Special issue of ADAM devoted to the International Workshop on Symmetries of Graph and Networks 2018

We are delighted to present this special issue of the *Art of Discrete and Applied Mathematics* (ADAM), on topics presented or related to topics covered at the TSIMF workshop on 'Symmetries of Graphs and Networks', held at Sanya, on the beautiful semi-tropical island province of Hainan (China), in January 2018.

This workshop added to the series of conferences and workshops on symmetries of graphs and networks initiated at BIRS (Canada) in 2008 and progressed in Slovenia every two years from 2010 to 2016.

It was attended by 50 mathematicians from China and other parts of the world (including Australia, Canada, New Zealand, Slovakia, Slovenia, South Korea and the USA), many of whom gave lectures on a range of topics involving the symmetries of graphs and maps, including Cayley graphs, arc-transitive graphs and digraphs, covering graphs, regular maps on surfaces, and regular Cayley maps, plus related topics such as graph embeddings and skew morphisms of groups.

Participants very much enjoyed the venue, which is similar in style to the BIRS facilities in Banff and Oaxaca and the institute at Oberwolfach, giving plenty of opportunity for interactions between participants, and stimulating further research on the topics covered.

This issue contains a number of interesting papers resulting from or associated with the workshop. We would like to thank the authors for their valuable contributions. Also we are very grateful to the TSIMF for its generous financial support and administrative assistance for the workshop, and to the chief editors of ADAM for making this special issue possible.

Marston Conder (University of Auckland, New Zealand) and
Yan-Quan Feng (Beijing Jiaotong University, China)

Principal organisers of the workshop and Editors of this issue

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Digraphs with small automorphism groups that are Cayley on two nonisomorphic groups*

Luke Morgan [†]

Centre for the Mathematics of Symmetry and Computation, Department of Mathematics and Statistics (M019), The University of Western Australia, 35 Stirling Highway, Crawley, 6009, Australia
Current address: *University of Primorska, FAMNIT, Glagoljaška 8, 6000 Koper, Slovenia,* and *University of Primorska, IAM, Muzejski trg 2, 6000 Koper, Slovenia*

Joy Morris [‡]

Department of Mathematics and Computer Science, University of Lethbridge, Lethbridge, AB T1K 3M4, Canada

Gabriel Verret

Department of Mathematics, The University of Auckland, Private Bag 92019, Auckland 1142, New Zealand

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Abstract

Let $\Gamma = \mathrm{Cay}(G, S)$ be a Cayley digraph on a group G and let $A = \mathrm{Aut}(\Gamma)$. The *Cayley index* of Γ is $|A : G|$. It has previously been shown that, if p is a prime, G is a cyclic p-group and A contains a noncyclic regular subgroup, then the Cayley index of Γ is superexponential in p.

We present evidence suggesting that cyclic groups are exceptional in this respect. Specifically, we establish the contrasting result that, if p is an odd prime and G is abelian but not cyclic, and has order a power of p at least p^3, then there is a Cayley digraph Γ on G whose Cayley index is just p, and whose automorphism group contains a nonabelian regular subgroup.

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1 Introduction

Every digraph and group in this paper is finite. A *digraph* Γ consists of a set of *vertices* $\mathrm{V}(\Gamma)$ and a set of *arcs* $\mathrm{A}(\Gamma)$, each arc being an ordered pair of distinct vertices. (Our digraphs do not have loops.) We say that Γ is a *graph* if, for every arc (u, v) of Γ, (v, u) is also an arc. Otherwise, Γ is a *proper* digraph.

The *automorphisms* of Γ are the permutations of $\mathrm{V}(\Gamma)$ that preserve $\mathrm{A}(\Gamma)$. They form a group under composition, denoted $\mathrm{Aut}(\Gamma)$.

Let G be a group and let S be a subset of G that does not contain the identity. The *Cayley digraph* on G with connection set S is $\Gamma = \mathrm{Cay}(G, S)$, the digraph with vertex-set G and where $(u, v) \in \mathrm{A}(\Gamma)$ whenever $vu^{-1} \in S$. The index of G in $\mathrm{Aut}(\Gamma)$ is called the *Cayley index* of Γ.

It is well-known that a digraph is a Cayley digraph on G if and only if its automorphism group contains the right regular representation of G. A digraph may have more than one regular subgroup in its automorphism group and hence more than one representation as a Cayley digraph. This is an interesting situation that has been studied in [2, 9, 12, 13, 15], for example.

Let p be a prime. Joseph [7] proved that if Γ has order p^2 and $\mathrm{Aut}(\Gamma)$ has two regular subgroups, one of which is cyclic and the other not, then Γ has Cayley index at least p^{p-1}. The second author generalised this in [11], showing that if $p \geqslant 3$, Γ has order p^n and $\mathrm{Aut}(\Gamma)$ has two regular subgroups, one of which is cyclic and the other not, then Γ has Cayley index at least $p^{np-p-n+1}$. A simpler proof of this was later published in [1]. Kovács and Servatius [8] proved the analogous result when $p = 2$.

The theme of the results above is that if $\mathrm{Aut}(\Gamma)$ has two regular subgroups, one of which is cyclic and the other not, then Γ must have "large" Cayley index. The goal of this paper is to show that cyclic p-groups are exceptional with respect to this property, at least among abelian p-groups. More precisely, we prove the following.

Theorem 1.1. *Let p be an odd prime and let G be an abelian p-group. If G has order at least p^3 and is not cyclic, then there exists a proper Cayley digraph on G with Cayley index p and whose automorphism group contains a nonabelian regular subgroup.*

It would be interesting to generalise Theorem 1.1 to nonabelian p-groups and to 2-groups. More generally, we expect that "most" groups admit a Cayley digraph of "small" Cayley index such that the automorphism group of the digraph contains another (or even a nonisomorphic) regular subgroup. At the moment, we do not know how to approach this problem in general, or even what a sensible definition of "small" might be. (Lemma 3.2 shows that the smallest index of a proper subgroup of either of the regular subgroups is a lower bound – and hence that the Cayley index of p in Theorem 1.1 is best possible.) As an example, we prove the following.

E-mail addresses: luke.morgan@famnit.upr.si (Luke Morgan), joy.morris@uleth.ca (Joy Morris), g.verret@auckland.ac.nz (Gabriel Verret)

Proposition 1.2. *Let G be a group generated by an involution x and an element y of order 3, and such that $\mathbb{Z}_6 \not\cong G \not\cong \mathbb{Z}_3 \wr \mathbb{Z}_2$. If G has a subgroup H of index 2, then there is a Cayley digraph Γ with Cayley index 2 such that $\mathrm{Aut}(\Gamma)$ contains a regular subgroup distinct from G and isomorphic to $H \times \mathbb{Z}_2$.*

This paper is laid out as follows. Section 2 includes structural results on cartesian products of digraphs that will be required in the proofs of our main results, while in Section 3 we collect results about automorphism groups of digraphs. Section 4 consists of the proof of Theorem 1.1. Finally, in Section 5 we prove Proposition 1.2 and consider the case of symmetric groups.

2 Cartesian products

The main result of this section is a version of a result about cartesian products of graphs due to Imrich [6, Theorem 1] that is adapted to the case of proper digraphs. Imrich's proof can be generalised directly to all digraphs, but his proof involves a detailed case-by-case analysis for small graphs, which can be avoided by restricting attention to proper digraphs.

Generally, there are two notions of connectedness for digraphs: a digraph is *weakly connected* if its underlying graph is connected, and *strongly connected* if for every ordered pair of vertices there is a directed path from the first to the second. In a finite Cayley digraph, these notions coincide (see [4, Lemma 2.6.1] for example). For this reason, we will refer to Cayley digraphs as simply being *connected* or *disconnected*.

The *complement* of a digraph Γ, denoted by $\overline{\Gamma}$, is the digraph with vertex-set $\mathrm{V}(\Gamma)$, with $(u, v) \in \mathrm{A}(\overline{\Gamma})$ if and only if $(u, v) \notin \mathrm{A}(\Gamma)$, for every two distinct vertices u and v of Γ. It is easy to see that a digraph and its complement have the same automorphism group.

Given digraphs Γ and Δ, the *cartesian product* $\Gamma \,\square\, \Delta$ is the digraph with vertex-set $\mathrm{V}(\Gamma) \times \mathrm{V}(\Delta)$ and with $((u, v), (u', v'))$ being an arc if and only if either $u = u'$ and $(v, v') \in \mathrm{A}(\Delta)$, or $v = v'$ and $(u, u') \in \mathrm{A}(\Gamma)$. For each $u \in \mathrm{V}(\Gamma)$, we obtain a *copy* Δ^u of Δ in $\Gamma \,\square\, \Delta$, the induced digraph on $\{(u, v) \mid v \in \mathrm{V}(\Delta)\}$. Similarly, for each $v \in \mathrm{V}(\Delta)$, we obtain a copy Γ^v of Γ in $\Gamma \,\square\, \Delta$ (defined analogously).

A digraph Γ is *prime* with respect to the cartesian product if the existence of an isomorphism from Γ to $\Gamma_1 \,\square\, \Gamma_2$ implies that either Γ_1 or Γ_2 has order 1, so that Γ is isomorphic to either Γ_1 or Γ_2.

It is well known that, with respect to the cartesian product, graphs can be factorised uniquely as a product of prime factors. Digraphs also have this property.

Theorem 2.1 (Walker, [14])**.** *Let $\Gamma_1, \dots, \Gamma_k, \Gamma'_1, \dots, \Gamma'_\ell$ be weakly connected prime digraphs. If α is an isomorphism from $\Gamma_1 \,\square \cdots \square\, \Gamma_k$ to $\Gamma'_1 \,\square \cdots \square\, \Gamma'_\ell$, then $k = \ell$ and there exist a permutation π of $\{1, \dots, k\}$ and isomorphisms α_i from Γ_i to $\Gamma'_{\pi(i)}$ such that α is the product of the α_is ($1 \leqslant i \leqslant k$).*

Theorem 2.1 is a corollary of [14, Theorem 10], as noted in the "Applications" section of [14], but is more commonly proved by replacing a digraph Γ by its shadow $S(\Gamma)$ (by replacing arcs and digons with edges), noting that $S(\Gamma_1 \,\square\, \Gamma_2) = S(\Gamma_1) \,\square\, S(\Gamma_2)$, and using the unique prime factorisation for graphs with respect to the cartesian product.

We now present the version of Imrich's result that applies to proper digraphs. During the refereeing process of this article, we were made aware of [5, Theorem 1], which is similar to the theorem below.

Theorem 2.2. *If Γ is a proper digraph, then at least one of Γ or $\overline{\Gamma}$ is prime with respect to cartesian product and is weakly connected.*

Proof. We first make a key observation. Let $u = (x, y)$ and $v = (x', y)$ be distinct vertices of a cartesian product $X \,\square\, Y$ lying in the same copy X^y of X. If $w = (z, y')$ is a vertex adjacent to *both* u and v (with no specification on the direction of the arcs), we claim that $w \in X^y$. Indeed, since $w \sim u$, if $y \neq y'$, then $z = x$ and $y \sim y'$ in Y. It follows that $w = (x, y')$ cannot be adjacent to $v = (x', y)$ in $X \,\square\, Y$ since $x \neq x'$. Hence $y = y'$ and $w \in X^y$.

Suppose that either Γ or $\overline{\Gamma}$ is not weakly connected, say Γ. Then $\overline{\Gamma}$ is weakly connected. We may assume that $\overline{\Gamma}$ admits a non-trivial factorisation as $\overline{\Gamma} = X \,\square\, Y$ (so that X and Y have at least two vertices).

Suppose there exist a weakly connected component $\mathcal{C}$ of Γ and a copy of X, X^y say, such that X^y has two vertices that are not in $\mathcal{C}$, u and v say. By the key observation applied to $\overline{\Gamma}$, every vertex of $\mathcal{C}$ is also in X^y. Let $c \in \mathcal{C}$ and let $y' \in Y$ with $y' \neq y$. Then, since $\mathcal{C}$ is contained in X^y, there are at least two vertices of $X^{y'}$ with no arc in Γ in common with c. Applying the key observation yields that c is in $X^{y'}$, which is a contradiction. We may thus assume that every weakly connected component of Γ misses at most one vertex from each copy of X. In particular, Γ has exactly two weakly connected components, and therefore X has exactly two vertices. A symmetric argument yields that Y also has two vertices and thus Γ has four vertices, and the result can be checked by brute force.

Now we may assume that both Γ and $\overline{\Gamma}$ are weakly connected. Towards a contradiction, assume that $\Gamma = \overline{A \,\square\, B}$ and that φ is an isomorphism from Γ to $C \,\square\, D$, where A, B, C, and D all have at least 2 vertices.

Since Γ is a proper digraph, without loss of generality so is A, and A has an arc (a, a') such that (a', a) is not an arc of A. Let b be a vertex of B.

Pick b' to be a vertex of B distinct from b. We claim that $\varphi((a, b))$, $\varphi((a', b))$, $\varphi((a, b'))$, $\varphi((a', b'))$ all lie in some copy of either C or D. The digraph in Figure 1 is the subdigraph of Γ under consideration.

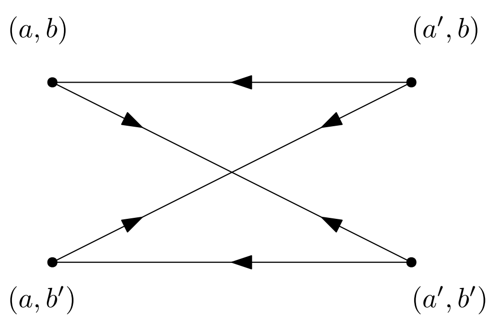

Figure 1: A subdigraph of Γ.

Since every arc in $C \,\square\, D$ lies in either a copy of C or D, we may assume that the arc from $\varphi((a, b))$ to $\varphi((a', b'))$ lies in some copy C^d of C, say $\varphi((a, b)) = (c, d)$ and $\varphi((a', b')) = (c', d)$, with $c, c' \in \mathrm{V}(C)$ and $d \in \mathrm{V}(D)$. Towards a contradiction, suppose that $\varphi((a', b)) \notin C^d$. Then the arc from $\varphi((a', b))$ to (c, d) must lie in D^c, so $\varphi((a', b)) = (c, d')$ for some vertex d' of D. Since there is a path of length 2 via $\varphi((a, b'))$ from (c', d)

to (c, d') and since $\varphi((a, b')) \neq (c, d) = \varphi((a, b))$, we must have $\varphi((a, b')) = (c', d')$. But now we have an arc from (c, d') to (c, d) and an arc from (c', d) to (c', d'), so arcs in both directions between d and d' in D. This implies that there are arcs in both directions between $(c, d) = \varphi((a, b))$ and $(c, d') = \varphi((a', b))$, a contradiction. Hence $\varphi((a', b)) \in C^d$, and by the observation in the first paragraph, we have $\varphi((a, b')) \in C^d$ also. This proves the claim.

By repeatedly applying the claim, all elements of $\varphi(\{a, a'\} \times \mathrm{V}(B))$ lie in some copy of C or D, say, C^d. Let $a'' \in \mathrm{V}(A) - \{a, a'\}$ and let b and b' be distinct vertices of B. By the definitions of cartesian product and complement, there are arcs in both directions between (a'', b) and (a, b') and between (a'', b) and (a', b'). Thus, by the observation in the first paragraph, $\varphi((a'', b))$ also lies in C^d. This shows that every vertex of Γ lies in C^d, so D is trivial. This is the desired contradiction. □

Remark 2.3. Imrich's Theorem [6, Theorem 1] states that, for every graph Γ, either Γ or $\overline{\Gamma}$ is prime with respect to the cartesian product, with the following exceptions: $\mathrm{K}_2 \,\Box\, \mathrm{K}_2$, $\mathrm{K}_2 \,\Box\, \overline{\mathrm{K}_2}$, $\mathrm{K}_2 \,\Box\, \mathrm{K}_2 \,\Box\, \mathrm{K}_2$, $\mathrm{K}_4 \,\Box\, \mathrm{K}_2$, $\mathrm{K}_2 \,\Box\, \mathrm{K}_4^-$, and $\mathrm{K}_3 \,\Box\, \mathrm{K}_3$, where K_n denotes the complete graph on n vertices and K_4^- denotes K_4 with an edge deleted. These would therefore be the complete list of exceptions to Theorem 2.2 if we removed the word 'proper' from the hypothesis.

Remark 2.4. While most of our results apply only to finite digraphs, Theorem 2.2 also applies to infinite ones (as does Imrich's Theorem). The proof is the same.

Corollary 2.5. *Let Γ_1 be a proper Cayley digraph on G with Cayley index i_1 and let Γ_2 be a connected Cayley digraph on H with Cayley index i_2. If $i_1 > i_2$, then at least one of $\Gamma_1 \,\Box\, \Gamma_2$ or $\overline{\Gamma_1} \,\Box\, \Gamma_2$ has automorphism group equal to $\mathrm{Aut}(\Gamma_1) \times \mathrm{Aut}(\Gamma_2)$ and, in particular, is a proper Cayley digraph on $G \times H$ with Cayley index $i_1 i_2$.*

Proof. By Theorem 2.2, one of Γ_1 and $\overline{\Gamma_1}$ is connected and prime with respect to the cartesian product, say Γ_1 without loss of generality. Clearly, we have $\mathrm{Aut}(\Gamma_1) \times \mathrm{Aut}(\Gamma_2) \leqslant \mathrm{Aut}(\Gamma_1 \,\Box\, \Gamma_2)$. Since $i_1 > i_2$, Γ_1 cannot be a cartesian factor of Γ_2. It follows by Theorem 2.1 that every automorphism of $\Gamma_1 \,\Box\, \Gamma_2$ is a product of an automorphism of Γ_1 and an automorphism of Γ_2, so that $\mathrm{Aut}(\Gamma_1) \times \mathrm{Aut}(\Gamma_2) = \mathrm{Aut}(\Gamma_1 \,\Box\, \Gamma_2)$. □

3 Additional background

The following lemma is well known and easy to prove.

Lemma 3.1. *Let G be a group, let $S \subseteq G$ and let $\alpha \in \mathrm{Aut}(G)$. If $S^\alpha = S$, then α induces an automorphism of $\mathrm{Cay}(G, S)$ which fixes the vertex corresponding to the identity.*

The next lemma is not used in any of our proofs, but it shows that the Cayley indices in Theorem 1.1 and Theorem 1.2 are as small as possible.

Lemma 3.2. *If $\mathrm{Cay}(G, S)$ has Cayley index i and $\mathrm{Aut}(\mathrm{Cay}(G, S))$ has at least two regular subgroups, then G has a proper subgroup of index at most i.*

Proof. Let $A = \mathrm{Aut}(\mathrm{Cay}(G, S))$ and let H be a regular subgroup of A different from G. Clearly, $G \cap H$ is a proper subgroup of G and we have $|A| \geqslant |GH| = \frac{|G||H|}{|G \cap H|}$ hence $i = |A : G| = |A : H| \geqslant |G : G \cap H|$. □

If v is vertex of a digraph Γ, then $\Gamma^+(v)$ denotes the *out-neighbourhood* of v, that is, the set of vertices w of Γ such that (v, w) is an arc of Γ.

Let A be a group of automorphisms of a digraph Γ. For $v \in \mathrm{V}(\Gamma)$ and $i \geqslant 1$, we use $A_v^{+[i]}$ to denote the subgroup of A_v that fixes every vertex u for which there is a directed path of length at most i from v to u.

Lemma 3.3. *Let Γ be a connected digraph, let v be a vertex of Γ and let A be a transitive group of automorphisms of Γ. If $A_v^{+[1]} = A_v^{+[2]}$, then $A_v^{+[1]} = 1$.*

Proof. By the transitivity of A, we have $A_u^{+[1]} = A_u^{+[2]}$ for every vertex u. Using induction on i, it easily follows that, for every $i \geqslant 1$, we have $A_v^{+[i]} = A_v^{+[i+1]}$. By connectedness, this implies that $A_v^{+[1]} = 1$. □

Lemma 3.4. *Let p be a prime and let A be a permutation group whose order is a power of p. If A has a regular abelian subgroup G of index p and G has a subgroup M of index p that is normalised but not centralised by a point-stabiliser in A, then A has a regular nonabelian subgroup.*

Proof. Let A_v be a point-stabiliser in A. Note that $A = G \rtimes A_v$ and that $|A_v| = p$. Since M is normal in G and normalised by A_v, it is normal in A and has index p^2. Clearly, $M \rtimes A_v \neq G$ hence A/M contains at least two subgroups of order p and must therefore be elementary abelian.

Let α be a generator of A_v and let $g \in G - M$. By the previous paragraph, we have $(g\alpha)^p \in M$. Let $H = \langle M, g\alpha \rangle$. Since M is centralised by g but not by α, it is not centralised by $g\alpha$ hence H is nonabelian. Further, we have $|H| = p|M| = |G|$, so that H is normal in A. If H was non-regular, it would contain all point-stabilisers of A, and thus would contain α and hence also g. This would give $G = \langle M, g \rangle \leqslant H$, a contradiction. Thus H is a regular nonabelian subgroup of A. □

4 Proof of Theorem 1.1

Throughout this section, p denotes an odd prime. In Section 4.1, we show that Theorem 1.1 holds when $G \cong \mathbb{Z}_p^3$. In Sections 4.2 and 4.3, we subdivide abelian groups of rank 2 and order at least p^3 into two families, and show that the theorem holds for all such groups. Finally, in Section 4.4, we explain how these results can be applied to show that the theorem holds for all abelian groups of order at least p^3.

4.1 $G \cong \mathbb{Z}_p^3$

Write $G = \langle x, y, z \rangle$, let α be the automorphism of G that maps (x, y, z) to (xy, yz, z), let $S = \{x^{\alpha^i}, y^{\alpha^i} : i \in \mathbb{Z}\}$, let $\Gamma = \mathrm{Cay}(G, S)$, and let $A = \mathrm{Aut}(\Gamma)$. Note that Γ is a proper digraph (this will be needed in Section 4.4).

It is easy to see that, for $i \in \mathbb{N}$, we have $x^{\alpha^i} = xy^i z^{\binom{i}{2}}$, $y^{\alpha^i} = yz^i$ and $z^{\alpha^i} = z$. In particular, α has order p and $|S| = 2p$. By Lemma 3.1, $G \rtimes \langle \alpha \rangle \leqslant A$. We will show that equality holds.

Using the formulas above, it is not hard to see that the induced digraph on S has exactly $2p$ arcs: $(x^{\alpha^i}, x^{\alpha^{i+1}})$ and $(y^{\alpha^i}, x^{\alpha^{i+1}})$, where $i \in \mathbb{Z}_p$. Thus, for every $s \in S$, $A_{1,s} = A_1^{+[1]}$. By vertex-transitivity, $A_{u,v} = A_u^{+[1]}$ for every arc (u, v).

Let $s \in S$. We have already seen that $A_{1,s} = A_1^{+[1]}$. Let $t \in S$. From the structure of the induced digraph on $S = \Gamma^+(1)$, we see that t has an out-neighbour in S, so that both t and this out-neighbour are fixed by $A_{1,s}$. It follows that $A_{1,s}$ fixes all out-neighbours of t. We have shown that $A_{1,s} = A_1^{+[2]}$. By Lemma 3.3, it follows that $A_{1,s} = 1$. Since the induced digraph on S is not vertex-transitive and $\alpha \in A_1$, the A_1-orbits on S have length p. Hence $|A_1| = p|A_{1,s}| = p$. Thus, Γ has Cayley index p and $A = G \rtimes \langle\alpha\rangle$. Finally, we apply Lemma 3.4 with $M = \langle y, z\rangle$ to deduce that A contains a nonabelian regular subgroup.

4.2 $G \cong \mathbb{Z}_{p^n} \times \mathbb{Z}_p$ with $n \geqslant 2$

Choose $x, y \in G$ of order p^n and p respectively so that $G = \langle x, y\rangle$. Let $x_0 = x^{p^{n-1}}$, let α be the automorphism of G that maps (x, y) to (xy, x_0y), let $S = \{x^{\alpha^i}, y^{\alpha^i} : i \in \mathbb{Z}\}$ and let $\Gamma = \mathrm{Cay}(G, S)$. Again, note that Γ is a proper digraph.

Since $n \geqslant 2$, x_0 is fixed by α. It follows that, for $i \in \mathbb{N}$, we have $x^{\alpha^i} = xy^ix_0^{\binom{i}{2}}$ and $y^{\alpha^i} = yx_0^i$. In particular, α has order p and $|S| = 2p$.

Using these formulas, it is not hard to see that the induced digraph on S has exactly $2p$ arcs: $(x^{\alpha^i}, x^{\alpha^{i+1}})$ and $(y^{\alpha^i}, x^{\alpha^{i+1}})$, where $i \in \mathbb{Z}_p$. The proof is now exactly as in the previous section, except that we use $M = \langle x^p, y\rangle$ when applying Lemma 3.4.

4.3 $G \cong \mathbb{Z}_{p^n} \times \mathbb{Z}_{p^m}$ with $n \geqslant m \geqslant 2$

Choose $x, y \in G$ of order p^n and p^m respectively so that $G = \langle x, y\rangle$. Let $x_0 = x^{p^{n-1}}$, let $y_0 = y^{p^{m-1}}$, let α be the automorphism of G that maps (x, y) to (xy_0, yx_0), let $S = \{x^{\alpha^i}, y^{\alpha^i}, (xy^{-1})^{\alpha^i} : i \in \mathbb{Z}\}$, let $\Gamma = \mathrm{Cay}(G, S)$, and let $A = \mathrm{Aut}(\Gamma)$. Again, note that Γ is a proper digraph.

Since $n \geqslant m \geqslant 2$, x_0 and y_0 are both fixed by α. It follows that, for $i \in \mathbb{N}$, we have $x^{\alpha^i} = xy_0^i$, and $y^{\alpha^i} = yx_0^i$. In particular, α has order p and $|S| = 3p$. By Lemma 3.1, $G \rtimes \langle\alpha\rangle \leqslant A$.

Using the formulas above, it is not hard to see that the induced digraph on S has exactly $2p$ arcs: $((xy^{-1})^{\alpha^i}, x^{\alpha^i})$ and $(y^{\alpha^i}, x^{\alpha^i})$, where $i \in \mathbb{Z}_p$. It follows that $|A_1 : A_{1,x}| = p$. We will show that $A_{1,x} = 1$, which will imply that $A = G \rtimes \langle\alpha\rangle$.

Let $X = \{x^{\alpha^i} : i \in \mathbb{Z}\} = x\langle y_0\rangle$, $Y = \{x^{\alpha^i} : i \in \mathbb{Z}\} = y\langle x_0\rangle$ and $Z = \{(xy^{-1})^{\alpha^i} : i \in \mathbb{Z}\} = xy^{-1}\langle x_0^{-1}y_0\rangle$. It follows from the previous paragraph that X is an orbit of A_1 on S.

Note that the p elements of $Y^2 = y^2\langle x_0\rangle$ are out-neighbours of every element of Y. Similarly, the p elements of Z^2 are out-neighbours of every element of Z. On the other hand, one can check that an element of Y and an element of Z have a unique out-neighbour in common, namely their product. This shows that Y and Z are blocks for A_1. We claim that Y and Z are orbits of A_1.

Let $Y_1 = Y$ and, for $i \geqslant 2$, inductively define $Y_i = \bigcap_{x \in Y_{i-1}} \Gamma^+(x)$. Define Z_i analogously. Let $g \in A_1$. By induction, $Y_i^g \in \{Y_i, Z_i\}$, and $Y_i^g = Z_i$ if and only if $Y^g = Z$. Note that $Y_i = Y^i = y^i\langle x_0\rangle$ and $Z_i = Z^i = x^iy^{-i}\langle x_0^{-1}y_0\rangle$. If $n = m$, then $1 \in x_0y_0^{-1}\langle x_0^{-1}y_0\rangle = Z_{p^{m-1}}$, but $1 \notin y_0\langle x_0\rangle = Y_{p^{m-1}}$, so Y and Z are orbits for A_1. We may thus assume that $n > m$. Note that $y_0 \in y_0\langle x_0\rangle = Y_{p^{m-1}}$, and y_0 is an in-neighbour of $x \in X$. However, $Z_{p^{m-1}} = x^{p^{m-1}}y_0^{-1}\langle x_0^{-1}y_0\rangle$. Since $n > m$, we see that no vertex of

$Z_{p^{m-1}}$ is an in-neighbour of a vertex of X. Again it follows that Y and Z are orbits for A_1.

Considering the structure of the induced digraph on S, it follows that, for every $s \in S$, $A_{1,s} = A_1^{+[1]}$. By vertex-transitivity, $A_{u,v} = A_u^{+[1]}$ for every arc (u, v). Since elements of Y and Z have an out-neighbour in S, $A_{1,x}$ fixes the out-neighbours of elements of Y and Z. Furthermore, for every $i \in \mathbb{Z}$, $xy_0^i y$ is a common out-neighbour of xy_0^i and y, hence it is fixed by $A_{1,x}$. Thus, every element of X has an out-neighbour fixed by $A_{1,x}$. It follows that $A_{1,x}$ fixes all out-neighbours of elements of X and thus $A_{1,x} = A_1^{+[1]} = A_1^{+[2]}$. By Lemma 3.3, it follows that $A_{1,x} = 1$. As in Section 4.1, we can also conclude $|A_1| = p$, Γ has Cayley index p and $A = G \rtimes \langle\alpha\rangle$. Finally, applying Lemma 3.4 with $M = \langle x^p, y\rangle$ implies that A contains a nonabelian regular subgroup.

4.4 General case

Recall that G is an abelian p-group that has order at least p^3 and is not cyclic. By the Fundamental Theorem of Finite Abelian Groups, we can write $G = G_1 \times G_2$, where G_1 falls into one of the three cases that have already been dealt with in this section.

(More explicitly, if G is not elementary abelian, then we can take G_1 isomorphic to $\mathbb{Z}_{p^n} \times \mathbb{Z}_{p^m}$ with $n \geqslant 2$ and $m \geqslant 1$. If G is elementary abelian, then, since $|G| \geqslant p^3$, we can take G_1 isomorphic to $\mathbb{Z}_p^3$.)

We showed in the previous three sections that there exists a proper Cayley digraph Γ_1 on G_1 with Cayley index equal to p and whose automorphism group contains a nonabelian regular subgroup.

Note that every cyclic group admits a prime, connected Cayley digraph whose Cayley index is 1. (For example, the directed cycle of the corresponding order.) Since G_2 is a direct product of cyclic groups, applying Corollary 2.5 iteratively yields a proper connected Cayley digraph Γ on $G_1 \times G_2$ with automorphism group $\mathrm{Aut}(\Gamma_1) \times G_2$. In particular, Γ has Cayley index p and its automorphism group contains a nonabelian regular subgroup. This concludes the proof of Theorem 1.1.

In fact, the proof above yields the following stronger result.

Theorem 4.1. *Let G be an abelian group. If there is an odd prime p such that the Sylow p-subgroup of G is neither cyclic nor elementary abelian of rank 2, then G admits a proper Cayley digraph with Cayley index p whose automorphism group contains a nonabelian regular subgroup.*

5 Proof of Proposition 1.2

We begin with a lemma that helps to establish the existence of regular subgroups.

Lemma 5.1. *Let G be a group with nontrivial subgroups H and B such that $G = HB$ and $H \cap B = 1$, and let $\Gamma = \mathrm{Cay}(G, S)$ be a Cayley digraph on G. If S is closed under conjugation by B, then $\mathrm{Aut}(\Gamma)$ has a regular subgroup distinct from the right regular representation of G and isomorphic to $H \times B$.*

Proof. Let $A = \mathrm{Aut}(\Gamma)$. For $g \in G$, let ℓ_g and r_g denote the permutations of G induced by left and right multiplication by g, respectively. Similarly, for $g \in G$, let c_g denote the permutation of G induced by conjugation by g. For $X \leqslant G$, let $R_X = \langle r_x : x \in X\rangle$. Let $L_B = \langle \ell_b : b \in B\rangle$ and $C_B = \langle c_b : b \in B\rangle$. Note that $R_H \leqslant A$. For every $g \in G$,

$r_g c_{g^{-1}} = \ell_g$. For all $b \in B$, we have $r_b \in A$ and, since S is closed under conjugation by B, $c_{b^{-1}} \in A$ hence $L_B \leqslant A$.

Let $K = \langle L_B, R_H \rangle$. If $K = R_G$, then $L_B \leqslant R_G$ which implies that $C_B \leqslant R_G$, contradicting the fact that R_G is regular. Thus $K \neq R_G$. Note that L_B and R_H commute. Suppose that $k \in R_H \cap L_B$, so $k = r_h = \ell_b$ for some $h \in H$ and some $b \in B$. Thus

$$h = 1^{r_h} = 1^k = 1^{\ell_b} = b.$$

Since $H \cap B = 1$, this implies $k = 1$. It follows that $R_H \cap L_B = 1$ and hence $K = R_H \times L_B \cong H \times B$. Finally, suppose that some $k = r_h \ell_b \in K$ fixes 1. It follows that $1^{r_h \ell_b} = 1 = bh$ so that $b \in H$, a contradiction. This implies that K is regular, which concludes the proof. □

We now prove a general result, which together with Lemma 5.1 will imply Proposition 1.2.

Proposition 5.2. *Let G be a group generated by an involution x and an element y of order 3, let $S = \{x, y, y^x\}$ and let $\Gamma = \mathrm{Cay}(G, S)$. If G is isomorphic to neither $\mathbb{Z}_6$ nor $\mathbb{Z}_3 \wr \mathbb{Z}_2 \cong \mathbb{Z}_3^2 \rtimes \mathbb{Z}_2$, then Γ has Cayley index 2.*

Proof. Clearly, Γ is connected. Since G is not isomorphic to $\mathbb{Z}_6$, we have $y^x \neq y$. In particular, we have $|S| = 3$. If $y^x = y^{-1}$, then $G \cong \mathrm{Sym}(3)$ and the result can be checked directly. We therefore assume that $y^x \neq y^{-1}$. Since $G \ncong \mathbb{Z}_3 \wr \mathbb{Z}_2$, we have $y^x y \neq y y^x$.

We have that $\Gamma^+(x) = \{1, yx, xy\}$, $\Gamma^+(y) = \{xy, y^2, y^x y\}$ and $\Gamma^+(y^x) = \{yx, yy^x, (y^2)^x\}$. One can check that the only equalities between elements of these sets are the ones between elements having the same representation. In other words,

$$|\{1, yx, xy, y^2, y^x y, yy^x, (y^2)^x\}| = 7.$$

(For example, if $yx = y^x y$, then $x^{y^{-1}} = y^x$, contradicting the fact that x and y have different orders.)

Let $A = \mathrm{Aut}(\Gamma)$ and let c_x denote conjugation by x. Note that $c_x \in A_1$. We first show that $A_1^{+[1]} = 1$. It can be checked that y^2 is the unique out-neighbour of y that is also an in-neighbour of 1, hence it is fixed by $A_1^{+[1]}$, and so is $(y^2)^x$ by analogous reasoning. We have seen earlier that xy is the unique common out-neighbour of x and y, hence it too is fixed by $A_1^{+[1]}$, and similarly for yx. Being the only remaining out-neighbours of y, $y^x y$ must be also fixed, and similarly for yy^x. Thus $A_1^{+[1]} = A_1^{+[2]}$. Since Γ is connected, Lemma 3.3 implies that $A_1^{+[1]} = 1$.

Note that x is the only out-neighbour of 1 that is also an in-neighbour, hence it is fixed by A_1, whereas c_x interchanges y and y^x. It follows that $|A_1| = |A_1 : A_1^{+[1]}| = 2$ and Γ has Cayley index 2, as desired. □

Proof of Proposition 1.2. Let $\Gamma = \mathrm{Cay}(G, \{x, y, y^x\})$. By Proposition 5.2, Γ has Cayley index 2. Since $|G : H| = 2$ and y has order 3, we have $y \in H$. As $\langle x, y \rangle = G$, we have $x \notin H$ and $G = H \rtimes \langle x \rangle$. Clearly, $\{x, y, y^x\}$ is closed under conjugation by x. It follows by Lemma 5.1 that $\mathrm{Aut}(\Gamma)$ has a regular subgroup distinct from G and isomorphic to $H \times \langle x \rangle \cong H \times \mathbb{Z}_2$. □

It was shown by Miller [10] that, when $n \geqslant 9$, $\mathrm{Sym}(n)$ admits a generating set consisting of an element of order 2 and one of order 3; this is also true when $n \in \{3, 4\}$. In these cases, we can apply Proposition 1.2 with $H = \mathrm{Alt}(n)$ to obtain a Cayley digraph on $\mathrm{Sym}(n)$ that has Cayley index 2 and whose automorphism group contains a regular subgroup isomorphic to $\mathrm{Alt}(n) \times \mathbb{Z}_2$.

A short alternate proof of this fact can be derived from a result of Feng [3]. This yields a Cayley graph and is valid for $n \geqslant 5$.

Proposition 5.3. *If $n \geqslant 5$, then there is a Cayley graph on $\mathrm{Sym}(n)$ with Cayley index 2, whose automorphism group contains a regular subgroup isomorphic to $\mathrm{Alt}(n) \times \mathbb{Z}_2$.*

Proof. Let $T = \{(1\ 2), (2\ 3), (2\ 4)\} \cup \{(i\ i+1) : 4 \leqslant i \leqslant n-1\}$ and let $\Gamma = \mathrm{Cay}(\mathrm{Sym}(n), T)$. Note that all elements of T are transpositions. Let $\mathrm{Tra}(T)$ be the transposition graph of T, that is, the graph with vertex-set $\{1, \ldots, n\}$ and with an edge $\{i, j\}$ if and only if $(i\ j) \in T$. Note that $\mathrm{Tra}(T)$ is a tree and thus T is a minimal generating set for $\mathrm{Sym}(n)$ (see for example [4, Section 3.10]). Let $B = \langle (1\ 3) \rangle$. Since $n \geqslant 5$, $\mathrm{Aut}(\mathrm{Tra}(T)) = B$. It follows by [3, Theorem 2.1] that $\mathrm{Aut}(\Gamma) \cong \mathrm{Sym}(n) \rtimes B$. In particular, Γ has Cayley index 2.

Note that $\mathrm{Sym}(n) = \mathrm{Alt}(n) \rtimes B$ and that T is closed under conjugation by B. Applying Lemma 5.1 with $H = \mathrm{Alt}(n)$ shows that $\mathrm{Aut}(\Gamma)$ has a regular subgroup isomorphic to $\mathrm{Alt}(n) \times \mathbb{Z}_2$. □

ORCID iDs

Luke Morgan https://orcid.org/0000-0003-2396-5430
Gabriel Verret https://orcid.org/0000-0003-1766-4834

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On strongly sequenceable abelian groups

Brian Alspach, Georgina Liversidge

School of Mathematical and Physical Sciences, University of Newcastle, Callaghan, NSW 2308, Australia

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Abstract

A group is strongly sequenceable if every connected Cayley digraph on the group admits an orthogonal directed cycle or an orthogonal directed path. This paper deals with the problem of whether finite abelian groups are strongly sequenceable. A method based on posets is used to show that if the connection set for a Cayley digraph on an abelian group has cardinality at most nine, then the digraph admits either an orthogonal directed path or an orthogonal directed cycle.

Keywords: Strongly sequenceable, abelian group, diffuse poset, sequenceable poset.

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1 Introduction

The *Cayley digraph* $\overrightarrow{\mathrm{Cay}}(G;S)$ on the group G has the elements of G for the vertex set and an arc (g,h) from g to h whenever $h = gs$ for some $s \in S$, where $S \subset G$ and $1 \notin S$. The set S is called the *connection set*. It is easy to see that left-multiplication by any element of G is an automorphism of $\overrightarrow{\mathrm{Cay}}(G;S)$ which implies that the automorphism group of $\overrightarrow{\mathrm{Cay}}(G;S)$ contains the left-regular representation of G.

A given $s \in S$ generates a spanning digraph of $\overrightarrow{\mathrm{Cay}}(G;S)$ composed of vertex-disjoint directed cycles of length $|s|$, where $|s|$ denotes the order of s. We call this subdigraph a $(1,1)$-*directed factor* because the in-valency and out-valency at each vertex is 1. Hence, there is a natural factorization of $\overrightarrow{\mathrm{Cay}}(G;S)$ into $|S|$ arc-disjoint $(1,1)$-directed factors. This is the *Cayley factorization* of $\overrightarrow{\mathrm{Cay}}(G;S)$ and is denoted $\mathcal{F}(G;S)$.

Let $\overrightarrow{\mathrm{Cay}}(G;S)$ be a Cayley digraph on a group G. A subdigraph $\overrightarrow{Y}$ of $\overrightarrow{\mathrm{Cay}}(G;S)$ of size $|S|$ (the *size* is the number of arcs in $\overrightarrow{Y}$), is *orthogonal* to $\mathcal{F}(G;S)$ if $\overrightarrow{Y}$ has one arc

E-mail addresses: brian.alspach@newcastle.edu.au (Brian Alspach), gliv560@aucklanduni.ac.nz (Georgina Liversidge)

from each $(1,1)$-directed factor of $\mathcal{F}(G;S)$. In order to simplify the language, we simply say that $\overrightarrow{\mathrm{Cay}}(G;S)$ *admits* an orthogonal $\overrightarrow{Y}$.

The complete digraph $\overrightarrow{K}_n$ may be viewed as a Cayley digraph $\overrightarrow{K}(G)$ on any group G of order n by choosing the connection set to be $G \setminus \{1\}$. B. Gordon [13] defined a group G to be *sequenceable* if $\overrightarrow{K}(G)$ admits an orthogonal Hamilton directed path (he used different language). Gordon was motivated by looking for methods to produce row-complete Latin squares and a sequenceable group gives rise to a row-complete Latin square.

From his work on the Heawood map coloring problem, G. Ringel [18] asked when does $\overrightarrow{K}(G)$ admit an orthogonal directed cycle of length $|G|-1$ (he also used different language). A group G for which this holds was called *R-sequenceable* in [12].

So the two notions of a sequenceable group and an R-sequenceable group were motivated by quite disparate mathematical problems, but as we have seen they are closely related. The topic of sequenceable and R-sequenceable groups has generated, and continues to generate, a considerable amount of research. There have been surveys [11, 17] and many papers including [1, 2, 3, 4, 5, 6, 7, 12, 13, 14, 16].

The following definition is a natural extension of sequenceable and R-sequenceable groups.

Definition 1.1. A group G is *strongly sequenceable* if every connected Cayley digraph on G admits either an orthogonal directed path or an orthogonal directed cycle.

An abelian group cannot be both sequenceable and R-sequenceable, but by allowing either an orthogonal directed path or an orthogonal directed cycle in the definition of strongly sequenceable, we guarantee that when an abelian group is strongly sequenceable, it is either sequenceable or R-sequenceable.

The first author and T. Kalinowski have posed the following problem.

Research problem 1.2. Determine the strongly sequenceable groups.

2 Abelian groups

It is not difficult to verify that the non-abelian group of order 6 is not strongly sequenceable. The only connection set for which it fails is the one giving $\overrightarrow{K}_6$.

There has been some work on the preceding problem for abelian groups. We use additive notation for abelian groups which is the case for the remainder of this paper. The first author asked whether cyclic groups are strongly sequenceable in 2000. Bode and Harborth [9] showed that the answer is yes for the cyclic group Z_n whenever the the sum of the elements in the connection set S is not 0 and either $|S| = n-1$ or $|S| = n-2$.

The same problem was discovered independently by Archdeacon, also restricted to cyclic groups, and studied in [8]. The authors prove that all cyclic groups of order at most 25 are strongly sequenceable. They also show that there is an orthogonal directed path or orthogonal directed cycle whenever the connection set S has cardinality at most 6 for all cyclic groups..

Costa, Morini, Pasotti and Pellegrini [10] observed that almost all the methods employed for the previously cited work do not depend on the group being cyclic. Consequently, their paper deals with abelian groups. They use computer verification to show that all abelian groups of orders at most 23 are strongly sequenceable. They also look at the problem in terms of the cardinality of S but with two restrictions, namely, they do not allow

S to contain any inverse pairs, that is, if $g \in S$, then $-g \notin S$, and they insist the elements sum to 0. With these restrictions in place, they show that if $|S| \leq 9$, the Cayley digraph admits either an orthogonal directed cycle.

In some new work, Hicks, Ollis and Schmitt [15] restrict themselves to the case that the group has prime order. They improve the Bode and Harborth result to include $|S| = p - 3$, and they improve the cardinality of the connection set result to $|S| \leq 10$. Thus, a circulant digraph of prime order admits an orthogonal directed path or an orthogonal directed cycle whenever its out-valency (and in-valency) is at most 10.

There is one obvious fact about a Cayley digraph on an abelian group we now observe. For the connection set S, let ΣS denote the sum of the elements in S.

Proposition 2.1. *Let $\overrightarrow{X} = \overrightarrow{\mathrm{Cay}}(G; S)$ be a Cayley digraph on an abelian group G. When $\overrightarrow{X}$ admits an orthogonal directed cycle or directed path $\overrightarrow{Y}$, then $\overrightarrow{Y}$ is a directed cycle if $\Sigma S = 0$; otherwise, it is a directed path.*

Proof. If we use one arc of each length $s \in S$ and we start at vertex g, the directed trail formed terminates at $g + \Sigma S$ no matter in which order we choose the lengths because G is abelian. From this it is easy to see that the proposition follows. □

3 The associated poset

We use $\subseteq$ for subset inclusion so that $A \subset B$ means that A is a proper subset of B.

We define a poset $\mathcal{P}$ to be *diffuse* if the following properties hold:

- The elements of $\mathcal{P}$ are subsets of a ground set Ω and the order relation is set inclusion;
- $\emptyset \in \mathcal{P}$;
- Every non-empty element of $\mathcal{P}$ has cardinality at least 2;
- If $A, B \in \mathcal{P}$ are disjoint, then $A \cup B \in \mathcal{P}$;
- If $A, B \in \mathcal{P}$ and $A \subset B$, then $B \setminus A \in \mathcal{P}$; and
- If $A, B \in \mathcal{P}$ and A and B are not comparable, then $|A \triangle B| \geq 3$.

In order to simplify the discussion, if the ground set has cardinality at least 1 and the empty set is the only element in the poset, we shall say this poset is diffuse.

Definition 3.1. *Let $\overrightarrow{X} = \overrightarrow{\mathrm{Cay}}(G; S)$ be a Cayley digraph on the abelian group G. The* associated poset *$\mathcal{P}(\overrightarrow{X})$ is defined as follows. The ground set is S and and the elements are any non-empty subsets S' of S such that $\Sigma S' = 0$ plus the empty set.*

Theorem 3.2. *If $\overrightarrow{X} = \overrightarrow{\mathrm{Cay}}(G; S)$ is a Cayley digraph on the abelian group G, then the associated poset $\mathcal{P}(\overrightarrow{X})$ is diffuse.*

Proof. If S' is a non-empty subset of $G \setminus \{0\}$ whose elements sum to 0, then clearly S' has at least two elements of S. If $S', S'' \in \mathcal{P}(\overrightarrow{X})$, then the sum of the elements in each of the subsets is 0. If the two subsets are disjoint, then the sum of the elements in their union also is 0 implying that $S' \cup S'' \in \mathcal{P}(\overrightarrow{X})$. If $S'' \subset S'$ and both belong to $\mathcal{P}(\overrightarrow{X})$, then clearly the elements of $S' \setminus S''$ also sum to 0. This implies $S' \setminus S'' \in \mathcal{P}(\overrightarrow{X})$. Finally, if $S', S'' \in \mathcal{P}(\overrightarrow{X})$ and they are not comparable, there must be at least one element of S' not in S'' and vice versa. If the symmetric difference $S' \triangle S''$ has exactly two elements $x, y \in S$, then $x = y$ would hold because S is a subset of an abelian group. This is a contradiction and the conclusion follows. □

Given a sequence $s_1, s_2, \ldots, s_n$, a *segment* denotes a subsequence of consecutive entries. The notation $[s_i, s_j]$ is used for the segment $s_i, s_{i+1}, \ldots, s_j$, where $i \leq j$.

Definition 3.3. *Let $\mathcal{P}$ be a poset on a groundset $\Omega = \{s_1, s_2, \ldots, s_k\}$ with set inclusion as the order relation. We say that $\mathcal{P}$ is* sequenceable *if there is a sequence $a_1, a_2, \ldots, a_k$ of all the elements of Ω such that no proper segment of the sequence is an element of $\mathcal{P}$. The sequence is called an* admissible sequence.

We only require that proper segments are not elements of $\mathcal{P}$ in the preceding definition because we wish to allow all of Ω to be an element of the poset and still have the poset possibly be sequenceable.

Corollary 3.4. *Let $\overrightarrow{X} = \overrightarrow{\mathrm{Cay}}(G; S)$ be a Cayley digraph on the abelian group G. If the associated poset $\mathcal{P}(\overrightarrow{X})$ is sequenceable, then $\overrightarrow{X}$ admits either an orthogonal directed path or an orthogonal directed cycle.*

Proof. Let $s_1, s_2, \ldots, s_k$ be an admissible sequence for $\mathcal{P}(\overrightarrow{X})$. If we take a directed trail of arcs of lengths $s_1, s_2, \ldots, s_k$ in that order, it is easy to see that we obtain an orthogonal directed path of length k if $\Sigma S \neq 0$, whereas, we obtain an orthogonal directed cycle of length k when $\Sigma S = 0$. □

Conjecture 3.5. Diffuse posets are sequenceable.

Because of Theorem 3.2 and Corollary 3.4, the truth of Conjecture 3.5 would imply that abelian groups are strongly sequenceable. We do not prove the conjecture here, but it does shift the work to looking at a restricted family of posets and getting away from the structure of the groups.

4 The poset approach

Recall that an *atom* in a poset is an element that covers a minimal element of the poset. When the empty set is an element, it is the unique minimal element so that the atoms are the sets not containing any non-empty proper subset in the poset. As we are considering posets whose elements are sets, we shall refer to an atom of cardinality t as a t-atom. Because of the properties possessed by diffuse posets, once we have a list of the atoms we know all the elements of the poset. The elements are all possible unions of mutually disjoint atoms. Note that the same element may arise in more than one way as a union of atoms.

Given a poset $\mathcal{P}$ whose elements are subsets of a ground set Ω, then the poset *induced* on a subset $\Omega' \subseteq \Omega$ is the collection of all members of $\mathcal{P}$ that lie entirely in Ω'. This poset is denoted by $\mathcal{P}\langle\Omega'\rangle$. Note that an induced subposet of a diffuse poset is itself diffuse.

Lemma 4.1. *If every atom of a diffuse poset $\mathcal{P}$ is a 2-atom, then $\mathcal{P}$ is sequenceable.*

Proof. The 2-atoms of $\mathcal{P}$ are mutually disjoint because $\mathcal{P}$ is diffuse. Let $\{a_1, b_1\}, \{a_2, b_2\}, \ldots, \{a_t, b_t\}$ be the 2-atoms of $\mathcal{P}$, and let $x_1, x_2, \ldots, x_r$ be any elements not belonging to atoms. Note that none of the elements $x_1, x_2, \ldots, x_r$ belong to any element of $\mathcal{P}$ because its non-empty elements are disjoint unions of atoms. The sequence

$$a_1, a_2, \ldots, a_t, x_1, x_2, \ldots, x_r, b_1, b_2, \ldots, b_t$$

is admissible and the proof is complete. □

We shall use the language of a segment belonging or not belonging to a diffuse poset $\mathcal{P}$ and this refers to the set of elements in the segment belonging to $\mathcal{P}$.

Lemma 4.2. *Let $\mathcal{P}$ be a diffuse poset with ground set Ω. If $\Omega \in \mathcal{P}$ and every diffuse poset on a ground set of cardinality $|\Omega| - 1$ is sequenceable, then for each $s \in \Omega$, there is an admissible sequence whose first term is s.*

Proof. Let s be any element of the ground set Ω. The set $\Omega \setminus \{s\}$ does not belong to the induced poset $\mathcal{P}' = \mathcal{P}\langle\Omega \setminus \{s\}\rangle$ because $\Omega \in \mathcal{P}$. The poset $\mathcal{P}'$ is diffuse and has an admissible sequence $a_1, a_2, \ldots, a_t$ by hypothesis. We claim the sequence $s, a_1, a_2, \ldots, a_t$ is admissible for $\mathcal{P}$.

Any proper segment of the latter sequence not containing s is not in $\mathcal{P}$ because $\mathcal{P}'$ is an induced poset. If there is a proper subsequence containing s belonging to $\mathcal{P}$, then by complementation and the fact that $\Omega \in \mathcal{P}$, the rest of the sequence belongs to $\mathcal{P}$. But this contradicts the fact that the sequence $a_1, a_2, \ldots, a_t$ is admissible for $\mathcal{P}'$. This concludes the proof. □

Lemma 4.3. *Let $\mathcal{P}$ be a diffuse poset with ground set Ω. If there exists an element $s \in \Omega$ such that $\Omega \setminus \{s\} \in \mathcal{P}$, s belongs to a single atom, and all diffuse posets on ground sets of cardinality $|\Omega| - 2$ are sequenceable, then $\mathcal{P}$ is sequenceable.*

Proof. Let s and $\mathcal{P}$ satisfy the hypotheses. Let a_1 be an element of the atom containing s such that $a_1 \neq s$. By Lemma 4.2, there is an admissible sequence $a_1, a_2, \ldots, a_t$ for the induced poset $\mathcal{P}\langle\Omega \setminus \{s\}\rangle$. Consider the sequence $a_1, a_2, \ldots, a_{t-1}, s, a_t$.

The set composed of the entire sequence is not in $\mathcal{P}$ because the latter is diffuse. Any segment not containing s is not in $\mathcal{P}$ as the segment is part of an admissible sequence for $\mathcal{P}\langle\Omega \setminus \{s\}\rangle$. Thus, if there is a proper segment in $\mathcal{P}$, it must contain s which implies it must contain a_1 because s belongs to only one atom. But the segment $[a_1, s]$ cannot be in $\mathcal{P}$ because the cardinality of the symmetric difference of $[a_1, s]$ and $\Omega \setminus \{s\}$ is 2. The result follows. □

Lemma 4.4. *Let $\mathcal{P}$ be a diffuse poset with ground set Ω, where $|\Omega| \geq 3$. If there exists an element $s \in \Omega$ such that $\Omega \setminus \{s\}$ is an atom, then $\mathcal{P}$ is sequenceable.*

Proof. If s belongs to no atoms, then any sequence of the elements of Ω such that s is at neither end is admissible. The preceding is the case when $|\Omega| = 3$. If s belongs only to a single atom A, then there must be $x, y \in \Omega$ such that $x, y \in \Omega \setminus A$ by the symmetric difference condition. It is straightforward to verify that any sequence beginning x, s, y is admissible by observing that neither x nor y can be in the atom A. Thus, we may assume s belongs to at least two atoms.

Hence, we have that $|\Omega| > 4$ and s belongs to an r-atom A with $r \geq 3$ because an element belongs to at most one 2-atom. Choose A so that r is maximum among all atoms containing s. Note that $|A| < |\Omega| - 1$ because $\mathcal{P}$ is diffuse. Let the elements of A be $s, s_2, \ldots, s_r$ and let $y \neq s$ be an element of Ω not belonging to A.

We claim the sequence $\pi = s_2, s, s_3, \ldots, s_{r-1}, y, s_r, \ldots$ completed by any permutation of the remaining elements is admissible for $\mathcal{P}$. To verify this, first observe that no segment beginning from the third entry or later belongs to $\mathcal{P}$ because the entries form a proper subset of $\Omega \setminus \{s\}$ which is an atom. The elements of the entire sequence do not belong to $\mathcal{P}$ because the poset is diffuse.

Any segment of the form $[s_2, x]$, where $x \in \{s, s_3, \ldots, s_{r-1}\}$, is a proper subset of A so that it does not belong to $\mathcal{P}$. The segment $[s_2, y]$ does not belong to $\mathcal{P}$ because the symmetric difference with A has cardinality 2. Finally, any segment of the form $[s_2, x]$, where x is any element from s_r or later in π, cannot be an atom because this contradicts the choice of A. Thus, if it is in $\mathcal{P}$, there would be an atom properly contained in $\Omega \setminus \{s\}$.

The only segments remaining to check are those beginning with s. The argument for these is essentially the same as for those beginning with s_2. One difference is the segment $[s, s_r]$ but it has cardinality 2 symmetric difference with A so cannot be in $\mathcal{P}$. Another difference is the segment $[s, y]$. If this segment is in $\mathcal{P}$, then interchange s_{r-1} and s_r in the sequence and the new segment $[s, y]$ cannot be in $\mathcal{P}$ because of the symmetric difference condition. The switching argument just used requires that $r \geq 4$, and when this holds the rest of the argument is the same as the preceding paragraph completing the proof.

When $r = 3$, the sequence π begins s_2, s, y, s_3. The segment $[s, y] \in \mathcal{P}$ implies that $\{s, y\}$ is a 2-atom. However, there are at least two elements of $\Omega \setminus \{s\}$ not in A. So choose one that does not form a 2-atom with s. □

Lemma 4.5. *Let $\mathcal{P}$ be a diffuse poset with ground set Ω, where $|\Omega| \geq 4$. If there exist $s_1, s_2 \in \Omega$ such that $\Omega \setminus \{s_1, s_2\}$ is an atom, then $\mathcal{P}$ is sequenceable.*

Proof. If $|\Omega| = 4$, then either $\{s_1, s_2\}$ also is a 2-atom in which case $\mathcal{P}$ is sequenceable by Lemma 4.1, or neither s_1 nor s_2 are in atoms in which case $\mathcal{P}$ is easily seen to be sequenceable. If $|\Omega| = 5$, then either $\{s_1, s_2\}$ is a 2-atom, just one of s_1, s_2 belongs to a 2-atom, both s_1, s_2 belong to 2-atoms, neither s_1 nor s_2 belong to a 2-atom, s_1, s_2 belong to a 3-atom or s_1, s_2 belong to a 4-atom. It is easy to find an admissible sequence in all six situations.

We assume $|\Omega| > 5$ for the rest of the proof. Let A denote the atom $\Omega \setminus \{s_1, s_2\}$. Let $M(s_1), M(s_2)$ and $M(s_1, s_2)$ denote the collections of atoms containing s_1 and not s_2, s_2 and not s_1, and both s_1, s_2, respectively. We assume that at least one of $M(s_1)$ and $M(s_2)$ contains a k-atom for $k \geq 3$ as it is easy to verify that $\mathcal{P}$ is sequenceable when this is not the case.

Given an atom A_1 in $M(s_1)$ of maximum cardinality $r + 1$, $r \geq 2$, *stretching* the atom refers to a sequence of the form $a_1, s_1, a_2, \ldots, a_{r-1}, x, a_r$, where x is an element to be named later and $a_1, a_2, \ldots, a_r$ is any sequence of the distinct elements of A_1 different from s_1. Let $B = A \setminus A_1 = \{b_1, b_2, \ldots, b_q\}$ and note that $q \geq 2$ because $\mathcal{P}$ is diffuse. We first consider the case $M(s_2) = \emptyset$.

Start a sequence π by stretching the atom A_1 and choose $x = b_1$. Complete the sequence as $a_r, b_2, b_3, \ldots, b_q, s_2$. We now verify that π is admissible.

Because $M(s_2)$ is empty, the only possible proper segment ending with s_2 that can be in $\mathcal{P}$ is $[s_1, s_2]$. If it is in $\mathcal{P}$, then it must be an atom and the theorem holds by Lemma 4.4.

So we consider only segments not ending with s_2.

Any such segment beginning with an element different from a_1 or s_1 is a proper subset of A which implies it is not in $\mathcal{P}$. Almost all proper segments beginning with a_1 or s_1 are not in $\mathcal{P}$ because they either are proper subsets of A_1 or violate the maximality of $|A_1|$. The exceptional segments are $[a_1, b_1]$, $[s_1, a_r]$ and $[s_1, b_1]$. The segments $[a_1, b_1]$ and $[s_1, a_r]$ are not in $\mathcal{P}$ because the symmetric difference with A_1 has cardinality 2 in both cases. If $[s_1, b_1] \in \mathcal{P}$, then interchange b_1 and b_2 in π. The resulting sequence is then admissible.

Thus, we now consider the case that $M(s_2) \neq \emptyset$. Because both $M(s_1)$ and $M(s_2)$ are non-empty, we may assume that no atom in $M(s_2)$ has cardinality bigger than $|A_1|$. Over all atoms in $M(s_2)$, let ℓ be the largest cardinality of the intersections with B. Let A_2 be an atom of $M(s_2)$ of maxium cardinality intersecting B in ℓ elements.

Partition A into four subsets as follows:

- $B_1 = A_1 \setminus A_2$;
- $B_2 = A_1 \cap A_2$;
- $B_3 = B \setminus A_2$; and
- $B_4 = B \cap A_2$.

We present the argument for the case $B_2 = B_3 = \emptyset$ in detail and use it to dispose of the remaining cases fairly quickly. In this case we see that the atoms A_1 and A_2 are disjoint and $A_1 \cup A_2 = \Omega$. This implies that $\Omega \in \mathcal{P}$ which, in turn, implies that $\{s_1, s_2\}$ is a 2-atom.

Using the same notation for the elements as above, define the sequence

$$\pi = a_1, s_1, a_2, \ldots, a_{r-1}, b_1, a_r, b_2, \ldots, b_q, s_2.$$

We use π to find an admissible sequence.

Consider segments of the form $[a_1, x]$. If $x \in \{s_1, a_2, \ldots, a_{r-1}\}$, then $[a_1, x]$ is not in $\mathcal{P}$ because it is a proper subset of the atom A_1. If $x = b_1$, then the symmetric difference of A_1 and $[a_1, b_1]$ has cardinality 2 which implies $[a_1, b_1] \notin \mathcal{P}$.

For $x \in \{a_r, b_2, \ldots, b_q\}$, if $[a_1, x] \in \mathcal{P}$, then because the segment contains the elements of A_1, the elements of the segment not belonging to A_1 would have to be in $\mathcal{P}$. But this is impossible because they form a proper subset of A_2. Hence, no proper segment beginning with a_1 belongs to $\mathcal{P}$.

We move to segments beginning with s_1, that is, of the form $[s_1, x]$. If $x \in \{a_2, a_3, \ldots, a_{r-1}\}$, then it is a proper subset of A_1 so that it is not in $\mathcal{P}$. If $[s_1, b_1] \in \mathcal{P}$, then interchange b_1 and b_2 (and their labels too). The new interval $[s_1, b_1]$ does not belong to $\mathcal{P}$. This interchange is possible because $q \geq 2$ as noted earlier.

