GCD-Graphs and NEPS of Complete Graphs

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Abstract

A gcd-graph is a Cayley graph over a finite abelian group defined by greatest common divisors. Such graphs are known to have integral spectrum. A non-complete extended p-sum, or NEPS in short, is well-known general graph product. We show that the class of gcd-graphs and the class of NEPS of complete graphs coincide. Thus, a relation between the algebraically defined Cayley graphs and the combinatorially defined NEPS of complete graphs is established. We use this link to show that gcd-graphs have a particularly simple eigenspace structure, to be precise, that every eigenspace of the adjacency matrix of a gcd-graph has a basis with entries $-1, 0, 1$ only.

Keywords: Integral graphs, Cayley graphs, graph products.

Math. Subj. Class.: 05C25, 05C50

1 Introduction

Given a set $B \subseteq \{0,1\}^n$ and graphs $G_1, \ldots, G_n$, the NEPS (non-complete extended p-sum) of these graphs with respect to *basis B*, $G = \text{NEPS}(G_1, \ldots, G_n; B)$, has as its vertex set the Cartesian product of the vertex sets of the individual graphs, $V(G) = V(G_1) \times \cdots \times V(G_n)$. Distinct vertices $x = (x_1, \ldots, x_n)$, $y = (y_1, \ldots, y_n) \in V(G)$ are adjacent in G, if and only if there exists some n-tuple $(\beta_1, \ldots, \beta_n) \in B$ such that $x_i = y_i$, whenever $\beta_i = 0$, and x_i, y_i are distinct and adjacent in G_i, whenever $\beta_i = 1$. In particular, $\text{NEPS}(G_1; \{(1)\}) = G_1$ and $\text{NEPS}(G_1; \emptyset) = \text{NEPS}(G_1; \{(0)\})$ is the graph without edges on the vertices of G_1.

The NEPS operation generalizes a number of known graph products, all of which have in common that the vertex set of the resulting graph is the Cartesian product of the input vertex sets. For example, $\text{NEPS}(G_1, \ldots, G_n; \{(1, 1, \ldots, 1)\}) = G_1 \otimes \ldots \otimes G_n$ is the

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product of $G_1, \ldots, G_n$ (cf. [10], "direct product" in [15]). As can be seen, unfortunately, the naming of graph products is not standardized at all. The "Cartesian product" of graphs in [15] is even known as the "sum" of graphs in [10]. With respect to this seemingly arbitrary mixing of sum and product terminology, let us point out that here the term "sum" (and also the "p-sum" contained in the NEPS acronym) indicates that the adjacency matrix of the constructed product graph arises from a certain sum of matrices (involving the adjacency matrices of the input graphs). Refer to [10] or [11] for the history of the notion of NEPS. We remark that the NEPS operation can be generalized even further, see e.g. [12] and [21].

Next, we consider the important class of Cayley graphs [13]. These graphs have been and still are studied intensively because of their symmetry properties and their connections to communication networks, quantum physics and other areas [8], [13]. Let Γ be a finite, additive group. A subset $S \subseteq \Gamma$ is called a *symbol* (also: connection set, shift set) of Γ if $-S = \{-s : s \in S\} = S, 0 \notin S$. The undirected *Cayley graph over Γ with symbol S*, denoted by $\mathrm{Cay}(\Gamma, S)$, has vertex set Γ; two vertices $a, b \in \Gamma$ are adjacent if and only if $a - b \in S$.

