquad (11.8)
\end{aligned}$$

After the postulate of Eq. (19.6) no vector space of $\tilde{\gamma}^a$'s needs to be taken into account for the description of the internal space of either fermions or bosons, in agreement with the observed properties of fermions and bosons. Also the Grassmann algebra is reduced to only one of the Clifford subalgebras. The operators $\tilde{\gamma}^a$'s describe from now on properties of fermion and boson "basis vectors" determined by superposition of products of odd or even numbers of γ^a's, respectively.
$\tilde{\gamma}^a$'s equip each irreducible representation of the Lorentz group (with the infinitesimal generators $S^{ab} = \frac{i}{4}\{\gamma^a, \gamma^b\}_-$) when applying on the Clifford odd "basis vectors" (which are superposition of odd products of $\gamma^{a's}$) with the family quantum numbers (determined by $\tilde{S}^{ab} = \frac{i}{4}\{\tilde{\gamma}^a, \tilde{\gamma}^b\}_-$).
Correspondingly the Clifford odd "basis vectors" (they are superposition of an odd products of γ^a's) form $2^{\frac{d}{2}-1}$ families, with the quantum number f, each family have $2^{\frac{d}{2}-1}$ members, m. They offer the description of the second quantized fermion fields.
The Clifford even "basis vectors" (they are superposition of an even products of γ^a's) have no families as we shall see in what follows, but they do carry both quantum numbers, f and m. They offer the description of the second quantized boson fields as the gauge fields of the second quantized fermion fields. The generators of the Lorentz transformations in the internal space of the Clifford even "basis vectors" are $\mathcal{S}^{ab} = S^{ab} + \tilde{S}^{ab}$.
Properties of the Clifford odd and the Clifford even "basis vectors" are discussed in the next subsection.

11.2.1 "Basis vectors" of fermions and bosons

After the reduction of the two Clifford subalgebras to only one, Eq. (19.6), we only need to define "basis vectors" for the case that the internal space of second quantized fields is described by superposition of odd or even products γ^a's [6].
Let us use the technique which makes "basis vectors" products of nilpotents and projectors [2,3,13,14] which are eigenvectors of the (chosen) Cartan subalgebra

[6] In Ref. [5] the reader can find in Subsects. (3.2.1 and 3.2.2) definitions for the "basis vectors" for the Grassmann and the two Clifford subalgebras, which are products of nilpotents and projectors chosen to be eigenvactors of the corresponding Cartan subalgebra members of the Lorentz algebras presented in Eq. (19.4).

members, Eq. (19.4), of the Lorentz algebra in the space of γ^a's, either in the case of the Clifford odd or in the case of the Clifford even products of γ^a's .
There are $\frac{d}{2}$ members of the Cartan subalgebra, Eq. (19.4), in even dimensional spaces.
One finds for any of the $\frac{d}{2}$ Cartan subalgebra member, S^{ab} or $\tilde{S}^{ab}$, both applying on a nilpotent $\stackrel{ab}{(k)}$ or on projector $\stackrel{ab}{[k]}$

$$\stackrel{ab}{(k)}:= \frac{1}{2}(\gamma^a + \frac{\eta^{aa}}{ik}\gamma^b)\,, \quad (\stackrel{ab}{(k)})^2 = 0,$$

$$\stackrel{ab}{[k]}:= \frac{1}{2}(1 + \frac{i}{k}\gamma^a\gamma^b)\,, \quad (\stackrel{ab}{[k]})^2 = \stackrel{ab}{[k]}$$

the relations

$$\begin{aligned} S^{ab} \stackrel{ab}{(k)} &= \frac{k}{2} \stackrel{ab}{(k)}\,, \qquad & \tilde{S}^{ab} \stackrel{ab}{(k)} &= \frac{k}{2} \stackrel{ab}{(k)}\,, \\ S^{ab} \stackrel{ab}{[k]} &= \frac{k}{2} \stackrel{ab}{[k]}\,, \qquad & \tilde{S}^{ab} \stackrel{ab}{[k]} &= -\frac{k}{2} \stackrel{ab}{[k]}\,, \end{aligned} \tag{11.9}$$

with $k^2 = \eta^{aa}\eta^{bb}$, demonstrating that the eigenvalues of S^{ab} on nilpotents and projectors expressed with γ^a's differ from the eigenvalues of $\tilde{S}^{ab}$ on nilpotents and projectors expressed with γ^a's, so that $\tilde{S}^{ab}$ can be used to equip each irreducible representation of S^{ab} with the "family" quantum number. [7]
We define in even d the "basis vectors" as algebraic, $*_A$, products of nilpotents and projectors so that each product is eigenvector of all $\frac{d}{2}$ Cartan subalgebra members.
We recognize in advance that the superposition of an odd products of γ^a's, that is the Clifford odd "basis vectors", must include an odd number of nilpotents, at least one, while the superposition of an even products of γ^a"s, that is Clifford even "basis vectors", must include an even number of nilpotents or only projectors.
To define the Clifford odd "basis vectors", we shall see that they have properties appropriate to describe the internal space of the second quantized fermion fields, and the Clifford even "basis vectors", we shall see that they have properties appropriate to describe the internal space of the second quantized boson fields, we need to know the relations for nilpotents and projectors

$$\begin{aligned} \stackrel{ab}{(k)}&: = \frac{1}{2}(\gamma^a + \frac{\eta^{aa}}{ik}\gamma^b)\,, \quad & \stackrel{ab}{[k]}&:= \frac{1}{2}(1 + \frac{i}{k}\gamma^a\gamma^b)\,, \\ \stackrel{ab}{(\tilde{k})}&: = \frac{1}{2}(\tilde{\gamma}^a + \frac{\eta^{aa}}{ik}\tilde{\gamma}^b)\,, \quad & \stackrel{ab}{[\tilde{k}]}&: \frac{1}{2}(1 + \frac{i}{k}\tilde{\gamma}^a\tilde{\gamma}^b)\,, \end{aligned} \tag{11.10}$$

[7] The reader can find the proof of Eq. (19.7) in Ref. [5], App. (I).

which can be derived after taking into account Eq. (19.3)

$$
\begin{aligned}
&\gamma^a \stackrel{ab}{(k)} = \eta^{aa} \stackrel{ab}{[-k]}, \quad \gamma^b \stackrel{ab}{(k)} = -ik \stackrel{ab}{[-k]}, \quad \gamma^a \stackrel{ab}{[k]} = \stackrel{ab}{(-k)}, \quad \gamma^b \stackrel{ab}{[k]} = -ik\eta^{aa} \stackrel{ab}{(-k)}\,,\\
&\tilde{\gamma}^a \stackrel{ab}{(k)} = -i\eta^{aa} \stackrel{ab}{[k]}, \quad \tilde{\gamma}^b \stackrel{ab}{(k)} = -k \stackrel{ab}{[k]}, \quad \tilde{\gamma}^a \stackrel{ab}{[k]} = \; i \stackrel{ab}{(k)}, \quad \tilde{\gamma}^b \stackrel{ab}{[k]} = -k\eta^{aa} \stackrel{ab}{(k)}\,,\\
&\stackrel{ab}{(k)}^{\dagger} = \eta^{aa} \stackrel{ab}{(-k)}\,, \quad (\stackrel{ab}{(k)})^2 = 0\,, \quad \stackrel{ab}{(k)}\stackrel{ab}{(-k)} = \eta^{aa} \stackrel{ab}{[k]}\,,\\
&\stackrel{ab}{[k]}^{\dagger} = \stackrel{ab}{[k]}\,, \quad (\stackrel{ab}{[k]})^2 = \stackrel{ab}{[k]}\,, \quad \stackrel{ab}{[k]}\stackrel{ab}{[-k]} = 0\,,\\
&\stackrel{ab}{(k)}\stackrel{ab}{[k]} = 0\,, \quad \stackrel{ab}{[k]}\stackrel{ab}{(k)} = \stackrel{ab}{(k)}\,, \quad \stackrel{ab}{(k)}\stackrel{ab}{[-k]} = \stackrel{ab}{(k)}\,, \quad \stackrel{ab}{[k]}\stackrel{ab}{(-k)} = 0\,,\\
&\stackrel{ab}{(\tilde{k})}^{\dagger} = \eta^{aa} \stackrel{ab}{(\widetilde{-k})}\,, \quad (\stackrel{ab}{(\tilde{k})})^2 = 0\,, \quad \stackrel{ab}{(\tilde{k})}\stackrel{ab}{(\widetilde{-k})} = \eta^{aa} \stackrel{ab}{[\tilde{k}]}\,,\\
&\stackrel{ab}{[\tilde{k}]}^{\dagger} = \stackrel{ab}{[\tilde{k}]}\,, \quad (\stackrel{ab}{[\tilde{k}]})^2 = \stackrel{ab}{[\tilde{k}]}\,, \quad \stackrel{ab}{[\tilde{k}]}\stackrel{ab}{[\widetilde{-k}]} = 0\,,\\
&\stackrel{ab}{(\tilde{k})}\stackrel{ab}{[\tilde{k}]} = 0\,, \quad \stackrel{ab}{[\tilde{k}]}\stackrel{ab}{(\tilde{k})} = \stackrel{ab}{(\tilde{k})}\,, \quad \stackrel{ab}{(\tilde{k})}\stackrel{ab}{[\widetilde{-k}]} = \stackrel{ab}{(\tilde{k})}\,, \quad \stackrel{ab}{[\tilde{k}]}\stackrel{ab}{(\widetilde{-k})} = 0\,. \qquad (11.11)
\end{aligned}
$$

Looking at relations in Eq. (19.9) it is obvious that the properties of the "basis vectors" which include odd number of nilpotents differ essentially from the "basis vectors" which include even number of nilpotents.
One namely recognizes:

i. Since the Hermitian conjugated partner of a nilpotent $\stackrel{ab}{(k)}^{\dagger}$ is $\eta^{aa} \stackrel{ab}{(-k)}$ and since neither S^{ab} nor $\tilde{S}^{ab}$ nor both can transform odd products of nilpotents to belong to one of the $2^{\frac{d}{2}-1}$ members of one of $2^{\frac{d}{2}-1}$ irreducible representations (families), the Hermitian conjugated partners of the Clifford odd "basis vectors" must belong to a different group of $2^{\frac{d}{2}-1}$ members of $2^{\frac{d}{2}-1}$ families.

Since S^{ac} transforms $\stackrel{ab}{(k)} *_A \stackrel{cd}{(k')}$ into $\stackrel{ab}{[-k]} *_A \stackrel{cd}{[-k']}$, while $\tilde{S}^{ab}$ transforms $\stackrel{ab}{[-k]} *_A \stackrel{cd}{[-k']}$ into $\stackrel{ab}{(-k)} *_A \stackrel{cd}{(-k')}$ it is obvious that the Hermitian conjugated partners of the Clifford odd "basis vectors" must belong to the same group of $2^{\frac{d}{2}-1} \times 2^{\frac{d}{2}-1}$ members. Projectors are self adjoint.

ii. Since an odd products of γ^a's anticommute with another group of an odd product of γ^a, the Clifford odd "basis vectors" anticommute, manifesting in a tensor product with the basis in ordinary space together with the corresponding Hermitian conjugated partners properties of the anticommutation relations postulated by Dirac for the second quantized fermion fields.

The Clifford even "basis vectors" correspondingly fulfil the commutation relations for the second quantized boson fields.

iii. The Clifford odd "basis vectors" have all the eigenvalues of the Cartan subalgebra members equal to either $\pm\frac{1}{2}$ or to $\pm\frac{i}{2}$.

The Clifford even "basis vectors" have all the eigenvalues of the Cartan subalgebra members $\mathcal{S}^{ab}$ equal to either ± 1 and zero or to $\pm i$ and zero.

Let us define odd an even "basis vectors" as products of nilpotents and projectors in even dimensional spaces.

a. *Clifford odd "basis vectors"*

The Clifford odd "basis vectors" must be products of an odd number of nilpotents and the rest, up to $\frac{d}{2}$, of projectors, each nilpotent and projector must be the "eigenstate" of one of the members of the Cartan subalgebra, Eq. (19.4), correspondingly are the "basis vectors" eigenstates of all the members of the Lorentz algebras: S^{ab}'s determine $2^{\frac{d}{2}-1}$ members of one family, $\tilde{S}^{ab}$'s transform each member of one family to the same member of the rest of $2^{\frac{d}{2}-1}$ families.
Let us name the Clifford odd "basis vectors" $\hat{b}^{m\dagger}_f$, where m determines membership of 'basis vectors" in any family and f determines a particular family. The Hermitian conjugated partner of $\hat{b}^{m\dagger}_f$ is named by $\hat{b}^{m}_f = (\hat{b}^{m\dagger}_f)^{\dagger}$.
Let us start in $d = 2(2n+1)$ with the "basis vector" $\hat{b}^{1\dagger}_1$ which is the product of only nilpotents, all the rest members belonging to the $f = 1$ family follow by the application of $S^{01}, S^{03}, \dots, S^{0d}, S^{15}, \dots, S^{1d}, S^{5d} \dots, S^{d-2\,d}$. The algebraic product mark $*_A$ is skipped.

$$
\begin{aligned}
& d = 2(2n+1)\,, \\
& \hat{b}^{1\dagger}_1 = \overset{03}{(+i)}\overset{12}{(+)}\overset{56}{(+)} \cdots \overset{d-1\,d}{(+)}\,, \\
& \hat{b}^{2\dagger}_1 = \overset{03}{[-i]}\overset{12}{[-]}\overset{56}{(+)} \cdots \overset{d-1\,d}{(+)}\,, \\
& \dots \\
& \hat{b}^{2^{\frac{d}{2}-1}\dagger}_1 = \overset{03}{[-i]}\overset{12}{[-]}\overset{56}{(+)} \dots \overset{d-3\,d-2}{[-]}\;\overset{d-1\,d}{[-]}\,, \\
& \cdots\,.
\end{aligned}
\tag{11.12}
$$

The Hermitian conjugated partners of the Clifford odd "basis vector" $\hat{b}^{m\dagger}_1$, presented in Eq. (11.12), are

$$
\begin{aligned}
& d = 2(2n+1)\,, \\
& \hat{b}^{1}_1 = \overset{03}{(-i)}\overset{12}{(-)} \cdots \overset{d-1\,d}{(-)}\,, \\
& \hat{b}^{2}_1 = \overset{03}{[-i]}\overset{12}{[-]}\overset{56}{(-)} \cdots \overset{d-1\,d}{(-)}\,, \\
& \dots \\
& \hat{b}^{2^{\frac{d}{2}-1}\dagger}_1 = \overset{03}{[-i]}\overset{12}{[-]}\overset{56}{(-)}\overset{78}{[-]} \dots \overset{d-3\,d-2}{[-]}\;\overset{d-1\,d}{[-]}\,, \\
& \cdots\,.
\end{aligned}
\tag{11.13}
$$

In $d = 4n$ the choice of the starting "basis vector"with maximal number of nilpotents must have one projector

$$
\begin{aligned}
& d = 4n\,, \\
& \hat{b}^{1\dagger}_1 = \overset{03}{(+i)}\overset{12}{(+)} \cdots \overset{d-1\,d}{[+]}\,, \\
& \hat{b}^{2\dagger}_1 = \overset{03}{[-i]}\overset{12}{[-]}\overset{56}{(+)} \cdots \overset{d-1\,d}{[+]}\,, \\
& \dots \\
& \hat{b}^{2^{\frac{d}{2}-1}\dagger}_1 = \overset{03}{[-i]}\overset{12}{[-]}\overset{56}{(+)} \dots \overset{d-3\,d-2}{[-]}\;\overset{d-1\,d}{[+]}\,, \\
& \cdots\,.
\end{aligned}
\tag{11.14}
$$

The Hermitian conjugated partners of the Clifford odd "basis vector" $\hat{b}^{m\dagger}_{1}$, presented in Eq. (11.14), follow if all nilpotents $\overset{ab}{(k)}$ are transformed into $\eta^{aa}\ \overset{ab}{(-k)}$. For either $d = 2(2n+1)$ or for $d = 4n$ all the $2^{\frac{d}{2}-1}$ families follow by applying $\tilde{S}^{ab}$'s on all the members of the starting family. (Or one can find the starting $\hat{b}^{1}_{f}$ for all families f and then generate all the members $\hat{b}^{m}_{f}$ from $\hat{b}^{1}_{f}$ by the application of $\tilde{S}^{ab}$ on the starting member.)
It is not difficult to see that all the "basis vectors" within any family as well as the "basis vectors" among families are orthogonal, that is their algebraic product is zero, and the same is true for the Hermitian conjugated partners, what can be proved by the algebraic multiplication using Eq.(19.9).