The interval $[s_1, a_r]$ cannot belong to $\mathcal{P}$ because the cardinality of the symmetric difference with A_1 is 2. If $x \in \{b_2, b_3, \ldots, b_q\}$, the interval $[s_1, x]$ is not an atom because it has cardinality bigger than $|A_1|$. But the elements of the interval not belonging to an atom containing s_1 form a proper subset of A and cannot belong to $\mathcal{P}$. Finally, the interval $[s_1, s_2]$ is not in $\mathcal{P}$ because $\Omega \in \mathcal{P}$.

Of the remaining intervals, the only ones which are not proper subsets of A are those ending in s_2 so that we now examine intervals of the form $[y, s_2]$. Any such interval belonging to $\mathcal{P}$ must be an atom otherwise the elements of the segment not in the atom containing

s_2 is a proper subset of A. Thus, when $y \in \{b_2, b_3, \dots, b_q\}$, $[y, s_2]$ is not in $\mathcal{P}$ because it is a proper subset of A_2.

The interval $[a_r, s_2]$ is not in $\mathcal{P}$ because the cardinality of the symmetric difference with A_2 is 2. The intervals $[y, s_2]$, for $y \in \{a_2, a_3, \dots, a_{r-1}, b_1\}$, are not in $\mathcal{P}$ as they would contradict the choice of A_2 .

We see that most intervals are trivially eliminated as possible elements of $\mathcal{P}$ in the preceding argument. There are several crucial intervals and they are all we discuss in the remaining cases.

Now let $B_2 = \emptyset$ and $B_3 \neq \emptyset$. Moreover, label the elements of B so that $B_3 = \{b_1, b_2, \dots, b_{q-\ell}\}$ and $B_4 = \{b_{q-\ell+1}, \dots, b_q\}$. We modify the sequence π slightly depending on the value of $q - \ell$. Here are the three scenarios. When $q - \ell = 1$, let π be the same through a_{r-1} and end the sequence as

$$a_{r-1}, b_2, a_r, b_1, b_3, \dots, b_q, s_2.$$

When $q - \ell = 2$, end the sequence as

$$a_{r-1}, b_1, a_r, b_3, b_2, b_4, \dots, b_q, s_2.$$

When $q - \ell > 2$, end the sequence as

$$a_{r-1}, b_1, a_r, b_2, \dots, b_{q-\ell-1}, b_{q-\ell+1}, b_{q-\ell}, b_{q-\ell+2}, \dots, s_2.$$

First consider segments of the form $[y, s_2]$ for $y \notin \{a_1, s_1\}$ and for all three scenarios. The interval $[b_{q-\ell}, s_2]$ has symmetric difference of cardinality 2 with A_2 so is not in $\mathcal{P}$. The interval $[y, s_2]$ for $y \in \{b_{q-\ell+2}, \dots, b_q\}$ is a proper subset of A_2 implying it is not in $\mathcal{P}$.

The interval $[b_{q-\ell+1}, s_2]$ has bigger intersection with B than A_2 so that it cannot be an atom. This implies it is not in $\mathcal{P}$ as this would imply the existence of an atom properly contained in A. We obtain essentially the same contradiction for all other values of y distinct from a_1 and s_1. Notice that these intervals are eliminated independent of the choice of $a_r \in A_1 \setminus \{s_1\}$.

No segment of the form $[a_1, x]$ belongs to $\mathcal{P}$ in any of the three preceding scenarios for the same reasons discussed earlier. This conclusion holds independent of the choice of $a_1 \in A_1 \setminus \{s_1\}$. The only problematic segments beginning with s_1 are $[s_1, s_2]$, $[s_1, b_2]$ in the first scenario, and $[s_1, b_1]$ in the second and third scenarios.

If both $[s_1, s_2]$ and $[s_1, b_2]$, or both $[s_1, s_2]$ and $[s_1, b_1]$ are in $\mathcal{P}$, then interchanging a_1 and a_2 results in an admissible sequence. If just $[s_1, s_2] \in \mathcal{P}$, then interchange a_1 and a_r to obtain an admiisible sequence. Finally, if just $[s_1, b_2]$ or $[s_1, b_1]$ belongs to $\mathcal{P}$, then interchange a_{r-1} and a_r to obtain an admissible sequence.

The preceding interchanges require $r \geq 3$ to hold so we consider the special subcase $r = 2$ separately. This case means that A is a 3-atom $\{s_1, a_1, a_2\}$. If s_2 is in a 3-atom $\{s_2, b_{q-1}, b_q\}$ and $b_1, \dots, b_{q-2}$ are the remaining elements of B, then $a_1, s_1, b_{q-1}, a_2, b_{q-2}, b_{q-3}, \dots, b_1, b_q, s_2$ is an admissible sequence. When s_2 is not in a 3-atom, then it is easy to find an admissible sequence.

Now we examine the case that $B_2 \neq \emptyset$ and $B_3 = \emptyset$. Label the elements of A_1 so that $B_1 = \{a_1, a_2, \dots, a_t\}$ and $B_2 = \{a_{t+1}, a_{t+2}, \dots, a_r\}$. Note that $t \geq 2$ because A_2 contains all of B and $|A_2| \leq |A_1|$. When $t \geq 3$, modify the original sequence π by interchanging the positions of a_t and a_{t+1}. The previous arguments for segments beginning with a_1 and s_1 are valid and we look at segments of the form $[y, s_2]$. The interval $[a_t, s_2]$ has symmetric difference of cardinality 2 with A_2 so it is not in $\mathcal{P}$.

When $t = 2$, $A_1 = \{s_1, a_1, a_2, \dots, a_r\}$ and $A_2 = \{s_2, b_1, b_2, a_3, \dots, a_r\}$. The sequence $a_1, s_1, a_3, a_2, \dots, a_{r-1}, b_1, a_r, b_2, s_2$ has only $[s_1, b_1]$ and $[s_1, s_2]$ as possible inadmissible segments. If both are inadmissible, then interchanging a_1 and a_2 produces an admissible sequence. If just $[s_1, b_1]$ is inadmissible, then interchanging b_1 and b_2 does the job. If just $[s_1, s_2]$ is inadmissible, then interchange a_1 and a_2 making the new $[s_1, s_2]$ admissible. If the new $[s_1, b_1]$ still is admissible, then we are done. However, if the new $[s_1, b_1]$ is inadmissible, then interchanging b_1 and b_2 finally achieves an admissible sequence.

The preceding argument works when $r \geq 2$. When $r = 1$, the sequence a_1, s_1, a_3, b_1, a_2, b_2, s_2 has only the segments $[s_1, b_1]$, $[a_2, s_2]$ and $[s_1, s_2]$ that may be inadmissible. When the 6-segment is inadmissible, it either is a 6-atom or there is a unique partition into two 3-atoms. If it is a 6-atom, there is an admissible sequence by Lemma 4.4. In the other situation, it is a straightforward, though tedious, exercise to find an interchange of elements that achieves an admissible sequence for the various partitions into two 3-atoms.

When the 6-segment is admissible, it is easy to fix any problems with the two 3-segments. This completes this case.

We now consider the final case that both B_2 and B_3 are non-empty. We first provide a general argument and then examine any special cases arising because certain B_i sets are too small.

Label elements of B as: $B_3 = \{b_1, \dots, b_{q-\ell}\}$ and $B_4 = \{b_{q-\ell+1}, \dots, b_q\}$. Consider the sequence

$$\pi = a_1, s_1, a_2, \dots, a_{r-1}, b_1, a_r, b_2, \dots, b_{q-\ell-1}, b_{q-\ell+1}, b_{q-\ell}, \dots, b_q, s_2.$$

Proper segments beginning with a_1 do not belong to $\mathcal{P}$ for the same reasons given earlier. Segments of the form $[s_1, y]$, $y \notin \{b_1, s_2\}$, fail to be in $\mathcal{P}$ for the same reasons as before.

If the segment $[s_1, s_2] \in \mathcal{P}$, then we may assume it is not an atom as Lemma 4.4 implies $\mathcal{P}$ is sequenceable otherwise. Then $[s_1, s_2]$ is not a disjoint union of three or more atoms because this would give an atom properly contained in A. Hence, because the segment contains A_2, $[s_1, b_{q-\ell-1}] \cup \{b_{q-\ell}\}$ must be an atom. But the cardinality of the latter set of elements either has cardinality bigger than A_1 or has symmetric difference with A_1 of cardinality 2. In either case we see that it cannot be an atom. Thus, $[s_1, s_2]$ is not in $\mathcal{P}$.

The segment $[s_1, b_1]$ could belong to $\mathcal{P}$ and if it does, this is fixed by interchanging a_1 and a_2. This results in an admissible sequence. Now we consider situations for parameters being too small to let π breathe.

The proof requires $r \geq 3$ in order to make all segments beginning with a_1 not be members of $\mathcal{P}$. Because $r > 1$, we are considering $r = 2$ which means $A_1 = \{s_1, a_1, a_2\}$. So we begin a sequence with a_1, s_1 but now cannot use a_2 as the next element. Because B_2 is non-empty and $|A_2| \leq |A_1|$, we may assume $A_2 = \{s_1, a_2\}$ or $\{s_2, a_2, b_q\}$.

We need to make certain the sequence does not end with the elements of A_2. We know that $B_3 \neq \emptyset$. If it has at least two elements, then we may choose b_1 so that $\{s_1, b_1\}$ is not an atom. In this case, we start the sequence $a_1, s_1, b_1, a_2,$. The completion then depends on q. When $q \geq 4$, we complete the sequence so that it ends b_q, b_{q-1}, s_2. The sequence is admissible independent of whether or not $b_q \in A_2$.

There several other cases to check for $1 \leq q \leq 3$ and $|B_3| = 1$ and they can be checked similarly. This completes the proof. □

5 Small cardinality posets

There are several items worth mentioning before stating the main theorem. First, when discussing a sequenceable poset, we tacitly assume the order relation is set inclusion for subsets of a ground set Ω. Second, we now use lower case letters from the beginning of the alphabet for the elements of Ω.

Third, it is clear that if a poset P is sequenceable, then any subposet is sequenceable as well as any order-isomorphic poset which has arisen via a permutation of Ω. The latter comment means we may relabel elements for some of the subsequent conclusions.

Theorem 5.1. *If $\mathcal{P}$ is a diffuse poset whose ground set has cardinality at most 9, then $\mathcal{P}$ is sequenceable.*

Proof. It is easy to see that a diffuse poset whose ground set has cardinality 1 or 2 is sequenceable. If the ground set is $\{a, b, c\}$, then a diffuse poset has either no elements, a single 2-atom or a single 3-atom. The sequence a, c, b is admissible assuming the 2-atom is $\{a, b\}$ when there is a single 2-atom.

If the ground set is $\{a, b, c, d\}$, then $\mathcal{P}$ is sequenceable if $\{a, b, c, d\}$ belongs to $\mathcal{P}$ by Lemma 4.2, in particular, when there are two 2-atoms. If there is a 3-atom, then there is an admissible sequence by Lemma 4.4. The situation is trivial if there is a single 2-atom or no atoms at all.

So we see that diffuse posets with ground sets of cardinality at most 4 are sequenceable. We next consider ground sets of cardinality 5.

If $\Omega = \{a, b, c, d, e\}$ and Ω belongs to the diffuse poset $\mathcal{P}$, then $\mathcal{P}$ is sequenceable by Lemma 4.2. If $\Omega \notin \mathcal{P}$ but there is an atom of cardinality 4, then $\mathcal{P}$ is sequenceble by Lemma 4.4. If there are no 4-atoms but there is a 3-atom, then $\mathcal{P}$ is sequenceable by Lemma 4.5. If the only atoms are 2-atoms, then $\mathcal{P}$ is sequenceable by Lemma 4.1. Hence, all diffuse posets with ground sets of cardinality 5 are sequenceable.

This takes us to ground sets of cardinality 6. Let $\Omega = \{a, b, c, d, e, f\}$. As before, Lemma 4.4 implies $\mathcal{P}$ is sequenceable if there is a 5-atom and Lemma 4.5 implies $\mathcal{P}$ is sequenceable when there is a 4-atom. Lemma 4.2 implies that a diffuse poset $\mathcal{P}$ with ground set Ω is sequenceable whenever $\Omega \in \mathcal{P}$. Thus, we may assume that every atom is either a 2-atom or a 3-atom and there are neither two disjoint 3-atoms nor three 2-atoms.

It is not difficult to verify that to within order-isomorphism there is a unique maximal diffuse poset on Ω with only 2- and 3-atoms, that is, every diffuse poset with this restriction on the atoms is order-isomorphic to a subposet. The atoms of this unique poset are $\{be, cd, abc, ade, bdf, cef\}$ and an admissible sequence is d, b, c, f, a, e.

For ground sets of cardinalities 7 and 8, the results are displayed in Tables 1 and 2 in the appendix. We have verified the result for ground sets of cardinality 9, but the number of pages to display the table is about 30 and we have chosen to not include the table. This concludes the proof. □

Corollary 5.2. *Let $\overrightarrow{X}$ be a Cayley digraph on an abelian group. If the connection S set for $\overrightarrow{X}$ has at most nine elements, then $\overrightarrow{X}$ admits an orthogonal directed path when $\Sigma S \neq 0$ or an orthogonal directed cycle when $\Sigma S = 0$.*

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ORCID iDs

Brian Alspach https://orcid.org/0000-0002-1034-3993
Georgina Liversidge https://orcid.org/0000-0002-4467-4328

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Appendix

Table 1 below provides admissible sequences for all diffuse posets with a ground set of cardinality 7 and Table 2 does the same for ground sets of cardinality 8. However, two conventions are the following:

(1) No poset having the ground set as an element is included because they are sequenceable by Lemma 4.2; and

(2) only posets with k-atoms for $k \in \{2, 3, 4\}$ and $k \in \{2, 3, 4, 5\}$ are listed in Tables 1 and 2, respectively, as the others are sequenceable by the lemmas of Section 4.

included atoms	excluded atoms	sequence
ab,cd,ef,ace,adg	cfg	b,d,a,e,g,c,f
ab,cd,ef,ace	**adg**,bdeg	a,c,b,e,d,g,f
ab,cd,ef,acg	ade,adf,adfg	b,e,d,a,f,g,c
ab,cd,ef	xyz,acfg	a,c,g,f,d,e,b
ab cd,aef,ceg,bdf	xy	c,a,e,b,d,g,f
ab,cd,aef,ceg	xy,**bdf**	a,d,f,e,g,b,c
ab,cd,aef,bce	xy,cfg,deg,dfg	a,d,f,c,g,e,b
ab,cd,aef	xy,**ceg**,**bce**,acfg	e,b,d,a,f,c,g
ab,cd,ace	xy,**bef**,**afg**,**bfg**,adg	b,d,a,g,c,e,f
ab,cd	xy,xyz,bcef	e,b,f,c,a,d,g
ab,acd,cef,beg	xy	a,d,g,e,f,b,c
ab,acd,cef	xy,**beg**,acfg	b,d,a,f,c,g,e
ab,acd,bef	xy,**ceg**,acfg	a,c,g,f,d,e,b
ab,acd,bce	xy,**def**,cfg,dfg,bfg,bdef	a,f,b,e,d,c,g
ab,acd	xy,**cef**,**bef**,**bce**,adef	e,d,f,a,c,b,g
ab,cde	xy,axy,bxy,acef,acdg	b,f,e,a,c,d,g
ab,acde	xy,xyz	b,c,f,a,d,e,g
ab	xy,xyz,axyz,bxyz,defg	a,d,e,b,f,g,c
abc,ade,bdf	xy,bdeg	c,a,e,b,d,g,f
abc,ade,abdf	xy,**bef**,cef,bdg,**beg**,ceg	a,c,g,b,f,d,e
abc,ade	xy,**bdf**,**abdf**	f,b,d,a,c,e,g
abc,def	xy,xyz,abeg,abfg,acdg bcdg,adeg,adfg,bdeg,bdfg	a,c,g,d,b,e,f
abc,abde	xy,xyz	c,a,f,b,d,e,g
abc	xy,xyz,abxy,acxy,bcxy	a,d,e,b,c,f,g
abcd	xy,xyz,acef	b,g,d,a,c,e,f

Table 1: Ground set of cardinality 7

We alter the notation for the tables in two ways. First, we use words rather than set notation because it saves considerable space. Second, we use roman letters rather than italics because the appearance of the words is better. In summary, the atom $\{a, b, c\}$ in the main body of the paper appears as abc in the tables.

The tables have been compacted to an extent that makes it necessary to describe how to read them.

The column headed “included atoms” contains a list of atoms that definitely belong to the poset under discussion . The only convention to keep in mind here is that an entry in parentheses—such as in row 12 in Table 1—indicates that precisely one of the words is an atom but not both. So in this situation, one of bf or cg is a 2-atom but it is not the case that both 2-atoms are in the poset.

The column headed “excluded atoms” indicates which atoms are definitely not in the poset, but there are some conventions being followed. These conventions are now listed.

included atoms	excluded atoms	sequence
ab,cd,ef,acg,beh	-	a,d,g,c,e,b,f,h
ab,cd,ef,acg,beg	bfh,deh,dfh	a,h,c,g,e,d,f,b
ab,cd,ef,acg,bgh	**deh**,**deg**	c,h,a,g,e,b,f,d
ab,cd,ef,acg,egh	**bfh**,**bfg**,cfh	a,g,e,c,h,f,d,b
ab,cd,ef,acg	adf,**beh**,**beg**,**bgh**,**egh**	b,d,a,f,c,g,h,e
ab,cd,ef,agh,ace	gxy,hxy	c,a,d,g,f,b,e,h
ab,cd,ef,agh	xyz,acfg	d,b,e,a,g,f,c,h
ab,cd,ef,ace,bdf	xyz	a,c,b,g,d,f,h,e
ab,cd,ef	xyz,acfg	b,d,c,a,f,d,e,h
ab,cd,aef,bcg,egh	xy	a,d,e,f,g,h,c,b
ab,cd,aef,bcg	xy,**egh**	g,e,b,c,f,d,a,h
ab,cd,aef,egh	xy,**bcg**,bcf	a,d,f,e,g,c,h,b
ab,cd,aef,ceg	xy,bdg,bch bdh,fgh,beh	e,h,b,g,c,a,d,f

Table 2: Ground set of cardinality 8 (Continued)

(1) Any atom that violates the definition of a diffuse poset because of an included atom, certainly is not in the poset and it is not listed as an excluded atom. For example, if abc is an included atom, then no other 3-atom may contain ab, ac or bc so they simply are not listed in the excluded atoms column. Similarly, if abcd is an included atom, then neither abce nor abc can be atoms and they are not listed in the excluded atoms column.

(2) If an excluded atom uses letters from the ground set, then that particular atom is not in the poset.

(3) If an excluded atom uses letters from the end of the alphabet, then it means that all atoms of that cardinality different from any included atoms are excluded. For example, in line 7 of Table 1, xy indicates that there are no 2-atoms other than ae, and xyzw indicates that abcd is the only 4-atom in the poset.

(4) If an excluded atom uses both letters from the ground set and the end of the alphabet, it means all atoms of that form different from any included atoms are not in the poset. For example, in poset 51 of Table 2, ab is an included atom and both axy and bxy are excluded. This means there are no 3-atoms containing a and no 3-atoms containing b. Of course, there are no 3-atoms containing both a and b because ab is a 2-atom.

(5) Excluded atoms in boldface indicate all atoms that can be formed via label changes allowed because of the included atoms are excluded. The following example should

included atoms	excluded atoms	sequence
ab,cd,aef	xy,**bcg**,**egh**,**ceg**,acfg	e,g,c,f,a,d,b,h
ab,cd,ace,efg	xy,**afg**,**afh**,**bef**,**beh**,**bfg**,**bfh**	a,e,f,c,h,d,g,b
ab,cd,ace	xy,**afg**,**bef**,**bfg**,**efg**,adh	f,b,d,a,h,c,e,g
ab,cd,efg,aefh	xy,xyz	c,d,h,a,e,d,f,g
ab,cd,efg	xy,xyz,**aefh**,adef,befh,cefh aegh,afgh,begh,bfgh,defh cegh,cfgh,degh,dfgh,adef	b,e,a,d,f,c,h,g
ab,cd,aefg	xy,xyz	e,b,f,a,c,g,d,h
ab,cd	xy,xyz,**aefg**	e,b,g,a,c,h,d,f
ab,acd,bef,ceg,agh	xy,dfg	c,e,h,f,b,d,a,g
ab,acd,bef,ceg	xy,**agh**	a,d,g,c,f,e,h,b
ab,acd,bef,aeg	xy,**cfg**,ceh,**cfh**	b,g,f,a,e,c,d,h
ab,acd,bef,agh	xy,**ceg**,**bcg**	e,b,c,a,g,d,h,f
ab,acd,bef,cgh	xy,**deg**,**aeg**,**bdg**	a,e,d,c,g,f,h,b
ab,acd,bef	xy,xyz,adgh	a,c,g,d,e,f,h,b
ab,acd,cef,deg,bch	xy,bfg	a,d,f,c,h,e,b,g
ab,acd,cef,deg	xy,beh,bfg **bfh**,**bch**,bceh	d,a,h,c,b,e,f,g
ab,acd,cef,egh	xy,**bfg**,**dfg**,aceg	b,g,a,e,c,d,f,h
ab,acd,cef,aeg	xy,bfg,beh,bfh,bgh dfg,deh,**dfh**	b,g,f,a,e,c,d,h
ab,acd,cef,bdg	xy,**aeh**,**beh**,**deh**,**egh**,**aeg**,bcfg	d,a,g,c,b,f,e,h
ab,acd,cef,bde	xy,**bfg**,bgh,**dfg**,egh fgh,aeg,afg,aeg,afh,bcdf	g,a,c,b,d,f,e,h
ab,acd,cef,aceg	xy,beg,bfg,beh,bfh,bgh,deg,dfg deh,dfh,egh,fgh,afg,aeh afh,bdg,bdh,bde,bdf	b,d,a,f,c,g,e,h
ab,acd,cef	xy,**beg**,bgh,**deg**,**egh**,**aeg**,**bdg** **bde**,**aceg**,bcfg	d,a,g,c,f,b,e,h
ab,acd,bce,afg	xy,bfg,**bfh**,cgh **def**,deh,dfg,**dfh**	b,d,e,a,c,f,g,h
ab,acd,bce,efg	xy,**afh**,**bfh**,cgh,deh,**dfh**,adef	c,b,d,e,f,g,h
ab,acd,bce,aef,adfg	xy,bfg,bfh,bgh,**cfg** deh,dfh,**cgh**,dgh,agh	a,c,b,h,e,f,g,d
ab,acd,bce,aef	xy,bfg,bgh,cfg,deg,deh **dfg**,**cgh**,dgh,agh,**adfg**	b,g,a,d,f,c,e,h
ab,acd,bce,fgh	xy,**def**,**aef**,adfg	d,c,f,e,b,g,a,h

Table 2: Ground set of cardinality 8 (Continued)

included atoms	excluded atoms	sequence
ab,acd,bce,adfg	xy,xyz	a,c,b,h,e,f,g,d
ab,acd,bce	xy,xyz,**adfg**,bfgh	e,f,b,c,a,g,d,h
ab,acd,bcdg	xy,**bef**,**bfg**,**cef**,def,ceh deh,dfh,**ceg**,**bce**,adeg	b,d,a,g,e,f,h,c
ab,acd	xy,**bef**,**cef**,def,ceg,cfg deg,dfg,ceh,deh,dfh,cgh dgh,**bce**,bcdg,bcdh	g,b,c,a,e,d,f,h
ab,acd,efg,acef	xy,xyz	b,e,f,a,h,c,d,g
ab,acd,efg	xy,xyz,**acef**,begh	f,b,e,a,c,g,d,h
ab,acd,acef	xy,xyz	b,d,a,g,c,e,f,h
ab,acd,bcef	xy,xyz,adef,**aceg**,**adeg**,acgh,adgh	d,g,c,a,e,b,f,h
ab,acd,cefg	xy,xyz,**acef**,**adef**,**aceh**,**adeh** **bceh**,**bdef**,**bcef**,**bdeh**,bcgh,begh	b,f,a,d,e,c,g,h
ab,acd	xy,xyz,**acef**,**bcef**,**cefg**,befh,begh	f,b,e,a,c,h,d,g
ab,cde,cfg,dfh,acef	xy,axy,bxy	b,d,a,e,f,h,c,g
ab,cde,cfg,dfh	xy,axy,bxy,**acef**,**bcdf**	e,a,d,b,f,c,h,g
ab,cde,cfg,acdf	xy,xyz	b,h,d,a,c,e,f,g
ab,cde,cfg	xy,xyz,**acdf**,bfgh	d,a,e,c,f,b,g,h
ab,cde,acdf	xy,xyz,aceh	b,e,a,h,c,d,f,g
ab,cde,acfg	xy,xyz,**adef**,**bcdf**,**bdef** **acdh**,adeh,**bcdh**,bdeh	b,d,a,e,f,c,g,h
ab,cde	xy,xyz,**acdf**,**acfg**,cefg	f,a,g,c,b,e,d,h
ab,acde,bcdf	xy,xyz	a,e,g,c,d,f,h,b
ab,acde,acfg	xy,xyz,**bcdf**,bdef bdeg,bcdh,bceh,bdeh	b,d,a,e,f,c,g,h
ab,acde	xy,xyz,**bcdf**,**acfg** acefg,acdfh,acdgh	b,e,f,c,a,d,h,g
ab,cdef	xy,xyz,axyz,bxyz,acdeh,acdfg	b,g,f,a,c,d,e,h
ab,acdef	xy,xyz,xyzw	b,c,g,a,d,e,f,h
ab	xy,xyz,xyzw,**axyzw**,defgh	a,d,e,b,f,g,h,c
abc,ade,bdf,afg,beh	xy,bcfg	c,b,d,a,f,e,g,h
abc,ade,bdf,afg,ceh	xy,bgh,dgh,cdef	b,a,d,c,e,f,h,g
abc,ade,bdf afg,bcfh	xy,beh,cdh,efh,bgh dgh,ceh,cgh,egh	a,c,g,f,b,h,d,e
abc,ade,bdf,afg	xy,**beh** **cgh**,bcfh,bdeh,dfgh	c,a,e,b,d,h,f,g

Table 2: Ground set of cardinality 8 (Continued)

included atoms	excluded atoms	sequence
abc,ade,bdf,ceg,cfh	xy,**afg**,bcgh	a,f,d,e,g,h,c,b
abc,ade,bdf,ceg,bfgh	xy,afg,afh,beh cdh,cfh,efh,dgh	a,b,f,c,g,h,e,d
abc,ade,bdf,ceg	xy,afg,afh,**beh** **cfh**,**bfgh**,acfg	b,f,c,g,a,e,h,d
abc,ade,bdf,cgh,abeg	xy,afg,afh,beh,efg,efh	a,b,d,e,g,f,h,c
abc,ade,bdf,cgh,acdg	xy,afg,beg,beh,afh,efg efh,abeg,abeh,abfh,abfg	b,a,f,c,d,g,h,e
abc,ade,bdf,cgh	xy,**afg**,**efg**,**abeg** **acdg**,bcfg	e,g,a,c,d,b,h,f
abc,ade,bdf,agh,bcfg	xy,beg,cdg,beh,cdh ceg,efg,ceh,cfh,efh	a,c,g,b,h,f,d,e
abc,ade,bdf,agh	xy,**beg**,**ceg**,**cfg** **bcfg**,abdg	c,g,a,b,d,e,f,h
abc,ade,bdf,abeg	xy,afg,cdg,afh,beh,cdh ceg,cfg,efg,ceh,cfh,efh cgh,egh,fgh,agh,bgh,dgh	c,b,f,a,e,g,d,h
abc,ade,bdf,acfg	xy,beg,cdg,afh,beh,cdh ceg,efg,ceh,cfh,efh,cgh egh,fgh,agh,bgh,dgh,abeg bcdg,bdeg,abeh,abfh acdh,adfh,bcdh,bdeh	b,h,c,a,g,e,f,d
abc,ade,bdf,bcfg	xy,beg,beh,cfh,egh,bgh bfg,**afg,afh,ceg,ceh,cgh** **agh,abeg,abfg,acdg** **aefg,acfh,bceh,abeh** **abfh,acdh,acfg,bceg,aefh**	a,c,g,b,h,f,d,e
abc,ade,bdf	xy,**afg**,**ceg**,**cgh** **agh**,**abeg**,**acfg**,bcfg,bcfh	g,c,a,d,b,e,f,h
abc,ade,bfg,dfh,cgh	xy,xyz,adfg	b,a,f,d,g,h,e,c
abc,ade,bfg	xy,cef,cdg,ceg,beh,ceh	c,a,d,b,e,f,h,g
abc,ade,bfg dfh,acef	xy,bdf,**bef**,**beg**,**bdgcdg** **ceg**,**adfg**,**abdg**,**bcdf**	b,c,g,a,f,e,d,h
abc,ade,bfg,dfh	xy,cef,cdg,ceg beh,ceh,**abef**,**acef**	c,a,h,b,d,e,f,g

Table 2: Ground set of cardinality 8 (Continued)

included atoms	excluded atoms	sequence
abc,ade,bfg,bcfh	xy,**cdf**,**cdg**,**bdh** **cdh**, **dfh**,**dgh**	a,c,d,b,f,h,g,e
abc,ade,bfg,acdf	xy,cef,cdg,ceg,bdh,beh cdh,ceh,dfh,efh,dgh,egh bcfh,bcgh,dfgh,aceh	b,c,f,a,g,d,e,h
abc,ade,bfg	xy,**cdf**,bdh,beh,cdh,ceh **dfh** **bcfh**,**acdf**,abeh	d,h,a,e,b,c,f,g
abc,ade,abdf	xy,**bef**,cef,**beg** **ceg**,**bdh**,cdeg	g,e,d,c,a,f,b,h
abc,ade,bcdf	xy,**bef**,**bdg**,**beg** **abef**,**abdg**,**abeg**,aefh	g,b,c,d,a,f,e,h
abc,ade,abfg	xy,**bdf**,**cdf**,**bdh** **cdh**,**acdf**,**abdh**,**acdh**,**bcdh** **bcdf**,**bdef**,**cdef**,bdeh,cdeh	d,f,g,a,e,b,c,h
abc,ade	xy,**bdf,abdf,bcdf,abfg** bcfh,bcgh	f,b,c,d,a,h,e,g
abc,def,abdg	xy,axy,bxy,cxy,acfh	e,g,b,f,a,c,h
abc,def,adgh	xy,axy,bxy,cxy,**abeg** **bcdg**,**bceg**,abdfg	b,a,g,d,f,h,e,c
abc,def,abde	xy,axy,bxy,cxy,egh,**abfg** **acdg**,**acfg**,**adgh**,**afgh**,cfgh	f,d,g,a,e,b,c,h
abc,def,abgh	xy,axy,bxy,cxy,xyzw,acdeg	b,a,g,d,c,e,f,h
abc,def	xy,axy,bxy,cxy,xyzw,abdfg	c,a,g,d,b,f,e,h
abc,abde,adefg	xy,xyz	c,b,f,a,d,e,h,g
abc,abde	xy,xyz,**adefg**,cdefg	c,a,f,b,d,e,g,h
abc,adef	xy,xyz,abxy,acxy,bcxy,abdfg	g,d,f,a,b,e,c,h
abc,abdef	xy,xyz,axyz,bxyz,cxyz	c,a,g,b,d,e,f,h
abc	xy,xyz,axyz,bxyz,cxyz,abxyz acxyz,bcxyz,adefh,adegh bdefg,cdefg	b,f,d,a,e,h,g,c
abcd,abef,abceg	xy,xyz	c,d,g,a,b,e,h,f
abcd,abef	xy,xyz,**abceg**	c,g,d,a,b,e,h,f
abcd,abcef	xy,xyz,abxy,acxy,adxy bcxy,bdxy,cdxy,befg	d,a,h,c,b,e,f,g
abcd	xy,xyz,**abef**,**abcef**	a,e,f,b,c,d,g
abcde	xy,xyz,xyzw,abcfg	d,h,e,a,b,c,f,g

Table 2: Ground set of cardinality 8

make this subtle concept clear. Start with the template obtained from the included atoms and observe which label changes are allowed. For example, consider the sixteenth entry from the end of Table 2. The included atoms are abc, ade so that the template is two 3-atoms intersecting at a single point. The label a is fixed because it is the only point belonging to both 3-atoms. The labels b and c may be switched because they lie in the same 3-atom. The same holds for the labels d and e. Furthermore, the two sets {b,c} and {d,e} may be switched. Finally, the labels f,g,h may be switched with each other as they belong to neither 3-atom. Thus, excluding the atom **bdf** means that all of the 3-atoms bdf, bef, cdf, cef, bdg, beg, cdg, ceg, bdh, beh, cdh and ceh are excluded. Similarly, excluding the atom **abdf** means all the atoms abdf, abef, acdf, acef, abdg, abeg, acdg, aceg, abdh, abeh, acdh and aceh are excluded.

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Self-dual, self-Petrie-dual and Möbius regular maps on linear fractional groups*

Grahame Erskine

School of Mathematics and Statistics, Faculty of Science, Technology, Engineering and Mathematics, The Open University, Walton Hall, Milton Keynes, MK7 6AA, U.K.

Katarína Hriňáková†

Department of Mathematics, Faculty of Civil Engineering, Slovak University of Technology, Radlinského 11, 810 05, Bratislava, Slovakia

Olivia Reade Jeans‡

School of Mathematics and Statistics, Faculty of Science, Technology, Engineering and Mathematics, The Open University, Walton Hall, Milton Keynes, MK7 6AA, U.K.

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Abstract

Regular maps on linear fractional groups $\mathrm{PSL}(2, q)$ and $\mathrm{PGL}(2, q)$ have been studied for many years and the theory is well-developed, including generating sets for the associated groups. This paper studies the properties of self-duality, self-Petrie-duality and Möbius regularity in this context, providing necessary and sufficient conditions for each case. We also address the special case for regular maps of type $(5, 5)$. The final section includes an enumeration of the $\mathrm{PSL}(2, q)$ maps for $q \leq 81$ and a list of all the $\mathrm{PSL}(2, q)$ maps which have any of these special properties for $q \leq 49$.

Keywords: Regular map, external symmetry, self-dual, self-Petrie-dual, Möbius regular.

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‡Corresponding author.

E-mail addresses: grahame.erskine@open.ac.uk (Grahame Erskine), hrinakova@math.sk (Katarína Hriňáková), olivia.jeans@open.ac.uk (Olivia Reade Jeans)

1 Introduction

Regular maps always display inherent symmetry by virtue of their definition. A regular map can have further symmetry properties which are called *external symmetries*. These occur when a map is isomorphic to its image under a particular operation. The best known example of this is the tetrahedron, a Platonic solid which is self-dual.

A *map* is a cellular embedding of a graph on a surface and is made up of vertices, edges and faces. A *flag* is a triple incidence of edge-end, edge-side and face centre. Informally we can visualise each flag as a triangle with its corners at the vertex, the centre of the face and the midpoint of the edge. Thus there are four flags incident to any edge and the whole surface is covered by flags. We consider the symmetries of a map by reference to its flags. An *automorphism* of a map is an arbitrary permutation of its flags such that all adjacency relationships of the flags are preserved. The map is *regular* if the group of automorphisms acts regularly on the flags, that is the group is fixed-point-free and transitive. An implication of this is that each vertex of a regular map has a given valency, say k, and the face lengths are all equal, say to l. Henceforth we will refer to maps of type (k, l) where k is the vertex degree and l is the face length of the regular map.

For further details about the theory of regular maps see [3, 9, 12, 15, 16].

Every regular map has an associated *dual map* which is also a regular map. Informally, the dual map is created by forming a vertex at the centre of each original face and considering each of the original vertices as the centre of a face. Each edge of the dual map is thereby formed by linking a pair of neighbouring vertices across one of the original edges.

A different type of dual, the *Petrie dual* of a map has the same edges and vertices as the original map but the faces are different. That is, the underlying graph is the same, but the embedding is different. The boundary walk of a face of the Petrie dual map can be described informally as follows:

1. Starting from a vertex on the original map, trace along one side of an incident edge until you get to the midpoint of that edge;
2. Cross over to the other side of the edge and continue tracing along the edge in the same direction as before. When you approach a vertex, sweep the corner and continue along the next edge until you reach its midpoint;
3. Repeat step 2 until you rejoin the face boundary walk where you started.

When the associated dual or Petrie dual map is isomorphic to the original map, we call the map *self-dual* or *self-Petrie-dual* respectively. This paper explores necessary and sufficient conditions for a regular map with automorphism group $\mathrm{PSL}(2, q)$ or $\mathrm{PGL}(2, q)$, where q is odd, to have each of these external symmetries.

Another property of interest in the theory of regular maps is *Möbius regularity*. This concept was introduced by S. Wilson in [17] who originally named them *cantankerous*. A regular map is Möbius regular if any two distinct adjacent vertices are joined by exactly two edges and any open set supporting these edges contains a Möbius strip. Clearly such a map must have even vertex degree k and we will establish the further conditions under which a regular map on $\mathrm{PSL}(2, q)$ or $\mathrm{PGL}(2, q)$ is Möbius regular.

In Section 2 we state some of the background material and results which we will need. Section 3 investigates regular maps of type (p, p), (k, p) and (p, l) when k and l are coprime to p. Type (p, p) is self-dual but not self-Petrie-dual nor Möbius regular, and type (p, l) can be self-Petrie-dual but not Möbius regular. Type (k, p) can be self-Petrie-dual or Möbius regular. In Section 4 we address the necessary and sufficient conditions for a map of type (k,l) to be self-dual, self-Petrie-dual and Möbius regular respectively. Section 5 highlights a special case, namely maps of type $(5,5)$ whose orientation-preserving automorphism groups turn out to be isomorphic to A_5 and Section 6 comments on and lists examples of maps with some or all of these properties.

2 Background information, notes and notation

This paper is founded on work done by M. Conder, P. Potočnik and J. Širáň in [4] which provides a detailed analysis of reflexible regular hypermaps for triples (k,l,m) on projective two-dimensional linear groups including explicit generating sets for the associated groups. In particular this paper is concerned only with maps, not hypermaps, and so, without loss of generality, we let $m=2$.

The group of automorphisms of a regular map is generated by three involutions, two of which commute, where the three involutions can be thought of as local reflections in the boundary lines of a given flag which preserve all the adjacency relationships between flags. As shown in Figure 1 the involutions act locally on the given flag as follows: X as a reflection in the edge bisector; Y as a reflection across the edge; Z as a reflection in the angle bisector at the vertex. The dots on the diagram indicate where there may be further vertices, edges and faces while the dashed lines outline each of the flags of this part of the map.

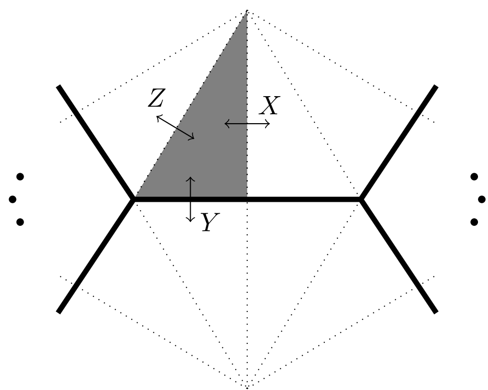

Figure 1: The action of automorphisms X, Y and Z on the shaded flag.

The study of regular maps is equivalent to the study of group presentations of the form

$$G \cong \langle X,Y,Z \mid X^2, Y^2, Z^2, (YZ)^k, (ZX)^l, (XY)^2, \ldots\rangle,$$

see [16]. The dots indicate the potential for further relations not listed, and we assume the orders shown are indeed the true orders of those elements in the group.

The surface on which a regular map is embedded could be orientable or non-orientable. If the regular map is on an orientable surface then G has a subgroup of index two which corresponds to the orientation preserving automorphisms. Instead of the group generated by these three involutions X,Y and Z, we can consider the group of orientation-preserving automorphisms which is generated by the two rotations $R=YZ$ and $S=ZX$. On a non-orientable surface these two elements will still generate the full automorphism group and we can say that studying these maps is equivalent to studying groups which have presentaions of the form $\langle R, S \mid R^k, S^l, (RS)^2, \ldots\rangle$.

We focus on regular maps of type (k, l) where the associated group $G \cong \langle X, Y, Z\rangle$ is isomorphic to $\mathrm{PSL}(2, q)$ or $\mathrm{PGL}(2, q)$ where q is a power of a given odd prime p. By [4], both k and l are either equal to p or divide $q - 1$ or $q + 1$. Following the convention and notation of [4], in the latter two cases we let ξ_κ and/or ξ_λ be primitive $2k$th or $2l$th roots of unity respectively. Note that in the case where k or l divides $q - 1$ then the corresponding primitive root is in the field $\mathrm{GF}(q)$; otherwise it is in the unique quadratic extension $\mathrm{GF}(q^2)$. We also define $\omega_i = \xi_i + \xi_i^{-1}$ for $i \in \{\kappa, \lambda\}$. Note that ω_i is thus in the field $\mathrm{GF}(q)$. We too assume that (k, l) is a *hyperbolic* pair, that is $1/k + 1/l < 1/2$. This implies that $k \geq 3$ and $l \geq 3$. The conditions in this paragraph are what we refer to as *the usual setup*.

We can consider duality and Petrie duality as operators on a map. Since the dual of a map is obtained by swapping the vertices for faces and vice versa, in terms of the involutions X, Y, Z the dual operator would fix Z and interchange X and Y. The Petrie dual operator would replace X with XY and fix Y and Z. The automorphism associated with any type of duality is an involution. This is because it acts on our map to produce the dual map, and when this automorphism is repeated we get back to the original map. Self-duality and self-Petrie-duality are therefore equivalent to the existence of precisely such involutory automorphisms of G, the group associated with the regular map.

Our paper is devoted in large part to finding conditions for the existence of involutory automorphisms which imply self-duality and/or self-Petrie-duality. The automorphism group for G is $\mathrm{P\Gamma L}(2, q)$, the semidirect product $\mathrm{PGL}(2, q) \rtimes C_e$ where $q = p^e$, [13]. Observe that, when $e = 1$, this group is essentially $\mathrm{PGL}(2, p)$ and so, in the case where $G \cong \mathrm{PGL}(2, p)$, all automorphisms are inner automorphisms. Elements $(A, j) \in \mathrm{P\Gamma L}(2, q)$ act as follows: $(A, j)(T) = A\phi_j(T)A^{-1}$ where ϕ_j is the repeated Frobenius field automorphism of the finite field, $\phi_j : x \to x^r$ with $r = p^j$. The function ϕ_j acts element-wise on a matrix and we use the general rule for composition in $\mathrm{P\Gamma L}(2, q)$ which is $(B, j)(A, i) = (B\phi_j(A), i + j)$.

When $(A, j) \in \mathrm{P\Gamma L}(2, p^e)$ is an involution, it must be such that $(A, j)(A, j) = (A\phi_j(A), 2j)$ is the identity, so $2j \equiv 0 \pmod{e}$. One case is when there is no field automorphism involved, that is $j = 0$ and $A^2 = I$. Alternatively $e = 2j$ is even, and then we need $\phi_j(A) = A^{-1}$. This is summarised in Lemma 2.1.

Lemma 2.1. $(A, j) \in \mathrm{P\Gamma L}(2, p^e)$ *is an involution if and only if one of the following conditions holds:*

1. $j = 0$ *and* $A^2 = I$
2. $2j = e$ *and* $\phi_j(A) = A^{-1}$.

Explicit generating sets are known for regular maps with automorphism group G isomorphic to $\mathrm{PSL}(2, q)$ or $\mathrm{PGL}(2, q)$, and for details we refer the interested reader to [4]. We present the results for maps of each type as required.

We will need to consider performing operations on the elements X, Y and Z of G. As such we denote elements of the group $\mathrm{PSL}(2, q)$ or $\mathrm{PGL}(2, q)$, by a representative matrix with square brackets. This allows us to perform the necessary calculations. We can then determine whether or not two resulting matrices are equivalent within G, that is whether or not they correspond to the same element of the group G. A pair of matrices are in the same equivalence class, that is they represent the same *element* of G, if one is a scalar multiple of the other. We use curved brackets for matrix representatives for X, Y and Z.

Lemma 2.1 can then be used to find conditions for the elements of the matrix part A of an involutory automorphism (A, j) as follows.

Lemma 2.2. *Let $A = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$. The automorphism denoted $(A, j) \in \mathrm{P\Gamma L}(2, p^e)$ is an involution, if and only if a, b, c, d satisfy the following equations, with $r = p^j$ for $j = 0$ or $2j = e$.*

1. $a^{r+1} = d^{r+1}$,
2. $bc^r = cb^r$,
3. $ab^r + bd^r = 0$,
4. $ca^r + dc^r = 0$.

Proof. By Lemma 2.1, and letting $r = p^j$ we have

$$A\phi_j(A) = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} a^r & b^r \\ c^r & d^r \end{bmatrix} = \begin{bmatrix} a^{r+1} + bc^r & ab^r + bd^r \\ ca^r + dc^r & cb^r + d^{r+1} \end{bmatrix} = I.$$

By comparing the leading diagonal entries we see that $a^{r+1} + bc^r = cb^r + d^{r+1}$. Applying the field automorphism ϕ_j yields $a^{1+r} + b^r c = c^r b + d^{1+r}$. Subtracting these two equations, and remembering that q is odd, we get the first two equations, while looking at the off-diagonal immediately gives rise to the final two equations. $\square$

When we are establishing the conditions under which a regular map is Möbius regular we will rely on the following group-theoretic result, proved in [11] by Li and Širáň. Note that implicit in this necessary and sufficient condition is that for a map of type (k, l) to be Möbius regular k must be even.

Lemma 2.3. *A regular map is Möbius regular if and only if $XR^{\frac{k}{2}}X = R^{\frac{k}{2}}Y$ where $R = YZ$.*

3 Regular maps on linear fractional groups of type (p, p), (k, p) and (p, l) where p is an odd prime

For odd prime p, by Proposition 3.1 in [4], maps of the type (p, p) have the following representatives for X, Y and Z, where $\alpha^2 = -1$:

$$X_1 = -\alpha \begin{pmatrix} 1 & 0 \\ 2 & -1 \end{pmatrix}, \quad Y_1 = -\alpha \begin{pmatrix} 1 & -1 \\ 0 & -1 \end{pmatrix} \quad \text{and} \quad Z_1 = \alpha \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Proposition 3.1. *With the usual setup, a map of type (p, p) is self-dual.*

Proof. For self duality we need $G \cong \langle X, Y, Z \rangle$ to admit an automorphism such that X and Y are interchanged, and Z is fixed. So the question is: can we find an automorphism $(A, j) \in \mathrm{P\Gamma L}(2, q)$ such that $A\phi_j(X)A^{-1} = Y$, $A\phi_j(Y)A^{-1} = X$, and $A\phi_j(Z)A^{-1} = Z$. It is easy to verify that $(A, 0)$, where A has the form $A = \left[\begin{smallmatrix} 0 & 1 \\ 2 & 0 \end{smallmatrix}\right]$ satisfies these conditions, so this type of map is self-dual. $\square$

Proposition 3.2. *With the usual setup, a map of type (p, p) is not self-Petrie-dual.*

Proof. In order to be self-Petrie-dual, the group G needs to admit an involutory automorphism (B, j) which fixes Z and Y, and exchanges X with XY.

First notice that $\phi_j(Z) = Z$ and $\phi_j(Y) = Y$ so if B exists, it must be of a form which commutes with both Z and Y. To commute with Z, the necessarily non-identity element B must be either $B_1 = \left[\begin{smallmatrix} 0 & b \\ c & 0 \end{smallmatrix}\right]$ or $B_2 = \left[\begin{smallmatrix} a & 0 \\ 0 & d \end{smallmatrix}\right]$. Note that $0 \notin \{a, b, c, d\}$ and $a \neq d$. As shown below, neither of these commute with Y.

$$B_1 Y = -\alpha \begin{bmatrix} 0 & -b \\ c & -c \end{bmatrix} \neq Y B_1 = -\alpha \begin{bmatrix} -c & b \\ -c & 0 \end{bmatrix}$$

$$B_2 Y = -\alpha \begin{bmatrix} a & -a \\ 0 & -d \end{bmatrix} \neq Y B_2 = -\alpha \begin{bmatrix} a & -d \\ 0 & -d \end{bmatrix}$$

Hence this type of map is not self-Petrie-dual. □

Remark 3.3. Maps of type (k, p) and (p, l) where k and l are coprime to p clearly cannot be self-dual since the vertex degree and face lengths differ.

Proposition 3.4. *With the usual setup, and for k coprime to p, a map of type (k, p) is self-Petrie-dual if and only if $k \mid 2(r \pm 1)$ and $\pm\omega_\kappa^{(r+1)} = 4\xi_\kappa^{(r\pm1)}$ when the corresponding signs in each $(r \pm 1)$ are read simultaneously, and where $r = p^j$ and $j = 0$ or $2j = e$.*

Proof. When k is coprime to p, [8] tells us that a map of type (k, p) has the following triple of generating matrices corresponding to X, Y, and Z:

$$X_2 = \eta\alpha \begin{pmatrix} -\omega_\kappa & -2\xi_\kappa \\ 2\xi_\kappa^{-1} & \omega_\kappa \end{pmatrix}, \quad Y_2 = -\alpha \begin{pmatrix} 0 & \xi_\kappa \\ \xi_\kappa^{-1} & 0 \end{pmatrix}, \quad Z_2 = \alpha \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

where $\alpha^2 = -1$ and $\eta = (\xi_\kappa - \xi_\kappa^{-1})^{-1}$.

Suppose the map is self-Petrie-dual and $(B, j) = (\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right], j)$ is the associated involutory automorphism. In order to fix Z we must have $a = d$ and $b = c$ or $a = -d$ and $b = -c$. In order to fix Y we find that either $a = 0$ or $c = 0$ in which case we need $k \mid 2(p^j + 1)$ or $k \mid 2(p^j - 1)$ respectively. When $a = 0$, the involution then interchanges X with XY if and only if $\pm\omega_\kappa^{(r+1)} = 4\xi_\kappa^{(r+1)}$. When $c = 0$, the involution then interchanges X with XY if and only if $\pm\omega_\kappa^{(r+1)} = 4\xi_\kappa^{(r-1)}$. □

Proposition 3.5. *Under the usual setup, and with l coprime to p, a regular map of type (p, l) is self-Petrie-dual if and only if $\omega_\lambda^2 = -\omega_\lambda^{2r}$ where $r = p^j$ and $2j = e$.*

Proof. Using a similar argument to the above applied to the appropriate matrix triple from [8], namely

$$X_3 = \alpha \begin{pmatrix} 0 & \omega_\lambda^{-1} \\ \omega_\lambda & 0 \end{pmatrix}, \quad Y_3 = -\alpha \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad Z_3 = \alpha \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix},$$

we find that the only allowable form for B is the identity. The necessary non-trivial field automorphism applied to the X and XY interchange then yields the stated condition. □

Remark 3.6. For odd p, a regular map of type (p, p) or (p, l) is not Möbius regular. This is immediate from the fact that each pair of adjacent vertices in a Möbius regular map is joined by exactly two edges, hence the vertex degree must be even.

Proposition 3.7. *Under the usual setup for even k, a map of type (k,p) is Möbius regular if and only if $\omega_\kappa^2+4=0$*

Proof. A regular map is Möbius regular if and only if the equation $XR^{\frac{k}{2}}X = R^{\frac{k}{2}}Y$ is satisfied. We assume k is even, since if k is odd then the map is certainly not Möbius regular. In this case

$$R=[Y_2Z_2]=\begin{bmatrix}\xi_\kappa & 0\\ 0 & \xi_\kappa^{-1}\end{bmatrix}$$

so we have $R^{\frac{k}{2}}=\left[\begin{smallmatrix}\alpha & 0\\ 0 & \alpha^{-1}\end{smallmatrix}\right]$, where $\alpha^2=-1$. Hence the map is Möbius regular if and only if these matrices are equivalent:

$$XR^{\frac{k}{2}}X=-\eta^2\alpha\begin{bmatrix}\omega_\kappa^2+4 & 4\omega_\kappa\xi_\kappa\\ -4\omega_\kappa\xi_\kappa^{-1} & -(\omega_\kappa^2+4)\end{bmatrix} \quad\text{and}\quad R^{\frac{k}{2}}Y=\begin{bmatrix}0 & \xi_\kappa\\ -\xi_\kappa^{-1} & 0\end{bmatrix}.$$

These matrices are equivalent if and only if $\omega_\kappa^2+4=0$. □

4 Regular maps on linear fractional groups of type (k,l) where both k and l are coprime to p

In this case we have different generating triples for the group G. As per Proposition 3.2 in [4], the triple (X,Y,Z) has representatives as defined below where $D=\omega_\kappa^2+\omega_\lambda^2-4$, $\beta=-1/\sqrt{-D}$ and $\eta=(\xi_\kappa-\xi_\kappa^{-1})^{-1}$.

$X_4=\eta\beta\begin{pmatrix}D & D\omega_\lambda\xi_\kappa\\ -\omega_\lambda\xi_\kappa^{-1} & -D\end{pmatrix}$, $Y_4=\beta\begin{pmatrix}0 & \xi_\kappa D\\ \xi_\kappa{}^{-1} & 0\end{pmatrix}$, and $Z_4=\beta\begin{pmatrix}0 & D\\ 1 & 0\end{pmatrix}$

We will also consider the pair of matrices which represent R and S, the rotations around a vertex and a face respectively, which by Proposition 2.2 in [4] are:

$R_4=\begin{pmatrix}\xi_\kappa & 0\\ 0 & \xi_\kappa^{-1}\end{pmatrix}$ and $S_4=\eta\begin{pmatrix}-\omega_\lambda\xi_\kappa^{-1} & -D\\ 1 & \omega_\lambda\xi_\kappa\end{pmatrix}$.

At this point we note that there is an exception for maps of type $(5,5)$, which is addressed in Section 5.

Theorem 4.1. *Under the usual setup, a regular map of type (k,k) is self-dual if and only if $\omega_\lambda=\pm\omega_\kappa^r$ where $r=p^j$, and $j=0$ or $2j=e$.*

Proof. Suppose the map is self-dual.

There is an involutory automorphism of G which fixes Z and interchanges X and Y. This is equivalent to interchanging the rotations $R^{-1}=(YZ)^{-1}=ZY$ and $S=ZX$ around a vertex and a face respectively. That is, there is an automorphism (A,j) which interchanges $\pm R^{-1}$ with S. Here the $\pm$ takes into account both representative elements for R. So $A(\pm\phi_j(R^{-1}))A^{-1}=S$. Remembering that conjugation preserves traces this implies $\pm\phi_j\mathrm{tr}(R^{-1})=\mathrm{tr}(S)$ which immediately yields the condition $\pm\omega_\kappa^r=\omega_\lambda$.

Conversely suppose $\pm\omega_\kappa^r=\omega_\lambda$.

We note that $\omega_\kappa^{2r}=\omega_\lambda^2 \iff \omega_\kappa^2=\omega_\lambda^{2r}$ and so $D^r=(\omega_\kappa^2+\omega_\lambda^2-4)^r=\omega_\kappa^{2r}+\omega_\lambda^{2r}-4=\omega_\lambda^2+\omega_\kappa^2-4=D$.

We aim to find an involutory automorphism (A,j) which demonstrates this map is self-dual. Consider $A=\begin{bmatrix}a & D\\ -1 & -a\end{bmatrix}$ which, by Lemma 2.2, so long as $a^r=a$, satisfies all the equations necessary for the element (A,j) to be involutory. Notice that (A,j) also fixes Z. We also need X and Y to be interchanged by the automorphism in which case the following matrices are equivalent.

$$A\phi_j(X) = \eta^r\beta^r \begin{bmatrix} D(a-\omega_\lambda^r\xi_\kappa^{-r}) & D(a\omega_\lambda^r\xi_\kappa^r - D) \\ \omega_\lambda^r\xi_\kappa^{-r}a - D & D(a-\omega_\lambda^r\xi_\kappa^r) \end{bmatrix} \text{ and } YA = \beta \begin{bmatrix} -\xi_\kappa D & -a\xi_\kappa D \\ a\xi_\kappa^{-1} & D\xi_\kappa^{-1} \end{bmatrix}$$

Ratio of elements in the leading diagonal: $-\xi_\kappa^2 = (a-\omega_\lambda^r\xi_\kappa^{-r})/(a-\omega_\lambda^r\xi_\kappa^r)$

Ratio of elements in the left column: $-D\xi_\kappa^2/a = D(a-\omega_\lambda^r\xi_\kappa^{-r})/(\omega_\lambda^r\xi_\kappa^{-r}a - D)$

Ratio of elements in the top row: $1/a = (a-\omega_\lambda^r\xi_\kappa^{-r})/(a\omega_\lambda^r\xi_\kappa^r - D)$

The last ratio listed yields the following quadratic in a: $0 = a^2 - a\omega_\lambda^r(\xi_\kappa^{-r}+\xi_\kappa^r) + D$, that is $0 = a^2 - a\omega_\lambda^r\omega_\kappa^r + D$. This is consistent with all the necessary ratios. All that remains is to check that a value of a satisfying this quadratic is invariant under the repeated Frobenius field automorphism. The discriminant $\Delta = \omega_\kappa^2\omega_\kappa^{2r} - 4(\omega_\kappa^2 + (\omega_\kappa^r)^2 - 4) = ((\omega_\kappa^r)^2 - 4)(\omega_\kappa^2 - 4)$. Furthermore the expression for $a = (\omega_\lambda^r\omega_\kappa^r \pm \sqrt{\Delta})/2$ is invariant under the transformation $x \to x^r$ as required. Hence the map is self-dual. □

Theorem 4.2. *With the usual setup, where k, l are coprime to p, a map of type (k, l) is self-Petrie-dual if and only if one of the following conditions is fulfilled:*

1. $\omega_\lambda^2 = -D$

2. $q = r^2 = p^{2j}$, $\omega_\lambda^{2r} = -D$ and $k|(r \pm 1)$.

Proof. First suppose the map is self-Petrie-dual. So there exists $(B, j) \in P\Gamma L(2, q)$ such that $B\phi_j(X)B^{-1} = XY$, $B\phi_j(Y)B^{-1} = Y$, and $B\phi_j(Z)B^{-1} = Z$. By comparing the traces of $\phi_j(ZX)$ and ZXY we get the necessary conditon: $\omega_\lambda^{2r} = -D$.

For the rest of the proof we split the situation into two cases: the first when $j = 0$ and we do not consider any field automorphism, and the second case where a field automorphism is included.

Case 1: $j = 0$.

Suppose $\omega_\lambda^2 = -D$. Notice that $B = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ fixes both Y and Z. The map is self-Petrie-dual if $BX = \eta\beta \begin{bmatrix} D & D\omega_\lambda\xi_\kappa \\ \omega_\lambda\xi_\kappa^{-1} & D \end{bmatrix}$ and $XYB = \eta\beta^2 \begin{bmatrix} D\omega_\lambda & -D^2\xi_\kappa \\ -D\xi_\kappa^{-1} & D\omega_\lambda \end{bmatrix}$ are also equivalent. Comparing these and applying our assumption that $\omega_\lambda^2 = -D$ we conclude this map is self-Petrie-dual.

Case 2: $2j = e$. By Lemma 2.1 we include the repeated Frobenius automorphism.

Suppose $\omega_\lambda^{2r} = -D$ and $k|(r \pm 1)$.

The map is self-Petrie-dual if there is an involutory automorphism which not only fixes Z but also fixes Y and interchanges XY with X. We hope to find $(B, j) = (\begin{bmatrix} a & b \\ c & d \end{bmatrix}, j)$, the associated element of $P\Gamma L$. In addition to the conditions for a, b, c, d established in Lemma 2.2, we require $B\phi_j(Z)B^{-1} = Z$ and $B\phi_j(Y)B^{-1} = Y$. In order to fix Z we must have $bd = acD^r$ and $D(d^2 - c^2D^r) = a^2D^r - b^2$. Fixing Y yields two further equations: $bd = ac\xi_\kappa^{2r}D^r$ and $\xi_\kappa^2 D(d^2\xi_\kappa^{-r} - c^2\xi_\kappa^r D^r) = a^2\xi_\kappa^r D^r - b^2\xi_\kappa^{-r}$. Since $\xi_\kappa^{2r} \neq 1$ and $D \neq 0$, notice that $bd = ac\xi_\kappa^{2r}D^r = acD^r$ tells us that either $a = 0$ or $c = 0$.

If $a = 0$, we immediately see $d = 0$ too, and so we can assume $b = 1$ without loss of generality. The equations for a, b, c, d tell us that to fix Z we have $c^2 = \frac{1}{D^{r+1}}$ and to fix Y we have $c^2\xi^{2r+2} = \frac{1}{D^{r+1}}$. So this automorphism exists only if $\xi_\kappa^{2r+2} = 1$. By definition ξ_κ is a primitive $2k$th root of unity and $\xi_\kappa^{2r+2} = 1 \iff 2k|(2r+2) \iff k|(r+1)$, which is the case by our assumption.

$B\phi_j(XY) = \eta^r\beta^{2r}\begin{bmatrix} -D^r\xi^{-r} & \mp D^r\sqrt{-D} \\ \pm cD^r\sqrt{-D} & cD^{2r}\xi_\kappa^r \end{bmatrix}$ and $XB = \eta\beta\begin{bmatrix} cD\omega_\lambda\xi_\kappa & D \\ -cD & -\omega_\lambda\xi_\kappa^{-1} \end{bmatrix}$ are also equivalent if the map is self-Petrie-dual so we compare the ratios of the elements in turn. This yields $\pm c = \frac{1}{\omega_\lambda\xi_\kappa^{r+1}\sqrt{-D}} = \frac{\omega_\lambda\sqrt{-D}}{D^{r+1}\xi_\kappa^{r+1}}$ which is true only if $D^r = -\omega_\lambda^2$, which is again the case by our assumption. These conditions are consistent with our other requirements for the value of c, (namely that $c^r = c$) so we have an automorphism demonstrating that this map is self-Petrie-dual.

If on the other hand $c = 0$ then we have $b = 0$ and we assume $a = 1$ without loss of generality. Fixing Z yields $d^2D = D^r$. Fixing Y yields $d^2D = \xi_\kappa^{2r-2}D^r$. So the map is self-Petrie-dual only if $\xi_\kappa^{2r-2} = 1$, which is the case by our assumption.

Now $B\phi_j(XY) = \eta^r\beta^{2r}D^r\begin{bmatrix} \omega_\lambda^r & D^r\xi_\kappa^r \\ -d\xi_\kappa^{-r} & -d\omega_\lambda^r \end{bmatrix}$ and $XB = \eta\beta\begin{bmatrix} D & dD\omega_\lambda\xi_\kappa \\ -\omega_\lambda\xi_\kappa^{-1} & -dD \end{bmatrix}$.

