Recipes for edge-transitive tetravalent graphs[*]

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Abstract

This paper presents all constructions known to the authors which result in tetravalent graphs whose symmetry groups are large enough to be transitive on the edges of the graph.

Keywords: Graph, automorphism group, symmetry.

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1 The census

This paper is to accompany the Census of Edge-Transitive Tetravalent Graphs, available at

http://jan.ucc.nau.edu/∼swilson/C4FullSite/index.html,

which is a collection of all known edge-transitive graphs of valence 4 up to 512 vertices.

The Census contains information for each graph. This information includes parameters such as group order, diameter, girth etc., all known constructions, relations to other graphs in the Census (coverings, constructions, etc.), and interesting substructures such as colorings, cycle structures, and dissections.

We try to present most graphs as members of one or more parameterized families, and one purpose of this paper is to gather together, here in one place, descriptions of each of these families, to show how each is constructed, what the history of each is and how one family is related to another. We also discuss in this paper the theory and techniques behind computer searches leading to many entries in the Census.

We should point out that similar censi exist for edge transitive graphs of valence 3 [8, 10]. Unlike our census, these censi are complete in the sense that they contain all the graphs up to a given order. The method used in these papers relies on the fact that in the case of prime valence, the order of the automorphism group can be bounded by a linear function of the order of the graph, making exhaustive computer searches possible – see [9] for details.

Even though our census is not proved to be complete, it is complete in some segments. In particular, the census contains all dart-transitive tetravalent graphs up to 512 vertices (see [32, 36]) and all $\frac{1}{2}$-arc-transitive tetravalent graphs up to 512 vertices (see [33, 38]). Therefore, if a graph is missing from our census, then it is semisymmetric (see Section 2).

2   Basic notions

A *graph* is an ordered pair $\Gamma = (\mathcal{V}, \mathcal{E})$, where $\mathcal{V}$ is an arbitrary set of things called vertices, and $\mathcal{E}$ is a collection of subsets of $\mathcal{V}$ of size two; these are called *edges*. We let $\mathcal{V}(\Gamma) = \mathcal{V}$ and $\mathcal{E}(\Gamma) = \mathcal{E}$ in this case. If $e = \{u, v\} \in \mathcal{E}$, we say u is a *neighbor* of v, that u and v are *adjacent*, and that u is *incident* with e and vice versa. A *dart* or *directed edge* is an ordered pair (u, v) where $\{u, v\} \in \mathcal{E}$. Let $\mathcal{D}(\Gamma)$ be the set of darts of Γ. The *valence* or *degree* of a vertex v is the number of edges to which v belongs. A graph is *regular* provided that every vertex has the same valence, and then we refer to that as the valence of the graph.

A *digraph* is an ordered pair $\Delta = (\mathcal{V}, \mathcal{E})$, where $\mathcal{V}$ is an arbitrary set of things called vertices, and $\mathcal{E}$ is a collection of ordered pairs of distinct elements of $\mathcal{V}$. We think of the pair (u, v) as being an edge directed from u to v. An *orientation* is a digraph in which for all $u, v \in \mathcal{V}$, if $(u, v) \in \mathcal{E}$ then $(v, u) \notin \mathcal{E}$.

A *symmetry*, or *automorphism*, of a graph or a digraph $\Gamma = (\mathcal{V}, \mathcal{E})$ is a permutation of $\mathcal{V}$ which preserves $\mathcal{E}$. If $v \in \mathcal{V}(\Gamma)$ and σ is a symmetry of Γ, then we denote the image of v under σ by $v\sigma$, and if ρ is also a symmetry of Γ, then the product $\sigma\rho$ is a symmetry that maps v to $(v\sigma)\rho$. Together with this product, the set of symmetries of Γ forms a group, the *symmetry group* or *automorphism group* of Γ, denoted $\mathrm{Aut}(\Gamma)$. We are interested in those graphs for which $G = \mathrm{Aut}(\Gamma)$ is big enough to be transitive on $\mathcal{E}$. Such a graph is called *edge-transitive*. Within the class of edge-transitive graphs of a given valence, there are three varieties:

(1) A graph is *symmetric* or *dart-transitive* provided that G is transitive on $\mathcal{D} = \mathcal{D}(\Gamma)$.

(2) A graph is $\frac{1}{2}$-*arc-transitive* provided that G is transitive on $\mathcal{E}$ and on $\mathcal{V}$, but not on $\mathcal{D}$. A $\frac{1}{2}$-arc-transitive graph must have even valence [48]. The G-orbit of one dart is then an orientation Δ of Γ such that every vertex has k in-neighbours and k out-neighbors, where $2k$ is the valence of Γ. The symmetry group $\mathrm{Aut}(\Delta)$ is then transitive on vertices and on edges. We call such a Δ a *semitransitive orientation* and we say that a graph which has such an orientation is *semitransitive* [54].

(3) Finally, Γ is *semisymmetric* provided that G is transitive on $\mathcal{E}$ but not on $\mathcal{V}$ (and hence not on $\mathcal{D}$). In this case, the graph must be bipartite, with each edge having one vertex

from each class. More generally, we say that a graph Γ is *bi-transitive* provided that Γ is bipartite, and its group of color-preserving symmetries is transitive on edges (and so on vertices of each color); a bi-transitive graph is thus either semisymmetric or dart-transitive.

There is a fourth important kind of symmetricity that a tetravalent graph might have, in which $\mathrm{Aut}(\Gamma)$ is transitive on vertices but has two orbits on edges. Graphs satisfying this and certain other conditions are called *LR structures*. These are introduced and defined in Section 18 but referred to in several places before that. These graphs, though not edge-transitive themselves, can be used directly to construct semisymmetric graphs.

For $\sigma \in \mathrm{Aut}(\Gamma)$, if there is a vertex $v \in \mathcal{V}$ such that $v\sigma$ is adjacent to v and $v\sigma^2 \neq v$ we call σ a *shunt* and then σ induces a directed cycle $[v, v\sigma, v\sigma^2, \ldots, v\sigma^m]$, with $v\sigma^{m+1} = v$, called a *consistent* cycle. The remarkable theorem of Biggs and Conway [4, 30] says that if Γ is dart-transitive and regular of degree d, then there are exactly $d - 1$ orbits of consistent cycles. A later result [5] shows that a $\frac{1}{2}$-arc-transitive graph of degree $2e$ must have exactly e orbits of consistent cycles. For tetravalent graphs, the dart-transitive ones have 3 orbits of consistent cycles and the $\frac{1}{2}$-arc-transitive ones have two such orbits. An LR structure might have 1 or 2 orbits of consistent cycles.

3 Computer generated lists of graphs

3.1 Semisymmetric graphs arising from amalgams of index $(4, 4)$

Let L and R be two finite groups intersecting in a common subgroup B and assume that no non-trivial subgroup of B is normal in both L and R. Then the triple (L, B, R) is called an *amalgam*. For example, if we let $L = A_4$, the alternating group of degree 4, $B \cong C_3$, viewed as a point-stabiliser in L, and $R \cong C_{12}$ containing B as a subgroup of index 4, then (L, B, R) is an amalgam.

If G is a group that contains both L and R and is generated by them, then G is called a *completion* of the amalgam. It is not too difficult to see that there exists a completion that is universal in the sense that every other completion is isomorphic to a quotient thereof. This universal completion is sometimes called the *free product of L and R amalgamated over B* (usually denoted by $L *_B R$) and can be constructed by merging together (disjoint) presentations of L and R and adding relations that identify copies of the same element of B in both L and R. For example, if the amalgam (L, B, R) is as above, then we can write

$$L = \langle x, y, b | x^2, y^2, [x, y], b^3, x^b y, y^b xy \rangle, \quad R = \langle z | z^{12} \rangle,$$

yielding

$$L *_B R = \langle x, y, b, z | x^2, y^2, [x, y], b^3, z^{12}, x^b y, y^b xy, z^4 = b \rangle.$$

Completions of a given amalgam (L, B, R) up to a given order, say M, can be computed using a LOWINDEXNORMALSUBGROUPS routine, developed by Firth and Holt [12] and implemented in MAGMA [6].

Given a completion G of an amalgam (L, B, R), one can construct a bipartite graph, called *the graph of the completion*, with white and black vertices being the cosets of L and R in G, respectively, and two cosets Lg and Rh adjacent whenever they intersect. Note that white (black) vertices are of valence $[L : B]$ ($[R : B]$, respectively). In particular, if B is of index 4 in both L and R, then the graph is tetravalent. We shall say in this case that the amalgam is of index $(4, 4)$.

The group G acts by right multiplication faithfully as an edge-transitive group of automorphisms of the graph and so the graph of a completion of an amalgam always admits an edge- but not vertex-transitive group of automorphisms, and so the graph is bi-transitive (and thus either dart-transitive or semisymmetric). This now gives us a good strategy for constructing tetravalent semisymmetric graphs of order at most M: Choose your favorite amalgams (L, B, R) of index $(4, 4)$, find their completions up to order $2M|B|$ and construct the corresponding graphs.

We have done this for several amalgams of index $(4, 4)$ and the resulting graphs appear in the census under the name SS. The graph $\mathrm{SS}[n, i]$ is the i-th graph in the list of semisymmetric graphs of order n. These graphs are available in magma code at [33].

3.2 Dart-transitive graphs from amalgams

If Γ is a tetravalent dart-transitive graph, then its *subdivision*, obtained from Γ by inserting a vertex of valence 2 on each edge, is edge-transitive but not vertex-transitive. This process is reversible, by removing vertices of degree 2 in a bi-transitive graph of valence $(4, 2)$ one obtains a tetravalent dart-transitive graph.

In the spirit of the previous section, each such graph can be obtained from an amalgam of index $(4, 2)$. Amalgams of index $(4, 2)$ were fully classified in [11, 34, 53], however, unlike in the case of amalgams of index $(3, 2)$, giving rise to cubic dart-transitive graphs, the number of these amalgams is infinite. This fact, together with existence of relatively small tetravalent dart-transitive graphs with very large automorphism groups (see Section 14) made the straightforward approach used in [9, 8] in the case of cubic graphs impossible in the case of valence 4. This obstacle was finally overcome in [37] and now a complete list of dart-transitive tetravalent graphs of order up 640 is described in [36] and available in magma code at [33]. We use $\mathrm{AT}[n, i]$ for these graphs to indicate the i-th graph of order n in this magma file.

3.3 Semitransitive graphs from universal groups

Every semitransitive tetravalent graph arises from an infinite 4-valent tree T_4 and a group G acting on T_4 semitransitively and having a finite stabiliser by quotienting out a normal semiregular subgroup of G. All such groups G were determined in [27]. This result in principle enables the same approach as used in the case of tetravalent dart-transitive graphs, and indeed, by overcoming the issue of semitransitive graphs with large automorphism groups (see [45]), a complete list of semitransitive tetravalent graphs (and in particular $\frac{1}{2}$-arc-transitive graphs) of order up to 1000 was obtained in [38] and is available at [33]. We include the $\frac{1}{2}$-arc-transitive graphs from this list in our census with the designation $\mathrm{HT}[n, i]$.

4 Wreaths, unworthy graphs

A general *Wreath graph*, denoted $\mathrm{W}(n, k)$, has n bunches of k vertices each, arranged in a circle; every vertex of bunch i is adjacent to every vertex in bunches $i + 1$ and $i - 1$. More

precisely, its vertex set is $\mathbb{Z}_n \times \mathbb{Z}_k$; edges are all pairs of the form $\{(i, r), (i + 1, s)\}$. The graph $W(n, k)$ is regular of degree $2k$. If $n = 4$, then $W(n, k)$ is isomorphic to $K_{2k,2k}$ and its symmetry group is the semidirect product of $S_{2k} \times S_{2k}$ with $\mathbb{Z}_2$. If $n \neq 4$, then its symmetry group is the semidirect product of S_k^n with D_n; this group is often called the *wreath product* of D_n over S_k.