$$\hat{b}^{m\dagger}_{f} *_A \hat{b}^{m`\dagger}_{f`} = 0\,, \quad \hat{b}^{m}_{f} *_A \hat{b}^{m`}_{f`} = 0\,, \quad \forall m, m', f, f`\,. \tag{11.15}$$

If we require that each family of "basis vectors", determined by nilpotents and projectors described by γ^a's, carries the family quantum number determined by $\tilde{S}^{ab}$ and define the vacuum state on which "basis vectors" apply as

$$|\psi_{oc}>= \sum_{f=1}^{2^{\frac{d}{2}-1}} \hat{b}^{m}_{f}{}_{*A}\hat{b}^{m\dagger}_{f}\,|\,1\,>\,, \tag{11.16}$$

it follows that the Clifford odd "basis vectors" obey the relations

$$\begin{aligned}
\hat{b}^{m}_{f}{}_{*A}|\psi_{oc}> &= 0.\,|\psi_{oc}>\,,\\
\hat{b}^{m\dagger}_{f}{}_{*A}|\psi_{oc}> &= |\psi^{m}_{f}>\,,\\
\{\hat{b}^{m}_{f}, \hat{b}^{m'}_{f`}\}_{*A+}|\psi_{oc}> &= 0.\,|\psi_{oc}>\,,\\
\{\hat{b}^{m\dagger}_{f}, \hat{b}^{m'\dagger}_{f`}\}_{*A+}|\psi_{oc}> &= 0.\,|\psi_{oc}>\,,\\
\{\hat{b}^{m}_{f}, \hat{b}^{m'\dagger}_{f}\}_{*A+}|\psi_{oc}> &= \delta^{mm'}\,\delta_{ff`}|\psi_{oc}>\,,
\end{aligned} \tag{11.17}$$

while the normalization $<\psi_{oc}|\hat{b}^{m'\dagger}_{f'} *_A \hat{b}^{m\dagger}_{f} *_A |\psi_{oc}>= \delta^{mm'}\delta_{ff'}$ is used and the anticommutation relation mean $\{\hat{b}^{m\dagger}_{f}, \hat{b}^{m'\dagger}_{f`}\}_{*A+} = \hat{b}^{m\dagger}_{f} *_A \hat{b}^{m'\dagger}_{f`} + \hat{b}^{m'\dagger}_{f`} *_A \hat{b}^{m\dagger}_{f}$.
If we write the creation and annihilation operators as the tensor, $*_T$, products of "basis vectors" and the basis in ordinary space, the creation and annihilation operators fulfil the Dirac's anticommutation postulates since the "basis vectors" transfer their anticommutativity to creation and annihilation operators. It turns out that not only the Clifford odd "basis vectors" offer the description of the internal space of fermions, they offer the explanation for the second quantization postulates for fermions as well.
Table 11.1, presented in Subsect. 11.2.2, illustrates the properties of the Clifford odd "basis vectors" on the case of $d = (5+1)$.

b. *Clifford even "basis vectors"*

The Clifford even "basis vectors" must be products of an even number of nilpotents and the rest, up to $\frac{d}{2}$, of projectors, each nilpotent and projector in a product must be the "eigenstate" of one of the members of the Cartan subalgebra, Eq. (19.4),

correspondingly are the "basis vectors" eigenstates of all the members of the Lorentz algebra: S^{ab}'s and $\tilde{S}^{ab}$'s generate from the starting "basis vector" all the $2^{\frac{d}{2}-1} \times 2^{\frac{d}{2}-1}$ members of one group which includes as well the Hermitian conjugated partners of any member. $2^{\frac{d}{2}-1}$ members of the group are products of projectors only. They are self adjoint.
There are two groups of Clifford even "basis vectors"with $2^{\frac{d}{2}-1} \times 2^{\frac{d}{2}-1}$ members each. The members of one group are not connected with the members of another group by either by S^{ab}'s or $\tilde{S}^{ab}$'s or both.
Let us name the Clifford even "basis vectors" ${}^{i}\hat{\mathcal{A}}^{m\dagger}_{f}$, where $i = (I, II)$ denotes that there are two groups of Clifford even "basis vectors", while m and f determine membership of 'basis vectors" in any of the two groups, I or II. Let me repeat that the Hermitian conjugated partner of any "basis vector" appears either in the case of ${}^{I}\hat{\mathcal{A}}^{m\dagger}_{f}$ or in the case of ${}^{II}\hat{\mathcal{A}}^{m\dagger}_{f}$ within the same group.
Let us write down the Clifford even "basis vectors" as a product of an even number of nilpotents and the rest of projectors, so that the Clifford even "basis vectors" are eigenvectors of all the Cartan subalgebra members, and let us name them as follows

$$
\begin{array}{ll}
 & d = 2(2n+1) \\
{}^{I}\hat{\mathcal{A}}^{1\dagger}_{1} = \stackrel{03}{(+i)}\stackrel{12}{(+)} \cdots \stackrel{d-1\, d}{[+]}\,, & {}^{II}\hat{\mathcal{A}}^{1\dagger}_{1} = \stackrel{03}{(-i)}\stackrel{12}{(+)} \cdots \stackrel{d-1\, d}{[+]}\,, \\
{}^{I}\hat{\mathcal{A}}^{2\dagger}_{1} = \stackrel{03}{[-i]}\stackrel{12}{[-]}\stackrel{56}{(+)} \cdots \stackrel{d-1\, d}{[+]}\,, & {}^{II}\hat{\mathcal{A}}^{2\dagger}_{1} = \stackrel{03}{[+i]}\stackrel{12}{[-]}\stackrel{56}{(+)} \cdots \stackrel{d-1\, d}{[+]}\,, \\
{}^{I}\hat{\mathcal{A}}^{3\dagger}_{1} = \stackrel{03}{(+i)}\stackrel{12}{(+)}\stackrel{56}{(+)} \cdots \stackrel{d-3\, d-2}{[-]}\;\stackrel{d-1\, d}{(-)}\,, & {}^{II}\hat{\mathcal{A}}^{3\dagger}_{1} = \stackrel{03}{(-i)}\stackrel{12}{(+)}\stackrel{56}{(+)} \cdots \stackrel{d-3\, d-2}{[-]}\;\stackrel{d-1\, d}{(-)}\,, \\
\cdots & \cdots \\
 & d = 4n \\
{}^{I}\hat{\mathcal{A}}^{1\dagger}_{1} = \stackrel{03}{(+i)}\stackrel{12}{(+)} \cdots \stackrel{d-1\, d}{(+)}\,, & {}^{II}\hat{\mathcal{A}}^{1\dagger}_{1} = \stackrel{03}{(-i)}\stackrel{12}{(+)} \cdots \stackrel{d-1\, d}{(+)}\,, \\
{}^{I}\hat{\mathcal{A}}^{2\dagger}_{1} = \stackrel{03}{[-i]}\stackrel{12}{[-i]}\stackrel{56}{(+)} \cdots \stackrel{d-1\, d}{(+)}\,, & {}^{II}\hat{\mathcal{A}}^{2\dagger}_{1} = \stackrel{03}{[+i]}\stackrel{12}{[-i]}\stackrel{56}{(+)} \cdots \stackrel{d-1\, d}{(+)}\,, \\
{}^{I}\hat{\mathcal{A}}^{3\dagger}_{1} = \stackrel{03}{(+i)}\stackrel{12}{(+)}\stackrel{56}{(+)} \cdots \stackrel{d-3\, d-2}{[-]}\;\stackrel{d-1\, d}{[-]}\,, & {}^{II}\hat{\mathcal{A}}^{3\dagger}_{1} = \stackrel{03}{(-i)}\stackrel{12}{(+)}\stackrel{56}{(+)} \cdots \stackrel{d-3\, d-2}{[-]}\;\stackrel{d-1\, d}{[-]} \\
\cdots & \cdots
\end{array}
\tag{11.18}
$$

There are $2^{\frac{d}{2}-1} \times 2^{\frac{d}{2}-1}$ Clifford even "basis vectors" of the kind ${}^{I}\hat{\mathcal{A}}^{m\dagger}_{f}$ and there are $2^{\frac{d}{2}-1} \times 2^{\frac{d}{2}-1}$ Clifford even "basis vectors" of the kind ${}^{II}\hat{\mathcal{A}}^{m\dagger}_{f}$.
Table 11.1, presented in Subsect. 11.2.2, illustrates properties of the Clifford odd and Clifford even "basis vectors" on the case of $d = (5+1)$. Looking at this particular case it is easy to evaluate properties of either even or odd "basis vectors". I shall present here the general results which follow after careful inspection of properties of both kinds of "basis vectors".
The Clifford even "basis vectors" belonging to two different groups are orthogonal due to the fact that they differ in the sign of one nilpotent or one projectors, or the algebraic products of members of one group give zero according to Eq. (19.9).

$$
{}^{I}\hat{\mathcal{A}}^{m\dagger}_{f} *_{A} {}^{II}\hat{\mathcal{A}}^{m\dagger}_{f} = 0 = {}^{II}\hat{\mathcal{A}}^{m\dagger}_{f} *_{A} {}^{I}\hat{\mathcal{A}}^{m\dagger}_{f}\,. \tag{11.19}
$$

The members of each of this two groups have the property

$$
{}^{I,II}\hat{\mathcal{A}}^{m\dagger}_{f} *_A {}^{I,II}\hat{\mathcal{A}}^{m'\dagger}_{f`} \rightarrow \begin{cases} {}^{I,II}\hat{\mathcal{A}}^{m\dagger}_{f`}\,, \text{only one for } \forall f`\,, \\ \qquad\qquad \text{or zero}\,. \end{cases} \tag{11.20}
$$

Two "basis vectors" ${}^{I}\hat{\mathcal{A}}^{m\dagger}_{f}$ and ${}^{I}\hat{\mathcal{A}}^{m'\dagger}_{f'}$, the algebraic product, $*_A$, of which gives nonzero contribution, "scatter" into the third one ${}^{I}\hat{\mathcal{A}}^{m\dagger}_{f`}$. The same is true also for the "basis vectors" ${}^{II}\hat{\mathcal{A}}^{m\dagger}_{f}$.
Let us write the commutation relations for Clifford even "basis vectors" taking into account Eq. (11.20).

i. In the case that ${}^{I}\hat{\mathcal{A}}^{m\dagger}_{f} *_A {}^{I}\hat{\mathcal{A}}^{m'\dagger}_{f`} \rightarrow {}^{I}\hat{\mathcal{A}}^{m\dagger}_{f`}$ and ${}^{I}\hat{\mathcal{A}}^{m'\dagger}_{f`} *_A {}^{I}\hat{\mathcal{A}}^{m\dagger}_{f} = 0$ it follows

$$
\{{}^{I}\hat{\mathcal{A}}^{m\dagger}_{f}\,, {}^{I}\hat{\mathcal{A}}^{m'\dagger}_{f`}\}_{*_A -} \rightarrow \begin{cases} {}^{I}\hat{\mathcal{A}}^{m\dagger}_{f`}\,, \quad (\text{if} \quad {}^{I}\hat{\mathcal{A}}^{m\dagger}_{f} *_A {}^{I}\hat{\mathcal{A}}^{m'\dagger}_{f`} \rightarrow {}^{I}\hat{\mathcal{A}}^{m\dagger}_{f`} \\ \qquad\qquad \text{and } {}^{I}\hat{\mathcal{A}}^{m'\dagger}_{f`} *_A {}^{I}\hat{\mathcal{A}}^{m\dagger}_{f} = 0)\,, \end{cases} \tag{11.21}
$$

ii. In the case that ${}^{I}\hat{\mathcal{A}}^{m\dagger}_{f} *_A {}^{I}\hat{\mathcal{A}}^{m'\dagger}_{f`} \rightarrow {}^{I}\hat{\mathcal{A}}^{m\dagger}_{f`}$ and ${}^{I}\hat{\mathcal{A}}^{m'\dagger}_{f`} *_A {}^{I}\hat{\mathcal{A}}^{m\dagger}_{f} \rightarrow {}^{I}\hat{\mathcal{A}}^{m'\dagger}_{f}$ it follows

$$
\{{}^{I}\hat{\mathcal{A}}^{m\dagger}_{f}\,, {}^{I}\hat{\mathcal{A}}^{m'\dagger}_{f`}\}_{*_A -} \rightarrow \begin{cases} {}^{I}\hat{\mathcal{A}}^{m\dagger}_{f`} - {}^{I}\hat{\mathcal{A}}^{m'\dagger}_{f}\,, \quad (\text{if} \quad {}^{I}\hat{\mathcal{A}}^{m\dagger}_{f} *_A {}^{I}\hat{\mathcal{A}}^{m'\dagger}_{f`} \rightarrow {}^{I}\hat{\mathcal{A}}^{m\dagger}_{f`} \\ \qquad\qquad \text{and } {}^{I}\hat{\mathcal{A}}^{m'\dagger}_{f`} *_A {}^{I}\hat{\mathcal{A}}^{m\dagger}_{f} \rightarrow {}^{I}\hat{\mathcal{A}}^{m'\dagger}_{f})\,, \end{cases} \tag{11.22}
$$

iii. In all other cases we have

$$
\{{}^{I}\hat{\mathcal{A}}^{m\dagger}_{f}\,, {}^{I}\hat{\mathcal{A}}^{m'\dagger}_{f`}\}_{*_A -} = 0\,. \tag{11.23}
$$

$\{{}^{I}\hat{\mathcal{A}}^{m\dagger}_{f}\,, {}^{I}\hat{\mathcal{A}}^{m'\dagger}_{f`}\}_{*_A -}$ means ${}^{I}\hat{\mathcal{A}}^{m\dagger}_{f} *_A {}^{I}\hat{\mathcal{A}}^{m'\dagger}_{f`} - {}^{I}\hat{\mathcal{A}}^{m'\dagger}_{f`} *_A {}^{I}\hat{\mathcal{A}}^{m\dagger}_{f}$.

It remains to evaluate the algebraic application, $*_A$, of the Clifford even "basis vectors" ${}^{I}\hat{\mathcal{A}}^{m\dagger}_{f}$ on the Clifford odd "basis vectors" $\hat{b}^{m'\dagger}_{f`}$. One finds

$$
{}^{I}\hat{\mathcal{A}}^{m\dagger}_{f`} *_A \hat{b}^{m'\dagger}_{f} \rightarrow \begin{cases} \hat{b}^{m\dagger}_{f}\,, \\ \text{or zero}\,. \end{cases} \tag{11.24}
$$

For each ${}^{I}\hat{\mathcal{A}}^{m\dagger}_{f}$ there are among $2^{\frac{d}{2}-1} \times 2^{\frac{d}{2}-1}$ members of the Clifford odd "basis vectors" (describing the internal space of fermion fields) $2^{\frac{d}{2}-1}$ members, $\hat{b}^{m'\dagger}_{f`}$, fulfilling the relation of Eq. (11.24). All the rest $(2^{\frac{d}{2}-1} \times (2^{\frac{d}{2}-1} - 1)$, give zero contributions.

Eq. (11.24) clearly demonstrates that ${}^{I}\hat{\mathcal{A}}^{m\dagger}_{f}$ transforms the Clifford odd "basis vector" in general into another Clifford odd "basis vector", transfering to the Clifford odd "basis vector" an integer spin.
We can obviously conclude that the Clifford even "basis vectors" offer the description of the gauge fields to the corresponding fermion fields.

While the Clifford odd "basis vectors" offer the description of the internal space of the second quantized anticommuting fermion fields, appearing in families, the Clifford even "basis vectors" offer the description of the internal space of the second quantized commuting boson fields, having no families and manifesting as the gauge fields of the corresponding fermion fields.

11.2.2 Example demonstrating properties of Clifford odd and even "basis vectors" for $d = (1+1)$, $d = (3+1)$, $d = (5+1)$

Subsect. 11.2.2 demonstrates properties of the Clifford odd and even "basis vectors" in special cases when $d = (1+1)$, $d = (3+1)$, and $d = (5+1)$.