Again, the map is self-Petrie-dual if these two elements are equivalent, that is if both $D^r\xi_\kappa^{r-1} = d\omega_\lambda^{r+1}$ and $\omega_\lambda^{r+1}\xi_\kappa^{r-1} = dD$. Applying our assumption $\omega_\kappa^{2r} = -D$, we conclude the map is self-Petrie-dual. □

Using the fact that when $q = p$ the conditions are often much simpler to state, the preceding two results, Theorem 4.1 and Theorem 4.2, indicate a *sufficient* condition for a regular map of type (k, k) to be both self-dual and self-Petrie-dual, namely $\omega_\kappa^2 = \omega_\lambda^2 = -D$. Corollary 4.3 shows this becomes a tractable sufficient condition for both self-duality and self-Petrie-duality.

Corollary 4.3. *If $\omega = \omega_\kappa = \omega_\lambda$ and $3\omega^2 = 4$ then the associated map is both self-dual and self-Petrie-dual.*

We now turn our attention to the conditions for Möbius regularity.

Proposition 4.4. *With the usual setup, a regular map of type (k, l) is Möbius regular if and only if k is even and $\omega_\kappa^2 + 2\omega_\lambda^2 = 4$.*

Proof. By Lemma 2.3, a regular map is Möbius regular if and only if the equation $XR^{\frac{k}{2}}X = R^{\frac{k}{2}}Y$ is satisfied. In this case $R = \begin{bmatrix} \xi_\kappa & 0 \\ 0 & \xi_\kappa^{-1} \end{bmatrix}$. So $R^{\frac{k}{2}} = \begin{bmatrix} \alpha & 0 \\ 0 & -\alpha \end{bmatrix}$, where $\alpha^2 = -1$. Then $XR^{\frac{k}{2}}X = R^{\frac{k}{2}}Y$ is satisfied if and only if

$$\eta^2\beta^2\begin{bmatrix} \alpha D^2 + \alpha D\omega_\lambda^2 & 2\alpha D^2\omega_\lambda\xi_\kappa \\ -2\alpha D\omega_\lambda\xi_\kappa^{-1} & -\alpha D\omega_\lambda^2 - \alpha D^2 \end{bmatrix} = \beta\begin{bmatrix} 0 & \alpha\xi_\kappa D \\ -\alpha\xi_\kappa^{-1} & 0 \end{bmatrix}.$$

The elements on the leading diagonal must be zero, which yields just one equation: $\eta^2\beta^2\alpha D(D + \omega_\lambda^2) = 0$. The ratio between the non-zero entries is the same for both matrices and so no further conditions arise.

We conclude that for even k, the map is Möbius regular if and only if $D = -\omega_\lambda^2$, which is equivalent to $\omega_\kappa^2 + 2\omega_\lambda^2 = 4$. □

It is not surprising to see some similarity between conditions for self-Petrie-duality and Möbius regularity since we know all Möbius regular maps (with any automorphism group) are also self-Petrie-dual [17]. However, since there are alternative conditions which imply self-Petrie-duality, the converse is not true – not all self-Petrie-dual regular maps are Möbius regular.

5 Regular maps of type $(5,5)$ whose orientation-preserving automorphism group $\langle R,S\rangle$ is isomorphic to A_5

Adrianov's [1] enumeration of regular hypermaps on $\mathrm{PSL}(2,q)$ includes a constant which deals with the special case which occurs for maps of type $(5,5)$.

For us to be considering a map of type (k,l) we must have $2k|(q\pm 1)$ and $2l|(q\pm 1)$, and it is known, see [10], that $\mathrm{PSL}(2,q)$ has subgroup A_5 when $q \equiv \pm 1 \pmod{10}$. The constant in Adrianov's enumeration, which is 2 for maps of type $(5,5)$ and zero otherwise, is subtracted to account for the cases when the group $\langle R,S\rangle$ collapses into the subgroup $A_5 \leq \mathrm{PSL}(2,q)$.

The following result, with the usual definitions for ω_κ and ω_λ, indicates when the orientation-preserving automorphism group of a type $(5,5)$ map is not the linear fractional group that we might expect, and as such addresses an omission in [4].

Proposition 5.1. *The group $\langle R,S\rangle$ of a regular map of type $(5,5)$, generated by the representative matrices R_4 and S_4, is isomorphic to A_5 if and only if $\omega_\lambda \neq \omega_\kappa$.*

Proof. From [14] we know a presentation of the group A_5 is: $\langle a,b|a^5,b^5,(ab)^2,(a^4b)^3\rangle$.

Considering the group $\langle R,S|R^5,S^5,(RS)^2,\dots\rangle$, it is clear that this will be isomorphic to A_5 if and only if the condition $(R^4S)^3 = I$ is also satified. This is the case if and only if $R^{-1}S$ has order 3.

$$R^{-1}S = \eta\begin{bmatrix}\xi_\kappa^{-1} & 0\\ 0 & \xi_\kappa\end{bmatrix}\begin{bmatrix}-\omega_\lambda\xi_\kappa^{-1} & -D\\ 1 & \omega_\lambda\xi_\kappa\end{bmatrix} = \eta\begin{bmatrix}-\omega_\lambda\xi_\kappa^{-2} & -\xi_\kappa^{-1}D\\ \xi_\kappa & \omega_\lambda\xi_\kappa^2\end{bmatrix}$$

$$(R^{-1}S)^3 = \eta^3\begin{bmatrix}\omega_\lambda(2D\xi_\kappa^{-2}-\omega_\lambda^2\xi_\kappa^{-6}-D\xi_\kappa^2) & D\xi_\kappa^{-2}(D\xi_\kappa+\omega_\lambda^2(\xi_\kappa-\xi_\kappa^5-\xi_\kappa^{-3}))\\ -(D\xi_\kappa+\omega_\lambda^2(\xi_\kappa-\xi_\kappa^5-\xi_\kappa^{-3})) & \omega_\lambda(D\xi_\kappa^{-2}+\omega_\lambda^2\xi_\kappa^6-2D\xi_\kappa^2)\end{bmatrix}$$

The off diagonal elements are both zero if and only if $D\xi_\kappa+\omega_\lambda^2(\xi_\kappa-\xi_\kappa^5-\xi_\kappa^{-3})=0$. This condition is equivalent to $(\omega_\kappa+2)(\omega_\kappa-2)(1+\omega_\kappa\omega_\lambda)(1-\omega_\kappa\omega_\lambda)=0$ and we know that $\omega_\kappa \neq \pm 2$ so long as $k \neq p$.

The leading diagonal entries are equal if and only if $D(\xi_\kappa^{-2}+\xi_\kappa^2) = \omega_\lambda^2(\xi_\kappa^6+\xi_\kappa^{-6})$. Applying $\xi_\kappa^{10}=1$ and eliminating D shows this is equivalent to

$$(\omega_\kappa^2-4)(\omega_\kappa^2-\omega_\kappa^2\omega_\lambda^2+\omega_\lambda^2-2)=0.$$

Assume $\omega_\kappa^2\omega_\lambda^2=1$. The off-diagonals are clearly zero, and the leading diagonal entries are equal since $(\omega_\kappa^2-\omega_\kappa^2\omega_\lambda^2+\omega_\lambda^2-2)=\omega_\kappa^{-2}(\omega_\kappa^4-3\omega_\kappa^2+1)=0$ is always the case since the expression inside the bracket is the sum of powers of ξ_κ^2, a 5th root of unity. Then $R^{-1}S$ has order 3.

Conversely, assume $(R^{-1}S)^3=I$. Then we instantly have $\omega_\kappa^2\omega_\lambda^2=1$ since $p\neq 5$.

We conclude that $(R^{-1}S)^3=I$ if and only if $\omega_\kappa\omega_\lambda=\pm 1$. By considering the two possible values for ω_κ and ω_λ we see that this will happen if and only if $\omega_\kappa \neq \omega_\lambda$. □

6 Tables of results and comments

In the following tables, produced using the computer package GAP [7], we list for given $q \leq 49$, all the $\mathrm{PSL}(2,q)$ maps which have one or more of the properties we have addressed in the paper, the ticks indicating when the map has each property. The tables are ordered by the characteristic of the field, and the elements ξ_κ, ξ_λ, ω_κ and ω_λ are expressed as powers of a primitive element ξ in the field $\mathrm{GF}(q^2)$. For a given k, l, only one map is shown in each equivalence class under the action of the automorphism group. Extended tables of results detailing the $\mathrm{PSL}(2, q)$ regular maps for $q \leq 81$ are available in the ancillary file to [5].

For interest we also include an enumeration in Table 2 which shows how many $\mathrm{PSL}(2, q)$ maps there are with each of these combinations of properties for $q \leq 81$.

Table 1: All the $\mathrm{PSL}(2, q)$ maps which have one or more of the properties we have addressed in the paper.

q	k	l	$\log_\xi \xi_\kappa$	$\log_\xi \xi_\lambda$	$\log_\xi \omega_\kappa$	$\log_\xi \omega_\lambda$	SD	SPD	MR
3^2	5	5	8	8	30	30	✓		
3^3	7	7	52	52	420	420	✓		
3^3	13	7	140	156	28	532		✓	
3^3	13	13	28	28	560	560	✓		
3^3	13	13	28	252	560	672		✓	
3^3	13	13	140	140	28	28	✓		
3^3	14	14	26	26	476	476	✓		
5	5	5					✓		
5^2	3	13	104	72	0	494		✓	
5^2	4	13	78	168	390	26		✓	
5^2	6	13	52	24	546	260		✓	
5^2	12	12	26	26	416	416	✓		
5^2	12	13	26	216	416	130		✓	✓
5^2	13	13	24	24	260	260	✓		
5^2	13	13	24	120	260	52	✓		
5^2	13	13	72	72	494	494	✓		
5^2	13	13	72	264	494	598	✓		
5^2	13	13	168	168	26	26	✓		
5^2	13	13	168	216	26	130	✓		
7	7	7					✓		
7^2	4	25	300	144	400	1250		✓	
7^2	5	5	240	240	1350	1350	✓		
7^2	6	24	200	250	1400	2050		✓	
7^2	7	24		50		500		✓	
7^2	7	25		432		1500		✓	
7^2	7	25		816		100		✓	
7^2	8	25	150	48	2200	1450		✓	
7^2	8	25	450	528	600	950		✓	

q	k	l	$\log_\xi \xi_\kappa$	$\log_\xi \xi_\lambda$	$\log_\xi \omega_\kappa$	$\log_\xi \omega_\lambda$	SD	SPD	MR
7^2	12	12	100	100	750	750	✓		
7^2	12	25	100	144	750	1250		✓	✓
7^2	24	12	50	100	500	750		✓	✓
7^2	24	24	50	50	500	500	✓		
7^2	24	24	50	350	500	1100	✓		
7^2	24	24	250	250	2050	2050	✓		
7^2	24	24	250	550	2050	1150	✓		
7^2	24	25	250	912	2050	700		✓	✓
7^2	25	25	48	48	1450	1450	✓		
7^2	25	25	48	336	1450	550	✓		
7^2	25	25	144	144	1250	1250	✓		
7^2	25	25	144	1008	1250	1550	✓		
7^2	25	25	432	432	1500	1500	✓		
7^2	25	25	432	624	1500	900	✓		
7^2	25	25	528	528	950	950	✓		
7^2	25	25	528	1104	950	1850	✓		
7^2	25	25	816	816	100	100	✓		
7^2	25	25	816	912	100	700	✓		
11	5	5	12	12	36	36	✓		
11	5	5	36	36	24	24	✓	✓	
11	5	6	12	10	36	108		✓	
11	6	6	10	10	108	108	✓		
11	11	11					✓		
13	6	6	14	14	112	112	✓		
13	7	7	12	12	126	126	✓		
13	7	7	12	60	126	56		✓	
13	7	7	36	36	70	70	✓	✓	
13	7	7	60	60	56	56	✓		
13	7	13	60		56			✓	
13	13	13					✓		
17	8	8	18	18	36	36	✓		
17	8	8	54	54	90	90	✓		
17	8	9	54	80	90	54		✓	✓
17	8	17	18		36			✓	✓
17	9	9	16	16	216	216	✓		
17	9	9	80	80	54	54	✓		

q	k	l	$\log_\xi \xi_\kappa$	$\log_\xi \xi_\lambda$	$\log_\xi \omega_\kappa$	$\log_\xi \omega_\lambda$	SD	SPD	MR
17	9	9	112	112	18	18	✓		
17	17	17					✓		
19	3	9	60	20	0	300		✓	
19	5	5	36	36	320	320	✓		
19	5	5	108	108	220	220	✓		
19	9	5	20	36	300	320		✓	
19	9	9	20	20	300	300	✓		
19	9	9	100	100	80	80	✓		
19	9	9	140	100	340	80		✓	
19	9	9	140	140	340	340	✓		
19	9	10	100	18	80	60		✓	
19	10	10	18	18	60	60	✓		
19	10	10	54	54	100	100	✓		
19	19	19					✓		
23	3	11	88	72	0	168		✓	
23	6	6	44	44	192	192	✓		
23	6	11	44	216	192	240		✓	✓
23	11	11	24	24	360	360	✓		
23	11	11	72	72	168	168	✓		
23	11	11	120	120	480	480	✓		
23	11	11	168	168	336	336	✓		
23	11	11	216	216	240	240	✓		
23	12	11	22	24	144	360		✓	✓
23	12	12	22	22	144	144	✓		
23	12	12	110	110	384	384	✓	✓	✓
23	23	23					✓		
29	3	15	140	28	0	480		✓	
29	5	5	84	84	180	180	✓		
29	5	5	252	252	240	240	✓		
29	5	14	84	90	180	270		✓	
29	5	29	252		240			✓	
29	7	7	60	60	570	570	✓		
29	7	7	180	180	750	750	✓		
29	7	7	300	300	780	780	✓		
29	14	14	30	30	630	630	✓		
29	14	14	90	90	270	270	✓		
29	14	14	150	150	540	540	✓		

q	k	l	$\log_\xi \xi_\kappa$	$\log_\xi \xi_\lambda$	$\log_\xi \omega_\kappa$	$\log_\xi \omega_\lambda$	SD	SPD	MR
29	15	7	308	180	300	750		✓	
29	15	7	364	300	810	780		✓	
29	15	14	196	30	90	630		✓	
29	15	15	28	28	480	480	✓		
29	15	15	28	308	480	300		✓	
29	15	15	196	196	90	90	✓		
29	15	15	308	308	300	300	✓		
29	15	15	364	364	810	810	✓		
29	29	29					✓		
31	5	5	96	96	128	128	✓		
31	5	5	288	288	352	352	✓		
31	8	8	60	60	160	160	✓		
31	8	8	180	180	704	704	✓		
31	8	15	60	352	160	320		✓	✓
31	8	16	180	30	704	256		✓	✓
31	15	15	32	32	416	416	✓		
31	15	15	224	224	512	512	✓		
31	15	15	352	352	320	320	✓		
31	15	15	416	416	672	672	✓		
31	16	5	210	288	448	352		✓	✓
31	16	8	90	60	576	160		✓	✓
31	16	15	150	416	544	672		✓	✓
31	16	16	30	30	256	256	✓		
31	16	16	30	150	256	544		✓	✓
31	16	16	90	90	576	576	✓		
31	16	16	150	150	544	544	✓		
31	16	16	210	210	448	448	✓		
31	31	31					✓		
37	6	6	114	114	1178	1178	✓		
37	9	9	76	76	190	190	✓		
37	9	9	380	380	1292	1292	✓		
37	9	9	532	532	1254	1254	✓		
37	18	18	38	38	836	836	✓		
37	18	18	190	190	266	266	✓		
37	18	18	266	266	76	76	✓		
37	19	9	180	380	798	1292		✓	
37	19	9	468	76	646	190		✓	

q	k	l	$\log_\xi \xi_\kappa$	$\log_\xi \xi_\lambda$	$\log_\xi \omega_\kappa$	$\log_\xi \omega_\lambda$	SD	SPD	MR
37	19	18	252	38	1140	836		✓	
37	19	18	612	266	988	76		✓	
37	19	19	36	36	1026	1026	✓		
37	19	19	36	108	1026	418		✓	
37	19	19	108	108	418	418	✓		
37	19	19	108	252	418	1140		✓	
37	19	19	180	180	798	798	✓		
37	19	19	252	252	1140	1140	✓		
37	19	19	324	324	1064	1064	✓		
37	19	19	396	396	228	228	✓	✓	
37	19	19	468	468	646	646	✓		
37	19	19	540	324	532	1064		✓	
37	19	19	540	540	532	532	✓		
37	19	19	612	612	988	988	✓		
37	19	37	324		1064			✓	
37	37	37					✓		
41	5	5	168	168	1638	1638	✓		
41	5	5	504	504	882	882	✓		
41	5	7	168	600	1638	126		✓	
41	5	20	504	294	882	924		✓	
41	7	7	120	120	210	210	✓		
41	7	7	360	360	504	504	✓		
41	7	7	600	600	126	126	✓		
41	10	10	84	84	630	630	✓		
41	10	10	252	252	672	672	✓		
41	10	21	84	520	630	336		✓	✓
41	10	41	252		672			✓	✓
41	20	20	42	42	1302	1302	✓		
41	20	20	42	126	1302	714		✓	✓
41	20	20	126	126	714	714	✓		
41	20	20	294	294	924	924	✓		
41	20	20	378	378	420	420	✓		
41	20	21	126	680	714	1218		✓	✓
41	20	21	294	200	924	168		✓	✓
41	20	21	378	440	420	756		✓	✓
41	21	21	40	40	1428	1428	✓		
41	21	21	200	200	168	168	✓		

q	k	l	$\log_\xi \xi_\kappa$	$\log_\xi \xi_\lambda$	$\log_\xi \omega_\kappa$	$\log_\xi \omega_\lambda$	SD	SPD	MR
41	21	21	440	440	756	756	✓		
41	21	21	520	520	336	336	✓		
41	21	21	680	680	1218	1218	✓		
41	21	21	760	760	1134	1134	✓		
41	41	41					✓		
43	3	22	308	294	0	1276		✓	
43	7	7	132	132	792	792	✓		
43	7	7	396	396	220	220	✓		
43	7	7	660	660	1760	1760	✓		
43	7	21	132	484	792	1628		✓	
43	7	21	396	572	220	484		✓	
43	7	21	660	748	1760	1496		✓	
43	11	11	84	84	1452	1452	✓		
43	11	11	252	252	1012	1012	✓		
43	11	11	420	420	1540	1540	✓		
43	11	11	588	588	1584	1584	✓		
43	11	11	756	756	1804	1804	✓		
43	21	7	484	660	1628	1760		✓	
43	21	11	44	420	396	1540		✓	
43	21	11	220	588	1100	1584		✓	
43	21	11	748	84	1496	1452		✓	
43	21	11	836	252	440	1012		✓	
43	21	21	44	44	396	396	✓		
43	21	21	220	220	1100	1100	✓		
43	21	21	484	484	1628	1628	✓		
43	21	21	572	220	484	1100		✓	
43	21	21	572	572	484	484	✓		
43	21	21	748	748	1496	1496	✓		
43	21	21	836	836	440	440	✓		
43	22	22	42	42	132	132	✓		
43	22	22	126	126	308	308	✓		
43	22	22	210	210	968	968	✓		
43	22	22	294	294	1276	1276	✓		
43	22	22	378	378	1672	1672	✓		
43	43	43					✓		
47	3	23	368	528	0	1152		✓	
47	6	6	184	184	480	480	✓		

q	k	l	$\log_\xi \xi_\kappa$	$\log_\xi \xi_\lambda$	$\log_\xi \omega_\kappa$	$\log_\xi \omega_\lambda$	SD	SPD	MR
47	6	24	184	46	480	672		✓	✓
47	8	8	138	138	960	960	✓		
47	8	8	414	414	576	576	✓		
47	8	23	138	624	960	144		✓	✓
47	8	23	414	432	576	528		✓	✓
47	12	8	92	414	1200	576		✓	✓
47	12	12	92	92	1200	1200	✓		
47	12	12	460	460	1008	1008	✓		
47	12	23	460	816	1008	768		✓	✓
47	23	23	48	48	1344	1344	✓		
47	23	23	144	144	1056	1056	✓		
47	23	23	240	240	288	288	✓		
47	23	23	336	336	720	720	✓		
47	23	23	432	432	528	528	✓		
47	23	23	528	528	1152	1152	✓		
47	23	23	624	624	144	144	✓		
47	23	23	720	720	1296	1296	✓		
47	23	23	816	816	768	768	✓		
47	23	23	912	912	624	624	✓		
47	23	23	1008	1008	2016	2016	✓		
47	24	12	46	460	672	1008		✓	✓
47	24	23	322	144	1920	1056		✓	✓
47	24	23	506	48	336	1344		✓	✓
47	24	24	46	46	672	672	✓		
47	24	24	230	230	1488	1488	✓	✓	✓
47	24	24	322	322	1920	1920	✓		
47	24	24	506	506	336	336	✓		
47	47	47					✓		

Table 2: External symmetries of regular maps on $\mathrm{PSL}(2, q)$.

q	Maps	None	SD only	SP only	SD+SP	SP+MR	SD+SP+MR
3^2	3	2	1	0	0	0	0
3^3	54	48	4	2	0	0	0
3^4	381	356	15	7	0	3	0
5	1	0	1	0	0	0	0
5^2	63	52	7	3	0	1	0
7	5	4	1	0	0	0	0
7^2	264	238	16	7	0	3	0
11	16	11	3	1	1	0	0
13	33	26	4	2	1	0	0
17	58	50	6	0	0	2	0
19	70	58	8	4	0	0	0
23	113	101	8	1	0	2	1
29	183	163	13	7	0	0	0
31	209	190	13	0	0	6	0
37	315	290	16	8	1	0	0
41	382	356	18	2	0	6	0
43	430	400	20	10	0	0	0
47	515	485	20	1	0	8	1
53	663	625	25	13	0	0	0
59	820	779	27	13	1	0	0
61	879	836	28	14	1	0	0
67	1072	1024	32	16	0	0	0
71	1199	1151	32	4	0	11	1
73	1276	1227	33	3	1	12	0
79	1493	1438	37	2	0	16	0

The work in this paper has only addressed regular maps on linear fractional groups where the associated finite field has odd characteristic. Many of the calculations would look quite different if we were to consider the case when $p = 2$.

It has been an open problem for some time as to whether there exists a self-dual and self-Petrie-dual regular map for any given vertex degree k on some surface. In [2], Archdeacon, Conder and Širáň proved the existence of such a map for any even valency. The results in this paper allow Fraser, Jeans and Širáň [6] to prove the existence of a self-dual, self-Petrie-dual regular map for any given odd valency $k \geq 5$.

ORCID iDs

Grahame Erskine https://orcid.org/0000-0001-7067-6004
Katarína Hriňáková https://orcid.org/0000-0001-6137-2607
Olivia Reade Jeans https://orcid.org/0000-0002-7598-9680

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A new generalization of generalized Petersen graphs*

Katarína Jasenčáková
Faculty of Management Science and Informatics, University of Žilina, Žilina, Slovakia
Robert Jajcay†
Faculty of Mathematics, Physics and Informatics, Comenius University, Bratislava, Slovakia; also affiliated with Faculty of Mathematics, Natural Sciences and Information Technology, University of Primorska, Koper, Slovenia
Tomaž Pisanski‡
Faculty of Mathematics, Natural Sciences and Information Technology, University of Primorska, Koper, Slovenia; also affiliated with FMF and IMFM, University of Ljubljana, Jadranska 19, 1000 Ljubljana, Slovenia

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Abstract

We discuss a new family of cubic graphs, which we call group divisible generalised Petersen graphs ($GDGP$-graphs), that bears a close resemblance to the family of generalised Petersen graphs, both in definition and properties. The focus of our paper is on determining the algebraic properties of graphs from our new family. We look for highly symmetric graphs, e.g., graphs with large automorphism groups, and vertex- or arc-transitive graphs. In particular, we present arithmetic conditions for the defining parameters that guarantee that graphs with these parameters are vertex-transitive or Cayley, and we find one arc-transitive $GDGP$-graph which is neither a CQ graph of Feng and Wang, nor a generalised Petersen graph.

Keywords: Generalised Petersen graph, arc-transitive graph, vertex-transitive graph, Cayley graph, automorphism group.

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E-mail addresses: katarina.jasencakova@fri.uniza.sk (Katarína Jasenčáková), robert.jajcay@fmph.uniba.sk (Robert Jajcay), pisanski@upr.si (Tomaž Pisanski)

1 Introduction

Generalised Petersen graphs $GP(n,k)$ (the name and notation coined in 1969 by Watkins [18], with the subclass with n and k relatively prime considered already by Coxeter in 1950 [6]) constitute one of the central families of algebraic graph theory. While there are many reasons for the interest in this family, with a bit of oversimplification one could say that among the most important is the simplicity of their description (requiring just two parameters n and k) combined with the richness of the family that includes the well-known Petersen and dodecahedron graphs, as well as large families of vertex-transitive graphs and seven symmetric (arc-transitive) graphs.

Our motivation for studying the new family of *group divisible generalised Petersen graphs* ($GDGP$-graphs; introduced in [11] under the name SGP-graphs) lies in the fact that they share the above characteristics with the generalised Petersen graphs. They include all vertex-transitive generalised Petersen graphs but the dodecahedron as a proper subclass, are easily defined via a sequence of integral parameters, and contain graphs of various levels of symmetry.

Historically, ours is certainly not the first attempt at generalising generalised Petersen graphs. In 1988, the family of I-*graphs* was introduced in the Foster Census [2]. This family differs from the generalised Petersen graphs in allowing the span on the outer rim to be different from 1: The I-graph $I(n,j,k)$ is the cubic graph with vertex set $\{u_i, v_i \mid i \in \mathbb{Z}_n\}$ and edge set $\{\{u_i, u_{i+j}\}, \{u_i, v_i\}, \{v_i, v_{i+k}\} \mid i \in \mathbb{Z}_n\}$. However, the only I-graphs that are vertex-transitive are the original generalised Petersen graphs which are the graphs $I(n,1,k)$ [1, 14].

The family of GI-*graphs* introduced in [5] by Conder, Pisanski and Žitnik in 2014 is a further generalisation of I-graphs. For positive integers $n \geq 3$, $m \geq 1$, and a sequence K of elements in $\mathbb{Z}_n - \{0, \frac{n}{2}\}$, $K = (k_0, k_1, \ldots, k_{m-1})$, the GI-graph $GI(n; k_0, k_1, \ldots, k_{m-1})$ is the graph with vertex set $\mathbb{Z}_m \times \mathbb{Z}_n$ and edges of two types:

(i) an edge from (u,v) to (u',v), for all distinct $u, u' \in \mathbb{Z}_m$ and all $v \in \mathbb{Z}_n$,

(ii) edges from (u,v) to $(u, v \pm k_u)$, for all $u \in \mathbb{Z}_m$ and all $v \in \mathbb{Z}_n$.

The GI-graphs are $(m+1)$-regular, thus cubic when $m = 2$, which is the case that covers the I-graphs, with the subclass of the $GI(n; 1, k)$ graphs covering the generalised Petersen graphs.

Another generalisation is due to Lovrečič-Saražin, Pacco and Previtali, who extended the class of generalised Petersen graphs to the so-called *supergeneralised Petersen graphs* [15]. Let $n \geq 3$ and $m \geq 2$ be integers and $k_0, k_1, \ldots, k_{m-1} \in \mathbb{Z}_n - \{0\}$. The vertex-set of the supergeneralised Petersen graph $P(m, n; k_0, \ldots, k_{m-1})$ is $\mathbb{Z}_m \times \mathbb{Z}_n$ and its edges are of two types:

(i) an edge from (u,v) to $(u+1, v)$, for all $u \in \mathbb{Z}_m$ and all $v \in \mathbb{Z}_n$,

(ii) edges from (u,v) to $(u, v \pm k_u)$, for all $u \in \mathbb{Z}_m$ and all $v \in \mathbb{Z}_n$.

Note that $GP(n,k)$ is isomorphic to $P(2, n; 1, k)$.

Finally, in 2012, Zhou and Feng [20] modified the class of generalised Petersen graphs in order to classify cubic vertex-transitive non-Cayley graphs of order $8p$, for any prime p [20]. In their definition, the subgraph induced by the outer edges is a union of two n-cycles. Let $n \geq 3$ and $k \in \mathbb{Z}_n - \{0\}$. *The double generalised Petersen graph* $DP(n,k)$ is defined to have the vertex set $\{x_i, y_i, u_i, v_i \mid i \in \mathbb{Z}_n\}$ and the edge set equal to the union of the outer

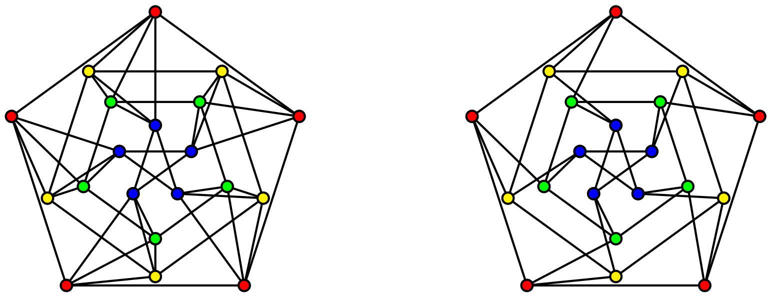

Figure 1: $GI(5;1,1,1,2)$ and $P(4,5;1,1,1,2)$.

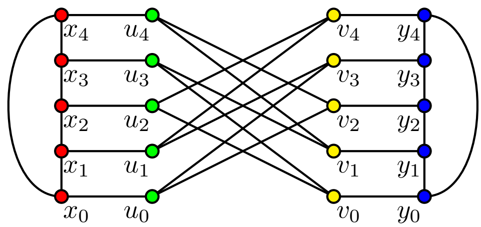

Figure 2: $DP(5,2)$.

edges $\{\{x_i, x_{i+1}\}, \{y_i, y_{i+1}\} \,|\, i \in \mathbb{Z}_n\}$, the inner edges $\{\{u_i, v_{i+k}\}, \{v_i, u_{i+k}\} \,|\, i \in \mathbb{Z}_n\}$, and the spokes $\{\{x_i, u_i\}, \{y_i, v_i\} \,|\, i \in \mathbb{Z}_n\}$.

Even though non-empty intersections exist between the above classes and the class of group divisible generalised Petersen graphs considered in our paper, none of these is significant, and we believe that ours is, in a way, the most natural generalisation of generalised Petersen graphs.

2 Generalised Petersen graphs

Let us review the basic properties of generalised Petersen graphs. A *generalised Petersen graph* $GP(n,k)$ is determined by integers n and k, $n \geq 3$ and $\frac{n}{2} > k \geq 1$. The vertex set $V(GP(n,k)) = \{u_i, v_i \,|\, i \in \mathbb{Z}_n\}$ is of order $2n$ and the edge set $E(GP(n,k))$ of size $3n$ consists of edges of the form

$$\{u_i, u_{i+1}\}, \;\; \{u_i, v_i\}, \;\; \{v_i, v_{i+k}\}, \tag{2.1}$$

where $i \in \mathbb{Z}_n$. Thus, $GP(n,k)$ is always a trivalent graph, the Petersen graph is the graph $GP(5,2)$, the dodecahedron is $GP(10,2)$, and the ADAM graph is $GP(24,5)$.

We will call the u_i vertices the *outer vertices*, the v_i vertices the *inner vertices*, and the three distinct forms of edges displayed in (2.1) *outer edges, spokes*, and *inner edges*, respectively. Graphs introduced in this paper will also contain vertices and edges of these types. We will use the symbols Ω, Σ and I, respectively, to denote the three n-sets of edges. The n-circuit induced in $GP(n,k)$ by Ω will be called the *outer rim*. If d denotes the greatest common divisor of n and k, then I induces a subgraph which is the union of d pairwise-disjoint $\frac{n}{d}$-circuits, called *inner rims*. The parameter k also denotes the *span* of

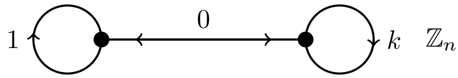

Figure 3: Voltage graph for the generalised Petersen graph $GP(n, k)$.

the inner rims (which is the distance, as measured on the outer rim, between the outer rim neighbors of two vertices adjacent on an inner rim).

The class of generalised Petersen graphs is well understood and has been studied by many authors. In 1971, Frucht, Graver and Watkins [9] determined their automorphism groups. They proved that $GP(n, k)$ *is vertex-transitive if and only if* $k^2 \equiv \pm 1 \mod n$ *or* $(n, k) = (10, 2)$. Later, Nedela and Škoviera [16], and (independently) Lovrečič-Saražin [13] proved that a generalised Petersen graph $GP(n, k)$ is Cayley if and only if $k^2 \equiv 1 \mod n$. Recall that a *Cayley graph* $Cay(G, X)$, where G is a group generated by the set X which does not contain the identity 1_G and is closed under taking inverses, is the graph whose vertices are the elements of G and edges are the pairs $\{g, xg\}$, $g \in G$, $x \in X$.

3 Polycirculants and voltage graphs

A non-identity automorphism of a graph is (m, n)-*semiregular* if its cycle decomposition consists of m cycles of length n. Graphs admitting (m, n)-semiregular automorphisms are called m-*circulants* (if one chooses to suppress the parameter m, they are sometimes called *polycirculants*). If $m = 1, 2, 3$, or 4, an m-circulant is said to be a *circulant*, a *bicirculant*, a *tricirculant*, or a *tetracirculant*, respectively. It is easy to see that generalised Petersen graphs are bicirculants; the corresponding automorphism consists of the two cycles $(u_0, u_1, u_2, \ldots, u_{n-1}), (v_0, v_1, v_2, \ldots, v_{n-1})$.

The reader is also most likely familiar with the fact that generalised Petersen graphs can be defined in a nice and compact way using the language of voltage graphs (a more detailed treatment may be found for example in [10]):

If Γ is an undirected graph, we associate each edge of Γ with a pair of opposite arcs and denote the set of all such arcs by $D(\Gamma)$. A *voltage assignment* on Γ is any mapping α from $D(\Gamma)$ into a group G that satisfies the condition $\alpha(e^{-1}) = (\alpha(e))^{-1}$ for all $e \in D(\Gamma)$ (with e^{-1} being the opposite arc of e, and $(\alpha(e))^{-1}$ being the inverse of $\alpha(e)$ in G). The *lift* (sometimes called the *derived regular cover*) of Γ with respect to a voltage assignment α on Γ is a graph denoted by Γ^α. The vertex set $V(\Gamma^\alpha)$ consists of $|V(\Gamma)| \cdot |G|$ vertices $u_g = (u, g)$, $(u, g) \in V(\Gamma) \times G$. Two vertices u_g and v_f are adjacent in Γ^α if $e = (u, v)$ is an arc of Γ and $f = g \cdot \alpha(e)$ in G.

All generalised Petersen graphs $GP(n, k)$ are lifts of the dumbbell graph $\mathcal{D}$ which consists of two vertices joined by an edge and loops attached to them. The corresponding voltage assignment $\alpha : D(\mathcal{D}) \longrightarrow \mathbb{Z}_n$ assigns 0 to the arcs connecting the two vertices, 1 and -1 to the arcs of the loop at one of the vertices, and k and $-k$ to the arcs of the loop at the other vertex (Figure 3).

Similarly, the I-graph $I(n, j, k)$ is a derived regular cover of the dumbbell graph with 0 assigned to the 'handle', and the values $j, -j$ and $k, -k \in \mathbb{Z}_n$ assigned to the loops (Figure 4).

The GI-graph $GI(n; a, b, c, d)$ is a lift of the complete graph K_4, and so is the super-generalised Petersen graph $P(4, n; a, b, c, d)$ (see Figures 5 and 6, respectively).

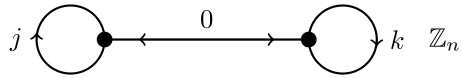

Figure 4: Voltage graph for the I-graph $I(n, j, k)$.

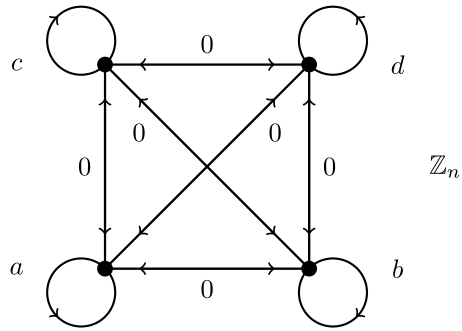

Figure 5: Voltage graph for the GI-graph $GI(n; a, b, c, d)$.

Recently, Conder, Estélyi, and Pisanski in [4] considered more general voltage assignments, and thus generalised the double generalised Petersen graphs even further.

There is another family of polycirculants that will prove useful later in our paper, introduced by Feng and Wang in 2003 [7]. Their graphs are called CQ graphs, and were originally introduced as octacirculants (for their voltage graph description see Figure 7). The definition of $CQ(k, n)$ used in [7] makes sense for any k, n such that $\gcd(k, n) = 1$, which is equivalent to $k \in \mathbb{Z}_n^*$. Frelih and Kutnar [8] later correctly showed that each $CQ(k, n)$ is in fact a tetracirculant. However, their voltage graph depiction is not correct. One needs two different voltage graphs, depending on the parity of m. The correct voltage graphs are depicted in Figures 8 and 9.

Moreover, in the definition of $CQ(k, m)$ used by Frelih and Kutnar the inverse k^{-1} is

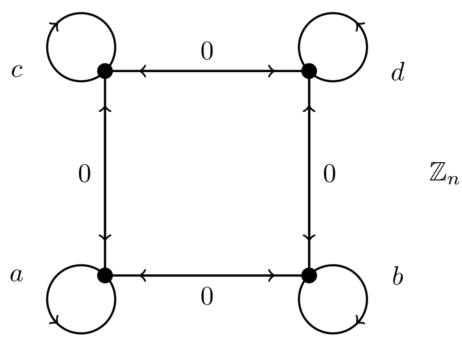

Figure 6: Voltage graph for the supergeneralised Petersen graph $P(4, n; a, b, c, d)$.

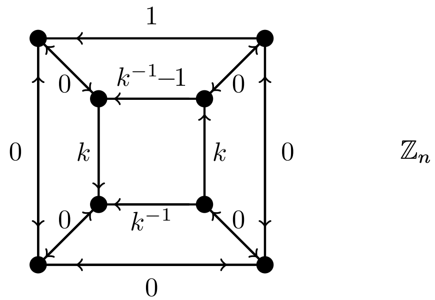

Figure 7: Original voltage graph for the octacirculant graph $CQ(k,n), \gcd(k,n) = 1$, which appeared in [7].

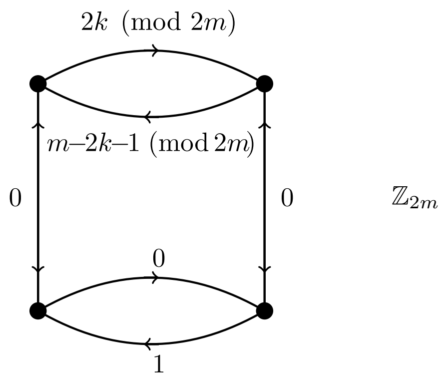

Figure 8: Corrected voltage graph for the tetracirculant graph $CQ(k,m)$, k odd, m even.

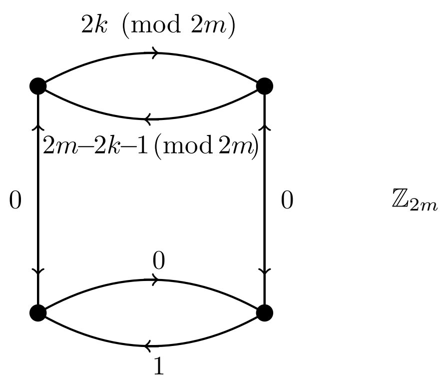

Figure 9: Corrected voltage graph for the tetracirculant graph $CQ(k,m)$, k odd, m odd.

not needed. Hence, their voltage graphs define a family of graphs that is more general than that of Feng and Wang [7]. In our paper, we shall use this more general definition.

4 $GDGP$-graphs

The graphs we shall focus on in this paper are defined as follows.

Definition 4.1. Let $n \geq 3$ and $m \geq 2$ be positive integers such that m divides n, let a be a non-zero element of $\mathbb{Z}_m$, and let $K = (k_0, k_1, \ldots, k_{m-1})$ be a sequence of elements from $\mathbb{Z}_n$ all of which are congruent to a modulo m and satisfy the requirement $k_j + k_{j-a} \not\equiv 0 \pmod{n}$, for all $j \in \mathbb{Z}_m$.

The graph $GDGP_m(n; K)$, or alternatively $GDGP_m(n; k_0, k_1, \ldots, k_{m-1})$, has the vertex set $\{u_i, v_i \,|\, i \in \mathbb{Z}_n\}$ of order $2n$ and the edge set of size $3n$:

$$\{u_i, u_{i+1}\}, \quad \{u_i, v_i\}, \quad \{v_{mi+j}, v_{mi+j+k_j}\}, \tag{4.1}$$

where $i \in \mathbb{Z}_{\frac{n}{m}}$, $j \in \mathbb{Z}_m$, and the arithmetic operations are performed modulo n.

While the $GDGP$-graphs defined above share the outer rim edges and the spokes with the generalised Petersen graphs (and are therefore all connected), the inner edges are determined by the more complicated rule $\{\{v_{mi+j}, v_{mi+j+k_j}\} \,|\, i \in \mathbb{Z}_{\frac{n}{m}}, j \in \mathbb{Z}_m\}$ applied in groups of size $m \geq 2$.

The choices made in our definition guarantee that the $GDGP$-graphs are cubic. This claim is obviously true for the outer vertices u_i. To prove the claim for the inner vertices v_{mi+j}, $i \in \mathbb{Z}_{\frac{n}{m}}$, $j \in \mathbb{Z}_m$, it is enough to show that each inner vertex v_{mi+j} of $GDGP_m(n; K)$ is incident with exactly two edges of the type determined by the third rule of (4.1). Thus, assume that $\{v_{mi'+j'}, v_{mi'+j'+k_{j'}}\}$ is incident with v_{mi+j}. Then, either $v_{mi+j} = v_{mi'+j'}$ or $v_{mi+j} = v_{mi'+j'+k_{j'}}$. If $v_{mi+j} = v_{mi'+j'}$, then $i = i', j = j'$, and $k_j = k_{j'}$ is uniquely determined; there is exactly one such edge. If $v_{mi+j} = v_{mi'+j'+k_{j'}}$, then $mi + j = mi' + j' + k_{j'}$, thus $j \equiv j' + k_{j'} \equiv j' + a \pmod{m}$, which uniquely determines $j' = j - a \in \mathbb{Z}_m$ as well as $k_{j'}$. The equation $mi + j = mi' + j' + k_{j'}$ then uniquely determines i', and therefore there is exactly one edge $\{v_{mi'+j'}, v_{mi'+j'+k_{j'}}\}$ for which $v_{mi+j} = v_{mi'+j'+k_{j'}}$. To complete the argument, note that the two edges $\{v_{mi+j}, v_{mi+j+k_j}\}$ and $\{v_{mi+j-k_{j-a}}, v_{mi+j}\}$ are necessarily different, since we assume that $k_j + k_{j-a} \not\equiv 0 \pmod{n}$, for all $j \in \mathbb{Z}_m$. Let us observe for future reference that the three neighbors of v_{mi+j} are the vertices u_{mi+j}, v_{mi+j+k_j} and $v_{mi+j-k_{j-a}}$.

Being 2-regular, the graph induced by the inner vertices v_i, $i \in \mathbb{Z}_n$ consists of disjoint cycles. If we denote the order of a in $\mathbb{Z}_m$ by $o_m(a)$, it is easy to see that the length of the inner cycle containing v_{mi+j} is the product of $o_m(a)$ with the order $o_n(k_j + k_{j+a} + k_{j+2a} + \ldots + k_{j+(o_m(a)-1)a})$ of the element $k_j + k_{j+a} + k_{j+2a} + \ldots + k_{j+(o_m(a)-1)a}$ in $\mathbb{Z}_n$ (with the indices calculated modulo m). Thus, if a is chosen to be a generator for $\mathbb{Z}_m$ (i.e., $o_m(a) = m$), all inner cycles in $GDGP_m(n; k_0, k_1, \ldots, k_{m-1})$ are of the same length $m \cdot o_n(k_0 + k_1 + \ldots + k_{m-1})$. In particular, if $m = 2$, a is by definition necessarily congruent to $1 \pmod{2}$, and is therefore a generator for $\mathbb{Z}_2$, hence all inner cycles of the graphs $GDGP_2(n; k_0, k_1)$ are of length $2 \cdot o_n(k_0 + k_1)$.

Example 4.2. Consider the graph $GDGP_2(8; 1, 3)$ in Figure 10. Both 1 and 3 are congruent to $1 \pmod{2}$, which is a generator for $\mathbb{Z}_2$. The order of the sum $1 + 3 = 4$ is 2 in $\mathbb{Z}_8$, and hence the inner edges of this graph form two disjoint 4-cycles.

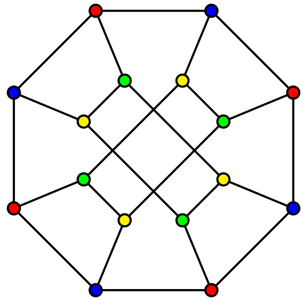

Figure 10: $GDGP_2(8;1,3)$.

It is easy to see that, for even n and odd k, $GP(n,k)$ is isomorphic to $GDGP_2(n;k,k)$, and for n divisible by 3 and $k \not\equiv 0 \pmod 3$, $GP(n,k)$ is isomorphic to $GDGP_3(n;k,k,k)$. We generalise this observation in the following lemma. A sequence $k_0,\ldots,k_{m-1}$ is said to be *periodic* if there exists an integer $0 < p < m$ that divides m and $k_i = k_{i+p}$, for all $i \in \mathbb{Z}_m$. The smallest p with this property is the *period* of the sequence. The proof of the following lemma is now obvious.

Lemma 4.3. *Let $GDGP_m(n;K)$ be a graph such that $K = (k_0,\ldots,k_{m-1})$ is a periodic sequence with a period p. If $p = 1$, the graph $GDGP_m(n;K)$ is isomorphic to the graph $GP(n,k)$, and if $p > 1$, $GDGP_m(n;K)$ is isomorphic to the graph $GDGP_p(n;K')$, where $K' = (k_0,\ldots,k_{p-1})$.*

Consequently, $GDGP_m(n;k,k,\ldots,k) \cong GP(n,k)$, for all divisors m of n and all $k \not\equiv 0 \pmod m$. To simplify our notation and arguments, *we will assume from now on that the sequence K used in the notation $GDGP_m(n;K)$ is aperiodic.*

Most importantly, not all $GDGP$-graphs are generalised Petersen graphs. Consider, for example, the graph $GDGP_2(8;1,3)$ constructed in Example 4.2. The subgraph induced by the inner vertices consists of two disjoint 4-cycles. While the same is true for $GP(8,2)$, nevertheless, $GDGP_2(8;1,3)$ is not isomorphic to any generalised Petersen graph. To see this, note that each of the outer vertices of this graph lies on exactly one 4-cycle, while each of the inner vertices lies on two 4-cycles. Thus, no automorphism of $GDGP_2(8;1,3)$ interchanges the outer and inner vertices (and $GDGP_2(8;1,3)$ is not a vertex-transitive graph). If $GDGP_2(8;1,3)$ were to be isomorphic to a generalised Petersen graph, it would have to be a bicirculant and would have to admit a $(2,8)$-semiregular automorphism. The two 8-orbits would thus necessarily consist of the outer and inner vertices. The automorphism group of the 8-cycle induced by the outer vertices is equal to the dihedral group $\mathbb{D}_8$ which contains only two 8-cycles: $(u_0,u_1,\ldots,u_7)$, and its inverse. Thus, the action of any $(2,8)$-semiregular automorphism of $GDGP_2(8;1,3)$ on the outer vertices would have to be equal to one of these cycles. Since automorphisms must preserve adjacency, this would necessarily force the action of this semiregular automorphism on the inner vertices to be the cycle $(v_0,v_1,\ldots,v_7)$ or its inverse. However, neither the permutation $(u_0,u_1,\ldots,u_7)(v_0,v_1,\ldots,v_7)$ nor its inverse are graph automorphisms of $GDGP_2(8;1,3)$. Hence, $GDGP_2(8;1,3)$ is not a bicirculant, and is therefore not isomorphic to any generalised Petersen graph.

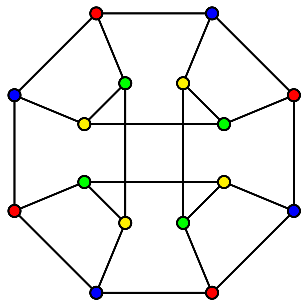

Figure 11: $GDGP_2(8;1,5)$.

On the other hand, it is not hard to see that the $(4,4)$-semiregular permutation $(u_0,u_2, u_4,u_6)(u_1,u_3,u_5,u_7)(v_0,v_2,v_4,v_6)(v_1,v_3,v_5,v_7)$ is an automorphism of $GDGP_2(8;1,3)$, and that the following theorem holds in general.

Theorem 4.4. *For all $m \geq 2$, the permutation*

$$\alpha : V(GDGP_m(n;K)) \to V(GDGP_m(n;K)),\ u_i \mapsto u_{i+m},\ v_i \mapsto v_{i+m},\ i \in \mathbb{Z}_n,$$

is a $(2m, \frac{n}{m})$-semiregular automorphism of $GDGP_m(n;K)$.
Thus, every $GDGP_m(n;K)$-graph is a $2m$-circulant.

We conclude the section with an easy but useful graph-theoretical property of the $GDGP$-graphs.

Lemma 4.5. *The $GDGP_m(n;K)$ is a bipartite graph if and only if n is even and all elements in K are odd.*

5 Automorphisms of $GDGP$-graphs

Many of the ideas of this and the forthcoming sections can be demonstrated with the use of the following family of $GDGP$-graphs.

Example 5.1. For each *even* $n \geq 4$, consider the graph $GDGP_2(n;1,n-3)$. The graphs in this family are also known as the *crossed prism graphs* [19]. In particular, $GDGP_2(4;1,1) \cong GP(4,1)$, and $GDGP_2(6;1,3)$ is the *Franklin graph*. The graph $GDGP_2(8;1,5)$ is depicted in Figure 11. All the graphs $GDGP_2(n;1,n-3)$ are vertex-transitive, which means, in particular, that they admit an automorphism mapping an outer vertex to an inner vertex.

Our first result follows already from our discussion of $GDGP_2(8;1,3)$.

Lemma 5.2. *If $\gamma \in \mathrm{Aut}(GDGP_m(n;K))$ fixes set-wise any of the sets Ω, Σ or I, then it either fixes all three sets or fixes Σ set-wise and interchanges Ω and I.*

We have observed already that the action on Ω of an Ω-preserving automorphism must belong to $\mathbb{D}_n$. Assume that an automorphism σ preserves Ω and acts on Ω as a reflection. Then one of the following occurs:

1. σ has no fixed points, in which case n is necessarily even and there exists an $s \in \mathbb{Z}_n$ such that σ swaps u_s and u_{s+1}, and, consequently, σ swaps all the pairs u_{s-i} and u_{s+1+i}, $i \in \mathbb{Z}_n$;
2. σ fixes at least one vertex, say u_s, and consequently swaps the pairs u_{s+i} and u_{s-i}, $i \in \mathbb{Z}_n$.

The same is necessarily true for the inner vertices, and in case (1) σ swaps v_{s-i} and v_{s+1+i}, $i \in \mathbb{Z}_n$, while in case (2) σ swaps the pairs v_{s+i}, v_{s-i}, $i \in \mathbb{Z}_n$.

In either case, σ is a bijection on the vertices and it preserves the outer edges and the spokes. Thus the image of an inner edge must again be an inner edge. Assume first that σ is of type (1), and consider an arbitrary inner edge $\{v_{mi+j}, v_{mi+j+k_j}\}$. Its image under σ is the pair $\{v_{2s-mi-j+1}, v_{2s-mi-j-k_j+1}\}$, which is an edge of $GDGP_m(n;K)$ if and only if $2s-mi-j-k_j+1 \equiv 2s-mi-j+1+k_{j'} \pmod n$, i.e., $-k_j \equiv k_{j'} \pmod n$, where $j' \equiv 2s-mi-j+1 \equiv 2s-j+1 \pmod m$, or, $2s-mi-j+1 \equiv 2s-mi-j-k_j+1+k_{j''} \pmod n$, i.e., $k_j \equiv k_{j''} \pmod n$, where $j'' \equiv 2s-mi-j-k_j+1 \equiv 2s-j-a+1 \pmod m$. Therefore, σ is a graph automorphism of $GDGP_m(n;K)$ if and only if (at least) one the above equalities holds for each $i \in \mathbb{Z}_{\frac{n}{m}}$, $j \in \mathbb{Z}_m$. In the special case when $m = 2$, a is necessarily 1, $j'' \equiv 2s+j+1+1 \equiv j \pmod 2$, and hence:

Lemma 5.3. *For every $GDGP_2(n;K)$, and for every $s \in \mathbb{Z}_n$, the reflection σ_s swapping the pairs u_{s-i} and u_{s+1+i}, and the pairs v_{s-i} and v_{s+1+i}, for all $i \in \mathbb{Z}_n$, is a graph automorphism of $GDGP_2(n;K)$. Consequently, $\mathrm{Aut}(GDGP_2(n;K))$ acts transitively on the two sets of outer and inner vertices of $GDGP_2(n;K)$.*

Proof. The graph automorphism $\alpha : V(GDGP_2(n;K)) \to V(GDGP_2(n;K))$ defined in Theorem 4.4 and sending $u_i \mapsto u_{i+2}$ and $v_i \mapsto v_{i+2}$, for all $i \in \mathbb{Z}_2$, has two orbits on the outer and two orbits on the inner vertices. The reflection automorphisms σ_s mix these two orbits. □

Both of our examples, $GDGP_2(8;1,3)$ and $GDGP_2(8;1,5)$, can be easily seen to be symmetric with respect to reflections about axes passing through the centers of a pair of opposing outer edges.

Next, let us consider σ of type (2). The image of an arbitrary edge $\{v_{mi+j}, v_{mi+j+k_j}\}$ is the pair $\{v_{2s-mi-j}, v_{2s-mi-j-k_j}\}$, which is an edge if and only if $2s-mi-j-k_j \equiv 2s-mi-j+k_{j'} \pmod n$, i.e., $-k_j \equiv k_{j'} \pmod n$, for $j' \equiv 2s-mi-j \equiv 2s-j \pmod m$, or $2s-mi-j \equiv 2s-mi-j-k_j+k_{j''} \pmod n$, i.e., $k_j \equiv k_{j''} \pmod n$, for $j'' \equiv 2s-mi-j-k_j \equiv 2s-j-a \pmod m$. Comparing this result to that of Lemma 5.3, assuming $m = 2$ would require $-k_j \equiv k_{j'} \pmod n$, for $j' \equiv j \pmod 2$, or $k_j \equiv k_{j''} \pmod n$, for $j'' \equiv j+a \pmod 2$. The first is impossible as that would require $k_0 = k_1 = \frac{n}{2}$, which would violate our agreement that we do not consider periodic sequences, while the second possibility is explicitely prohibited in the definition of the GDGP-graphs. Hence, no $GDGP_2(n;K)$ that is not a generalised Petersen graph admits automorphisms of type (2). Specifically, it is easy to see that neither $GDGP_2(8;1,3)$ nor $GDGP_2(8;1,5)$ admits such automorphisms. There are, on the other hand, infinitely many graphs that do admit at least one such automorphism. The proof of the next lemma follows from the above discussion.

Lemma 5.4. *Let $n \geq 3$, and let K have the property that $-k_j \equiv k_{j'} \pmod n$, for $j' \equiv 2s-j \pmod m$, or let K have the property $k_j \equiv k_{j''} \pmod n$, for $j'' \equiv 2s-j-a$*

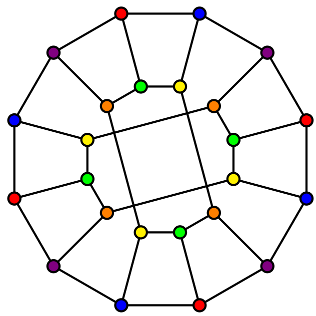

Figure 12: $GDGP_3(12; 1, 4, 1)$.

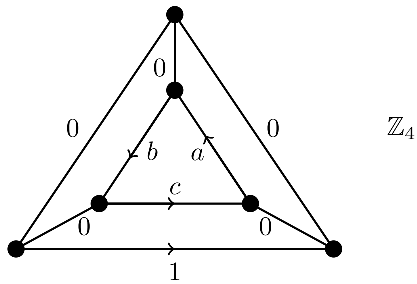

Figure 13: The voltage graph for $GDGP_3(12; 1, 4, 1)$, $a = 0, b = 0, c = 2$.

$\pmod m$. *Then $GDGP_m(n, K)$ admits an automorphism σ that fixes u_s and v_s, and swaps the pairs u_{s+i} and u_{s-i}, v_{s+i} and v_{s-i}, $i \in \mathbb{Z}_n$.*

Example 5.5. The graphs $GDGP_3(n; a, 1, n-1)$ as well as the graphs $GDGP_3(n; 1, a, 1)$, $a \in \mathbb{Z}_n$, all admit an automorphism fixing the vertex u_0. In particular, the graph $GDGP_3(12; 1, 4, 1)$ pictured in Figure 12 is symmetric with respect to the axes passing through the vertices u_0 and u_6, and through the vertices u_3 and u_9.

All the automorphisms considered so far preserve the outer and inner rim as well as the spokes. We conclude this section by considering the graphs we started the section with, namely with the graphs $GDGP_2(n; 1, n-3)$. As stated at the beginning of the section, all of them are vertex-transitive and admit an automorphism mapping an outer vertex to an inner vertex. Computational evidence collected in [12] suggests that the order of the full automorphism group of an $GDGP_2(n; 1, n-3)$ is $n \cdot 2^{\frac{n}{2}}$, and it is easy to see that these graphs are neither edge- nor arc-transitive. In what follows, we present automorphisms that do not preserve the set of spokes.

Lemma 5.6. *Let $n \geq 6$. Then $GDGP_2(n; 1, n-3)$ admits at least one automorphism which does not fix its set of spokes.*

Proof. The desired automorphism $\delta \in \mathrm{Aut}(GDGP_2(n; 1, n-3))$ consists of just two 2-cycles: $\delta = (u_1 v_0)(u_2 v_3)$. Since δ moves only four vertices, to show that it is indeed

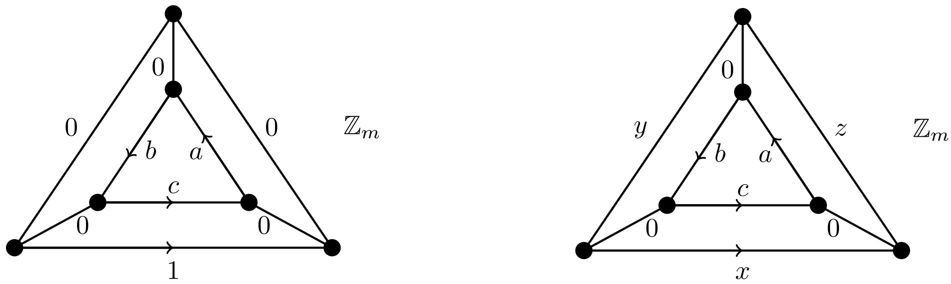

Figure 14: The voltage graphs for $GDGP_3(m; k_1, k_2, k_3)$ and $SI_3(m; l_1, l_2, l_3, k_1, k_2, k_3)$.

a graph automorphism, it suffices to show that it maps edges incident with the vertices u_1, v_0, u_2, v_3 to edges incident to the vertices u_1, v_0, u_2, v_3. This easy exercise is left to the reader. It is also easy to see that $(u_{i+1}v_i)(u_{i+2}v_{i+3})$ belongs to $\mathrm{Aut}(GDGP_2(n; 1, n-3))$ for all positive integers i divisible by 4 and smaller than $n - 3$. □

Lemma 5.3 together with Lemma 5.6 yield that the graphs $GDGP_2(n; 1, n - 3)$ are indeed vertex-transitive for all $n \geq 6$.

6 Vertex-transitive and Cayley $GDGP_2$-graphs

We continue searching for vertex-transitive graphs. Lemma 5.3 asserts that $\mathrm{Aut}(GDGP_2(n; k_0, k_1))$ acts transitively on the set of the outer and the set of the inner vertices of every $GDGP_2(n; k_0, k_1)$. Thus, an $GDGP_2(n; k_0, k_1)$ graph is vertex-transitive if and only if it admits a graph automorphism mapping at least one outer vertex to an inner vertex. In this section, we present some sufficient conditions for this to happen. Note that the graphs $GDGP_2(n; k_0, k_1)$, $GDGP_2(n; k_1, k_0)$, $GDGP_2(n; -k_0, -k_1)$ and $GDGP_2(n; -k_1, -k_0)$ are all isomorphic.

Since we only seek sufficient conditions, we will focus on the special case of graphs that admit automorphisms preserving the set of spokes and swapping the entire sets of outer and inner vertices. Obviously, these must be those $GDGP$-graphs in which the graphs induced by the inner vertices form a single cycle. As observed already in the discussion following the definition of the $GDGP$-graphs, this is the case if and only if the order of the element $k_0 + k_1$ in $\mathbb{Z}_n$ is equal to $\frac{n}{2}$. Thus, we shall assume from now on that $o_n(k_0 + k_1) = \frac{n}{2}$.

Suppose $\gamma \in \mathrm{Aut}(GDGP_2(n; K))$ swaps the outer and the inner rim and preserves the spokes. Because of Lemma 5.3, we may assume that γ maps u_0 to v_0. The outer rim can be mapped onto the inner rim in either the clockwise or in the counterclockwise direction. Hence, there might be two automorphisms which swap the outer and inner cycles and map u_0 to v_0.