Let us now construct the class of gcd-graphs. The greatest common divisor of nonnegative integers a and b is denoted by $\gcd(a, b)$, $\gcd(0, b) = \gcd(b, 0) = b$. If $x = (x_1, \ldots, x_r)$ and $m = (m_1, ..., m_r)$ are tuples of nonnegative integers, then we set

$$\gcd(x, m) = (d_1, \ldots, d_r) = d, \ \ d_i = \gcd(x_i, m_i) \text{ for } i = 1, \ldots, r.$$

For an integer $n \geq 1$ we denote by Z_n the additive group of integers modulo n, the ring of integers modulo n, or simply the set $\{0, 1, \ldots, n - 1\}$. The particular choice will be clear from the context. Let Γ be an (additive) finite abelian group represented as a direct sum of cyclic groups,

$$\Gamma = Z_{m_1} \oplus \ldots \oplus Z_{m_r} \, , \ m_i \geq 1 \text{ for } i = 1, \ldots, r.$$

Suppose that d_i is a divisor of m_i, $1 \leq d_i \leq m_i$, for $i = 1, \ldots, r$. For the divisor tuple $d = (d_1, \ldots, d_r)$ of $m = (m_1, \ldots, m_r)$ we define

$$S_\Gamma(d) \ = \ \{x = (x_1, \ldots, x_r) \in \Gamma : \gcd(x, m) = d\}.$$

Let $D = \{d^{(1)}, \ldots, d^{(k)}\}$ be a set of distinct divisor tuples of m and define

$$S_\Gamma(D) \ = \ \bigcup_{j=1}^{k} S_\Gamma(d^{(j)}).$$

Observe that the union is actually disjoint. The sets $S_\Gamma(D)$ shall be called *gcd-sets* of Γ. We define the class of *gcd-graphs* as the Cayley graphs $\mathrm{Cay}(\Gamma, S)$ over a finite abelian group Γ with symbol S a gcd-set of Γ. The most prominent members of this class are perhaps the unitary Cayley graphs $X_n = \mathrm{Cay}(Z_n, U_n)$, where $U_n = S_{Z_n}(1)$ is the multiplicative group of units of Z_n (cf. [16], [17], [22]).

The main goal of this paper is to show in Section 2 that every gcd-graph is isomorphic to a NEPS of complete graphs. Conversely, every NEPS of complete graphs is isomorphic to a gcd-graph over some abelian group. This relation is remarkable since it allows us to define gcd-graphs either algebraically (via Cayley graphs) or purely combinatorially (via NEPS). The characterization of gcd-graphs as NEPS of complete graphs reveals some new access to structural properties of gcd-graphs. As a first application, we show in Section

[3] that every gcd-graph has simply structured eigenspace bases for all of its eigenvalues. This means that for every eigenspace a basis can be found whose vectors only have entries from the set $\{0, 1, -1\}$. It is known that other graph classes exhibit a similar eigenspace structure, although not necessarily for all of their eigenspaces [9], [20], [25]. Finally, we present some open problems in Section 4.

2 Isomorphisms between NEPS of complete graphs and gcd-graphs

We are going to show in several steps that gcd-graphs and NEPS of complete graphs are the same.

Lemma 2.1. *Let* $\Gamma = Z_{m_1} \oplus \cdots \oplus Z_{m_r}$ *and* $d = (d_1, \ldots, d_r)$ *a tuple of positive divisors of* $m = (m_1, \ldots, m_r)$. *Define* $b = (b_i) \in \{0, 1\}^r$ *by*

$$b_i = \begin{cases} 1 & \text{if } d_i < m_i, \\ 0 & \text{if } d_i = m_i. \end{cases}$$

Then we have

$$\mathrm{Cay}(\Gamma, S_\Gamma(d)) = \mathrm{NEPS}(\mathrm{Cay}(Z_{m_1}, S_{Z_{m_1}}(d_1)), \ldots, \mathrm{Cay}(Z_{m_r}, S_{Z_{m_r}}(d_r)); \{b\}).$$

Proof. Both $\mathrm{Cay}(\Gamma, S_\Gamma(d))$ and the above NEPS have the same vertex set Γ. It remains to show that they have the same edge set.