Those of degree 4 are the graphs $W(n, 2)$. Here, for simplicity, we can also notate the vertex $(i, 0)$ as A_i, and $(i, 1)$ as B_i for $i \in \mathbb{Z}_n$, with edges then being of the forms $\{A_i, A_{i+1}\}, \{A_i, B_{i+1}\}, \{B_i, A_{i+1}\}, \{B_i, B_{i+1}\}$. For example, Figure 1 shows $W(7, 2)$.
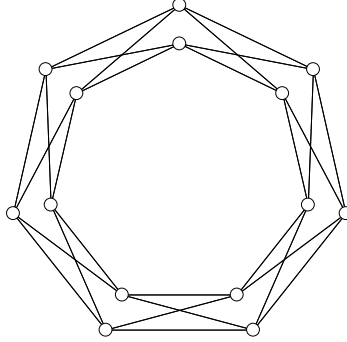

Figure 1: $W(7, 2)$

A wreath graph $W(n, 2)$ has dihedral symmetries ρ and μ, where ρ sends (i, j) to $(i + 1, j)$ and μ sends (i, j) to $(-i, j)$. The important aspect of this graph is that for each $i \in \mathbb{Z}_n$, there is a symmetry σ_i, called a "local" symmetry, which interchanges $(i, 0)$ with $(i, 1)$ and leaves every other vertex fixed. Notice that ρ acts as a shunt for a cycle of length n, and $\rho\sigma_0$ acts as a shunt for a cycle of length $2n$. The third orbit of consistent cycles are those of the form $[A_i, A_{i+1}, B_i, B_{i+1}]$ The symmetry $\rho\mu\sigma_0$ is a shunt for the cycle $[A_0, B_0, A_1, B_1]$ in this orbit.

Since every σ_i for $i \neq 0$ is in the stabilizer of A_0, we see that vertex stabilizers in these graphs can be arbitrarily large. In fact, the order of the vertex-stabilizer grows exponentially with respect to the order of the graph.

A graph Γ is *unworthy* provided that some two of its vertices have exactly the same neighbors. The graph $W(n, 2)$ is unworthy because for each i, the vertices A_i and B_i have the same neighbors. The symmetry groups of vertex-transitive unworthy graphs tend to be large due to the symmetries that fix all but two vertices sharing the same neighborhood.

The paper [39] shows that there are only two kinds of tetravalent edge-transitive graphs which are unworthy. One is the dart-transitive $W(n, 2)$ graphs. The other is the "subdivided double" of a dart-transitive graph; this is a semisymmetric graph given by this construction:

Construction 4.1. *Suppose that Λ is a tetravalent graph. We construct a bipartite graph $\Gamma = \mathrm{SDD}(\Lambda)$ in the following way. The white vertices of Γ correspond to edges of Λ. The black vertices correspond two-to-one to vertices of Λ; for each $v \in \mathcal{V}(\Lambda)$, there are two vertices v_0, v_1 in $\mathcal{V}(\Gamma)$. An edge of Γ joins each e to each v_i where v is a vertex of e in Λ.*

The Folkman graph on 20 vertices [13] is constructible as $\mathrm{SDD}(K_5)$. It is clear that $\mathrm{SDD}(\Lambda)$ is tetravalent, and that if Λ is dart-transitive, then $\mathrm{SDD}(\Lambda)$ is edge-transitive.

The paper [39] shows that any unworthy tetravalent edge-transitive graph is isomorphic to some $\mathrm{W}(n, 2)$ if it is dart-transitive, and to $\mathrm{SDD}(\Lambda)$ for some dart-transitive tetravalent Λ if it is semisymmetric. There are no tetravalent unworthy $\frac{1}{2}$-arc-transitive graphs.

5   Circulants

In general, the *circulant* graph $C_n(S)$ is the Cayley graph for $\mathbb{Z}_n$ with generating set S. Here S must be a subset of $\mathbb{Z}_n$ which does not include 0, but does, for each $x \in S$, include $-x$ as well. Explicitly, vertices are $0, 1, 2, \ldots, n-1$, considered as elements of $\mathbb{Z}_n$, and two numbers i, j are adjacent if their difference is in S. Thus the edge set consists of all pairs $\{i, i+s\}$ for $i \in \mathbb{Z}_n$, and $s \in S$. If $S = \{\pm a_1, \pm a_2, \ldots\}$, we usually abbreviate the name of each graph $C_n(S)$ as $C_n(a_1, a_2, \ldots)$. The numbers a_i are called *jumps* and the set of edges of the form $\{j, j+a_i\}$ for a fixed i is called a *jumpset*. Figure 2 shows an example, the graph $C_{10}(1, 3)$.
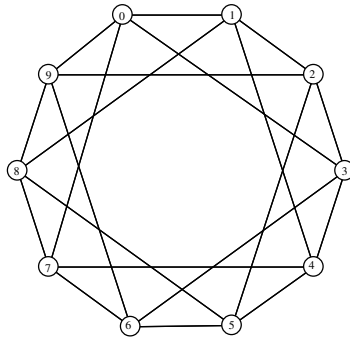

Figure 2: $C_{10}(1, 3)$

For general n, if S is a subgroup of the group $\mathbb{Z}_n^*$ of units mod n, then $C_n(S)$ is dart-transitive, though it may be so in many other circumstances as well. In the tetravalent case, there are two possibilities for an edge-transitive circulant:

Theorem 5.1. *If Γ is a tetravalent edge-transitive circulant graph with n vertices, then it is dart-transitive and either:*

(1) *Γ is isomorphic to $C_n(1, a)$ for some a such that $a^2 \equiv \pm 1 \pmod{n}$, or*

(2) *n is even, $n = 2m$, and Γ is isomorphic to $C_{2m}(1, m+1)$.*

Proof. Note first that every edge-transitive circulant $C_n(a, b)$ is dart-transitive, due to an automorphism which maps a vertex i to the vertex $a - i \bmod n$, and thus inverts the edge $\{0, a\}$.

In (1), the dihedral group D_n acts transitively on darts of each of the two jumpsets, and because $a^2 \equiv \pm 1 \pmod{n}$, multiplication by $a \bmod n$ induces a symmetry of the graph which interchanges the two jumpsets.

On the other hand, in (2), $C_{2m}(1, m+1)$ is isomorphic to the unworthy graph $W(m, 2)$, with i and $i+m$ playing the roles of A_i and B_i. Thus, the sufficiency of (1) or (2) for edge-transitivity is clear.

The necessity can be deduced either from a complete classification of dart-transitive circulants of arbitrary valence proved in [19] and [23] or by a careful examination of short cycles in dart-transitive circulants. □

6   Toroidal graphs

The tessellation of the plane into squares is known by its Schläfli symbol, $\{4, 4\}$. Let T be the group of translations of the plane that preserve the tessellation. Then T is isomorphic to $\mathbb{Z} \times \mathbb{Z}$ and acts regularly on the vertices of the tessellation. If U is a subgroup of finite index in T, let $\mathcal{M} = \{4, 4\}/U$ be the quotient space formed from $\{4, 4\}$ by identifying points if they are in the same orbit under U. The result is a finite map of type $\{4, 4\}$ on the torus, and the paper [17] shows that, even in a more general setting, every such map arises in this way. We will call U the *kernel* of $\mathcal{M}$. A symmetry α of $\{4, 4\}$ acts as a symmetry of $\mathcal{M}$ if and only if α normalizes U. Thus every such $\mathcal{M}$ has its symmetry group $\mathrm{Aut}(\mathcal{M})$ transitive on vertices, on horizontal edges, on vertical edges; further for each edge e of $\mathcal{M}$, there is a symmetry reversing the edge (and acting as a $180°$ rotation about its center). Thus, $\mathrm{Aut}(\mathcal{M})$ is transitive on the edges of $\mathcal{M}$ (and so must be dart-transitive) if and only if U is normalized by some symmetry interchanging the horizontal and vertical parallel classes. This must be a $90°$ rotation or a reflection about some axis at a $45°$ angle to the axes.

We will use the symbols $\{4, 4\}_{b,c}$, $\{4, 4\}_{[b,c]}$, $\{4, 4\}_{<b,c>}$, intoduced below, to stand for certain maps on the torus as well as their skeletons. As shown in [50], this can happen in three different ways:

(1) $\{4, 4\}_{b,c}$: For this graph and map, defined for $b \geq c \geq 0$, U is the group generated by the translations (b, c) and $(-c, b)$. These are the well-known *rotary* maps. Because this U is normalized by $90°$ rotations, the map admits these rotations as symmetries. $\{4, 4\}_{b,c}$ has $D = b^2 + c^2$ vertices, D faces and $2D$ edges. Figure 3 shows the case when $b = 3, c = 2$.

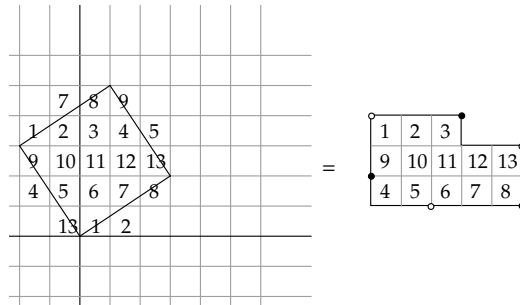

Figure 3: The map $\{4, 4\}_{3,2}$

(2) $\{4, 4\}_{<b,c>}$: This graph and map, defined for $b - 1 > c \geq 0$, uses for U the group generated by the translations (b, c) and (c, b). Because this U is normalized by re-

flections whose axes are at $45°$ to the axes, the map admits these reflections as symmetries. It has $E = b^2 - c^2$ vertices, E faces and $2E$ edges. Figure 4 shows the map $\{4,4\}_{<3,1>}$.
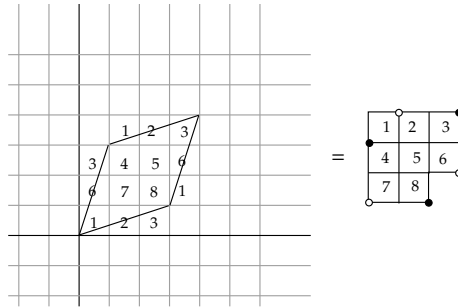

Figure 4: The map $\{4,4\}_{<3,1>}$

Notice that we exclude the case $b = c + 1$. The map $\{4,4\}_{<c+1,c>}$ exists, but its skeleton has parallel edges.

(3) $\{4,4\}_{[b,c]}$: For this graph and map, defined for $b \geq c \geq 0$, U is the group generated by the translations (b,b) and $(-c,c)$. Because this U is normalized by reflections whose axes are at $45°$ to the axes, the map admits these reflections as symmetries. It is defined only for $b \geq c > 1$. It has $F = 2bc$ vertices, F faces and $2F$ edges.
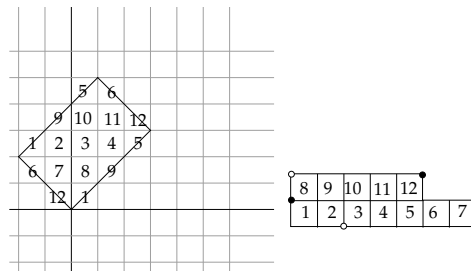

Figure 5: The map $\{4,4\}_{[3,2]}$

If $c = 1$, then the map $\{4,4\}_{[b,c]}$ exists, but, again, its skeleton has multiple edges and so is not a simple graph.

It is interesting to note that $\{4,4\}_{b+c,b-c}$ is a double cover of $\{4,4\}_{b,c}$, while $\{4,4\}_{[b+c,b-c]}$ is a double cover of $\{4,4\}_{<b,c>}$ and $\{4,4\}_{<b+c,b-c>}$ is a double cover of $\{4,4\}_{[b,c]}$. In each case, the covering is 2-fold because the kernel of the covering map has index 2 in the kernel of the covered map.

Because $\{4,4\}_{b,0}$ is isomorphic to $\{4,4\}_{<b,0>}$, this map is reflexible. Because $\{4,4\}_{b,b}$ is isomorphic to $\{4,4\}_{[b,b]}$, this map is also reflexible. All other $\{4,4\}_{b,c}$ are *chiral*: i.e., rotary but not reflexible.