Let us start with the simplest case:

d=(1+1)

There are 4 ($2^{d=2}$) "eigenvectors" of the Cartan subalgebra members S^{01} and $\mathcal{S}^{01}$ of the Lorentz algebra S^{ab} and $\mathcal{S}^{ab}$, Eq. (19.4), representing one Clifford odd "basis vector" $\hat{b}^{1\dagger}_{1} = \stackrel{01}{(+i)}$ (m=1), appearing in one family (f=1) and correspondingly one Hermitian conjugated partner $\hat{b}^{1}_{1} = \stackrel{01}{(-i)}$ [8] and two Clifford even "basis vector" ${}^{I}\mathcal{A}^{1\dagger}_{1} = \stackrel{01}{[+i]}$ and ${}^{II}\mathcal{A}^{1\dagger}_{1} = \stackrel{01}{[-i]}$, each of them is self adjoint.
Correspondingly we have two Clifford odd

$$\hat{b}^{1\dagger}_{1} = \stackrel{01}{(+i)}\,, \quad \hat{b}^{1}_{1} = \stackrel{01}{(-i)}$$

and two Clifford even

$${}^{I}\mathcal{A}^{1\dagger}_{1} = \stackrel{01}{[+i]}\,, \quad {}^{II}\mathcal{A}^{1\dagger}_{1} = \stackrel{01}{[-i]}$$

"basis vectors".
The first two Clifford odd "basis vectors" are Hermitian conjugated to each other. I make a choice that $\hat{b}^{1\dagger}_{1}$ is the "basis vector", the second Clifford odd object is its Hermitian conjugated partner. Defining the handedness as $\Gamma^{(1+1)} = \gamma^{0}\gamma^{1}$ it follows, using Eq. (19.5), that $\Gamma^{(1+1)}\, \hat{b}^{1\dagger}_{1} = \hat{b}^{1\dagger}_{1}$, which means that $\hat{b}^{1\dagger}_{1}$ is the right handed "basis vector".

We could make a choice of left handed "basis vector" if choosing $\hat{b}^{1\dagger}_{1} = \stackrel{01}{(-i)}$, but the choice of handedness would remain only one.
Each of the two Clifford even "basis vectors" is self adjoint ($({}^{I,II}\mathcal{A}^{1\dagger}_{1})^{\dagger} = {}^{I,II}\mathcal{A}^{1\dagger}_{1}$).

---

[8] It is our choice which one, $\stackrel{01}{(+i)}$ or $\stackrel{01}{(-i)}$, we chose as the "basis vector" $\hat{b}^{1\dagger}_{1}$ and which one is its Hermitian conjugated partner. The choice of the "basis vector" determines the vacuum state $|\psi_{oc}>$, Eq. (19.8). For $\hat{b}^{1\dagger}_{1} = \stackrel{01}{(+i)}$, the vacuum state is $|\psi_{oc}> = \stackrel{01}{[-i]}$ (due to the requirement that $\hat{b}^{1\dagger}_{1}|\psi_{oc}>$ is nonzero) which is the Clifford even object.

Let us notice, taking into account Eqs. (19.5, 19.9), that

$$\{\hat{b}^{1}_{1}(\equiv\stackrel{01}{(-i)}) *_A \hat{b}^{1\dagger}_{1}(\equiv\stackrel{01}{(+i)})\}|\psi_{oc}>= {}^{II}\mathcal{A}^{1\dagger}_{1}(\equiv\stackrel{01}{[-i]})|\psi_{oc}>= |\psi_{oc}>\,,$$

$$\{\hat{b}^{1\dagger}_{1}(\equiv\stackrel{01}{(+i)}) *_A \hat{b}^{1}_{1}(\equiv\stackrel{01}{(-i)})\}|\psi_{oc}>= 0\,,$$

$${}^{I}\mathcal{A}^{1\dagger}_{1}(\equiv\stackrel{01}{[+i]})\; *_A\; \hat{b}^{1}_{1}(\equiv\stackrel{01}{(+i)})|\psi_{oc}>= \hat{b}^{1}_{1}(\equiv\stackrel{01}{(+i)})|\psi_{oc}>\,,$$
$${}^{I}\mathcal{A}^{1\dagger}_{1}(\equiv\stackrel{01}{[+i]})\, \hat{b}^{1}_{1}(\equiv\stackrel{01}{(-i)})|\psi_{oc}>= 0\,.$$

We find that

$${}^{I}\mathcal{A}^{1\dagger}_{1} *_A {}^{II}\mathcal{A}^{1\dagger}_{1} = 0 = {}^{II}\mathcal{A}^{1\dagger}_{1} *_A {}^{I}\mathcal{A}^{1\dagger}_{1}\,.$$

From the case $d = (3 + 1)$ we can learn a little more:

d=(3+1)

There are 16 $(2^{d=4})$ "eigenvectors" of the Cartan subalgebra members (S^{03}, S^{12}) and $(\mathcal{S}^{03}, \mathcal{S}^{12})$ of the Lorentz algebras S^{ab} and $\mathcal{S}^{ab}$, Eq. (19.4), in $d = (3 + 1)$.
There are two families $(2^{\frac{4}{2}-1}$, f=(1,2)) with two $(2^{\frac{4}{2}-1}$, m=(1,2)) members each of the Clifford odd "basis vectors" $\hat{b}^{m\dagger}_{f}$, with $2^{\frac{4}{2}-1} \times 2^{\frac{4}{2}-1}$ Hermitian conjugated partners $\hat{b}^{m}_{f}$ in a separate group (not reachable by S^{ab}).
There are $2^{\frac{4}{2}-1} \times 2^{\frac{4}{2}-1}$ members of the group of ${}^{I}\mathcal{A}^{m\dagger}_{f}$, which are Hermitian conjugated to each other or are self adjoint, all reachable by $\mathcal{S}^{ab}$ from any starting "basis vector ${}^{I}\mathcal{A}^{1\dagger}_{1}$.
And there is another group of $2^{\frac{4}{2}-1} \times 2^{\frac{4}{2}-1}$ members of ${}^{II}\mathcal{A}^{m\dagger}_{f}$, again either Hermitian conjugated to each other or are self adjoint. All are reachable from the starting vector ${}^{II}\mathcal{A}^{1\dagger}_{1}$ by the application of $\mathcal{S}^{ab}$.
Again we can make a choice of either right or left handed Clifford odd "basis vectors", but not of both handedness. Making a choice of the right handed "basis vectors"

$$\begin{array}{ccccc}
f=1 & & f=2 & & \\
\tilde{S}^{03} = \frac{i}{2}, \tilde{S}^{12} = -\frac{1}{2}, & & \tilde{S}^{03} = -\frac{i}{2}, \tilde{S}^{12} = \frac{1}{2}, & S^{03}, & S^{12}\\
\hat{b}^{1\dagger}_{1} = \stackrel{03}{(+i)}\stackrel{12}{[+]} & & \hat{b}^{1\dagger}_{2} = \stackrel{03}{[+i]}\stackrel{12}{(+)} & \frac{i}{2} & \frac{1}{2}\\
\hat{b}^{2\dagger}_{1} = \stackrel{03}{[-i]}\stackrel{12}{(-)} & & \hat{b}^{2\dagger}_{2} = \stackrel{03}{(-i)}\stackrel{12}{[-]} & -\frac{i}{2} & -\frac{1}{2}\,,
\end{array}$$

we find for the Hermitian conjugated partners of the above "basis vectors"

$$\begin{array}{ccccc}
S^{03} = -\frac{i}{2}, S^{12} = \frac{1}{2}, & & S^{03} = \frac{i}{2}, S^{12} = -\frac{1}{2}, & \tilde{S}^{03}, & \tilde{S}^{12}\\
\hat{b}^{1}_{1} = \stackrel{03}{(-i)}\stackrel{12}{[+]} & & \hat{b}^{1}_{2} = \stackrel{03}{[+i]}\stackrel{12}{(-)} & -\frac{i}{2} & -\frac{1}{2}\\
\hat{b}^{2}_{1} = \stackrel{03}{[-i]}\stackrel{12}{(+)} & & \hat{b}^{2}_{2} = \stackrel{03}{(+i)}\stackrel{12}{[-]} & \frac{i}{2} & \frac{1}{2}\,.
\end{array}$$

Let us notice that if we look at the subspace $SO(1, 1)$ with the Clifford odd "basis vectors" with the Cartan subalgebra member S^{03} of the space $SO(3, 1)$, and neglect

the values of S^{12}, we do have $\hat{b}^{1\dagger}_1 = \stackrel{03}{(+i)}$ and $\hat{b}^{2\dagger}_2 = \stackrel{03}{(-i)}$, which have opposite handedness $\Gamma^{(1,1)}$ in $d=(1+1)$, but they have different "charges" S^{12} in $d=(3+1)$. In the whole internal space all the Clifford odd "basis vectors" have only one handedness.
We further find that $|\psi_{oc}>= \frac{1}{\sqrt{2}}(\stackrel{03}{[-i]}\stackrel{12}{[+]} + \stackrel{03}{[+i]}\stackrel{12}{[+]})$. All the Clifford odd "basis vectors" are orthogonal: $\hat{b}^{m\dagger}_f *_A \hat{b}^{m'\dagger}_{f'} = 0$.
For the Clifford even "basis vectors" we find two groups of either self adjoint members or with the Hermitian conjugated partners within the same group. The two groups are not reachable by $\mathcal{S}^{03}$. We have for ${}^{I}\mathcal{A}^{m\dagger}_f$, $m=(1,2)$, $f=(1,2)$

$$
\begin{array}{lcc l cc}
 & \mathcal{S}^{03} & \mathcal{S}^{12} & & \mathcal{S}^{03} & \mathcal{S}^{12}\\
{}^{I}\mathcal{A}^{1\dagger}_1 = \stackrel{03}{[+i]}\stackrel{12}{[+]} & 0 & 0 & , {}^{I}\mathcal{A}^{1\dagger}_2 = \stackrel{03}{(+i)}\stackrel{12}{(+)} & i & 1\\
{}^{I}\mathcal{A}^{2\dagger}_1 = \stackrel{03}{(-i)}\stackrel{12}{(-)} & -i & -1 & , {}^{I}\mathcal{A}^{2\dagger}_2 = \stackrel{03}{[-i]}\stackrel{12}{[-]} & 0 & 0\,,
\end{array}
$$

and for ${}^{II}\mathcal{A}^{m\dagger}_f$, $m=(1,2)$, $f=(1,2)$

$$
\begin{array}{lcc l cc}
 & \mathcal{S}^{03} & \mathcal{S}^{12} & & \mathcal{S}^{03} & \mathcal{S}^{12}\\
{}^{II}\mathcal{A}^{1\dagger}_1 = \stackrel{03}{[+i]}\stackrel{12}{[-]} & 0 & 0 & , {}^{II}\mathcal{A}^{1\dagger}_2 = \stackrel{03}{(+i)}\stackrel{12}{(-)} & i & 1\\
{}^{II}\mathcal{A}^{2\dagger}_1 = \stackrel{03}{(-i)}\stackrel{12}{(+)} & -i & 1 & , {}^{II}\mathcal{A}^{2\dagger}_2 = \stackrel{03}{[-i]}\stackrel{12}{[+]} & 0 & 0\,.
\end{array}
$$

The Clifford even "basis vectors" have no families. ${}^{I}\mathcal{A}^{m\dagger}_f *_A {}^{I}\mathcal{A}^{m'\dagger}_{f'} = 0$, for any (m, m', f, f').

$d = (5+1)$

In Table 11.1 the 64 $(=2^{d=6})$ "eigenvectors" of the Cartan subalgebra members of the Lorentz algebra S^{ab} and $\mathcal{S}^{ab}$, Eq. (19.4), are presented. The Clifford odd "basis vectors", they appear in 4 $(=2^{\frac{d=6}{2}-1})$ families, each family has 4 members, are products of an odd number of nilpotents, that is either of three nilpotents or of one nilpotent. They appear in Table 11.1 in the group named $odd\, I\, \hat{b}^{m\dagger}_f$. Their Hermitian conjugated partners appear in the second group named $odd\, II\, \hat{b}^{m}_f$. Within each of these two groups, the members are orthogonal, Eq. (11.15), which means that the algebraic product of $\hat{b}^{m\dagger}_f *_A \hat{b}^{m'\dagger}_{f'} = 0$ for all (m, m', f, f'). This can be checked by using relations in Eq. (19.9). Equivalently, the algebraic products of their Hermitian conjugated partners are also orthogonal among themselves. The "basis vectors" and their Hermitian conjugated partners are normalized as follows

$$
< \psi_{oc}|\hat{b}^{m}_f *_A \hat{b}^{m'\dagger}_{f'}|\psi_{oc} >= \delta^{mm'}\delta_{ff'}\,, \tag{11.25}
$$

since the vacuum state $|\psi_{oc}>= \frac{1}{\sqrt{2^{\frac{d=6}{2}-1}}}(\stackrel{03}{[-i]}\stackrel{12}{[-]}\stackrel{56}{[-]} + \stackrel{03}{[-i]}\stackrel{12}{[+]}\stackrel{56}{[+]} + \stackrel{03}{[+i]}\stackrel{12}{[-]}\stackrel{56}{[+]} + \stackrel{03}{[+i]}\stackrel{12}{[+]}\stackrel{56}{[-]})$ is normalized to one: $< \psi_{oc}|\psi_{oc} >= 1$.
The longer overview of the properties of the Clifford odd "basis vectors" and their Hermitian conjugated partners for the case $d=(5+1)$ can be found in Ref. [5].

The Clifford even "basis vectors" are products of an even number of nilpotents, of either two or none in this case. They are presented in Table 11.1 in two groups, each with $16 (= 2^{\frac{d=6}{2}-1} \times 2^{\frac{d=6}{2}-1})$ members, as $even\, \mathrm{I}\, \mathcal{A}^{m\dagger}_{f}$ and $even\, \mathrm{II}\, \mathcal{A}^{m\dagger}_{f}$. One can easily check, using Eq. (19.9), that the algebraic product ${}^{I}\mathcal{A}^{m\dagger}_{f} *_A {}^{II}\mathcal{A}^{m'\dagger}_{f`} = 0, \forall\, (m, m', f.f`)$, Eq. (11.19). The longer overview of the Clifford even "basis vectors" and their Hermitian conjugated partners for the case $d = (5+1)$- can be found in Ref. [9].
While the Clifford odd "basis vectors" are (chosen to be) right handed, $\Gamma^{(5+1)} = 1$, have their Hermitian conjugated partners opposite handedness [9]
While the Clifford odd "basis vectors" have half integer eigenvalues of the Cartan subalgebra members, Eq.(19.4), that is of S^{03}, S^{12}, S^{56} in this particular case of $d = (5+1)$, the Clifford even "basis vectors" have integer spins, obtained by $\mathcal{S}^{03} = S^{03} + \tilde{S}^{03}, \mathcal{S}^{12} = S^{12} + \tilde{S}^{12}, \mathcal{S}^{56} = S^{56} + \tilde{S}^{56}$.
Let us check what does the algebraic application, $*_A$, of ${}^{I}\hat{\mathcal{A}}^{m\dagger}_{f=3}, m = (1,2,3,4)$, presented in Table 11.1 in the third column of $even$ I, do on the Clifford odd "basis vectors" $\hat{b}^{m=1\dagger}_{f=1}$, presented as the first odd I "basis vector" in Table 11.1. This can easily be evaluated by taking into account Eq. (19.5) for any m.

$$
\begin{aligned}
&{}^{I}\hat{\mathcal{A}}^{m\dagger}_{3} *_A \hat{b}^{1\dagger}_{1} (\equiv \stackrel{03}{(+i)}\stackrel{12}{[+]}\stackrel{56}{[+]}) : \\
&{}^{I}\hat{\mathcal{A}}^{1\dagger}_{3} (\equiv \stackrel{03}{[+i]}\stackrel{12}{[+]}\stackrel{56}{[+]}) *_A \hat{b}^{1\dagger}_{1} (\equiv \stackrel{03}{(+i)}\stackrel{12}{[+]}\stackrel{56}{[+]}) \rightarrow \hat{b}^{1\dagger}_{1}\, , \text{selfadjoint} \\
&{}^{I}\hat{\mathcal{A}}^{2\dagger}_{3} (\equiv \stackrel{03}{(-i)}\stackrel{12}{(-)}\stackrel{56}{[+]}) *_A \hat{b}^{1\dagger}_{1} \rightarrow \hat{b}^{2\dagger}_{1} (\equiv \stackrel{03}{[-i]}\stackrel{12}{(-)}\stackrel{56}{[+]})\, , \\
&{}^{I}\hat{\mathcal{A}}^{3\dagger}_{3} (\equiv \stackrel{03}{(-i)}\stackrel{12}{[+]}\stackrel{56}{(-)}) *_A \hat{b}^{1\dagger}_{1} \rightarrow \hat{b}^{3\dagger}_{1} (\equiv \stackrel{03}{[-i]}\stackrel{12}{[+]}\stackrel{56}{(-)})\, , \\
&{}^{I}\hat{\mathcal{A}}^{4\dagger}_{3} (\equiv \stackrel{03}{[+i]}\stackrel{12}{(-)}\stackrel{56}{(-)}) *_A \hat{b}^{1\dagger}_{1} \rightarrow \hat{b}^{4\dagger}_{1} (\equiv \stackrel{03}{(+i)}\stackrel{12}{(-)}\stackrel{56}{(-)})\, .
\end{aligned}
\tag{11.26}
$$

The sign $\rightarrow$ means that the relation is valid up to the constant. ${}^{I}\hat{\mathcal{A}}^{1\dagger}_{3}$ is self adjoint, the Hermitian conjugated partner of ${}^{I}\hat{\mathcal{A}}^{2\dagger}_{3}$ is ${}^{I}\hat{\mathcal{A}}^{1\dagger}_{4}$, of ${}^{I}\hat{\mathcal{A}}^{3\dagger}_{3}$ is ${}^{I}\hat{\mathcal{A}}^{1\dagger}_{2}$ and of ${}^{I}\hat{\mathcal{A}}^{4\dagger}_{3}$ is ${}^{I}\hat{\mathcal{A}}^{1\dagger}_{1}$.