Let x, $y \in \Gamma$ with $x = (x_1, \ldots, x_r)$, $y = (y_1, \ldots, y_r)$ and suppose that $x \neq y$. Now x and y are adjacent in $\mathrm{Cay}(\Gamma, S_\Gamma(d))$ if and only if $\gcd(x_i - y_i, m_i) = d_i$ for $i = 1, \ldots, r$. The latter condition means that in case $d_i < m_i$ the vertices x_i and y_i are adjacent in $G_i = \mathrm{Cay}(Z_{m_i}, S_{Z_{m_i}}(d_i))$, and in case $d_i = m_i$ we have $x_i = y_i$. But this is exactly the condition for adjacency of x and y in $\mathrm{NEPS}(G_1, \ldots, G_r; \{b\})$. $\square$

The following lemma allows us to break down the Cayley graphs that form the factors of the NEPS mentioned in Lemma 2.1. Each factor can be transformed into a gcd-graph over a product of cyclic groups of prime power order. Using Lemma 2.1 once again, we obtain a representation of the original graph as a NEPS of NEPS of gcd-graphs over cyclic groups of prime power order.

Lemma 2.2. *Let the integer* $m \geq 2$ *and a proper divisor* $d \geq 1$ *of* m *be given as products of powers of distinct primes,*

$$m = \prod_{i=1}^r m_i, \quad m_i = p_i^{\alpha_i}, \ \alpha_i > 0 \text{ for } i = 1, \ldots, r,$$

$$d = \prod_{i=1}^r d_i, \quad d_i = p_i^{\beta_i}, \ 0 \leq \beta_i \leq \alpha_i \text{ for } i = 1, \ldots, r.$$

If we set $\Gamma = Z_{m_1} \oplus \cdots \oplus Z_{m_r}$ *and* $\tilde{d} = (d_1, \ldots, d_r)$, *then there exists an isomorphism*

$$\mathrm{Cay}(Z_m, S_{Z_m}(d)) \simeq \mathrm{Cay}(\Gamma, S_\Gamma(\tilde{d})).$$

Proof. By the Chinese remainder theorem [23] we know that every $z \in Z_m$ is uniquely determined by the congruences

$$z \equiv z_i \mod m_i, \ z_i \in Z_{m_i} \text{ for } i = 1, \dots, r.$$

This gives rise to a bijection $Z_m \to \Gamma$ by virtue of $z \mapsto (z_1, \dots, z_r) =: \tilde{z}$. We show that this bijection induces an isomorphism between $\mathrm{Cay}(Z_m, S_{Z_m}(d))$ and $\mathrm{Cay}(\Gamma, S_\Gamma(\tilde{d}))$.

Let $x, y \in Z_m$, $x \neq y$. Note that $\tilde{x}$ and $\tilde{y}$ are vertices of $\mathrm{Cay}(\Gamma, S_\Gamma(\tilde{d}))$. The vertices x and y are adjacent in $\mathrm{Cay}(Z_m, S_{Z_m}(d))$ if and only if $\gcd(x-y, m) = d$. This is equivalent to $\gcd(x_i - y_i, m_i) = d_i$ for every $i = 1, \dots, r$. Now this means $\gcd(\tilde{x} - \tilde{y}, \tilde{m}) = \tilde{d}$, with $\tilde{m} = (m_1, \dots, m_r)$, which is the condition for adjacency of $\tilde{x}$ and $\tilde{y}$ in $\mathrm{Cay}(\Gamma, S_\Gamma(\tilde{d}))$. □

Next we shall prove a lemma that helps us consolidate the nesting of NEPS operations into a single NEPS operation. As a result, we then know that every single-divisor tuple gcd-graph is isomorphic to a NEPS of gcd-graphs over cyclic groups of prime power order.

Lemma 2.3. *Let*

$$H = \mathrm{NEPS}(H^{(1)}, \dots, H^{(t)}; B) \tag{2.1}$$

be a NEPS of graphs $H^{(j)}$ with respect to basis B such that each graph $H^{(j)}$ is itself a NEPS of graphs $G_i^{(j)}$ with respect to basis $B^{(j)}$,

$$H^{(j)} = \mathrm{NEPS}(G_1^{(j)}, \dots, G_{r_j}^{(j)}; B^{(j)}) \text{ for } j = 1, \dots, t. \tag{2.2}$$