Now, every U of finite index in $\mathbb{Z}^2$ can be expressed in the form $U = \langle (d, e), (f, g) \rangle$, where (d, e) and (f, g) are linearly independent. In particular, we claim that U can also be expressed in the form $U = \langle (r, 0), (s, t) \rangle$, where $t = \text{GCD}(e, g)$. To see that, let $e = e't, g = g't$, and let m and n be Bezout multipliers, so that $me' + ng' = 1$. Now let $s = md + nf$ and $r = g'd - e'f$. Then $\langle (r, 0), (s, t) \rangle$ has a fundamental region which is a rectangle r squares wide, t squares high, with the left and right edges identified directly, and the bottom edges identified with the top after a shift s squares to the right, as in Figure 6.
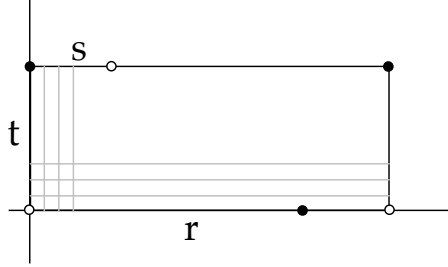

Figure 6: A standard form for maps of type $\{4, 4\}$

In the special case in which b and c are relatively prime, we have $t = 1$, and this forces the graph to be circulant; in fact, the circulant graph $C_r(1, s)$. Here, $s^2 \equiv -1$ for $\{4, 4\}_{b,c}$, and $s^2 \equiv 1$ for both $\{4, 4\}_{<b,c>}$ and $\{4, 4\}_{[b,c]}$. On the other hand, every tetravalent circulant graph is toroidal or has an embedding on the Klein bottle. More precisely, if $a^2 \equiv -1 \pmod{n}$, then $C_n(1, a) \cong \{4, 4\}_{b,c}$ for some b, c. If $a^2 \equiv 1 \pmod{n}$, then $C_n(1, a) \cong \{4, 4\}_{<b,c>}$ or $\{4, 4\}_{[b,c]}$ for some b, c. Of the graphs $C_{2m}(1, m + 1) \cong W(m, 2)$, if m is even then it is $\{4, 4\}_{[\frac{m}{2}, 2]}$. On the other hand, if m is odd, it has an embedding on the Klein bottle, though that embedding is not edge-transitive [55].
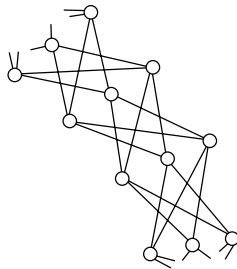
7 Depleted wreaths

The general *Depleted Wreath graph* $\text{DW}(n, k)$ is formed from $\text{W}(n, k)$ by removing the edges of k disjoint n-cycles, each of these cycles containing one vertex from each of the k bunches. More precisely, its vertex set is $\mathbb{Z}_n \times \mathbb{Z}_k$. Its edge set is the set of all pairs of the form $\{(i, r), (i + 1, s)\}$ for $i \in \mathbb{Z}_n$ and $r, s \in \mathbb{Z}_k, r \neq s$. Its vertices are of degree $2(k - 1)$. It is tetravalent when $k = 3$. Figure 7 shows part of $\text{DW}(n, 3)$.

It is not hard to see that if $n > 4$, then the group of symmetries of this graph acts imprimmitively in two different ways. One system of blocks is the collection of sets of the form $R_i = \{(i, r) | r \in \mathbb{Z}_3\}$, defined for each $i \in \mathbb{Z}_n$. Another system is the collection of sets of the form $Q_r = \{(i, r) | i \in \mathbb{Z}_n\}$, defined for each $r \in \mathbb{Z}_3$. Then for $n > 4$, the group of symmetries of $\text{DW}(n, 3)$ is isomorphic to $S_3 \times D_n$.

From this, we can see that the symmetry group has an element of order $3n$ exactly when 3 does not divide n. More precisely, if $n \equiv 1 \pmod 3$ then $\text{DW}(n, 3) \cong C_{3n}(1, n + 1)$ and if $n \equiv 2 \pmod 3$ then $\text{DW}(n, 3) \cong C_{3n}(1, n - 1)$; in the remaining case, $n \equiv 0 \pmod 3$, $\text{DW}(n, 3)$ is not a circulant.

A primary result in [39] is that, with one exception on 14 vertices, any edge-transitive tetravalent graph in which each edge belongs to at least two 4-cycles is toroidal. Because every edge of $\text{DW}(n, 3)$ belongs to two 4-cycles of the form $(i - 1, x) - (i, y) - (i + 1, x) -$

Figure 7: Part of $DW(n, 3)$

$(i, z) - (i-1, x)$, where $\{x, y, z\} = \{0, 1, 2\}$, the Depleted Wreaths are also toroidal. If n is even, then $DW(n, 3) \cong \{4, 4\}_{[\frac{n}{2}, 3]}$, while if n is odd, then $DW(n, 3) \cong \{4, 4\}_{<\frac{n+3}{2}, \frac{n-3}{2}>}$.

8   Spidergraphs

The *Power Spidergraph* $PS(k, n; r)$ is defined for $k \geq 3, n \geq 5$, and r such that $r^k \equiv \pm 1$ (mod n), but $r \not\equiv \pm 1$ (mod n). Its vertex set is $\mathbb{Z}_k \times \mathbb{Z}_n$, and vertex (i, j) is connected by edges to vertices $(i + 1, j \pm r^i)$. It may happen that this graph is not connected; if so, we re-assign the name to the connected component containing $(0, 0)$. Directing each edge from (i, j) to $(i + 1, j \pm r^i)$ gives a semitransitive orientation, and so the resulting graph is always semitransitive. (In this presentation, we wish to not include toroidal graphs as spidergraphs. This is what moves us to require $r \not\equiv \pm 1$ (mod n) and the consequent $n \geq 5$.)

Closely related is the *Mutant Power Spidergraph* $MPS(k, n; r)$. It is defined for $k \geq 3$, n even and $n \geq 8$, and r such that $r^k \equiv \pm 1$ (mod n), but $r \not\equiv \pm 1$ (mod n). Its vertex set is $\mathbb{Z}_k \times \mathbb{Z}_n$. For $0 \leq i < k - 1$, vertex (i, j) is connected by edges to vertices $(i + 1, j \pm r^i)$; vertex $(k-1, j)$ is connected to $(0, j \pm r^{k-1} + n/2)$. Marušič [25] and Šparl [51] have shown that every tetravalent tightly-attached graph is isomorphic to some PS or MPS graph, and that the graph is $\frac{1}{2}$-arc-transitive in all but a few cases: if $r^2 \equiv \pm 1$ (mod n), then the graph is dart-transitive. The very special graph $\Sigma = PS(3, 7; 2)$ is dart-transitive. If m is an integer not divisible by 7 and r is the unique solution mod $n = 7m$ to $r \equiv 5$ (mod 7), $r \equiv 1$ (mod m), then $PS(6, n; r)$ (which is a covering of Σ) is dart-transitive.

The paper [51] defines and notates these graphs in ways which differ from this paper, and the difference is worthy of note. If m, n, r, t are integers satisfying (1) m, n are even and at least 4, (2) $r^m \equiv 1$ (mod n) and (3) $s = 1 + r + r^2 + \ldots r^{m-1} + 2t$ is equivalent to 0 (mod n), then [51] defines the graph $X_e(m, n; r, t)$ ("e" stands for "even") to have vertices $[i, j]$ with $i \in \mathbb{Z}_m$ and $j \in \mathbb{Z}_n$ and edges from $[i, j]$ to $[i + 1, j]$ and $[i + 1, j + r^i]$ when $0 \leq i < m - 1$, while $[m - 1, j]$ is connected to $[0, j + t]$ and $[0, j + r^{m-1} + t]$.

The argument in [54] shows that if (1), (2) and (3) hold, then $r^m \equiv 1$ (mod $2n$). Then it is not hard to see that if the integer s is equivalent to 0 mod $2n$, then $X_e(m, n; r, t)$ is isomorphic to $PS(m, 2n; r)$; if s is equivalent to n mod $2n$, then it is isomorphic to $MPS(m, 2n; r)$.

9   Attebery graphs

The following quite general construction is due to Casey Attebery [3]. We will first define the digraph $\text{Att}[A, T, k; a, b]$ and then define the graph $\text{Att}(A, T, k; a, b)$ to be the underlying graph of the digraph. The parameters are: an abelian group A, an automorphism T of A, an integer k at least 3, and two elements a and b of A. Let $c = b - a$ and define a_i, b_i, c_i to be aT^i, bT^i, cT^i respectively for $i = 0, 1, 2, \ldots, k$. We require that:

(1) $\{a_k, b_k\} = \{a, b\}$,

(2) A is generated by $c_0, c_1, c_2, \ldots, c_{k-1}$ and $\Sigma_{i=0}^{k-1} a_i$; and

(3) $a + b$ is in the kernel of the endomorphism $T^* = \Sigma_{i=0}^{k-1} T^i$.

Then the vertex set of the digraph $\text{Att}[A, T, k; a, b]$ and of the graph $\text{Att}(A, T, k; a, b)$ is defined to be $A \times \mathbb{Z}_k$. In the digraph, edges lead from each (x, i) to $(x + a_i, i + 1)$ and $(x + b_i, i + 1)$. Then the digraph is a semitransitive orientation of the graph [3]. The graph is thus semitransitive, and it is often, but not always, $\frac{1}{2}$-arc-transitive.

Not all Attebery graphs are implemented in the Census. There are four special cases which are:

1. If $A = \mathbb{Z}_n$ and T is multiplication by r, the Attebery graph is just $\text{PS}(k, n; r)$, and thus the Attebery graphs are generalizations of the spidergraphs.

2. The graph called $C^{\pm 1}(p; st, s)$ in [14]. This is an Attebery graph with $A = \mathbb{Z}_p^s, k = st, T : (a_1, a_2, \ldots, a_s) \to (a_2, a_3, \ldots, a_s, a_1), -b = a = (1, 0, 0, \ldots, 0)$.

3. The graph called $C^{\pm e}(p; 2st, s)$ in [14] with $e^2 \equiv -1 \pmod{p}$. This is an Attebery graph with $A = \mathbb{Z}_p^s, k = 2st, T : (a_1, a_2, \ldots, a_s) \mapsto (ea_2, ea_3, \ldots, ea_s, ea_1), -b = a = (1, 0, 0, \ldots, 0)$.

4. Define $\text{AMC}(k, n, M)$ to be $\text{Att}(A, T, k; a, b)$ where A is $\mathbb{Z}_n \times \mathbb{Z}_n$, M is a 2×2 matrix over $\mathbb{Z}_n$ satisfying $M^k = \pm I$, T is multiplication by M, and $a = (1, 0), b = (-1, 0)$.

The second and third of these are generalized to the graph $\text{CPM}(n, s, t, r)$, defined in section 15 of this paper.

It is intriguing that even though the matrix

$$M = \begin{bmatrix} 1 & -4 \\ 4 & 1 \end{bmatrix}$$

does not satisfy the condition $M^4 = \pm I$, the graph $\text{AMC}(4, 12, M)$ is nevertheless edge-transitive, and in fact semisymmetric. Even more striking is that so far we have no other construction for this graph.

10   The separated box product

Suppose that Δ_1 and Δ_2 are digraphs in which every vertex has in- and out-valence 2. We allow Δ_1 and Δ_2 to be non-simple.

We form the *separated box product* $\Delta_1 \# \Delta_2$ (first defined in [41]) as the underlying graph of the orientation whose vertex set is $\mathcal{V}(\Delta_1) \times \mathcal{V}(\Delta_2) \times \mathbb{Z}_2$, and whose edge set contains two types of edges: "horizontal" edges join $(a, x, 0) \to (b, x, 1)$, and "vertical" edges $(b, x, 1) \to (b, y, 0)$, where $a \to b$ in Δ_1 and $x \to y$ in Δ_2.

An orientation is *reversible* provided that it is isomorphic to its reversal. As shown in Theorem 5.1 of [41], there are several useful cases of this construction:

- When $\Delta_1 = \Delta_2$ and Δ_1 is reversible, then $\Delta_1 \# \Delta_2$ is dart-transitive.