We can conclude that the algebraic, $*_A$, application of ${}^{I}\hat{\mathcal{A}}^{m\dagger}_{3} (\equiv \stackrel{03}{(-i)}\stackrel{12}{[+]}\stackrel{56}{(-)})$ on $\hat{b}^{1\dagger}_{1}$ leads to the same or another family member of the same family $f = 1$, namely to $\hat{b}^{m\dagger}_{1}$, $m = (1,2,3,4)$.
Calculating the eigenvalues of the Cartan subalgebra members, Eq. (19.4), before and after the algebraic multiplication, $*_A$, one sees that ${}^{I}\hat{\mathcal{A}}^{m\dagger}_{3}$ carry the integer eigenvalues of the Cartan subalgebra members, namely of $\mathcal{S}^{ab} = S^{ab} + \tilde{S}^{ab}$, since they transfer when applying on the Clifford odd "basis vector" to it the integer eigenvalues of the Cartan subalgebra members, changing the Clifford odd "basis vector" into another Clifford odd "basis vector".
We therefore find out that the algebraic application of ${}^{I}\hat{\mathcal{A}}^{m\dagger}_{3}$, $m = 1,2,3,4$, on $\hat{b}^{1\dagger}_{1}$ transforms $\hat{b}^{1\dagger}_{1}$ into $\hat{b}^{m\dagger}_{1}$, $m = (1,2,3,4)$. Similarly we find that the algebraic application of ${}^{I}\hat{\mathcal{A}}^{m}_{4}$, $m = (1,2,3,4)$ on $\hat{b}^{2\dagger}_{1}$ transforms $\hat{b}^{2\dagger}_{1}$ into $\hat{b}^{m\dagger}_{1}$, $m = (1,2,3,4)$.

[9] The handedness $\Gamma^{(d)}$, one of the invariants of the group $SO(d)$, with the infinitesimal generators of the Lorentz group S^{ab}, is defined as $\Gamma^{(d)} = \alpha\, \varepsilon_{a_1 a_2 \dots a_{d-1} a_d}\, S^{a_1 a_2} \cdot S^{a_3 a_4} \cdots S^{a_{d-1} a_d}$, with α chosen so that $\Gamma^{(d)} = \pm 1$.

Table 11.1: $2^d = 64$ "eigenvectors" of the Cartan subalgebra of the Clifford odd and even algebras — the superposition of odd and even products of γ^a's — in $d = (5+1)$-dimensional space are presented, divided into four groups. The first group, odd I, is chosen to represent "basis vectors", named $\hat{b}^{m\dagger}_f$, appearing in $2^{\frac{d}{2}-1} = 4$ "families" ($f = 1, 2, 3, 4$), each "family" with $2^{\frac{d}{2}-1} = 4$ "family" members ($m = 1, 2, 3, 4$). The second group, odd II, contains Hermitian conjugated partners of the first group for each family separately, $\hat{b}^m_f = (\hat{b}^{m\dagger}_f)^\dagger$. Either odd I or odd II are products of an odd number of nilpotents, the rest are projectors. The "family" quantum numbers of $\hat{b}^{m\dagger}_f$, that is the eigenvalues of $(\tilde{S}^{03}, \tilde{S}^{12}, \tilde{S}^{56})$, are for the first *odd I* group written above each "family", the quantum numbers of the members (S^{03}, S^{12}, S^{56}) are written in the last three columns. For the Hermitian conjugated partners of *odd I*, presented in the group *odd II*, the quantum numbers (S^{03}, S^{12}, S^{56}) are presented above each group of the Hermitian conjugated partners, the last three columns tell eigenvalues of $(\tilde{S}^{03}, \tilde{S}^{12}, \tilde{S}^{56})$. The two groups with the even number of γ^a's, *even I* and *even II*, each has their Hermitian conjugated partners within its own group, have the quantum numbers f, that is the eigenvalues of $(\tilde{S}^{03}, \tilde{S}^{12}, \tilde{S}^{56})$, written above column of four members, the quantum numbers of the members, (S^{03}, S^{12}, S^{56}), are written in the last three columns.

"basis vectors" $(\tilde{S}^{03}, \tilde{S}^{12}, \tilde{S}^{56})$	m →	f = 1 $(\frac{i}{2}, -\frac{1}{2}, -\frac{1}{2})$	f = 2 $(-\frac{i}{2}, -\frac{1}{2}, \frac{1}{2})$	f = 3 $(-\frac{i}{2}, \frac{1}{2}, -\frac{1}{2})$	f = 4 $(\frac{i}{2}, \frac{1}{2}, \frac{1}{2})$	S^{03}	S^{12}	S^{56}
odd I $\hat{b}^{m\dagger}_f$	1	03 12 56 (+i)[+][+]	03 12 56 [+i][+](+)	03 12 56 [+i](+)[+]	03 12 56 (+i)(+)(+)	$\frac{i}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
	2	[−i](−)[+]	(−i)(−)(+)	(−i)[−][+]	[−i][−](+)	$-\frac{i}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$
	3	[−i][+](−)	(−i)[+][−]	(−i)(+)(−)	[−i](+)[−]	$-\frac{i}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$
	4	(+i)(−)(−)	[+i](−)[−]	[+i][−](−)	(+i)[−][−]	$\frac{i}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$
(S^{03}, S^{12}, S^{56})	→	$(-\frac{i}{2}, \frac{1}{2}, \frac{1}{2})$ 03 12 56	$(\frac{i}{2}, \frac{1}{2}, -\frac{1}{2})$ 03 12 56	$(\frac{i}{2}, -\frac{1}{2}, \frac{1}{2})$ 03 12 56	$(-\frac{i}{2}, -\frac{1}{2}, -\frac{1}{2})$ 03 12 56	$\tilde{S}^{03}$	$\tilde{S}^{12}$	$\tilde{S}^{56}$
odd II $\hat{b}^{m}_f$	1	(−i)[+][+]	[+i][+](−)	[+i](−)[+]	(−i)(−)(−)	$-\frac{i}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$
	2	[−i](+)[+]	(+i)(+)(−)	(+i)[−][+]	[−i][−](−)	$\frac{i}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$
	3	[−i][+](+)	(+i)[+][−]	(+i)(−)(+)	[−i](−)[−]	$\frac{i}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$
5	−1	−1						
	4	(−i)(+)(+)	[+i](+)[−]	[+i][−](+)	(−i)[−][−]	$-\frac{i}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
$(\tilde{S}^{03}, \tilde{S}^{12}, \tilde{S}^{56})$	→	$(-\frac{i}{2}, \frac{1}{2}, \frac{1}{2})$ 03 12 56	$(\frac{i}{2}, -\frac{1}{2}, \frac{1}{2})$ 03 12 56	$(-\frac{i}{2}, -\frac{1}{2}, -\frac{1}{2})$ 03 12 56	$(\frac{i}{2}, \frac{1}{2}, -\frac{1}{2})$ 03 12 56	S^{03}	S^{12}	S^{56}
even I ${}^{I}\mathcal{A}^m_f$	1	[+i](+)(+)	(+i)[+](+)	[+i][+][+]	(+i)(+)[+]	$\frac{i}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
	2	(−i)[−](+)	[−i](−)(+)	(−i)(−)[+]	[−i][−][+]	$-\frac{i}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$
	3	(−i)(+)[−]	[−i][+][−]	(−i)[+](−)	[−i](+)(−)	$-\frac{i}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$
	4	[+i][−][−]	(+i)(−)[−]	[+i](−)(−)	(+i)[−](−)	$\frac{i}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$
$(\tilde{S}^{03}, \tilde{S}^{12}, \tilde{S}^{56})$	→	$(\frac{i}{2}, \frac{1}{2}, \frac{1}{2})$ 03 12 56	$(-\frac{i}{2}, -\frac{1}{2}, \frac{1}{2})$ 03 12 56	$(\frac{i}{2}, -\frac{1}{2}, -\frac{1}{2})$ 03 12 56	$(-\frac{i}{2}, \frac{1}{2}, -\frac{1}{2})$ 03 12 56	S^{03}	S^{12}	S^{56}
even II ${}^{II}\mathcal{A}^m_f$	1	[−i](+)(+)	(−i)[+](+)	[−i][+][+]	(−i)(+)[+]	$-\frac{i}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
	2	(+i)[−](+)	[+i](−)(+)	(+i)(−)[+]	[+i][−][+]	$\frac{i}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$
	3	(+i)(+)[−]	[+i][+][−]	(+i)[+](−)	[+i](+)(−)	$\frac{i}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$
	4	[−i][−][−]	(−i)(−)[−]	[−i](−)(−)	(−i)[−](−)	$-\frac{i}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$

The algebraic application of ${}^{I}\hat{\mathcal{A}}^{m}_{2}$, $m = (1,2,3,4)$ on $\hat{b}^{3\dagger}_{1}$ transforms $\hat{b}^{3\dagger}_{1}$ into $\hat{b}^{m\dagger}_{1}$, $m = (1,2,3,4)$. And the algebraic application of ${}^{I}\hat{\mathcal{A}}^{m}_{1}$, $m = (1,2,3,4)$ on $\hat{b}^{4\dagger}_{1}$ transforms $\hat{b}^{4\dagger}_{1}$ into $\hat{b}^{m\dagger}_{1}$, $m = (1,2,3,4)$.

The statement of Eq. (11.24) is therefore demonstrated on the case of $d = (5+1)$. It remains to stress and illustrate in the case of $d = (5+1)$ some general properties of the Clifford even "basis vector" ${}^{I}\hat{\mathcal{A}}^{m\dagger}_{f}$ when they apply on each other. Let us denote the self adjoint member in each group of "basis vectors" of particular f as ${}^{I}\hat{\mathcal{A}}^{m_0\dagger}_{f}$. We easily see that

$$\{{}^{I}\hat{\mathcal{A}}^{m\dagger}_{f}, {}^{I}\hat{\mathcal{A}}^{m'\dagger}_{f}\}_{-} = 0\,, \quad \text{if}\,(m,m') \neq m_0\, \text{or}\, m = m_0 = m'\,, \forall f\,,$$
$${}^{I}\hat{\mathcal{A}}^{m\dagger}_{f} *_{A} {}^{I}\hat{\mathcal{A}}^{m_0\dagger}_{f} \rightarrow {}^{I}\hat{\mathcal{A}}^{m\dagger}_{f}\,, \quad \forall m\,, \forall f\,. \tag{11.27}$$

In Table 11.1 we see that in each column of either *even* ${}^{I}\hat{\mathcal{A}}^{m\dagger}_{f}$ or of *even* ${}^{II}\hat{\mathcal{A}}^{m\dagger}_{f}$ there is one self adjoint ${}^{I,II}\hat{\mathcal{A}}^{m_0\dagger}_{f}$. We also see that two "basis vectors" ${}^{I}\hat{\mathcal{A}}^{m\dagger}_{f}$ and ${}^{I}\hat{\mathcal{A}}^{m'\dagger}_{f}$ of the same f and of $(m,m') \neq m_0$ are orthogonal. We only have to take into account Eq. (19.9), which tells that

$$\overset{ab}{(k)}\overset{ab}{[k]}= 0\,, \quad \overset{ab}{[k]}\overset{ab}{(k)}=\overset{ab}{(k)}\,, \quad \overset{ab}{(k)}\overset{ab}{[-k]}=\overset{ab}{(k)}\,, \quad \overset{ab}{[k]}\overset{ab}{(-k)}= 0.$$

These relations tell us that ${}^{I}\hat{\mathcal{A}}^{1\dagger}_{4} *_{A} {}^{I}\hat{\mathcal{A}}^{2\dagger}_{3} = {}^{I}\hat{\mathcal{A}}^{1\dagger}_{3}$, what illustrates Eq. (11.23), while ${}^{I}\hat{\mathcal{A}}^{2\dagger}_{3} *_{A} {}^{I}\hat{\mathcal{A}}^{1\dagger}_{4} = {}^{I}\hat{\mathcal{A}}^{2\dagger}_{4}$ illustrating Eq. (11.22), while ${}^{I}\hat{\mathcal{A}}^{1\dagger}_{3} *_{A} {}^{I}\hat{\mathcal{A}}^{2\dagger}_{4} = 0$ illustrates Eq. (11.21).

Table 11.2 presents the Clifford even "basis vectors" ${}^{I}\hat{\mathcal{A}}^{m\dagger}_{f}$ for $d = (5+1)$ with the properties: **i.**They are products of an even number of nilpotents, $\overset{ab}{(k)}$, with the rest up to $\frac{d}{2}$ of projectors, $\overset{ab}{[k]}$. **ii.** Nilpotents and projectors are eigenvectors of the Cartan subalgebra members $\mathcal{S}^{ab} = S^{ab} + \tilde{S}^{ab}$, Eq. (19.4), carrying the integer eigenvalues of the Cartan subalgebra members.
iii. They have their Hetmitian conjugated partners within the same group of ${}^{I}\hat{\mathcal{A}}^{m\dagger}_{f}$ with $2^{\frac{d}{2}-1} \times 2^{\frac{d}{2}-1}$ members.
iv. They have properties of the boson gauge fields. When applying on the Clifford odd "basis vectors" (offering the description of the fermion fields) they transform the Clifford odd "basis vectors" into another Clifford odd "basis vectors", transferring to the Clifford odd "basis vectors" the integer spins with respect to the $SO(d-1,1)$ group, while with respect to subgroups of the $SO(d-1,1)$ group they transfer appropriate superposition of the eigenvalues (manifesting the properties of the adjoint representations of the corresponding groups).

To demonstrate that the Clifford even "basis vectors" have properties of the gauge fields of the corresponding Clifford odd "basis vectors" we study properties of the $SU(3) \times U(1)$ subgroups of the Clifford odd and Clifford even "basis vectors".
We present in Eqs. (11.28, 11.29) the superposition of members of Cartan subalgebra, Eq. (19.4), for S^{ab} for the Clifford odd "basis vectors", for the subgroups $SO(3,1) \times U(1)$ $(N^{3}_{\pm}, \tau)$ and for the subgroups $SU(3) \times U(1)$: $(\tau', \tau^{3}, \tau^{8})$. The same relations can be used

also for the corresponding operators determining the "family" quantum numbers ($\tilde{N}^3_{\pm}, \tilde{\tau}$) of the Clifford odd "basis vectors', if S^{ab}'s are replaced by $\tilde{S}^{ab}$'s. For the Clifford even objects $\mathcal{S}^{ab} (= S^{ab} + \tilde{S}^{ab})$ must replace S^{ab}.

$$N^3_{\pm} (= N^3_{(L,R)}) := \frac{1}{2} (S^{12} \pm i S^{03}) , \quad \tau = S^{56} , \tag{11.28}$$

$$\begin{aligned} \tau^3 :=& \; \frac{1}{2} \, (-S^{1\,2} - i S^{0\,3}) , \qquad \tau^8 = \frac{1}{2\sqrt{3}} (-i S^{0\,3} + S^{1\,2} - 2 S^{5\,6}) , \\ \tau' =& \; -\frac{1}{3} (-i S^{0\,3} + S^{1\,2} + S^{5\,6}) . \end{aligned} \tag{11.29}$$

Let us, for example, algebraically apply ${}^{I}\hat{\mathcal{A}}^2_3$ $(\equiv \overset{03}{(-i)} \overset{12}{(-)} \overset{56}{[+]})$, denoted by ⊙⊙ on Table 11.2, carrying ($\tau^3 = 0, \tau^8 = -\frac{1}{\sqrt{3}}, \tau' = \frac{2}{3}$), represented also on Fig. 11.2 by ⊙⊙, on the Clifford odd "basis vector" $\hat{b}^{1\dagger}_1 (\equiv \overset{03}{(+i)} \overset{12}{(+)} \overset{56}{(+)})$, presented on Table 11.1, with ($\tau^3 = 0, \tau^8 = 0, \tau' = -\frac{1}{2}$), as we can calculate using Eq. (11.29) and which is represented on Fig. 11.1 by a square as a singlet. ${}^{I}\hat{\mathcal{A}}^2_3$ transforms $\hat{b}^{1\dagger}_1$ (by transferring to $\hat{b}^{1\dagger}_1$ ($\tau^3 = 0, \tau^8 = -\frac{1}{\sqrt{3}}, \tau' = \frac{2}{3}$)) to $\hat{b}^{1\dagger}_2$ with ($\tau^3 = 0, \tau^8 = -\frac{1}{\sqrt{3}}, \tau' = \frac{1}{6}$), belonging on Fig. 11.1 to the triplet, denoted by ◯. The corresponding gauge fields, presented on Fig. 11.2, if belonging to the sextet, would transform the triplet of quarks among themselves.