Then there exists a set $B' \subseteq \{0,1\}^r$, $r = r_1 + \dots + r_t$, such that

$$H \simeq \mathrm{NEPS}(G_1^{(1)}, \dots, G_{r_1}^{(1)}, \dots, G_1^{(t)}, \dots, G_{r_t}^{(t)}; B'). \tag{2.3}$$

Proof. We show that in (2.1) the graph $H^{(1)}$ can be replaced by $G_1^{(1)}, \dots, G_{r_1}^{(1)}$. More precisely, we construct a set $\tilde{B}$ such that

$$H \simeq \mathrm{NEPS}(G_1^{(1)}, \dots, G_{r_1}^{(1)}, H^{(2)}, \dots, H^{(t)}; \tilde{B}), \ \tilde{B} \subseteq \{0,1\}^{r_1+t-1}. \tag{2.4}$$

An analogous procedure can be repeated for $H^{(2)}, \dots, H^{(t)}$ until we end up with the representation (2.3) of H.

In the original representation (2.1) every vertex x of the vertex set $V(H)$ has the form

$$x = (x^{(1)}, \dots, x^{(t)}), \ x^{(j)} \in V(H^{(j)}) \text{ for } j = 1, \dots, t. \tag{2.5}$$

By (2.2) each coordinate $x^{(j)}$ is itself an r_j-tuple, in particular

$$x^{(1)} = (x_1^{(1)}, \dots, x_{r_1}^{(1)}), \ x_i^{(1)} \in V(G_i^{(1)}) \text{ for } i = 1, \dots, r_1.$$

Expansion of $x^{(1)}$ in (2.5) yields

$$\begin{aligned}
&\tilde{x} = (x_1^{(1)}, \dots, x_{r_1}^{(1)}, x^{(2)}, \dots, x^{(t)}), \\
&x_i^{(1)} \in V(G_i^{(1)}) \text{ for } i = 1, \dots, r_1, \ x^{(j)} \in V(H^{(j)}) \text{ for } j = 2, \dots, t.
\end{aligned} \tag{2.6}$$

This is the representation of vertices for (2.4).

Now we adapt the basis set B to the new representation of vertices of H such that adjacencies remain unchanged. Let the distinct vertices x and y of H be given in their original representation according to (2.5) and in their new representation $\tilde{x}$, $\tilde{y}$ according to (2.6).

$$x = (x^{(1)}, \dots, x^{(t)}), \ y = (y^{(1)}, \dots, y^{(t)}),$$
$$\tilde{x} = (x_1^{(1)}, \dots, x_{r_1}^{(1)}, x^{(2)}, \dots, x^{(t)}), \ \tilde{y} = (y_1^{(1)}, \dots, y_{r_1}^{(1)}, y^{(2)}, \dots, y^{(t)}).$$

For each $b = (b_1, \dots, b_t) \in B$ we define a set $\tilde{B}(b) \subseteq \{0,1\}^{r_1+t-1}$ such that

$$x, \ y \text{ adjacent with respect to } b \ \Leftrightarrow \ \tilde{x}, \ \tilde{y} \text{ adjacent with respect to } \tilde{B}(b). \qquad (2.7)$$

Case 1: $b_1 = 0$.
For x and y to be adjacent with respect to b we must have $x^{(1)} = y^{(1)}$. If this is satisfied, then x and y are adjacent, if and only if $(x^{(2)}, \dots, x^{(t)})$ and $(y^{(2)}, \dots, y^{(t)})$ are adjacent with respect to $(b_2, \dots, b_t)$. We achieve (2.7) by setting $\tilde{b} = (0, \dots, 0, b_2, \dots, b_t)$ (first r_1 entries equal to zero) and $\tilde{B}(b) = \{\tilde{b}\}$.
Case 2: $b_1 = 1$.
Now x and y are adjacent with respect to b, if and only if $x^{(1)}$ and $y^{(1)}$ are adjacent in $H^{(1)}$ and $x^{(2)}, \dots, x^{(t)}$ and $y^{(2)}, \dots, y^{(t)}$ are equal or adjacent with respect to $b_2, \dots, b_t$, respectively. By (2.2) vertices $x^{(1)}$ and $y^{(1)}$ of $H^{(1)}$ are adjacent, if and only if they are adjacent with respect to some $b^{(1)} = (b_1^{(1)}, \dots, b_{r_1}^{(1)}) \in B^{(1)}$. In this case we satisfy (2.7) by setting