- When $\Delta_1 = \Delta_2$ and Δ_1 is not reversible, $\Delta_1 \# \Delta_2$ has a semitransitive orientation and so might be dart-transitive or $\frac{1}{2}$-transitive.

- When Δ_1 is not isomorphic to Δ_2 or its reverse but both are reversible, then $\Delta_1 \# \Delta_2$ has an LR structure.

- When Δ_1 is not isomorphic to Δ_2 but is isomorphic to the reverse of Δ_2, then $\Delta_1 \# \Delta_2$ is at least bi-transitive and is semisymmetric in all known cases.

In the Census, we use for Δ_1 and Δ_2 directed graphs from the census of 2-valent dart-transitive digraphs [38, 33] with notation $\text{ATD}[n, i]$ for the ith digraph of order n from that census. We also allow the "sausage digraph" DCyc_n: an n-cycle with each edge replaced by two directed edges, one in each direction. See [41] for more details.

11   Rose windows

The *Rose Window graph* $\text{R}_n(a, r)$ has $2n$ vertices: A_i, B_i for $i \in \mathbb{Z}_n$. The graph has four kinds of edges:

$$\begin{aligned} \text{Rim:} &\quad A_i - A_{i+1} \\ \text{In-Spoke:} &\quad A_i - B_i \\ \text{Out-spoke:} &\quad B_i - A_{i+a} \\ \text{Hub:} &\quad B_i - B_{i+r} \end{aligned}$$

For example, Figure 8 shows $\text{R}_{12}(2, 5)$.

Figure 8: $\text{R}_{12}(2, 5)$
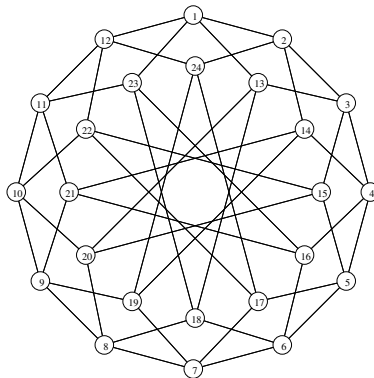
We will soon mention these graphs as examples of bicirculant graphs, and more generally later as examples of polycirculant graphs.

The paper [20] shows that every edge-transitive $\text{R}_n(a, r)$ is dart-transitive and is isomorphic to one of these:

a. $\text{R}_n(2, 1)$. (This graph is isomorphic to $\text{W}(n, 2)$.)

b. $R_{2m}(m+2, m+1)$.

c. $R_{12k}(3k \pm 2, 3k \mp 1)$.

d. $R_{2m}(2b, r)$, where $b^2 \equiv \pm 1 \pmod{m}$, r is odd and $r \equiv 1 \pmod{m}$.

12   Bicirculants

A graph Γ is *bicirculant* provided that it has a symmetry ρ which acts on $\mathcal{V}$ as two cycles of the same length. A rose window graph is bicirculant, as the symmetry sending A_i to A_{i+1} and B_i to B_{i+1} is the required ρ.

The paper [21] classifies all edge-transitive tetravalent bicirculant graphs. Besides the Rose Window graphs there is one other class of graphs, called BC4 in that paper, and called BC here and in the Census. The graph $BC_n(a, b, c, d)$ has $2n$ vertices: A_i, B_i for $i \in \mathbb{Z}_n$. The edges are all pairs of the form $\{A_i, B_{i+e}\}$ for $i \in \mathbb{Z}_n, e \in \{a, b, c, d\}$. It is easy to see that any such graph is isomorphic to one of the form $BC_n(0, a, b, c)$ where a divides n. The edge-transitive graphs in this class consist of three sporadic examples and three infinite families. The sporadics are:

$$BC_7(0, 1, 2, 4), \ BC_{13}(0, 1, 3, 9), \ BC_{14}(0, 1, 4, 6)$$

Of the three infinite families of graphs $BC_n(0, a, b, c)$, there are two in which we can choose $a = 1$, and a third, less easy to describe, in which none of the parameters is relatively prime to n:

(I) $BC_n(0, 1, m+1, m^2+m+1)$, where $(m+1)(m^2+1) = 0 \pmod{n}$

(II) $BC_n(0, 1, d, 1-d)$, where $2d(1-d) = 0 \pmod{n}$

(III) $BC_{krst}(0, r, rs's + st, rt't + st + rst)$, where

      (1) r, s, t are all integers greater than 1,

      (2) s, t are odd,

      (3) r, s, t are relatively prime in pairs,

      (4) $k \in \{1, 2\}$,

      (5) if $k = 2$, then r is even,

      (6) s' is an inverse of s mod krt, and t' is an inverse of t mod krs.

13   Semiregular symmetries and their diagrams

A symmetry σ is *semiregular* provided that it acts on the $|\mathcal{V}| = kn$ vertices as k cycles of length n. We can visually represent such a graph and symmetry with a *diagram*. This is a graph-like object, with labels. Each "node" represents one orbit under σ. If $\sigma = (u_0, u_1, \ldots, u_{n-1})(v_0, v_1, \ldots, v_{n-1}) \ldots (w_0, w_1, \ldots, w_{n-1})$ and there is an edge from u_0 to v_a, then there is an edge from each u_i to v_{a+i} (indices computed modulo n). This matching between $u_i's$ and $v_i's$ is represented in the diagram by a directed edge from node u to node v with label a (or one from v to u with label $-a$). If $a = 0$, then the label a and the direction of the edge in the diagram can be omitted.

If there is an edge from u_0 to u_b (and thus one from each u_i to u_{i+b}), we represent this by a loop at u with label b. In the special case in which n is even and $b = \frac{n}{2}$, there are only

$\frac{n}{2}$ edges in the orbit and we represent them with a semi-edge at u. This convention makes the valence of u in the diagram the same as the valences of all of the u_i's in the graph.

For example, the graph in Figure 9 has the symmetry

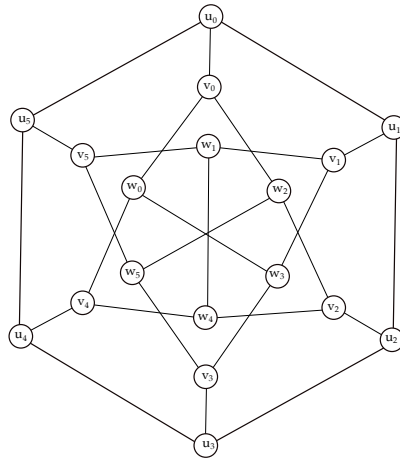$$\varphi = (u_0 u_1 u_2 u_3 u_4 u_5)(v_0 v_1 v_2 v_3 v_4 v_5)(w_0 w_1 w_2 w_3 w_4 w_5).$$

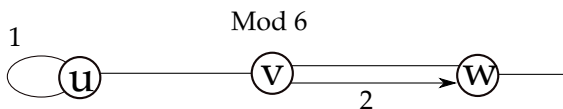Figure 9: A trivalent graph having a semiregular symmetry

Figure 10: The diagram of the semiregular symmetry φ

The corresponding diagram is shown in Figure 10. The label "Mod 6" in this diagram informs the reader that all numbers which label edges in the diagram are to be considered as elements of $\mathbb{Z}_6$.

It should be pointed out that what we were describing above is simply the notion of a quotient of a graph (as defined, for example, in [24]) by the cyclic group generated by the semiregular element σ, and the diagram which we obtain is the voltage graph describing the graph Γ.

Further, the graph can be reconstructed from the diagram. This is simply the ordinary voltage graph construction with voltage group $\mathbb{Z}_n$ and voltages of the darts as shown with the integers drawn at the darts in the diagram (if neither the dart nor its inverse dart have any integers drawn next to them, then the corresponding voltage is assumed to be 0).

In constructions, we often refer to the generic form of the diagram, in which the modulus is unspecified and parameter names are used as labels for some edges, as in Figure 11.

As an example, consider the bicirculant graphs $\mathrm{R}_n(a, r)$ and $\mathrm{BC}_n(0, a, b, c)$, which can be represented by the diagrams shown in Figure 12.

In the following, we use diagrams to define many families of graphs.
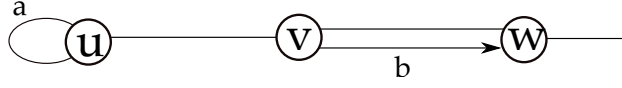
Figure 11: A generic diagram
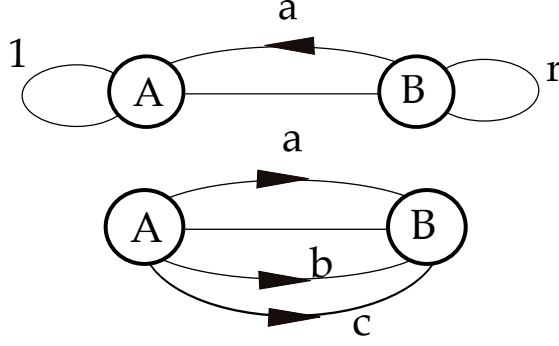
Figure 12: Diagrams of graphs

13.1   Propellors
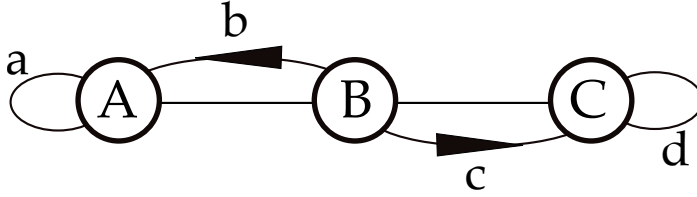
Figure 13: Diagram for the graph $\mathrm{Pr}_n(a, b, c, d)$

A *Propellor Graph* is a graph with the diagram shown in Figure 13. This means that the graph has $3n$ vertices: A_i, B_i, C_i for $i \in \mathbb{Z}_n$. There are 6 kinds of edges:

Tip:	$A_i - A_{i+a}$
	$C_i - C_{i+d}$
Flat:	$A_i - B_i$
	$B_i - C_i$
Blade:	$B_i - A_{i+b}$
	$B_i - C_{i+c}$

Propellor graphs have been investigated by Matthew Sterns. He conjectures in [46] that the edge-transitive propellor graphs are isomorphic to

I $\mathrm{Pr}_n(1, 2d, 2, d)$, where n is even and $d^2 \equiv \pm 1 \pmod{n}$

II $\mathrm{Pr}_n(1, b, b+4, 2b+3)$, where n is divisible by 4, $8b + 16 \equiv 0 \pmod{n}$, and $b \equiv 1 \pmod 4$.

III These five sporadic examples:

$$\mathrm{Pr}_5(1,1,2,2), \mathrm{Pr}_{10}(1,1,2,2),\ \mathrm{Pr}_{10}(1,4,3,2),\ \mathrm{Pr}_{10}(1,1,3,3),\ \mathrm{Pr}_{10}(2,3,1,4).$$

Notice first that in every case except the last of the sporadic cases, the A-tip edges form a single cycle. The first of the infinite families consists of the *2-weaving* graphs: those graphs in which some symmetry of the graphs sends this cycle to a cycle of the form AB AB AB. . . The second class consists of *4-weaving* graphs: those in which some symmetry sends the A-tip cycle to a cycle of the form ABCB ABCB AB. . . There is also a class of *5-weaving* graphs in which some image of the tip cycle is of the form AABCB AABCB AAB. . ., but the requirements for this class force n to divide 10, resulting in the first four of the sporadic cases. This leaves the last sporadic graph as something of a mystery.

A more recent paper [47] proves that if a or d is relatively prime to n, then the conjecture holds.

13.2 Metacirculants

In [29], Marušič and Šparl considered tetravalent graphs which are properly called *weak metacirculant*, though we will simply refer to them as *metacirculant* in this paper. A graph is metacirculant provided that it has a symmetry ρ which acts on the mn vertices as m cycles of length n and another symmetry σ which normalizes $\langle \rho \rangle$ and permutes the m ρ-orbits in a cycle of length m. That paper considers four classes of $\frac{1}{2}$-transitive tetravalent metacirculant graphs. The four classes together include all such graphs.