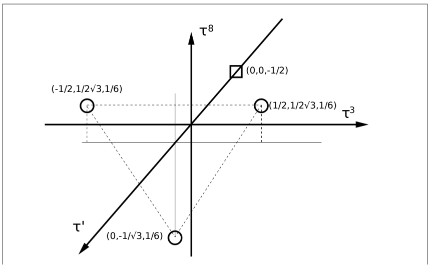

Fig. 11.1: Representations of the subgroups $SU(3)$ and $U(1)$ of the group $SO(5,1)$, the properties of which appear in Table 11.1, are presented. (τ^3, τ^8 and τ') can be calculated if using Eqs.(11.28, 11.29). On the abscissa axis, on the ordinate axis and on the third axis the eigenvalues of the superposition of the three Cartan subalgebra members, τ^3, τ^8, τ' are presented. One notices one triplet, denoted by ◯ with the values $\tau' = \frac{1}{6}$, ($\tau^3 = -\frac{1}{2}, \tau^8 = \frac{1}{2\sqrt{3}}, \tau' = \frac{1}{6}$), ($\tau^3 = \frac{1}{2}, \tau^8 = \frac{1}{2\sqrt{3}}, \tau' = \frac{1}{6}$), ($\tau^3 = 0, \tau^8 = -\frac{1}{\sqrt{3}}, \tau' = \frac{1}{6}$), respectively, and one singlet denoted by the square. ($\tau^3 = 0, \tau^8 = 0, \tau' = -\frac{1}{2}$). The triplet and the singlet appear in four families.

In the case of the group $SO(6)$ ($SO(5,1)$indeed), manifesting as $SU(3) \times U(1)$ and representing the $SU(3)$ colour group and $U(1)$ the "fermion" quantum number, embedded into $SO(13,1)$ the triplet would represent quarks and the singlet leptons. The corresponding gauge of the fields, presented on Fig. 11.2, if belonging to the sextet, would transform the triplet of quarks among themselves, changing the colour and leaving the "fermion" quantum number equal to $\frac{1}{6}$.

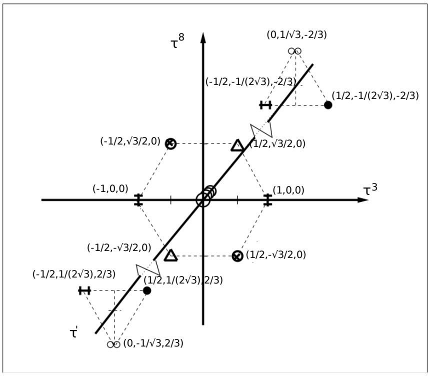

Fig. 11.2: The Clifford even "basis vectors" ${}^{I}\hat{\mathcal{A}}^{m}_{f}$, in the case that $d = (5+1)$, are presented with respect to the eigenvalues of the commuting operators of the subgroups $SU(3)$ and $U(1)$ of the group $SO(5,1)$: $\tau^3 = \frac{1}{2}\,(-\mathcal{S}^{12} - i\mathcal{S}^{03})$, $\tau^8 = \frac{1}{2\sqrt{3}}(\mathcal{S}^{12} - i\mathcal{S}^{03} - 2\mathcal{S}^{56})$, $\tau' = -\frac{1}{3}(\mathcal{S}^{12} - i\mathcal{S}^{03} + \mathcal{S}^{56})$. Their properties appear in Table 11.2. The abscissa axis carries the eigenvalues of τ^3, the ordinate axis of τ^8 and the third axis the eigenvalues of τ', One notices four singlets with $(\tau^3 = 0, \tau^8 = 0, \tau' = 0)$, denoted by $\bigcirc$, representing four self adjoint Clifford even "basis vectors" ${}^{I}\hat{\mathcal{A}}^{m}_{f}$, one sextet of three pairs with $\tau' = 0$, Hermitian conjugated to each other, denoted by $\triangle$ (with $(\tau' = 0, \tau^3 = -\frac{1}{2}, \tau^8 = -\frac{3}{2\sqrt{3}})$ and $(\tau' = 0, \tau^3 = \frac{1}{2}, \tau^8 = \frac{3}{2\sqrt{3}})$), respectively, by $\ddagger$ (with $(\tau' = 0, \tau^3 = -1, \tau^8 = 0)$ and $(\tau' = 0, \tau^3 = 1, \tau^8 = 0)$, respectively, and by $\otimes$ (with $(\tau' = 0, \tau^3 = \frac{1}{2}, \tau^8 = -\frac{3}{2\sqrt{3}})$ and $(\tau' = 0, \tau^3 = -\frac{1}{2}, \tau^8 = \frac{3}{2\sqrt{3}})$), respectively, and one triplet, denoted by $\star\star$ with $(\tau' = \frac{2}{3}, \tau^3 = \frac{1}{2}, \tau^8 = \frac{1}{2\sqrt{3}})$, by $\bullet$ with $(\tau' = \frac{2}{3}, \tau^3 = -\frac{1}{2}, \tau^8 = \frac{1}{2\sqrt{3}})$, and by $\odot\odot$ with $(\tau' = \frac{2}{3}, \tau^3 = 0, \tau^8 = -\frac{1}{\sqrt{3}})$, as well as one antitriplet, Hermitian conjugated to the triplet, denoted by $\star\star$ with $(\tau' = -\frac{2}{3}, \tau^3 = -\frac{1}{2}, \tau^8 = -\frac{1}{2\sqrt{3}})$, by $\bullet$ with $(\tau' = -\frac{2}{3}, \tau^3 = \frac{1}{2}, \tau^8 = -\frac{1}{2\sqrt{3}})$, and by $\odot\odot$ with $(\tau' = -\frac{2}{3}, \tau^3 = 0, \tau^8 = \frac{1}{\sqrt{3}})$.

We can see that ${}^{I}\hat{\mathcal{A}}^{m\dagger}_{3}$ with $(m = 2,3,4)$, if applied on the $SU(3)$ singlet $\hat{b}^{1\dagger}_{1}$ with $(\tau' = -\frac{1}{2}, \tau^3 = 0, \tau^8 = 0)$, transforms it to $\hat{b}^{m=2,3,4)\dagger}_{1}$, respectively, which are members of the $SU(3)$ triplet. All these Clifford even "basis vectors" have τ' equal to $\frac{2}{3}$, changing correspondingly $\tau' = -\frac{1}{2}$ into $\tau' = \frac{1}{6}$ and bringing the needed values of τ^3 and τ^8.
In Table 11.2 we find $(6+4)$ Clifford even "basis vectors" ${}^{I}\hat{\mathcal{A}}^{m\dagger}_{f}$ with $\tau‘ = 0$. Six of them are Hermitian conjugated to each other — the Hermitian conjugated partners are denoted by the same geometric figure on the third column. Four of them are self adjoint and correspondingly with $(\tau' = 0, \tau^3 = 0, \tau^8 = 0)$, denoted in the third column of Table 11.2 by $\bigcirc$. The rest 6 Clifford even "basis vectors" belong to one

triplet with $\tau' = \frac{2}{3}$ and (τ^3, τ^8) equal to $[(0, -\frac{1}{\sqrt{3}}), (-\frac{1}{2}, \frac{1}{2\sqrt{3}}), (\frac{1}{2}, \frac{1}{2\sqrt{3}})]$ and one antitriplet with $\tau' = -\frac{2}{3}$ and $((\tau^3, \tau^8)$ equal to $[(-\frac{1}{2}, -\frac{1}{2\sqrt{3}}), (\frac{1}{2}, -\frac{1}{2\sqrt{3}}), (0, \frac{1}{\sqrt{3}})]$.
Each triplet has Hermitian conjugated partner in antitriplet and opposite. In Table 11.2 the Hermitian conjugated partners of the triplet and antitriplet are denoted by the same signum: $({}^{I}\hat{\mathcal{A}}^{1\dagger}_{1}, {}^{I}\hat{\mathcal{A}}^{4\dagger}_{3})$ by $\star\star$, $({}^{I}\hat{\mathcal{A}}^{1\dagger}_{2}, {}^{I}\hat{\mathcal{A}}^{3\dagger}_{3})$ by $\bullet$, and $({}^{I}\hat{\mathcal{A}}^{2\dagger}_{3}, {}^{I}\hat{\mathcal{A}}^{1\dagger}_{4})$ by $\odot\odot$.
The octet and the two triplets are presented in Fig. 11.2.

Table 11.2: The Clifford even "basis vectors" ${}^{I}\hat{\mathcal{A}}^{m\dagger}_{f}$, each of them is the product of projectors and an even number of nilpotents, and each is the eigenvector of all the Cartan subalgebra members, $\mathcal{S}^{03}$, $\mathcal{S}^{12}$, $\mathcal{S}^{56}$, Eq. (19.4), are presented for $d = (5+1)$-dimensional case. Indexes m and f determine $2^{\frac{d}{2}-1} \times 2^{\frac{d}{2}-1}$ different members ${}^{I}\hat{\mathcal{A}}^{m\dagger}_{f}$. In the third column the "basis vectors" ${}^{I}\hat{\mathcal{A}}^{m\dagger}_{f}$ which are Hermitian conjugated partners to each other (and can therefore annihilate each other) are pointed out with the same symbol. For example, with $\star\star$ are equipped the first member with $m = 1$ and $f = 1$ and the last member of $f = 3$ with $m = 4$. The sign $\bigcirc$ denotes the Clifford even "basis vectors" which are self adjoint $({}^{I}\hat{\mathcal{A}}^{m\dagger}_{f})^{\dagger}$ $= {}^{I}\hat{\mathcal{A}}^{m'\dagger}_{f`}$. It is obvious that $^{\dagger}$ has no meaning, since ${}^{I}\hat{\mathcal{A}}^{m\dagger}_{f}$ are self adjoint or are Hermitian conjugated partner to another ${}^{I}\hat{\mathcal{A}}^{m'\dagger}_{f`}$. This table represents also the eigenvalues of the three commuting operators $\mathcal{N}^{3}_{L,R}$ and $\mathcal{S}^{56}$ of the subgroups $SU(2) \times SU(2) \times U(1)$ of the group $SO(5,1)$ and the eigenvalues of the three commuting operators τ^3, τ^8 and τ' of the subgroups $SU(3) \times U(1)$.

f	m	*	${}^{I}\hat{\mathcal{A}}^{m\dagger}_{f}$	$\mathcal{S}^{03}$	$\mathcal{S}^{12}$	$\mathcal{S}^{56}$	$\mathcal{N}^{3}_{L}$	$\mathcal{N}^{3}_{R}$	τ^3	τ^8	τ'
I	1	$\star\star$	$\stackrel{03}{[+i]}\stackrel{12}{(+)}\stackrel{56}{(+)}$	0	1	1.	$\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2\sqrt{3}}$	$-\frac{2}{3}$
	2	$\triangle$	$\stackrel{03}{(-i)}\stackrel{12}{[-]}\stackrel{56}{(+)}$	$-i$	0	1	$\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{3}{2\sqrt{3}}$	0
	3	$\ddagger$	$\stackrel{03}{(-i)}\stackrel{12}{(+)}\stackrel{56}{[-]}$	$-i$	1	0	1	0	-1	0	0
	4	$\bigcirc$	$\stackrel{03}{[+i]}\stackrel{12}{[-]}\stackrel{56}{[-]}$	0	0	0	0	0	0	0	0
II	1	$\bullet$	$\stackrel{03}{(+i)}\stackrel{12}{[+]}\stackrel{56}{(+)}$	i	0	1	$-\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2\sqrt{3}}$	$-\frac{2}{3}$
	2	$\otimes$	$\stackrel{03}{[-i]}\stackrel{12}{(-)}\stackrel{56}{(+)}$	0	-1	1	$-\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	$-\frac{3}{2\sqrt{3}}$	0
	3	$\bigcirc$	$\stackrel{03}{[-i]}\stackrel{12}{[+]}\stackrel{56}{[-]}$	0	0	0	0	0	0	0	0
	4	$\ddagger$	$\stackrel{03}{(+i)}\stackrel{12}{(-)}\stackrel{56}{[-]}$	i	-1	0	-1	0	1	0	0
III	1	$\bigcirc$	$\stackrel{03}{[+i]}\stackrel{12}{[+]}\stackrel{56}{[+]}$	0	0	0	0	0	0	0	0
	2	$\odot\odot$	$\stackrel{03}{(-i)}\stackrel{12}{(-)}\stackrel{56}{[+]}$	$-i$	-1	0	0	-1	0	$-\frac{1}{\sqrt{3}}$	$\frac{2}{3}$
	3	$\bullet$	$\stackrel{03}{(-i)}\stackrel{12}{[+]}\stackrel{56}{(-)}$	$-i$	0	-1	$\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2\sqrt{3}}$	$\frac{2}{3}$
	4	$\star\star$	$\stackrel{03}{[+i]}\stackrel{12}{(-)}\stackrel{56}{(-)}$	0	-1	-1	$-\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2\sqrt{3}}$	$\frac{2}{3}$
IV	1	$\odot\odot$	$\stackrel{03}{(+i)}\stackrel{12}{(+)}\stackrel{56}{[+]}$	i	1	0	0	1	0	$\frac{1}{\sqrt{3}}$	$-\frac{2}{3}$
	2	$\bigcirc$	$\stackrel{03}{[-i]}\stackrel{12}{[-]}\stackrel{56}{[+]}$	0	0	0	0	0	0	0	0
	3	$\otimes$	$\stackrel{03}{[-i]}\stackrel{12}{(+)}\stackrel{56}{(-)}$	0	1	-1	$\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{3}{2\sqrt{3}}$	0
	4	$\triangle$	$\stackrel{03}{(+i)}\stackrel{12}{[-]}\stackrel{56}{(-)}$	i	0	-1	$-\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{3}{2\sqrt{3}}$	0

Fig. 11.2 represents the $2^{\frac{d}{2}-1} \times 2^{\frac{d}{2}-1}$ members ${}^{I}\hat{\mathcal{A}}^{m}_{f}$ of the Clifford even "basis vectors" for the case that $d = (5+1)$. The properties of ${}^{I}\hat{\mathcal{A}}^{m}_{f}$ are presented also in Table 11.2. There are in this case again 16 members. Manifesting the structure of subgroups $SU(3) \times U(1)$ of the group $SO(5,1)$ they are represented as eigenvectors of the superposition of the Cartan subalgebra members $(\mathcal{S}^{03}, \mathcal{S}^{12}, \mathcal{S}^{56})$, that is with $\tau^3 = \frac{1}{2}(-\mathcal{S}^{12} - i\mathcal{S}^{03})$, $\tau^8 = \frac{1}{2\sqrt{3}}(\mathcal{S}^{12} - i\mathcal{S}^{03} - 2\mathcal{S}^{56})$, and $\tau' = -\frac{1}{3}(\mathcal{S}^{12} - i\mathcal{S}^{03} + \mathcal{S}^{56})$. There are four self adjoint Clifford even "basis vectors" with $(\tau^3 = 0, \tau^8 = 0, \tau' = 0)$, one sextet of three pairs Hermitian conjugated to each other, one triplet and one antitriplet with the members of the triplet Hermitian conjugated to the corresponding members of the antitriplet and opposite. These 16 members of the Clifford even "basis vectors" ${}^{I}\hat{\mathcal{A}}^{m}_{f}$ are the boson "partners" of the Clifford odd "basis vectors" $\hat{b}^{m\dagger}_{f}$, presented in Fig. 11.1 for one of four families, anyone. The reader can check that the algebraic application of ${}^{I}\hat{\mathcal{A}}^{m}_{f}$, belonging to the triplet, transforms the Clifford odd singlet, denoted on Fig. 11.1 by a square, to one of the members of the triplet, denoted on Fig. 11.1 by the circle ◯.
Looking at the boson fields ${}^{I}\hat{\mathcal{A}}^{m\dagger}_{f}$ from the point of view of subgroups $SU(3) \times U(1)$ of the group $SO(5+1)$ we will recognize in the part of fields forming the octet the colour gauge fields of quarks and leptons and antiquarks and antileptons.