$$\tilde{B}(b) = \{(b_1^{(1)}, \dots, b_{r_1}^{(1)}, b^{(2)}, \dots, b^{(t)}) : \ b^{(1)} \in B^{(1)}\}.$$

Finally, we collect the new basis tuples in $\tilde{B} = \cup\{\tilde{B}(b) : \ b \in B\}$ and thus achieve (2.4). $\qquad \square$

The next step towards our goal is to show that a single-divisor gcd-graph over a cyclic group of prime power order is actually isomorphic to a NEPS of complete graphs.

We denote the complete graph on n vertices by K_n. For our purposes, we assume that the vertex set of K_n is $Z_n = \{0, 1, \dots, n-1\}$.

Lemma 2.4. *Let $m = p^\alpha$ be a prime power, $d = p^\beta$ a divisor of m, $0 \le \beta \le \alpha$. Then the gcd-graph over Z_m with respect to d is isomorphic to a NEPS of α copies of the complete graph K_p, i.e.*

$$\mathrm{Cay}(Z_m, S_{Z_m}(d)) \simeq \mathrm{NEPS}(K_p, \dots, K_p; B) \text{ for some } B \subseteq \{0,1\}^\alpha.$$

Proof. In case $\beta = \alpha$ we have $\mathrm{Cay}(Z_m, S_{Z_m}(m)) \simeq \mathrm{NEPS}(K_p, \dots, K_p; \{(0, \dots, 0)\})$. So we may now assume $\beta < \alpha$.

Let us denote $G = \mathrm{Cay}(Z_m, S_{Z_m}(d))$ and $H = \mathrm{NEPS}(K_p, \dots, K_p; B)$ (where the basis B is not yet fixed). For every $x \in Z_m$ let $(x_0, \dots, x_{\alpha-1})$ be defined by the p-adic representation of x,

$$x = \sum_{i=0}^{\alpha-1} x_i p^i, \ 0 \le x_i < p \text{ for } i = 0, \dots, \alpha - 1.$$

We shall assume that the vertex set of K_p is Z_p. Define the bijection $\varphi : Z_m \to Z_p \oplus \cdots \oplus Z_p = Z_p^\alpha$ by $\varphi(x) = (x_0, \ldots, x_{\alpha-1})$. We now construct a basis set $B \subseteq \{0,1\}^\alpha$ such that φ induces an isomorphism between G and H. Observe that for every $z \in Z_m$,

$$\gcd(z, m) = d \Leftrightarrow z_i = 0 \text{ for every } i < \beta \text{ and } z_\beta \neq 0.$$

This leads to the definition of B as follows:

$$B = \{(b_0, \ldots, b_{\alpha-1}) \in \{0,1\}^\alpha : b_i = 0 \text{ for every } i < \beta, \ b_\beta = 1\}.$$

Let $x, y \in Z_m$, $x \neq y$, $\varphi(x) = (x_0, \ldots, x_{\alpha-1})$, $\varphi(y) = (y_0, \ldots, y_{\alpha-1})$. Now x and y are adjacent in G if and only if $\gcd(x - y, m) = d$, which means $x_i - y_i = 0$ for every $i < \beta$ and $x_\beta - y_\beta \neq 0$. Thanks to our choice of B, this is exactly the condition for $\varphi(x)$ and $\varphi(y)$ being adjacent in H. □