The Type I graphs are the Power Spidergraphs $\mathrm{PS}(m,n;r)$ and $\mathrm{MPS}(m,n;r)$. Papers [25, 29, 51, 54] completely determine which of these are $\frac{1}{2}$-transitive and which are dart-transitive. The Type II graphs are called Y there and will be called MSY here and in the census. These also have been classified, in unpublished work [2]. See Section 13.2.1 below.

The graphs of Type III we will call MC3 in this census. They have been studied with a few results. See Section 13.2.3. While the general Type IV graphs are very unruly, there are some results known; for instance, the paper [1] classifies Type IV $\frac{1}{2}$-arc-transitive metacirculants of girth 4. A subclass of Type IV graphs, called Z in [29] and MSZ in the Census, have received some concentrated study. See Section 13.2.2 for a description of the graph.

13.2.1 MSY

The graph $\mathrm{MSY}(m,n;r,t)$ has the diagram shown in Figure 14.

More precisely, its vertex set is $\mathbb{Z}_m \times \mathbb{Z}_n$, with two kinds of edges:

(1) $(i,j) - (i,j+r^i)$ for all i and j, and

(2) $(i,j) - (i+1,j)$ for $0 \le i < m-1$ and $(m-1,j) - (0,j+t)$ for all j.

The graph is seen to be metacirculant (with ρ sending (i,j) to $(i,j+1)$ and σ sending (i,j) to $(i+1,rj)$ for $i \neq m-1$ and sending $(m-1,j)$ to $(0,rj+t)$) if and only if $r^m = 1$ and $rt = t$. Here, all equalities are equivalences mod n. The paper [52] proves that $\mathrm{MSY}(m,n;r,t)$ is metacirculant and $\frac{1}{2}$-transitive if and only if it is isomorphic to one in which:

(1) $n = dm$ for some integer d at least 3,
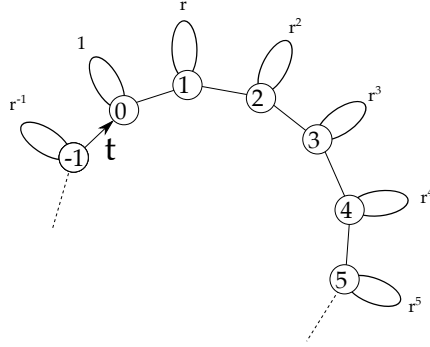
Figure 14: Diagram for the graph $\mathrm{MSY}(m, n; r, t)$

(2) $r^m = 1$,

(3) $r^2 \neq \pm 1$,

(4) $m(r-1) = t(r-1) = (r-1)^2 = 0$,

(5) $\langle m \rangle = \langle t \rangle$ in $\mathbb{Z}_n$,

(6) there is a unique c in $\mathbb{Z}_d$ satifying $cm = t$ and $ct = m$,

(7) there is a unique k in $\mathbb{Z}_d$ satifying $kt = -km = r - 1$, and

(8) either $m \neq 4$ or $t \neq 2 + 2r$.

The paper [2] shows that $\mathrm{MSY}(m, n; r, t)$ is metacirculant and edge-transitive if and only if it is isomorphic to one in which $r^m = 1, rt = t$ and one of three things happens:

(1) $m = GCD(t, n)$ (then $t = sm, n = n'm$ and $GCD(n', s) = 1$);

(2) $r \equiv 1 \pmod{m}$ and so $r = km + 1$ for some k;

(3) $st = m$;

(4) $kt = -km$.

or

(1) $m = GCD(t, n)$ (then $t = sm, n = n'm$ and $GCD(n', s) = 1$);

(2) $r \equiv 1 \pmod{m}$ and so $r = km + 1$ for some k;

(3) $st = -m$;

(4) $kt = km$.

or $[m, n, r, t]$ is one of these four sporadic examples:

$$[5, 11, 5, 0], \ [5, 22, 5, 11], \ [5, 33, 16, 0], \ [5, 66, 31, 33].$$

13.2.2   MSZ

The graph $\mathrm{MSZ}(m, n; k, r)$ has a diagram isomorphic to the circulant graph $C_m(1, k)$. It has vertex set $\mathbb{Z}_m \times \mathbb{Z}_n$. The vertex (i, j) is adjacent to $(i + 1, j)$ and to $(i + k, j + r^i)$.

13.2.3   MC3

The diagram for this family has an even number of nodes arranged in a circle. With $c = 1$ or 2, each node is joined by an edge to nodes that are c steps away in the circle. Each node is joined to the node opposite by two edges.

We call the graph $\Gamma = \mathrm{MC3}(m, n, a, b, r, t, c)$; here, m must be even, c is 1 or 2, and a, b, r, t are numbers mod n such that $rt = t, r^m = 1$, and $\{a+t, b+t\} = \{-ar^{\frac{m}{2}}, -br^{\frac{m}{2}}\}$. The vertex set is $\mathbb{Z}_m \times \mathbb{Z}_n$. One kind of edge connects each (i, j) to $(i + c, j)$ for $i = 0, 1, 2, \ldots, m - c - 1$, and (i, j) to $(i + c, j + t)$ for $i \in \{m - c, m - 1\}$. A second kind of edge joins each (i, j) to $(i + \frac{m}{2}, j + ar^i)$ and $(i + \frac{m}{2}, j + br^i)$.

Because it has been shown that each such graph which is $\frac{1}{2}$-arc-transitive is also a PS, MPS, MSY or MSZ, little attention has been given to it. However, many MC3's are dart-transitive and many are LR structures (see section 18, where we will refer to the two kinds of edges as *green* and *red*). Each of the following families is such an example:

1. m is divisible by 4, n is divisible by 2, $r^2 = \pm 1, a = 1, b = -1, t = 0$,

2. m is not divisible by 4, n is divisible by 4, $r^2 = 1, a = 1, b = -1, t = \frac{n}{2}$,

3. n is divisible by 4, $r^2 = 1, a = 1, b = n/2 - 1, t = n/2$.

These were found and proved to be LR strucures (see section 18) by Ben Lantz [22], and there are LR examples not covered by these families. Further, many of the MC3 graphs are dart-transitive. There are many open questions about this family.

13.3   Other diagrams

A number of other diagrams have been found to give what appears to be an infinite number of examples of edge-transitive graphs. The first of these is the Long Propellor, $\mathrm{LoPr}_n(a, b, c, d, e)$, shown in Figure 15.
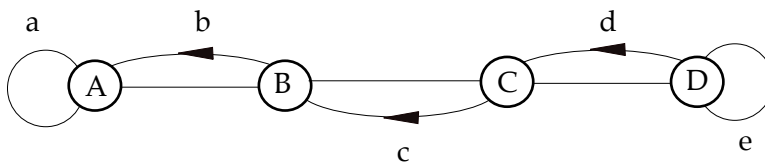
Figure 15: Diagram for the graph $\mathrm{LoPr}_n(a, b, c, d, e)$

Next is the Woolly Hat, $\mathrm{WH}_n(a, b, c, d)$, shown in Figure 16. This diagram appears to give no edge-transitive covers, but it does yield a family of LR structures (see Section 18), as yet unclassified.

The Kitten Eye, $\mathrm{KE}_n(a, b, c, d, e)$, shown in Figure 17, has dart-transitive covers.

The Curtain, $\mathrm{Curtain}_n(a, b, c, d, e)$, shown in Figure 18, has both dart-trasitive and LR covers.

14   Praeger-Xu constructions

The graph called $C(2, n, k)$ in [44] is also described in [14] in two different ways. In this Census, we will name the graph $\mathrm{PX}(n, k)$. In order to describe the graph, we need some
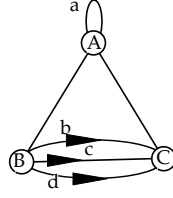
Figure 16: Diagram for the graph $\mathrm{WH}_n(a, b, c, d, e)$
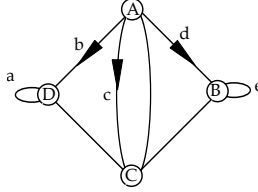
Figure 17: Diagram for the graph $\mathrm{KE}_n(a, b, c, d, e)$

notation about bit strings. A *bit string* of length k is the concatenation of k symbols, each of them a '0' or a '1'. For example $x = 0011011110$ is a bit string of length 10. If x is a bit string of length k, then x_i is its i-th entry, and x^i is the string identical to x in every place except the i-th. Also $1x$ is the string of length $k + 1$ formed from x by placing a '1' in front; similar definitions hold for the $(k + 1)$-strings $0x, x0, x1$. Finally, the string $\bar{x}$ is the reversal of x.

The vertices of the graph $\mathrm{PX}(n, k)$ are ordered pairs of the form (j, x), where $j \in \mathbb{Z}_n$ and x is a bit string of length k. Edges are all pairs of the form $\{(i, 0x), (i+1, x0)\}, \{(i, 0x), (i+1, x1)\}, \{(i, 1x), (i+1, x0)\}, \{(i, 1x), (i+1, x1)\}$, where x is any bit string of length $k-1$.

We first wish to consider some symmetries of this graph. First we note ρ and μ given by $(j, x)\rho = (j + 1, x)$ and $(j, x)\mu = (-j, \bar{x})$ These are clearly symmetries of the graph and act on it as D_r.

For $b \in \mathbb{Z}_n$, we define the symmetry τ_b to be the permutation which interchanges $(b - i, x)$ with $(b - i, x^i)$ for $i = 1, 2, 3, \ldots, k$ and leaves all other vertices fixed. If $n > k$, then the symmetries $\tau_0, \tau_1, \ldots, \tau_{n-1}$ commute with each other and thus generate an elementary abelian group of order 2^n, while the symmetries ρ, μ and τ_0 generate a semidirect product $\mathbb{Z}_2^n \rtimes D_n$ of order $n2^{n+1}$. Unless $n = 4$, this is also the full symmetry group of the graph (see [44]).

The Praeger-Xu graphs generalize two families of graphs: $\mathrm{PX}(n, 1) = \mathrm{W}(n, 2)$ and $\mathrm{PX}(n, 2) = \mathrm{R}_{2n}(n + 2, n + 1)$.

15   Gardiner-Praeger constructions

The paper [14] constructs two families of tetravalent graphs whose groups contain large normal subgroups such that the factor graph is a cycle. The first is $C^{\pm 1}(p, st, s)$, and the second is $C^{\pm e}(p, 2st, s)$. In the Census, we use a slight generalization of both, which we notate $\mathrm{CPM}(n, s, t, r)$, where n is any integer at least 3, s is an integer at least 2, t is a
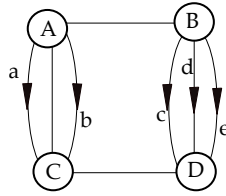
Figure 18: Diagram for the graph $\mathrm{Curtain}_n(a, b, c, d, e)$

positive integer, and r is a unit mod n. We first form the digraph $\mathrm{CPM}[n, s, t, r]$. Its vertex set is $\mathbb{Z}_n^s \times \mathbb{Z}_{st}$. Directed edges are of the form $((x, i), (x \pm r^i e_j, i + 1))$, where j is i mod s, and e_j is the j-th standard basis vector for $\mathbb{Z}_n^s$. If r^{st} is ± 1 mod n, then $\mathrm{CPM}[n, s, t, r]$ is a semitransitive orientation for its underlying graph $\mathrm{CPM}(n, s, t, r)$. When the graph is not connected, we re-assign the name $\mathrm{CPM}(n, s, t, r)$ to the component containing $(0, 0)$. If n is odd, then the graph has stn^s vertices. If n is even then it has $st(\frac{n}{2})^s$ vertices if t is even and twice that many if t is odd.