11.2.3 Second quantized fermion and boson fields the internal spaces of which are described by the Clifford basis vectors.

We learned in the previous subsection that in even dimensional spaces ($d = 2(2n+1)$ or $d = 4n$) the Clifford odd and the Clifford even "basis vectors", which are the superposition of the Clifford odd and the Clifford even products of γ^a's, respectively, offer the description of the internal spaces of fermion and boson fields.
The Clifford odd algebra offers $2^{\frac{d}{2}-1}$ "basis vectors" $\hat{b}^{m\dagger}_{f}$, appearing in $2^{\frac{d}{2}-1}$ families (with the family quantum numbers determined by $\tilde{S}^{ab} = \frac{i}{2}\{\tilde{\gamma}^a, \tilde{\gamma}^b\}_-$), which together with their $2^{\frac{d}{2}-1} \times 2^{\frac{d}{2}-1}$ Hermitian conjugated partners $\hat{b}^{m}_{f}$ fulfil the postulates for the second quantized fermion fields, Eq. (11.17) in this paper, Eq.(26) in Ref. [5], explaining the second quantization postulates of Dirac.
The Clifford even algebra offers $2^{\frac{d}{2}-1} \times 2^{\frac{d}{2}-1}$ "basis vectors" of ${}^{I}\hat{\mathcal{A}}^{m\dagger}_{f}$ (and the same number of ${}^{II}\hat{\mathcal{A}}^{m\dagger}_{f}$) with the properties of the second quantized boson fields manifesting as the gauge fields of fermion fields described by the Clifford odd "basis vectors" $\hat{b}^{m\dagger}_{f}$.
The Clifford odd and the Clifford even "basis vectors" are chosen to be products of nilpotents, $\stackrel{ab}{(k)}$ (with the odd number of nilpotents if describing fermions and the even number of nilpotents if describing bosons), and projectors, $\stackrel{ab}{[k]}$. Nilpotents and projectors are (chosen to be) eigenvectors of the Cartan subalgebra members of the Lorentz algebra in the internal space of S^{ab} for the Clifford odd "basis vectors" and of $\mathcal{S}^{ab} (= S^{ab} + \tilde{S}^{ab})$ for the Clifford even "basis vectors".

To define the creation operators, either for fermions or for bosons besides the "basis vectors" defining the internal space of fermions and bosons also the basis in ordinary space in momentum or coordinate representation is needed. Here Ref. [5], Subsect. 3.3 and App. J is overviewed.

Let us introduce the momentum part of the single particle states. The longer version is presented in Ref. [5] in Subsect. 3.3 and in App. J.

$$\begin{aligned}
|\vec{p}> &= \hat{b}^{\dagger}_{\vec{p}}\,|\,0_p\;>\,, \quad <\vec{p}\,| =< 0_p\,|\,\hat{b}_{\vec{p}}\,,\\
<\vec{p}\,|\,\vec{p}'> &= \delta(\vec{p}-\vec{p}') =< 0_p\,|\hat{b}_{\vec{p}}\;\hat{b}^{\dagger}_{\vec{p}'}|\,0_p\;>\,,\\
&\text{leading to}\\
\hat{b}_{\vec{p'}}, \hat{b}^{\dagger}_{\vec{p}} &= \delta(\vec{p'}-\vec{p})\,,
\end{aligned} \tag{11.30}$$

with the normalization $<\;0_p\,|\,0_p\;>= 1$. While the quantized operators $\hat{\vec{p}}$ and $\hat{\vec{x}}$ commute $\{\hat{p}^i\,,\hat{p}^j\}_- = 0$ and $\{\hat{x}^k\,,\hat{x}^l\}_- = 0$, it follows for $\{\hat{p}^i\,,\hat{x}^j\}_- = i\eta^{ij}$. One correspondingly finds

$$\begin{aligned}
&<\vec{p}\,|\,\vec{x}> = <0_{\vec{p}}\,|\,\hat{b}_{\vec{p}}\;\hat{b}^{\dagger}_{\vec{x}}|0_{\vec{x}}\;>= (<0_{\vec{x}}\,|\,\hat{b}_{\vec{x}}\;\hat{b}^{\dagger}_{\vec{p}}\,|0_{\vec{p}}\;>)^{\dagger}\\
&\{\hat{b}^{\dagger}_{\vec{p}}\,,\,\hat{b}^{\dagger}_{\vec{p}\,'}\}_- = 0\,, \qquad \{\hat{b}_{\vec{p}},\,\hat{b}_{\vec{p}\,'}\}_- = 0\,, \qquad \{\hat{b}_{\vec{p}},\,\hat{b}^{\dagger}_{\vec{p}\,'}\}_- = 0\,,\\
&\{\hat{b}^{\dagger}_{\vec{x}},\,\hat{b}^{\dagger}_{\vec{x}\,'}\}_- = 0\,, \qquad \{\hat{b}_{\vec{x}},\,\hat{b}_{\vec{x}\,'}\}_- = 0\,, \qquad \{\hat{b}_{\vec{x}},\,\hat{b}^{\dagger}_{\vec{x}\,'}\}_- = 0\,,\\
&\{\hat{b}_{\vec{p}},\,\hat{b}^{\dagger}_{\vec{x}}\}_- = e^{i\vec{p}\cdot\vec{x}}\frac{1}{\sqrt{(2\pi)^{d-1}}}\,, \qquad ,\{\hat{b}_{\vec{x}},\,\hat{b}^{\dagger}_{\vec{p}}\}_- = e^{-i\vec{p}\cdot\vec{x}}\frac{1}{\sqrt{(2\pi)^{d-1}}}\,,
\end{aligned} \tag{11.31}$$

.

The internal space of either fermion or boson fields has the finite number of "basis vectors", $2^{\frac{d}{2}-1}\times 2^{\frac{d}{2}-1}$, the momentum basis is continuously infinite.

The creation operators for either fermions or bosons must be a tensor product, $*_T$, of both contributions, the "basis vectors" describing the internal space of fermions or bosons and the basis in ordinary, momentum or coordinate, space.

The creation operators for a free massless fermion of the energy $p^0 = |\vec{p}|$, belonging to a family f and to a superposition of family members m applying on the vacuum state $|\psi_{oc}>\;*_T\,|0_{\vec{p}}>$ can be written as ([5], Subsect.3.3.2, and the references therein)

$$\hat{b}^{s\dagger}_{f}(\vec{p}) \;=\; \sum_{m} c^{sm}{}_{f}(\vec{p})\,\hat{b}^{\dagger}_{\vec{p}}\;*_T\;\hat{b}^{m\dagger}_{f}\;, \tag{11.32}$$

where the vacuum state for fermions $|\psi_{oc}>\;*_T\,|0_{\vec{p}}>$ includes both spaces, the internal part, Eq.(19.8), and the momentum part, Eq. (11.30) (in a tensor product for a starting single particle state with zero momentum, from which one obtains the other single fermion states of the same "basis vector" by the operator $\hat{b}^{\dagger}_{\vec{p}}$ which pushes the momentum by an amount $\vec{p}$ [10]).

[10] The creation operators and their Hermitian conjugated partners annihilation operators in the coordinate representation can be read in [5] and the references therein: $\hat{b}^{s\dagger}_{f}(\vec{x},x^0) = \sum_m \hat{b}^{m\dagger}_{f}\int^{+\infty}_{-\infty}\frac{d^{d-1}p}{(\sqrt{2\pi})^{d-1}}\,c^{ms}{}_{f}\;(\vec{p})\;\hat{b}^{\dagger}_{\vec{p}}\;e^{-i(p^0x^0-\varepsilon\vec{p}\cdot\vec{x})}$ ([5], subsect. 3.3.2., Eqs. (55,57,64) and the references therein).

The creation operators fulfil the anticommutation relations for the second quantized fermion fields

$$\begin{aligned}
\{\hat{b}^{s'}_{f'}(\vec{p'})\,,\,\hat{b}^{s\dagger}_{f}(\vec{p})\}_{+}\,|\psi_{oc}>|0_{\vec{p}}> &= \delta^{ss'}\delta_{ff'}\,\delta(\vec{p}'-\vec{p})\,|\psi_{oc}>|0_{\vec{p}}>\,,\\
\{\hat{b}^{s'}_{f'}(\vec{p'})\,,\,\hat{b}^{s}_{f}(\vec{p})\}_{+}\,|\psi_{oc}>|0_{\vec{p}}> &= 0\,.\,|\psi_{oc}>|0_{\vec{p}}>\,,\\
\{\hat{b}^{s'\dagger}_{f'}(\vec{p'})\,,\,\hat{b}^{s\dagger}_{f}(\vec{p})\}_{+}\,|\psi_{oc}>|0_{\vec{p}}> &= 0\,.\,|\psi_{oc}>|0_{\vec{p}}>\,,\\
\hat{b}^{s\dagger}_{f}(\vec{p})\,|\psi_{oc}>|0_{\vec{p}}> &= |\psi^{s}_{f}(\vec{p})>\\
\hat{b}^{s}_{f}(\vec{p})\,|\psi_{oc}>|0_{\vec{p}}> &= 0\,.\,|\psi_{oc}>|0_{\vec{p}}>\\
|p^{0}| &= |\vec{p}|\,.
\end{aligned} \tag{11.33}$$

The creation operators $\hat{b}^{s\dagger}_{f}(\vec{p}))$ and their Hermitian conjugated partners annihilation operators $\hat{b}^{s}_{f}(\vec{p})$, creating and annihilating the single fermion states, respectively, fulfil when applying on the vacuum state, $|\psi_{oc}> *_{T}|0_{\vec{p}}>$, the anticommutation relations for the second quantized fermions, postulated by Dirac (Ref. [5], Subsect. 3.3.1, Sect. 5). [11]

To write the creation operators for boson fields we must take into account that boson gauge fields have the space index α, describing the α component of the boson field in the ordinary space [12]. We therefore add the space index α as follows

$$ {}^{I}\hat{\mathcal{A}}^{m\dagger}_{f\alpha}(\vec{p}) = \hat{b}^{\dagger}_{\vec{p}} *_{T} \mathcal{C}^{m}{}_{f\alpha}\,{}^{I}\hat{\mathcal{A}}^{m\dagger}_{f}\,. \tag{11.34}$$

We treat free massless bosons of momentum $\vec{p}$ and energy $p^{0} = |\vec{p}|$ and of particular "basis vectors" ${}^{I}\hat{\mathcal{A}}^{m\dagger}_{f}$'s which are eigenvectors of all the Cartan subalgebra members [13], $\mathcal{C}^{m}{}_{f\alpha}$ carry the space index α of the boson field. Creation operators operate on the vacuum state $|\psi_{oc_{ev}}> *_{T}|0_{\vec{p}}>$ with the internal space part just a constant, $|\psi_{oc_{ev}}>=|\,1>$, and for a starting single boson state with a zero momentum from which one obtains the other single boson states with the same "basis vector" by the operators $\hat{b}^{\dagger}_{\vec{p}}$ which push the momentum by an amount $\vec{p}$, making also $\mathcal{C}^{m}{}_{f\alpha}$ depending on $\vec{p}$.
For the creation operators for boson fields in a coordinate representation we find using Eqs. (11.30, 11.31)

$$ {}^{I}\hat{\mathcal{A}}^{m\dagger}_{f\alpha}(\vec{x},x^{0}) = \int_{-\infty}^{+\infty} \frac{d^{d-1}p}{(\sqrt{2\pi})^{d-1}}\,{}^{I}\hat{\mathcal{A}}^{m\dagger}_{f\alpha}(\vec{p})\,e^{-i(p^{0}x^{0}-\varepsilon\vec{p}\cdot\vec{x})}|_{p^{0}=|\vec{p}|}\,. \tag{11.35}$$

[11] The anticommutation relations of Eq. (11.33) are valid also if we replace the vacuum state, $|\psi_{oc}>|0_{\vec{p}}>$, by the Hilbert space of Clifford fermions generated by the tensor product multiplication, $*_{T_{H}}$, of any number of the Clifford odd fermion states of all possible internal quantum numbers and all possible momenta (that is of any number of $\hat{b}^{s\,\dagger}_{f}(\vec{p})$ of any $(s, f, \vec{p})$), Ref. ([5], Sect. 5.).

[12] In the *spin-charge-family* theory the Higgs's scalars origin in the boson gauge fields with the vector index $(7, 8)$, Ref. ([5], Sect. 7.4.1, and the references therein).

[13] In general the energy eigenstates of bosons are in superposition of ${}^{I}\hat{\mathcal{A}}^{m\dagger}_{f}$. One example, which uses the superposition of the Cartan subalgebra eigenstates manifesting the $SU(3) \times U(1)$ subgroups of the group $SO(6)$, is presented in Fig. 11.2.

To understand what new does the Clifford algebra description of the internal space of fermion and boson fields, Eqs. (11.34, 11.35, 11.32), bring to our understanding of the second quantized fermion and boson fields and what new can we learn from this offer, we need to relate $\sum_{ab} c^{ab}\omega_{ab\alpha}$ and $\sum_{mf} {}^{I}\hat{\mathcal{A}}^{m\dagger}_{f}\mathcal{C}^{mf}_{\alpha}$, recognizing that ${}^{I}\hat{\mathcal{A}}^{m\dagger}_{f}\mathcal{C}^{mf}_{\alpha}$ are eigenstates of the Cartan subalgebra members, while $\omega_{ab\alpha}$ are not.
The gravity fields, the vielbeins and the two kinds of the spin connection fields, $f^{a}{}_{\alpha}$, $\omega_{ab\alpha}$, $\tilde{\omega}_{ab\alpha}$, respectively, are in the *spin-charge-family* theory (unifying spins, charges and families of fermions and offering not only the explanation for all the assumptions of the *standard model* but also for the increasing number of phenomena observed so far) the only boson fields in $d = (13+1)$, observed in $d = (3+1)$ besides as gravity also as all the other boson fields with the Higgs's scalars included [27]. We therefore need to relate

$$\{\frac{1}{2}\sum_{ab} S^{ab}\,\omega_{ab\alpha}\}\sum_{m}\beta^{mf}\,\hat{b}^{m\dagger}_{f}(\vec{p})\text{ relate to }\{\sum_{m'f'}{}^{I}\hat{\mathcal{A}}^{m'\dagger}_{f'}\,\mathcal{C}^{m'f'}_{\alpha}\}\sum_{m}\beta^{mf}\,\hat{b}^{m\dagger}_{f}(\vec{p})\,,$$
$$\forall f \text{ and } \forall\,\beta^{mf}\,,$$
$$\mathcal{S}^{cd}\sum_{ab}(c^{ab}{}_{mf}\,\omega_{ab\alpha})\text{ relate to }\mathcal{S}^{cd}\,({}^{I}\hat{\mathcal{A}}^{m\dagger}_{f}\,\mathcal{C}^{mf}_{\alpha})\,,$$
$$\forall\,(m,f),$$
$$\forall\text{ Cartan subalgebra members } \mathcal{S}^{cd} \tag{11.36}$$

Let be repeated that ${}^{I}\hat{\mathcal{A}}^{m\dagger}_{f}$ are chosen to be the eigenvectors of the Cartan subalgebra members, Eq. (19.4). Correspondingly we can relate a particular ${}^{I}\hat{\mathcal{A}}^{m\dagger}_{f}\mathcal{C}^{mf}_{\alpha}$ with such a superposition of $\omega_{ab\alpha}$'s which is the eigenvector with the same values of the Cartan subalgebra members as there is a particular ${}^{I}\hat{\mathcal{A}}^{m\dagger}_{f}\mathcal{C}^{mf}_{\alpha}$. We can do this in two ways:
i. Using the first relation in Eq. (11.36). On the left hand side of this relation S^{ab}'s apply on $\hat{b}^{m\dagger}_{f}$ part of $\hat{b}^{m\dagger}_{f}(\vec{p})$. On the right hand side ${}^{I}\hat{\mathcal{A}}^{m\dagger}_{f}$ apply as well on the same "basis vector" $\hat{b}^{m\dagger}_{f}$.
ii. Using the second relation, in which $\mathcal{S}^{cd}$ apply on the left hand side on $\omega_{ab\alpha}$'s

$$\mathcal{S}^{cd}\sum_{ab}c^{ab}{}_{mf}\,\omega_{ab\alpha} = \sum_{ab}c^{ab}{}_{mf}\,i\,(\omega_{cb\alpha}\eta^{ad} - \omega_{db\alpha}\eta^{ac} + \omega_{ac\alpha}\eta^{bd} - \omega_{ad\alpha}\eta^{bc}), \tag{11.37}$$

on each $\omega_{ab\alpha}$ separately; $c^{ab}{}_{mf}$ are constants to be determined from the second relation, where on the right hand side of this relation $\mathcal{S}^{cd}(= S^{cd} + \tilde{S}^{cd})$ apply on the "basis vector" ${}^{I}\hat{\mathcal{A}}^{m\dagger}_{f}$ of the corresponding gauge field.
Let us conclude this section by pointing out that either the Clifford odd "basis vectors" $\hat{b}^{m\dagger}_{f}$ or the Clifford even "basis vectors" ${}^{i}\hat{\mathcal{A}}^{m\dagger}_{f}, i = (I, II)$ have in any even d $2^{\frac{d}{2}-1}\times 2^{\frac{d}{2}-1}$ members, while $\omega_{ab\alpha}$ as well as $\tilde{\omega}_{ab\alpha}$ have each for each α $\frac{d}{2}(d-1)$ members. It is needed to find out what new does this difference bring into the - unifying theories of the Kaluza-Klein theories are.