Theorem 2.5. *Let G be an arbitrary gcd-graph, $G = \mathrm{Cay}(\Gamma, S_\Gamma(D))$, $\Gamma = Z_{m_1} \oplus \cdots \oplus Z_{m_r}$, $D = \{d^{(1)}, \ldots, d^{(k)}\}$ a set of divisor tuples of $m = (m_1, \ldots, m_r)$. If $n = p_1 \cdots p_t$ is the prime factorization of $n = m_1 \cdots m_r$, then*

$$G \simeq \mathrm{NEPS}(K_{p_1}, \ldots, K_{p_t}; B) = H \text{ for some } B \subseteq \{0,1\}^t.$$

Proof. Each divisor tuple in D gives rise to a graph $G^{(j)} = \mathrm{Cay}(\Gamma, S_\Gamma(d^{(j)}))$, $j = 1, \ldots, k$. By application of the preceding lemmas of this section we know that

$$G^{(j)} \simeq \mathrm{NEPS}(K_{p_1}, \ldots, K_{p_t}; B^{(j)}) = H^{(j)} \text{ for some } B^{(j)} \subseteq \{0,1\}^t.$$

The graphs $G^{(j)}$ constitute an edge disjoint decomposition of G. Now, for every divisor tuple $d^{(1)}, \ldots, d^{(k)} \in D$, we perform the decomposition process outlined by the lemmas in exactly the same way, in the sense that the vertex numberings of the resulting graphs $H^{(j)}$ are coherent. Then the graphs $H^{(j)}$ also constitute an edge disjoint decomposition of G:

$$E(G) = \bigcup_{j=1}^{k} E(G^{(j)}), \quad E(H) = \bigcup_{j=1}^{k} E(H^{(j)})$$

The binary sets $B^{(j)}$, $1 \leq j \leq k$, are also pairwise disjoint. The disjoint union of the edge sets $E(H^{(j)})$, $1 \leq j \leq k$, is generated in the NEPS of $K_{p_1}, \ldots, K_{p_t}$ by

$$B = \bigcup_{j=1}^{k} B^{(j)}.$$

With this choice of B the isomorphisms between the subgraphs $G^{(j)}$ and $H^{(j)}$, $1 \leq j \leq k$, extend to an isomorphism between G and H. □

Theorem 2.6. *Let G be a NEPS of complete graphs, $G = \mathrm{NEPS}(K_{m_1}, \ldots, K_{m_r}; B)$. Then G is isomorphic to a gcd-graph over $\Gamma = Z_{m_1} \oplus \cdots \oplus Z_{m_r}$.*

Proof. The vertex set of G can be represented by $\Gamma = Z_{m_1} \oplus \cdots \oplus Z_{m_r}$. Edges of G are generated by the binary r-tuples $b = (b_i)$ of the basis set B. Vertices $x = (x_1, \ldots, x_r) \neq y = (y_1, \ldots, y_r)$ are adjacent in G with respect to b, if $x_i = y_i$, whenever $b_i = 0$, and

$x_i \neq y_i$, whenever $b_i = 1$. Let the set $D(b)$ consist of all positive divisor tuples $d = (d_1, \ldots, d_r)$ of $m = (m_1, \ldots, m_r)$ such that $d_i = m_i$, whenever $b_i = 0$, and d_i a proper divisor of m_i, whenever $b_i = 1$. Then x and y are adjacent with respect to b, if and only if $\gcd(x - y, m) \in D(b)$. If we define $D = \cup\{D(b) : b \in B\}$, then the gcd-graph $\text{Cay}(\Gamma, S_\Gamma(D))$ is isomorphic to G.   □

Theorems 2.5 and 2.6 imply the following corollary.

Corollary 2.7. *Let $n = p_1 \cdots p_t$ be the prime factorization of the integer $n \geq 2$. Every gcd-graph with n vertices is isomorphic to a gcd-graph over $\Gamma = Z_{p_1} \oplus \cdots \oplus Z_{p_t}$.*

We conclude this section with some examples.

Example 2.8. We generalize the definition of a Hamming graph given in [15]. The Hamming graph $G = \text{Ham}(m_1, \ldots, m_r; D)$ has vertex set $V(G) = Z_{m_1} \oplus \ldots \oplus Z_{m_r}$. Distinct vertices are adjacent in G, if their Hamming distance is in D. It can be easily shown that G is a NEPS of the complete graphs $K_{m_1}, \ldots, K_{m_r}$.