Some special cases are known: For $r = 1$, $\mathrm{CPM}(n, s, t, 1) \cong C^{\pm 1}(n, st, s)$. If $r^2 = 1$, then $\mathrm{CPM}(n, s, 2t, r)$ is $C^{\pm r}(p, 2st, s)$. When $s = 1$, $\mathrm{CPM}(n, 1, t, r) \cong \mathrm{PS}(n, t; r)$. If $s = 1$ and $t = 4$, then the graph is a Wreath graph. When $s = 2$ and t is 1 or 2, then the graph is toroidal. Other special cases are conjectured:

The convention in the following conjectures is that q is a number whose square is one mod m and p is the parity function:

$$p(t) = \begin{cases} 1 & \text{if t is odd} \\ 2 & \text{if t is even} \end{cases} \tag{15.1}$$

With that said, we believe that:

1. If t is not divisible by 3, then $\mathrm{CPM}(3, 2, t, 1) \cong \mathrm{PS}(6, m; q)$ where $m = 3t$.

2. If t is not divisible by 5, then $\mathrm{CPM}(5, 2, t, 1) \cong \mathrm{CPM}(5, 2, t, 2) \cong \mathrm{PS}(10, m; q)$ where $m = 5tp(t)$.

3. If t is not divisible by 3, then $\mathrm{CPM}(6, 2, t, 1) \cong \mathrm{PS}(6, m; q)$ where $m = \frac{12t}{p(t)}$.

4. If t is not divisible by 4, then $\mathrm{CPM}(8, 2, t, 1) \cong \mathrm{CPM}(8, 2, t, 3) \cong \mathrm{MPS}(8, m; q)$ where $m = \frac{16t}{p(t)}$.

5. For all s, $\mathrm{CPM}(4, s, t, 1) \cong \mathrm{PX}(\frac{2st}{p(t)}, s)$.

16   Graphs $\Gamma^\pm$ of Spiga, Verret and Potočnik

It was proved in [37] that a tetravalent graph Γ whose automorphism group G is dart-transitive is either 2-arc-transitive (and then $|G_v| \le 2^4 3^6$), a PX-graph (see Section 14), one of eighteen exceptional graphs, or it satisfies the inequality

$$|V(\Gamma)| \ge 2|G_v| \log_2(|G_v|/2). \tag{$*$}$$

This result served as the basis for the construction of a complete list of dart-transitive tetravalent graphs (see [36] for details).

Moreover, in [35] it was proved that a graph attaining the bound $(*)$ has $t2^{t+2}$ vertices for some $t \geq 2$ and is isomorphic to one of the graphs (t, ϵ), for $\epsilon \in \{0, 1\}$, defined below as *coset graphs* of certain groups G_t^0 or G_t^1. Recall that the coset graph $\text{Cos}(G, H, a)$ on a group G relative to a subgroup $H \leq G$ and an element $a \in G$ is defined as the graph with vertex set the set of right cosets $G/H = \{Hg \mid g \in G\}$ and with edge set the set $\{\{Hg, Hag\} \mid g \in G\}$.

For $\epsilon \in \{0, 1\}$ and $t \geq 2$, let G_t^ϵ be the group defined as follows:

$$
\begin{aligned}
G_t^\epsilon = \langle x_0, \ldots, x_{2t-1}, z, a, b \quad | \quad & x_i^2 = z^2 = b^2 = z^\epsilon a^{2t} = (ab)^2 = 1, \\
& [x_i, z] = 1 \text{ for } 0 \leq i \leq 2t - 1, \\
& [x_i, x_j] = 1 \text{ for } |i - j| \neq t, \\
& [x_i, x_{t+i}] = z \text{ for } 0 \leq i \leq t - 1, \\
& x_i^a = x_{i+1} \text{ for } 0 \leq i \leq 2t - 1, \\
& x_i^b = x_{t-1-i} \text{ for } 0 \leq i \leq 2t - 1 \rangle.
\end{aligned}
$$

In either group, we let $H = \langle x_0, \ldots, x_{t-1}, b \rangle$, and define graphs

$$
\text{PPM}(t, \epsilon) = \text{Cos}(G_t^\epsilon, H, a),
$$

denoted Γ_t^+ (for $\epsilon = 0$) and Γ_t^- (for $\epsilon = 1$) in [35]. We should point out that a graph $\Gamma = \text{PPM}(t, \epsilon)$ is a 2-cover of the Praeger-Xu graph $\text{PX}(2t, t)$. Furthermore, the girth of $\text{PPM}(t, \epsilon)$ is generally 8, the only exceptions being that $\text{PPM}(2, 0)$ has girth 4, and $\text{PPM}(3, 0)$ has girth 6. Finally, $\text{PPM}(2, 0) \cong \text{PX}(4, 3)$, while in all other cases the graph Γ is not isomorphic to a PX graph.

The 2016 edition of the Census seems, mysteriously, to have missed implementing this construction. The next edition *will* include these graphs. Meanwhile, we know that exactly 6 of them fall in the range of sizes addressed in the Census. The resulting entries are:

PPM(2,0) = C4[32,2] = $\{4, 4\}_{4,4}$.
PPM(2,1) = C4[32,5] = $\text{MSY}(4, 8, 5, 4)$.
PPM(3,0) = C4[96,27] = $\text{KE}_{24}(1, 22, 8, 3, 7)$.
PPM(3,1) = C4[96,24] = $\text{KE}_{24}(1, 13, 4, 21, 5)$.
PPM(4,0) = C4[256,72] = $\text{UG}(\text{ATD}[256, 128])$.
PPM(4,1) = C4[256,78] = $\text{UG}(\text{ATD}[256, 146])$.

17   From cubic graphs

In this section, we describe five constructions, each of which constructs a tetravalent graph from a smaller cubic (i.e., trivalent) graph in such a way that the larger graph inherits many symmetries from the smaller graph. Throughout this section, assume that Λ is a cubic graph, and that it is dart-transitive. Our source of these graphs is Marston Conder's census of symmetric cubic graphs of up to 10,000 vertices [8].

17.1   Line graphs

The *line graph* of Λ is a graph $\Gamma = L(\Lambda)$ whose vertices are, or correspond to, the edges of Λ. Two vertices of Γ are joined by an edge exactly when the corresponding edges of Λ share a vertex. Every symmetry of Λ acts on Γ as a symmetry, though Γ may have other symmetries as well. Clearly, if Λ is edge-transitive, then Γ is vertex-transitive. If Λ is dart-transitive, then Γ is edge-transitive, and if Λ is 2-arc-transitive, then Γ is dart-transitive.

17.2 Dart graphs

The *Dart Graph* of Λ is a graph $\Gamma = \mathrm{DG}(\Lambda)$ whose vertices are, or correspond to, the darts of Λ. Edges join a dart (a, b) to the dart (b, c) whenever a and c are distinct neighbors of b. Clearly, $\mathrm{DG}(\Lambda)$ is a two-fold cover of $L(\Lambda)$.

17.3 Hill capping

For every vertex A of Λ, we consider the *symbols* $(A, 0), (A, 1)$, though we will ususaly write them as A_0, A_1. Vertices of $\Gamma = \mathrm{HC}(\Lambda)$ are all unordered pairs $\{A_i, B_j\}$ of symbols where $\{A, B\}$ is an edge of Λ. Edges join each vertex $\{A_i, B_j\}$ to $\{B_j, C_{1-i}\}$ where A and C are distinct neighbors of B.

If Λ is bipartite and 2-arc-transitive then Γ is dart-transitive. If Λ is bipartite and *not* 2-arc-transitive then Γ is semisymmetric. If Λ is *not* bipartite and is 2-arc-transitive then Γ is $\frac{1}{2}$-arc-transitive.

$\mathrm{HC}(\Lambda)$ is clearly a fourfold cover of $L(\Lambda)$; it is sometimes but not always a twofold cover of $\mathrm{DG}(\Lambda)$. The Hill Capping is described more fully in [15].

17.4 3-arc graph

The three-arc graph of Λ, called $A_3(\Lambda)$ in the literature [18] and called $\mathrm{TAG}(\Lambda)$in the Census, is a graph whose vertices are the darts of Λ, with (a, b) adjacent to (c, d) exactly when $[b, a, c, d]$ is a 3-arc in Λ. Thus, a and c are adjacent, $b \neq c$ and $a \neq d$. This graph is dart-transitive if Λ is 3-arc-transitive.

18 Cycle decompositions

A *cycle decomposition* of a tetravalent graph Λ is a partition $\mathcal{C}$ of its edges into cycles. Every edge belongs to exactly one cycle in $\mathcal{C}$ and each vertex belongs to exactly two cycles of $\mathcal{C}$. $\mathrm{Aut}(\mathcal{C})$ is the group of all symmetries of Λ which preserve $\mathcal{C}$. One possibility for such a symmetry is a *swapper*. If v is a vertex on the cycle C, a $C-$*swapper* at v is a symmetry which reverses C while fixing v and every vertex on the other cycle through v.

If $\mathcal{C}$ is a cycle decomposition of Λ, the *partial line graph* of $\mathcal{C}$, written $\mathbb{P}(\mathcal{C})$ and notated $\mathrm{PL}(\mathcal{C})$ in the Census, is a graph Γ whose vertices are (or correspond to) the edges of Λ, and whose edges are all $\{e, f\}$ where e and f are edges which share a vertex but belong to different cycles of $\mathcal{C}$.

Because $\mathrm{Aut}(\mathcal{C})$ acts on Γ as a group of its symmetries, the partial line graph is useful for constructing graphs having a large symmetry group. Almost all tetravalent dart-transitive graphs have cycle decompositions whose symmetry group is transitive on darts. These are called "cycle structures" in [31].

If Λ is $\frac{1}{2}$-arc-transitive, then it has a cycle decomposition $\mathcal{A}$ into 'alternating cycles' [25]. If the stabilizer of a vertex has order at least 4, then $\mathbb{P}(\mathcal{A})$ has a $\frac{1}{2}$-arc-transtive action and may actually be $\frac{1}{2}$-arc-transitive.

Many graphs in the census are constructed from smaller ones using the partial line graph. Important here are the Praeger-Xu graphs. Each $\mathrm{PX}(n, k)$ has a partition $\mathcal{C}$ of its edges into 4-cycles of the form:$(i, 0x), (i + 1, x0), (i, 1x), (i + 1, x1)$. Then $\mathbb{P}(\mathcal{C})$ is isomorphic to $\mathrm{PX}(n, k + 1)$. A special case of this is that family (b) of Rose Window graphs is $\mathbb{P}$ applied to family (a), the wreath graphs.

The toroidal graphs have a cycle decomposition in which each cycle consists entirely

of vertical edges or entirely of horizontal edges. The partial line graph of this cycle decomposition is another toroidal graph. For the rotary case, $\mathbb{P}(\{4,4\}_{b,c}) = \{4,4\}_{b+c,b-c}$. The other two families of toroidal maps need not have edge-transitive partial line graphs.

It may happen that a cycle decomposition $\mathcal{C}$ is a 'suitable LR structure' [42]; this means that $\mathcal{C}$ has a partition into $\mathcal{R}$ and $\mathcal{G}$ (the 'red' and the 'green' cycles) such that every vertex belongs to one cycle from each set, that the subgroup of $\mathrm{Aut}(\mathcal{C})$ which sends $\mathcal{R}$ to itself is transitive on the vertices of Λ, that $\mathrm{Aut}(\mathcal{C})$ has all possible swappers, that no element of $\mathrm{Aut}(\mathcal{C})$ interchanges $\mathcal{R}$ and $\mathcal{G}$ and, finally, that no 4-cycle alternates between $\mathcal{R}$ and $\mathcal{G}$. With all of that said, if $\mathcal{C}$ is a suitable LR structure, then $\mathbb{P}(\mathcal{C})$ is a semisymmetric tetrtavalent graph in which each edge belongs to a 4-cycle. Further, every such graph is constructed in this way. [42]

19   Some LR structures

The Census uses the partial line graph construction on a number of families of LR structures, shown here.

19.1   Barrels

The barrels are the most common of the suitable LR structures. The standard barrel is $\mathrm{Br}(k,n;r)$, where k is an even integer at least 4, n is an integer at least 5 and r is a number mod n such that $r^2 = \pm 1 \pmod{n}$ but $r \neq \pm 1 \pmod{n}$. The vertex set is $\mathbb{Z}_k \times \mathbb{Z}_n$. Green edges join each (i,j) to $(i,j+r^i)$. Red edges join each (i,j) to $(i+1,j)$.