11.3 Short overview and achievements of *spin-charge-family* theory

The *spin-chare-family* theory [1,2,23,25,27–32] is a kind of the Kaluza-Klein theories [27,38–45] since it is built on the assumption that the dimension of space-time is $\geq (13+1)$ [14], and that the only interaction among fermions is the gravitational one (vielbeins, the gauge fields of momenta, and two kinds of the spin connection fields, the gauge fields of S^{ab} and of $\tilde{S}^{ab}$ [15]).
This theory assumes as well that the internal space of fermion and boson fields are described by the Clifford odd and Clifford even algebra, respectively [6,7] [16].
The theory is offering the explanation for all the assumptions of the *standard model*, unifying not only charges, but also spins, charges and families, [36,37,46,48,51] and consequently offering the explanation for the appearance of families of quarks and leptons and antiquarks and antileptons, of vector gauge fields [27], of Higgs's scalar field and the Yukawa couplings [28,30,32,36], for the differences in masses among quarks and leptons [46,51], for the matter-antimatter asymmetry in the universe [51], for the *dark matter* [49], making several predictions.
The *spin-charge-family* theory shares with the Kaluza-Klein like theories their weak points, like: **a.** Not yet solved the quantization problem of the gravitational field [17]. **b.** The spontaneous symmetry breaking which would at low energies manifest the observed almost massless fermions [30,32,34,39]. The spontaneously break of the starting symmetry of $SO(13+1)$ with the condensate of the two right handed neutrinos (with the family quantum numbers of the group of four families, which does not include the observed three families ([19], Table III), ([5], Table 6) bringing masses of the scale $\propto 10^{16}$ GeV or higher to all the vector and scalar gauge fields, which interact with the condensate [25] is promising to show the right way [32–34].
The scalar fields (scalar fields are the spin connection fields with the space index α higher than $(0,1,2,3)$) with the space index $(7,8)$ offer, after gaining constant non zero vacuum values, the explanation for the Higgs's scalar and the Yukawa couplings. They namely determine the mass matrices of quarks and leptons and antiquarks and antileptons. In Refs. [24,27] it is pointed out that the spin connection

[14] $d = (13+1)$ is the smallest dimension for which the subgroups of the group $SO(13,1)$ offer the description of spins and charges of fermions assumed by the *standard model* and correspondingly also of boson gauge fields.

[15] If there are no fermions present both spin connection fields are expressible with vielbeins ([5], Eq. (103)).

[16] Fermions and bosons internal spaces are assumed to be superposition of odd products of γ^a's (fermion fields) or of even products of γ^a's (boson fields) what offers the explanation for the second quantized postulates of Dirac [16]. The "basis vectors" of the internal spaces namely determine anticommutativity or commutativity of the corresponding creation and annihilation operators.

[17] The description of the internal space of fermions and bosons as superposition of odd (for fermion fields) or even (for boson fields) products of the Clifford objects γ^a's seems very promising in looking for a new way to second quantization of all fields, with gravity included, as discussed in this talk.

gauge fields do manifest in $d = (3+1)$ as the ordinary gravity and all the observed vector and scalar gauge fields.
The *spin-charge-family* theory assumes a simple starting action for second quantized massless fermion and the corresponding gauge boson fields in $d = (13+1)$-dimensional space, presented in Eq. (19.1).
The fermion part of the action, Eq. (19.1), can be rewritten in the way that it manifests in $d = (3+1)$ in the low energy regime before the electroweak break by the *standard model* postulated properties of: **i.** Quarks and leptons and antiquarks and antileptons with the spins, handedness, charges and family quantum numbers. Their internal space is described by the Clifford odd "basis vectors" which are eigenvectors of the Cartan subalgebra of S^{ab} and $\tilde{S}^{ab}$, Eqs. (19.4, 11.29, 11.28).
ii. Couplings of fermions to the vector gauge fields, which are the superposition of gauge fields $\omega^{st}{}_{\alpha}$, Sect. 6.2 in Ref. [5], with the space index $\alpha = (0,1,2,3)$ and with the charges determined by the Cartan subalgebra of S^{ab} and $\tilde{S}^{ab}$ manifesting the symmetry of space $(d-4)$, and to the scalar gauge fields [1,2,23,24,26,29, 31,36,37,48–50] with the space index $\alpha \geq 5$ and the charges determined by the Cartan subalgebra of S^{ab} and $\tilde{S}^{ab}$ (as explained in the case of the vector gauge fields), and which are superposition of either $\omega^{st}{}_{\alpha}$ or $\tilde{\omega}^{ab}{}_{\alpha}$,

$$\begin{aligned}
\mathcal{L}_f = & \bar{\psi}\gamma^m (p_m - \sum_{A,i} g^{Ai}\tau^{Ai} A^{Ai}_m)\psi + \\
& \{\sum_{s=7,8} \bar{\psi}\gamma^s p_{0s}\ \psi\} + \\
& \{\sum_{t=5,6,9,\ldots,14} \bar{\psi}\gamma^t p_{0t}\ \psi\} ,
\end{aligned} \tag{11.38}$$

where $p_{0s} = p_s - \frac{1}{2}S^{s's"}\omega_{s's"s} - \frac{1}{2}\tilde{S}^{ab}\tilde{\omega}_{abs}$, $p_{0t} = p_t - \frac{1}{2}S^{t't"}\omega_{t't"t} - \frac{1}{2}\tilde{S}^{ab}\tilde{\omega}_{abt}$, with $p_{0s} = e^{\alpha}_{s} p_{0\alpha}$, $m \in (0,1,2,3)$, $s \in (7,8)$, $(s', s") \in (5,6,7,8)$, (a,b) (appearing in $\tilde{S}^{ab}$) run within either $(0,1,2,3)$ or $(5,6,7,8)$, t runs $\in (5,\ldots,14)$, $(t', t")$ run either $\in (5,6,7,8)$ or $\in (9,10,\ldots,14)$. The spinor function ψ represents all family members of all the $2^{\frac{7+1}{2}-1} = 8$ families.

The first line of Eq. (11.38) determines in $d = (3+1)$ the kinematics and dynamics of fermion fields coupled to the vector gauge fields [23,27,31]. The vector gauge fields are the superposition of the spin connection fields ω_{stm}, $m = (0,1,2,3)$, $(s,t) = (5,6,\cdots,13,14)$, and are the gauge fields of S^{st}, Subsect. (6.2.1) of Ref. [5]. The reader can find in Sect. 6 of Ref. [5] a quite detailed overview of the properties which the massless fermion and boson fields appearing in the simple starting action, Eq. (19.1), (the later only as gravitational fields) manifest in $d = (3+1)$ as all the observed fermions — quarks and leptons and antiquarks and antileptons in each family — appearing in twice four families, with the lower four families including the observed three families of quarks and leptons and antiquarks and antileptons. The higher four families offer the explanation for the dark matter [49]. Table 5 and Eq. (110) of Ref. [5] explain that the scalar fields with the space index $\alpha = (7,8)$ carry the weak charge $\tau^{13} = \pm\frac{1}{2}$ and the hyper charge $Y = \mp\frac{1}{2}$, just as assumed by the *standard model*.

Masses of families of quarks and leptons are determined by the superposition of the scalar fields, Eq. (108-120) of Ref. [5], appearing in two groups, each of them manifesting the symmetry $SU(2)\times SU(2)\times U(1)$ [18].
The scalar gauge fields with the space index $(7,8)$ determine correspondingly the symmetry of mass matrices of quarks and leptons ([5], Eq. (111)) which appear in two groups as the scalar fields do [49,51]. In Table 5 in Ref. [5]) the symmetry $SU(2)\times SU(2)\times U(1)$ for each of the two groups is presented and explained.
Although spontaneous symmetry braking of the starting symmetry has not (yet consistently enough) been studied and the coupling constants of the scalar fields among themselves and with quarks and leptons are not yet known, the known symmetry of mass matrices, presented in Eq. (111) of Ref. [5], enables to determine parameters of mass matrices from the measured data of the 3×3 sub mixing matrices and the masses of the measured three families of quarks and leptons.
Although the known 3×3 submatrix of the unitary 4×4 matrix enables to determine 4×4 matrix, the measured 3×3 mixing sub matrix is even for quarks far accurately enough measured, so that we only can predict the matrix elements of the 4×4 mixing matrix for quarks if assuming that masses (times c^2) of the fourth family quarks are heavy enough, that is above one TeV [46,49]. The new measurements of the matrix elements among the observed 3 families agree better with the predictions obtained by the *sspin-charge-family* theory than the old measurements. The reader can find predictions in Refs. ([50,51]) and the overview in Ref. ([5], Subsect. 7.3.1).
The upper group of four families offers the explanation for the *dark matter*, to which the quarks and leptons from the (almost) stable of the upper four families mostly contribute. The reader can find the report on this proposal for the *dark matter* origin in Ref. [49] and a short overview in Subsect. 7.3.1 of [5], where the appearance, development and properties of the *dark matter* are discussed. The upper four families predict nucleons of very heavy quarks with the nuclear force among nucleons which is correspondingly very different from the known one [49,52].

Besides the scalar fields with the space index $\alpha=(7,8)$, which manifest in $d=(3+1)$ as scalar gauge fields with the weak and hyper charge $\pm\frac{1}{2}$ and $\mp\frac{1}{2}$, respectively, and which gaining at low energies constant values make families of quarks and leptons and the weak gauge field massive, there are in the starting action, Eqs. (19.1), additional scalar gauge fields with the space index $\alpha=(9,10,11,12,13,14)$. They are with respect to the space index α either triplets or antitriplets causing transitions from antileptons into quarks and from antiquarks into quarks and back.

[18] The assumption that the symmetry $SO(13,1)$ first breaks into $SU(3)\times U(1)\times SO(7,1)$ makes that quarks and leptons distinguish only in the part $SU(3)\times U(1)$, while the $SO(7,1)$ part is identical separately for quarks and leptons and separately for antiquarks and antileptons. Table 7 of Ref. [5], presenting one family, which includes quarks and leptons and antiquarks and antileptons, manifests these properties. The $\omega_{ab\alpha}$, with the space index $(7,8)$ carry with respect to the flat index ab only quantum numbers Q, Y, τ^4, (Q $(=\tau^{13}+Y)$,τ^{13} $(=\frac{1}{2}(\mathcal{S}^{56}-\mathcal{S}^{78})$, Y $(=\tau^4+\tau^{23})$ and $\tau^4=-\frac{1}{3}(\mathcal{S}^{9\,10}+\mathcal{S}^{11\,12}+\mathcal{S}^{13\,14})$, the flat index (ab) of $\tilde{\omega}_{ab\alpha}$, with the space index $(7,8)$, includes all $(0,1,\ldots,8)$ correspondingly forming the symmetry $SU(2)\times SU(2)\times U(1)$.

Their properties are presented in Ref. [25] and briefly in Table 9 and Fig. 1 of Ref. [5].
Concerning this second point we proved on the toy model of $d = (5+1)$ that the break of symmetry can lead to (almost) massless fermions [34].
In $d = (3+1)$-dimensional space — at low energies — the gauge gravitational fields manifest as the observed vector gauge fields [27], which can be quantized in the usual way.
The author is in mean time trying to find out (together with the collaborators) how far can the *spin-charge-family* theory — starting in $d = (13+1)$-dimensional space with a simple and "elegant" action, Eq. (19.1) — reproduce in $d = (3+1)$ the observed properties of quarks and leptons [23,25,27–32], the observed vector gauge fields, the scalar field and the Yukawa couplings, the appearance of the *dark matter* and of the matter-antimatter asymmetry, as well as the other open questions, connecting elementary fermion and boson fields and cosmology.
The work done so far on the *spin-charge-family* theory seems promising.

11.4 Conclusions

In the *spin-charge-family* theory [1,2,5,23,25,27–32] the Clifford odd algebra is used to describe the internal space of fermion fields. The Clifford odd "basis vectors" — the superposition of odd products of γ^a's — in a tensor product with the basis in ordinary space form the creation and annihilation operators, in which the anticommutativity of the "basis vectors" is transferred to the creation and annihilation operators for fermions, offering the explanation for the second quantization postulates for fermion fields.
The Clifford odd "basis vectors" have all the properties of fermions: Half integer spins with respect to the Cartan subalgebra members of the Lorentz algebra in the internal space of fermions in even dimensional spaces ($d = 2(2n+1)$ or $d = 4n$), as discussed in Subsects. (11.2.1, 11.2.3).
With respect to the subgroups of the $SO(d-1,1)$ group the Clifford odd "basis vectors" appear in the fundamental representations, as illustrated in Subsects. 11.2.2.
In this article it is demonstrated that the Clifford even algebra is offering the description of the internal space of boson fields. The Clifford even "basis vectors" — the superposition of even products of γ^a's — in a tensor product with the basis in ordinary space form the creation and annihilation operators which manifest the commuting properties of the second quantized boson fields, offering explanation for the second quantization postulates for boson fields [9]. The Clifford even "basis vectors" have all the properties of bosons: Integer spins with respect to the Cartan subalgebra members of the Lorentz algebra in the internal space of bosons, as discussed in Subsects. (11.2.1, 11.2.3).

With respect to the subgroups of the $SO(d-1,1)$ group the Clifford even "basis vectors" manifest the adjoint representations, as illustrated in Subsect. 11.2.2.
There are two kinds of anticommuting algebras [2]: The Grassmann algebra, offering in d-dimensional space $2\,.\,2^d$ operators (2^d θ^a's and 2^d $\frac{\partial}{\partial\theta_a}$'s, Hermitian

conjugated to each other, Eq. (11.3)), and the two Clifford subalgebras, each with 2^d operators named γ^a's and $\tilde{\gamma}^a$'s, respectively, [2,13,14], Eqs. (11.2-19.3).
The operators in each of the two Clifford subalgebras appear in two groups of $2^{\frac{d}{2}-1}\times 2^{\frac{d}{2}-1}$ of the Clifford odd operators (the odd products of either γ^a's in one subalgebra or of $\tilde{\gamma}^a$'s in the other subalgebra), which are Hermitian conjugated to each other: In each Clifford odd group of any of the two subalgebras there appear $2^{\frac{d}{2}-1}$ irreducible representation each with the $2^{\frac{d}{2}-1}$ members and the group of their Hermitian conjugated partners.
There are as well the Clifford even operators (the even products of either γ^a's in one subalgebra or of $\tilde{\gamma}^a$'s in another subalgebra) which again appear in two groups of $2^{\frac{d}{2}-1}\times 2^{\frac{d}{2}-1}$ members each. In the case of the Clifford even objects the members of each group of $2^{\frac{d}{2}-1}\times 2^{\frac{d}{2}-1}$ members have the Hermitian conjugated partners within the same group, Subsect. 11.2.1, Table 11.1.
The Grassmann algebra operators are expressible with the operators of the two Clifford subalgebras and opposite, Eq. (11.5). The two Clifford subalgebras are independent of each other, Eq. (19.3), forming two independent spaces.
Either the Grassmann algebra [15,20] or the two Clifford subalgebras can be used to describe the internal space of anticommuting objects, if the superposition of odd products of operators (θ^a's or γ^a's, or $\tilde{\gamma}^a$'s) are used to describe the internal space of these objects. The commuting objects must be superposition of even products of operators (θ^a's or γ^a's or $\tilde{\gamma}^a$'s).