Let γ be an automorphism which maps the outer cycle to the inner cycle in the same direction. Thus,

$$\gamma(u_{2i}) = v_{i(k_0+k_1)}, \qquad \gamma(u_{2i+1}) = v_{i(k_0+k_1)+k_0}, \tag{6.1}$$

for all $i \in \mathbb{Z}_{\frac{n}{2}}$. Since γ is assumed to preserve the set of spokes, the image of a spoke must be a spoke again and thus it must be the case that

$$\gamma(v_{2i}) = u_{i(k_0+k_1)}, \qquad \gamma(v_{2i+1}) = u_{i(k_0+k_1)+k_0}, \tag{6.2}$$

for all $i \in \mathbb{Z}_{\frac{n}{2}}$. On the other hand, γ must map the inner cycle to the outer cycle, and, in particular, γ must map inner edges to outer edges. For any $i \in \{0, 1, \dots, n-1\}$, because of (6.2), the vertex v_{2i}, which is adjacent to vertices v_{2i+k_0} and v_{2i-k_1}, is mapped to the vertex $u_{i(k_0+k_1)}$, adjacent to the vertices $u_{i(k_0+k_1)+1}$ and $u_{i(k_0+k_1)-1}$. Thus, γ maps the 2-set $\{v_{2i+k_0}, v_{2i-k_1}\}$ onto the 2-set $\{u_{i(k_0+k_1)+1}, u_{i(k_0+k_1)-1}\}$. However, invoking (6.2) again,

$$\gamma(v_{2i+k_0}) = u_{(i+\frac{k_0-1}{2})(k_0+k_1)+k_0}, \text{ while } \gamma(v_{2i-k_1}) = u_{(i-\frac{k_1+1}{2})(k_0+k_1)+k_0}.$$

This gives us the congruences:

$$(i + \frac{k_0-1}{2})(k_0+k_1) + k_1 \equiv i(k_0+k_1) \pm 1 \pmod{n},$$
$$(i - \frac{k_1+1}{2})(k_0+k_1) + k_1 \equiv i(k_0+k_1) \mp 1 \pmod{n},$$

which are equivalent to the system of congruencies:

$$\begin{aligned} \tfrac{k_0-1}{2}(k_0+k_1) \quad + \quad k_1 \quad &\equiv \quad \pm 1 \qquad \pmod{n}, \\ -\tfrac{k_1+1}{2}(k_0+k_1) \quad + \quad k_1 \quad &\equiv \quad \mp 1 \qquad \pmod{n}. \end{aligned} \tag{6.3}$$

Moreover, applying the same ideas to the vertices v_{2i+1} yields conditions equivalent to the conditions (6.3).

Similarly, one can define an automorphism $\bar{\gamma}$ which maps the outer rim onto the inner rim in the opposite direction. In this case:

$$\begin{aligned} \bar{\gamma}(u_{2i}) &= v_{-i(k_0+k_1)}, & \bar{\gamma}(v_{2i}) &= u_{-i(k_0+k_1)}, \\ \bar{\gamma}(u_{2i+1}) &= v_{-i(k_0+k_1)-k_1}, & \bar{\gamma}(v_{2i+1}) &= u_{-i(k_0+k_1)-k_1}, \end{aligned} \tag{6.4}$$

for all $i \in \mathbb{Z}_{\frac{n}{2}}$. Inner edges are preserved if and only if the system of congruencies

$$\begin{aligned} \tfrac{1-k_0}{2}(k_0+k_1) \quad - \quad k_1 \quad &\equiv \quad \pm 1 \qquad \pmod{n}, \\ \tfrac{k_1+1}{2}(k_0+k_1) \quad - \quad k_1 \quad &\equiv \quad \mp 1 \qquad \pmod{n} \end{aligned} \tag{6.5}$$

is satisfied.

Note that k_0, k_1 that satisfy the system (6.3) also necessarily satisfy the congruence

$$(k_0+k_1)^2 \equiv \pm 4 \pmod{n}, \tag{6.6}$$

and parameters k_0, k_1 that satisfy (6.5) also satisfy

$$(k_0+k_1)^2 \equiv \mp 4 \pmod{n}. \tag{6.7}$$

Recall that a necessary and sufficient condition for a vertex-transitivity of the generalised Petersen graphs $GP(n,k)$ is $k^2 \equiv \pm 1 \pmod{n}$ (except for $GP(10,2)$)[9], which implies the congruence $(k+k)^2 \equiv \pm 4 \pmod{n}$. In this sense, the conditions (6.6) and (6.7) are generalisations of the well-known characterization of vertex-transitive generalised Petersen graphs.

We have proved the following:

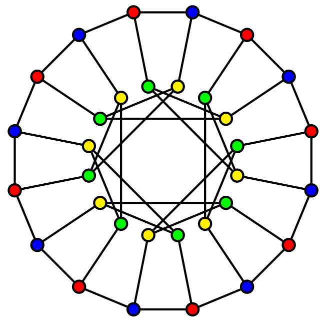

Figure 15: Graph $GDGP_2(16;3,11)$ which admits the automorphism γ.

Lemma 6.1. *The graph $GDGP_2(n;k_0,k_1)$ admits an automorphism that preserves the set of spokes and swaps the outer and inner vertices if and only if the order $o_n(k_0+k_1)=\frac{n}{2}$ in $\mathbb{Z}_n$ and the parameters n,k_0,k_1 satisfy one of the systems of congruencies (6.3) or (6.5).*

Proof. We have proved in detail that at least one of the conditions is necessary. Their sufficiency follows from the fact that the vertex map of $GDGP_2(n;k_0,k_1)$ whose parameters satisfy (6.3) and which is defined by equations (6.1) and (6.2) fixes the spokes and swaps the outer and inner edges. The same holds true for the vertex map whose parameters satisfy (6.5) and which is defined via (6.4). □

It is easy to observe that the parameters of a fixed $GDGP_2(n;k_0,k_1)$ can satisfy at most one of the systems (6.3) or (6.5). Therefore, graphs $GDGP_2(n;k_0,k_1)$ admit at most one of the automorphisms γ or $\bar{\gamma}$.

The following theorem provides the sufficient condition promised at the beginning of the section. Its proof follows from Lemma 5.3 and Lemma 6.1.

Theorem 6.2. *Let $GDGP_2(n;k_0,k_1)$ be a graph whose parameters satisfy $o_n(k_0+k_1)=\frac{n}{2}$ and one of the systems (6.3) or (6.5). Then $GDGP_2(n;k_0,k_1)$ is a vertex-transitive graph.*

Example 6.3. The parameters of the graphs $GDGP_2(n;a,n-a+2)$ satisfy the condition $o_n(a+n-a+2)=o_n(2)=\frac{n}{2}$ as well as the system of congruencies (6.3). Hence, they all admit the automorphism γ defined by formulas (6.1) and (6.2).

Example 6.4. The parameters of the graphs $GDGP_2(n;1,n-3)$ satisfy neither of the systems (6.3) or (6.5). They are nevertheless vertex-transitive and their inner edges form a single cycle. Thus, conditions (6.3) or (6.5) are sufficient but not necessary.

Example 6.5. The thesis [11] contains yet another family of graphs whose parameters do not satisfy (6.3) or (6.5), but nevertheless includes vertex-transitive graphs. These are the graphs $GDGP_2(8a+4;1,4a-1)$, with a being a positive integer. The inner rim of these graphs does not form a single cycle. The smallest graph in this family is the graph $GDGP_2(12;1,3)$ isomorphic to the truncated octahedral graph. It is known that the truncated octahedral graph is the Cayley graph $Cay(G,X)$, where $G=S_4$ and $X=\{(1234),(1432),(12)\}$, and the group of automorphisms has order 48. Another member of the family is the graph $GDGP_2(20;1,7)$.

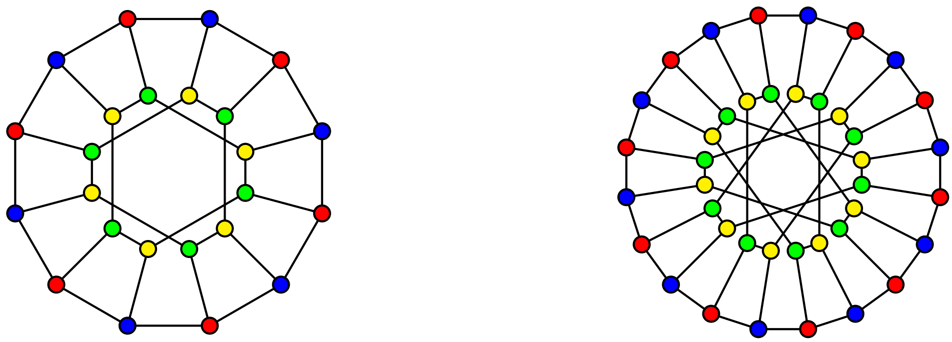

Figure 16: Graphs $GDGP_2(12; 1, 3)$ and $GDGP_2(20; 1, 7)$.

Example 6.6. Based on computational evidence, there are other families of vertex-transitive $GDGP_2$-graphs with parameters that do not satisfy (6.3) or (6.5). In the thesis [12], relying on exhaustive search of all $GDGP_2(n; k_0, k_1)$ with $n \leq 300$, six more graphs have been found whose parameters do not satisfy (6.3) or (6.5) but are vertex-transitive. These are the graphs

$$GDGP_2(96, 21, 49), GDGP_2(96, 27, 47), GDGP_2(192, 45, 97),$$

$$GDGP_2(192, 51, 95), GDGP_2(288, 69, 145), GDGP_2(288, 75, 143).$$

Since all their orders are multiples of 96, their existence suggests a possible infinite family.

Once again referring to the characterization of vertex-transitive generalised Petersen graphs, we observe $GP(n, k)$ is a Cayley graph if and only if $k^2 \equiv 1 \pmod{n}$, and thus vertex-transitive generalised Petersen graphs whose parameters satisfy the congruence relation $k^2 \equiv -1 \pmod{n}$ are not Cayley [9]. We show that this is not the case for the graphs considered in this section. Namely, we show that all graphs $GDGP_2(n; k_0, k_1)$ whose parameters satisfy (6.3) or (6.5) are Cayley graphs. We will somewhat abbreviate our arguments. Detailed proofs of the claims made in this part can be found in [11].

Let $GDGP_2(n; k_0, k_1)$ be a graph admitting one of the automorphisms γ or $\bar{\gamma}$, α be the automorphism from Theorem 4.4, and β be the automorphism of $GDGP_2(n; k_0, k_1)$ that maps u_i to u_{1-i} and v_i to v_{1-i}, $0 \leq i \leq n-1$. The groups $G_\Sigma = \langle \alpha, \beta, \gamma \rangle$ and $\bar{G}_\Sigma = \langle \alpha, \beta, \bar{\gamma} \rangle$ are subgroups of $\mathrm{Aut}(GDGP_2(n; k_0, k_1))$ that preserve the set of spokes. It is easy to verify that if $(k_0 + k_1)^2 \equiv 4 \pmod{n}$, then

$$G_\Sigma = \langle \alpha, \beta, \gamma \,|\, \alpha^{\frac{n}{2}} = \beta^2 = \gamma^2 = 1,\ \beta\alpha\beta = \alpha^{-1},\ \gamma\alpha\gamma = \alpha^{\frac{k_0+k_1}{2}},\ \beta\gamma = \gamma\beta\alpha^{\frac{k_0-1}{2}} \rangle,$$

and

$$\bar{G}_\Sigma = \langle \alpha, \beta, \bar{\gamma} \,|\, \alpha^{\frac{n}{2}} = \beta^2 = \bar{\gamma}^2 = 1,\ \beta\alpha\beta = \alpha^{-1},\ \bar{\gamma}\alpha\bar{\gamma} = \alpha^{-\frac{k_0+k_1}{2}},\ \beta\bar{\gamma} = \bar{\gamma}\beta\alpha^{-\frac{k_1+1}{2}} \rangle.$$

On the other hand, if $(k_0+k_1)^2 \equiv -4 \pmod{n}$, then $GDGP_2(n; k_0, k_1)$ is isomorphic to the generalised Petersen graph $GP(n, k_0)$.

Theorem 6.7. *The following statements are true for all $GDGP_2(n; k_0, k_1)$:*

1. *If γ is an automorphism of the graph $GDGP_2(n; k_0, k_1)$, then $GDGP_2(n; k_0, k_1)$ is isomorphic to the Cayley graph $Cay(G_\Sigma, \{\beta, \alpha\beta, \gamma\})$.*
2. *If $\bar{\gamma}$ is an automorphism of the graph $GDGP_2(n; k_0, k_1)$, then $GDGP_2(n; k_0, k_1)$ is isomorphic to the Cayley graph $Cay(\bar{G}_\Sigma, \{\beta, \alpha\beta, \bar{\gamma}\})$.*

Proof. Leaving out the technical details, we claim that the map

$$\varphi : GDGP_2(n; k_0, k_1) \longrightarrow Cay(G_\Sigma, \{\beta, \alpha\beta, \gamma\}),$$

defined on the vertices of $GDGP_2(n; k_0, k_1)$ via the formulas

$$\begin{array}{ll} u_{2i} \mapsto \alpha^i, & u_{2i+1} \mapsto \beta\alpha^i, \\ v_{2i} \mapsto \gamma\alpha^i, & v_{2i+1} \mapsto \gamma\beta\alpha^i, \end{array}$$

is an isomorphism between $GDGP_2(n; K)$ and $Cay(G_\Sigma, \{\beta, \alpha\beta, \gamma\})$.

Similarly, the map

$$\bar{\varphi} : GDGP_2(n; k_0, k_1) \longrightarrow Cay(\bar{G}_\Sigma, \{\beta, \alpha\beta, \bar{\gamma}\}),$$

defined via

$$\begin{array}{ll} u_{2i} \mapsto \alpha^i, & u_{2i+1} \mapsto \beta\alpha^i, \\ v_{2i} \mapsto \bar{\gamma}\alpha^i, & v_{2i+1} \mapsto \bar{\gamma}\beta\alpha^i, \end{array}$$

is an isomorphism between the graphs $GDGP_2(n; k_0, k_1)$ and $Cay(\bar{G}_\Sigma, \{\beta, \alpha\beta, \bar{\gamma}\})$. □

7 Symmetric $GDGP_2$-graphs

One of the main goals of our paper is to determine which of the $GDGP$-graphs are highly symmetric. In the previous section, we have presented sufficient conditions for $GDGP_2$-graphs being vertex-transitive. For the rest of our paper, we are going to consider even a higher level of symmetry, namely, we are going to address the question which $GDGP_2$-graphs are symmetric (arc-transitive), i.e., which $GDGP_2$-graphs possess enough automorphisms to map any arc of the graph to any other arc.

We have already established in Theorem 4.4 that all $GDGP_2(n; k_0, k_1)$ are tetracirculants. Since all cubic symmetric tetracirculants have been classified by Frelih and Kutnar in [8], in order to classify the symmetric $GDGP_2$-graphs (which are cubic tetracirculants), it is enough to determine which of the cubic symmetric tetracirculants listed in [8] are $GDGP_2$-graphs. Since the symmetric graphs in [8] are described in the form of the lifts, we shall achieve this goal by viewing the $GDGP_2$-graphs as lifts as well.

Recall that generalised Petersen graphs are the lifts of the dumbbell graphs from Figure 3. Note also that the dumbbell graph may be viewed as mono-gonal prism (which makes the generalised Petersen graphs bicirculant). Since the $GDGP_2$-graphs are tetracirculants, in order to view them as lifts, we need to consider base graphs of order 4. Consider the voltage graph in Figure 17 which is a di-gonal prism. In both Figures 3 and 17, the voltages along one basis cycle add up to 1. Since the voltages on the other edges must be integers, both k_0 and k_1 in Figure 17 must be odd. If we recall that the definition of the $GDGP_2(n; k_0, k_1)$-graphs also requires that the parameters k_0 and k_1 be odd, it is not hard to see that every $GDGP_2(n; k_0, k_1)$ is isomorphic to the lift described in Figure 17.

Furthermore, the voltage graph in Figure 17 is a rather special case of the more general voltage graph of Figure 7. As is well-known, the voltages along any spanning tree of the

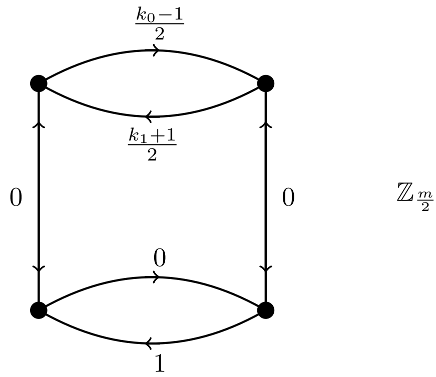

Figure 17: Voltage graph for $GDGP_2(m; k_0, k_1)$.

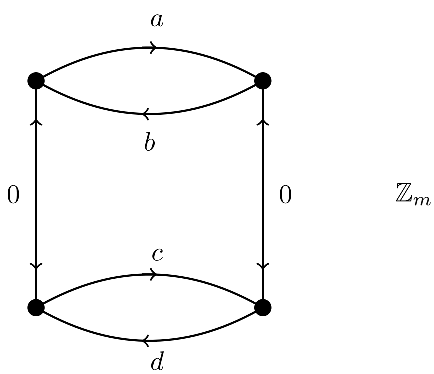

Figure 18: Voltage graph for the tetracirculant graph $C_1(m; a, b, c, d)$.

base graph may be chosen to be equal to 0, and still, after appropriate changes of the other voltages, produce the same graph. Hence, c may always be chosen to be 0. Such graphs have been studied before, for instance in [1], under the name of *C-graphs*. However, the spanning tree used in [1] differs from our choice here. Nevertheless, each $GDGP_2$ graph is a C-graph, while the converse is not true.

Let us now continue with the task of classifying symmetric $GDGP_2$-graphs. As mentioned above, the paper [8] contains a complete classification of cubic symmetric (i.e. arc-transitive) tetracirculants.

Theorem 7.1 ([8, Theorem 1.1])**.** *A connected cubic symmetric graph is a tetracirculant if and only if it is isomorphic to one of the following graphs:*

(i) $F008A, F020A, F020B, F024A, F028A, F032A, F040A,$

(ii) $F016A, F048A, F056C, F060A, F080A, F096A, F112B, F120B, F224C, F240C,$

(iii) $CQ(t, m)$ *for* $2 \le t \le m - 3$ *satisfying* $m | (t^2 + t + 1)$,

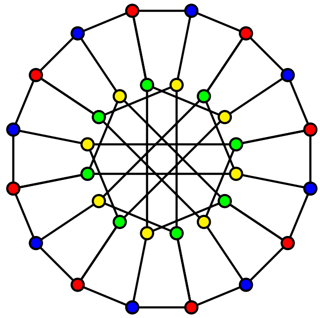

Figure 19: The Dyck graph $GDGP_2(16; 3, 7)$.

(iv) $CQ(2t-1, 2m)$ *for* $2 \leq t \leq m-1$ *satisfying* $m|(4t^2 - 2t + 1)$.

The notation FnA, FnB, etc., refers to the corresponding graphs in the Foster census [2], [3]. The graphs $CQ(t, m)$ are the lifts from Figures 8 and 9 introduced in [7].

Since all $GDGP_2$-graphs are bipartite (Lemma 4.5), we can easily rule out the tetracirculants that are not bipartite, such as $GP(5, 2)$ or $GP(10, 2)$. Graphs $F008A$, $F016A$, $F020B$, $F024A$, and $F048A$ are the generalised Petersen graphs $GP(4, 1)$, $GP(8, 3)$, $GP(10, 3)$, $GP(12, 5)$, and $GP(24, 5)$, respectively. Therefore, these graphs are not isomorphic to a non-periodic $GDGP_2(n; k_0, k_1)$. Furthermore, we checked all the sporadic cases in (i) and (ii) by our program in *SAGE-math*. That showed that the two graphs $F040A$ and $F080A$ are also not isomorphic to any non-periodic $GDGP_2(n; k_0, k_1)$.

Finally, the graph $F032A$ is isomorphic to $GDGP_2(16; 3, 7)$. It is also known under the name of the *Dyck graph*; Figure 19.

Summing up the above observations yields the following.

Theorem 7.2. *The only symmetric* $GDGP_2(n; k_0, k_1)$ *graph not isomorphic to a generalised Petersen graph* $GP(n, k)$ *or one of the graphs* $CQ(t, m)$*,* $2 \leq t \leq m-3$*,* $m|(t^2 + t + 1)$ *or* $CQ(2t-1, 2m)$*,* $2 \leq t \leq m-1$*,* $m|(4t^2 - 2t + 1)$*, is the Dyck graph* $GDGP_2(16; 3, 7) = F032A$*.*

Note that our computer program indicates that the only arc-transitive C-graph that is not a $GDGP_2$ graph is the graph $F040A$. It can alternately be described as an SI_2-graph, i.e., a further generalisation in which one allows for spans other than 1 in both rims; see Figure 20.

Our computer experiments also indicate the following:

Conjecture 7.3.

1. *The girth of* $CQ(t, m)$*,* m *odd,* $\gcd(t, m) = 1$*, is equal to 6. For* m *even, the girth may be* 6,8 *or* 10*.*
2. *Every* $CQ(t, m)$*-graph is a* $GDGP_2$ *graph. Every graph* $CQ(t, m)$ *with* $\gcd(t, m) = 1$ *is vertex transitive.*

We close our paper with two **open questions**:

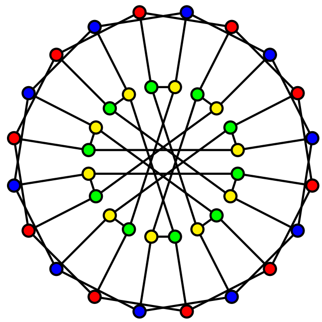

Figure 20: F040A as $SI_2(10; 2, 2, 1, 11)$.

1. *Which of the graphs* $CQ(t, m)$*,* $2 \leq t \leq m-3$*,* $m|(t^2+t+1)$*, and* $CQ(2t-1, 2m)$*,* $2 \leq t \leq m-1$*,* $m|(4t^2-2t+1)$*, are isomorphic to a* $GDGP_2(n; k_0, k_1)$*?*
2. *Which of the graphs* $GDGP_m(n; K)$*,* $m > 2$*, are symmetric?*

ORCID iDs

Katarína Jasenčáková https://orcid.org/0000-0001-7615-0038
Robert Jajcay https://orcid.org/0000-0002-2166-2092
Tomaž Pisanski https://orcid.org/0000-0002-1257-5376

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Regular balanced Cayley maps on nonabelian metacyclic groups of odd order*

Kai Yuan, Yan Wang†
School of Mathematics and Information Science, Yan Tai University, Yan Tai, P.R.C.

Haipeng Qu
School of Mathematics and Computer Science, Shan Xi Normal University, Shan Xi, P.R.C.

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Abstract

In this paper, we show that nonabelian metacyclic groups of odd order do not have regular balanced Cayley maps.

Keywords: Regular balanced Cayley map, metacyclic group.

Math. Subj. Class. (2020): 05C25, 05C30

1 Introduction

Groups are often studied in terms of their action on the elements of a set or on particular objects within a structure. The aim of this article is to study metacyclic groups of odd order acting on maps. A *Cayley graph* $\Gamma = \mathrm{Cay}(G, X)$ will be a graph based on a group G and a generating set $X = \{x_1, x_2, \dots, x_k\}$ which does not contain 1_G, is closed under the operation of taking inverses. In this paper, we call X a *Cayley subset* of G. The vertices of the Cayley graph Γ are the elements of G, and two vertices g and h are joined by an edge if and only if $g = hx_i$ for some $x_i \in X$. The ordered pairs (h, hx) for $h \in G$ and $x \in X$ are called the darts of Γ. For a cyclic permutation ρ of the set X, the Cayley map $\mathcal{M} = \mathrm{CM}(G, X, \rho)$ is the 2-cell embedding of the Cayley graph $\mathrm{Cay}(G, X)$

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E-mail address: pktide@163.com (Kai Yuan), wang_yan@pku.org.cn (Yan Wang), orcawhale@163.com (Haipeng Qu)

in some orientable surface such that the local rotation of darts emanating from every vertex is induced by the same cyclic permutation ρ of X.

An (orientation preserving) *automorphism* of a Cayley map $\mathcal{M}$ is a permutation of the set of darts of $\mathcal{M}$ which preserves the incidence relation of the vertices, edges, faces, and the orientation of the map. The full automorphism group of $\mathcal{M}$, denoted by $\mathrm{Aut}(\mathcal{M})$, is the group of all such automorphisms of $\mathcal{M}$ under the operation of composition. This group always acts semi-regularly on the set of darts of $\mathcal{M}$, that is, the stabilizer in $\mathrm{Aut}(\mathcal{M})$ of each dart of $\mathcal{M}$ is trivial. If the action of $\mathrm{Aut}(\mathcal{M})$ on the darts of $\mathcal{M}$ is transitive (and therefore regular), we say that the Cayley map $\mathcal{M}$ is a regular Cayley map. As the left regular multiplication action of the underlying group G lifts naturally into the full automorphism group of any Cayley map $\mathrm{CM}(G, X, \rho)$, Cayley maps are a very good source of regular maps. There are many papers on the topic of regular Cayley maps, we refer the readers to [2], [5] and [6] and the references therein. Furthermore, A Cayley map $\mathrm{CM}(G, X, \rho)$ is called *balanced* if $\rho(x)^{-1} = \rho(x^{-1})$ for every $x \in X$. In [6], the authors showed that a Cayley map $\mathrm{CM}(G, X, \rho)$ is regular and balanced if and only if there exists a group automorphism σ such that $\sigma|_X = \rho$, where $\sigma|_X$ denotes the restricted action of σ on X. Therefore, determining all the regular balanced Cayley maps of a group is equivalent to determining all the orbits of its automorphisms that can be Cayley subsets.

In [2], it was shown that all odd order abelian groups possess at least one regular balanced Cayley map [2]. Wang and Feng [7] classified all regular balanced Cayley maps for cyclic, dihedral and generalized quaternion groups. In [4], the author proved the non-existence of regular balanced Cayley maps with semi-dihedral groups. In [8], Yuan, Wang and Qu proved that a nonabelian metacyclic p-group for an odd prime number p does not have regular balanced Cayley maps. This was the first work on regular balanced Cayley map of nonabelian groups of odd order. We will take a step further and show that non-abelian metacyclic groups of odd order do not have regular balanced Cayley maps.

2 Preliminaries

We use the standard notation for group theory; see [3]. We denote by (r, s) the greatest common divisor of two positive integers r and s. By $|x|$, $|H|$, we denote the order of element x and subgroup H of a group G, respectively. By $N : H$, we denote a semidirect product of the group N by the group H. Set $[x, y] = x^{-1}y^{-1}xy$, the commutator of x and y and set $[H, K] = \langle [h, k] \bigm| h \in H, k \in K\rangle$, where $H, K \leq G$. For $\alpha \in \mathrm{Aut}(G)$ and $g \in G$, denote the orbit of g under $\langle\alpha\rangle$ by $g^{\langle\alpha\rangle}$.

Define $G_1 = G$, and, proceeding recursively, define $G_n = [G_{n-1}, G]$ for $1 < n \in \mathbb{Z}$. Then G is said to be *nilpotent* if $G_n = 1$ for some $1 \leq n \in \mathbb{Z}$. The following lemma is basic for nilpotent groups.

Lemma 2.1 ([3, Kapitel III.2.3])**.** *A finite group is nilpotent if and only if it is the direct product of its Sylow groups.*

Lemma 2.2 ([3, Kapitel III.7.2])**.** *Let G be a p-group, $N \trianglelefteq G$ and $|N| = p$. Then $N \leq Z(G)$.*

Define $M_{p,q}(m, r) = \langle a, b \mid a^p = 1, b^{q^m} = 1, b^{-1}ab = a^r\rangle$, where p and q are distinct prime numbers, m is a positive integer and $r \not\equiv 1 \pmod{p}$ but $r^q \equiv 1 \pmod{p}$. The following lemma is about the automorphism group of $M_{p,q}(m, r)$.

Lemma 2.3 ([8, Lemma 2.2]). *The automorphism group of $M_{p,q}(m,r)$ is*

$$\mathrm{Aut}(M_{p,q}(m,r)) = \{\sigma \mid a^{\sigma} = a^{i},\ b^{\sigma} = b^{j}a^{k}, 1 \leqslant i \leqslant p-1,\ 1 \leqslant j \leqslant q^{m}-1, q \mid (j-1)\}.$$

The following lemma shows that G/N has a regular balanced Cayley map whenever G has.

Lemma 2.4 ([8, Lemma 2.5]). *Let G be a finite group and N be a nontrivial characteristic subgroup of G. Take $\alpha \in \mathrm{Aut}(G)$ and $g \in G$. If $X = g^{\langle\alpha\rangle}$ is a Cayley subset of G, then $\overline{X} = \overline{g^{\langle\alpha\rangle}} = \bar{g}^{\langle\bar{\alpha}\rangle}$ is a Cayley subset of $\overline{G} = G/N$.*

Proposition 2.5 ([8, Corollary 4.7]). *For any odd prime number p, the nonabelian metacyclic p-group does not have regular balanced Cayley maps.*

3 Regular balanced Cayley maps on nonabelian metacyclic group of odd order

A metacyclic group is an extension of a cyclic group by a cyclic group. That is, it is a group G having a cyclic normal subgroup $\langle a\rangle$ such that the quotient $G/\langle a\rangle$ is also cyclic. We now prove our main theorem.

Theorem 3.1. Nonabelian metacyclic groups of odd order do not have regular balanced Cayley maps.

Proof. Let G be a counterexample of minimal order. Then G is a metacyclic group of odd order which has regular balanced Cayley maps. Set $G = \langle a\rangle\langle b\rangle$ where $\langle a\rangle \trianglelefteq G$. We will get a contradiction through the following seven steps.

Step 1. The order of the derived subgroup G' of G is a prime number p:

Clearly, the derived subgroup G' is a subgroup of $\langle a\rangle$. Thus G' is cyclic. If the order of G' is not a prime number, then there is a proper subgroup N of G' which has prime order. Since N is characteristic in G' and G' is characteristic in G, we find that N is characteristic in G. Now consider the quotient group G/N. On one hand, by Lemma 2.4, G/N has a regular balanced Cayley map. On the other hand, from the choice of G, G/N does not have regular balanced Cayley maps; a contradiction. Thus $|G'|$ is a prime number p.

Step 2. G does not have a nontrivial normal p'-subgroup. In particular, $Z(G) = 1$ or $Z(G)$ is a p-group:

If not, let N be a nontrivial normal p'-subgroup of G. We consider G/N. Since $|G'| = p$, we know that G/N is still nonabelian. By the minimality of G, it follows that G/N does not have regular balanced Cayley maps. However, by Lemma 2.4, G/N has a regular balanced Cayley map. This is a contradiction.

Step 3. The subgroup $\langle a\rangle$ is a p-group:

Otherwise, suppose $\langle a\rangle = P \times Q$, where P is a p-group and Q is a non-trivial p'-group. Since Q is a characteristic subgroup of $\langle a\rangle$, it is normal in G. This contradicts Step 2.

Step 4. $G' \nleq Z(G)$:

Otherwise, assume $G' \leq Z(G)$. Then G is nilpotent. By Lemma 2.1, we have $G = P_1 \times Q_1$, where P_1 is a p-group and Q_1 is a p'-group. By Step 2, $Q_1 = 1$. Thus G is a p-group. This contradicts Proposition 2.5.

Now, we assume $\langle b\rangle = P_2 \times Q_2$, where P_2 is a p-group and Q_2 is a non-trivial p'-group. We will show $P_2 = 1$, so that $\langle b\rangle$ is a p'-group.

Step 5. The order of $\langle a \rangle$ is p:

Set $|a| = p^n$ and $Q_2 = \langle c \rangle$. By Step 1, $[a, c] = a^{ip^{n-1}}$. Hence $a^c = a^{1+ip^{n-1}}$. Notice that $(1 + ip^{n-1})^p \equiv 1 \bmod p^n$ when $n > 1$. If $n > 1$, then $a^{c^p} = a$. Hence $[c^p, a] = 1$. Since $(|c|, p) = 1$, we find that $c \in \langle c^p \rangle$. Thus $[a, c] = 1$. This gives $Q_2 \leq Z(G)$, contradicting Step 2. Thus $n = 1$.

Step 6. $Z(G) = 1$ and $P_2 = 1$:

Since G is nonabelian, $\langle a \rangle \nleq Z(G)$. By Step 5, $\langle a \rangle \cap Z(G) = 1$. Thus $G/Z(G)$ is still nonabelian. By the choice of G, we find that $Z(G) = 1$.

Since $\langle a \rangle$ is a normal subgroup of $\langle a \rangle P_2$ and the order of $\langle a \rangle$ is p, from Lemma 2.2, we have $\langle a \rangle \leq Z(\langle a \rangle P_2)$. It follows that $P_2 \leq Z(G)$. So $P_2 = 1$.

Step 7. G does not have regular balanced Cayley maps:

We can assume $G = \langle a \rangle : \langle b \rangle$, where $\langle a \rangle \cong \mathbb{Z}_p$ and $\langle b \rangle$ is a p'-group. Since $Z(G) = 1$, we have $C_{\langle b \rangle}(a) = 1$. By the Normalizer-Centralizer (N/C) Theorem, $N_{\langle b \rangle}(a)/C_{\langle b \rangle}(a) \lesssim \mathrm{Aut}(\langle a \rangle)$, and so $\langle b \rangle \lesssim \mathrm{Aut}(\langle a \rangle)$. Take a prime factor q of $|b|$. If G has a regular balanced Cayley map, then we may take a corresponding Cayley subset X of G. Without loss of generality, we assume $b \in X$. Then there is some $\sigma \in \mathrm{Aut}(G)$ such that $b^\sigma = b^{-1}$. Let $b_1 = b^{|b|/q}$. Then $b_1^\sigma = b_1^{-1}$. Set $K = \langle a \rangle : \langle b_1 \rangle$ and let σ_1 be the restricted action of σ on K. Then $\sigma_1 \in \mathrm{Aut}(K)$. However, from Lemma 2.3, the metacyclic group K of order pq does not have any automorphism that can reverse b_1, a contradiction. Hence nonabelian metacyclic groups of odd order do not have regular balanced Cayley maps. □

From Lemma 2.4 and Theorem 3.1, we get the following:

Corollary 3.2. Let H be a characteristic subgroup of G. If the quotient group G/H is isomorphic to a nonabelian metacyclic group of odd order, then G does not admit regular balanced Cayley maps. In particular, a direct product of a nonabelian metacyclic group of odd order and a 2-group does not admit regular balanced Cayley maps.

Remark 3.1. Since the only nonabelian group with odd order less than 26 is metacyclic, we know from Corollary 3.2 that the minimal odd order of a nonabelian group that admits a regular balanced Cayley map is at least 27. The only non-metacyclic and nonabelian group of order 27 is $M_3(1,1,1) = \langle a, b, c \mid a^3 = b^3 = c^3 = 1, [a, b] = c, [c, a] = [c, b] = 1 \rangle$. With the help of MAGMA [1], we can easily see that $M_3(1,1,1)$ has regular balanced Cayley maps, and the corresponding Cayley graphs have valency $4, 6$ and 8, respectively.

ORCID iDs

Kai Yuan https://orcid.org/0000-0003-1858-3083
Yan Wang https://orcid.org/0000-0002-0148-2932
Haipeng Qu https://orcid.org/0000-0002-3858-5767

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Automorphism groups of maps, hypermaps and dessins

Gareth Aneurin Jones*

School of Mathematical Sciences, University of Southampton, Southampton SO17 1BJ, UK

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Abstract

A detailed proof is given of a theorem describing the centraliser of a transitive permutation group, with applications to automorphism groups of objects in various categories of maps, hypermaps, dessins, polytopes and covering spaces, where the automorphism group of any object is the centraliser of its monodromy group. An alternative form of the theorem, valid for finite objects, is discussed, with counterexamples based on Baumslag–Solitar groups to show how it can fail in the infinite case. The automorphism groups of objects with primitive monodromy groups are described, as are those of non-connected objects.

Keywords: Permutation group, centraliser, automorphism group, map, hypermap, dessin d'enfant.

Math. Subj. Class. (2020): 05C10, 14H57, 20B25, 20B27, 52B15, 57M10

1 Introduction

In certain categories $\mathfrak{C}$, such as those consisting of maps or hypermaps, oriented or unoriented, or of dessins d'enfants (regarded as finite oriented hypermaps), each object $\mathcal{O}$ can be identified with a permutation representation $\theta : \Gamma \to S := \mathrm{Sym}(\Omega)$ of a 'parent group' $\Gamma = \Gamma_{\mathfrak{C}}$ on some set Ω, and the morphisms $\mathcal{O}_1 \to \mathcal{O}_2$ can be identified with the functions $\Omega_1 \to \Omega_2$ which commute with the actions of Γ on the corresponding sets Ω_1 and Ω_2. These are the 'permutational categories' defined and discussed in [9] (see also §2). The automorphism group $\mathrm{Aut}_{\mathfrak{C}}(\mathcal{O})$ of an object $\mathcal{O}$ within $\mathfrak{C}$ is then identified with the centraliser $C := C_S(G)$ in S of the monodromy group $G := \theta(\Gamma)$ of $\mathcal{O}$. Now $\mathcal{O}$ is connected if and only if G is transitive on Ω, as we will assume unless otherwise stated (see §6). In this situation, an important result is the following, where N denotes the normaliser of a subgroup:

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E-mail address: g.a.jones@maths.soton.ac.uk (Gareth Aneurin Jones)

Theorem 1.1. *If $\mathcal{O}$ is connected then*

$$\mathrm{Aut}_{\mathfrak{C}}(\mathcal{O}) \cong N_G(H)/H \cong N_\Gamma(M)/M,$$

where H and M are the stabilisers in G and Γ of some $\alpha \in \Omega$, and $N_G(H)$ and $N_\Gamma(M)$ are their normalisers.

For instance, this theorem has recently been used in [10] to show that in various categories of maps and hypermaps, every countable group can be realised as the automorphism group of uncountably many non-isomorphic objects, infinitely many of which can be chosen to be finite if the group is. This is an analogue of the well-known theorems of Frucht [5] and Sabidussi [20] for graphs and their automorphism groups.

There are analogues of Theorem 1.1 in other contexts, ranging from abstract polytopes to covering spaces. Proofs of Theorem 1.1 for particular categories can be found in the literature: for instance, in [11] it is deduced for oriented maps from a more general result about morphisms in that category; in [12, Theorem 2.2 and Corollary 2.1] a proof for dessins is briefly outlined; analogous results for covering spaces are proved in [15, Appendix] and [17, Theorem 81.2], and for abstract polytopes in [16, Propositions 2D8 and 2E23(a)]. In §3 we give a detailed proof of the following 'folklore' theorem about permutation groups (see also Theorem 3.2 in the recent [19]), which immediately implies Theorem 1.1 for all permutational categories, including the special cases listed above.

Theorem 1.2. *Let G be a transitive permutation group on a set Ω, with H the stabiliser of some $\alpha \in \Omega$, and let $C := C_S(G)$ be the centraliser of G in the symmetric group $S := \mathrm{Sym}(\Omega)$. Then*

1. *$C \cong N_G(H)/H$,*
2. *C acts regularly on the set Φ of elements of Ω with stabiliser H.*

One sometimes finds proofs or statements of particular cases of Theorem 1.2 which include the assertion that C acts regularly on the set Φ of fixed points of H in Ω; while this is valid if H is finite, in §4 we give counter-examples, based on Baumslag–Solitar groups [1], to show that if H is infinite then Φ must be redefined more precisely as in (2). It follows from Theorem 1.1 that if the monodromy group G of an object $\mathcal{O}$ acts primitively on Ω, then either $\mathrm{Aut}_{\mathfrak{C}}(\mathcal{O})$ is trivial, or G is a cyclic group of prime order, acting regularly on Ω; in §5 we describe the objects with the latter property in various categories $\mathfrak{C}$. In §6 we briefly consider the structure and cardinality of the automorphism groups of non-connected objects in permutational categories, and in §7 we extend Theorem 1.1 to cover morphisms between connected objects.

2 Permutational categories

A *permutational category* $\mathfrak{C}$ is defined in [9] to be a category in which the objects $\mathcal{O}$ can be identified with the permutation representations $\theta : \Gamma \to S := \mathrm{Sym}(\Omega)$ of a *parent group* $\Gamma = \Gamma_{\mathfrak{C}}$, and the morphisms $\mathcal{O}_1 \to \mathcal{O}_2$ with the Γ-invariant functions $\Omega_1 \to \Omega_2$, those commuting with the actions of Γ on Ω_1 and Ω_2. The *automorphism group* $\mathrm{Aut}(\mathcal{O}) = \mathrm{Aut}_{\mathfrak{C}}(\mathcal{O})$ of $\mathcal{O}$ in the category $\mathfrak{C}$ is then the group of all permutations of Ω commuting with Γ, that is, the centraliser $C_S(G)$ of the *monodromy group* $G = \theta(\Gamma)$ of $\mathcal{O}$ in the symmetric group S. Here we will restrict our attention to the *connected* objects, those for which Γ acts transitively on Ω.

We will concentrate mainly on five particular examples of permutational categories, outlined briefly below (for full details, and other examples, see [9]). In each case, the parent group Γ is either an extended triangle group

$$\Delta[p,q,r] = \langle R_0, R_1, R_2 \mid R_i^2 = (R_1R_2)^p = (R_2R_0)^q = (R_0R_1)^r = 1\rangle,$$

or its orientation-preserving subgroup of index 2, the triangle group

$$\Delta(p,q,r) = \langle X, Y, Z \mid X^p = Y^q = Z^r = XYZ = 1\rangle,$$

where $X = R_1R_2$, $Y = R_2R_0$ and $Z = R_0R_1$. Here $p, q, r \in \mathbb{N} \cup \{\infty\}$, and we ignore any relations of the form $W^\infty = 1$. In what follows, $*$ denotes a free product, C_n denotes a cyclic group of order $n \in \mathbb{N} \cup \{\infty\}$, while V_n is an elementary abelian group of order n, and F_r is a free group of rank r.

1. The category $\mathfrak{M}$ of maps on surfaces (possibly non-orientable or with boundary), that is, embeddings of graphs with simply connected faces, has parent group

$$\Gamma = \Gamma_{\mathfrak{M}} = \Delta[\infty, 2, \infty] \cong V_4 * C_2.$$

This permutes the set Ω of incident vertex-edge-face flags of a map (equivalently, the faces of its barycentric subdivision), with each involution R_i $(i = 0, 1, 2)$ changing the i-dimensional component of each flag (whenever possible) while preserving the other two.

2. The category $\mathfrak{M}^+$ of oriented maps, those in which the underlying surface is oriented and without boundary, has parent group

$$\Gamma = \Gamma_{\mathfrak{M}^+} = \Delta(\infty, 2, \infty) \cong C_\infty * C_2.$$

This group permutes directed edges, with X usng the local orientation to rotate them about their target vertices, and Y reversing their direction, so that Z rotates them around incident faces.

3. There are several ways of defining or representing hypermaps. For our purposes, the most convenient is the Walsh representation as a bipartite map [21], in which the black and white vertices of the embedded graph correspond to the hypervertices and hyperedges of the hypermap, the edges correspond to incidences between them, and the faces correspond to its hyperfaces. The category $\mathfrak{H}$ of all hypermaps, where the underlying surface is unoriented and possibly with boundary, has parent group

$$\Gamma = \Gamma_{\mathfrak{H}} = \Delta[\infty, \infty, \infty] \cong C_2 * C_2 * C_2.$$

This permutes incident edge-face pairs of the bipartite map, with involutions R_0 and R_1 preserving the face and the incident white and black vertex respectively, while R_2 preserves the edge.

4. The category $\mathfrak{H}^+$ of oriented hypermaps, those in which the underlying surface is oriented and without boundary, has parent group

$$\Gamma = \Gamma_{\mathfrak{H}^+} = \Delta(\infty, \infty, \infty) \cong C_\infty * C_\infty \cong F_2.$$

This permutes the edges of the embedded graph, with X and Y using the local orientation to rotate them around their incident black and white vertices, so that Z rotates them around incident faces.

5. The category $\mathfrak{D}$ of dessins d'enfants can be identified with the subcategory of $\mathfrak{H}^+$ consisting of its finite objects, those in which the embedded bipartite graph is finite and the surface is compact. It has the same parent group $\Gamma = \Delta(\infty, \infty, \infty) \cong F_2$ as $\mathcal{H}^+$, permuting edges as before.

If we wish to restrict any of these categories to the subcategory of objects of a particular type (p, q, r), we replace the parent group given above with the corresponding triangle or extended triangle group of that type.

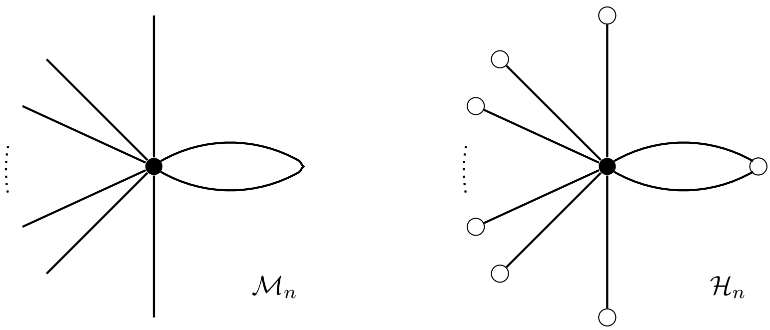

Figure 1: The map $\mathcal{M}_n$ and the hypermap $\mathcal{H}_n$

Example 2.1. The planar map $\mathcal{M}_n \in \mathfrak{M}^+$ $(n \geq 2)$, shown on the left in Figure 1, has one vertex, of valency n, incident with one loop and $n-2$ half edges. It corresponds to the epimorphism $\theta : \Delta(\infty, 2, \infty) \to S_n$ given by

$$X \mapsto x = (1, 2, \dots, n), \qquad Y \mapsto y = (1, 2), \qquad Z \mapsto z = (n, n-1, \dots, 2).$$

This map can be regarded as a hypermap $\mathcal{H}_n$, shown on the right in Figure 1, by adding a white vertex to each edge; this corresponds to composing θ with the natural epimorphism $\Delta(\infty, \infty, \infty) \to \Delta(\infty, 2, \infty)$. The hypermap $\mathcal{H}_n$ has type $(n, 2, n-1)$, and it can be regarded as a member of the category of oriented hypermaps of this type by factoring θ through $\Delta(n, 2, n-1)$. In all cases the monodromy group G is S_n, in its natural representation, and the automorphism group is trivial. However, if we regard $\mathcal{M}_n$ or $\mathcal{H}_n$ as an unoriented map or hypermap, then its monodromy group is $S_n \times S_2$ in its natural product action of degree $2n$, and the automorphism group has order 2, generated by the obvious reflection.

We briefly mention two other classes of permutational categories in which Theorem 1.1 applies. The first concerns abstract polytopes [16], regarded as higher-dimensional generalisations of maps. Those n-polytopes of a particular type, associated with the Schläfli symbol $\{p_1, \dots, p_{n-1}\}$, can be identified with transitive permutation representations of the Coxeter group Γ with presentation

$$\langle R_0, \dots, R_n \mid R_i^2 = (R_{i-1}R_i)^{p_i} = (R_iR_j)^2 = 1 \ (|i-j| > 1)\rangle.$$

For instance, $\Gamma_{\mathfrak{M}}$ is associated with the Schläfli symbol $\{\infty, \infty\}$. However, in higher dimensions, not all transitive representations correspond to abstract polytopes, since the monodromy groups must satisfy the intersection property [16, Proposition 2B10].

The second class of examples concerns covering spaces [15, 17]. Under suitable connectedness assumptions, the (connected, unbranched) coverings $Y \to X$ of a topological space X can be identified with the transitive permutation representations $\theta : \Gamma \to S = \mathrm{Sym}(\Omega)$ of its fundamental group $\Gamma = \pi_1 X$, acting by unique path-lifting on the fibre Ω over a base-point in X. In this case the automorphism group of an object $Y \to X$ in this category is the group of covering transformations, the centraliser in S of the monodromy group $\theta(\Gamma)$ of the covering.

This last example helps to explain the importance of the fifth category listed above, the category $\mathfrak{D}$ of dessins d'enfants. By Belyi's Theorem [2], a compact Riemann surface R is defined (as a projective algebraic curve) over the field $\overline{\mathbb{Q}}$ of algebraic numbers if and only if it admits a non-constant meromorphic function β branched over at most three points of the complex projective line (the Riemann sphere) $\mathbb{P}^1(\mathbb{C}) = \hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. Composing β with a Möbius transformation if necessary, we may assume that its critical values are contained in $\{0, 1, \infty\}$. Such *Belyi functions* β correspond to unbranched finite coverings $R \setminus \beta^{-1}(\{0, 1, \infty\}) \to X$ of the thrice-punctured sphere $X = \hat{\mathbb{C}} \setminus \{0, 1, \infty\}$, and hence to transitive finite permutation representations of its fundamental group $\Gamma = \pi_1 X$; this is a free group of rank 2, freely generated by the homotopy classes of small loops around 0 and 1. The unit interval $[0, 1] \subset \hat{\mathbb{C}}$ lifts to a bipartite graph embedded in R, with black and white vertices over 0 and 1, and face-centres over ∞. Conversely, any finite oriented hypermap, after suitable uniformisation, yields a Riemann surface R defined over $\overline{\mathbb{Q}}$; see [6, 7, 12, 13] for details of these connections, and of the action of the absolute Galois group $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ on dessins.

3 Proof of Theorems 1.1 and 1.2

Let $\mathcal{O}$ be a connected object in a permutational category $\mathfrak{C}$, identified with a transitive permutation representation $\Gamma \to G \le S := \mathrm{Sym}(\Omega)$, so that its automorphism group $\mathrm{Aut}_{\mathfrak{C}}(\mathcal{O})$ is identified with the centraliser $C_S(G)$ of G in S. Then Theorem 1.1 asserts that

$$\mathrm{Aut}_{\mathfrak{C}}(\mathcal{O}) \cong N_G(H)/H \cong N_\Gamma(M)/M,$$

where G is the monodromy group of $\mathcal{O}$, and H and M are point-stabilisers in G and Γ. The second isomorphism follows immediately from the first, and this in turn follows from part (1) of Theorem 1.2, both parts of which we will now prove.

Proof. The centraliser C of G acts semi-regularly (i.e. freely) on Ω. To see this, suppose that $c \in C$ and $\beta c = \beta$ for some $\beta \in \Omega$. Given any $\omega \in \Omega$, there is some $g \in G$ such that $\omega = \beta g$, since G is transitive on Ω. Then $\omega c = (\beta g)c = (\beta c)g = \beta g = \omega$. Thus $c = 1$, as required.

Let $\Phi = \{\beta \in \Omega \mid G_\beta = H\}$, so in particular $\alpha \in \Phi$. Then C leaves Φ invariant, since if $\beta \in \Phi$ and $c \in C$ then for all $h \in H$ we have $(\beta c)h = (\beta h)c = \beta c$, so that $\beta c \in \Phi$.

Let us identify Ω with the set of cosets of H in G in the usual way, identifying each $\omega \in \Omega$ with the unique coset Hx such that $x \in G$ and $\alpha x = \omega$. Thus α is identified with H itself, and G acts on the cosets by $g : Hx \mapsto Hxg$.

Then Φ is identified with the set of cosets of H in $N := N_G(H)$. To see this, let $\omega \in \Omega$ correspond to a coset Hx of H in G. First suppose that $x = n \in N$. Then $(Hn)h = (nH)h = n(Hh) = nH = Hn$ for all $h \in H$, so the coset Hn is fixed by H, giving $H \le G_\omega$, while if $g \in G_\omega$ then $Hng = Hn$, so $H = H^n = H^n g = Hg$ and hence $g \in H$, giving $G_\omega \leq H$. Thus $G_\omega = H$ and hence $\omega \in \Phi$. Conversely, suppose that $\omega \in \Phi$. Then $G_\omega = H$, so $Hxg = Hx$ if and only if $g \in H$, that is, $H^x g = H^x$ if and only if $g \in H$, so $H^x = H$ and hence $x \in N$.

Let us define a new action of N on Ω (now regarded as the set of cosets of H in G) by $n : Hx \mapsto n^{-1}Hx = Hn^{-1}x$ for all $n \in N$ and $x \in G$. If $n_1, n_2 \in N$ then n_1, followed by n_2, sends Hx to $n_2^{-1}n_1^{-1}Hx = (n_1n_2)^{-1}Hx$, as does n_1n_2, so this is indeed a group action of N. It commutes with the action of G on Ω, since $n^{-1}(Hxg) = (n^{-1}Hx)g$ for all $n \in N$ and $x, g \in G$, so it defines a homomorphism $\theta : N \to C$. In particular, this action preserves Φ, since C does.

The induced action of N on Φ is transitive, since if $n', n'' \in N$ then the element $n = n'(n'')^{-1} \in N$ sends Hn' to $n^{-1}Hn' = Hn^{-1}n' = Hn''$. Thus $\theta(N)$ acts on Φ as a transitive subgroup of C. But C acts semi-regularly on Φ, so it has no transitive proper subgroups. Therefore θ is an epimorphism, and C acts regularly on Φ, giving (2).

In this action of N, we have $n^{-1}Hx = Hx$ if and only if $n \in H$, so the subgroup stabilising any coset Hx is H, which is therefore the kernel $\ker(\theta)$ of this action of N. The First Isomorphism Theorem therefore gives $N/H \cong C$, so (1) is proved. □

Remark 3.1. The most symmetric objects in $\mathfrak{C}$ are the *regular* objects, those for which $\mathrm{Aut}_{\mathfrak{C}}(\mathcal{O})$ acts transitively on Ω. By Theorem 1.2(2) this is equivalent to $\Phi = \Omega$, that is, to $H = 1$, meaning that G acts regularly on Ω. This is also equivalent to M being a normal subgroup of Γ. Then $G \cong C \cong \Gamma/M$, and G and C can be identified with the right and left regular representations of the same group. (In fact, if G is abelian then $C = G$.)

Remark 3.2. A variety of groups is a class of groups defined by identical relations between their elements (see [18]); simple examples include groups of exponent dividing n, solvable groups of derived length at most l, and nilpotent groups of class at most c. The fact that the centraliser C of a transitive permutation group G can be realised within G as $N_G(H)/H$ means that if G is a member of a variety $\mathfrak{V}$, then so is C. This may seem surprising since, as subgroups of the symmetric group S, the groups C and G could have a very small intersection: for instance, $C \cap G = 1$ if, as in many cases, G has a trivial centre. Of course, this fact applies to permutational categories: if the parent group Γ is in $\mathfrak{V}$ then the automorphism group of each connected object is also in $\mathfrak{V}$. However, for most of the examples we are interested in, Γ generates the variety of all groups, and the only restrictions on automorphism groups are the obvious ones imposed by cardinality, as shown in [10].

4 An alternative form of Theorem 1.2(2)

One sometimes finds part (2) of Theorem 1.2 stated in the following alternative form:

$(2')$ C acts regularly on the set of fixed points of H in Ω.

This is equivalent to (2) in cases where Ω is finite, or more generally where H is finite, so that an inclusion $H = H_\alpha \leq H_\beta$ between conjugate subgroups is equivalent to their equality. However, $(2')$ can be false if H is infinite, as shown by the following example.

Example 4.1. Although our aim here is mainly group-theoretic, to construct a permutation group G on a set Ω such that condition $(2')$ fails, the original motivation, as in much of this paper, is combinatorial. It may therefore be useful, throughout this construction, to think of G as the monodromy group of an oriented hypermap $\mathcal{H}$, regarded as a bipartite map, with elements of Ω as edges, and cycles of the generators a and b of G as black and white vertices (see Remark 4.2).

Let G be the Baumslag–Solitar group [1]

$$G = BS(1,2) = \langle a, b \mid a^b = a^2 \rangle.$$

This is a semidirect product $G = A \rtimes B$, where $B = \langle b \rangle \cong C_\infty$, and A is the normal closure of a in G, an abelian group of countably infinite rank, generated by the conjugates

$$a_i := a^{2^i} = a^{b^i} \quad (i \in \mathbb{Z})$$

of a, with $a_i^2 = a_{i+1}$ for all $i \in \mathbb{Z}$. This subgroup A can be identified with the additive group of the ring $\mathbb{Z}[\frac{1}{2}]$, with each finite product $\prod_i a_i^{e_i}$ in A corresponding to $\sum_i e_i 2^i \in \mathbb{Z}[\frac{1}{2}]$. Thus $a = a_0$ corresponds to the element $1 \in \mathbb{Z}[\frac{1}{2}]$, and b, acting by conjugation on A as the automorphism $a_i \mapsto a_{i+1}$, acts on $\mathbb{Z}[\frac{1}{2}]$ by $t \mapsto 2t$. In particular, the subgroup $H := \langle a \rangle$ has conjugate subgroups $H_i := H^{b^{-i}} = \langle a^{2^{-i}} \rangle$ for all $i \in \mathbb{Z}$, with a chain of index 2 inclusions

$$\cdots < H_{-2} < H_{-1} < H_0 \ (= H) < H_1 < H_2 < \cdots .$$

Now let $\mathbb{T}$ be the Sylow 2-subgroup of $\mathbb{Q}/\mathbb{Z}$, that is,

$$\mathbb{T} := \mathbb{Z}[\tfrac{1}{2}]/\mathbb{Z} \cong C_{2^\infty} := \bigcup_{e \geq 0} C_{2^e}$$

where C_{2^e} corresponds to the subgroup generated by the image of 2^{-e} in $\mathbb{Z}[\frac{1}{2}]/\mathbb{Z}$. Let

$$\Omega = \mathbb{T} \times \mathbb{Z} = \bigcup_{i \in \mathbb{Z}} \Omega_i,$$

the disjoint union of countably many copies $\Omega_i = \mathbb{T} \times \{i\}$ of $\mathbb{T}$. Let the generators a and b of G act on Ω by

$$a : (t, i) \mapsto (t + 2^i, i) \quad \text{and} \quad b : (t, i) \mapsto (t, i - 1)$$

for $t \in \mathbb{T}$ and $i \in \mathbb{Z}$, where we interpret $t + 2^i$ as an element of $\mathbb{T}$ in the obvious way. Thus a preserves each set Ω_i, fixing it pointwise for $i \geq 0$, and with cycles of length 2^{-i} on it for $i < 0$, while b induces the obvious bijection $\Omega_i \to \Omega_{i-1}$. Then the element $a^b = b^{-1}ab$ acts on Ω by a composition of three permutations

$$a^b : (t, i) \mapsto (t, i + 1) \mapsto (t + 2^{i+1}, i + 1) \mapsto (t + 2^{i+1}, i).$$

This has the same effect as

$$a^2 : (t, i) \mapsto (t + 2^{i+1}, i),$$

so we have a group action of G on Ω.

It is easy to see from the decomposition $G = A \rtimes B$ that G acts faithfully and transitively on Ω, that the stabiliser of the element $\alpha = (0, 0) \in \Omega$ is $H = \langle a \rangle$, and that $N_G(H) = A$, so that

$$N_G(H)/H = A/H \cong \mathbb{T}.$$

We now calculate

$$C := C_S(G) = C_S(A) \cap C_S(B),$$

where $S := \mathrm{Sym}(\Omega)$. First note that $C_S(A)$ must permute the orbits of A, which are the sets Ω_i, and must do so trivially since A has a different representation, with kernel $H_i = \langle a^{2^{-i}} \rangle$, on each Ω_i. Since A induces the regular representation of the abelian group A/H_i on Ω_i, it follows that $C_S(A)$ must be the cartesian product $\prod_{i \in \mathbb{Z}} A/H_i$ of the groups A/H_i, each factor A/H_i acting regularly on Ω_i and fixing $\Omega \setminus \Omega_i$. Even though the subgroups H_i are all distinct, we have $A/H_i \cong \mathbb{T}$ for all $i \in \mathbb{Z}$, so $C_S(A) \cong \mathbb{T}^{\mathbb{Z}}$. The only elements of $C_S(A)$ commuting with b are those corresponding to elements of the diagonal subgroup of $\mathbb{T}^{\mathbb{Z}}$, inducing the same permutation on each subset Ω_i. These form a group isomorphic to $\mathbb{T}$, proving that

$$C \cong N_G(H)/H.$$

Thus G satisfies Theorem 1.2(1), but what about statements (2) and $(2')$? In this example, the orbits of C are the sets Ω_i, each permuted regularly by C, while the subset of Ω fixed by H is the disjoint union

$$\bigcup_{i \geq 0} \Omega_i$$

of infinitely many of these orbits. Thus G does not satisfy statement $(2')$. However, it does satisfy (2) since the points with stabiliser H are those in Ω_0, forming a regular orbit of C; the points in Ω_i for $i > 0$, although they are also fixed by H, have stabiliser H_i properly containing H.

Remark 4.2. This example also gives a construction of the oriented hypermap $\mathcal{H}$ with monodromy group G, corresponding to the epimorphism $\theta : F_2 \to G$, $X \mapsto a$, $Y \mapsto b$. We can identify the edges with the elements (t, i) of $\Omega = \mathbb{T} \times \mathbb{Z} = \cup_i \Omega_i$. The white vertices, corresponding to the cycles of b, can be identified with the elements t of $\mathbb{T}$, each incident with the edges (t, i) for $i \in \mathbb{Z}$ in decreasing order of i as one follows the local orientation. The black vertices correspond to the cycles of a; those in Ω_i for $i \geq 0$ are fixed points (t, i) of a, so they have valency 1, each of them connected by the edge (t, i) to the white vertex t; those in Ω_i for $i < 0$, also denoted by (t, i), correspond to cycles $\{(t + k2^i, i) \mid k = 0, 1, \ldots, 2^{-i} - 1\}$ of length 2^{-i}, so they have valency 2^{-i} and are connected by edges $(t + k2^i, i)$ to the white vertices $t + k2^i$ in cyclic order of increasing k. The two faces incident with an edge (t, i) have vertices, following the orientation and in alternate colours, given by

$$\ldots,\ (t + 2^i, i - 1),\ t + 2^i,\ (t, i),\ t,\ (t, i + 1),\ t - 2^{i+1},\ \ldots$$

and

$$\ldots, t + 2^{i-1},\ (t, i - 1),\ t,\ (t, i),\ t - 2^i,\ (t - 2^i, i + 1),\ \ldots .$$

The group $\mathrm{Aut}_{\mathfrak{H}^+}(\mathcal{H})$ can be identified with $\mathbb{T}$: each $t_0 \in \mathbb{T}$ acts on edges of $\mathcal{H}$ by $(t, i) \mapsto (t + t_0, i)$, with the obvious induced actions on black and white vertices and faces. Thus it acts transitively on white vertices and on faces, whereas its orbits on black vertices and on edges consist of those in each subset Ω_i for $i \in \mathbb{Z}$. In particular, it acts regularly on edges (and on black vertices) in Ω_0, those with stabiliser H in G, but it is intransitive on those with $i \geq 0$, fixed by H.