The mutant barrel is $\mathrm{MBr}(k,n;r)$, where k is an even integer at least 2, n is an *even* integer ≥ 8 and r is a number mod n such that $r^2 = \pm 1 \pmod{n}$ but $r \neq \pm 1 \pmod{n}$. The vertex set is $\mathbb{Z}_k \times \mathbb{Z}_n$. Green edges join each (i,j) to $(i,j+r^i)$. Red edges join each (i,j) to

$$\begin{cases} (i+1,j) & \text{if } i \neq k-1 \\ (0,j+\frac{n}{2}) & \text{if } i = k-1 \end{cases}.$$

19.2   Cycle structures

If Λ is a tetravalent graph admitting a cycle structure $\mathcal{C}$, we can form an LR struture from it in two steps:

1. Replace each vertex v with two vertices, each incident with the two edges of one of the two cycles in $\mathcal{C}$ containing v; think of these as green edges. Join the two vertices corresponding to v with two parallel red edges. Call this C'. We can cover C' in different ways.

2a. Double cover C'. We assign weights or voltages to red edges so that each pair has one 0 and one 1. Voltages for green edges are assigned in one of two ways: (0) every green edge gets voltage 0 or (1) one edge in each green cycle gets voltage 1, and the rest get 0. The double covers corresponding to these two assignments are called $\mathrm{CS}(\Lambda,\mathcal{C},0)$ and $\mathrm{CS}(\Lambda,\mathcal{C},1)$, respectively, and they are, in most cases, suitable LR structures, as shown in [43].

2b. Cover C' k-fold for some k at least 3. We assign weights or voltages to red edges so that each pair has a 1 in each direction. All green edges are assigned voltage 0. The k-cover corresponding to this assignment is called $\mathrm{CSI}(\Lambda,\mathcal{C},k)$, and [43] shows that each

such is a suitable LR structure.

19.3 Bicirculants

Consider a bicoloring of the edges of the bicirculant $\mathrm{BC}_n(0, a, b, c)$ with green edges linking A_i to B_i and B_{i+a} and red edges linking A_i to B_{i+b} and B_{i+c}. We call this coloring $\mathrm{BC}_n(\{0, a\}, \{b, c\})$. The paper [40] shows several cases in which $\mathrm{BC}_n(\{0, a\}, \{b, c\})$ is a suitable LR structure:

1. $a = 1 - r, b = 1, c = s$, where $r, s \in \mathbb{Z}_n^* \setminus \{-1, 1\}$, $r^2 = s^2 = 1$, $r \notin \{-s, s\}$, and $(r-1)(s-1) = 0$.

2. $n = 2m, a = m, b = 1, c \in \mathbb{Z}_{2m} \setminus \{1, -1, m+1, m-1\}$ such that $c^2 \in \{1, m+1\}$.

3. $n = 4k, a = 2k, b = 1, c = k + 1$, for $k \geq 3$

Moreover, it is conjectured in that paper that every suitable $\mathrm{BC}_n(\{0, a\}, \{b, c\})$ is isomorphic to at least one of these three.

19.4 MSY's and MSZ's

The graph $\mathrm{MSY}(m, n; r, t)$ has an LR structure, with edges of the first kind being red and those of the second kind being green, if and only if $2t = 0$ and $r^2 = \pm 1$. Many examples of MSZ graphs being suitable LR structures are known, but no general classification has been attempted.

19.5 Stack of pancakes

The structure is called $\mathrm{SoP}(4m, 4n)$. Let $r = 2n + 1$. The vertex set is $\mathbb{Z}_{4m} \times \mathbb{Z}_{4n} \times \mathbb{Z}_2$. Red edges join (i, j, k) to $(i, j \pm r^k, k)$; for a fixed i and j, green edges join the two vertices $(2i, j, 0)$ and $(2i, j, 1)$ to the two vertices $(2i + 1, j, 0)$ and $(2i + 1, j, 1)$ if j is even, to the two vertices $(2i - 1, j, 0)$ and $(2i - 1, j, 1)$ if j is odd.

The paper [43] shows that this is a suitable LR structure for all m and n, and that the symmetry group of it and of its partial line graph, can have arbitrarily large vertex-stabilizers.

19.6 Rows and columns

The LR structure $\mathrm{RC}(n, k)$ has as vertices all ordered pairs $(i, (r, j))$ and $((i, r), j)$, where i and j are in $\mathbb{Z}_n$, and r is in $\mathbb{Z}_k$, where k and n are integers at least 3. Green edges join $(i, (r, j))$ to $(i \pm 1, (r, j))$ and $((i, r), j)$ to $((i, r), j \pm 1)$, while red edges join $(i, (r, j))$ to $((i, r \pm 1), j)$ and so $((i, r), j)$ to $(i, (r \pm 1, j))$.

This structure is referred to in both [40] and [43].

19.7 Cayley constructions

Suppose a group A is generated by two sets, R and G, of size two, neither containing the identity, and each containing the inverse of each of its elements. Then we let $\mathrm{Cay}(A; R, G)$ be the structure whose vertex set is A, whose red edges join each a to sa for $s \in R$ and whose green edges join each a to sa for $s \in G$. The paper [40] shows that if A admits two

automorphisms, one fixing each element of R but interchanging the two element of G and the other vice versa, then $\text{Cay}(A; R, G)$ is an LR structure. The condition $RG \neq GR$ is equivalent to the structure not having alternating 4-cycles.

Many examples occur in the case where A is the dihedral group D_n. One family of this type is the first group of bicirculants $BC_n(\{0, 1 - r\}, \{1, s\})$ shown in subsection 19.3.

The paper [40] shows several other algebraically defined structures. First, there are examples for the group D_n where the swappers do not arise from group automorphisms. Second, there is a Cayley construction for the structure $RC(n, k)$ of the previous subsection.

Further, the body of the paper shows six 'linear' constructions in which A is an extension of some $\mathbb{Z}_n^k$:

Construction 19.1. *For n and k both at least 3, let A be a semidirect product of $\mathbb{Z}_n^k$ with the group generated by the permutation $\sigma = (123 \ldots k - 1k)$ acting on the coordinates. Let e_1 be the standard basis element $(100 \ldots 0)$, let $R = \{e_1, -e_1\}$ and $G = \{\sigma, \sigma^{-1}\}$. We define the LR structure $\text{AffLR}(n, k)$ to be $\text{Cay}(A; R, G)$.*

Construction 19.2. *Let $\text{ProjLR}(k, n)$ be $\text{AffLR}(k, n)$ factored out by the cyclic group generated by $(1, 1, \ldots, 1)$.*

Construction 19.3. *Let $\text{ProjLR}^\circ(2k, n)$ be $\text{AffLR}(2k, n)$ factored out by the group generated by all $d_i = e_i - e_{i+k}$, where e_i is the standard basis element having a 1 in position i and zeroes elsewhere.*

Construction 19.4. *Let A be a semidirect product of $\mathbb{Z}_2^{2k}$ with the group generated by the permutation $\gamma = (1, 2, 3, \ldots, k)(k + 1, k + 2, \ldots, 2k)$. acting on the coordinates. Let $R = \{e_1, e_{k+1}\}$ and $G = \{\gamma, \gamma^{-1}\}$. We define the LR structure $\text{AffLR}_2(k)$ to be $\text{Cay}(A; R, G)$.*

Let d_1 be the $2k$-tuple in which the first k entries are 1 and the last k entries are 0; let d_2 be the $2k$-tuple in which the first k entries are 0 and the last k entries are 1; let $d = d_1 + d_2$.

Construction 19.5. *Let $\text{ProjLR}'(k)$ be $\text{AffLR}_2(k)$ factored out by the group generated by d_1 and d_2.*

Construction 19.6. *Let $\text{ProjLR"}(k)$ be $\text{AffLR}_2(k)$ factored out by the group generated by d.*

The paper [40] proves that all six of these constructions lead to suitable LR structures (except for a few cases).

20 Base graph-connection graph constructions

This section deals with a family of constructions called *base graph - connection graph* (BGCG) constructions. In these, we construct an edge-transitive bipartite graph Γ from a set of copies of the subdivision (see section 3.2) of a *base graph B*, connected according to a *connection graph C* and some other information.

For a tetravalent graph B, let B^* be its subdivision, i.e., the graph resulting by replacing each edge of B with a path of length 2. We refer to its 4-valent vertices as *black* and its valence-2 vertices as *white*. We will often refer to an edge of B and the white vertex on the path that replaces it in B^* as being equivalent or corresponding.

In the constructions we often define a partition $\mathcal{P}$ of the edges of a graph (equivalently, of the white vertices of its subdivision) into sets of size 2, and refer to such a $\mathcal{P}$ as a *pairing*. We can regard the partition as a coloring of the edges of the graph. We can also think of the pairing as an involutory permutation by writing $y = x^{\mathcal{P}}$ when $\{x, y\} \in \mathcal{P}$.

Given simple graphs B and C, let X be a disjoint union of copies of B, one copy B_r for each vertex r of C. Refer to this X as B^C. A pairing $\mathcal{P}$ of X is *compatible* with C provided that it satisfies these two conditions:

1. if $\{x, y\} \in \mathcal{P}$ with x in B_r and y in B_s, $r \neq s$, then $\{r, s\}$ is an edge of C, and
2. every edge of C is so represented.

These ingredients are combined in the following construction:

Construction 20.1. *Given a tetravalent base graph B, a connection graph C, and a pairing $\mathcal{P}$ on $X = B^C$ which is compatible with C, form a graph Γ by identifying each pair of white vertices of X^* which corresponds to an element of $\mathcal{P}$. We refer to Γ as a BGCG of B and C with respect to $\mathcal{P}$, and write $\Gamma = \mathrm{BGCG}(B, C, \mathcal{P})$.*

Theorem 20.2 ([49])**.** *If B is tetravalent, C is a graph, and $\mathcal{P}$ is a pairing of B^C which is compatible with C, then $\Gamma = \mathrm{BGCG}(B, C, \mathcal{P})$ is a bipartite tetravalent graph.*

Definition 20.3. *If X is a (connected or disconnected) tetravalent graph and $\mathcal{P}$ is a pairing on X, we will call $\mathcal{P}$ a *dart-transitive pairing* provided that the subgroup of $\mathrm{Aut}(X)$ which preserves $\mathcal{P}$ is transitive on the darts of X.*

Theorem 20.4 ([49])**.** *If B and C are each dart-transitive and if $\mathcal{P}$ is a dart-transitive pairing on $X = B^C$ which is compatible with C, and if $\Gamma = \mathrm{BGCG}(B, C, \mathcal{P})$ is defined, then Γ is a bipartite edge-transitive graph.*

Theorem 20.4 is the basis for several BGCG constructions, each depending on the nature of C.

Theorem 20.5 ($C = K_1$)**.** *Suppose that $\mathcal{Q}$ is a dart-transitive pairing on B. Then $\mathrm{BGCG}(B, K_1, \mathcal{Q})$ is an edge-transitive tetravalent graph.*

Theorem 20.6 ($C = K_2$)**.** *Suppose that $\mathcal{Q}$ is a dart-transitive pairing on B, and consider the vertex set of K_2 to be $\{1, 2\}$. If e is an edge of B, let $(e, 1)$ and $(e, 2)$ be the corresponding edges in $X = B^{K_2}$. Define $\mathcal{P}$ on X by $(e, 1)^{\mathcal{P}} = (e^{\mathcal{Q}}, 2)$. Then $\mathrm{BGCG}(B, K_2, \mathcal{P})$ is an edge-transitive tetravalent graph.*

No truly general technique yet exists in the case when C is the n-cycle C_n. We give here two constructions from [56] and remark that there are many more for $C = C_n$.

Construction 20.7. *Suppose that*

1. *$\mathcal{Q}$ is a dart-transitive pairing of B invariant under a dart-transitive group G,*
2. *there is a partition $\{\mathcal{R}, \mathcal{G}\}$ of the edges of B into two 'colors' ($\mathcal{R}$ = 'red', $\mathcal{G}$ = 'green') which is also invariant under G,*
3. *each pair in $\mathcal{Q}$ meets both $\mathcal{R}$ and $\mathcal{G}$,*
4. *the vertex set of C_n is $\mathbb{Z}_n$.*

Then we can form $\mathcal{P}_1$ and $\mathcal{P}_2$ on $X = B^{C_n}$ as follows: for each pair $\{e, f\}$ in $\mathcal{Q}$, with $e \in \mathcal{R}$, and for each $i \in \mathbb{Z}_n$, we let $(e, i)^{\mathcal{P}_1} = (f, i + 1)$. If n is even, then for i even, we let $(e, i)^{\mathcal{P}_2} = (e, i + 1)$, and for i odd, we let $(f, i)^{\mathcal{P}_2} = (f, i + 1)$.