No integer spin anticommuting objects have been observed so far, and to describe the internal space of the so far observed fermions only one of the two Clifford odd subalgebras are needed.

The problem can be solved by reducing the two Clifford sub algebras to only one, the one (chosen to be) determined by γ^{ab}'s. The decision that $\tilde{\gamma}^a$'s apply on γ^a as follows: $\{\tilde{\gamma}^a B = (-)^B\, i\, B\gamma^a\}|\psi_{oc}>$, Eq. (19.6), (with $(-)^B = -1$, if B is a function of an odd products of γ^a's, otherwise $(-)^B = 1$) enables that $2^{\frac{d}{2}-1}$ irreducible representations of $S^{ab} = \frac{i}{2}\{\gamma^a\,,\,\gamma^b\}_-$ (each with the $2^{\frac{d}{2}-1}$ members) obtain the family quantum numbers determined by $\tilde{S}^{ab} = \frac{i}{2}\{\tilde{\gamma}^a\,,\,\tilde{\gamma}^b\}_-$.

The decision to use in the *spin-charge-family* theory in $d = 2(2n+1)$, $n \geq 3$ ($d \geq (13+1)$ indeed), the superposition of the odd products of the Clifford algebra elements γ^a's to describe the internal space of fermions which interact with the gravity only (with the vielbeins, the gauge fields of momenta, and the two kinds of the spin connection fields, the gauge fields of S^{ab} and $\tilde{S}^{ab}$, respectively), Eq. (19.1), offers not only the explanation for all the assumed properties of fermions and bosons in the *standard model*, with the appearance of the families of quarks and leptons and antiquarks and antileptons ([5] and the references therein) and of the corresponding vector gauge fields and the Higgs's scalars included [27], but also for the appearance of the *dark matter* [49] in the universe, for the explanation of the matter/antimatter asymmetry in the universe [25], and for several other observed phenomena, making several predictions [37,47,48,50].

Recognition that the use of the superposition of the even products of the Clifford algebra elements γ^a's to describe the internal space of boson fields, what appear

to manifest all the properties of the observed boson fields, as demonstrated in this articles, makes clear that the Clifford algebra offers not only the explanation for the postulates of the second quantized anticommuting fermion fields but also for the postulates of the second quantized boson fields.
The relations in Eq. (11.36)

$$\{\frac{1}{2}\sum_{ab} S^{ab}\,\omega_{ab\alpha}\}\sum_{m}\beta^{mf}\,\hat{b}^{m\dagger}_{f}(\vec{p}) \text{ relate to } \{\sum_{m'f'}{}^{I}\hat{\mathcal{A}}^{m'\dagger}_{f'}\,\mathcal{C}^{m'f'}_{\alpha}\}\sum_{m}\beta^{mf}\,\hat{b}^{m\dagger}_{f}(\vec{p})\,,$$
$$\forall f \text{ and } \forall\,\beta^{mf}\,,$$
$$\mathcal{S}^{cd}\,\sum_{ab}(c^{ab}{}_{mf}\,\omega_{ab\alpha}) \text{ relate to } \mathcal{S}^{cd}\,({}^{I}\hat{\mathcal{A}}^{m\dagger}_{f}\,\mathcal{C}^{mf}_{\alpha})\,,$$
$$\forall\,(m,f),$$
$$\forall \text{ Cartan subalgebra member } \mathcal{S}^{cd}\,,$$

offers the possibility to replace the covariant derivative $p_{0\alpha}$

$$p_{0\alpha} = p_{\alpha} - \frac{1}{2}S^{ab}\omega_{ab\alpha} - \frac{1}{2}\tilde{S}^{ab}\tilde{\omega}_{ab\alpha}$$

in Eq. (19.1) with

$$p_{0\alpha} = p_{\alpha} - \sum_{mf}{}^{I}\hat{\mathcal{A}}^{m\dagger}_{f}\,{}^{I}\mathcal{C}^{m}_{f\alpha} - \sum_{mf}{}^{I}\hat{\tilde{\mathcal{A}}}^{m\dagger}_{f}\,{}^{I}\tilde{\mathcal{C}}^{m}_{f\alpha}\,,$$

where the relation among ${}^{I}\hat{\tilde{\mathcal{A}}}^{m\dagger}_{f}\,{}^{I}\tilde{\mathcal{C}}^{m}_{f\alpha}$ and ${}^{II}\hat{\tilde{\mathcal{A}}}^{m\dagger}_{f}\,{}^{II}\tilde{\mathcal{C}}^{m}_{f\alpha}$ with respect to $\omega_{ab\alpha}$ and $\tilde{\omega}_{ab\alpha}$, not discussed directly in this article, needs additional study and explanation.
Although the properties of the Clifford odd and even "basis vectors" and correspondingly of the creation and annihilation operators for fermion and boson fields are, hopefully, clearly demonstrated in this article, yet the proposed way of the second quantization of fields, the fermion and the boson ones, needs further study to find out what new can the description of the internal space of fermions and bosons bring in understanding of the second quantized fields.
Let be added that in even dimensional spaces the Clifford odd "basis vectors" carry only one handedness, either right or left, depending on the definition of handedness and the choice of the "basis vectors". Their Hermitian conjugated partners carry opposite handedness. The "basis vectors" in the subspace of the whole space do have both handedness. In odd dimensional spaces ($d = (2n+1)$) the operator of handedness is a superposition of an odd products of γ^{a}'s. The eigenstates of the operator of handedness must be therefore the superposition of the Clifford odd and the Clifford even "basis vectors". These eigenstates can have either right or left handed. The properties of "basis vectors" in odd dimensional spaces are demonstrated in the App. 11.5 of this contribution for $d = 1$ and $d = (2+1)$ spaces.
It looks like that this study, showing up that the Clifford algebra can be used to describe the internal spaces of fermion and boson fields in an equivalent way, offering correspondingly the explanation for the second quantization postulates

for fermion and boson fields, is opening the new insight into the quantum field theory, since studies of the interaction of fermion fields with boson fields and of boson fields with boson fields so far looks very promising.
The study of properties of the second quantized boson fields, the internal space of which is described by the Clifford even algebra, has just started and needs further consideration. Studying properties of "basis vectors" in odd dimensional spaces might help to understand anomalies of quantum fields.

11.5 Examples demonstrating properties of Clifford odd and even "basis vectors" in odd dimensional spaces for $d = (1)$, $d = (2+1)$

The *spin-charge-family* theory, using even dimensional spaces, $d = (13+1)$ indeed, offers the explanation for all the assumptions of the *standard model*, explaining as well the postulates for the second quantization of fermion and boson fields. The internal space of fermions is in this theory described by "basis vectors" which are superposition of odd products of γ^a's while the internal space of bosons is described by "basis vectors" which are superposition of even products of γ^a's.
Subsect. 11.2.2 demonstrates properties of the Clifford odd and even "basis vectors" in special cases when $d = (1+1)$, $d = (3+1)$, and $d = (5+1)$.

Let us discuss here odd dimensional spaces, which have very different properties:
i. While in even dimensional spaces the Clifford odd "basis vectors" have $2^{\frac{d}{2}-1}$ members m in $2^{\frac{d}{2}-1}$ families f, $\hat{b}^{m\dagger}_f$, and their Hermitian conjugated partners appear in a separate group of $2^{\frac{d}{2}-1}$ members in $2^{\frac{d}{2}-1}$ families, there are in odd dimensional spaces some of the $2^{\frac{d}{2}-1} \times 2^{\frac{d}{2}-1} = 2^{d-2}$ Clifford odd "basis vectors" self adjoint and have correspondingly some of the Hermitian conjugated partners in another group with 2^{d-2} members.
ii. In even dimensional spaces the Clifford even "basis vectors" ${}^{i}\hat{\mathcal{A}}^{m\dagger}_f, i = (1,2)$, appear in two orthogonal groups, each with $2^{\frac{d}{2}-1} \times 2^{\frac{d}{2}-1}$ members and each with the Hermitian conjugated partners within the same group, $2^{\frac{d}{2}-1}$ of them are self adjoint. In odd dimensional spaces the Clifford even "basis vectors" appear in two groups, each with $2^{\frac{d}{2}-1} \times 2^{\frac{d}{2}-1} = 2^{d-2}$ members, which are either self adjoint or have their Hermitian conjugated partners in another group. Not all the members of one group are orthogonal to the members of another group, only the self adjoint ones are orthogonal.
iii. While $\hat{b}^{m\dagger}_f$ have in even dimensional spaces one handedness only (either right or left, depending on the definition of handedness), in odd dimensional spaces the operator of handedness is a Clifford odd object, still commuting with S^{ab}, which is the product of odd number of γ^a's and correspondingly transforms the Clifford odd "basis vectors" into Clifford even "basis vectors" and opposite. Correspondingly are the eigenvectors of handedness the superposition of the Clifford odd and the Clifford even "basis vectors". Correspondingly there are in odd dimensional

spaces right handed and left handed eigenvectors of the operator of handedness.

Let us illustrate the above mentioned properties of the "basis vectors" in odd dimensional spaces, starting with the simplest case:

d=(1)

There is one Clifford odd "basis vector"

$$\hat{b}^{1\dagger}_{1} = \gamma^0$$

and one Clifford even "basis vectors"

$$^{i}\hat{\mathcal{A}}^{1\dagger}_{1} = 1\,.$$

The operator of handedness $\Gamma^{(0+1)} = \gamma^0$ transforms $\hat{b}^{1\dagger}_{1}$ into identity $^{i}\hat{\mathcal{A}}^{1\dagger}_{1}$ and $^{i}\hat{\mathcal{A}}^{1\dagger}_{1}$ into $\hat{b}^{1\dagger}_{1}$.
The two eigenvectors of the operator of handedness are

$$\frac{1}{\sqrt{2}}(\gamma^0 + 1)\,, \quad \frac{1}{\sqrt{2}}(\gamma^0 - 1)\,,$$

with the handedness $(+1,-1)$, that is of right and left handedness. respectively.

d=(2+1)

There are twice $2^{d=3-2} = 2$ Clifford odd "basis vectors". We chose as the Cartan subalgebra member S^{01} of S^{ab}: $\hat{b}^{1\dagger}_{1} = \stackrel{01}{[-i]}\,\gamma^2$, $\hat{b}^{2\dagger}_{1} = \stackrel{01}{(+i)}$, $\hat{b}^{1\dagger}_{2} = \stackrel{01}{(-i)}$, $\hat{b}^{2\dagger}_{2} = \stackrel{01}{[+i]}\,\gamma^2$, with the properties

$$\begin{array}{ccc}
f=1 & f=2 & \\
\tilde{S}^{01} = \frac{i}{2} & \tilde{S}^{01} = -\frac{i}{2}, & S^{01} \\
\hat{b}^{1\dagger}_{1} = \stackrel{01}{[-i]}\,\gamma^2 & \hat{b}^{1\dagger}_{2} = \stackrel{01}{(-i)} & -\frac{i}{2} \\
\hat{b}^{2\dagger}_{1} = \stackrel{01}{(+i)} & \hat{b}^{2\dagger}_{2} = \stackrel{03}{[+i]}\,\gamma^2 & \frac{i}{2},
\end{array}$$

$\hat{b}^{1\dagger}_{1}$ and $\hat{b}^{2\dagger}_{2}$ are self adjoint (up to a sign), $\hat{b}^{2\dagger}_{1} = \stackrel{01}{(+i)}$ and $\hat{b}^{1\dagger}_{2} = \stackrel{01}{(-i)}$ are Hermitian conjugated to each other.

In odd dimensional spaces the "basis vectors" are not separated from their Hermitian conjugated partners and are correspondingly not well defined.

The operator of handedness is (chosen up to a sign to be) $\Gamma^{(2+1)} = i\gamma^1\gamma^2\gamma^2$.

There are twice $2^{(d=3)-2} = 2$ Clifford even "basis vectors". We choose as the Cartan subalgebra member $\mathcal{S}^{01}$: $^{I}\hat{\mathcal{A}}^{1\dagger}_{1} = \stackrel{01}{[+i]}$, $^{I}\hat{\mathcal{A}}^{2\dagger}_{1} = \stackrel{01}{(-i)}\,\gamma^2$, $^{II}\hat{\mathcal{A}}^{1\dagger}_{2} = \stackrel{01}{[-i]}$, $^{II}\hat{\mathcal{A}}^{2\dagger}_{2} = \stackrel{01}{(+i)}\,\gamma^2$, with the properties

$$\begin{array}{llll} & \mathcal{S}^{01} & & \mathcal{S}^{01} \\ {}^{I}\hat{\mathcal{A}}^{1\dagger}_{1} = \stackrel{01}{[+i]} & 0 & {}^{II}\hat{\mathcal{A}}^{1\dagger}_{2} = \stackrel{01}{[-i]} & 0 \\ {}^{I}\hat{\mathcal{A}}^{2\dagger}_{1} = \stackrel{01}{(-i)}\,\gamma^2 & -i & {}^{II}\hat{\mathcal{A}}^{2\dagger}_{2} = \stackrel{03}{(+i)}\,\gamma^2 & i, \end{array}$$

${}^{I}\hat{\mathcal{A}}^{1\dagger}_{1} = \stackrel{01}{[+i]}$ and ${}^{II}\hat{\mathcal{A}}^{1\dagger}_{2} = \stackrel{01}{[-i]}$ are self adjoint, ${}^{I}\hat{\mathcal{A}}^{2\dagger}_{1} = \stackrel{01}{(-i)}\,\gamma^2$ and ${}^{II}\hat{\mathcal{A}}^{2\dagger}_{2} = \stackrel{03}{(+i)}\,\gamma^2$ are Hermitian conjugated to each other.

In odd dimensional spaces the two groups of the Clifford even "basis vectors" are not orthogonal.

Let us find the eigenvectors of the operator of handedness $\Gamma^{(2+1)} = i\gamma^0\gamma^1\gamma^2$. Since it is the Clifford odd object its eigenvectors are superposition of Clifford odd and Clifford even "basis vectors".
It follows

$$\begin{aligned} \Gamma^{(2+1)}\{\stackrel{01}{[-i]} \pm i\,\stackrel{01}{[-i]}\,\gamma^2\} &= \mp\{\stackrel{01}{[-i]} \pm i\,\stackrel{01}{[-i]}\,\gamma^2\}\,, \\ \Gamma^{(2+1)}\{\stackrel{01}{(+i)} \pm i\,\stackrel{01}{(+i)}\,\gamma^2\} &= \mp\{\stackrel{01}{(+i)} \pm i\,\stackrel{01}{(+i)}\,\gamma^2\}\,, \\ \Gamma^{(2+1)}\{\stackrel{01}{[+i]} \pm i\,\stackrel{01}{[+i]}\,\gamma^2\} &= \pm\{\stackrel{01}{[+i]} \pm i\,\stackrel{01}{[+i]}\,\gamma^2\}\,, \\ \Gamma^{(2+1)}\{\stackrel{01}{(-i)}\,\gamma^2 \pm i\,\stackrel{01}{(-i)}\} &= \pm\{\stackrel{01}{(-i)}\,\gamma^2 \pm i\,\stackrel{01}{(-i)}\}\,, \end{aligned}$$

We can conclude that neither Clifford odd nor Clifford even "basis vectors" have in odd dimensional spaces the properties which they demonstrate in even dimensional spaces.
i. In odd dimensional spaces the "basis vectors" are not separated from their Hermitian conjugated partners and are correspondingly not well defined, that is we can not define creation and annihilation operators as a tensor products of "basis vectors" and basis in momentum space.
In odd dimensional spaces the two groups of the Clifford even "basis vectors" are not orthogonal, only self adjoint "basis vectors" are orthogonal, the rest of "basis vectors" have their Hermitian conjugated partners in another group.
ii. The Clifford odd operator of handedness allows left and right handed superposition of Clifford odd and Clifford even "basis vectors".

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