It is not easy to visualise this hypermap $\mathcal{H}$. An alternative approach is to regard it as a regular branched covering, with covering group $\mathbb{T}$, of the quotient hypermap $\overline{\mathcal{H}} = \mathcal{H}/\mathrm{Aut}_{\mathfrak{H}^+}(\mathcal{H})$ corresponding to the epimorphism $F_2 \to G^{\mathrm{ab}} = G/A \cong \mathbb{Z}$. As a bipartite map this is planar, with one face and one white vertex incident with 1-valent black vertices

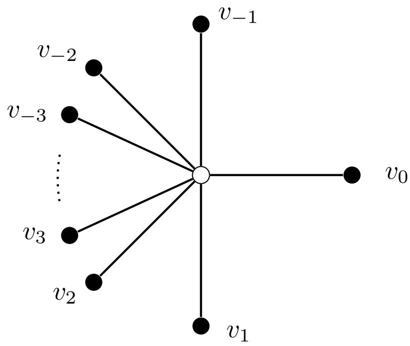

Figure 2: The hypermap $\overline{\mathcal{H}}$

v_i in decreasing order of $i \in \mathbb{Z}$ (see Figure 2). The covering $\mathcal{H} \to \overline{\mathcal{H}}$ is branched only at the vertices v_i for $i < 0$, where the local monodromy permutation has infinitely many cycles of length 2^{-i}, corresponding to the cycles of b on Ω_i. The automorphism group of $\mathcal{H}$ can be identified with the group of covering transformations, acting regularly on the sheets of the covering.

Remark 4.3. There is an obvious generalisation of this example based on the Baumslag–Solitar group $G = BS(1, q) = \langle a, b \mid a^b = a^q \rangle$ for an arbitrary integer $q \neq 0, \pm 1$. See [8] for a discussion of the oriented hypermaps associated with the Baumslag–Solitar groups $BS(p, q) = \langle a, b \mid (a^p)^b = a^q \rangle$ for arbitrary $p, q \neq 0$.

5 Primitive monodromy groups

If G is a permutation group on a set Ω, then the relation $G_\alpha = G_\beta$, appearing in Theorem 1.2 via the definition of Φ, is a G-invariant equivalence relation on Ω, and its equivalence classes are the orbits of the centraliser C of G. Recall that a permutation group is *primitive* if it preserves no non-trivial equivalence relation; equivalently, the point-stabilisers are maximal subgroups. As an immediate consequence of Theorem 1.2, we hav

Corollary 5.1. *If G is a primitive permutation group, then either $G \cong C_p$, acting regularly, for some prime p, with centraliser $C = G$, or the centraliser C of G is the trivial group.*

Proof. The equivalence relation $G_\alpha = G_\beta$ on Ω must be either the identity or the universal relation. In the first case the equivalence classes are singletons, so $|C| = 1$. In the second case $G_\alpha = \{1\}$; this is a maximal subgroup of G, so $G \cong C_p$ for some prime p, with $C = G$. □

Corollary 5.2. *In a permutational category $\mathfrak{C}$, if the monodromy group G of an object $\mathcal{O}$ is a primitive permutation group, then either $\mathcal{O}$ is regular, with $\mathrm{Aut}_{\mathfrak{C}}(\mathcal{O}) = G \cong C_p$ for some prime p, or $\mathrm{Aut}_{\mathfrak{C}}(\mathcal{O})$ is the trivial group.*

Of course, there are many examples of primitive permutation groups, either sporadic or members of infinite families: just represent a group on the cosets of a maximal subgroup. On the other hand, Cameron, Neumann and Teague [3] have shown that for a set of integers

$n \in \mathbb{N}$ of asymptotic density 1 the only primitive groups of degree n are the alternating and symmetric groups A_n and S_n.

Example 5.3. The symmetric and alternating groups, in their natural representationas, arise quite frequently as monodromy groups in various categories. For instance, let $\mathfrak{C} = \mathfrak{H}^+$, the category of oriented hypermaps, with parent group $\Gamma = F_2 = \langle X, Y \mid - \rangle$. A theorem of Dixon [4] states that a randomly chosen pair of permutations $x, y \in S_n$ generate S_n or A_n with probability approaching $3/4$ or $1/4$ as $n \to \infty$, so in that sense 'most' finite objects in this category have a symmetric or alternating monodromy group, and a trivial automorphism group.

In most of the permutational categories of current interest, it is simple to describe the regular objects with automorphism group C_p for each prime p; apart from these exceptions, objects with a primitive monodromy group have a trivial automorphism group. The exceptions correspond to the normal subgroups of index p in the parent group Γ, or equivalently to the maximal subgroups in the elementary abelian p-group $\Gamma/\Gamma'\Gamma^p$, where Γ' and Γ^P are the subgroups of Γ generated by the commutators and p-th powers. In the categories listed in §2, they are as follows.

If $\mathfrak{C} = \mathfrak{H}^+$ or $\mathfrak{D}$ then $\Gamma = F_2$, so $\Gamma/\Gamma'\Gamma^p \cong C_p \times C_p$, with $p+1$ maximal subgroups. Of the corresponding oriented hypermaps, three have type a permutation of $(p, p, 1)$ and are planar, while the remaining $p-2$ have type (p, p, p) and genus $(p-1)/2$. As dessins, the former are on the Riemann sphere, with Belyi functions $\beta : z \mapsto z^p$, $1/(1-z^p)$ and $1 - z^{-p}$, and automorphisms $z \mapsto \zeta z$ where $\zeta^p = 1$. The latter are on Lefschetz curves $y^p = x^u(x-1)$ for $u = 1, \ldots, p-2$, each with a Belyi function $\beta : (x, y) \mapsto x$ and automorphisms $(x, y) \mapsto (x, \zeta y)$ where $\zeta^p = 1$ (see [12, Example 5.6]). The four dessins for $p = 3$ are shown in Figure 3; in the dessin on the right, opposite sides of the hexagon are identified to form a torus.

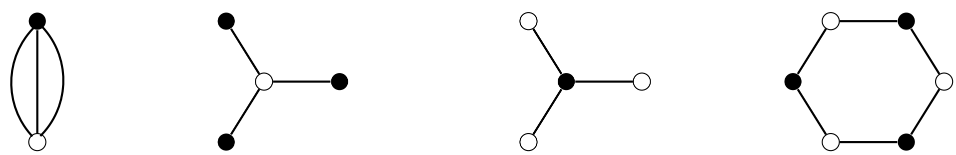

Figure 3: The four dessins with primitive monodromy group C_p, $p = 3$

If $\mathfrak{C} = \mathfrak{M}^+$ then $\Gamma = C_\infty * C_2$, so $\Gamma/\Gamma'\Gamma^p \cong V_4$ or C_p as $p = 2$ or $p > 2$, giving three oriented maps or one, all planar. Their types are the three permutations of $(2, 2, 1)$, together with $(p, 1, p)$. They are shown, for $p = 2$ and 3, in Figure 4.

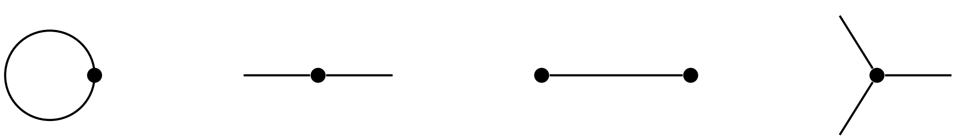

Figure 4: The four oriented maps with primitive monodromy group C_p, $p = 2, 3$

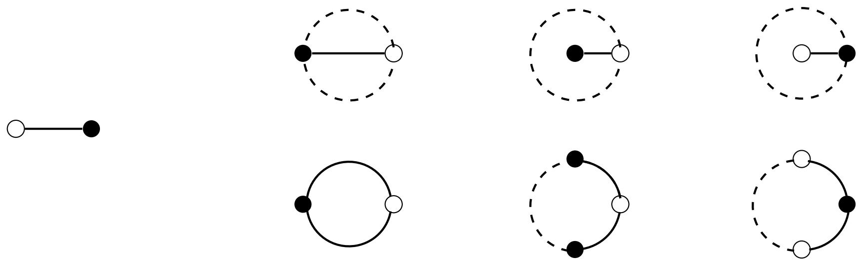

Figure 5: The seven hypermaps with primitive monodromy group C_2

If $\mathfrak{C} = \mathfrak{H}$ or $\mathfrak{M}$ then $\Gamma = C_2 * C_2 * C_2$ or $V_4 * C_2$, so in either case $\Gamma/\Gamma'\Gamma^p \cong V_8$ or 1 as $p = 2$ or $p > 2$. If $p = 2$ there are seven hypermaps and seven maps; if $p > 2$ there are none. The hypermaps are shown in Figure 5. The hypermap on the left is planar, while the other six are on the closed disc, shown by a broken line. The seven maps can be obtained from these hypermaps by ignoring all the white vertices.

If X is a compact orientable surface of genus g then there are $(p^{2g}-1)/(p-1)$ regular coverings $Y \to X$ with monodromy group C_p, corresponding to the normal subgroups of index p in $\pi_1 X = \langle A_i, B_i \ (i = 1, \ldots, g) \mid \prod_i [A_i, B_i] = 1\rangle$; the surfaces Y are orientable, of genus $1 + p(g-1)$. In the non-orientable case, with $\pi_1 X = \langle R_i \ (i = 1, \ldots, g) \mid \prod_i R_i^2 = 1\rangle$ there are $(p^{g-1}-1)/(p-1)$ or $2^g - 1$ such coverings as $p > 2$ or $p = 2$; the surfaces Y are all non-orientable, of genus $2 + p(g-2)$, apart from the orientable double cover of X for $p = 2$, which has genus $g - 1$.

6 Automorphism groups of non-connected objects

It is often convenient to restrict attention to the connected objects in a category, as we have done so far. Here we will briefly show how Theorem 1.1 extends to non-connected objects.

If $\mathcal{C}$ is a permutational category with parent group Γ, then the connected components $\mathcal{O}_i$ $(i \in I)$ of an object $\mathcal{O}$ in $\mathfrak{C}$ correspond bijectively to the orbits Ω_i $(i \in I)$ of Γ on the set Ω associated with $\mathcal{O}$. As before, $\mathrm{Aut}_{\mathfrak{C}}(\mathcal{O})$ is isomorphic to the centraliser C of Γ in $S = \mathrm{Sym}(\Omega)$. In order to describe the structure of C in general, we first consider two extreme cases.

Suppose first that the components $\mathcal{O}_i$ are mutually non-isomorphic. This is equivalent to the point stabilisers $M_i \le \Gamma$ for different orbits Ω_i being mutually non-conjugate in Γ. Then C is the cartesian product of the centralisers $C_i \le \mathrm{Sym}(\Omega_i)$ of Γ on the sets Ω_i. By the transitivity of Γ on Ω_i, we have $C_i \cong N_{G_i}(H_i) \cong N_\Gamma(M_i)/M_i$ for each $i \in I$, where G_i is the permutation group induced by Γ on Ω_i, and H_i is a point stabiliser in G_i for this action.

At the other extreme, suppose that the components $\mathcal{O}_i$ are all isomorphic, or equivalently the point stabilisers M_i are conjugate to each other. Then C is the wreath product $C_i \wr \mathrm{Sym}(I)$ of C_i by $\mathrm{Sym}(I)$. This is a semidirect product, in which the normal subgroup is the cartesian product of the mutually isomorphic groups C_i $(\cong N_{G_i}(H_i) \cong N_\Gamma(M_i)/M_i)$ for $i \in I$, and the complement is $\mathrm{Sym}(I)$, acting on this normal subgroup by permuting the factors C_i via isomorphisms between them.

We can now describe the general form of C by combining these two constructions.

We first partition the set of components of $\mathcal{O}$ into maximal sets $\{\mathcal{O}_{ij} | i \in I_j\}$ $(j \in J)$ of mutually isomorphic objects $\mathcal{O}_{ij}$, each subset indexed by a set I_j. We then define C_{ij} $(\cong N_{G_{ij}}(H_{ij}) \cong N_\Gamma(M_{ij})/M_{ij}$ with obvious notation) to be the centraliser of Γ in $\mathrm{Sym}(\Omega_{ij})$. Then C, and hence also $\mathrm{Aut}_{\mathfrak{C}}(\mathcal{O})$, is the cartesian product over all $j \in J$ of the wreath products $C_{ij} \wr \mathrm{Sym}(I_j)$ where $i \in I_j$. This is again a semidirect product, where the normal subgroup is the cartesian product of all the groups C_{ij} $(i \in I_j, j \in J)$, and the complement is the cartesian product of the groups $\mathrm{Sym}(I_j)$ $(j \in J)$, each factor $\mathrm{Sym}(I_j)$ of the latter acting on the normal subgroup by permuting the factors C_{ij} for $i \in I_j$ while fixing all other factors.

This description can be used to determine the cardinality of $\mathrm{Aut}_{\mathfrak{C}}(\mathcal{O})$. We will restrict attention to categories where the parent group Γ is countable (for instance, where it is finitely generated), since this condition is satisfied by most of the examples studied; the modifications required for an uncountable parent group are straightforward. By Theorem 1.1 this implies that $\mathrm{Aut}_{\mathfrak{C}}(\mathcal{O})$ is also countable for each connected object $\mathcal{O}$. Since a cartesian product of infinitely many non-trivial groups is uncountable, as is the symmetric group on any infinite set, the following is clear:

Theorem 6.1. *Let $\mathfrak{C}$ be a permutational category for which the parent group Γ is countable, and let $\mathcal{O}$ be an object in $\mathfrak{C}$ with connected components $\mathcal{O}_{ij}$, indexed by sets I_j $(j \in J)$ as above. Then*

1. $|\mathrm{Aut}_{\mathfrak{C}}(\mathcal{O})| > \aleph_0$ *if and only if either $\mathcal{O}$ has infinitely many components $\mathcal{O}_{ij}$ with $|\mathrm{Aut}_{\mathfrak{C}}(\mathcal{O}_{ij})| > 1$, or at least one set I_j is infinite;*
2. $|\mathrm{Aut}_{\mathfrak{C}}(\mathcal{O})| = \aleph_0$ *if and only if $\mathcal{O}$ has only finitely many components $\mathcal{O}_{ij}$ with $|\mathrm{Aut}_{\mathfrak{C}}(\mathcal{O}_{ij})| > 1$, each set I_j is finite, and $\mathrm{Aut}_{\mathfrak{C}}(\mathcal{O}_{ij})$ is infinite for some component $\mathcal{O}_{ij}$;*
3. $|\mathrm{Aut}_{\mathfrak{C}}(\mathcal{O})| < \aleph_0$ *if and only if $\mathcal{O}$ has only finitely many components $\mathcal{O}_{ij}$ with $|\mathrm{Aut}_{\mathfrak{C}}(\mathcal{O}_{ij})| > 1$, each set I_j is finite, and $\mathrm{Aut}_{\mathfrak{C}}(\mathcal{O}_{ij})$ is finite for each component $\mathcal{O}_{ij}$.*

Corollary 6.2. *In Case (3), where $\mathrm{Aut}_{\mathfrak{C}}(\mathcal{O})$ is finite, it has order*

$$\prod_{j \in J} |\mathrm{Aut}_{\mathfrak{C}}(\mathcal{O}_{ij})|^{|I_j|} |I_j|!\,.$$

Example 6.3. Let $\mathfrak{C} = \mathfrak{M}^+$, the category of oriented maps, which has parent group $\Gamma = \Delta(\infty, 2, \infty) = \langle X, Y \mid Y^2 = 1 \rangle$. For each integer $n \geq 2$ let $\tilde{\mathcal{M}}_n$ be the minimal regular cover of the map $\mathcal{M}_n \in \mathfrak{M}^+$ in Figure 1. This is a regular oriented map with automorphism and monodromy group S_n (in its regular representation) corresponding to the epimorphism $\Gamma \to S_n, X \mapsto (1, 2, \ldots, n), Y \mapsto (1, 2)$. If we take $\mathcal{M}$ to be the disjoint union of these maps $\tilde{\mathcal{M}}_n$, then $\mathrm{Aut}_{\mathfrak{M}^+}(\mathcal{M})$ is the cartesian product $\prod_{n \geq 2} S_n$. This uncountable group is very rich in subgroups: for instance, every finitely generated residually finite group (such as every finitely generated linear group, by Mal'cev's Theorem [14]) can be embedded in a cartesian product of finite groups of distinct orders, and hence (by Cayley's Theorem) can be embedded in $\mathrm{Aut}_{\mathfrak{M}^+}(\mathcal{M})$.

7 Morphisms

Although this paper has concentrated on automorphisms, Theorem 1.1 can be extended to describe the morphisms $\mathcal{O}_1 \to \mathcal{O}_2$ between connected objects. If each $\mathcal{O}_i$ corresponds to an action of the parent group Γ on Ω_i, with stabiliser M_i, then such a morphism exists if and only if M_1 is conjugate to a subgroup of M_2. The set $B := \{b \in \Gamma \mid M_1^b \leq M_2\}$ is a union of cosets bM_2 $(b \in B)$, and these correspond bijectively to the morphisms $\mathcal{O}_1 \to \mathcal{O}_2$. Specifically, if we identify elements of Ω_i with cosets $M_i g$ $(g \in \Gamma)$, then each $b \in B$ corresponds to the morphism $\phi_b : M_1 g \mapsto M_2 b^{-1} g$, where $\phi_b = \phi_{b'}$ if and only if $b' \in bM_2$. The proof of this in [11, Theorem 3.5] for oriented maps extends easily to all permutational categories. This shows that morphisms between connected objects are always surjective, but Example 5 shows that endomorphisms $\mathcal{O} \to \mathcal{O}$ of infinite objects need not be automorphisms. This also shows that the number of morphisms $\mathcal{O}_1 \to \mathcal{O}_2$ is bounded above by $|\Omega_2| = |\Gamma : M_2|$, attained if (and only if, when $\mathcal{O}_2$ is finite) M_1 is contained in the core of M_2.

Example 7.1. If $\mathcal{O}_i$ is regular for $i = 1$ or 2, so that M_i is a normal subgroup of Γ, then $B = \Gamma$ or $\emptyset$ as $M_1 \leq M_2$ or not, and there are respectively $|\Omega_2|$ morphisms $\mathcal{O}_1 \to \mathcal{O}_2$ or none.

There is an action of $\mathrm{Aut}_{\mathfrak{C}}(\mathcal{O}_1) \times \mathrm{Aut}_{\mathfrak{C}}(\mathcal{O}_2)$ on the set $\mathrm{Mor}_{\mathfrak{C}}(\mathcal{O}_1, \mathcal{O}_2)$ of morphisms $\mathcal{O}_1 \to \mathcal{O}_2$, given by $(\theta_1, \theta_2) : \phi \mapsto \theta_1^{-1} \circ \phi \circ \theta_2$. To realise this action concretely, note that the action $(n_1, n_2) : b \mapsto n_1^{-1} b n_2$ of $N_\Gamma(M_1) \times N_\Gamma(M_2)$ on B yields an induced action on the set of orbits $bM_2 \subseteq B$ of its normal subgroup $1 \times M_2$, with $M_1 \times M_2$ in the kernel; the resulting action of $(N_\Gamma(M_1) \times N_\Gamma(M_2))/(M_1 \times M_2) \cong N_\Gamma(M_1)/M_1 \times N_\Gamma(M_2)/M_2$ on these orbits is equivalent to the action of $\mathrm{Aut}_{\mathfrak{C}}(\mathcal{O}_1) \times \mathrm{Aut}_{\mathfrak{C}}(\mathcal{O}_2)$ on $\mathrm{Mor}_{\mathfrak{C}}(\mathcal{O}_1, \mathcal{O}_2)$. Thus if θ_i is given by $M_i g \mapsto n_i^{-1} M_i g$ for $i = 1, 2$, then $(\theta_1, \theta_2) : \phi_b \mapsto \theta_1^{-1} \circ \phi_b \circ \theta_2 = \phi_{b'}$, where $b' = n_1^{-1} b n_2 \in B$.

The subgroup of $\mathrm{Aut}_{\mathfrak{C}}(\mathcal{O}_1) \times \mathrm{Aut}_{\mathfrak{C}}(\mathcal{O}_2)$ fixing a morphism ϕ consists of those pairs (θ_1, θ_2) such that $\theta_1 \circ \phi = \phi \circ \theta_2$, that is, ϕ lifts θ_2 to θ_1. In particular, the subgroup of $\mathrm{Aut}_{\mathfrak{C}}(\mathcal{O}_1)$ fixing a morphism ϕ consists of the covering transformations of ϕ. Similarly, the action of $\mathrm{Aut}_{\mathfrak{C}}(\mathcal{O}_2)$ on $\mathrm{Mor}_{\mathfrak{C}}(\mathcal{O}_1, \mathcal{O}_2)$ is equivalent to its action on Ω_2: it is always semi-regular, and regular if and only if $\mathcal{O}_2$ is regular.

ORCID iDs

Gareth Aneurin Jones https://orcid.org/0000-0002-7082-7025

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Reflexible complete regular dessins and antibalanced skew morphisms of cyclic groups

Kan Hu*

Department of Mathematics, Zhejiang Ocean University, Zhoushan, Zhejiang 316022, P.R. China

Young Soo Kwon†

Department of Mathematics, Yeungnam University, Kyeongsan, 712-749, Republic of Korea

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Abstract

A skew morphism of a finite group A is a bijection φ on A fixing the identity element of A and for which there exists an integer-valued function π on A such that $\varphi(ab) = \varphi(a)\varphi^{\pi(a)}(b)$, for all $a, b \in A$. In addition, if $\varphi^{-1}(a) = \varphi(a^{-1})^{-1}$, for all $a \in A$, then φ is called antibalanced. In this paper we develop a general theory of antibalanced skew morphisms and establish a one-to-one correspondence between reciprocal pairs of antibalanced skew morphisms of the cyclic additive groups and isomorphism classes of reflexible regular dessins with complete bipartite underlying graphs. As an application, reflexible complete regular dessins are classified.

Keywords: Graph embedding, antibalanced skew morphism, reciprocal pair.

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1 Introduction

A *map* $\mathcal{M}$ is a 2-cell embedding $i : \Gamma \hookrightarrow \mathcal{S}$ of a connected graph Γ, possibly with loops or multiple edges, into a closed surface $\mathcal{S}$ such that each component of $\mathcal{S} \setminus i(\Gamma)$ is homeomorphic to an open disc. A map is *orientable* if its supporting surface $\mathcal{S}$ is orientable,

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E-mail addresses: hukan@zjou.edu.cn (Kan Hu), ysookwon@ynu.ac.kr (Young Soo Kwon)

otherwise it is *non-orientable*. Throughout the paper, maps considered are all orientable. A map with a 2-coloured bipartite underlying graph is called a *dessin*. An automorphism of a dessin $\mathcal{D}$ is a permutation of the edges of the underlying bipartite graph which preserves the graph structure and vertex colouring, and extends to an orientation-preserving self-homeomorphism of the supporting surface. It is well known that the automorphism group of a dessin acts semi-regularly on its edges. In the case where this action is transitive, and hence regular, the dessin is called *regular* as well.

A regular dessin is *reflexible* if it is isomorphic to its mirror image, otherwise it is called *chiral*. Moreover, a regular dessin is *symmetric* if it has an external symmetry transposing the vertex colors. Thus, a symmetric regular dessin may be viewed as a regular map, that is, a map whose orientation-preserving automorphism group acts transitively on the arcs.

A regular dessin is *complete* if its underlying graph is the complete bipartite graph $K_{m,n}$. Due to its important connection to generalized Fermat curves, the classification problem of complete regular dessins has attracted much attention. A full classification of the symmetric complete regular dessins was obtained in a series of papers [8, 9, 17, 18, 19, 22]. For the general case, complete bipartite graphs which underly a unique regular dessin were determined by Fan and Li [10], and complete regular dessins of odd prime power order have been recently classified by Hu, Nedela and Wang [13]. These results were proved by group-theoretic methods through a translation of complete regular dessins to exact bicyclic groups with two distinguished generators.

Recently, Feng et al discovered an alternative approach to this problem by establishing a surprising correspondence between complete regular dessins and reciprocal pairs of skew morphisms of cyclic groups [12]. A skew morphism of a finite group A is a bijection φ on A fixing the identity element of A and for which there exists an integer function π on A such that $\varphi(ab) = \varphi(a)\varphi^{\pi(a)}(b)$, for all $a, b \in A$. Suppose that φ and $\tilde{\varphi}$ are a pair of skew morphisms of the cyclic additive groups $\mathbb{Z}_n$ and $\mathbb{Z}_m$, and π and $\tilde{\pi}$ are associated power functions, respectively. The skew morphism pair $(\varphi, \tilde{\varphi})$ is called *reciprocal* if they satisfy the following conditions:

(a) the order of φ divides m and the order of $\tilde{\varphi}$ divides n,

(b) $\pi(x) \equiv \tilde{\varphi}^x(1) \pmod{|\varphi|}$ and $\tilde{\pi}(y) \equiv \varphi^y(1) \pmod{|\tilde{\varphi}|}$ for all $x \in \mathbb{Z}_n$ and $y \in \mathbb{Z}_m$.

In [12, Theorem 5] the authors proved that the isomorphism classes of complete regular dessins with underlying graphs $K_{m,n}$ are in one-to-one correspondence with the reciprocal pairs of skew morphisms of the cyclic groups $\mathbb{Z}_n$ and $\mathbb{Z}_m$.

The aim of this paper is to classify the reflexible complete regular dessins. Employing methods used in [12] we are led naturally to introduce a new concept of antibalanced skew morphism. More precisely, a skew morphism of a finite group A is *antibalanced* if $\varphi^{-1}(a) = \varphi(a^{-1})^{-1}$, for all $a \in A$. We note that there is a big difference between antibalanced skew morphisms and skew morphisms arising from antibanlanced regular Cayley maps, because the latter have a generating orbit which is closed under taking inverses [4, 16], while the former do not have such a restriction.

In Section 3 we develop a general theory of antibalanced skew morphisms, and present a classification and enumeration of antibalanced skew morphisms of cyclic groups, extending the results obtained by Conder, Jajcay and Tucker [4, Theorem 7.1]. In Section 4, we establish a one-to-one correspondence between reflexible complete regular dessins and reciprocal pairs of antibalanced skew morphisms of cyclic groups. In Section 5 all reciprocal pairs of antibalanced skew morphisms of cyclic groups are completely determined.

2 Preliminaries

The theory of skew morphisms has been developed and expanded by various authors. In this section we summarize and prove some preliminary results for future reference.

Let φ be a skew morphism of a finite group A, and let π be a power function of φ. In general, the function π is not uniquely determined by φ. However, if φ has order k, then π may be regarded as a function $\pi : A \to \mathbb{Z}_k$, which is unique. In this case we will refer to π as *the power function of* φ. A subgroup N of A is *φ-invariant* if $\varphi(N) = N$, in which case the restriction of φ to N is a skew morphism of N. Moreover, it is well known [16] that $\operatorname{Ker}\varphi$ and $\operatorname{Fix}\varphi$ defined by

$$\operatorname{Ker}\varphi = \{a \in A \mid \pi(a) = 1\} \quad \text{and} \quad \operatorname{Fix}\varphi = \{a \in A \mid \varphi(a) = a\}$$

are subgroups of A, and in particular, $\operatorname{Fix}\varphi$ is φ-invariant. Note that, for any two elements $a, b \in A$, $\pi(a) = \pi(b)$ if and only if $ab^{-1} \in \operatorname{Ker}\varphi$, so the power function π of φ takes exactly $|A : \operatorname{Ker}\varphi|$ distinct values in $\mathbb{Z}_k$. The index $|A : \operatorname{Ker}\varphi|$ will be called the *skew type* of φ. It follows that a skew morphism of A is an automorphism if and only if it has skew type 1. A skew morphism which is not an automorphism will be called a *proper* skew morphism.

Moreover, define

$$\operatorname{Core}\varphi = \bigcap_{i=1}^{k} \varphi^i(\operatorname{Ker}\varphi).$$

Then $\operatorname{Core}\varphi$ is a φ-invariant normal subgroup of A, and it it is the largest φ-invariant subgroup of A contained in $\operatorname{Ker}\varphi$. In particular, if A is abelian, then $\operatorname{Core}\varphi = \operatorname{Ker}\varphi$ [4, Lemma 5.1].

The following properties of skew morphisms are fundamental.

Proposition 2.1 ([16]). *Let φ be a skew morphism of a finite group A, let π be the power function of φ, and let k be the order of φ. Then the following hold:*

(a) *for any positive integer ℓ and for any $a, b \in A$, $\varphi^\ell(ab) = \varphi^\ell(a)\varphi^{\sigma(a,\ell)}(b)$, where $\sigma(a, \ell) = \sum_{i=1}^{\ell} \pi(\varphi^{i-1}(a))$;*

(b) *for all $a, b \in A$, $\pi(ab) \equiv \sum_{i=1}^{\pi(a)} \pi(\varphi^{i-1}(b)) \pmod{k}$.*

Proposition 2.2 ([1]). *Let φ be a skew morphism of a finite group A, let π be the power function of φ, and let k be the order of φ. Then $\mu = \varphi^\ell$ is a skew morphism of A if and only if the congruence $\ell x \equiv \sigma(a, \ell) \pmod{k}$ is soluble for every $a \in A$, in which case $\pi_\mu(a)$ is the solution*

Proposition 2.3 ([15]). *If φ is a skew morphism of a finite group A, then $O_a^{-1} = O_{a^{-1}}$ for any $a \in A$, where O_a and $O_{a^{-1}}$ denote the orbits of φ containing a and a^{-1}, respectively.*

Proposition 2.4 ([1, 25]). *Let φ be a skew morphism of a finite group A, and let π be the power function of φ. Then for any automorphism θ of A, $\mu = \theta^{-1}\varphi\theta$ is a skew morphism of A with power function $\pi_\mu = \pi\theta$.*

Proposition 2.5 ([26, 25])**.** *Let φ be a skew morphism of a finite group A, and let π be the power function of φ. If $A = \langle a_1, a_2, \dots, a_r\rangle$, then $|\varphi| = \operatorname{lcm}(|O_{a_1}|, |O_{a_2}|, \dots, |O_{a_r}|)$. Moreover, the skew morphism φ and its power function π are completely determined by the action of φ and the values of π on the generating orbits $O_{a_1}, O_{a_2}, \dots, O_{a_r}$, respectively.*

Proposition 2.6 ([26])**.** *Let φ be a skew morphism of a finite group A, and let π be the power function of φ. If N is a φ-invariant normal subgroup of A, then $\bar{\varphi}$ defined by $\bar{\varphi}(\bar{x}) = \overline{\varphi(x)}$ is a skew morphism of $\bar{A} := A/N$ and the power function $\bar{\pi}$ of $\bar{\varphi}$ is determined by $\bar{\pi}(\bar{x}) \equiv \pi(x) \pmod{|\bar{\varphi}|}$.*

Since $\operatorname{Core}\varphi$ is a φ-invariant normal subgroup of A, by Proposition 2.6, φ induces a skew morphism $\bar{\varphi}$ of $\bar{A} = A/\operatorname{Core}\varphi$. It is shown in [25] that the $\bar{\varphi}$-invariant subgroup $\operatorname{Fix}\bar{\varphi}$ of $\bar{A}$ lifts to a φ-invariant subgroup $\operatorname{Smooth}\varphi$ of A, namely,

$$\operatorname{Smooth}\varphi = \{a \in A \mid \bar{\varphi}(\bar{a}) = \bar{a}\}.$$

In particular, if $\operatorname{Smooth}\varphi = A$, then the skew morphism φ is called a *smooth* skew morphism.

Proposition 2.7 ([25])**.** *Let φ be a skew morphism of a finite group A, and let π be the power function of φ. Then φ is smooth if and only if $\pi(\varphi(a)) = \pi(a)$ for all $a \in A$.*

The most important properties of smooth skew morphisms are summarized as follows.

Proposition 2.8 ([25])**.** *Let φ be a smooth skew morphism of A, $|\varphi| = k$, and let π be the power function of φ. Then the following hold:*

(a) *$\varphi(\operatorname{Ker}\varphi) = \operatorname{Ker}\varphi$;*

(b) *$\pi : A \to \mathbb{Z}_k$ is a group homomorphism of A into the multiplicative group $\mathbb{Z}_k^*$ with $\operatorname{Ker}\pi = \operatorname{Ker}\varphi$;*

(c) *for any φ-invariant normal subgroup N of A, the induced skew morphism $\bar{\varphi}$ on A/N is also smooth, and in particular, if $N = \operatorname{Ker}\varphi$ then $\bar{\varphi}$ is the identity permutation;*

(d) *for any positive integer ℓ, $\mu = \varphi^\ell$ is a smooth skew morphism of A;*

(e) *for any automorphism θ of A, $\mu = \theta^{-1}\varphi\theta$ is a smooth skew morphism of A.*

Lemma 2.9. *Let φ be a skew morphism of a finite group A. Then φ is smooth if and only if there exists a φ-invariant normal subgroup N of A contained in $\operatorname{Ker}\varphi$ such that the induced skew morphism $\bar{\varphi}$ of $\bar{A} = A/N$ is the identity permutation.*

Proof. If φ is smooth, then by Proposition 2.8(a), φ is kernel-preserving, and so $\operatorname{Ker}\varphi = \operatorname{Core}\varphi$. Take $N = \operatorname{Ker}\varphi$, then by Proposition 2.8(c) the induced skew morphism $\bar{\varphi}$ of A/N is the identity permutation.

Conversely, suppose that there exists a φ-invariant normal subgroup N of A contained in $\operatorname{Ker}\varphi$ such that the induced skew morphism $\bar{\varphi}$ of $\bar{A} = A/N$ is the identity permutation. Then, for each $a \in A$, there is an element $u \in N \le \operatorname{Ker}\varphi$ such that $\varphi(a) = ua$. Thus, $\pi(\varphi(a)) = \pi(a)$, and therefore φ is smooth by Proposition 2.7. □

There is a fundamental correspondence between skew morphisms and groups with cyclic complements.

Proposition 2.10 ([5]). *If $G = AC$ is a factorisation of a finite group G into a product of a subgroup A and a cyclic subgroup $C = \langle c \rangle$ with $A \cap C = 1$, then c induces a skew morphism φ of A via the commuting rule $ca = \varphi(a)c^{\pi(a)}$, for all $a \in A$; in particular $|\varphi| = |C : C_G|$, where $C_G = \cap_{g \in G} C^g$.*

Conversely, if φ is a skew morphism of a finite group A, then $G = L_A \langle \varphi \rangle$ is a transitive permutation group on A with $L_A \cap \langle \varphi \rangle = 1$ and $\langle \varphi \rangle$ core-free in G, where L_A is the left regular representation of A.

3 Antibalanced skew morphisms

In this section we develop a theory of antibalanced skew morphisms and classify all antibalanced skew morphisms of cyclic groups.

A skew morphism φ of a finite group A will be called *antibalanced* if

$$\varphi^{-1}(a) = \varphi(a^{-1})^{-1}, \quad \text{for all } a \in A.$$

Since $1 = \varphi(aa^{-1}) = \varphi(a)\varphi^{\pi(a)}(a^{-1})$, we have $\varphi(a)^{-1} = \varphi^{\pi(a)}(a^{-1})$. Thus, φ is antibalanced if and only if $\varphi^{\pi(a)}(a^{-1}) = \varphi^{-1}(a^{-1})$, or equivalently, $\pi(a) \equiv -1 \pmod{|O_{a^{-1}}|}$, for all $a \in A$. By Proposition 2.3, $|O_a| = |O_{a^{-1}}|$. It follows that φ is antiblanced if and only if $\pi(a) \equiv -1 \pmod{|O_a|}$ for all $a \in A$. Note that for any $a \in \operatorname{Ker}\varphi$, $|O_a|$ is 1 or 2.

Remark 3.1. It was proved in [16, Theorem 1] that a Cayley map $\mathrm{CM}(A, X, p)$ is regular (on the arcs) if and only if there is a skew morphism φ of A such that the restriction of φ to X is equal to p. Since X is a generating set of A and is closed under taking inverses, the associated skew morphism φ has a generating orbit which is closed under taking inverses. For brevity, such a skew morphism will be called a *Cayley* skew morphism.

Moreover, a regular Cayley map $\mathrm{CM}(A, X, p)$ was termed antibalanced if $p^{-1}(x) = p(x^{-1})^{-1}$ for all $x \in X$ [24]. It follows that a regular Cayley map is antibalanced if and only if the associated Cayley skew morphism is antibalanced. However, neither generating orbit, nor inverse-closed orbit are assumed in the preceding definition of antibalanced skew morphisms. Therefore, antibalanced skew morphisms may be regarded as a natural generalization of the skew morphisms arising from antibalanced regular Cayley maps.

We give an example to clarify the concept.

Example 3.2. The cyclic group $\mathbb{Z}_{12}$ has exactly eight skew morphisms, four of which are proper:

$$\begin{aligned}
\varphi &= (0)(2)(4)(6)(8)(10)(1,3,5,7,9,11), & \pi_\varphi &= [1][1][1][1][1][1][5,5,5,5,5,5];\\
\psi &= (0)(2)(4)(6)(8)(10)(1,11,9,7,5,3), & \pi_\psi &= [1][1][1][1][1][1][5,5,5,5,5,5];\\
\mu &= (0)(2)(4)(6)(8)(10)(1,5,9)(3,7,11), & \pi_\mu &= [1][1][1][1][1][1][2,2,2][2,2,2];\\
\gamma &= (0)(2)(4)(6)(8)(10)(1,9,5)(3,11,7), & \pi_\gamma &= [1][1][1][1][1][1][2,2,2][2,2,2].
\end{aligned}$$

It is easily seen that all the above skew morphisms are antibalanced. Note that the first two skew morphisms contain a generating orbit closed under taking inverses, but the last two skew morphisms do not contain such an orbit. Therefore, φ and ψ are antibalanced Cayley skew morphism, and μ and γ are antibalanced non-Cayley skew morphisms.

We summarise some properties of antibalanced skew morphisms as follows.

Lemma 3.3. *Let φ be an antibalanced skew morphism of a finite group A, and let π be the associated power function. Then the following hold:*

(a) *for any positive integer ℓ, $\varphi^{-\ell}(a) = \varphi^{\ell}(a^{-1})^{-1}$ for all $a \in A$;*

(b) *for any automorphism θ of A, the skew morphism $\mu = \theta^{-1}\varphi\theta$ is antibalanced;*

(c) *for any φ-invariant normal subgroup N of A, the induced skew morphism $\bar{\varphi}$ of A/N is antibalanced;*

(d) *for any $c \in \operatorname{Ker}\varphi$ and $a \in A$, $\pi(a) \equiv 1 \pmod{|O_c|}$.*

Proof. (a) The case $\ell = 1$ is the definition. Assume the result for ℓ, i.e. $\varphi^{-\ell}(a) = \varphi^{\ell}(a^{-1})^{-1}$ for all $a \in A$. Then

$$\varphi^{-(\ell+1)}(a) = \varphi^{-1}(\varphi^{-\ell}(a)) = \varphi^{-1}(\varphi^{\ell}(a^{-1})^{-1}) = \varphi(\varphi^{\ell}(a^{-1}))^{-1} = \varphi^{\ell+1}(a^{-1})^{-1},$$

and the result follows by induction.

(b) For any $a \in A$, we have

$$\mu^{-1}(a) = \theta^{-1}\varphi^{-1}\theta(a) = \theta^{-1}(\varphi(\theta(a)^{-1}))^{-1}) = \big(\theta^{-1}(\varphi(\theta(a^{-1})))\big)^{-1} = \mu(a^{-1})^{-1},$$

so μ is antibalanced.

(c) Since $\varphi^{-1}(a) = \varphi(a^{-1})^{-1}$, we have $\bar{\varphi}^{-1}(\bar{a}) = \bar{\varphi}(\bar{a}^{-1})^{-1}$, and so $\bar{\varphi}$ is antibalanced.

(d) For any $c \in \operatorname{Ker}\varphi$ and any $a \in A$, we have

$$\begin{aligned}\varphi(c)a^{-1} &= \varphi(c)[\varphi^{-1}(\varphi(a))]^{-1} = \varphi(c)\varphi(\varphi(a)^{-1}) = \varphi(c\varphi(a)^{-1})\\ &= \varphi^{-1}(\varphi(a)c^{-1})^{-1} = \big(\varphi^{-1}(\varphi(a))\varphi^{-\pi\varphi^{-1}(\varphi(a))}(c^{-1})\big)^{-1}\\ &= (a\varphi^{-\pi(a)}(c^{-1}))^{-1} = (a\varphi^{\pi(a)}(c)^{-1})^{-1} = \varphi^{\pi(a)}(c)a^{-1},\end{aligned}$$

so $\varphi^{\pi(a)}(c) = \varphi(c)$, and hence $\pi(a) \equiv 1 \pmod{|O_c|}$. □

Lemma 3.4. *Let φ be an automorphism of a finite group A. Then φ is antibalanced if and only if $\varphi^2 = 1$, that is, φ is an involutory automorphism.*

Proof. By definition, φ is antibalanced if and only if for all $a \in A$, $\varphi^{-1}(a) = \varphi(a^{-1})^{-1}$. Since φ is an automorphism, $\varphi(a^{-1})^{-1} = \varphi(a)$, and hence φ is antibalanced if and only if for all $a \in A$, $\varphi^{-1}(a) = \varphi(a)$, that is, $\varphi^2(a) = a$. □

Corollary 3.5. *Every antibalanced automorphism of the cyclic additive group $\mathbb{Z}_n$ is of the form $\varphi(x) = sx$, $x \in \mathbb{Z}_n$, where $s^2 \equiv 1 \pmod{n}$.*

Let φ be a skew morphism of a finite group A. Suppose that φ has an orbit X generating A. The words of even length in the generators from X form a subgroup of A, which will be called the *even word subgroup* of A with respect to X and denoted by A_X^+. Note that the index of A_X^+ in A is 1 or 2.

The following results generalize the properties of antibalanced Cayley skew morphisms (or more precisely, antibalanced regular Cayley maps) obtained in [4]

Lemma 3.6. *Let φ be a skew morphism of a finite group A containing an orbit X which generates A, and let π be the associated power function. Then φ is antibalanced if and only if $\pi(x) \equiv -1 \pmod{|X|}$ for all $x \in X$ and φ restricted to A_X^+ is an involutory automorphism. Furthermore, if φ is antibalanced, then φ is a smooth skew morphism of skew type 1 or 2.*

Proof. Since X is a generating orbit of φ, we have $|\varphi| = |X|$ by Proposition 2.5, and the value of π on A is completely determined by the value of π on X. Suppose that $\pi(x) \equiv -1 \pmod{|X|}$ for all $x \in X$. Then by Proposition 2.1, for any $x, y \in X$,

$$\pi(xy) = \sum_{i=1}^{\pi(x)} \pi(\varphi^{i-1}(y)) \equiv \pi(x)\pi(y) \equiv 1 \pmod{|X|}.$$

Since $A = \langle X \rangle$, every element of A is expressible as a word of finite length in the elements of X. By induction, $\pi(a) \equiv 1 \pmod{|X|}$ if a is an even word, and $\pi(a) \equiv -1 \pmod{|X|}$ if a is an odd word. Note that if a is an even word (resp. an odd word), then both $\varphi(a)$ and a^{-1} are also even words (resp. odd words). Thus, φ is a smooth skew morphism of skew type 1 or 2 and φ restricted to A_X^+ is an automorphism of A_X^+.

If φ is antibalanced, then it is evident that $\pi(x) \equiv -1 \pmod{|X|}$ for all $x \in X$. This implies that φ restricted to A_X^+ is an automorphism of A_X^+, and hence, for any $a \in A_X^+$, $\varphi(a^{-1}) = \varphi(a)^{-1}$. Since φ is antibalanced, we have $\varphi(a^{-1})^{-1} = \varphi^{-1}(a)$, and hence, for any $a \in A_X^+$, $\varphi^{-1}(a) = \varphi(a)$. Therefore, φ restricted to A_X^+ is an involutory automorphism.

Conversely, assume that $\pi(x) \equiv -1 \pmod{|X|}$ for all $x \in X$ and φ restricted to A_X^+ is an involutory automorphism. For any even word $a \in A_X^+$, $\varphi^{-1}(a) = \varphi(a) = \varphi(a^{-1})^{-1}$. For any odd word b,

$$1 = \varphi(b^{-1}b) = \varphi(b^{-1})\varphi^{\pi(b^{-1})}(b) = \varphi(b^{-1})\varphi^{-1}(b),$$

and so $\varphi^{-1}(b) = \varphi(b^{-1})^{-1}$. Therefore, φ is antibalanced. □

Remark 3.7. Let φ be a skew morphism of a finite group A containing an orbit X which generates A, and let π be the associated power function. Lemma 3.6 implies that

(a) if $|A : A_X^+| = 1$, then the skew morphism φ is antibalanced if and only if φ is an involutory automorphism of A;

(b) if $|A : A_X^+| = 2$, then φ is antibalanced if and only if $\pi(a) \equiv 1 \pmod{|\varphi|}$ for all $a \in A_X^+$, $\pi(a) \equiv -1 \pmod{|\varphi|}$ for all $a \in A \setminus A_X^+$ and φ restricted to A_X^+ is an involutory automorphism.

The following lemma deals with antibalanced skew morphisms of abelian groups.

Lemma 3.8. *Let φ be a skew morphism of a finite abelian group A containing an orbit X which generates A, and let π be the associated power function. Then φ is antibalanced if and only if $\pi(x) \equiv -1 \pmod{|X|}$ for all $x \in X$.*

Proof. By Lemma 3.6, it suffices to prove the sufficient part. Assume that $\pi(x) \equiv -1 \pmod{|X|}$ for all $x \in X$. Then, $\pi(a) \equiv 1 \pmod{|X|}$ if a is an even word, and $\pi(a) \equiv -1$

$\pmod{|X|}$ if a is an odd word. Furthermore φ restricted to A_X^+ is an automorphism of A_X^+. For any $a \in A_X^+$ and for any odd word b,

$$\varphi(b)\varphi^{-1}(a) = \varphi(ba) = \varphi(ab) = \varphi(a)\varphi(b) = \varphi(b)\varphi(a),$$

and hence $\varphi^2(a) = a$. Thus, by Lemma 3.6, φ is antibalanced. □

Now we are ready to determine antibalanced skew morphisms of cyclic groups.

Theorem 3.9. *Let φ be an antibalanced skew morphism of the cyclic additive group $\mathbb{Z}_n$.*

(a) *If n is odd, then φ is an involutory automorphism of the form $\varphi(x) = sx$, $x \in \mathbb{Z}_n$, where $s^2 \equiv 1 \pmod{n}$.*

(b) *If n is even, then φ and the associated power function π are of the form*

$$\varphi(x) = \begin{cases} xs, & x \text{ is even}, \\ (x-1)s + 2r + 1, & x \text{ is odd}, \end{cases} \quad \text{and} \quad \pi(x) = \begin{cases} 1, & x \text{ is even}, \\ -1, & x \text{ is odd}, \end{cases} \tag{3.1}$$

where r, s are integers in $\mathbb{Z}_{n/2}$ such that

$$s^2 \equiv 1 \pmod{n/2} \quad \text{and} \quad (r+1)(s-1) \equiv 0 \pmod{n/2}. \tag{3.2}$$

In this case, the order of φ is equal to $n/\gcd(n, r(s+1))$, and in particular φ is an automorphism if and only if $s \equiv 2r+1 \pmod{n/2}$ and $(2r+1)^2 \equiv 1 \pmod{n}$.

Proof. First suppose that φ is an antibalanced skew morphism of $\mathbb{Z}_n$ with the associated power function π. Note that the orbit X of φ containing 1 generates $\mathbb{Z}_n$. Let $\mathbb{Z}_n^+$ be the even word subgroup of $\mathbb{Z}_n$ with respect to X. Then $|\mathbb{Z}_n : \mathbb{Z}_n^+| = 1$ or 2. By Lemma 3.6, φ restricted to $\mathbb{Z}_n^+$ is an involutory automorphism.

If n is odd, then $\mathbb{Z}_n^+ = \mathbb{Z}_n$, so φ is an involutory automorphism of $\mathbb{Z}_n$, and the result follows from Corollary 3.5.

Now assume that n is an even number. By Lemma 3.6, φ is a smooth skew morphism of skew type 1 or 2, so $\langle 2\rangle$ is a φ-invariant normal subgroup of $\mathbb{Z}_n$ contained in $\operatorname{Ker}\varphi$, and the induced skew morphism $\bar{\varphi}$ of $\mathbb{Z}_n/\langle 2\rangle$ is the identity permutation. Thus, there are integers $r, s \in \mathbb{Z}_{n/2}$ such that

$$\varphi(1) \equiv 2r+1 \pmod{n} \quad \text{and} \quad \varphi(2) \equiv 2s \pmod{n},$$

where $\gcd(s, n/2) = 1$. By Lemma 3.6,

$$\pi(x) = 1 \text{ if } x \text{ is even}, \quad \pi(x) = -1 \text{ if } x \text{ is odd}.$$

It follows that

$$\varphi(x) = \begin{cases} xs, & x \text{ is even}, \\ \varphi(x-1) + \varphi(1) = (x-1)s + 2r + 1, & x \text{ is odd}. \end{cases} \tag{3.3}$$

Since φ restricted to $\mathbb{Z}_n^+$ is an involutory automorphism, we have $s^2 \equiv 1 \pmod{n/2}$. Furthermore, we have

$$2s = \varphi(2) = \varphi(1) + \varphi^{-1}(1) = 2r + 1 - 2rs + 1 \pmod{n},$$

and hence $(r+1)(s-1) \equiv 0 \pmod{n/2}$. Moreover, by induction we have

$$\varphi^j(1) \equiv 1 + 2r\sum_{i=1}^{j} s^{i-1} \pmod{n}.$$

Let k be the smallest positive integer such that $\varphi^k(1) = 1$. By Proposition 2.5, $k = |\varphi|$. If k is odd, then $s \equiv 1 \pmod{n/2}$, since the length of the orbit of φ containing 2 divides k. Upon substitution we get $k = n/\gcd(n, 2r)$. If k is even, then the congruence $2r\sum_{i=1}^{k} s^{i-1} \equiv 0 \pmod{n}$ reduces to $rk(s+1) \equiv 0 \pmod{n}$, so $k = n/\gcd(n, r(s+1))$. Note that in either cases $k = n/\gcd(n, r(s+1))$. In particular, if φ is an automorphism, then for any $x \in \mathbb{Z}_n$, $\varphi(x) = x\varphi(1) = x(2r+1)$ and $(2r+1)^2 \equiv 1 \pmod{n}$.

Conversely, we need to verify that φ given by (3.1) is an antibalanced skew morphism of $\mathbb{Z}_n$, provided that the stated numerical conditions in (3.2) are fulfilled. It is easily seen that $\varphi(0) = 0$ and φ is a bijection on $\mathbb{Z}_n$. Now for any $x \in \mathbb{Z}_n$ and for any $y \in \mathbb{Z}_n$, if x is even, then one can easily show that

$$\varphi(x) + \varphi(y) = \varphi(x+y).$$

If x is odd and y is even, then

$$\varphi(x) + \varphi^{-1}(y) = (x-1)s + 2r + 1 + ys = (x+y-1)s + 2r + 1 = \varphi(x+y).$$

Finally, if both x and y are odd, then $\varphi(-2rs + (y-1)s + 1) = y$, and so $\varphi^{-1}(y) = -2rs + (y-1)s + 1$. From the condition $(r+1)(s-1) \equiv 0 \pmod{n/2}$ we deduce that $-2rs \equiv 2s - 2r - 2 \pmod{n}$ and hence $\varphi^{-1}(y) = (y+1)s - 2r - 1$. Consequently,

$$\varphi(x) + \varphi^{-1}(y) = (x-1)s + 2r + 1 + (y+1)s - 2r - 1 = (x+y)s = \varphi(x+y).$$

Therefore, φ is a skew morphism of $\mathbb{Z}_n$. By Lemma 3.8, it is antibalanced. □

From the proof of Theorem 3.9 we obtain the following corollary.

Corollary 3.10. *Let φ be an antibalanced skew morphism of $\mathbb{Z}_n$. If φ is of odd order, then the restriction of φ to* $\operatorname{Ker}\varphi$ *is the identity automorphism of* $\operatorname{Ker}\varphi$.

Theorem 3.11. *Let $n = 2^{\alpha} p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_\ell^{\alpha_\ell}$ be the prime power factorization of a positive integer n. Then the number $\nu(n)$ of antibalanced skew morphisms of the cyclic additive group $\mathbb{Z}_n$ is determined by the following formula:*

$$\nu(n) = \begin{cases} 2^{\ell}, & \alpha = 0, \\ \prod_{i=1}^{\ell}(p_i^{\alpha_i}+1), & \alpha = 1, \\ 2\prod_{i=1}^{\ell}(p_i^{\alpha_i}+1), & \alpha = 2, \\ 6\prod_{i=1}^{\ell}(p_i^{\alpha_i}+1), & \alpha = 3, \\ (4 + 2^{\alpha-2} + 2^{\alpha-1})\prod_{i=1}^{\ell}(p_i^{\alpha_i}+1), & \alpha \geq 4. \end{cases}$$

Proof. If $\alpha = 0$, then n is odd. By Theorem 3.9(a), every antibalanced skew morphism of $\mathbb{Z}_n$ is an automorphism of the form $\varphi(x) = xs$, $x \in \mathbb{Z}_n$, where $s^2 \equiv 1 \pmod{n}$. It follows that the number $\nu(n)$ is equal to the number of solutions of the quadratic congruence $s^2 \equiv 1 \pmod{n}$, which is equal to 2^ℓ.

Now assume $\alpha \geq 1$, so n is an even number. By Theorem 3.9(b), the number of antibalanced skew morphisms of $\mathbb{Z}_n$ is equal to the number of integer solutions (r, s) in $\mathbb{Z}_{n/2}$ of the system

$$\begin{cases} s^2 \equiv 1 \pmod{n/2}, \\ (r+1)(s-1) \equiv 0 \pmod{n/2}. \end{cases}$$

By the Chinese Remainder Theorem, (r, s) is a solution of the system if and only if it is a solution of each of the following $\ell + 1$ systems

$$\begin{cases} s^2 \equiv 1 \pmod{2^{\alpha-1}}, \\ (r+1)(s-1) \equiv 0 \pmod{2^{\alpha-1}} \end{cases} \tag{3.4}$$

and

$$\begin{cases} s^2 \equiv 1 \pmod{p_i^{\alpha_i}}, \\ (r+1)(s-1) \equiv 0 \pmod{p_i^{\alpha_i}}, \end{cases} \quad i = 1, 2, \dots, \ell. \tag{3.5}$$

We first determine the solutions of (3.5). By assumption, for each i, $i = 1, 2, \dots, \ell$, p_i is an odd prime. It follows from the congruence $s^2 \equiv 1 \pmod{p_i^{\alpha_i}}$ that either $s \equiv 1 \pmod{p_i^{\alpha_i}}$ or $s \equiv -1 \pmod{p_i^{\alpha_i}}$. If $s \equiv 1 \pmod{p_i^{\alpha_i}}$, then upon substitution the congruence $(r+1)(s-1) \equiv 0 \pmod{p_i^{\alpha_i}}$ holds for every $r \in \mathbb{Z}_{p_i^{\alpha_i}}$. On the other hand, if $s \equiv -1 \pmod{p_i^{\alpha_i}}$, then upon substitution the congruence $(r+1)(s-1) \equiv 0 \pmod{p_i^{\alpha_i}}$ reduces to $r \equiv -1 \pmod{p_i^{\alpha_i}}$. Therefore, for each odd prime p_i, the system (3.5) has precisely $(p_i^{\alpha_i} + 1)$ solutions in $\mathbb{Z}_{p_i^{\alpha_i}}$.

Now we turn to solutions of (3.4). If $\alpha = 1$, then it only has the trivial solution $(r, s) = (1, 1)$. If $\alpha = 2$, then $(r, s) = (0, 1), (1, 1)$ in $\mathbb{Z}_2$. If $\alpha = 3$, then $(r, s) = (0, 1), (1, 1), (2, 1), (3, 1), (1, 3), (3, 3)$ in $\mathbb{Z}_4$. If $\alpha \geq 4$, then by the congruence $s^2 \equiv 1 \pmod{2^{\alpha-1}}$ we have $s \equiv \pm 1, 2^{\alpha-2} \pm 1 \pmod{2^{\alpha-1}}$. Combining this with the congruence $(r+1)(s-1) \equiv 0 \pmod{2^{\alpha-1}}$ we obtain the following solutions (r, s) in $\mathbb{Z}_{2^{\alpha-1}}$: (a) $r \in \mathbb{Z}_{2^{\alpha-1}}$ and $s = 1$; (b) $r = 2^{\alpha-1} - 1, 2^{\alpha-2} - 1$ and $s = -1$; (c) $r = 2^{\alpha-1} - 1, 2^{\alpha-2} - 1$ and $s = 2^{\alpha-2} - 1$; (d) $r \equiv 1 \pmod{2}$ and $s = 2^{\alpha-2} + 1$.

Finally, multiplying the numbers of solutions for the prime power cases we obtain the number $\nu(n)$, as required. □

4 Correspondence

A correspondence between complete regular dessins and pairs of certain skew morphisms of cyclic groups has been established in [12, Theorem 5]. In this section we extend the correspondence to reflexible complete regular dessins.

Definition 4.1. Let $\varphi : \mathbb{Z}_n \to \mathbb{Z}_n$ and $\tilde{\varphi} : \mathbb{Z}_m \to \mathbb{Z}_m$ be a pair of skew morphisms of the cyclic additive groups $\mathbb{Z}_n$ and $\mathbb{Z}_m$, associated with power functions $\pi : \mathbb{Z}_n \to \mathbb{Z}_{|\varphi|}$ and $\tilde{\pi} : \mathbb{Z}_m \to \mathbb{Z}_{|\tilde{\varphi}|}$, respectively. The pair $(\varphi, \tilde{\varphi})$ will be called *reciprocal* if they satisfy the following conditions:

(a) $|\varphi|$ divides m and $|\tilde{\varphi}|$ divides n,

(b) $\pi(x) \equiv \tilde{\varphi}^x(1) \pmod{|\varphi|}$ and $\tilde{\pi}(y) \equiv \varphi^y(1) \pmod{|\tilde{\varphi}|}$ for all $x \in \mathbb{Z}_n$ and $y \in \mathbb{Z}_m$.

Suppose that $\mathcal{D}$ is a complete regular dessin with underling graph $K_{m,n}$. Take an arbitrary pair of vertices u and v of valency m and n, respectively. Then the stabilizers G_u and G_v of $G = \mathrm{Aut}(\mathcal{D})$ are cyclic of orders m and n, respectively. Assume $G_u = \langle a \rangle$ and $G_v = \langle b \rangle$. Then by the regularity we have $G = \langle a, b \rangle$ and $|G| = mn$. Since the underlying graph $K_{m,n}$ is simple, $\langle a \rangle \cap \langle b \rangle = 1$. Consequently, from the product formula we deduce that $G = \langle a \rangle \langle b \rangle$. Thus each complete regular dessin determines a triple (G, a, b) such that $G = \langle a \rangle \langle b \rangle$ and $\langle a \rangle \cap \langle b \rangle = 1$.

Now each of the cyclic factors $\langle a \rangle$ and $\langle b \rangle$ of G can be taken as the complement of the other, so in the spirit of Proposition 2.10, there are a pair of skew morphisms φ and $\tilde{\varphi}$ of the cyclic additive groups $\mathbb{Z}_n$ and $\mathbb{Z}_m$ such that

$$a^{-1}b^x = b^{\varphi(x)}a^{-\pi(x)} \quad \text{and} \quad b^{-1}a^y = a^{\tilde{\varphi}(y)}b^{-\tilde{\pi}(y)} \tag{4.1}$$

where $x \in \mathbb{Z}_n$ and $y \in \mathbb{Z}_m$. By induction we deduce that

$$a^{-k}b^x = b^{\varphi^k(x)}a^{-\sigma(x,k)} \quad \text{and} \quad b^{-l}a^y = a^{\tilde{\varphi}^l(y)}b^{-\tilde{\sigma}(y,l)}, \tag{4.2}$$

where

$$\sigma(x,k) = \sum_{i=1}^{k} \pi(\varphi^{i-1}(x)) \quad \text{and} \quad \tilde{\sigma}(y,l) = \sum_{i=1}^{l} \tilde{\pi}(\tilde{\varphi}^{i-1}(y)).$$

Inverting the above identities yields

$$b^{-x}a^k = a^{\sigma(x,k)}b^{-\varphi^k(x)} \quad \text{and} \quad a^{-y}b^l = b^{\tilde{\sigma}(y,l)}a^{-\tilde{\varphi}^l(y)}. \tag{4.3}$$

Substituting for $x = 1$ and $k = y$ we obtain $b^{-1}a^y = a^{\sigma(1,y)}b^{-\varphi^y(1)}$. By comparing this with the second identity in (4.1) we obtain

$$\tilde{\pi}(y) \equiv \varphi^y(1) \pmod{n} \quad \text{and} \quad \tilde{\varphi}(y) \equiv \sigma(1,y) \pmod{m}.$$

Similarly, inserting $y = 1$ and $l = x$ into the second identity in (4.3) we have $a^{-1}b^x = b^{\tilde{\sigma}(1,x)}a^{-\tilde{\varphi}^x(1)}$. A similar comparison with the first identity in (4.1) yields

$$\pi(x) \equiv \tilde{\varphi}^x(1) \pmod{m} \quad \text{and} \quad \varphi(x) \equiv \tilde{\sigma}(1,x) \pmod{n}.$$

By Proposition 2.10, $|\varphi| = |\langle a \rangle : \langle a \rangle_G|$ and $|\tilde{\varphi}| = |\langle b \rangle : \langle b \rangle_G|$. Thus $|\varphi|$ divides m and $|\tilde{\varphi}|$ divides n. In particular, the above four congruences are reduced to

$$\pi(x) \equiv \tilde{\varphi}^x(1) \pmod{|\varphi|}, \quad \tilde{\pi}(y) \equiv \varphi^y(1) \pmod{|\tilde{\varphi}|}$$

and

$$\varphi(x) \equiv \tilde{\sigma}(1,x) \pmod{|\tilde{\varphi}|}, \quad \tilde{\varphi}(y) \equiv \sigma(1,y) \pmod{|\varphi|}. \tag{4.4}$$

It follows that every complete regular dessin with underlying graph $K_{m,n}$ determines a pair of reciprocal skew morphisms $(\varphi, \tilde{\varphi})$ of the cyclic additive groups $\mathbb{Z}_n$ and $\mathbb{Z}_m$. Conversely, it is shown in [12, Proposition 4] that given a pair of reciprocal skew morphisms of the cyclic groups $\mathbb{Z}_n$ and $\mathbb{Z}_m$, a complete regular dessin with underlying graph $K_{m,n}$ may be reconstructed in a canonical way. Therefore, we obtain a correspondence between complete regular dessins and pairs of reciprocal skew morphisms of cyclic groups. See [12, Theorem 5] for details.