Theorem 20.8 ($C = C_n$). *With $\mathcal{P}_1$ and $\mathcal{P}_2$ as above, $\mathrm{BGCG}(B, C_n, \mathcal{P}_1)$ and $\mathrm{BGCG}(B, C_n, \mathcal{P}_2)$ are edge-transitive tetravalent graphs.*

In the Census, we have no efficient way to describe the diferent pairs, $(\mathcal{Q}, \{\mathcal{R}, \mathcal{G}\})$, and so we simply number them. If the pair is numbered i, the Census reports $\mathrm{BGCG}(B, C_n, \mathcal{P}_1)$ as "$\mathrm{BGCG}(B, C_n, i)$" and $\mathrm{BGCG}(B, C_n, \mathcal{P}_2)$ as "$\mathrm{BGCG}(B, C_n, i')$".

21   From regular maps

A *map* is an embedding of a graph or multigraph on a compact connected surface such that each component of the complement of the graph (these are called *faces*) is topologically a disk. A *symmetry* of a map is a symmetry of the graph which extends to a homeomorphism of the surface. A map $\mathcal{M}$ is *rotary* provided that for some face and some vertex of that face, there is a symmetry R which acts as rotation one step about the face and a symmetry S which acts as rotation one step about the vertex. A map $\mathcal{M}$ is *reflexible* provided that it is rotary and has a symmetry X acting as a reflection fixing that face and vertex. If $\mathcal{M}$ is rotary but not reflexible, we call it *chiral*. See [15] for more details.

If $\mathcal{M}$ is rotary, its symmetry group, $\mathrm{Aut}(\mathcal{M})$, is transitive on faces and on vertices. Thus all faces have the same number p of sides and all vertices have the same degree q. We then say that $\mathcal{M}$ has *type* $\{p, q\}$.

21.1   Underlying graphs

The underlying graph of a rotary map $\mathcal{M}$ is called $\mathrm{UG}(\mathcal{M})$ and is always dart-transitive. If $q = 4$, it belongs in this census.

21.2   Medial graphs

The vertices of the medial graph, $\mathrm{MG}(\mathcal{M})$, are the edges of $\mathcal{M}$. Two are joined by an edge if they are consecutive in some face (and so in some vertex). If $\mathcal{M}$ is rotary, $\mathrm{MG}(\mathcal{M})$ is always edge-transitive. If $\mathcal{M}$ is reflexible or if it is self-dual in such a way that some orientation-preserving homeomorphism of the surface interchanges $\mathcal{M}$ with its dual, $D(\mathcal{M})$, then $\mathrm{MG}(\mathcal{M})$ is dart-transitive. If not, it is quite often, but not quite always, $\frac{1}{2}$-transitive. No one seems to know a good criterion for this distinction.

21.3   Dart graphs

The vertices of the dart graph, $\mathrm{DG}(\mathcal{M})$, are the darts of $\mathcal{M}$. Two are joined by an edge if they are head-to-tail consecutive in some face. The graph $\mathrm{DG}(\mathcal{M})$ is a twofold cover of $\mathrm{MG}(\mathcal{M})$ and is often the medial graph of some larger rotary map. It can be dart-transitive or $\frac{1}{2}$-transitive; again, no good criterion is known.

21.4   HC of maps

The Hill Capping of a rotary map $\mathcal{M}$ is defined in a way completely analogous to the capping of a cubic graph Λ: we join $\{A_i, B_j\}$ to $\{B_j, C_{1-i}\}$ where A, B and C are vertices which are consecutive around some face. The graph $\mathrm{HC}(\mathcal{M})$ is a 4-fold covering

of $\mathrm{MG}(\mathcal{M})$ and can be dart-transitive or semisymmetric or $\frac{1}{2}$-transitive or even not edge-transitive.

21.5 XI of maps

Suppose that $\mathcal{M}$ is a rotary map of type $\{p, q\}$ for some even $q = 2n$. Then each corner of the map (formed by two consecutive edges in one face) is opposite at that vertex to another corner; we will call such a pair of corners an 'X'. As an example, consider Figure 19, which shows one vertex, of degree 6, in a map. The X's are pairs a, b, c of opposite corners.
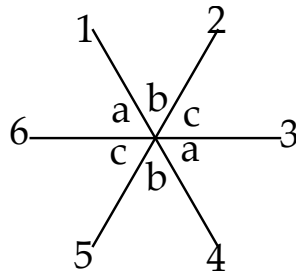

Figure 19: X's and I's in a map

We form the bipartite graph $\mathrm{XI}(\mathcal{M})$ in this way: The black vertices are the X's, the white vertices are the edges of $\mathcal{M}$, and edges of $\mathrm{XI}(\mathcal{M})$ are all pairs $\{x, e\}$ where x is an X and e is one of the four edges of x. Continuing our example, the black vertex a is adjacent to white vertices 1, 3, 4, 6, while b is adjacent to 1, 2, 4, 5 and c to 2, 3, 5, 6.

It is interesting to see the construction introduced here as a special case of two previous constructions. First it is made from $\mathrm{MG}(\mathcal{M})$ using a BGCG construction in which each edge of $\mathrm{MG}(\mathcal{M})$ is paired with the one opposite it at the vertex of $\mathcal{M}$ containing the corresponding corner.

Secondly, it is $\mathbb{P}$ of an LR structure called a *locally dihedral cycle structure* as outlined at the end of [43].

It is clear that if $\mathcal{M}$ is reflexible, then $\mathrm{XI}(\mathcal{M})$ is edge-transitive; it is surprising, though, that sometimes (criteria still unknown) XI of a chiral map can also be edge-transitive.

22 Sporadic graphs

There are a few graphs in the Census which are given familiar names rather than a parametric form. These are: $K_5 = C_5(1, 2)$, the Octahedron $= K_{2,2,2}$. Also there is the graph $\mathrm{Odd}(4)$; its vertices are subsets of $\{1, 2, 3, 4, 5, 6, 7\}$ of size 3. Two are joned by an edge when the sets are disjoint. Finally, there is the graph denoted $\mathrm{Gray}(4)$ due to a construction by Bouwer [7] which generalizes the Gray graph to make a semisymmetric graph of valence n on $2n^n$ vertices, in this case, 512. In fact we wanted to extend the Census to 512 vertices in order to include this graph.

Out of more than 7000 graphs in this Census, 400 of them have as their listed names one of the tags $\mathrm{AT}[n, i]$, $\mathrm{HT}[n, i]$ or $\mathrm{SS}[n, i]$ from the computer-generated censi. These graphs, then, must have no other known constructions. Each of these might be truly sporadic or, perhaps, might belong to some interesting family not yet recognized.

Each of these is a research project in its own right, a single example waiting to be meaningfully generalized.

23   Open questions

1. In compiling the Census, whenever we wanted to include a parameterized family $(C_n(1, a)$ (section 5), $\mathrm{BC}_n(a, b, c, d)$ (section 12), $\mathrm{Pr}_n(a, b, c, d)$ (section 13.1), etc.), it was very helpful to have some established theorems which either completely classified which values of the parameters gave edge-transitive graphs or restricted those parameters in some way. In families for which no such theorems were known, we were forced into brute-force searches, trying all possible values of the parameters. In many cases this caused our computers to run out of time or space before finishing the search. There are many families for which no such results or only partial results exist. So, our first and most pressing question is :
   For which values of parameters are the following graphs edge-transitive (or LR):

   $\mathrm{AMC}(k, n, M)$ (section 9),
   $\mathrm{MSZ}(m, n; k, r)$ (section 13.2.2),
   $\mathrm{MC3}(m, n, a, b, r, t, c)$ (section 13.2.3),
   $\mathrm{LoPr}_n(a, b, c, d, e)$ (section 13.3),
   $\mathrm{WH}_n(a, b, c, d)$ (section 13.3),
   $\mathrm{KE}_n(a, b, c, d, e)$ (section 13.3),
   $\mathrm{Curtain}_n(a, b, c, d, e)$ (section 13.3),
   $\mathrm{CPM}(n, s, t, r)$ (section 15).

2. The general class IV metacirculants (of which the graphs MSZ are merely a part) has begun to be explored. It is parameterized in [1], where the authors point out that not all values of the parameters which make the graph metacirculent make it $\frac{1}{2}$-arc-transitive. Then the same questions as above are relevant: When are these graphs isomorphic? dart-transitive? $\frac{1}{2}$-arc-transitive? LR structures?

3. Given a and n with $a^2 \equiv \pm 1 \pmod{n}$, which toroidal graph is isomorphic to $C_n(1, a)$? See sections 5 and 6. The toroidal graphs are very common and almost every family includes some as special cases. In researching a family, we often point out that certain values of the parameters give toroidal graphs and so will not be studied with this family. However, it is often difficult to say exactly which toroidal graph is given by the indicated parameters. This is simply the first of many such questions.

4. Under what conditions on their parameters can two spidergraphs be isomorphic? This is a question mentioned in [54]. Many examples of isomorphism theorems are given there as well as examples which show that not all isomorphisms have been found.

5. The Attebery construction presents many challenges. First, for the general construction, under what conditions are the graphs $\mathrm{Att}(A, T, k; a, b)$ and $\mathrm{Att}(A, T', k; a', b')$ isomorphic? It is clear that if P is any automorphism of A, then $\mathrm{Att}(A, T, k; a, b) \cong \mathrm{Att}(A, P^{-1}TP, k; aP, bP)$, but it is almost certain that there are other isomorphisms. Even in the special AMC construction, we do not know when $\mathrm{AMC}(k, n, M)$ can be isomorphic to $\mathrm{AMC}(k', n', M')$. This question must be answered before we can make much progress on other aspects of the Attebery graphs.

6. Some graphs with the same diagrams as the metacirculants PS, MPS, MSY, MSZ, MC3 are not themselves metacirculants but are nevertheless edge-transitive. For example consider the graph $\mathrm{KE}_{12}(1, 3, 8, 5, 1)$ It is isomorphic to the graph whose vertices are $\mathbb{Z}_4 \times \mathbb{Z}_{12}$, with each $(1, i)$ adjacent to $(2, i)$ and $(2, i + 1)$, each $(2, i)$ to $(3, i)$ and $(3, i + 4)$, each $(3, i)$ to $(4, i + 4)$ and $(4, i + 5)$, and each $(4, i)$ to $(1, i)$ and $(1, i + 10)$. This diagram is the same 'sausage graph' that characterizes the PS and MPS graphs and yet the graph is not isomorphic to any PS or MPS graph. This happens rarely enough that the exceptional cases might be classifiable.

7. The Praeger-Xu graphs, including $\mathrm{W}(n, 2)$ and $\mathrm{R}_{2m}(m + 2, m + 1)$, present many problems computationally. Some aspects, such as semitransitive orientations, cycle structures and regular maps have been addressed in [16]. Can we establish theorems determining their dart-transitive colorings, their BGCG dissections, and other properties?

8. When do the constructions DG, HC, TAG, applied to some cubic graph Λ or some rotary map $\mathcal{M}$, simply result in the line graph or medial graph of some larger graph or map?

9. The BGCG constructions we have used here are only the beginning of this topic. We have given some constructions for cases where the connection graph is K_1, K_2, or C_k, but for other connection graphs, we have no general techniques at all.

10. How can XI of a chiral map be edge-transitive?

11. The 3-arc graph of a cubic graph (see section 17.4) is the partial line graph of some cycle decomposition; *what* decomposition? ... of *what* graph?

12. In section 19, we use cycle structures to construct LR structures. What is an efficient way to find all isomorphism classes of cycle structures for a given dart-transitive tetravalent graph?

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