The following theorem extends this correspondence to reflexible complete regular dessins.

Theorem 4.2. *The isomorphism classes of reflexible regular dessins with complete bipartite underlying graphs $K_{m,n}$ are in one-to-one correspondence with pairs of reciprocal antibalanced skew morphisms $(\varphi, \tilde{\varphi})$ of the cyclic groups $\mathbb{Z}_n$ and $\mathbb{Z}_m$.*

Proof. It is proved in [12, Theorem 5] that the isomorphism classes of regular dessins $\mathcal{D} = (G, a, b)$ with complete bipartite underlying graphs $K_{m,n}$ are in one-to-one correspondence with the pairs $(\varphi, \tilde{\varphi})$ of reciprocal skew morphisms of the cyclic groups $\mathbb{Z}_n$ and $\mathbb{Z}_m$. It remains to show that $\mathcal{D}$ is reflexible if and only if the corresponding pair of skew morphisms $(\varphi, \tilde{\varphi})$ are both antibalanced.

First suppose that $\mathcal{D} = (G, a, b)$ is reflexible, then the identities in (4.1) determine a pair of reciprocal skew morphisms $(\varphi, \tilde{\varphi})$ of the cyclic groups $\mathbb{Z}_n$ and $\mathbb{Z}_m$. Since $\mathcal{D}$ is reflexible, the assignment $\theta : a \mapsto a^{-1}, b \mapsto b^{-1}$ extends to an automorphism of G. By the identities in (4.2) derived from (4.1) we have

$$ab^{-x} = b^{\varphi^{-1}(-x)}a^{-\sigma(-x,-1)} \quad \text{and} \quad ba^{-y} = a^{\tilde{\varphi}^{-1}(-y)}b^{-\tilde{\sigma}(-y,-1)}.$$

Applying the automorphism θ of G to the above identities we obtain

$$a^{-1}b^{x} = \theta(ab^{-x}) = \theta(b^{\varphi^{-1}(-x)}a^{-\sigma(-x,-1)}) = b^{-\varphi^{-1}(-x)}a^{\sigma(-x,-1)}$$

and

$$b^{-1}a^{y} = \theta(ba^{-y}) = \theta(a^{\tilde{\varphi}^{-1}(-y)}b^{-\tilde{\sigma}(-y,-1)}) = a^{-\tilde{\varphi}^{-1}(-y)}b^{\tilde{\sigma}(-y,-1)}.$$

By comparing these with the identities in (4.1) we get $\varphi(x) = -\varphi^{-1}(-x)$ and $\tilde{\varphi}(y) = -\tilde{\varphi}^{-1}(-y)$. Thus both φ and $\tilde{\varphi}$ are antibalanced.

Conversely, suppose that $\varphi : \mathbb{Z}_n \to \mathbb{Z}_n$ and $\tilde{\varphi} : \mathbb{Z}_m \to \mathbb{Z}_m$ form a pair of antibalanced reciprocal skew morphisms. Denote

$$\mathbb{Z}_n = \{0, 1, \dots, (n-1)\} \quad \text{and} \quad \mathbb{Z}_m = \{0', 1', \dots, (m-1)'\},$$

so that $\mathbb{Z}_n$ and $\mathbb{Z}_m$ are disjoint sets. Define two cyclic permutations ρ and $\tilde{\rho}$ on the sets $\mathbb{Z}_n$ and $\mathbb{Z}_m$ by setting

$$\rho = (0, 1, \dots, (n-1)) \quad \text{and} \quad \tilde{\rho} = (0', 1', \dots, (m-1)').$$

We extend the permutations $\varphi, \tilde{\varphi}, \rho$ and $\tilde{\rho}$ to permutations on $\Omega = \mathbb{Z}_n \cup \mathbb{Z}_m$ in a natural way, still denoted by $\varphi, \tilde{\varphi}, \rho$ and $\tilde{\rho}$. Set $a = \varphi\tilde{\rho}$, $b = \tilde{\varphi}\rho$ and $G = \langle a, b\rangle$. It is proved in [12, Proposition 4] that $|a| = m$, $|b| = n$, $\langle a\rangle \cap \langle b\rangle = 1$ and $G = \langle a\rangle\langle b\rangle$, so $\mathcal{D} = (G, a, b)$ is a complete regular dessin with underlying graph $K_{m,n}$. Now define a bijection $\gamma : \Omega \to \Omega$ on Ω to be $\gamma(x) = -x$ and $\gamma(y') = -y'$ for all $x \in \mathbb{Z}_n$ and $y' \in \mathbb{Z}_m$. Since both φ and $\tilde{\varphi}$ are antibalanced, we have

$$\begin{aligned}\gamma a(x) &= \gamma\varphi\tilde{\rho}(x) = \gamma\varphi(x) = -\varphi(x) = \varphi^{-1}(-x)\\ &= \varphi^{-1}\gamma(x) = \varphi^{-1}\tilde{\rho}^{-1}(\gamma(x)) = a^{-1}\gamma(x)\end{aligned}$$

and

$$\begin{aligned}\gamma a(y') &= \gamma\varphi\tilde{\rho}(y') = \gamma\varphi((y+1)') = \gamma((y+1)') = -(y+1)'\\ &= (-y-1)' = \varphi^{-1}\tilde{\rho}^{-1}(-y') = a^{-1}\gamma(y').\end{aligned}$$

Thus $\gamma a = a^{-1}\gamma$. Similarly, $\gamma b = b^{-1}\gamma$. Hence, $(G, a, b) \cong (G, a^{-1}, b^{-1})$, where (G, a^{-1}, b^{-1}) denotes the mirror image of $\mathcal{D}$. Therefore, (G, a, b) is reflexible, as required. □

We summarize two properties of reciprocal skew morphisms.

Lemma 4.3 ([12, 14]). *Let $\varphi : \mathbb{Z}_n \to \mathbb{Z}_n$ and $\tilde{\varphi} : \mathbb{Z}_m \to \mathbb{Z}_m$ be a pair of reciprocal skew morphisms of the cyclic additive groups $\mathbb{Z}_n$ and $\mathbb{Z}_m$. Then*

(a) $\varphi(x) \equiv \sum_{i=1}^{x} \tilde{\pi}(\tilde{\varphi}^{i-1}(1)) \pmod{|\tilde{\varphi}|}$ *and* $\tilde{\varphi}(y) \equiv \sum_{i=1}^{y} \pi(\varphi^{i-1}(1)) \pmod{|\varphi|}$,

(b) $|\mathbb{Z}_m : \operatorname{Ker}\tilde{\varphi}|$ *divides* $|\varphi|$ *and* $|\mathbb{Z}_n : \operatorname{Ker}\varphi|$ *divides* $|\tilde{\varphi}|$.

Lemma 4.4 ([14]). *Let $\varphi : \mathbb{Z}_n \to \mathbb{Z}_n$ and $\tilde{\varphi} : \mathbb{Z}_m \to \mathbb{Z}_m$ be a pair of reciprocal skew morphisms of the cyclic additive groups $\mathbb{Z}_n$ and $\mathbb{Z}_m$. If one of the skew morphisms is an automorphism, then the other is smooth. In particular, if one of the skew morphism is the identity automorphism, then the other is an automorphism.*

5 Classification

By Theorem 4.2, the classification of reflexible complete regular dessins is reduced to the classification of reciprocal pairs of antibalanced skew morphisms of cyclic groups. The aim of this section is to present a classification of the latter.

Proposition 5.1. *Let $\varphi : \mathbb{Z}_n \to \mathbb{Z}_n$ and $\tilde{\varphi} : \mathbb{Z}_m \to \mathbb{Z}_m$ be a reciprocal pair of antibalanced skew morphisms of the cyclic additive groups $\mathbb{Z}_n$ and $\mathbb{Z}_m$, respectively. If both n and m are odd, then $(\varphi, \tilde{\varphi}) = (\mathrm{id}_n, \mathrm{id}_m)$ where id_k denotes the identity automorphism of $\mathbb{Z}_k$, $k = n, m$.*

Proof. By Theorem 3.9(a), both φ and $\tilde{\varphi}$ are involutory automorphisms. The divisibility condition on reciprocality implies that both φ and $\tilde{\varphi}$ are the identity automorphisms. □

Theorem 5.2. *Let $\varphi : \mathbb{Z}_n \to \mathbb{Z}_n$ and $\tilde{\varphi} : \mathbb{Z}_m \to \mathbb{Z}_m$ be a reciprocal pair of antibalanced skew morphisms of the cyclic additive groups $\mathbb{Z}_n$ and $\mathbb{Z}_m$, respectively. If n is odd and m is even, then φ is an automorphism of the form $\varphi(x) \equiv sx \pmod{n}$, and $\tilde{\varphi}$ is a skew morphism of the form*

$$\tilde{\varphi}(y) = \begin{cases} y, & y \text{ is even,} \\ y + 2u, & y \text{ is odd} \end{cases} \quad \textit{and} \quad \tilde{\pi}(y) = \begin{cases} 1, & y \text{ is even,} \\ -1, & y \text{ is odd} \end{cases}$$

where $s \in \mathbb{Z}_n$ and $u \in \mathbb{Z}_{m/2}$ are integers such that

$$\gcd(n, s+1)\gcd(m/2, u) \equiv 0 \pmod{m/2} \quad \textit{and} \quad s^2 \equiv 1 \pmod{n}. \tag{5.1}$$

Proof. By assumption, both φ and $\tilde{\varphi}$ are antibalanced. Since n is odd and m is even, by Theorem 3.9, φ is an automorphism of the form $\varphi(x) = sx$, where $s^2 \equiv 1 \pmod{n}$ and $\tilde{\varphi}$ is a skew morphism of the form

$$\tilde{\varphi}(y) = \begin{cases} ty, & y \text{ is even,} \\ t(y-1) + 2u + 1, & y \text{ is odd,} \end{cases}$$

for some $t, u \in \mathbb{Z}_{m/2}$ satisfying the following conditions:

$$t^2 \equiv 1 \pmod{m/2} \quad \text{and} \quad (u+1)(t-1) \equiv 0 \pmod{m/2}.$$

Note that the order of φ is equal to the multiplicative order of s in $\mathbb{Z}_n$, which is a divisor of 2, and the order of $\tilde{\varphi}$ is the smallest positive integer ℓ such that

$$2u\sum_{i=1}^{\ell} t^{i-1} \equiv 0 \pmod{m}.$$

Now we employ the reciprocality to simplify these numerical conditions. By Definition 4.1(a), $|\tilde{\varphi}|$ divides n. Since n is odd, $|\tilde{\varphi}|$ is also odd, so by Corollary 3.10, $t = 1$, and consequently, $\tilde{\varphi}$ reduces to the stated form and $|\tilde{\varphi}| = m/\gcd(m, 2u)$. By Definition 4.1(b),

$$-1 \equiv \tilde{\pi}(1) \equiv \varphi(1) = s \pmod{\frac{m}{\gcd(m, 2u)}}.$$

Thus, $|\tilde{\varphi}| = m/\gcd(m, 2u)$ is a common divisor of $(s + 1)$ and n. Since m is even, $\gcd(m, 2u) = 2\gcd(m/2, u)$, and we obtain the first condition in (5.1), as required. $\square$

By exchanging the roles of φ and $\tilde{\varphi}$, and the associated parameters, we obtain all reciprocal pairs of antibalanced skew morphisms of $\mathbb{Z}_n$ and $\mathbb{Z}_m$ where n is even and m is odd. The details are left to the reader.

Theorem 5.3. *Let $\varphi : \mathbb{Z}_n \to \mathbb{Z}_n$ and $\tilde{\varphi} : \mathbb{Z}_m \to \mathbb{Z}_m$ be a reciprocal pair of antibalanced skew morphisms of the cyclic additive groups $\mathbb{Z}_n$ and $\mathbb{Z}_m$, respectively. If both n and m are even, then*

$$\varphi(x) = \begin{cases} sx, & x \text{ is even}, \\ s(x-1) + 2r + 1, & x \text{ is odd}, \end{cases} \qquad \pi(x) = \begin{cases} 1, & x \text{ is even}, \\ -1, & x \text{ is odd} \end{cases}$$

and

$$\tilde{\varphi}(y) = \begin{cases} ty, & y \text{ is even}, \\ t(y-1) + 2u + 1, & y \text{ is odd}, \end{cases} \qquad \tilde{\pi}(y) = \begin{cases} 1, & y \text{ is even}, \\ -1, & y \text{ is odd} \end{cases}$$

where $r, s \in \mathbb{Z}_{n/2}$ and $u, t \in \mathbb{Z}_{m/2}$ are integers such that

$$\begin{cases} s^2 \equiv 1 \pmod{n/2}, \\ (r+1)(s-1) \equiv 0 \pmod{n/2}, \\ \gcd(m/2, u+1)\gcd(n, r(s+1)) \equiv 0 \pmod{n/2} \end{cases} \tag{5.2}$$

and

$$\begin{cases} t^2 \equiv 1 \pmod{m/2}, \\ (u+1)(t-1) \equiv 0 \pmod{m/2}, \\ \gcd(n/2, r+1)\gcd(m, u(t+1)) \equiv 0 \pmod{m/2}. \end{cases} \tag{5.3}$$

Proof. By Theorem 3.9(b), the skew morphisms φ and $\tilde{\varphi}$ may be represented by the stated form, where the parameters $r, s \in \mathbb{Z}_{n/2}$ and $u, t \in \mathbb{Z}_{m/2}$ are integers such that

$$s^2 \equiv 1 \pmod{n/2}, \quad (r+1)(s-1) \equiv 0 \pmod{n/2}$$

and

$$t^2 \equiv 1 \pmod{m/2}, \quad (u+1)(t-1) \equiv 0 \pmod{m/2}.$$

In particular,

$$|\varphi| = n/\gcd(n, r(s+1)) \quad \text{and} \quad |\tilde{\varphi}| = m/\gcd(m, u(t+1)).$$

We now employ the reciprocality to simplify the numerical conditions. By Definition 4.1, we have $|\varphi| = n/\gcd(n, r(s+1))$ divides m, $|\tilde{\varphi}| = m/\gcd(m, u(t+1))$ divides n,

$$-1 \equiv \pi(1) \equiv \tilde{\varphi}(1) \equiv 2u+1 \pmod{n/\gcd(n, r(s+1))}$$

and

$$-1 \equiv \tilde{\pi}(1) \equiv \varphi(1) \equiv 2r+1 \pmod{m/\gcd(m, u(t+1))}.$$

Thus, $n/\gcd(n, r(s+1))$ divides $\gcd(m, 2(u+1))$ and $m/\gcd(m, u(t+1))$ divides $\gcd(n, 2(r+1))$, or equivalently,

$$\gcd(n, r(s+1))\gcd(m/2, u+1) \equiv 0 \pmod{n/2}$$

and

$$\gcd(m, u(t+1))\gcd(n/2, r+1) \equiv 0 \pmod{m/2},$$

as required. □

ORCID iDs

Kan Hu https://orcid.org/0000-0003-4775-7273
Young Soo Kwon https://orcid.org/0000-0002-1765-0806

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Recipes for edge-transitive tetravalent graphs*

Primož Potočnik†
University of Ljubljana, Faculty of Mathematics and Physics, Jadranska 19, SI-1000 Ljubljana, Slovenia

Stephen E. Wilson‡
Department of Mathematics and Statistics, Northern Arizona University, Box 5717, Flagstaff, AZ 86011, USA

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Abstract

This paper presents all constructions known to the authors which result in tetravalent graphs whose symmetry groups are large enough to be transitive on the edges of the graph.

Keywords: Graph, automorphism group, symmetry.

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1 The census

This paper is to accompany the Census of Edge-Transitive Tetravalent Graphs, available at

http://jan.ucc.nau.edu/∼swilson/C4FullSite/index.html,

which is a collection of all known edge-transitive graphs of valence 4 up to 512 vertices.

The Census contains information for each graph. This information includes parameters such as group order, diameter, girth etc., all known constructions, relations to other graphs in the Census (coverings, constructions, etc.), and interesting substructures such as colorings, cycle structures, and dissections.

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‡Corresponding author.

E-mail addresses: primoz.potocnik@fmf.uni-lj.si (Primož Potočnik), stephen.wilson@nau.edu (Stephen E. Wilson)

We try to present most graphs as members of one or more parameterized families, and one purpose of this paper is to gather together, here in one place, descriptions of each of these families, to show how each is constructed, what the history of each is and how one family is related to another. We also discuss in this paper the theory and techniques behind computer searches leading to many entries in the Census.

We should point out that similar censi exist for edge transitive graphs of valence 3 [8, 10]. Unlike our census, these censi are complete in the sense that they contain all the graphs up to a given order. The method used in these papers relies on the fact that in the case of prime valence, the order of the automorphism group can be bounded by a linear function of the order of the graph, making exhaustive computer searches possible – see [9] for details.

Even though our census is not proved to be complete, it is complete in some segments. In particular, the census contains all dart-transitive tetravalent graphs up to 512 vertices (see [32, 36]) and all $\frac{1}{2}$-arc-transitive tetravalent graphs up to 512 vertices (see [33, 38]). Therefore, if a graph is missing from our census, then it is semisymmetric (see Section 2).

2 Basic notions

A *graph* is an ordered pair $\Gamma = (\mathcal{V}, \mathcal{E})$, where $\mathcal{V}$ is an arbitrary set of things called vertices, and $\mathcal{E}$ is a collection of subsets of $\mathcal{V}$ of size two; these are called *edges*. We let $\mathcal{V}(\Gamma) = \mathcal{V}$ and $\mathcal{E}(\Gamma) = \mathcal{E}$ in this case. If $e = \{u, v\} \in \mathcal{E}$, we say u is a *neighbor* of v, that u and v are *adjacent*, and that u is *incident* with e and vice versa. A *dart* or *directed edge* is an ordered pair (u, v) where $\{u, v\} \in \mathcal{E}$. Let $\mathcal{D}(\Gamma)$ be the set of darts of Γ. The *valence* or *degree* of a vertex v is the number of edges to which v belongs. A graph is *regular* provided that every vertex has the same valence, and then we refer to that as the valence of the graph.

A *digraph* is an ordered pair $\Delta = (\mathcal{V}, \mathcal{E})$, where $\mathcal{V}$ is an arbitrary set of things called vertices, and $\mathcal{E}$ is a collection of ordered pairs of distinct elements of $\mathcal{V}$. We think of the pair (u, v) as being an edge directed from u to v. An *orientation* is a digraph in which for all $u, v \in \mathcal{V}$, if $(u, v) \in \mathcal{E}$ then $(v, u) \notin \mathcal{E}$.

A *symmetry*, or *automorphism*, of a graph or a digraph $\Gamma = (\mathcal{V}, \mathcal{E})$ is a permutation of $\mathcal{V}$ which preserves $\mathcal{E}$. If $v \in \mathcal{V}(\Gamma)$ and σ is a symmetry of Γ, then we denote the image of v under σ by $v\sigma$, and if ρ is also a symmetry of Γ, then the product $\sigma\rho$ is a symmetry that maps v to $(v\sigma)\rho$. Together with this product, the set of symmetries of Γ forms a group, the *symmetry group* or *automorphism group* of Γ, denoted $\mathrm{Aut}(\Gamma)$. We are interested in those graphs for which $G = \mathrm{Aut}(\Gamma)$ is big enough to be transitive on $\mathcal{E}$. Such a graph is called *edge-transitive*. Within the class of edge-transitive graphs of a given valence, there are three varieties:

(1) A graph is *symmetric* or *dart-transitive* provided that G is transitive on $\mathcal{D} = \mathcal{D}(\Gamma)$.

(2) A graph is $\frac{1}{2}$*-arc-transitive* provided that G is transitive on $\mathcal{E}$ and on $\mathcal{V}$, but not on $\mathcal{D}$. A $\frac{1}{2}$-arc-transitive graph must have even valence [48]. The G-orbit of one dart is then an orientation Δ of Γ such that every vertex has k in-neighbours and k out-neighbors, where $2k$ is the valence of Γ. The symmetry group $\mathrm{Aut}(\Delta)$ is then transitive on vertices and on edges. We call such a Δ a *semitransitive orientation* and we say that a graph which has such an orientation is *semitransitive* [54].

(3) Finally, Γ is *semisymmetric* provided that G is transitive on $\mathcal{E}$ but not on $\mathcal{V}$ (and hence not on $\mathcal{D}$). In this case, the graph must be bipartite, with each edge having one vertex

from each class. More generally, we say that a graph Γ is *bi-transitive* provided that Γ is bipartite, and its group of color-preserving symmetries is transitive on edges (and so on vertices of each color); a bi-transitive graph is thus either semisymmetric or dart-transitive.

There is a fourth important kind of symmetricity that a tetravalent graph might have, in which $\mathrm{Aut}(\Gamma)$ is transitive on vertices but has two orbits on edges. Graphs satisfying this and certain other conditions are called *LR structures*. These are introduced and defined in Section 18 but referred to in several places before that. These graphs, though not edge-transitive themselves, can be used directly to construct semisymmetric graphs.

For $\sigma \in \mathrm{Aut}(\Gamma)$, if there is a vertex $v \in \mathcal{V}$ such that $v\sigma$ is adjacent to v and $v\sigma^2 \neq v$ we call σ a *shunt* and then σ induces a directed cycle $[v, v\sigma, v\sigma^2, \ldots, v\sigma^m]$, with $v\sigma^{m+1} = v$, called a *consistent* cycle. The remarkable theorem of Biggs and Conway [4, 30] says that if Γ is dart-transitive and regular of degree d, then there are exactly $d-1$ orbits of consistent cycles. A later result [5] shows that a $\frac{1}{2}$-arc-transitive graph of degree $2e$ must have exactly e orbits of consistent cycles. For tetravalent graphs, the dart-transitive ones have 3 orbits of consistent cycles and the $\frac{1}{2}$-arc-transitive ones have two such orbits. An LR structure might have 1 or 2 orbits of consistent cycles.

3 Computer generated lists of graphs

3.1 Semisymmetric graphs arising from amalgams of index $(4, 4)$

Let L and R be two finite groups intersecting in a common subgroup B and assume that no non-trivial subgroup of B is normal in both L and R. Then the triple (L, B, R) is called an *amalgam*. For example, if we let $L = A_4$, the alternating group of degree 4, $B \cong C_3$, viewed as a point-stabiliser in L, and $R \cong C_{12}$ containing B as a subgroup of index 4, then (L, B, R) is an amalgam.

If G is a group that contains both L and R and is generated by them, then G is called a *completion* of the amalgam. It is not too difficult to see that there exists a completion that is universal in the sense that every other completion is isomorphic to a quotient thereof. This universal completion is sometimes called the *free product of L and R amalgamated over B* (usually denoted by $L *_B R$) and can be constructed by merging together (disjoint) presentations of L and R and adding relations that identify copies of the same element of B in both L and R. For example, if the amalgam (L, B, R) is as above, then we can write

$$L = \langle x, y, b | x^2, y^2, [x, y], b^3, x^b y, y^b xy \rangle, \quad R = \langle z | z^{12} \rangle,$$

yielding

$$L *_B R = \langle x, y, b, z | x^2, y^2, [x, y], b^3, z^{12}, x^b y, y^b xy, z^4 = b \rangle.$$

Completions of a given amalgam (L, B, R) up to a given order, say M, can be computed using a LowIndexNormalSubgroups routine, developed by Firth and Holt [12] and implemented in Magma [6].

Given a completion G of an amalgam (L, B, R), one can construct a bipartite graph, called *the graph of the completion*, with white and black vertices being the cosets of L and R in G, respectively, and two cosets Lg and Rh adjacent whenever they intersect. Note that white (black) vertices are of valence $[L : B]$ ($[R : B]$, respectively). In particular, if B is of index 4 in both L and R, then the graph is tetravalent. We shall say in this case that the amalgam is of index $(4, 4)$.

The group G acts by right multiplication faithfully as an edge-transitive group of automorphisms of the graph and so the graph of a completion of an amalgam always admits an edge- but not vertex-transitive group of automorphisms, and so the graph is bi-transitive (and thus either dart-transitive or semisymmetric). This now gives us a good strategy for constructing tetravalent semisymmetric graphs of order at most M: Choose your favorite amalgams (L, B, R) of index $(4, 4)$, find their completions up to order $2M|B|$ and construct the corresponding graphs.

We have done this for several amalgams of index $(4, 4)$ and the resulting graphs appear in the census under the name SS. The graph $\mathrm{SS}[n, i]$ is the i-th graph in the list of semisymmetric graphs of order n. These graphs are available in magma code at [33].

3.2 Dart-transitive graphs from amalgams

If Γ is a tetravalent dart-transitive graph, then its *subdivision*, obtained from Γ by inserting a vertex of valence 2 on each edge, is edge-transitive but not vertex-transitive. This process is reversible, by removing vertices of degree 2 in a bi-transitive graph of valence $(4, 2)$ one obtains a tetravalent dart-transitive graph.

In the spirit of the previous section, each such graph can be obtained from an amalgam of index $(4, 2)$. Amalgams of index $(4, 2)$ were fully classified in [11, 34, 53], however, unlike in the case of amalgams of index $(3, 2)$, giving rise to cubic dart-transitive graphs, the number of these amalgams is infinite. This fact, together with existence of relatively small tetravalent dart-transitive graphs with very large automorphism groups (see Section 14) made the straightforward approach used in [9, 8] in the case of cubic graphs impossible in the case of valence 4. This obstacle was finally overcome in [37] and now a complete list of dart-transitive tetravalent graphs of order up 640 is described in [36] and available in magma code at [33]. We use $\mathrm{AT}[n, i]$ for these graphs to indicate the i-th graph of order n in this magma file.

3.3 Semitransitive graphs from universal groups

Every semitransitive tetravalent graph arises from an infinite 4-valent tree T_4 and a group G acting on T_4 semitransitively and having a finite stabiliser by quotienting out a normal semiregular subgroup of G. All such groups G were determined in [27]. This result in principle enables the same approach as used in the case of tetravalent dart-transitive graphs, and indeed, by overcoming the issue of semitransitive graphs with large automorphism groups (see [45]), a complete list of semitransitive tetravalent graphs (and in particular $\frac{1}{2}$-arc-transitive graphs) of order up to 1000 was obtained in [38] and is available at [33]. We include the $\frac{1}{2}$-arc-transitive graphs from this list in our census with the designation $\mathrm{HT}[n, i]$.

We now begin to show the notation and details of each construction used for graphs in the Census.

4 Wreaths, unworthy graphs

A general *Wreath graph*, denoted $\mathrm{W}(n, k)$, has n bunches of k vertices each, arranged in a circle; every vertex of bunch i is adjacent to every vertex in bunches $i + 1$ and $i - 1$. More precisely, its vertex set is $\mathbb{Z}_n \times \mathbb{Z}_k$; edges are all pairs of the form $\{(i, r), (i + 1, s)\}$. The graph $\mathrm{W}(n, k)$ is regular of degree $2k$. If $n = 4$, then $\mathrm{W}(n, k)$ is isomorphic to $K_{2k,2k}$ and its symmetry group is the semidirect product of $S_{2k} \times S_{2k}$ with $\mathbb{Z}_2$. If $n \neq 4$, then its symmetry group is the semidirect product of S_k^n with D_n; this group is often called the *wreath product* of D_n over S_k.

Those of degree 4 are the graphs $\mathrm{W}(n, 2)$. Here, for simplicity, we can also notate the vertex $(i, 0)$ as A_i, and $(i, 1)$ as B_i for $i \in \mathbb{Z}_n$, with edges then being of the forms $\{A_i, A_{i+1}\}, \{A_i, B_{i+1}\}, \{B_i, A_{i+1}\}, \{B_i, B_{i+1}\}$. For example, Figure 1 shows $W(7, 2)$.

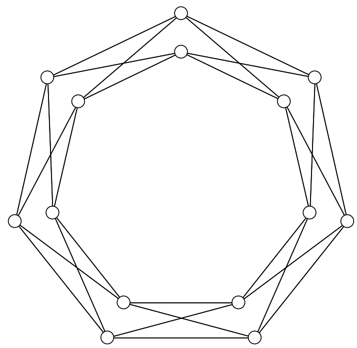

Figure 1: $W(7, 2)$

A wreath graph $\mathrm{W}(n, 2)$ has dihedral symmetries ρ and μ, where ρ sends (i, j) to $(i + 1, j)$ and μ sends (i, j) to $(-i, j)$. The important aspect of this graph is that for each $i \in \mathbb{Z}_n$, there is a symmetry σ_i, called a "local" symmetry, which interchanges $(i, 0)$ with $(i, 1)$ and leaves every other vertex fixed. Notice that ρ acts as a shunt for a cycle of length n, and $\rho\sigma_0$ acts as a shunt for a cycle of length $2n$. The third orbit of consistent cycles are those of the form $[A_i, A_{i+1}, B_i, B_{i+1}]$ The symmetry $\rho\mu\sigma_0$ is a shunt for the cycle $[A_0, B_0, A_1, B_1]$ in this orbit.

Since every σ_i for $i \neq 0$ is in the stabilizer of A_0, we see that vertex stabilizers in these graphs can be arbitrarily large. In fact, the order of the vertex-stabilizer grows exponentially with respect to the order of the graph.

A graph Γ is *unworthy* provided that some two of its vertices have exactly the same neighbors. The graph $W(n, 2)$ is unworthy because for each i, the vertices A_i and B_i have the same neighbors. The symmetry groups of vertex-transitive unworthy graphs tend to be large due to the symmetries that fix all but two vertices sharing the same neighborhood.

The paper [39] shows that there are only two kinds of tetravalent edge-transitive graphs which are unworthy. One is the dart-transitive $\mathrm{W}(n, 2)$ graphs. The other is the "subdivided double" of a dart-transitive graph; this is a semisymmetric graph given by this construction:

Construction 4.1. *Suppose that Λ is a tetravalent graph. We construct a bipartite graph $\Gamma = \mathrm{SDD}(\Lambda)$ in the following way. The white vertices of Γ correspond to edges of Λ. The black vertices correspond two-to-one to vertices of Λ; for each $v \in \mathcal{V}(\Lambda)$, there are two vertices v_0, v_1 in $\mathcal{V}(\Gamma)$. An edge of Γ joins each e to each v_i where v is a vertex of e in Λ.*

The Folkman graph on 20 vertices [13] is constructible as $\mathrm{SDD}(K_5)$. It is clear that $\mathrm{SDD}(\Lambda)$ is tetravalent, and that if Λ is dart-transitive, then $\mathrm{SDD}(\Lambda)$ is edge-transitive.

The paper [39] shows that any unworthy tetravalent edge-transitive graph is isomorphic to some $\mathrm{W}(n, 2)$ if it is dart-transitive, and to $\mathrm{SDD}(\Lambda)$ for some dart-transitive tetravalent Λ if it is semisymmetric. There are no tetravalent unworthy $\frac{1}{2}$-arc-transitive graphs.

5 Circulants

In general, the *circulant* graph $C_n(S)$ is the Cayley graph for $\mathbb{Z}_n$ with generating set S. Here S must be a subset of $\mathbb{Z}_n$ which does not include 0, but does, for each $x \in S$, include $-x$ as well. Explicitly, vertices are $0, 1, 2, \ldots, n-1$, considered as elements of $\mathbb{Z}_n$, and two numbers i, j are adjacent if their difference is in S. Thus the edge set consists of all pairs $\{i, i+s\}$ for $i \in \mathbb{Z}_n$, and $s \in S$. If $S = \{\pm a_1, \pm a_2, \ldots\}$, we usually abbreviate the name of each graph $C_n(S)$ as $C_n(a_1, a_2, \ldots)$. The numbers a_i are called *jumps* and the set of edges of the form $\{j, j + a_i\}$ for a fixed i is called a *jumpset*. Figure 2 shows an example, the graph $C_{10}(1, 3)$.

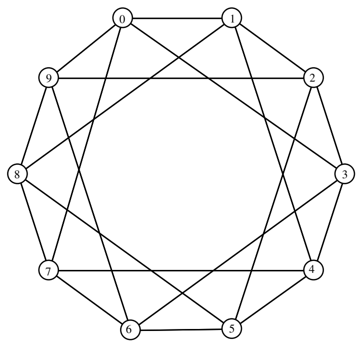

Figure 2: $C_{10}(1, 3)$

For general n, if S is a subgroup of the group $\mathbb{Z}_n^*$ of units mod n, then $C_n(S)$ is dart-transitive, though it may be so in many other circumstances as well. In the tetravalent case, there are two possibilities for an edge-transitive circulant:

Theorem 5.1. *If Γ is a tetravalent edge-transitive circulant graph with n vertices, then it is dart-transitive and either:*

(1) *Γ is isomorphic to $C_n(1, a)$ for some a such that $a^2 \equiv \pm 1 \pmod n$, or*

(2) *n is even, $n = 2m$, and Γ is isomorphic to $C_{2m}(1, m+1)$.*

Proof. Note first that every edge-transitive circulant $C_n(a, b)$ is dart-transitive, due to an automorphism which maps a vertex i to the vertex $a - i$ mod n, and thus inverts the edge $\{0, a\}$.

In (1), the dihedral group D_n acts transitively on darts of each of the two jumpsets, and because $a^2 \equiv \pm 1 \pmod n$, multiplication by a mod n induces a symmetry of the graph which interchanges the two jumpsets.

On the other hand, in (2), $C_{2m}(1, m+1)$ is isomorphic to the unworthy graph $\mathrm{W}(m, 2)$, with i and $i+m$ playing the roles of A_i and B_i. Thus, the sufficiency of (1) or (2) for edge-transitivity is clear.

The necessity can be deduced either from a complete classification of dart-transitive circulants of arbitrary valence proved in [19] and [23] or by a careful examination of short cycles in dart-transitive circulants. □

6 Toroidal graphs

The tessellation of the plane into squares is known by its Schläfli symbol, $\{4, 4\}$. Let T be the group of translations of the plane that preserve the tessellation. Then T is isomorphic to $\mathbb{Z} \times \mathbb{Z}$ and acts regularly on the vertices of the tessellation. If U is a subgroup of finite index in T, let $\mathcal{M} = \{4, 4\}/U$ be the quotient space formed from $\{4, 4\}$ by identifying points if they are in the same orbit under U. The result is a finite map of type $\{4, 4\}$ on the torus, and the paper [17] shows that, even in a more general setting, every such map arises in this way. We will call U the *kernel* of $\mathcal{M}$. A symmetry α of $\{4, 4\}$ acts as a symmetry of $\mathcal{M}$ if and only if α normalizes U. Thus every such $\mathcal{M}$ has its symmetry group $\mathrm{Aut}(\mathcal{M})$ transitive on vertices, on horizontal edges, on vertical edges; further for each edge e of $\mathcal{M}$, there is a symmetry reversing the edge (and acting as a $180°$ rotation about its center). Thus, $\mathrm{Aut}(\mathcal{M})$ is transitive on the edges of $\mathcal{M}$ (and so must be dart-transitive) if and only if U is normalized by some symmetry interchanging the horizontal and vertical parallel classes. This must be a $90°$ rotation or a reflection about some axis at a $45°$ angle to the axes.

We will use the symbols $\{4, 4\}_{b,c}$, $\{4, 4\}_{[b,c]}$, $\{4, 4\}_{<b,c>}$, intoduced below, to stand for certain maps on the torus as well as their skeletons. As shown in [50], this can happen in three different ways:

(1) $\{4, 4\}_{b,c}$: For this graph and map, defined for $b \geq c \geq 0$, U is the group generated by the translations (b, c) and $(-c, b)$. These are the well-known *rotary* maps. Because this U is normalized by $90°$ rotations, the map admits these rotations as symmetries. $\{4, 4\}_{b,c}$ has $D = b^2 + c^2$ vertices, D faces and $2D$ edges. Figure 3 shows the case when $b = 3, c = 2$.

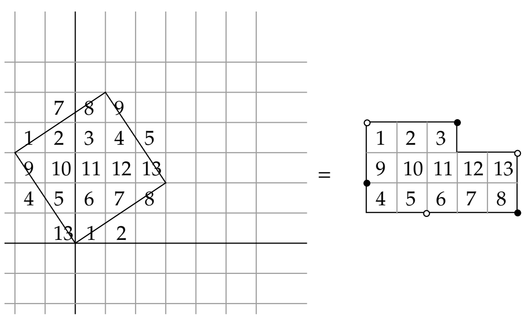

Figure 3: The map $\{4, 4\}_{3,2}$

(2) $\{4, 4\}_{<b,c>}$: This graph and map, defined for $b - 1 > c \geq 0$, uses for U the group generated by the translations (b, c) and (c, b). Because this U is normalized by re-

flections whose axes are at $45°$ to the axes, the map admits these reflections as symmetries. It has $E = b^2 - c^2$ vertices, E faces and $2E$ edges. Figure 4 shows the map $\{4,4\}_{<3,1>}$.

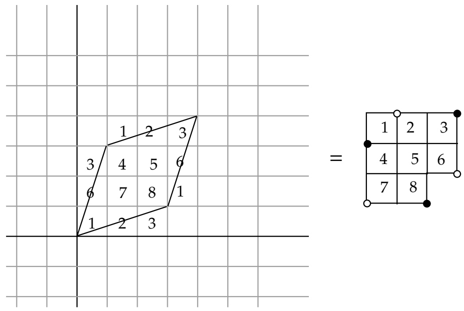

Figure 4: The map $\{4,4\}_{<3,1>}$

Notice that we exclude the case $b = c + 1$. The map $\{4,4\}_{<c+1,c>}$ exists, but its skeleton has parallel edges.

(3) $\{4,4\}_{[b,c]}$: For this graph and map, defined for $b \geq c \geq 0$, U is the group generated by the translations (b,b) and $(-c,c)$. Because this U is normalized by reflections whose axes are at $45°$ to the axes, the map admits these reflections as symmetries. It is defined only for $b \geq c > 1$. It has $F = 2bc$ vertices, F faces and $2F$ edges.

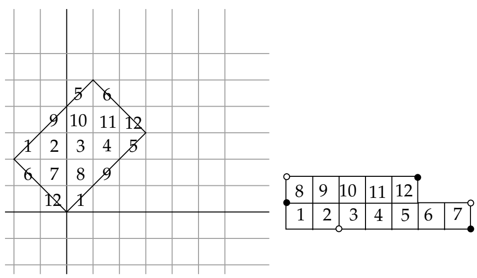

Figure 5: The map $\{4,4\}_{[3,2]}$

If $c = 1$, then the map $\{4,4\}_{[b,c]}$ exists, but, again, its skeleton has multiple edges and so is not a simple graph.

It is interesting to note that $\{4,4\}_{b+c,b-c}$ is a double cover of $\{4,4\}_{b,c}$, while $\{4,4\}_{[b+c,b-c]}$ is a double cover of $\{4,4\}_{<b,c>}$ and $\{4,4\}_{<b+c,b-c>}$ is a double cover of $\{4,4\}_{[b,c]}$. In each case, the covering is 2-fold because the kernel of the covering map has index 2 in the kernel of the covered map.

Because $\{4,4\}_{b,0}$ is isomorphic to $\{4,4\}_{<b,0>}$, this map is reflexible. Because $\{4,4\}_{b,b}$ is isomorphic to $\{4,4\}_{[b,b]}$, this map is also reflexible. All other $\{4,4\}_{b,c}$ are *chiral*: i.e., rotary but not reflexible.

Now, every U of finite index in $\mathbb{Z}^2$ can be expressed in the form $U = \langle (d,e),(f,g)\rangle$, where (d,e) and (f,g) are linearly independent. In particular, we claim that U can also be expressed in the form $U = \langle (r,0),(s,t)\rangle$, where $t = \mathrm{GCD}(e,g)$. To see that, let $e = e't, g = g't$, and let m and n be Bezout multipliers, so that $me' + ng' = 1$. Now let $s = md + nf$ and $r = g'd - e'f$. Then $\langle (r,0),(s,t)\rangle$ has a fundamental region which is a rectangle r squares wide, t squares high, with the left and right edges identified directly, and the bottom edges identified with the top after a shift s squares to the right, as in Figure 6.

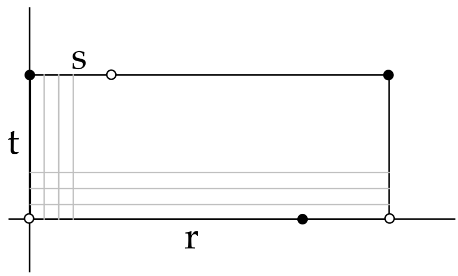

Figure 6: A standard form for maps of type $\{4,4\}$

In the special case in which b and c are relatively prime, we have $t = 1$, and this forces the graph to be circulant; in fact, the circulant graph $C_r(1,s)$. Here, $s^2 \equiv -1$ for $\{4,4\}_{b,c}$, and $s^2 \equiv 1$ for both $\{4,4\}_{<b,c>}$ and $\{4,4\}_{[b,c]}$. On the other hand, every tetravalent circulant graph is toroidal or has an embedding on the Klein bottle. More precisely, if $a^2 \equiv -1 (\mathrm{mod}\ n)$, then $C_n(1,a) \cong \{4,4\}_{b,c}$ for some b,c. If $a^2 \equiv 1 (\mathrm{mod}\ n)$, then $C_n(1,a) \cong \{4,4\}_{<b,c>}$ or $\{4,4\}_{[b,c]}$ for some b,c. Of the graphs $C_{2m}(1,m+1) \cong \mathrm{W}(m,2)$, if m is even then it is $\{4,4\}_{[\frac{m}{2},2]}$. On the other hand, if m is odd, it has an embedding on the Klein bottle, though that embedding is not edge-transitive [55].

7 Depleted wreaths

The general *Depleted Wreath graph* $\mathrm{DW}(n,k)$ is formed from $\mathrm{W}(n,k)$ by removing the edges of k disjoint n-cycles, each of these cycles containing one vertex from each of the k bunches. More precisely, its vertex set is $\mathbb{Z}_n \times \mathbb{Z}_k$. Its edge set is the set of all pairs of the form $\{(i,r),(i+1,s)\}$ for $i \in \mathbb{Z}_n$ and $r,s \in \mathbb{Z}_k$, $r \neq s$. Its vertices are of degree $2(k-1)$. It is tetravalent when $k = 3$. Figure 7 shows part of $\mathrm{DW}(n,3)$.

It is not hard to see that if $n > 4$, then the group of symmetries of this graph acts imprimmitively in two different ways. One system of blocks is the collection of sets of the form $R_i = \{(i,r) | r \in \mathbb{Z}_3\}$, defined for each $i \in \mathbb{Z}_n$. Another system is the collection of sets of the form $Q_r = \{(i,r) | i \in \mathbb{Z}_n\}$, defined for each $r \in \mathbb{Z}_3$. Then for $n > 4$, the group of symmetries of $\mathrm{DW}(n,3)$ is isomorphic to $S_3 \times D_n$.

From this, we can see that the symmetry group has an element of order $3n$ exactly when 3 does not divide n. More precisely, if $n \equiv 1$ (mod 3) then $\mathrm{DW}(n,3) \cong C_{3n}(1,n+1)$ and if $n \equiv 2$ (mod 3) then $\mathrm{DW}(n,3) \cong C_{3n}(1,n-1)$; in the remaining case, $n \equiv 0$ (mod 3), $\mathrm{DW}(n,3)$ is not a circulant.

A primary result in [39] is that, with one exception on 14 vertices, any edge-transitive tetravalent graph in which each edge belongs to at least two 4-cycles is toroidal. Because every edge of $\mathrm{DW}(n,3)$ belongs to two 4-cycles of the form $(i-1,x)-(i,y)-(i+1,x)-$

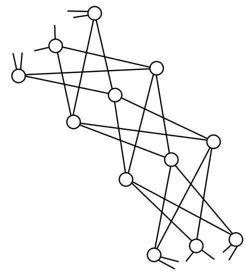

Figure 7: Part of $\mathrm{DW}(n, 3)$

$(i, z)-(i-1, x)$, where $\{x, y, z\} = \{0, 1, 2\}$, the Depleted Wreaths are also toroidal. If n is even, then $\mathrm{DW}(n, 3) \cong \{4, 4\}_{[\frac{n}{2}, 3]}$, while if n is odd, then $\mathrm{DW}(n, 3) \cong \{4, 4\}_{<\frac{n+3}{2}, \frac{n-3}{2}>}$.

8 Spidergraphs

The *Power Spidergraph* $\mathrm{PS}(k, n; r)$ is defined for $k \geq 3, n \geq 5$, and r such that $r^k \equiv \pm 1 \pmod{n}$, but $r \not\equiv \pm 1 \pmod{n}$. Its vertex set is $\mathbb{Z}_k \times \mathbb{Z}_n$, and vertex (i, j) is connected by edges to vertices $(i + 1, j \pm r^i)$. It may happen that this graph is not connected; if so, we re-assign the name to the connected component containing $(0, 0)$. Directing each edge from (i, j) to $(i + 1, j \pm r^i)$ gives a semitransitive orientation, and so the resulting graph is always semitransitive. (In this presentation, we wish to not include toroidal graphs as spidergraphs. This is what moves us to require $r \not\equiv \pm 1 \pmod{n}$ and the consequent $n \geq 5$.)

Closely related is the *Mutant Power Spidergraph* $\mathrm{MPS}(k, n; r)$. It is defined for $k \geq 3$, n even and $n \geq 8$, and r such that $r^k \equiv \pm 1 \pmod{n}$, but $r \not\equiv \pm 1 \pmod{n}$. Its vertex set is $\mathbb{Z}_k \times \mathbb{Z}_n$. For $0 \leq i < k - 1$, vertex (i, j) is connected by edges to vertices $(i + 1, j \pm r^i)$; vertex $(k-1, j)$ is connected to $(0, j \pm r^{k-1} + n/2)$. Marušič [25] and Šparl [51] have shown that every tetravalent tightly-attached graph is isomorphic to some PS or MPS graph, and that the graph is $\frac{1}{2}$-arc-transitive in all but a few cases: if $r^2 \equiv \pm 1 \pmod{n}$, then the graph is dart-transitive. The very special graph $\Sigma = \mathrm{PS}(3, 7; 2)$ is dart-transitive. If m is an integer not divisible by 7 and r is the unique solution mod $n = 7m$ to $r \equiv 5 \pmod{7}$, $r \equiv 1 \pmod{m}$, then $\mathrm{PS}(6, n; r)$ (which is a covering of Σ) is dart-transitive.

The paper [51] defines and notates these graphs in ways which differ from this paper, and the difference is worthy of note. If m, n, r, t are integers satisfying (1) m, n are even and at least 4, (2) $r^m \equiv 1 \pmod{n}$ and (3) $s = 1 + r + r^2 + \ldots r^{m-1} + 2t$ is equivalent to 0 $\pmod{n}$, then [51] defines the graph $X_e(m, n; r, t)$ ("e" stands for "even") to have vertices $[i, j]$ with $i \in \mathbb{Z}_m$ and $j \in \mathbb{Z}_n$ and edges from $[i, j]$ to $[i + 1, j]$ and $[i + 1, j + r^i]$ when $0 \leq i < m - 1$, while $[m - 1, j]$ is connected to $[0, j + t]$ and $[0, j + r^{m-1} + t]$.

The argument in [54] shows that if (1), (2) and (3) hold, then $r^m \equiv 1 \pmod{2n}$. Then it is not hard to see that if the integer s is equivalent to 0 mod $2n$, then $X_e(m, n; r, t)$ is isomorphic to $\mathrm{PS}(m, 2n; r)$; if s is equivalent to n mod $2n$, then it is isomorphic to $\mathrm{MPS}(m, 2n; r)$.

9 Attebery graphs

The following quite general construction is due to Casey Attebery [3]. We will first define the digraph $\mathrm{Att}[A, T, k; a, b]$ and then define the graph $\mathrm{Att}(A, T, k; a, b)$ to be the underlying graph of the digraph. The parameters are: an abelian group A, an automorphism T of A, an integer k at least 3, and two elements a and b of A. Let $c = b - a$ and define a_i, b_i, c_i to be aT^i, bT^i, cT^i respectively for $i = 0, 1, 2, \ldots, k$. We require that:

(1) $\{a_k, b_k\} = \{a, b\}$,

(2) A is generated by $c_0, c_1, c_2, \ldots, c_{k-1}$ and $\Sigma_{i=0}^{k-1} a_i$; and

(3) $a + b$ is in the kernel of the endomorphism $T^* = \Sigma_{i=0}^{k-1} T^i$.

Then the vertex set of the digraph $\mathrm{Att}[A, T, k; a, b]$ and of the graph $\mathrm{Att}(A, T, k; a, b)$ is defined to be $A \times \mathbb{Z}_k$. In the digraph, edges lead from each (x, i) to $(x + a_i, i + 1)$ and $(x + b_i, i + 1)$. Then the digraph is a semitransitive orientation of the graph [3]. The graph is thus semitransitive, and it is often, but not always, $\frac{1}{2}$-arc-transitive.

Not all Attebery graphs are implemented in the Census. There are four special cases which are:

1. If $A = \mathbb{Z}_n$ and T is multiplication by r, the Attebery graph is just $\mathrm{PS}(k, n; r)$, and thus the Attebery graphs are generalizations of the spidergraphs.
2. The graph called $C^{\pm 1}(p; st, s)$ in [14]. This is an Attebery graph with $A = \mathbb{Z}_p^s, k = st, T : (a_1, a_2, \ldots, a_s) \to (a_2, a_3, \ldots, a_s, a_1), -b = a = (1, 0, 0, \ldots, 0)$.
3. The graph called $C^{\pm e}(p; 2st, s)$ in [14] with $e^2 \equiv -1$ (mod p). This is an Attebery graph with $A = \mathbb{Z}_p^s, k = 2st, T : (a_1, a_2, \ldots, a_s) \mapsto (ea_2, ea_3, \ldots, ea_s, ea_1), -b = a = (1, 0, 0, \ldots, 0)$.
4. Define $\mathrm{AMC}(k, n, M)$ to be $\mathrm{Att}(A, T, k; a, b)$ where A is $\mathbb{Z}_n \times \mathbb{Z}_n$, M is a 2×2 matrix over $\mathbb{Z}_n$ satisfying $M^k = \pm I$, T is multiplication by M, and $a = (1, 0), b = (-1, 0)$.

The second and third of these are generalized to the graph $\mathrm{CPM}(n, s, t, r)$, defined in section 15 of this paper.

It is intriguing that even though the matrix

$$M = \begin{bmatrix} 1 & -4 \\ 4 & 1 \end{bmatrix}$$

does not satisfy the condition $M^4 = \pm I$, the graph $\mathrm{AMC}(4, 12, M)$ is nevertheless edge-transitive, and in fact semisymmetric. Even more striking is that so far we have no other construction for this graph.

10 The separated box product

Suppose that Δ_1 and Δ_2 are digraphs in which every vertex has in- and out-valence 2. We allow Δ_1 and Δ_2 to be non-simple.

We form the *separated box product* $\Delta_1 \# \Delta_2$ (first defined in [41]) as the underlying graph of the orientation whose vertex set is $\mathcal{V}(\Delta_1) \times \mathcal{V}(\Delta_2) \times \mathbb{Z}_2$, and whose edge set contains two types of edges: "horizontal" edges join $(a, x, 0) \to (b, x, 1)$, and "vertical" edges $(b, x, 1) \to (b, y, 0)$, where $a \to b$ in Δ_1 and $x \to y$ in Δ_2.

An orientation is *reversible* provided that it is isomorphic to its reversal. As shown in Theorem 5.1 of [41], there are several useful cases of this construction:

- When $\Delta_1 = \Delta_2$ and Δ_1 is reversible, then $\Delta_1\#\Delta_2$ is dart-transitive.
- When $\Delta_1 = \Delta_2$ and Δ_1 is not reversible, $\Delta_1\#\Delta_2$ has a semitransitive orientation and so might be dart-transitive or $\frac{1}{2}$-transitive.
- When Δ_1 is not isomorphic to Δ_2 or its reverse but both are reversible, then $\Delta_1\#\Delta_2$ has an LR structure.
- When Δ_1 is not isomorphic to Δ_2 but is isomorphic to the reverse of Δ_2, then $\Delta_1\#\Delta_2$ is at least bi-transitive and is semisymmetric in all known cases.

In the Census, we use for Δ_1 and Δ_2 directed graphs from the census of 2-valent dart-transitive digraphs [38, 33] with notation $\mathrm{ATD}[n, i]$ for the i^{th} digraph of order n from that census. We also allow the "sausage digraph" DCyc_n: an n-cycle with each edge replaced by two directed edges, one in each direction. See [41] for more details.

11 Rose windows

The *Rose Window graph* $\mathrm{R}_n(a, r)$ has $2n$ vertices: A_i, B_i for $i \in \mathbb{Z}_n$. The graph has four kinds of edges:

Rim: $A_i - A_{i+1}$
In-Spoke: $A_i - B_i$
Out-spoke: $B_i - A_{i+a}$
Hub: $B_i - B_{i+r}$

For example, Figure 8 shows $\mathrm{R}_{12}(2, 5)$.

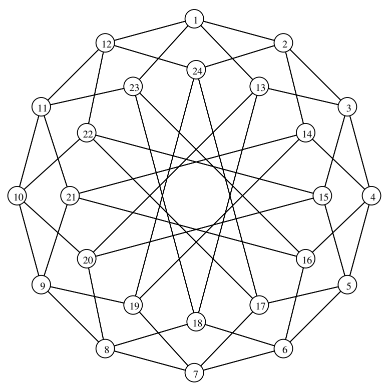

Figure 8: $\mathrm{R}_{12}(2, 5)$

We will soon mention these graphs as examples of bicirculant graphs, and more generally later as examples of polycirculant graphs.

The paper [20] shows that every edge-transitive $\mathrm{R}_n(a, r)$ is dart-transitive and is isomorphic to one of these:

a. $\mathrm{R}_n(2, 1)$. (This graph is isomorphic to $\mathrm{W}(n, 2)$.)

b. $\mathrm{R}_{2m}(m+2, m+1)$.

c. $\mathrm{R}_{12k}(3k \pm 2, 3k \mp 1)$.

d. $\mathrm{R}_{2m}(2b, r)$, where $b^2 \equiv \pm 1 \pmod{m}$, r is odd and $r \equiv 1 \pmod{m}$.

12 Bicirculants

A graph Γ is *bicirculant* provided that it has a symmetry ρ which acts on $\mathcal{V}$ as two cycles of the same length. A rose window graph is bicirculant, as the symmetry sending A_i to A_{i+1} and B_i to B_{i+1} is the required ρ.

The paper [21] classifies all edge-transitive tetravalent bicirculant graphs. Besides the Rose Window graphs there is one other class of graphs, called BC4 in that paper, and called BC here and in the Census. The graph $\mathrm{BC}_n(a, b, c, d)$ has $2n$ vertices: A_i, B_i for $i \in \mathbb{Z}_n$. The edges are all pairs of the form $\{A_i, B_{i+e}\}$ for $i \in \mathbb{Z}_n, e \in \{a, b, c, d\}$. It is easy to see that any such graph is isomorphic to one of the form $\mathrm{BC}_n(0, a, b, c)$ where a divides n. The edge-transitive graphs in this class consist of three sporadic examples and three infinite families. The sporadics are:

$$\mathrm{BC}_7(0,1,2,4),\ \mathrm{BC}_{13}(0,1,3,9),\ \mathrm{BC}_{14}(0,1,4,6)$$

Of the three infinite families of graphs $\mathrm{BC}_n(0, a, b, c)$, there are two in which we can choose $a = 1$, and a third, less easy to describe, in which none of the parameters is relatively prime to n:

(I) $\mathrm{BC}_n(0, 1, m+1, m^2+m+1)$, where $(m+1)(m^2+1) = 0 \pmod{n}$

(II) $\mathrm{BC}_n(0, 1, d, 1-d)$, where $2d(1-d) = 0 \pmod{n}$

(III) $\mathrm{BC}_{krst}(0, r, rs's + st, rt't + st + rst)$, where

- (1) r, s, t are all integers greater than 1,
- (2) s, t are odd,
- (3) r, s, t are relatively prime in pairs,
- (4) $k \in \{1, 2\}$,
- (5) if $k = 2$, then r is even,
- (6) s' is an inverse of s mod krt, and t' is an inverse of t mod krs.

13 Semiregular symmetries and their diagrams

A symmetry σ is *semiregular* provided that it acts on the $|\mathcal{V}| = kn$ vertices as k cycles of length n. We can visually represent such a graph and symmetry with a *diagram*. This is a graph-like object, with labels. Each "node" represents one orbit under σ. If $\sigma = (u_0, u_1, \ldots, u_{n-1})(v_0, v_1, \ldots, v_{n-1}) \ldots (w_0, w_1, \ldots, w_{n-1})$ and there is an edge from u_0 to v_a, then there is an edge from each u_i to v_{a+i} (indices computed modulo n). This matching between $u_i's$ and $v_i's$ is represented in the diagram by a directed edge from node u to node v with label a (or one from v to u with label $-a$). If $a = 0$, then the label a and the direction of the edge in the diagram can be omitted.

If there is an edge from u_0 to u_b (and thus one from each u_i to u_{i+b}), we represent this by a loop at u with label b. In the special case in which n is even and $b = \frac{n}{2}$, there are only

$\frac{n}{2}$ edges in the orbit and we represent them with a semi-edge at u. This convention makes the valence of u in the diagram the same as the valences of all of the u_i's in the graph.

For example, the graph in Figure 9 has the symmetry

$$\varphi = (u_0u_1u_2u_3u_4u_5)(v_0v_1v_2v_3v_4v_5)(w_0w_1w_2w_3w_4w_5).$$

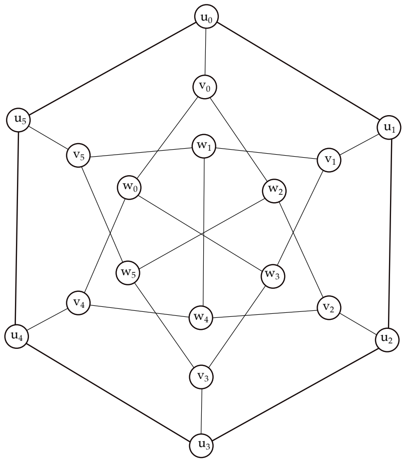

Figure 9: A trivalent graph having a semiregular symmetry

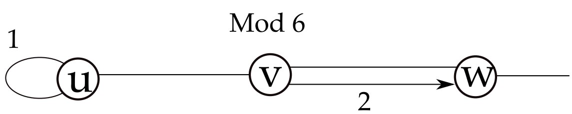

Figure 10: The diagram of the semiregular symmetry φ

The corresponding diagram is shown in Figure 10. The label "Mod 6" in this diagram informs the reader that all numbers which label edges in the diagram are to be considered as elements of $\mathbb{Z}_6$.

It should be pointed out that what we were describing above is simply the notion of a quotient of a graph (as defined, for example, in [24]) by the cyclic group generated by the semiregular element σ, and the diagram which we obtain is the voltage graph describing the graph Γ.

Further, the graph can be reconstructed from the diagram. This is simply the ordinary voltage graph construction with voltage group $\mathbb{Z}_n$ and voltages of the darts as shown with the integers drawn at the darts in the diagram (if neither the dart nor its inverse dart have any integers drawn next to them, then the corresponding voltage is assumed to be 0).

In constructions, we often refer to the generic form of the diagram, in which the modulus is unspecified and parameter names are used as labels for some edges, as in Figure 11.

As an example, consider the bicirculant graphs $\mathrm{R}_n(a, r)$ and $\mathrm{BC}_n(0, a, b, c)$, which can be represented by the diagrams shown in Figure 12.

In the following, we use diagrams to define many families of graphs.

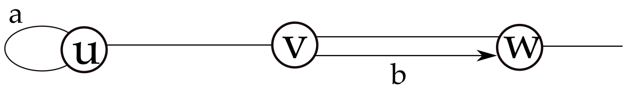

Figure 11: A generic diagram

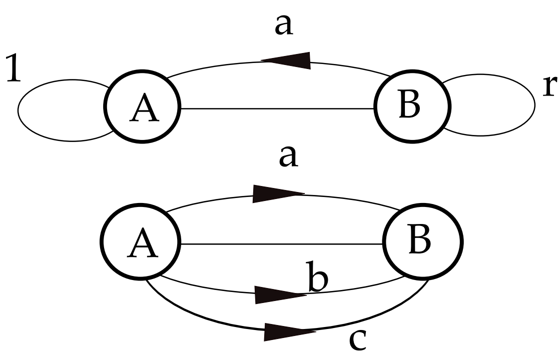

Figure 12: Diagrams of graphs

13.1 Propellors

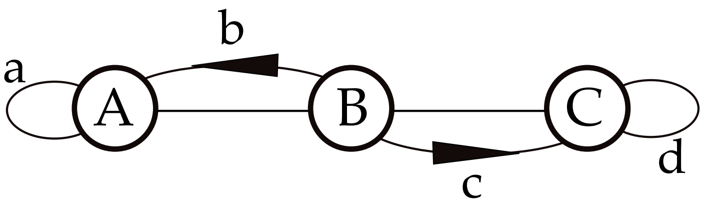

Figure 13: Diagram for the graph $\mathrm{Pr}_n(a, b, c, d)$

A *Propellor Graph* is a graph with the diagram shown in Figure 13. This means that the graph has $3n$ vertices:A_i, B_i, C_i for $i \in \mathbb{Z}_n$. There are 6 kinds of edges:

Tip:	$A_i - A_{i+a}$
	$C_i - C_{i+d}$
Flat:	$A_i - B_i$
	$B_i - C_i$
Blade:	$B_i - A_{i+b}$
	$B_i - C_{i+c}$

Propellor graphs have been investigated by Matthew Sterns. He conjectures in [46] that the edge-transitive propellor graphs are isomorphic to

I $\mathrm{Pr}_n(1, 2d, 2, d)$, where n is even and $d^2 \equiv \pm 1 \pmod{n}$

II $\mathrm{Pr}_n(1, b, b + 4, 2b + 3)$, where n is divisible by 4, $8b + 16 \equiv 0 \pmod{n}$, and $b \equiv 1 \pmod{4}$.

III These five sporadic examples:

$$\mathrm{Pr}_5(1,1,2,2), \mathrm{Pr}_{10}(1,1,2,2),\ \mathrm{Pr}_{10}(1,4,3,2),\ \mathrm{Pr}_{10}(1,1,3,3),\ \mathrm{Pr}_{10}(2,3,1,4).$$

Notice first that in every case except the last of the sporadic cases, the A-tip edges form a single cycle. The first of the infinite families consists of the *2-weaving* graphs: those graphs in which some symmetry of the graphs sends this cycle to a cycle of the form AB AB AB... The second class consists of *4-weaving* graphs: those in which some symmetry sends the A-tip cycle to a cycle of the form ABCB ABCB AB... There is also a class of *5-weaving* graphs in which some image of the tip cycle is of the form AABCB AABCB AAB..., but the requirements for this class force n to divide 10, resulting in the first four of the sporadic cases. This leaves the last sporadic graph as something of a mystery.

A more recent paper [47] proves that if a or d is relatively prime to n, then the conjecture holds.

13.2 Metacirculants

In [29], Marušič and Šparl considered tetravalent graphs which are properly called *weak metacirculant*, though we will simply refer to them as *metacirculant* in this paper. A graph is metacirculant provided that it has a symmetry ρ which acts on the mn vertices as m cycles of length n and another symmetry σ which normalizes $\langle\rho\rangle$ and permutes the m ρ-orbits in a cycle of length m. That paper considers four classes of $\frac{1}{2}$-transitive tetravalent metacirculant graphs. The four classes together include all such graphs.

The Type I graphs are the Power Spidergraphs $\mathrm{PS}(m, n; r)$ and $\mathrm{MPS}(m, n; r)$. Papers [25, 29, 51, 54] completely determine which of these are $\frac{1}{2}$-transitive and which are dart-transitive. The Type II graphs are called Y there and will be called MSY here and in the census. These also have been classified, in unpublished work [2]. See Section 13.2.1 below.

The graphs of Type III we will call MC3 in this census. They have been studied with a few results. See Section 13.2.3. While the general Type IV graphs are very unruly, there are some results known; for instance, the paper [1] classifies Type IV $\frac{1}{2}$-arc-transitive metacirculants of girth 4. A subclass of Type IV graphs, called Z in [29] and MSZ in the Census, have received some concentrated study. See Section 13.2.2 for a description of the graph.

13.2.1 MSY

The graph $\mathrm{MSY}(m, n; r, t)$ has the diagram shown in Figure 14.

More precisely, its vertex set is $\mathbb{Z}_m \times \mathbb{Z}_n$, with two kinds of edges:

(1) $(i, j) - (i, j + r^i)$ for all i and j, and

(2) $(i, j) - (i + 1, j)$ for $0 \leq i < m - 1$ and $(m - 1, j) - (0, j + t)$ for all j.

The graph is seen to be metacirculant (with ρ sending (i, j) to $(i, j + 1)$ and σ sending (i, j) to $(i + 1, rj)$ for $i \neq m - 1$ and sending $(m - 1, j)$ to $(0, rj + t)$) if and only if $r^m = 1$ and $rt = t$. Here, all equalities are equivalences mod n. The paper [52] proves that $\mathrm{MSY}(m, n; r, t)$ is metacirculant and $\frac{1}{2}$-transitive if and only if it is isomorphic to one in which:

(1) $n = dm$ for some integer d at least 3,

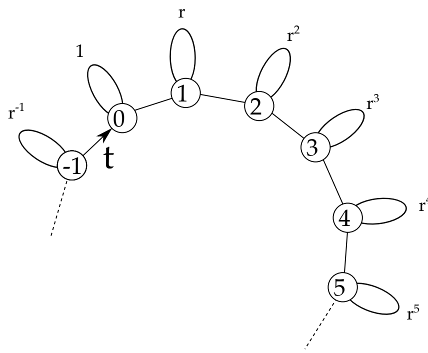

Figure 14: Diagram for the graph $\mathrm{MSY}(m,n;r,t)$

(2) $r^m = 1$,

(3) $r^2 \neq \pm 1$,

(4) $m(r-1) = t(r-1) = (r-1)^2 = 0$,

(5) $\langle m \rangle = \langle t \rangle$ in $\mathbb{Z}_n$,

(6) there is a unique c in $\mathbb{Z}_d$ satisfying $cm = t$ and $ct = m$,

(7) there is a unique k in $\mathbb{Z}_d$ satisfying $kt = -km = r-1$, and

(8) either $m \neq 4$ or $t \neq 2+2r$.

The paper [2] shows that $\mathrm{MSY}(m,n;r,t)$ is metacirculant and edge-transitive if and only if it is isomorphic to one in which $r^m = 1, rt = t$ and one of three things happens:

(1) $m = GCD(t,n)$ (then $t = sm, n = n'm$ and $GCD(n',s) = 1$);

(2) $r \equiv 1 \pmod{m}$ and so $r = km+1$ for some k;

(3) $st = m$;

(4) $kt = -km$.

or

(1) $m = GCD(t,n)$ (then $t = sm, n = n'm$ and $GCD(n',s) = 1$);

(2) $r \equiv 1 \pmod{m}$ and so $r = km+1$ for some k;

(3) $st = -m$;

(4) $kt = km$.

or $[m,n,r,t]$ is one of these four sporadic examples:

$$[5,11,5,0],\ [5,22,5,11],\ [5,33,16,0],\ [5,66,31,33].$$

13.2.2 MSZ

The graph $\mathrm{MSZ}(m,n;k,r)$ has a diagram isomorphic to the circulant graph $C_m(1,k)$. It has vertex set $\mathbb{Z}_m \times \mathbb{Z}_n$. The vertex (i,j) is adjacent to $(i+1,j)$ and to $(i+k, j+r^i)$.

13.2.3 MC3

The diagram for this family has an even number of nodes arranged in a circle. With $c = 1$ or 2, each node is joined by an edge to nodes that are c steps away in the circle. Each node is joined to the node opposite by two edges.

We call the graph $\Gamma = \mathrm{MC3}(m, n, a, b, r, t, c)$; here, m must be even, c is 1 or 2, and a, b, r, t are numbers mod n such that $rt = t, r^m = 1$, and $\{a+t, b+t\} = \{-ar^{\frac{m}{2}}, -br^{\frac{m}{2}}\}$. The vertex set is $\mathbb{Z}_m \times \mathbb{Z}_n$. One kind of edge connects each (i, j) to $(i + c, j)$ for $i = 0, 1, 2, \ldots, m - c - 1$, and (i, j) to $(i + c, j + t)$ for $i \in \{m - c, m - 1\}$. A second kind of edge joins each (i, j) to $(i + \frac{m}{2}, j + ar^i)$ and $(i + \frac{m}{2}, j + br^i)$.

Because it has been shown that each such graph which is $\frac{1}{2}$-arc-transitive is also a $\mathrm{PS}, \mathrm{MPS}, \mathrm{MSY}$ or MSZ, little attention has been given to it. However, many MC3's are dart-transitive and many are LR structures (see section 18, where we will refer to the two kinds of edges as *green* and *red*). Each of the following families is such an example:

1. m is divisible by 4, n is divisible by 2, $r^2 = \pm 1, a = 1, b = -1, t = 0$,
2. m is not divisible by 4, n is divisible by 4, $r^2 = 1, a = 1, b = -1, t = \frac{n}{2}$,
3. n is divisible by 4, $r^2 = 1, a = 1, b = n/2 - 1, t = n/2$.

These were found and proved to be LR strucures (see section 18) by Ben Lantz [22], and there are LR examples not covered by these families. Further, many of the MC3 graphs are dart-transitive. There are many open questions about this family.

13.3 Other diagrams

A number of other diagrams have been found to give what appears to be an infinite number of examples of edge-transitive graphs. The first of these is the Long Propellor, $\mathrm{LoPr}_n(a, b, c, d, e)$, shown in Figure 15.

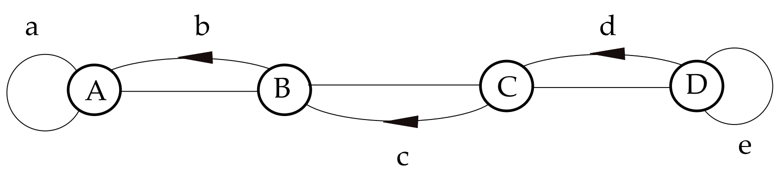

Figure 15: Diagram for the graph $\mathrm{LoPr}_n(a, b, c, d, e)$

Next is the Woolly Hat, $\mathrm{WH}_n(a, b, c, d)$, shown in Figure 16. This diagram appears to give no edge-transitive covers, but it does yield a family of LR structures (see Section 18), as yet unclassified.

The Kitten Eye, $\mathrm{KE}_n(a, b, c, d, e)$, shown in Figure 17, has dart-transitive covers.

The Curtain, $\mathrm{Curtain}_n(a, b, c, d, e)$, shown in Figure 18, has both dart-trasitive and LR covers.

14 Praeger-Xu constructions

The graph called $C(2, n, k)$ in [44] is also described in [14] in two different ways. In this Census, we will name the graph $\mathrm{PX}(n, k)$. In order to describe the graph, we need some

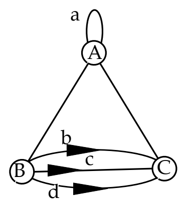

Figure 16: Diagram for the graph $\mathrm{WH}_n(a, b, c, d, e)$

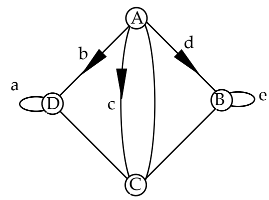

Figure 17: Diagram for the graph $\mathrm{KE}_n(a, b, c, d, e)$

notation about bit strings. A *bit string* of length k is the concatenation of k symbols, each of them a '0' or a '1'. For example x = 0011011110 is a bit string of length 10. If x is a bit string of length k, then x_i is its i-th entry, and x^i is the string identical to x in every place except the i-th. Also $1x$ is the string of length $k + 1$ formed from x by placing a '1' in front; similar definitions hold for the $(k + 1)$-strings $0x, x0, x1$. Finally, the string $\bar{x}$ is the reversal of x.

The vertices of the graph $\mathrm{PX}(n, k)$ are ordered pairs of the form (j, x), where $j \in \mathbb{Z}_n$ and x is a bit string of length k. Edges are all pairs of the form $\{(i, 0x), (i+1, x0)\}, \{(i, 0x), (i+1, x1)\}, \{(i, 1x), (i+1, x0)\}, \{(i, 1x), (i+1, x1)\}$, where x is any bit string of length $k-1$.

We first wish to consider some symmetries of this graph. First we note ρ and μ given by $(j, x)\rho = (j + 1, x)$ and $(j, x)\mu = (-j, \bar{x})$ These are clearly symmetries of the graph and act on it as D_r.

For $b \in \mathbb{Z}_n$, we define the symmetry τ_b to be the permutation which interchanges $(b - i, x)$ with $(b - i, x^i)$ for $i = 1, 2, 3, \ldots, k$ and leaves all other vertices fixed. If $n > k$, then the symmetries $\tau_0, \tau_1, \ldots, \tau_{n-1}$ commute with each other and thus generate an elementary abelian group of order 2^n, while the symmetries ρ, μ and τ_0 generate a semidirect product $\mathbb{Z}_2^n \rtimes D_n$ of order $n2^{n+1}$. Unless $n = 4$, this is also the full symmetry group of the graph (see [44]).

The Praeger-Xu graphs generalize two families of graphs: $\mathrm{PX}(n, 1) = \mathrm{W}(n, 2)$ and $\mathrm{PX}(n, 2) = \mathrm{R}_{2n}(n + 2, n + 1)$.

15 Gardiner-Praeger constructions

The paper [14] constructs two families of tetravalent graphs whose groups contain large normal subgroups such that the factor graph is a cycle. The first is $C^{\pm 1}(p, st, s)$, and the second is $C^{\pm e}(p, 2st, s)$. In the Census, we use a slight generalization of both, which we notate $\mathrm{CPM}(n, s, t, r)$, where n is any integer at least 3, s is an integer at least 2, t is a

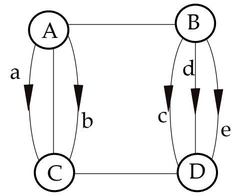

Figure 18: Diagram for the graph $\mathrm{Curtain}_n(a, b, c, d, e)$

positive integer, and r is a unit mod n. We first form the digraph $\mathrm{CPM}[n, s, t, r]$. Its vertex set is $\mathbb{Z}_n^s \times \mathbb{Z}_{st}$. Directed edges are of the form $((x, i), (x \pm r^i e_j, i + 1))$, where j is i mod s, and e_j is the j-th standard basis vector for $\mathbb{Z}_n^s$. If r^{st} is ± 1 mod n, then $\mathrm{CPM}[n, s, t, r]$ is a semitransitive orientation for its underlying graph $\mathrm{CPM}(n, s, t, r)$. When the graph is not connected, we re-assign the name $\mathrm{CPM}(n, s, t, r)$ to the component containing $(0, 0)$. If n is odd, then the graph has stn^s vertices. If n is even then it has $st(\frac{n}{2})^s$ vertices if t is even and twice that many if t is odd.

Some special cases are known: For $r = 1, \mathrm{CPM}(n, s, t, 1) \cong C^{\pm 1}(n, st, s)$. If $r^2 = 1$, then $\mathrm{CPM}(n, s, 2t, r)$ is $C^{\pm r}(p, 2st, s)$. When $s = 1, \mathrm{CPM}(n, 1, t, r) \cong \mathrm{PS}(n, t; r)$. If $s = 1$ and $t = 4$, then the graph is a Wreath graph. When $s = 2$ and t is 1 or 2, then the graph is toroidal. Other special cases are conjectured:

The convention in the following conjectures is that q is a number whose square is one mod m and p is the parity function:

$$p(t) = \begin{cases} 1 & \text{if t is odd} \\ 2 & \text{if t is even} \end{cases} \tag{15.1}$$

With that said, we believe that:

1. If t is not divisible by 3, then $\mathrm{CPM}(3, 2, t, 1) \cong \mathrm{PS}(6, m; q)$ where $m = 3t$.
2. If t is not divisible by 5, then $\mathrm{CPM}(5, 2, t, 1) \cong \mathrm{CPM}(5, 2, t, 2) \cong \mathrm{PS}(10, m; q)$ where $m = 5tp(t)$.
3. If t is not divisible by 3, then $\mathrm{CPM}(6, 2, t, 1) \cong \mathrm{PS}(6, m; q)$ where $m = \frac{12t}{p(t)}$.
4. If t is not divisible by 4, then $\mathrm{CPM}(8, 2, t, 1) \cong \mathrm{CPM}(8, 2, t, 3) \cong \mathrm{MPS}(8, m; q)$ where $m = \frac{16t}{p(t)}$.
5. For all s, $\mathrm{CPM}(4, s, t, 1) \cong \mathrm{PX}(\frac{2st}{p(t)}, s)$.

16 Graphs $\Gamma^{\pm}$ of Spiga, Verret and Potočnik

It was proved in [37] that a tetravalent graph Γ whose automorphism group G is dart-transitive is either 2-arc-transitive (and then $|G_v| \leq 2^4 3^6$), a PX-graph (see Section 14), one of eighteen exceptional graphs, or it satisfies the inequality

$$|V(\Gamma)| \geq 2|G_v| \log_2(|G_v|/2). \tag{$*$}$$

This result served as the basis for the construction of a complete list of dart-transitive tetravalent graphs (see [36] for details).

Moreover, in [35] it was proved that a graph attaining the bound $(*)$ has $t2^{t+2}$ vertices for some $t \geq 2$ and is isomorphic to one of the graphs (t, ϵ), for $\epsilon \in \{0, 1\}$, defined below as *coset graphs* of certain groups G_t^0 or G_t^1. Recall that the coset graph $\mathrm{Cos}(G, H, a)$ on a group G relative to a subgroup $H \leq G$ and an element $a \in G$ is defined as the graph with vertex set the set of right cosets $G/H = \{Hg \mid g \in G\}$ and with edge set the set $\{\{Hg, Hag\} \mid g \in G\}$.

For $\epsilon \in \{0, 1\}$ and $t \geq 2$, let G_t^ϵ be the group defined as follows:

$$\begin{aligned} G_t^\epsilon = \langle x_0, \ldots, x_{2t-1}, z, a, b \mid\; & x_i^2 = z^2 = b^2 = z^\epsilon a^{2t} = (ab)^2 = 1, \\ & [x_i, z] = 1 \text{ for } 0 \leq i \leq 2t-1, \\ & [x_i, x_j] = 1 \text{ for } |i-j| \neq t, \\ & [x_i, x_{t+i}] = z \text{ for } 0 \leq i \leq t-1, \\ & x_i^a = x_{i+1} \text{ for } 0 \leq i \leq 2t-1, \\ & x_i^b = x_{t-1-i} \text{ for } 0 \leq i \leq 2t-1 \rangle. \end{aligned}$$

In either group, we let $H = \langle x_0, \ldots, x_{t-1}, b \rangle$, and define graphs

$$\mathrm{PPM}(t, \epsilon) = \mathrm{Cos}(G_t^\epsilon, H, a),$$

denoted Γ_t^+ (for $\epsilon = 0$) and Γ_t^- (for $\epsilon = 1$) in [35]. We should point out that a graph $\Gamma = \mathrm{PPM}(t, \epsilon)$ is a 2-cover of the Praeger-Xu graph $\mathrm{PX}(2t, t)$. Furthermore, the girth of $\mathrm{PPM}(t, \epsilon)$ is generally 8, the only exceptions being that $\mathrm{PPM}(2, 0)$ has girth 4, and $\mathrm{PPM}(3, 0)$ has girth 6. Finally, $\mathrm{PPM}(2, 0) \cong \mathrm{PX}(4, 3)$, while in all other cases the graph Γ is not isomorphic to a PX graph.

The 2016 edition of the Census seems, mysteriously, to have missed implementing this construction. The next edition *will* include these graphs. Meanwhile, we know that exactly 6 of them fall in the range of sizes addressed in the Census. The resulting entries are:

PPM(2,0) = C4[32,2] = $\{4, 4\}_{4,4}$.
PPM(2,1) = C4[32,5] = $\mathrm{MSY}(4, 8, 5, 4)$.
PPM(3,0) = C4[96,27] = $\mathrm{KE}_{24}(1, 22, 8, 3, 7)$.
PPM(3,1) = C4[96,24] = $\mathrm{KE}_{24}(1, 13, 4, 21, 5)$.
PPM(4,0) = C4[256,72] = UG(ATD[256, 128]).
PPM(4,1) = C4[256,78] = UG(ATD[256, 146]).

17 From cubic graphs

In this section, we describe five constructions, each of which constructs a tetravalent graph from a smaller cubic (i.e., trivalent) graph in such a way that the larger graph inherits many symmetries from the smaller graph. Throughout this section, assume that Λ is a cubic graph, and that it is dart-transitive. Our source of these graphs is Marston Conder's census of symmetric cubic graphs of up to 10,000 vertices [8].

17.1 Line graphs

The *line graph* of Λ is a graph $\Gamma = L(\Lambda)$ whose vertices are, or correspond to, the edges of Λ. Two vertices of Γ are joined by an edge exactly when the corresponding edges of Λ share a vertex. Every symmetry of Λ acts on Γ as a symmetry, though Γ may have other symmetries as well. Clearly, if Λ is edge-transitive, then Γ is vertex-transitive. If Λ is dart-transitive, then Γ is edge-transitive, and if Λ is 2-arc-transitive, then Γ is dart-transitive.

17.2 Dart graphs

The *Dart Graph* of Λ is a graph $\Gamma = \mathrm{DG}(\Lambda)$ whose vertices are, or correspond to, the darts of Λ. Edges join a dart (a, b) to the dart (b, c) whenever a and c are distinct neighbors of b. Clearly, $\mathrm{DG}(\Lambda)$ is a two-fold cover of $L(\Lambda)$.

17.3 Hill capping

For every vertex A of Λ, we consider the *symbols* $(A, 0), (A, 1)$, though we will ususally write them as A_0, A_1. Vertices of $\Gamma = \mathrm{HC}(\Lambda)$ are all unordered pairs $\{A_i, B_j\}$ of symbols where $\{A, B\}$ is an edge of Λ. Edges join each vertex $\{A_i, B_j\}$ to $\{B_j, C_{1-i}\}$ where A and C are distinct neighbors of B.

If Λ is bipartite and 2-arc-transitive then Γ is dart-transitive. If Λ is bipartite and *not* 2-arc-transitive then Γ is semisymmetric. If Λ is *not* bipartite and is 2-arc-transitive then Γ is $\frac{1}{2}$-arc-transitive.

$\mathrm{HC}(\Lambda)$ is clearly a fourfold cover of $L(\Lambda)$; it is sometimes but not always a twofold cover of $\mathrm{DG}(\Lambda)$. The Hill Capping is described more fully in [15].

17.4 3-arc graph

The three-arc graph of Λ, called $A_3(\Lambda)$ in the literature [18] and called $\mathrm{TAG}(\Lambda)$in the Census, is a graph whose vertices are the darts of Λ, with (a, b) adjacent to (c, d) exactly when $[b, a, c, d]$ is a 3-arc in Λ. Thus, a and c are adjacent, $b \neq c$ and $a \neq d$. This graph is dart-transitive if Λ is 3-arc-transitive.

18 Cycle decompositions

A *cycle decomposition* of a tetravalent graph Λ is a partition $\mathcal{C}$ of its edges into cycles. Every edge belongs to exactly one cycle in $\mathcal{C}$ and each vertex belongs to exactly two cycles of $\mathcal{C}$. $\mathrm{Aut}(\mathcal{C})$ is the group of all symmetries of Λ which preserve $\mathcal{C}$. One possibility for such a symmetry is a *swapper*. If v is a vertex on the cycle C, a $C-$*swapper* at v is a symmetry which reverses C while fixing v and every vertex on the other cycle through v.

If $\mathcal{C}$ is a cycle decomposition of Λ, the *partial line graph* of $\mathcal{C}$, written $\mathbb{P}(\mathcal{C})$ and notated $\mathrm{PL}(\mathcal{C})$ in the Census, is a graph Γ whose vertices are (or correspond to) the edges of Λ, and whose edges are all $\{e, f\}$ where e and f are edges which share a vertex but belong to different cycles of $\mathcal{C}$.

Because $\mathrm{Aut}(\mathcal{C})$ acts on Γ as a group of its symmetries, the partial line graph is useful for constructing graphs having a large symmetry group. Almost all tetravalent dart-transitive graphs have cycle decompositions whose symmetry group is transitive on darts. These are called "cycle structures" in [31].

If Λ is $\frac{1}{2}$-arc-transitive, then it has a cycle decomposition $\mathcal{A}$ into 'alternating cycles' [25]. If the stabilizer of a vertex has order at least 4, then $\mathbb{P}(\mathcal{A})$ has a $\frac{1}{2}$-arc-transtive action and may actually be $\frac{1}{2}$-arc-transitive.

Many graphs in the census are constructed from smaller ones using the partial line graph. Important here are the Praeger-Xu graphs. Each $\mathrm{PX}(n, k)$ has a partition $\mathcal{C}$ of its edges into 4-cycles of the form:$(i, 0x), (i+1, x0), (i, 1x), (i+1, x1)$. Then $\mathbb{P}(\mathcal{C})$ is isomorphic to $\mathrm{PX}(n, k+1)$. A special case of this is that family (b) of Rose Window graphs is $\mathbb{P}$ applied to family (a), the wreath graphs.

The toroidal graphs have a cycle decomposition in which each cycle consists entirely of vertical edges or entirely of horizontal edges. The partial line graph of this cycle decomposition is another toroidal graph. For the rotary case, $\mathbb{P}(\{4,4\}_{b,c}) = \{4,4\}_{b+c,b-c}$. The other two families of toroidal maps need not have edge-transitive partial line graphs.

It may happen that a cycle decomposition $\mathcal{C}$ is a 'suitable LR structure' [42]; this means that $\mathcal{C}$ has a partition into $\mathcal{R}$ and $\mathcal{G}$ (the 'red' and the 'green' cycles) such that every vertex belongs to one cycle from each set, that the subgroup of $\mathrm{Aut}(\mathcal{C})$ which sends $\mathcal{R}$ to itself is transitive on the vertices of Λ, that $\mathrm{Aut}(\mathcal{C})$ has all possible swappers, that no element of $\mathrm{Aut}(\mathcal{C})$ interchanges $\mathcal{R}$ and $\mathcal{G}$ and, finally, that no 4-cycle alternates between $\mathcal{R}$ and $\mathcal{G}$. With all of that said, if $\mathcal{C}$ is a suitable LR structure, then $\mathbb{P}(\mathcal{C})$ is a semisymmetric tetrtavalent graph in which each edge belongs to a 4-cycle. Further, every such graph is constructed in this way. [42]

19 Some LR structures

The Census uses the partial line graph construction on a number of families of LR structures, shown here.

19.1 Barrels

The barrels are the most common of the suitable LR structures. The standard barrel is $\mathrm{Br}(k, n; r)$, where k is an even integer at least 4, n is an integer at least 5 and r is a number mod n such that $r^2 = \pm 1 \pmod n$ but $r \neq \pm 1 \pmod n$. The vertex set is $\mathbb{Z}_k \times \mathbb{Z}_n$. Green edges join each (i, j) to $(i, j + r^i)$. Red edges join each (i, j) to $(i + 1, j)$.

The mutant barrel is $\mathrm{MBr}(k, n; r)$, where k is an even integer at least 2, n is an *even* integer ≥ 8 and r is a number mod n such that $r^2 = \pm 1 \pmod n$ but $r \neq \pm 1 \pmod n$. The vertex set is $\mathbb{Z}_k \times \mathbb{Z}_n$. Green edges join each (i, j) to $(i, j + r^i)$. Red edges join each (i, j) to

$$\begin{cases} (i+1, j) & \text{if } i \neq k-1 \\ (0, j + \frac{n}{2}) & \text{if } i = k-1 \end{cases} .$$

19.2 Cycle structures

If Λ is a tetravalent graph admiting a cycle structure $\mathcal{C}$, we can form an LR struture from it in two steps:

1. Replace each vertex v with two vertices, each incident with the two edges of one of the two cycles in $\mathcal{C}$ containing v; think of these as green edges. Join the two vertices corresponding to v with two parallel red edges. Call this C'. We can cover C' in different ways.

2a. Double cover C'. We assign weights or voltages to red edges so that each pair has one 0 and one 1. Voltages for green edges are assigned in one of two ways: (0) every green edge gets voltage 0 or (1) one edge in each green cycle gets voltage 1, and the rest get 0. The double covers corresponding to these two assignments are called $\mathrm{CS}(\Lambda, \mathcal{C}, 0)$ and $\mathrm{CS}(\Lambda, \mathcal{C}, 1)$, respectively, and they are, in most cases, suitable LR structures, as shown in [43].

2b. Cover C' k-fold for some k at least 3. We assign weights or voltages to red edges so that each pair has a 1 in each direction. All green edges are assigned voltage 0. The k-cover corresponding to this assignment is called $\mathrm{CSI}(\Lambda, \mathcal{C}, k)$, and [43] shows that each such is a suitable LR structure.

19.3 Bicirculants

Consider a bicoloring of the edges of the bicirculant $\mathrm{BC}_n(0, a, b, c)$ with green edges linking A_i to B_i and B_{i+a} and red edges linking A_i to B_{i+b} and B_{i+c}. We call this coloring $\mathrm{BC}_n(\{0, a\}, \{b, c\})$. The paper [40] shows several cases in which $\mathrm{BC}_n(\{0, a\}, \{b, c\})$ is a suitable LR structure:

1. $a = 1 - r, b = 1, c = s$, where $r, s \in \mathbb{Z}_n^* \setminus \{-1, 1\}$, $r^2 = s^2 = 1$, $r \notin \{-s, s\}$, and $(r - 1)(s - 1) = 0$.
2. $n = 2m, a = m, b = 1, c \in \mathbb{Z}_{2m} \setminus \{1, -1, m+1, m-1\}$ such that $c^2 \in \{1, m+1\}$.
3. $n = 4k, a = 2k, b = 1, c = k + 1$, for $k \geq 3$

Moreover, it is conjectured in that paper that every suitable $\mathrm{BC}_n(\{0, a\}, \{b, c\})$ is isomorphic to at least one of these three.

19.4 MSY's and MSZ's

The graph $\mathrm{MSY}(m, n; r, t)$ has an LR structure, with edges of the first kind being red and those of the second kind being green, if and only if $2t = 0$ and $r^2 = \pm 1$. Many examples of MSZ graphs being suitable LR structures are known, but no general classification has been attempted.

19.5 Stack of pancakes

The structure is called $\mathrm{SoP}(4m, 4n)$. Let $r = 2n + 1$. The vertex set is $\mathbb{Z}_{4m} \times \mathbb{Z}_{4n} \times \mathbb{Z}_2$. Red edges join (i, j, k) to $(i, j \pm r^k, k)$; for a fixed i and j, green edges join the two vertices $(2i, j, 0)$ and $(2i, j, 1)$ to the two vertices $(2i + 1, j, 0)$ and $(2i + 1, j, 1)$ if j is even, to the two vertices $(2i - 1, j, 0)$ and $(2i - 1, j, 1)$ if j is odd.

The paper [43] shows that this is a suitable LR structure for all m and n, and that the symmetry group of it and of its partial line graph, can have arbitrarily large vertex-stabilizers.

19.6 Rows and columns

The LR structure $\mathrm{RC}(n, k)$ has as vertices all ordered pairs $(i, (r, j))$ and $((i, r), j)$, where i and j are in $\mathbb{Z}_n$, and r is in $\mathbb{Z}_k$, where k and n are integers at least 3. Green edges join $(i, (r, j))$ to $(i \pm 1, (r, j))$ and $((i, r), j)$ to $((i, r), j \pm 1)$, while red edges join $(i, (r, j))$ to $((i, r \pm 1), j)$ and so $((i, r), j)$ to $(i, (r \pm 1, j))$.

This structure is referred to in both [40] and [43].

19.7 Cayley constructions

Suppose a group A is generated by two sets, R and G, of size two, neither containing the identity, and each containing the inverse of each of its elements. Then we let $\mathrm{Cay}(A; R, G)$ be the structure whose vertex set is A, whose red edges join each a to sa for $s \in R$ and whose green edges join each a to sa for $s \in G$. The paper [40] shows that if A admits two automorphisms, one fixing each element of R but interchanging the two element of G and the other vice versa, then $\text{Cay}(A; R, G)$ is an LR structure. The condition $RG \neq GR$ is equivalent to the structure not having alternating 4-cycles.

Many examples occur in the case where A is the dihedral group D_n. One family of this type is the first group of bicirculants $BC_n(\{0, 1-r\}, \{1, s\})$ shown in subsection 19.3.

The paper [40] shows several other algebraically defined structures. First, there are examples for the group D_n where the swappers do not arise from group automorphisms. Second, there is a Cayley construction for the structure $RC(n, k)$ of the previous subsection.

Further, the body of the paper shows six 'linear' constructions in which A is an extension of some $\mathbb{Z}_n^k$:

Construction 19.1. *For n and k both at least 3, let A be a semidirect product of $\mathbb{Z}_n^k$ with the group generated by the permutation $\sigma = (123\ldots k-1k)$ acting on the coordinates. Let e_1 be the standard basis element $(100\ldots0)$, let $R = \{e_1, -e_1\}$ and $G = \{\sigma, \sigma^{-1}\}$. We define the LR structure $\text{AffLR}(n, k)$ to be $\text{Cay}(A; R, G)$.*

Construction 19.2. *Let $\text{ProjLR}(k, n)$ be $\text{AffLR}(k, n)$ factored out by the cyclic group generated by $(1, 1, \ldots, 1)$.*

Construction 19.3. *Let $\text{ProjLR}^\circ(2k, n)$ be $\text{AffLR}(2k, n)$ factored out by the group generated by all $d_i = e_i - e_{i+k}$, where e_i is the standard basis element having a 1 in position i and zeroes elsewhere.*

Construction 19.4. *Let A be a semidirect product of $\mathbb{Z}_2^{2k}$ with the group generated by the permutation $\gamma = (1, 2, 3, \ldots, k)(k+1, k+2, \ldots, 2k)$. acting on the coordinates. Let $R = \{e_1, e_{k+1}\}$ and $G = \{\gamma, \gamma^{-1}\}$. We define the LR structure $\text{AffLR}_2(k)$ to be $\text{Cay}(A; R, G)$.*

Let d_1 be the $2k$-tuple in which the first k entries are 1 and the last k entries are 0; let d_2 be the $2k$-tuple in which the first k entries are 0 and the last k entries are 1; let $d = d_1 + d_2$.

Construction 19.5. *Let $\text{ProjLR}'(k)$ be $\text{AffLR}_2(k)$ factored out by the group generated by d_1 and d_2.*

Construction 19.6. *Let $\text{ProjLR}"(k)$ be $\text{AffLR}_2(k)$ factored out by the group generated by d.*

The paper [40] proves that all six of these constructions lead to suitable LR structures (except for a few cases).

20 Base graph-connection graph constructions

This section deals with a family of constructions called *base graph - connection graph* (BGCG) constructions. In these, we construct an edge-transitive bipartite graph Γ from a set of copies of the subdivision (see section 3.2) of a *base graph* B, connected according to a *connection graph* C and some other information.

For a tetravalent graph B, let B^* be its subdivision, i.e., the graph resulting by replacing each edge of B with a path of length 2. We refer to its 4-valent vertices as *black* and its valence-2 vertices as *white*. We will often refer to an edge of B and the white vertex on the path that replaces it in B^* as being equivalent or corresponding.

In the constructions we often define a partition $\mathcal{P}$ of the edges of a graph (equivalently, of the white vertices of its subdivision) into sets of size 2, and refer to such a $\mathcal{P}$ as a *pairing*. We can regard the partition as a coloring of the edges of the graph. We can also think of the pairing as an involutory permutation by writing $y = x^{\mathcal{P}}$ when $\{x, y\} \in \mathcal{P}$.

Given simple graphs B and C, let X be a disjoint union of copies of B, one copy B_r for each vertex r of C. Refer to this X as B^C. A pairing $\mathcal{P}$ of X is *compatible* with C provided that it satisfies these two conditions:

1. if $\{x, y\} \in \mathcal{P}$ with x in B_r and y in B_s, $r \neq s$, then $\{r, s\}$ is an edge of C, and
2. every edge of C is so represented.

These ingredients are combined in the following construction:

Construction 20.1. *Given a tetravalent base graph B, a connection graph C, and a pairing $\mathcal{P}$ on $X = B^C$ which is compatible with C, form a graph Γ by identifying each pair of white vertices of X^* which corresponds to an element of $\mathcal{P}$. We refer to Γ as a* BGCG *of B and C with respect to $\mathcal{P}$, and write $\Gamma = \mathrm{BGCG}(B, C, \mathcal{P})$.*

Theorem 20.2 ([49])**.** *If B is tetravalent, C is a graph, and $\mathcal{P}$ is a pairing of B^C which is compatible with C, then $\Gamma = \mathrm{BGCG}(B, C, \mathcal{P})$ is a bipartite tetravalent graph.*

Definition 20.3. If X is a (connected or disconnected) tetravalent graph and $\mathcal{P}$ is a pairing on X, we will call $\mathcal{P}$ a *dart-transitive pairing* provided that the subgroup of $\mathrm{Aut}(X)$ which preserves $\mathcal{P}$ is transitive on the darts of X.

Theorem 20.4 ([49])**.** *If B and C are each dart-transitive and if $\mathcal{P}$ is a dart-transitive pairing on $X = B^C$ which is compatible with C, and if $\Gamma = \mathrm{BGCG}(B, C, \mathcal{P})$ is defined, then Γ is a bipartite edge-transitive graph.*

Theorem 20.4 is the basis for several BGCG constructions, each depending on the nature of C.

Theorem 20.5 ($C = K_1$)**.** *Suppose that $\mathcal{Q}$ is a dart-transitive pairing on B. Then* $\mathrm{BGCG}(B, K_1, \mathcal{Q})$ *is an edge-transitive tetravalent graph.*

Theorem 20.6 ($C = K_2$)**.** *Suppose that $\mathcal{Q}$ is a dart-transitive pairing on B, and consider the vertex set of K_2 to be $\{1, 2\}$. If e is an edge of B, let $(e, 1)$ and $(e, 2)$ be the corresponding edges in $X = B^{K_2}$. Define $\mathcal{P}$ on X by $(e, 1)^{\mathcal{P}} = (e^{\mathcal{Q}}, 2)$. Then $\mathrm{BGCG}(B, K_2, \mathcal{P})$ is an edge-transitive tetravalent graph.*

No truly general technique yet exists in the case when C is the n-cycle C_n. We give here two constructions from [56] and remark that there are many more for $C = C_n$.

Construction 20.7. *Suppose that*

1. *$\mathcal{Q}$ is a dart-transitive pairing of B invariant under a dart-transitive group G,*
2. *there is a partition $\{\mathcal{R}, \mathcal{G}\}$ of the edges of B into two 'colors' ($\mathcal{R}$ = 'red', $\mathcal{G}$= 'green') which is also invariant under G,*
3. *each pair in $\mathcal{Q}$ meets both $\mathcal{R}$ and $\mathcal{G}$,*
4. *the vertex set of C_n is $\mathbb{Z}_n$.*

Then we can form $\mathcal{P}_1$ and $\mathcal{P}_2$ on $X = B^{C_n}$ as follows: for each pair $\{e, f\}$ in $\mathcal{Q}$, with $e \in \mathcal{R}$, and for each $i \in \mathbb{Z}_n$, we let $(e, i)^{\mathcal{P}_1} = (f, i + 1)$. If n is even, then for i even, we let $(e, i)^{\mathcal{P}_2} = (e, i + 1)$, and for i odd, we let $(f, i)^{\mathcal{P}_2} = (f, i + 1)$.

Theorem 20.8 ($C = C_n$)**.** *With $\mathcal{P}_1$ and $\mathcal{P}_2$ as above,* $\mathrm{BGCG}(B, C_n, \mathcal{P}_1)$ *and* $\mathrm{BGCG}(B, C_n, \mathcal{P}_2)$ *are edge-transitive tetravalent graphs.*

In the Census, we have no efficient way to describe the diferent pairs, $(\mathcal{Q}, \{\mathcal{R}, \mathcal{G}\})$, and so we simply number them. If the pair is numbered i, the Census reports $\mathrm{BGCG}(B, C_n, \mathcal{P}_1)$ as "$\mathrm{BGCG}(B, C_n, i)$" and $\mathrm{BGCG}(B, C_n, \mathcal{P}_2)$ as "$\mathrm{BGCG}(B, C_n, i')$".

21 From regular maps

A *map* is an embedding of a graph or multigraph on a compact connected surface such that each component of the complement of the graph (these are called *faces*) is topologically a disk. A *symmetry* of a map is a symmetry of the graph which extends to a homeomorphism of the surface. A map $\mathcal{M}$ is *rotary* provided that for some face and some vertex of that face, there is a symmetry R which acts as rotation one step about the face and a symmetry S which acts as rotation one step about the vertex. A map $\mathcal{M}$ is *reflexible* provided that it is rotary and has a symmetry X acting as a reflection fixing that face and vertex. If $\mathcal{M}$ is rotary but not reflexible, we call it *chiral*. See [15] for more details.

If $\mathcal{M}$ is rotary, its symmetry group, $\mathrm{Aut}(\mathcal{M})$, is transitive on faces and on vertices. Thus all faces have the same number p of sides and all vertices have the same degree q. We then say that $\mathcal{M}$ has *type* $\{p, q\}$.

21.1 Underlying graphs

The underlying graph of a rotary map $\mathcal{M}$ is called $\mathrm{UG}(\mathcal{M})$ and is always dart-transitive. If $q = 4$, it belongs in this census.

21.2 Medial graphs

The vertices of the medial graph, $\mathrm{MG}(\mathcal{M})$, are the edges of $\mathcal{M}$. Two are joined by an edge if they are consecutive in some face (and so in some vertex). If $\mathcal{M}$ is rotary, $\mathrm{MG}(\mathcal{M})$ is always edge-transitive. If $\mathcal{M}$ is reflexible or if it is self-dual in such a way that some orientation-preserving homeomorphism of the surface interchanges $\mathcal{M}$ with its dual, $D(\mathcal{M})$, then $\mathrm{MG}(\mathcal{M})$ is dart-transitive. If not, it is quite often, but not quite always, $\frac{1}{2}$-transitive. No one seems to know a good criterion for this distinction.

21.3 Dart graphs

The vertices of the dart graph, $\mathrm{DG}(\mathcal{M})$, are the darts of $\mathcal{M}$. Two are joined by an edge if they are head-to-tail consecutive in some face. The graph $\mathrm{DG}(\mathcal{M})$ is a twofold cover of $\mathrm{MG}(\mathcal{M})$ and is often the medial graph of some larger rotary map. It can be dart-transitive or $\frac{1}{2}$-transitive; again, no good criterion is known.

21.4 HC of maps

The Hill Capping of a rotary map $\mathcal{M}$ is defined in a way completely analogous to the capping of a cubic graph Λ: we join $\{A_i, B_j\}$ to $\{B_j, C_{1-i}\}$ where A, B and C are vertices which are consecutive around some face. The graph $\mathrm{HC}(\mathcal{M})$ is a 4-fold covering of $\mathrm{MG}(\mathcal{M})$ and can be dart-transitive or semisymmetric or $\frac{1}{2}$-transitive or even not edge-transitive.

21.5 XI of maps

Suppose that $\mathcal{M}$ is a rotary map of type $\{p, q\}$ for some even $q = 2n$. Then each corner of the map (formed by two consecutive edges in one face) is opposite at that vertex to another corner; we will call such a pair of corners an 'X'. As an example, consider Figure 19, which shows one vertex, of degree 6, in a map. The X's are pairs a, b, c of opposite corners.

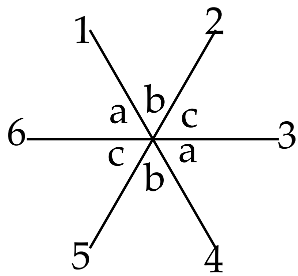

Figure 19: X's and I's in a map

We form the bipartite graph $\mathrm{XI}(\mathcal{M})$ in this way: The black vertices are the X's, the white vertices are the edges of $\mathcal{M}$, and edges of $\mathrm{XI}(\mathcal{M})$ are all pairs $\{x, e\}$ where x is an X and e is one of the four edges of x. Continuing our example, the black vertex a is adjacent to white vertices 1, 3, 4, 6, while b is adjacent to 1, 2, 4, 5 and c to 2, 3, 5, 6.

It is interesting to see the construction introduced here as a special case of two previous constructions. First it is made from $\mathrm{MG}(\mathcal{M})$ using a BGCG construction in which each edge of $\mathrm{MG}(\mathcal{M})$ is paired with the one opposite it at the vertex of $\mathcal{M}$ containing the corresponding corner.

Secondly, it is $\mathbb{P}$ of an LR structure called a *locally dihedral cycle structure* as outlined at the end of [43].

It is clear that if $\mathcal{M}$ is reflexible, then $\mathrm{XI}(\mathcal{M})$ is edge-transitive; it is surprising, though, that sometimes (criteria still unknown) XI of a chiral map can also be edge-transitive.

22 Sporadic graphs

There are a few graphs in the Census which are given familiar names rather than a parametric form. These are: $K_5 = C_5(1, 2)$, the Octahedron = $K_{2,2,2}$. Also there is the graph $\mathrm{Odd}(4)$; its vertices are subsets of $\{1, 2, 3, 4, 5, 6, 7\}$ of size 3. Two are joned by an edge when the sets are disjoint. Finally, there is the graph denoted $\mathrm{Gray}(4)$ due to a construction by Bouwer [7] which generalizes the Gray graph to make a semisymmetric graph of valence n on $2n^n$ vertices, in this case, 512. In fact we wanted to extend the Census to 512 vertices in order to include this graph.

Out of more than 7000 graphs in this Census, 400 of them have as their listed names one of the tags $\mathrm{AT}[n, i]$, $\mathrm{HT}[n, i]$ or $\mathrm{SS}[n, i]$ from the computer-generated censi. These graphs, then, must have no other known constructions. Each of these might be truly sporadic or, perhaps, might belong to some interesting family not yet recognized.

Each of these is a research project in its own right, a single example waiting to be meaningfully generalized.

23 Open questions

1. In compiling the Census, whenever we wanted to include a parameterized family ($C_n(1, a)$ (section 5), $\mathrm{BC}_n(a, b, c, d)$ (section 12), $\mathrm{Pr}_n(a, b, c, d)$ (section 13.1), etc.), it was very helpful to have some established theorems which either completely classified which values of the parameters gave edge-transitive graphs or restricted those parameters in some way. In families for which no such theorems were known, we were forced into brute-force searches, trying all possible values of the parameters. In many cases this caused our computers to run out of time or space before finishing the search. There are many families for which no such results or only partial results exist. So, our first and most pressing question is :
   For which values of parameters are the following graphs edge-transitive (or LR):

   $\mathrm{AMC}(k, n, M)$ (section 9),
   $\mathrm{MSZ}(m, n; k, r)$ (section 13.2.2),
   $\mathrm{MC3}(m, n, a, b, r, t, c)$ (section 13.2.3),
   $\mathrm{LoPr}_n(a, b, c, d, e)$ (section 13.3),
   $\mathrm{WH}_n(a, b, c, d)$ (section 13.3),
   $\mathrm{KE}_n(a, b, c, d, e)$ (section 13.3),
   $\mathrm{Curtain}_n(a, b, c, d, e)$ (section 13.3),
   $\mathrm{CPM}(n, s, t, r)$ (section 15).

2. The general class IV metacirculants (of which the graphs MSZ are merely a part) has begun to be explored. It is parameterized in [1], where the authors point out that not all values of the parameters which make the graph metacirculent make it $\frac{1}{2}$-arc-transitive. Then the same questions as above are relevant: When are these graphs isomorphic? dart-transitive? $\frac{1}{2}$-arc-transitive? LR structures?

3. Given a and n with $a^2 \equiv \pm 1 \pmod{n}$, which toroidal graph is isomorphic to $\mathcal{C}_n(1, a)$? See sections 5 and 6. The toroidal graphs are very common and almost every family includes some as special cases. In researching a family, we often point out that certain values of the parameters give toroidal graphs and so will not be studied with this family. However, it is often difficult to say exactly which toroidal graph is given by the indicated parameters. This is simply the first of many such questions.

4. Under what conditions on their parameters can two spidergraphs be isomorphic? This is a question mentioned in [54]. Many examples of isomorphism theorems are given there as well as examples which show that not all isomorphisms have been found.

5. The Attebery construction presents many challenges. First, for the general construction, under what conditions are the graphs $\mathrm{Att}(A, T, k; a, b)$ and $\mathrm{Att}(A, T', k; a', b')$ isomorphic? It is clear that if P is any automorphism of A, then $\mathrm{Att}(A, T, k; a, b) \cong \mathrm{Att}(A, P^{-1}TP, k; aP, bP)$, but it is almost certain that there are other isomorphisms. Even in the special AMC construction, we do not know when $\mathrm{AMC}(k, n, M)$ can be isomorphic to $\mathrm{AMC}(k', n', M')$. This question must be answered before we can make much progress on other aspects of the Attebery graphs.

6. Some graphs with the same diagrams as the metacirculants $\mathrm{PS}, \mathrm{MPS}, \mathrm{MSY}, \mathrm{MSZ}, \mathrm{MC3}$ are not themselves metacirculants but are nevertheless edge-transitive. For example consider the graph $\mathrm{KE}_{12}(1, 3, 8, 5, 1)$ It is isomorphic to the graph whose vertices are $\mathbb{Z}_4 \times \mathbb{Z}_{12}$, with each $(1, i)$ adjacent to $(2, i)$ and $(2, i + 1)$, each $(2, i)$ to $(3, i)$ and $(3, i + 4)$, each $(3, i)$ to $(4, i + 4)$ and $(4, i + 5)$, and each $(4, i)$ to $(1, i)$ and $(1, i + 10)$. This diagram is the same 'sausage graph' that characterizes the PS and MPS graphs and yet the graph is not isomorphic to any PS or MPS graph. This happens rarely enough that the exceptional cases might be classifiable.
7. The Praeger-Xu graphs, including $\mathrm{W}(n, 2)$ and $\mathrm{R}_{2m}(m + 2, m + 1)$, present many problems computationally. Some aspects, such as semitransitive orientations, cycle structures and regular maps have been addressed in [16]. Can we establish theorems determining their dart-transitive colorings, their BGCG dissections, and other properties?
8. When do the constructions DG, HC, TAG, applied to some cubic graph Λ or some rotary map $\mathcal{M}$, simply result in the line graph or medial graph of some larger graph or map?
9. The BGCG constructions we have used here are only the beginning of this topic. We have given some constructions for cases where the connection graph is K_1, K_2, or C_k, but for other connection graphs, we have no general techniques at all.
10. How can XI of a chiral map be edge-transitive?
11. The 3-arc graph of a cubic graph (see section 17.4) is the partial line graph of some cycle decomposition; *what* decomposition? . . . of *what* graph?
12. In section 19, we use cycle structures to construct LR structures. What is an efficient way to find all isomorphism classes of cycle structures for a given dart-transitive tetravalent graph?

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Observations and answers to questions about edge-transitive maps*

Marston D.E. Conder†, Isabel Holm

Department of Mathematics, University of Auckland
Private Bag 92019, Auckland 1142, New Zealand

Thomas W. Tucker‡

Department of Mathematics, Colgate University, Hamilton, NY, USA

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Abstract

A map is a 2-cell embedding of a connected graph or multigraph on a closed surface, and a map is called edge-transitive if its automorphism group has a single orbit on edges. There are 14 classes of edge-transitive maps, determined by the effect of the automorphism group. In this paper we make some observations about these classes, and answer three open questions from a 2001 paper by Širán, Tucker and Watkins, by showing that (a) in each of the classes 1, 2^P, 2^Pex, 3, 4^P and 5^P, there exists a self-dual edge-transitive map, (b) there exists an edge-transitive map with simple underlying graph on an orientable surface of genus g for every integer $g \geq 0$, and (c) there exists an orientable surface that carries an edge-transitive map of each of the 14 classes, and indeed that these three things still hold when we insist that both the map and its dual have simple underlying graph. We also give the maximum number of automorphisms of an edge-transitive map on an orientable surface of given genus $g > 1$, and consider some special cases in which the automorphism group (or its subgroup of orientation-preserving automorphisms) is prescribed. For example, we show that a certain soluble group of order 576 is the smallest group that occurs as the automorphism group of some edge-transitive map in each of the 14 classes.

Keywords: Graph embedding, regular map, automorphism, edge-transitive, simple underlying graph, duality.

Math. Subj. Class. (2020): 57M15, 05C10, 05E18, 57M60

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E-mail address: m.conder@auckland.ac.nz (Marston D.E. Conder), ihol325@aucklanduni.ac.nz (Isabel Holm), ttucker@colgate.edu (Thomas W. Tucker)

1 Introduction

A regular map is a symmetric embedding of a connected graph or multigraph on a closed surface: the automorphism group of the embedding has a single orbit on 'flags' (which are like incident vertex-edge-face triples), or more loosely on 'arcs' (which are incident vertex-edge pairs). The theory of such discrete objects has a long and interesting history, dating back to the work of Brahana, Burnside, Dyck and others, and recent work has produced infinite families of examples [10] as well as complete lists of those on hyperbolic surfaces of small genera [3, 5, 6].

In contrast, relatively little is known about the more general case of *edge-transitive* maps, for which the automorphism group has a single orbit on edges. By work of Graver and Watkins [11], it is known that these can be divided into 14 classes according to certain properties (determined by automorphisms preserving a given edge or one of the vertices or faces incident with it). That work was taken further in [18] by Širán, Tucker and Watkins, who showed there exist finite maps in each class. They also posed a number of questions, some of which were answered by Alen Orbanić in his 2006 PhD thesis. Further questions were also posed by Orbanić et al in [17]. Some of the remaining questions and related ones were discussed at a BIRS workshop (on Symmetries of surfaces, maps and dessins) at Banff in September 2017, and in this paper we present the answers to many of those.

In particular, we answer Questions 3, 4 and 6 from [18], by showing the following:

(a) In each of the classes 1, 2^P, 2^Pex, 3, 4^P and 5^P, there exists a self-dual edge-transitive map, indeed one for which the map is *non-degenerate*, in that both the map and its dual have simple underlying graph;

(b) There exists a non-degenerate edge-transitive map on an orientable surface of genus g for every integer $g \geq 0$, and

(c) There exists an orientable surface that carries a edge-transitive map of each of the 14 classes, and indeed a non-degenerate one of each class.

We also give the maximum number of automorphisms of an edge-transitive map on an orientable surface of given genus $g > 1$, and consider some special cases in which the automorphism group (or its subgroup of orientation-preserving automorphisms) is prescribed. For example, we show that a certain soluble group of order 576 is the smallest group that occurs as the automorphism group of some edge-transitive map in each of the 14 classes.

Before doing that in Sections 3 to 8, we provide some further information about the 14 classes, including 'universal' groups that determine the effect of the automorphism group of every map in the class, and then we conclude the papers with some remarks and further questions.

Many of the findings we describe in this paper resulted from computations involving the universal groups and their quotients, using the MAGMA system, or were guided by them. In cases where the outcomes depended almost entirely on computations, we summarise them in tables in an Appendix at the end.

2 Details and properties of the 14 classes

In this section we present some background information on each class that can be helpful in constructing or analysing examples of edge-transitive maps. Much of this information can also be found in references [11] or [18] or [16] or [17], but unfortunately the table in [11] that was copied as Table I in [18] contains a number of shortcomings that make it difficult to follow. First, the columns headed G_v and G_f refer to the stabiliser of a particular vertex v or face f, but for some classes the stabilisers of the vertex u and/or face g should be given as well. And in fact the column headed G_v looks more appropriate for G_u, but even then, its entry for class 4^* should be $\langle \sigma_u^4, \theta_{ug} \rangle$, not $\langle \sigma_u^4, \theta_{uf} \rangle$. Also in Table II of [18], some relations are missing, namely $\tau^2 = 1$ for type 4, and $\lambda^2 = 1$ for type 4^*, and $\phi^2 = 1$ for type 4^P. The information below is more accurate and comprehensive.

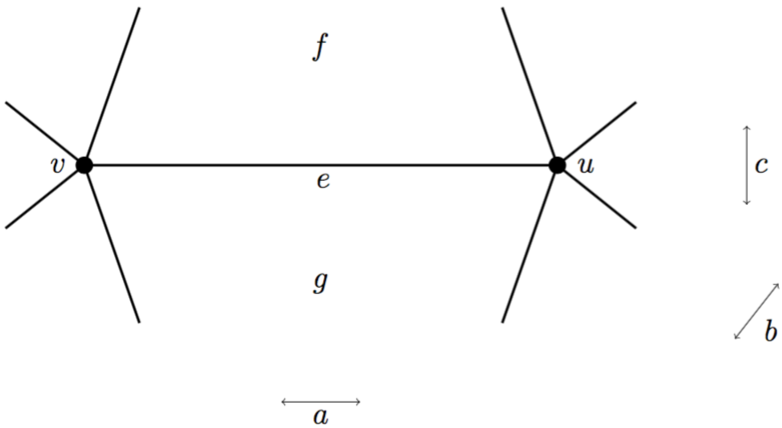

Figure 1: Reflecting generators for the universal group of class 1

In each case we let $e = \{u, v\}$ be a given edge, and let f and g be the faces incident with e, as illustrated in Figure 1.

Associated with each class is a universal group U, which has the property that if M is any ET map of the given class, then its automorphism group $\mathrm{Aut}(M)$ is a quotient of U. This group U is generated by particular elements that can be described in terms of their effect on the vertices, edges and faces labelled in Figure 1. Here we compose automorphisms from left to right.

Conversely, if A is any quotient of U in which the images of the generators and certain other elements have the appropriate orders (to avoid collapse), then there exists an ET map M on which A acts edge-transitively as a group of automorphisms, in the appropriate way. This map can be constructed using cosets of the stabilisers of the vertices u and v, the edge e and the faces f and g coming from Figure 1, in a similar way to the well-known construction for regular maps from groups (see [3], for example).

Class 1 consists of the fully regular maps. In this case, the universal group is

$$U_1 = \langle\, a, b, c \mid a^2 = b^2 = c^2 = (ac)^2 = 1 \,\rangle$$

where a, b and c are automorphisms that act like local reflections, such that the stabilisers of the vertex u, edge e and face f are the subgroups generated by $\{b, c\}$, $\{a, c\}$ and $\{a, b\}$ respectively. (These correspond to the elements λ_e, θ_{uf} and τ_e in the notation of [18].)

The universal group U for any one of the 14 classes can be embedded into the group U_1 above, as a subgroup of index dividing 4, with transversal a subgroup of $\langle a, c \rangle$. We give this embedding, which is unique except for classes 4, 4^* and 4^P, where there are two possibilities (that can be interchanged under conjugation by a, c and either a or c, respectively).

In each case, we also give generators for the orientation-preserving subgroup (which is the intersection of U with the subgroup $\langle ab, bc\rangle$ of U_1) in the orientable case, and state whether or not the automorphism group of a map in the given class acts transitively on vertices, on faces, and/or on Petrie polygons of the map. Note that when the action (on vertices, faces, or Petrie polygons) is not transitive, there are two orbits (since each edge is incident with at most two vertices, and most two faces, and at most two Petrie polygons). Then we describe the stabilisers of the vertices u and v, the edge e, and the faces f and g, in terms of both generators for U and generators for U_1. We also indicate the effect of an orientation-reversing element (usually but not always b) by conjugation on the generators for the orientation-preserving subgroup, when such an element exists, and similarly, the effect of a map duality on the chosen set of generators for U, in the orientable case.

Finally, for each element $g \in \{a, c, ac\}$ in the stated transversal for U in U_1 (but lying outside U), we describe the effect of the automorphism ψ_g of U induced by conjugation by g, taking each generator h of U to $g^{-1}hg$. These ψ_g are what we may call 'barred' automorphisms, in the sense that if A is the automorphism group of an ET map in the given class, then A has no automorphism that has the same effect on the images in A of the generators of U as ψ_g has on the generators themselves.

Class 1

Universal group: $U = \langle\, x, y, z \mid x^2 = y^2 = z^2 = (xz)^2 = 1\,\rangle$;
Embedding of generators in $U_1 : (x, y, z) \mapsto (a, b, c)$, with transversal $\{1\}$;
Orientation-preserving subgroup U^+ is generated by $r = xy = ab$ and $s = xz = ac$ (and $r^{-1}s = yz = bc$), subject to the single relation $s^2 = 1$;
Automorphism group of map is transitive on vertices, faces and Petrie polygons;
Stabiliser of vertex u is generated by $\{y, z\} = \{b, c\}$;
Stabiliser of vertex v is generated by $\{xyx, z\} = \{aba, c\}$;
Stabiliser of edge e is generated by $\{x, z\} = \{a, c\}$;
Stabiliser of face f is generated by $\{x, y\} = \{a, b\}$;
Stabiliser of face g is generated by $\{x, zyz\} = \{a, cbc\}$;
A reflection (by a) takes $(r, s) = (ab, ac) \mapsto (ba, ca) = (r^{-1}, s^{-1})$;
An orientable duality takes $(a, b, c) \mapsto (c, b, a)$, preserving class 1;
There are no barred automorphisms.

Class 2

Universal group: $U = \langle\, x, y, z \mid x^2 = y^2 = z^2 = 1\,\rangle$;
Embedding of generators in $U_1 : (x, y, z) \mapsto (b, c, aba)$, with transversal $\{1, a\}$;
Orientation-preserving subgroup U^+ is generated by $r = xy = bc$ and $s = zy = abac = abca$, and is free of rank 2;
Automorphism group of map is transitive on faces and Petrie polygons but not on vertices;
Stabiliser of vertex u is generated by $\{x, y\} = \{b, c\}$;
Stabiliser of vertex v is generated by $\{y, z\} = \{c, aba\}$;
Stabiliser of edge e is generated by $\{y\} = \{c\}$;
Stabiliser of face f is generated by $\{x, z\} = \{b, aba\}$;
Stabiliser of face g is generated by $\{yxy, yzy\} = \{cbc, acbca\} = \{cbc, cabac\}$;
A reflection (by c) takes $(r, s) = (bc, abac) \mapsto (cb, caba) = (r^{-1}, s^{-1})$;
An orientable map duality takes $(b, c, aba) \mapsto (b, a, cbc)$, interchanging classes 2 and 2*;

Extending automorphism ψ_a takes $(x, y, z) = (b, c, aba) \mapsto (aba, c, b) = (z, y, x)$.

Class 2^*

Universal group: $U = \langle\, x, y, z \mid x^2 = y^2 = z^2 = 1 \,\rangle$;
Embedding of generators in $U_1 : (x, y, z) \mapsto (a, b, cbc)$, with transversal $\{1, c\}$;
Orientation-preserving subgroup U^+ is generated by $r = xy = ab$ and $s = yz = (bc)^2$, and is free of rank 2;
Automorphism group of map is transitive on vertices and Petrie polygons but not on faces;
Stabiliser of vertex u is generated by $\{y, z\} = \{b, cbc\}$;
Stabiliser of vertex v is generated by $\{xyx, xzx\} = \{aba, acbca\} = \{aba, cabac\}$;
Stabiliser of edge e is generated by $\{x\} = \{a\}$;
Stabiliser of face f is generated by $\{x, y\} = \{a, b\}$;
Stabiliser of face g is generated by $\{x, z\} = \{a, cbc\}$;
A reflection (by b) takes $(r, s) = (ab, (bc)^2) \mapsto (ba, (cb)^2) = (r^{-1}, s^{-1})$;
An orientable map duality takes $(a, b, cbc) \mapsto (c, b, aba)$, interchanging classes 2^* and 2;
Extending automorphism ψ_c takes $(x, y, z) = (a, b, cbc) \mapsto (a, cbc, b) = (x, z, y)$.

Class 2^P

Universal group: $U = \langle\, x, y, z \mid x^2 = y^2 = z^2 = 1 \,\rangle$;
Embedding of generators in $U_1 : (x, y, z) \mapsto (ac, b, cbc)$, with transversal $\{1, a\}$ or $\{1, c\}$;
Orientation-preserving subgroup U^+ is generated by $r = x = ac$, $s = yz = (bc)^2$ and $t = yxz = babc$, subject to the two relations $r^2 = (st^{-1})^2 = 1$;
Automorphism group of map is transitive on vertices and faces but not on Petrie polygons;
Stabiliser of vertex u is generated by $\{y, z\} = \{b, cbc\}$;
Stabiliser of vertex v is generated by $\{xzx, xyx\} = \{aba, acbca\} = \{aba, cabac\}$;
Stabiliser of edge e is generated by $\{x\} = \{ac\}$;
Stabiliser of face f is generated by $\{y, xzx\} = \{b, aba\}$;
Stabiliser of face g is generated by $\{z, xyx\} = \{cbc, acbca\} = \{cbc, cabac\}$;
A reflection (by b) takes
$(r, s, t) = (ac, (bc)^2, babc) \mapsto (bacb, (cb)^2, abcb) = (ts^{-1}, s^{-1}, rs^{-1})$;
An orientable map duality takes $(ac, b, cbc) \mapsto (ac, b, aba)$, preserving class 2^P;
Extending automorphism ψ_c takes $(x, y, z) = (ac, b, cbc) \mapsto (ca, cbc, b) = (x, z, y)$.

Class $2\mathrm{ex}$

Universal group: $U = \langle\, x, y \mid x^2 = 1 \,\rangle$;
Embedding of generators in $U_1 : (x, y) \mapsto (c, ab)$, with transversal $\{1, a\}$ or $\{1, b\}$;
Orientation-preserving subgroup U^+ is generated by $r = y = ab$ and $s = xyx = cabc$, and is free of rank 2;
Automorphism group of map is transitive on vertices, faces and Petrie polygons;
Stabiliser of vertex u is generated by $\{x, y^{-1}xy\} = \{c, bcb\}$;
Stabiliser of vertex v is generated by $\{x, yxy^{-1}\} = \{c, abcba\} = \{c, (aba)c(aba)\}$;
Stabiliser of edge e is generated by $\{x\} = \{c\}$;
Stabiliser of face f is generated by $\{y\} = \{ab\}$;
Stabiliser of face g is generated by $\{xyx\} = \{cabc\} = \{acbc\}$;
A reflection (by c) takes $(r, s) = (ab, cabc) \mapsto (cabc, ab) = (s, r)$;

An orientable map duality takes $(c, ab) \mapsto (a, cb)$, interchanging classes 2ex and 2*ex;
Extending automorphism ψ_a takes $(x, y) = (c, ab) \mapsto (c, ba) = (x^{-1}, y^{-1})$.

Class 2^*ex

Universal group: $U = \langle\, x, y \mid x^2 = 1 \,\rangle$;
Embedding of generators in $U_1 : (x, y) \mapsto (a, bc)$, with transversal $\{1, b\}$ or $\{1, c\}$;
Orientation-preserving subgroup U^+ is generated by $r = y = bc$ and $s = xyx = abca$, and is free of rank 2;
Automorphism group of map is transitive on vertices, faces and Petrie polygons;
Stabiliser of vertex u is generated by $\{y\} = \{bc\}$;
Stabiliser of vertex v is generated by $\{xyx\} = \{abca\} = \{abac\}$;
Stabiliser of edge e is generated by $\{x\} = \{a\}$;
Stabiliser of face f is generated by $\{x, yxy^{-1}\} = \{a, bab\}$;
Stabiliser of face g is generated by $\{x, y^{-1}xy\} = \{a, cbabc\}$;
A reflection (by a) takes $(r, s) = (bc, abca) \mapsto (abca, bc) = (s, r)$;
An orientable map duality takes $(a, bc) \mapsto (c, ba)$, interchanging classes 2^*ex and 2ex;
Extending automorphism ψ_c takes $(x, y) = (a, bc) \mapsto (a, cb) = (x^{-1}, y^{-1})$.

Class 2^Pex

Universal group: $U = \langle\, x, y \mid x^2 = 1 \,\rangle$;
Embedding of generators in $U_1 : (x, y) \mapsto (ac, ab)$, with transversal $\{1, a\}$;
Orientation-preserving subgroup U^+ is equal to U;
Automorphism group of map is transitive on vertices, faces and Petrie polygons;
Stabiliser of vertex u is generated by $\{y^{-1}x\} = \{bc\}$;
Stabiliser of vertex v is generated by $\{yx\} = \{abac\} = \{abca\}$;
Stabiliser of edge e is generated by $\{x\} = \{ac\}$;
Stabiliser of face f is generated by $\{y\} = \{ab\}$;
Stabiliser of face g is generated by $\{xyx\} = \{cbac\} = \{cbca\}$;
There is no orientation-reversing automorphism in U;
An orientable map duality takes $(ac, ab) \mapsto (ac, bc)$, preserving class 2^Pex;
Extending automorphism ψ_a takes $(x, y) = (ac, ab) \mapsto (ca, ba) = (x^{-1}, y^{-1})$.

Class 3

Universal group: $U = \langle\, x, y, z, w \mid x^2 = y^2 = z^2 = w^2 = 1 \,\rangle$;
Embedding of generators in $U_1 : (x, y, z, w) \mapsto (b, cbc, acbca, aba)$, with transversal $\{1, a, c, ac\}$;
This group U is also a subgroup of index 2 in the universal groups for classes 2, 2^* and 2^P (but not in the universal groups for classes 2ex, 2^*ex or 2^Pex);
Orientation-preserving subgroup U^+ is generated by $r = wx = (ab)^2$ and $s = xy = (bc)^2$ and $t = zx = (acb)^2 = (cab)^2$, and is free of rank 3;
Automorphism group of map is not transitive on vertices, faces or Petrie polygons;
Stabiliser of vertex u is generated by $\{x, y\} = \{b, cbc\}$;
Stabiliser of vertex v is generated by $\{w, z\} = \{aba, acbca\} = \{aba, cabac\}$;
Stabiliser of edge e is trivial;
Stabiliser of face f is generated by $\{x, w\} = \{b, aba\}$;

Stabiliser of face g is generated by $\{y, z\} = \{cbc, acbca\}$;
A reflection (by b) takes
$(r, s, t) = ((ab)^2, (bc)^2, (acb)^2) \mapsto ((ba)^2, (cb)^2, (bac)^2) = (r^{-1}, s^{-1}, t^{-1})$;
An orientable duality takes $(b, cbc, acbca, aba) \mapsto (b, aba, acbca, cbc)$ or $(acbca, cbc, b, aba)$, preserving class 3;
Extending automorphisms ψ_a, ψ_c and ψ_{ac} respectively take $(x, y, z, w) = (b, b^c, b^{ac}, b^a)$ to (b^a, b^{ac}, b^c, b), (b^c, b, b^a, b^{ac}) and (b^{ac}, b^a, b, b^c),
that is, to (w, z, y, x), (y, x, w, z), and (z, w, x, y).

Class 4

Universal group: $U = \langle\, x, y, z \mid x^2 = y^2 = 1 \,\rangle$
Embedding of generators in $U_1 : (x, y, z) \mapsto (b, cbc, acba)$, with transversal $\{1, a, c, ac\}$;
This group U is also a subgroup of index 2 in the universal group for class 2
(but not in the universal groups for classes 2^*, 2^P, 2ex, 2^*ex or 2^Pex);
Orientation-preserving subgroup U^+ is generated by
$r = yx = (cb)^2$ and $s = z = acba$ and $t = yzx = cbabab$, and is free of rank 3;
Automorphism group of map is transitive on faces and Petrie polygons but not on vertices;
Stabiliser of vertex u is generated by $\{x, y\} = \{b, cbc\}$;
Stabiliser of vertex v is generated by $\{z\} = \{acba\} = \{caba\}$;
Stabiliser of edge e is trivial;
Stabiliser of face f is generated by $\{x, z^{-1}yz\} = \{b, abababa\}$;
Stabiliser of face g is generated by
$\{y, zxz^{-1}\} = \{cbc, acbababca\} = \{cbc, a(cbc)a(cbc)a(cbc)a\}$;
A reflection (by b) takes
$(r, s, t) = ((cb)^2, acba, cbabab) \mapsto ((bc)^2, bacbab, bcbaba) = (r^{-1}, r^{-1}t, r^{-1}s)$;
An orientable duality takes $(b, cbc, acba) \mapsto (b, aba, cabc)$, interchanging classes 4 and 4^*;
Extending automorphism ψ_c takes $(x, y, z) = (b, cbc, acba) \mapsto (cbc, b, abac) = (y, x, z^{-1})$.

Class 4^*

Universal group: $U = \langle\, x, y, z \mid x^2 = y^2 = 1 \,\rangle$;
Embedding of generators in $U_1 : (x, y, z) \mapsto (b, aba, cabc)$, with transversal $\{1, a, c, ac\}$;
This group U is also a subgroup of index 2 in the universal group for class 2^*
(but not in the universal groups for classes 2, 2^P, 2ex, 2^*ex or 2^Pex);
Orientation-preserving subgroup U^+ is generated by
$r = yx = (ab)^2$ and $s = z = cabc$ and $t = yzx = abcbcb$, and is free of rank 3;
Automorphism group of map is transitive on vertices and Petrie polygons but not on faces;
Stabiliser of vertex u is generated by $\{x, z^{-1}yz\} = \{b, cbcbcbc\}$;
Stabiliser of vertex v is generated by
$\{y, zxz^{-1}\} = \{aba, cabcbcbac\} = \{aba, c(aba)c(aba)c(aba)c\}$;
Stabiliser of edge e is trivial;
Stabiliser of face f is generated by $\{x, y\} = \{b, aba\}$;
Stabiliser of face g is generated by $\{z\} = \{cabc\} = \{acbc\}$;
A reflection (by b) takes
$(r, s, t) = ((ab)^2, cabc, abcbcb) \mapsto ((ba)^2, bacbcb, babcbc) = (r^{-1}, r^{-1}t, r^{-1}s)$;
An orientable duality takes $(b, aba, cabc) \mapsto (b, cbc, acba)$, interchanging classes 4^* and 4;

Extending automorphism ψ_a takes $(x, y, z) = (b, aba, cabc) \mapsto (aba, b, cbca) = (y, x, z^{-1})$.

Class 4^P

Universal group: $U = \langle\, x, y, z \mid x^2 = y^2 = 1 \,\rangle$;
Embedding of generators in $U_1 : (x, y, z) \mapsto (b, acbca, abc)$, with transversal $\{1, a, c, ac\}$;
This group U is also a subgroup of index 2 in the universal group for class 2^P (but not in the universal groups for classes 2, 2^*, 2ex, 2^*ex or 2^Pex);
Orientation-preserving subgroup U^+ is generated by $r = zx = abcb$ and $s = xz = babc$ and $t = zy = ababac$, and is free of rank 3;
Automorphism group of map is transitive on vertices and faces but not on Petrie polygons;
Stabiliser of vertex u is generated by $\{x, z^{-1}yz\} = \{b, cbcbcbc\}$;
Stabiliser of vertex v is generated by $\{y, zxz^{-1}\} = \{acbca, abcbcba\} = \{cabac, (aba)c(aba)c(aba)\}$;
Stabiliser of edge e is trivial;
Stabiliser of face f is generated by $\{x, zyz^{-1}\} = \{b, abababa\}$;
Stabiliser of face g is generated by $\{y, z^{-1}xz\} = \{acbca, cbababc\}$;
A reflection (by b) takes $(r, s, t) = (abcb, babc, ababac) \mapsto (babc, abcb, bababacb) = (s, r, st^{-1}r)$;
An orientable duality takes $(b, acbca, abc) \mapsto (b, acbca, cba)$ or $(acbca, b, abc)$, preserving class 4^P;
Extending automorphism ψ_{ac} takes $(x, y, z) = (b, acbca, abc) \mapsto (acbca, b, cba) = (y, x, z^{-1})$.

Class 5

Universal group: $U = \langle\, x, y \mid - \,\rangle$ (free of rank 2);
Embedding of generators in $U_1 : (x, y) \mapsto (bc, abca)$, with transversal $\{1, a, c, ac\}$;
This group U is also a subgroup of index 2 in the universal groups for classes 2, 2^*ex and 2^Pex (but not in the universal groups for classes 2^*, 2^P or 2ex);
Orientation-preserving subgroup U^+ is equal to U;
Automorphism group of map is transitive on faces and Petrie polygons but not on vertices;
Stabiliser of vertex u is generated by $\{x\} = \{bc\}$;
Stabiliser of vertex v is generated by $\{y\} = \{abca\} = \{abac\}$;
Stabiliser of edge e is trivial;
Stabiliser of face f is generated by $\{yx^{-1}\} = \{(ab)^2\}$;
Stabiliser of face g is generated by $\{y^{-1}x\} = \{c(ab)^2c\} = \{a(cbc)a(cbc)\}$;
There is no orientation-reversing automorphism in U;
An orientable duality takes $(bc, abca) \mapsto (ba, cbac)$, interchanging classes 5 and 5^*;
Extending automorphisms ψ_a, ψ_c and ψ_{ac} respectively take $(x, y) = (bc, abca)$ to $(abca, bc) = (y, x)$, $(cb, caba) = (x^{-1}, y^{-1})$ and $(acba, cb) = (y^{-1}, x^{-1})$.

Class 5^*

Universal group: $U = \langle\, x, y \mid - \,\rangle$ (free of rank 2);
Embedding of generators in $U_1 : (x, y) \mapsto (ab, cabc)$, with transversal $\{1, a, c, ac\}$;
This group U is also a subgroup of index 2 in the universal groups for classes 2^*, 2ex and 2^Pex (but not in the universal groups for classes 2, 2^P or 2^*ex);

Orientation-preserving subgroup U^+ is equal to U;
Automorphism group of map is transitive on vertices and Petrie polygons but not on faces;
Stabiliser of vertex u is generated by $\{x^{-1}y\} = \{(bc)^2\}$;
Stabiliser of vertex v is generated by $\{xy^{-1}\} = \{a(bc)^2a\} = \{(aba)c(aba)c\}$;
Stabiliser of edge e is trivial;
Stabiliser of face f is generated by $\{x\} = \{ab\}$;
Stabiliser of face g is generated by $\{y\} = \{cabc\} = \{acbc\}$;
There is no orientation-reversing automorphism in U;
An orientable duality takes $(ab, cabc) \mapsto (cb, acba)$, interchanging classes 5^* and 5;
Extending automorphisms ψ_a, ψ_c and ψ_{ac} respectively take $(x, y) = (ab, cabc)$ to $(ba, cbca) = (x^{-1}, y^{-1})$, $(cabc, ab) = (y, x)$ and $(cbac, ba) = (y^{-1}, x^{-1})$.

Class 5^P

Universal group: $U = \langle\, x, y \mid - \,\rangle$ (free of rank 2);
Embedding of generators in $U_1 : (x, y) \mapsto (abc, acb)$, with transversal $\{1, a, c, ac\}$;
This group U is also a subgroup of index 2 in the universal groups for classes 2^P, 2ex and 2^*ex (but not in the universal groups for classes 2, 2^* or 2^Pex);
Orientation-preserving subgroup U^+ is generated by $r = xy = (ab)^2$ and $s = y^{-1}x = (bc)^2$ and $t = xy^{-1} = a(bc)^2a$, and is free of rank 3;
Automorphism group of map is transitive on vertices and faces but not on Petrie polygons;
Stabiliser of vertex u is generated by $\{y^{-1}x\} = \{(bc)^2\}$;
Stabiliser of vertex v is generated by $\{xy^{-1}\} = \{a(bc)^2a\} = \{(aba)c(aba)c\}$;
Stabiliser of edge e is trivial;
Stabiliser of face f is generated by $\{xy\} = \{(ab)^2\}$;
Stabiliser of face g is generated by $\{yx\} = \{c(ab)^2c\} = \{a(cbc)a(cbc)\}$;
Conjugation by the orientation-reversing automorphism $x = abc$ takes $(r, s, t) = ((ab)^2, (bc)^2, a(bc)^2a) \mapsto (cababc, cbabcbcabc, bcbc) = (t^{-1}rs, s^{-1}r^{-1}trs, s)$;
An orientable duality takes $(abc, acb) \mapsto (cba, acb)$ or (abc, bca), preserving class 5^P;
Extending automorphisms ψ_a, ψ_c and ψ_{ac} respectively take $(x, y) = (abc, acb)$ to $(bca, cba) = (y^{-1}, x^{-1})$, $(cab, abc) = (y, x)$ and $(cba, bac) = (x^{-1}, y^{-1})$.

It can also be helpful to see how the 14 universal subgroups can be embedded not just in U_1 but also in each other. These inclusions are illustrated in Figure 2.

Indeed here we may note that the 14 classes correspond precisely to the 14 conjugacy classes of subgroups of U_1 that are complementary to some subgroup of the edge-stabiliser $\langle a, c\rangle$, and that this gives a purely algebraic way of finding them, much more easily than in the approach taken in [11]. It is also easy to determine these classes using the combinatorial approach of 'symmetry type graphs' in [17].

3 Maximum orders of automorphism groups for ET maps of genus greater than 1

Theorem 3.1. *Let A be a group of automorphisms of an edge-transitive map on some orientable surface of genus $g > 1$, or some non-orientable surface of genus $p > 2$. Then $|A| \le |A|_{\rm maxo}$ or $|A| \le |A|_{\rm maxnono}$, respectively, where $|A|_{\rm maxo}$ and $|A|_{\rm maxnono}$ are given in Table 1 for each of the 14 classes of edge-transitive maps. Moreover, these bounds are*

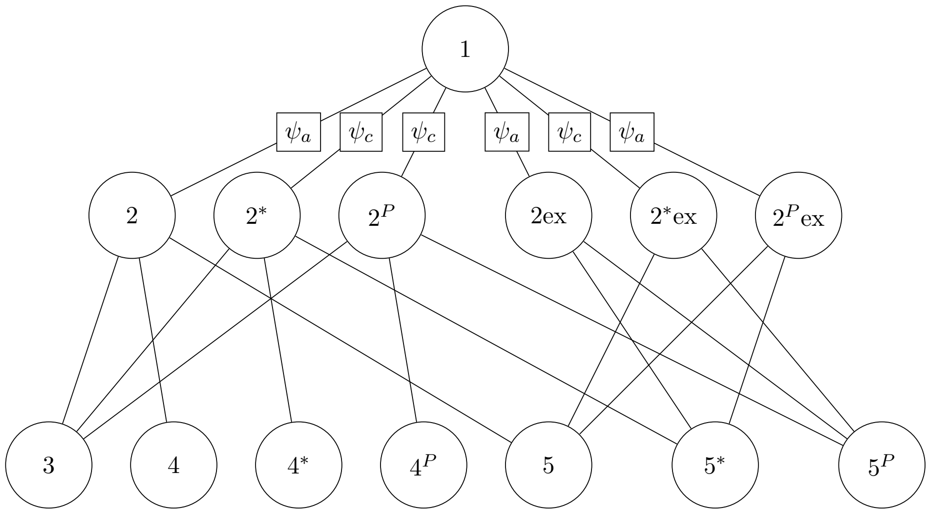

Figure 2: Inclusions among the universal groups of the 14 classes

sharp for certain values of g and p in each class.

Proof. First, we might as well take A as the full automorphism group of the map in each case, and then the bounds can be proved easily using the orders of stabilisers of vertices, edges and faces, and the Euler-Poincaré formula.

For example, in class 1 we know that if $k = o(xy) = o(ab)$ and $m = o(yz) = o(bc)$, then $|V| = |A : A_u| = |A|/k$ and $|E| = |A : A_e| = |A|/4$ and $|F| = |A : A_f| = |A|/m$, so the Euler characteristic is $\chi = |V| - |E| + |F| = |A|(1/k - 1/4 + 1/m)$. Then the maximum possible negative value of this is $-|A|/84$, achievable when $\{k, m\} = \{3, 7\}$, and giving $|A| \leq -84\chi = 168(g - 1)$ or $84(p - 2)$ in the orientable and non-orientable cases, respectively. Similarly, for class 4, we have $|V| = |A : A_u| + |A : A_v| = |A|/o(z) + |A|/(2o(xy))$ and $|E| = |A|$ and $|F| = |A : A_f| = |A|/o(xzyz^{-1})$, and this gives the maximum possible negative value of $\chi = |V| - |E| + |F|$ as $-|A|/24$, achievable when $(o(z), o(xy), o(xzyz^{-1})) = (3, 4, 1)$ or $(3, 1, 4)$. The other cases are similar, and are left as an exercise for the reader.

Sharpness can be proved by exhibiting examples for which the bounds are attained.

If M is a fully regular map of type $\{3, 7\}$ or $\{7, 3\}$, such as Klein's map of genus 3, then $\mathrm{Aut}(M)$ is generated by elements a, b, c satisfying $a^2 = b^2 = c^2 = (ab)^3 = (bc)^7 = (ac)^2 = 1$, and $|\mathrm{Aut}(M)| = 168(g-1)$ or $84(p-2)$, and this proves sharpness for class 1. But then also the triple $(x, y, z) = (a, b, c)$ satisfies the defining relations for the universal group of class 2 maps, but there exists no automorphism of the group that takes (x, y, z) to (z, y, x), since $o(xy) = 3$ while $o(zy) = o(yz) = 7$. Hence this triple gives an ET map in class 2, with $\chi = (|A|/6 + |A|/14) - |A|/2 + |A|/4 = -|A|/84$, and therefore the same genus and the same number of automorphisms as M. Also its dual is in class 2^* and has the same genus and same automorphism group as well.

Similarly, if M is an orientably-regular but chiral map of type $\{3, 7\}$ or $\{7, 3\}$, then it has class 2^Pex and $84(g - 1)$ automorphisms. Also any irreflexible generating pair

Classes	$\lvert A\rvert_{\text{maxo}}$	$\lvert A\rvert_{\text{maxnono}}$	Sufficient conditions for achieving maximum
1	$168(g-1)$	$84(p-2)$	$(o(xy), o(yz)) = (3, 7)$ or $(7, 3)$
2, 2^*	$168(g-1)$	$84(p-2)$	$\{o(xy), o(yz), o(xz)\} = \{2, 3, 7\}$
2^P	$24(g-1)$	$12(p-2)$	$(o(yz), o(xyxz)) = (2, 3)$ or $(3, 2)$
2ex, 2^*ex	$48(g-1)$	$24(p-2)$	$(o(y), o(xy^{-1}xy)) = (3, 4)$
2^Pex	$84(g-1)$	N/A	$\{o(y), o(xy)\} = \{3, 7\}$
3	$24(g-1)$	$12(p-2)$	$\{o(xy), o(zw), o(yz), o(xw)\} = \{2, 2, 2, 3\}$
4, 4^*	$48(g-1)$	$24(p-2)$	$(o(z), o(xy), o(xzyz^{-1}))$ $= (3, 4, 1)$ or $(3, 1, 4)$
4^P	$8(g-1)$	$4(p-2)$	$(o(xz^{-1}yz), o(xzyz^{-1})) = (1, 2)$ or $(2, 1)$
5, 5^*	$84(g-1)$	N/A	$\{o(x), o(y), o(xy^{-1})\} = \{2, 3, 7\}$
5^P	$12(g-1)$	$6(p-2)$	$\{o(xy), o(xy^{-1)}\} = \{2, 3\}$

Table 1: Bounds on the number of automorphisms

(x, y) for $\mathrm{Aut}(M)$ such that $x^3 = y^7 = (x^{-1}y)^2 = 1$ satisfies the conditions for a canonical generating pair for the automorphism group of a map of class 5 or 5^*, with $\chi = (|A|/3 + |A|/7) - |A| + |A|/2 = -|A|/42$, and hence the same genus and same number of automorphisms as M.

Small examples of maps in classes 2^P, 3 and 4^P attaining the upper bound on the number automorphisms can be found in Sections 4.5 of Alen Orbanić's thesis [16]. In particular, his lists include orientable maps with $(g, |A|) = (6, 60)$, $(2, 24)$ and $(15, 112)$, respectively, and non-orientable maps with $(p, |A|) = (30, 168)$, $(3, 12)$ and $(16, 56)$, respectively.

Additional computations (using MAGMA [1]) give examples attaining the bounds for the remaining five classes. In class 2ex, there exists an orientable map of genus $g = 26$ with $|A| = 1200 = 48(g-1)$, and a non-orientable map of genus $p = 1346$ with $|A| = 32256 = 24(p-2)$, and their duals give corresponding examples in class 2^*ex. Also the same thing happens in classes 4 and 4^*. Finally, in class 5^P there exists an orientable map of genus $g = 14$ with $|A| = 156 = 12(g-1)$, as given also in [16, §4.5], and a non-orientable map of genus $p = 226$ with $|A| = 1344 = 6(p-2)$. This completes the proof. □

Many other examples than those mentioned in the last two paragraphs of the above proof can be found. Further details are available form the first author upon request.

4 Self-dual non-degenerate orientable ET maps

In this brief section, we answer Question 6 of [18], about self-duality for ET maps in classes 1, 2^P, 2^Pex, 3, 4^P and 5^P. In fact we can do even more, by proving the following:

Theorem 4.1. *In each of the classes* 1, 2^P, 2^Pex, 3, 4^P *and* 5^P, *there exist self-dual edge-transitive orientable maps such that the map and its dual have simple underlying graph.*

Proof. Computations using MAGMA produce the non-degenerate self-dual maps summarised in Table 3, with defining relations for their automorphism groups given in Table 4, in the Appendix. □

5 Edge-transitive maps with simple underlying graphs

Question 3 in [18] asked the following: *Does every closed orientable surface support some non-degenerate, edge-transitive map?* (It is not difficult to construct degenerate regular maps (of class 1) on orientable surfaces of all possible genera.)

Examples of small genera are already widely known, such as the five Platonic maps on the sphere (genus 0) and the regular maps $\{4,4\}_{2q}$ on the torus (see [10]).

We can complete an answer to the above question positively by proving the following:

Theorem 5.1. *For every integer $g \geq 2$, there exists an edge-transitive map of class 2 on the orientable surface of genus g, such that both the map and its dual have simple underlying graph (indeed with underlying graph being a complete bipartite graph in each case).*

We have two proofs of this theorem. We give one of them, and then give a brief description of the other, based on an alternative construction for the family of maps involved.

Proof. We start by taking the following group, which is a quotient of the universal group for ET maps of class 2 (obtained by adding two extra relations):

$$G = \langle\, x, y, z \mid x^2 = y^2 = z^2 = [x,z] = (xyzy)^2 = 1 \,\rangle.$$

In this group G, the subgroup N generated by $a = (xy)^2 = [x,y]$ and $b = (yz)^2 = [y,z]$ is normal, with

$$a^x = a^{-1},\ a^y = a^{-1} \ \text{ and }\ a^z = zxyxyz = zx(yxyz)^{-1} = zxzyxy = (xy)^2 = a,$$
$$\text{and }\ b^x = xyzyzx = (xyzy)^{-1}zx = yzyxzx = (yz)^2 = b,\ b^y = b^{-1} \ \textit{and}\ b^z = b^{-1},$$

and the quotient G/N is elementary abelian of order 8. Moreover, by Reidemeister-Schreier theory (explained in [12, §12 & §13] and implemented as the `Rewrite` command in MAGMA [1]), the subgroup N is free abelian of rank 2.

Now for any even positive integers k and m, let $N^{(k,m)}$ be the subgroup of N generated by $a^{k/2} = (xy)^k$ and $b^{m/2} = (yz)^m$. Then $N^{(k,m)}$ is normal in G, with conjugation of its generators by x, $t = y$ and z following the same pattern as given for a and b above, and the quotient $Q^{(k,m)} = G/N^{(k,m)}$ is isomorphic to an extension of $N/N^{(k,m)} \cong C_{k/2} \times C_{m/2}$ by $G/N \cong C_2 \times C_2 \times C_2$. In particular, $Q^{(k,m)} = G/N^{(k,m)}$ has order $2km$.

We now use these quotients $Q^{(k,m)}$ of G to construct bipartite ET maps of class 2 with the required properties, one for each choice of the pair (k,m) with $k \neq m$. For notational convenience, from now on we will let x, y, z, a and b denote the images in $Q^{(k,m)}$ of the elements given above. Equivalently, we simply assume that the elements $a = (xy)^2$ and $b = (yz)^2$ have orders $k/2$ and $m/2$ respectively. Also we now define $Q = Q^{(k,m)}$, and let N be the normal subgroup generated by a and b in Q, isomorphic to $C_{k/2} \times C_{m/2}$, with quotient $Q/N \cong C_2 \times C_2 \times C_2$.

For the underlying graph of the map for a given pair (k,m), we take the vertices of one part as the m right cosets of the dihedral subgroup V_1 of order $2k$ generated by x and y, and the vertices of the other part as the k right cosets of the dihedral subgroup V_2 of order $2m$ generated by y and z, and define adjacency by non-trivial intersection.

For example, the vertex $V_1 = \langle x, y\rangle$ is adjacent to the k cosets of $V_2 = \langle y, z\rangle$ of the form $V_2(xy)^i$ for $i \in \mathbb{Z}_k$, with $V_2(xy)^i x = V_2(yx)^{-i}x = V_2(xy)^{-i-1}$ for all such i. Note that the coset intersections are given by $V_1 \cap V_2(xy)^i = \{(xy)^i, y(xy)^i\} = \{(xy)^i, (xy)^{-i-1}x\}$ for $i \in \mathbb{Z}_k$. In particular, the valency of the vertex V_1 is k, and hence

V_1 is adjacent to every vertex of the second part. Similarly, the vertex $V_2 = \langle y, z\rangle$ has neighbours $V_1(zy)^j$ for $0 \le j < m$, and hence it is adjacent to every one of the m vertices of the first part.

For the faces of the map, we use the right cosets of the subgroup H of order 4 generated by x and z, and define incidence again by non-empty intersection. (We preserve the symbol F for the set of faces.) For example, since $H = \{1, x, z, xz\}$, the face H itself is incident with the vertices V_1 and $V_1 z$ in the first part, and the vertices V_2 and $V_2 x$ in the second part. Here we note that $z \notin V_1 = \langle x, y\rangle$, for otherwise $Q = \langle x, y, z\rangle = \langle x, y\rangle$ which has order $2k < 2km$, and similarly $x \notin V_2 = \langle y, z\rangle$, because the latter has order $2m < 2km$. In particular, the face H has four different vertices, and hence also four different edges.

Also the group Q acts by right multiplication on this map, with two orbits of sizes m and k on vertices, namely the two parts of the graph, and a single orbit on faces, and a single orbit on edges. Indeed the edges can be identified with right cosets of the subgroup K of order 2 generated by y, with $K(xy)^i = \{(xy)^i, y(xy)^i\} = \{(xy)^i, (xy)^{-i-1}x\}$ for all $i \in \mathbb{Z}_k$. It follows that every vertex of the first part has valency k, and every vertex of the second part has valency m, and hence the graph is isomorphic to the complete bipartite graph $K_{m,k}$. Similarly, the map has $2km/4 = km/2$ faces, all of which have length 4 (with four distinct vertices), and so the dual of the map is simple too.

Next, if $k \neq m$ then the parts of the underlying graph have different sizes, so the map cannot lie in class 1, and hence it has class 2. (This also follows from the fact that Q has no automorphism taking (x, y, z) to (z, y, x), since the orders of xy and zy are k and m.)

Finally, the map is orientable, since the subgroup generated by xy and zy $(= (yz)^{-1})$ has index 2 in Q (with image $C_2 \times C_2$ in $Q/N \cong C_2 \times C_2 \times C_2$), and its Euler characteristic is $\chi = |V| - |E| + |F| = (m + k) - mk + (mk/2) = m + k - mk/2$, so its genus g is $(2 - \chi)/2 = (4 - 2m - 2k + mk)/4 = (k - 2)(m - 2)/4$. Taking $m = 4$ gives genus $g = (k - 2)/2 = k/2 - 1$, which can be any integer greater than 1 when $k/2 > 2$.

This completes the construction and proof. □

An alternative way of constructing these maps is to add six extra relations to the universal group for ET maps of class 3, to give the group with presentation

$$\begin{aligned} \langle\, x, y, z, w \mid x^2 = y^2 = z^2 = w^2 = [x, z] = \\ = [y, z] = [x, w] = [y, w] = (xy)^k = (zw)^m = 1\,\rangle, \end{aligned}$$

which is isomorphic to the direct product of $\langle x, y\rangle \cong D_k$ and $\langle z, w\rangle \cong D_m$.

The above map with underlying graph $K_{m,k}$ can be constructed from this group, but the group admits an automorphism of order 2 that takes (x, y, z, w) to (y, x, w, z), and another taking (x, y, z, w) to (z, w, x, y) when $k = m$, and hence the map has class 1 or class 2, depending on whether or not $k = m$. The proof is straightforward.

Next, a natural question related to Question 3 in [18] (but not posed in [18]) is the following: *Does there exist a simple graph X that is the underlying graph of a map in each of the* 14 *classes of edge-transitive maps?*

Note that any such graph must be arc-transitive and hence regular (because it underlies an ET map of class 1), with valency divisible by 4 (because it underlies an ET map of class 5), and also bipartite (because it underlies an ET map of class 2). Accordingly, there are natural candidates to check. In an early search we found that the complete bipartite graph $K_{8,8}$ underlies ET maps of 11 of the 14 classes, namely all of them except classes 2^*ex, 2^Pex and 5. A little further work led us to a positive answer to the question:

Theorem 5.2. *In each of the* 14 *classes of edge-transitive maps, there exists an orientable map with underlying graph isomorphic to the complete bipartite graph* $K_{16,16}$.

Proof. Computations using MAGMA produce the maps with underlying graph $K_{16,16}$ summarised in Table 5, with defining relations for their automorphism groups given in Table 6, in the Appendix. □

6 Orientable ET maps of genus 14

Question 4 in [18, Section 6] is the following: *What is the largest number of automorphism-group types for edge-transitive maps contained by one surface? Is there some surface that supports all 14 types?* We can now answer both parts of this.

Theorem 6.1. *The orientable surface of genus* 14 *carries edge-transitive maps of all* 14 *classes.*

Proof. Computations using MAGMA produce the maps summarised in Table 7, with defining relations for their automorphism groups given in Table 8, in the Appendix. □

In particular, the answer to the second part of Question 4 in [18, Section 6] is "Yes", and the answer to the first part is 14. Also we can prove that 14 is the smallest genus for which this happens, by MAGMA computations and Theorem 3.1 (to bound the order of the groups required for consideration in a search for examples). Again, further details are available from the first author upon request. On the other hand, there are other surfaces of higher genus that carry ET maps of all 14 classes, as explained in the next section.

7 Non-degenerate orientable ET maps of genus 17

A natural extension of Question 4 in [18, Section 6] is the following: *Is there some surface that supports at least one* non-degenerate *ET map of each of the 14 classes?* The answer to this question is also "Yes":

Theorem 7.1. *The orientable surface of genus* 17 *carries edge-transitive maps of all* 14 *classes, with the property that the map and its dual have simple underlying graph.*

Proof. Computations using MAGMA produce the non-degenerate maps summarised in Table 9, with defining relations for their automorphism groups given in Table 10, in the Appendix. □

8 Edge-transitive maps with prescribed automorphism group(s)

Another very natural question is this: *Which finite groups occur as the automorphism group of at least one ET map in each of the 14 classes?* This same question can be asked with a restriction to simple or non-degenerate maps.

Clearly a necessary condition is that the group can be generated by two elements. On the other hand, for orientable ET maps with more than two edges, it cannot be cyclic, because in [18] it was shown that if an orientable ET map M has at least three edges, then $\mathrm{Aut}(M)$ is a non-abelian group, except in the case where M has class 4^P, and $\mathrm{Aut}(M)$ is isomorphic to the direct product $C_n \times C_2$ where $n \equiv 2 \bmod 4$. In that exceptional case the stabiliser of every vertex and every face is the subgroup of order 4 generated by the images of the two involutory generators x and y of the universal group for class 4^P, and the underlying graph of the map has double edges. Similarly it can be shown easily that if a non-orientable ET map M has at least three edges and $\mathrm{Aut}(M)$ is abelian, then either M has class 3 (with two vertices and eight edges) and $\mathrm{Aut}(M) \cong C_2 \times C_2 \times C_2$, or M has class 4 or 4^* and $\mathrm{Aut}(M) \cong C_n \times C_2$ for some even n, or M has class 4^P and $\mathrm{Aut}(M) \cong C_n \times C_2$ for some n divisible by 4. Again in these cases the underlying graph of the map has multiple edges.

We can also extend these observations to abelian groups that occur as the orientation-preserving subgroup of $\mathrm{Aut}(M)$ for some orientable ET map M. It is not difficult to show that there are no such maps in classes 2^P, 2ex, 2^*ex, 2^Pex, 4, 4^*, 5, 5^* and 5^P (mainly because the abelian groups involved admit outer automorphisms that put the map into a higher class), but there exist infinite families of examples in the other five classes, namely 1, 2, 2^*, 3 and 4^P.

In class 1 these maps have just one or two vertices, and just one or two faces, with multiple edges. In class 4^P the maps have $(|V|, |E|, |F|) = (k, 2kn, k)$ for arbitrary integers $k \geq 1$ and $n \geq 2$ or 3, but again none of them is simple. In class 2, some of the maps are simple but all have non-simple dual, while in class 2^*, all of the maps are non-simple (but some have simple dual). The only class containing non-degenerate orientable ET maps is class 3, and in these case there are four infinite families of examples with $(|V|, |E|, |F|) = (4k+4, 16k, 4k+4)$ and $\mathrm{Aut}(M) \cong C_{2k} \times C_2 \times C_2$ for all $k \geq 2$, with two infinite families containing self-dual examples, and the other two consisting of duals of each other.

Details are available from the first author upon request.

A major contribution in the non-abelian case for the above question was made by Gareth Jones in [13], where he considered this for non-abelian simple groups, the symmetric groups S_n, soluble groups, and nilpotent groups. In particular, he showed in [13, Theorem 1.2] that a given non-abelian simple group is the automorphism group of some ET map of a given class unless it appears in a list of known exceptions, copied in Table 2 below.

Class	Non-abelian simple groups that do not occur
1	$L_3(q)$, $U_3(q)$, $L_4(2^e)$, $U_4(3)$, $U_5(2)$, A_6, A_7, M_{11}, M_{22}, M_{23}, McL
2, 2^*, 2^P	$U_3(3)$
2ex, 2^*ex, 2^Pex	$L_2(q)$, $L_3(q)$, $U_3(q)$, A_7
3	–
4, 4^*, 4^P	–
5, 5^*, 5^P	$L_2(q)$

Table 2: Non-occurrences of non-abelian simple groups for a given class of ET map

Note that in 11 of the 14 classes (all except 2^Pex, 5 and 5^*), the maps are non-orientable, for the obvious reason that a non-abelian simple group has no subgroup of index 2.

It follows easily from Table 2 that the smallest simple group that occurs for all 14 classes is the Suzuki group $Sz(8)$, of order 29120. In fact, $Sz(8)$ is the automorphism group of some non-degenerate ET map in every one of the 14 classes; details are available from the first author on request. The same holds for the next smallest examples, which are M_{12}, J_1 and A_9, and in particular, A_5, A_6, A_7 and A_8 do not occur in this way.

Indeed by [13, Theorem 1.1], the alternating group A_n of degree n is the automorphism group of some ET map in any given class, except in the following cases: $n \in \{3, 4, 6, 7, 8\}$ for class 1; $n \in \{1, 2, 3, 4\}$ for classes 2, 2^*, 2^P and 3; $n \in \{1, 2, 3, 4, 5, 6, 7\}$ for classes 2ex, 2^*ex and 2^Pex; $n \in \{1, 2, 3\}$ for classes 4, 4^* and 4^P; and $n \in \{1, 2, 3, 4, 5, 6\}$ for classes 5, 5^* and 5^P. In particular, A_9 is the smallest alternating group that occurs for all 14 classes. Moreover, non-degenerate ET maps occur in all classes for A_9 (and A_{10}); details are available from the first author on request.

Similarly, by [13, Theorem 1.1], the symmetric group S_n of degree n is the automorphism group of some ET map in any given class, except in the following cases: $n = 1$ for classes 2, 2^*, 2^P, 3, 4, 4^* and 4^P; and $n \in \{1, 2, 3, 4, 5\}$ for classes 2ex, 2^*ex, 2^Pex, 5, 5^* and 5^P. In particular, S_6 is the smallest symmetric group that occurs for all 14 classes. Moreover, non-degenerate ET maps occur in all classes for S_6 (and S_7, S_8, S_9 and S_{10}); details are available from the first author on request.

Incidentally, the symmetric group S_6 is the smallest insoluble group that occurs as the automorphism group of some ET map in every one of the 14 classes, and again, the same holds under the assumption that the map is simple or non-degenerate.

Finally, we have the following:

Theorem 8.1. *The smallest finite group G that occurs as the automorphism group of some ET map in every one of the* 14 *classes is the* 8654*th group of order* 576 (*in the database of all groups of order up to* 2000).

Proof. Computations using MAGMA produce maps of each class for this group, indeed non-degenerate maps of all 14 classes, and these are summarised in Table 11, with defining relations for their automorphism groups given in Table 12, in the Appendix. The same computations show that no group of smaller or equal order (necessarily divisible by 4) has the required property. □

Note that just one of the maps given in Table 11 is non-orientable, namely the one in class 2^P. There are also orientable examples in class 2^P with simple underlying graph, but none with simple dual. Again, further details about these maps are available from the first author on request.

9 Some final remarks and questions

Answers always raise more questions. We ask a few of them for this topic below.

For each of the 14 classes of ET maps, define the *non-degenerate genus spectrum* for that class to be the set of genera of orientable surfaces that carry a non-degenerate map of that class. By our Theorem 5.1 (and knowledge of ET maps of genus 0 and 1), the non-degenerate genus spectrum for class 2 (and therefore also 2^*) is the set of all non-negative integers. The corresponding question for classes 1 (regular maps) and/or 2^Pex (chiral maps), however, is a challenging open question (see [7, 8]).

Indeed, questions about genus spectra questions are notoriously difficult. Another example comes from group actions. Given a finite group G, its *symmetric genus* $\sigma(G)$ is defined as the smallest non-negative integer g for which G has a faithful action on some compact Riemann surface of genus g. Much but not all of the spectrum for σ is known; see [9]. In contrast, the *strong symmetric genus* $\sigma^o(G)$ is the smallest g such that G has a faithful action on such a surface of genus g, preserving orientation, and the spectrum for σ^o contains all non-negative integers, by a theorem of May and Zimmerman in [15], which incidentally uses the groups $C_k \times D_m$ in a similar way to the way we used the groups $D_k \times D_m$ in proving Theorem 5.1. The following question(s) could constitute a long-term project:

Question 9.1. What are the genus spectra for the 14 classes of ET maps, under the restriction to non-degenerate maps, or to simple maps, or to ET maps in general?

The symmetric groups S_n play a prominent role in [18], producing ET maps in each class, but with restrictions on the congruence class n modulo 12. Those restrictions are removed in [13], where it is also shown that S_n realises all classes for every $n \geq 6$, and A_n realises all classes for every $n \geq 9$.

Similarly, the first author of this paper showed in [4] that S_n and A_n realise the upper bounds $|A|_{\text{maxo}}$ and $|A|_{\text{maxnono}}$ given in Table 1 for class 1 (and hence also for classes 2 and 2^*), for all but finitely many n, and many years later the same in [2, Theorem 6.3] for A_n for class 2^Pex (and hence also classes 5 and 5^*), again for all but finitely many n.

Question 9.2. For which other classes do A_n and/or S_n realise the upper bounds $|A|_{\text{maxo}}$ and $|A|_{\text{maxnono}}$ in *Table 1*, possibly with restrictions on n?

Edge-transitive maps on non-orientable surfaces have been largely ignored in the literature. Even in this paper, most of our theorems, examples and tables concern maps on orientable surfaces, and yet there should be non-orientable versions:

Question 9.3. What are the analogues of *Theorems 4.1*, *5.1*, *5.2*, *6.1* and *7.1* for non-orientable ET maps (in the 11 classes other than 2^Pex, 5 and 5^*)? How might the information given in the tables in the Appendix differ for non-orientable maps?

Finally, for a map on a non-orientable surface S with automorphism group G, a standard technique is to pass to the orientable double cover of S, where $G \times C_2$ acts, with G preserving orientation and C_2 orientation-reversing and fixed-point free (in effect, an antipodal symmetry). If the given map (on S) is edge-transitive, then so is the lifted map under the group $G \times C_2$. On the other hand, the lifted map may actually be 'unstable', in that it has a symmetry group larger than $G \times C_2$, which means that the lifted map may be in a different class, with larger edge-stabiliser. Many examples of this phenomenon have been provided recently by Gareth Jones [14].

Question 9.4. How common is instability for edge-transitive maps on non-orientable surfaces?

ORCID iDs

Marston D.E. Conder https://orcid.org/0000-0002-0256-6978
Isabel Holm https://orcid.org/0000-0002-6806-7514
Thomas W. Tucker https://orcid.org/0000-0002-7868-6925

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Appendix

Below are the tables mentioned in the proofs of Theorems 5.1, 5.2, 6.1, 7.1 and 4.1. Note that there are two tables for each theorem: the first gives details about a map from each of the relevant classes, and the second gives defining relations for the automorphism groups of those maps (in terms of the canonical generators for the relevant universal group).
The following notation is used in the odd-numbered tables. First 'SD' and 'NSD' indicate that the map is self-dual or non-self-dual, respectively. (We do not indicate 'NSD' in the obvious cases, where $|V| \neq |F|$.) Next, C_n, D_n, A_n and S_n denote the cyclic group of order n, the dihedral group of degree n (and order $2n$), and the alternating and symmetric groups of degree n, while $\mathrm{Group}(n,k)$ denotes the kth group in the database of all groups of order up to 2000 (except 1024), available in the MAGMA system [1]. Finally, V_4 denotes the direct product $C_2 \times C_2$ (the Sylow 2-subgroup of A_4), while $K \rtimes H$ denotes a semi-direct product with kernel K and complement H, and $C_n \rtimes_k C_m$ denotes the semi-direct product $\langle\, a, b \mid a^m = b^n = 1, a^{-1}ba = b^k \,\rangle$.

Class	$\lvert V\rvert$	$\lvert E\rvert$	$\lvert F\rvert$	Genus	$\lvert A\rvert$	A_u, A_v	A_f, A_g	Comments
1	4	6	4	0	24	D_3	D_3	$\mathrm{Aut}(M) \cong S_4$
2^P	8	16	8	1	32	V_4	V_4	$\mathrm{Aut}(M) \cong$ $\mathrm{Group}(32, 43)$
2^Pex	5	10	5	1	20	C_4	C_4	$\mathrm{Aut}(M) \cong C_5 \rtimes_2 C_4$
3	12	32	12	5	32	D_4, V_4	D_4, V_4	$\mathrm{Aut}(M) \cong D_4 \times V_4$
4^P	12	48	12	13	48	V_4	V_4	$\mathrm{Aut}(M) \cong A_4 \times V_4$
5^P	14	42	14	8	42	C_3	C_3	$\mathrm{Aut}(M) \cong C_7 \rtimes_2 C_6$

Table 3: Non-degenerate self-dual maps with the smallest number of edges in six of the 14 classes

Class	Defining relations for $A = \mathrm{Aut}(M)$
1	$x^2 = y^2 = z^2 = (xz)^2 = (xy)^3 = (yz)^3 = 1$
2^P	$x^2 = y^2 = z^2 = (yz)^2 = (xz)^4 = (xyxz)^2 = (xy)^3 zxzy = 1$
2^Pex	$x^2 = y^4 = xyxy^2xy^{-1} = 1$
3	$x^2 = y^2 = z^2 = w^2 = (xz)^2 = (yz)^2 = (yw)^2 = (zw)^2 = (xy)^4 = xywxwy = 1$
4^P	$x^2 = y^2 = (xy)^2 = (xz)^3 = [x, z]^2 = [xy, z] = 1$
5^P	$x^6 = xy^2x^2y^{-1} = xy^{-2}x^2y = 1$

Table 4: Defining relations for the automorphism groups of the maps in Table 3

Class	$\lvert F\rvert$	Genus	$\lvert\mathrm{Aut}(M)\rvert$	A_u, A_v	A_f, A_g	Comments
1	16	105	1024	D_{16}	D_{32}	$\mathrm{Aut}(M)$ not in database
2	128	49	512	D_{16}, D_{16}	V_4	$\mathrm{Aut}(M) \cong \mathrm{Group}(512, 30471)$
2^*	80	201	512	D_8	D_{16}, D_4	$\mathrm{Aut}(M) \cong \mathrm{Group}(512, 32917)$
2^P	32	225	512	D_8	D_8	SD, $\mathrm{Aut}(M) \cong \mathrm{Group}(512, 32917)$
2ex	16	105	512	D_8	C_{32}	$\mathrm{Aut}(M) \cong \mathrm{Group}(512, 1056)$
2^*ex	16	105	512	C_{16}	D_{16}	$\mathrm{Aut}(M) \cong \mathrm{Group}(512, 955)$
2^Pex	16	105	512	C_{16}	C_{32}	$\mathrm{Aut}(M) \cong \mathrm{Group}(512, 955)$
3	32	97	256	D_8, D_8	D_8, D_8	SD, $\mathrm{Aut}(M) \cong \mathrm{Group}(256, 722)$
4	32	97	256	D_8, C_{16}	D_4	$\mathrm{Aut}(M) \cong \mathrm{Group}(256, 56)$
4^*	48	89	256	D_4	D_4, C_{16}	$\mathrm{Aut}(M) \cong \mathrm{Group}(256, 95)$
4^P	32	97	256	D_4	D_4	SD, $\mathrm{Aut}(M) \cong \mathrm{Group}(256, 95)$
5	16	105	256	C_{16}, C_{16}	C_{16}	$\mathrm{Aut}(M) \cong \mathrm{Group}(256, 41)$
5^*	96	65	256	C_8	C_4, C_8	$\mathrm{Aut}(M) \cong \mathrm{Group}(256, 117)$
5^P	32	97	256	C_8	C_8	SD, $\mathrm{Aut}(M) \cong \mathrm{Group}(256, 117)$

Table 5: ET maps of all 14 types with underlying graph $K_{16,16}$

Class	Defining relations for $A = \mathrm{Aut}(M)$
1	$x^2 = y^2 = z^2 = [x,z] = (xyxzyzy)^2 = (xyxyxyxyz)^2 = (xy)^{13}zyxyzy = (yz)^{16} = 1$
2	$x^2 = y^2 = z^2 = (xz)^2 = (xyxyz)^2 = (xy)^{13}zyxyzy = (yz)^{16} = 1$
2^*	$x^2 = y^2 = z^2 = (xz)^4 = (yz)^8 = xyxyxzyxyz = xyxzxzyzxz = 1$
2^P	Same defining relations as for class 2^* above
2ex	$x^2 = (xy^4)^2 = (xy^{-2})^2(xy^2)^2 = (xyxyxy^{-2})^2 = xyxyxy^{-1}xyxy^{-1}xyxy^{-1}xy^3 = 1$
2^*ex	$x^2 = xy^{-1}xyxy^{-3}xy^3 = (xy^2)^2(xy^{-2})^2 = y^{16} = 1$
2^Pex	$x^2 = xy^4xy^{12} = xyxy^{-2}xy^3xy^{-2} = xyxy^{-4}xy^{-1}xy^4 = xy^{-1}xy(xy^{-1})^3(xy)^3 = 1$
3	$x^2 = y^2 = z^2 = w^2 = (xz)^2 = (yw)^4 = (yz)^8 = (xwy)^2 = (xy)^2(xw)^2 = (xy)^2zywz = 1$
4	$x^2 = y^2 = xyxzxz^{-1} = xyxz^{-1}xz = xyxyzyzyz^{-1}yz^{-1}y = z^{16} = 1$
4^*	$x^2 = y^2 = (xy)^4 = xyzxyz^{-1} = xz(yz)^3 = xz^{-1}xz^{-2}xzxz^2 = (xy)^2z^8 = 1$
4^P	Same defining relations as for class 4^* above
5	$xyx^3y^{-1} = x^{-1}y^{-1}x^5y = y^{16} = 1$
5^*	$x^4 = y^8 = [x^2, y^2] = xyxy^{-1}x^{-1}yxy^{-1} = x^{-1}y^{-3}(xy)^3 = 1$
5^P	Same defining relations as for class 5^* above

Table 6: Defining relations for the automorphism groups of the maps in Table 5

Class	$\|V\|$	$\|E\|$	$\|F\|$	$\|\mathrm{Aut}(M)\|$	A_u, A_v	A_f, A_g	Comments
1	2	35	7	140	D_{35}	D_{10}	$\mathrm{Aut}(M) \cong D_5 \times D_7$
2	2	29	1	58	D_{29}, D_{29}	D_{29}	$\mathrm{Aut}(M) \cong D_{29}$
2^*	1	29	2	58	D_{29}	D_{29}, D_{29}	Map dual to the one above
2^P	2	30	2	60	D_{15}	D_{15}	SD, $\mathrm{Aut}(M) \cong D_{15} \times C_2$
2ex	4	40	10	80	D_{10}	C_8	$\mathrm{Aut}(M) \cong C_5 \rtimes (C_8 \rtimes_5 C_2)$
2^*ex	10	40	4	80	C_8	D_{10}	Map dual to the one above
2^Pex	26	78	26	156	C_6	C_6	NSD, $\mathrm{Aut}(M) \cong (C_{13} \rtimes_{10} C_6) \times C_2$
3	2	30	2	30	D_{15}, D_{15}	D_{15}, D_{15}	SD, $\mathrm{Aut}(M) \cong D_{15}$
4	12	40	2	40	D_{10}, C_4	D_{10}	$\mathrm{Aut}(M) \cong D_5 \times C_4$
4^*	2	40	12	40	D_{10}	D_{10}, C_4	Map dual to the one above
4^P	13	52	13	52	V_4	V_4	SD, $\mathrm{Aut}(M) \cong C_{26} \times C_2$
5	9	40	5	40	C_8, C_{10}	C_8	$\mathrm{Aut}(M) \cong C_5 \rtimes_2 C_8$
5^*	5	40	9	40	C_8	C_8, C_{10}	Map dual to the one above
5^P	26	78	26	78	C_3	C_3	SD, $\mathrm{Aut}(M) \cong C_{13} \rtimes_3 C_6$

Table 7: ET maps of all 14 types on the orientable surface of genus 14

Class	Defining relations for $A = \mathrm{Aut}(M)$
1	$x^2 = y^2 = z^2 = (xz)^2 = (xyxyz)^2 = (xyz)^2(yz)^5 = 1$
2	$x^2 = y^2 = z^2 = xzyz = (xy)^{14}xz = 1$
2^*	$x^2 = y^2 = z^2 = yzxz = (yx)^{14}yz = 1$
2^P	$x^2 = y^2 = z^2 = (xyz)^2 = (xz)^2(yz)^3 = (xy)^6 = 1$
2ex	$x^2 = y^8 = [x, y^4] = (xy)^4y^4 = xyxy^2xy^2xy^{-1} = 1$
2^*ex	Same defining relations as for class 2ex above
2^Pex	$x^2 = y^6 = (xyxy^2)^2 = (xy)^3(xy^{-1})^3 = xy^{-2}xy^2xy^3xy^3 = 1$
3	$x^2 = y^2 = z^2 = w^2 = xyxzwy = xyxwxw = (xz)^3 = 1$
4	$x^2 = y^2 = z^4 = [x, z] = [xz, y] = (xy)^3(zy)^2 = 1$
4^*	Same defining relations as for class 4 above
4^P	$x^2 = y^2 = (xy)^2 = [x, z] = [y, z] = yz^{13} = 1$
5	$x^2yx^{-2}y = xy^3x^{-1}y^{-1} = xy^{-1}x^3y^2 = 1$
5^*	Same defining relations as for class 5 above
5^P	$x^6 = (xy)^3 = (xy^{-1})^3 = xy^{-1}x^{-1}y^3 = 1$

Table 8: Defining relations for the automorphism groups of the maps in Table 7

Class	$\lvert V\rvert$	$\lvert E\rvert$	$\lvert F\rvert$	$\lvert\mathrm{Aut}(M)\rvert$	A_u, A_v	A_f, A_g	Comments
1	32	96	32	384	D_6	D_6	SD, $\mathrm{Aut}(M) \cong \mathrm{Group}(384, 5602)$
2	40	96	24	192	D_4, D_6	D_4	$\mathrm{Aut}(M) \cong \mathrm{Group}(192, 591)$
2^*	24	96	40	192	D_4	D_4, D_6	Map dual to the one above
2^P	16	64	16	128	D_4	D_4	SD, $\mathrm{Aut}(M) \cong \mathrm{Group}(128, 332)$
2ex	32	128	64	256	D_4	C_4	$\mathrm{Aut}(M) \cong \mathrm{Group}(256, 511)$
2^*ex	64	128	32	256	C_4	D_4	Map dual to the one above
2^Pex	32	96	32	192	C_6	C_6	SD, $\mathrm{Aut}(M) \cong \mathrm{Group}(192, 1008)$
3	16	64	16	64	V_4, V_4	V_4, V_4	$\mathrm{Aut}(M) \cong \mathrm{Group}(64, 73)$
4	40	96	24	96	V_4, C_6	V_4	$\mathrm{Aut}(M) \cong \mathrm{Group}(96, 118)$
4^*	24	96	40	96	V_4	V_4, C_6	Map dual to the one above
4^P	16	64	16	64	V_4	V_4	SD, $\mathrm{Aut}(M) \cong \mathrm{Group}(64, 42)$
5	48	96	16	96	C_3, C_6	C_6	$\mathrm{Aut}(M) \cong \mathrm{Group}(96, 71)$
5^*	16	96	48	96	C_6	C_3, C_6	Map dual to the one above
5^P	16	64	16	64	C_4	C_4	SD, $\mathrm{Aut}(M) \cong \mathrm{Group}(64, 46)$

Table 9: Non-degenerate ET maps of all 14 types on the orientable surface of genus 17

Class	Defining relations for $A = \mathrm{Aut}(M)$
1	$x^2 = y^2 = z^2 = (xz)^2 = (xy)^6 = (xyzy)^3 = (yz)^6 = [xyxyx, zyzyz] = 1$
2	$x^2 = y^2 = z^2 = (xy)^4 = (xz)^4 = (yz)^6 = (xyz)^2(xzy)^2 = x(yz)^2x(zy)^2 = 1$
2^*	$x^2 = y^2 = z^2 = (xy)^4 = (yz)^4 = (xz)^6 = (yxz)^2(yzx)^2 = y(xz)^2y(zx)^2 = 1$
2^P	$x^2 = y^2 = z^2 = (xz)^4 = (yz)^4 = (xy)^3zxzy = xyxzyz(xyz)^2 = 1$
2ex	$x^2 = y^4 = (xy)^2(xy^2)^2(xy^{-1})^2 = 1$
2^*ex	Same defining relations as for class 2ex above
2^Pex	$x^2 = y^6 = (xy^3)^4 = xyxy^2xy^{-2}xy^{-1} = xyxy^{-1}xy^{-1}xy^2xyxy^{-2} = 1$
3	$x^2 = y^2 = z^2 = w^2 = (yw)^2 = (xy)^4 = (yz)^4 = xyzxzw = yzywzw = 1$
4	$x^2 = y^2 = z^6 = (xy)^2 = [x, z^2] = (yz^2)^2 = (xzyz)^2 = (yz)^2(yz^{-1})^2 = xzxz^{-1}xz^{-1}yxz^{-1}y = 1$
4^*	Same defining relations as for class 4 above
4^P	$x^2 = y^2 = xyz^{-1}xyz = xyz^{-1}yz^{-1}y = xyxz^3xz^{-1} = 1$
5	$x^3 = y^6 = (xy)^4 = xy^{-1}x^{-1}yx^{-1}y^2 = xy^2x^{-1}y^{-1}xy^3 = 1$
5^*	Same defining relations as for class 5 above
5^P	$x^2y^{-1}x^2y = x^{-1}y^{-1}xy^3 = 1$

Table 10: Defining relations for the automorphism groups of the maps in Table 9

Class	$\lvert V\rvert$	$\lvert E\rvert$	$\lvert F\rvert$	Genus	A_u, A_v	A_f, A_g	Comments
1	48	144	48	25	D_6	D_6	NSD, Orientable
2	120	288	48	61	D_4, D_6	D_6	Orientable
2^*	48	288	120	61	D_6	D_4, D_6	Map dual to the one above
2^P	48	288	48	97	D_6	D_6	Non-orientable
2ex	48	288	96	73	D_6	C_6	Orientable
2^*ex	96	288	48	73	C_6	D_6	Map dual to the one above
2^Pex	96	288	96	49	C_6	C_6	SD, Orientable
3	240	576	120	109	V_4, D_3	D_6, D_4	Orientable
4	240	576	144	97	D_6, C_3	V_4	Orientable
4^*	144	576	240	97	V_4	D_6, C_3	Map dual to the one above
4^P	72	576	144	181	D_4	V_4	Orientable
5	240	576	96	121	C_4, C_6	C_6	Orientable
5^*	96	576	240	121	C_6	C_4, C_6	Map dual to the one above
5^P	96	576	96	193	C_6	C_6	SD, Orientable

Table 11: Non-degenerate ET maps of all 14 types with $\mathrm{Aut}(M) \cong \mathrm{Group}(576, 8654)$

Class	Defining relations for $A = \mathrm{Aut}(M)$
1	$x^2 = y^2 = z^2 = (xz)^2 = (xy)^6 = (yz)^6 = (xyxyzy)^3 = (xyz)^6 = (xyzy)^2xzyzyxyzyz = 1$
2	$x^2 = y^2 = z^2 = (xy)^4 = (xz)^6 = (yz)^6 = (xyxz)^3 = xyzyxyzyxzxz = 1$
2^*	$x^2 = y^2 = z^2 = (xy)^4 = (xz)^6 = (yz)^6 = (xyxz)^3 = xyzyxyzyxzxz = 1$
2^P	$x^2 = y^2 = z^2 = (zx)^3 = (xy)^4 = (yz)^6 = (xzxyzy)^3 = xyxzyxzyzxyz = 1$
2ex	$x^2 = y^6 = (xy)^6 = (xy^2)^4 = (xy)^2(xy^{-1})^2(xy^3)^2 = 1$
2^*ex	Same defining relations as for class 2ex above
2^Pex	Same defining relations as for class 2ex above
3	$x^2 = y^2 = z^2 = w^2 = (xy)^2 = (xz)^2 = (wz)^3 = (yz)^4 = (xzwyw)^2 = (xwxwz)^2$ $= xyzwxzywyzyw = xwyzwzyzwzyw = 1$
4	$x^2 = y^2 = z^3 = (yz)^4 = (xz^{-1}yz)^2 = (xz^{-1})^2(xz)^2 = xyxz^{-1}xyxzyz^{-1}y = 1$
4^*	Same defining relations as for class 4 above
4^P	$x^2 = y^2 = z^6 = (xy)^2 = (yz)^4 = [y, z^3] = (xzyz^{-1})^2 = xyzyz^{-2}xzxz^{-1} = (xz^3)^2xz^{-3} = 1$
5	$x^4 = y^6 = x^{-1}yx^{-2}yxy^2 = [x^2, y]^2 = (xy)^4x^{-1}y^{-1}xy^{-1} = 1$
5^*	Same defining relations as for class 5 above
5^P	Same defining relations as for class 5 above

Table 12: Defining relations for the automorphism groups of the maps in Table 11