Example 2.9. Sudoku graphs arise from the popular game of Sudoku. The Sudoku graph $\text{Sud}(n)$ models the number restrictions imposed when filling out an $n^2 \otimes n^2$ Sudoku puzzle. Each vertex represents a cell of the Sudoku puzzle. Two vertices are adjacent if the two corresponding cells are required to contain different numbers (which is the case when they lie in the same row, column or block of the puzzle). It has been shown that Sudoku graphs are NEPS of complete graphs [25].

Example 2.10. This is an example that demonstrates the application of Theorem 2.5. Let $\Gamma = Z_4 \oplus Z_{18}$ and $D = \{(1, 6), (4, 2), (2, 9)\}$. We want to represent the graph $\text{Cay}(\Gamma, S_\Gamma(D))$ as a NEPS of complete graphs. To start with, let us consider the graph $\text{Cay}(\Gamma, S_\Gamma((1, 6)))$. Application of Lemma 2.1, Lemma 2.2, once again Lemma 2.1, then Lemma 2.3, Lemma 2.4, and finally once again Lemma 2.3 gives us:

$$\text{Cay}(Z_4 \oplus Z_{18}, S((1, 6)))$$
$$\simeq \text{NEPS}(\text{Cay}(Z_4, S(1)), \text{Cay}(Z_{18}, S(6)); \{(1, 1)\})$$
$$\simeq \text{NEPS}(\text{Cay}(Z_4, S(1)), \text{Cay}(Z_2 \oplus Z_9, S((2, 3))); \{(1, 1)\})$$
$$\simeq \text{NEPS}(\text{Cay}(Z_4, S(1)), \text{NEPS}(\text{Cay}(Z_2, S(2)), \text{Cay}(Z_9, S(3)); \{(0, 1)\}); \{(1, 1)\})$$
$$\simeq \text{NEPS}(\text{Cay}(Z_4, S(1)), \text{Cay}(Z_2, S(2)), \text{Cay}(Z_9, S(3)); \{(1, 0, 1)\})$$
$$\simeq \text{NEPS}(\text{NEPS}(K_2, K_2; \{(1, 0), (1, 1)\}), \text{NEPS}(K_2; \{(0)\}),$$
$$\qquad \text{NEPS}(K_3, K_3; \{(0, 1)\}); \{(1, 0, 1)\})$$
$$\simeq \text{NEPS}(K_2, K_2, K_2, K_3, K_3; \{(1, 0, 0, 0, 1), (1, 1, 0, 0, 1)\}).$$

Note that for the sake of simplicity we have dropped the subscripts of the symbol sets since the respective groups are clear from the context. Regarding the application of Lemma 2.3 note that, trivially, $G \simeq \text{NEPS}(G; \{(1)\})$.

$$\text{Cay}(Z_4 \oplus Z_{18}, S((4, 2))) \quad \simeq \quad \text{NEPS}(K_2, K_2, K_2, K_3, K_3;$$
$$\{(0, 0, 0, 1, 0), (0, 0, 0, 1, 1)\}),$$
$$\text{Cay}(Z_4 \oplus Z_{18}, S((2, 9))) \quad \simeq \quad \text{NEPS}(K_2, K_2, K_2, K_3, K_3; \{(0, 1, 1, 0, 0)\}).$$

The graph $\mathrm{Cay}(\Gamma, S_\Gamma(D))$ is the disjoint union of the graphs $\mathrm{Cay}(\Gamma, S_\Gamma(d))$ with $d \in D$ which we have considered above, so we arrive at:

$$
\begin{aligned}
\mathrm{Cay}(\Gamma, S_\Gamma(D)) \quad \simeq \quad & \mathrm{NEPS}(K_2, K_2, K_2, K_3, K_3; \\
& \{(1,0,0,0,1), (1,1,0,0,1), (0,0,0,1,0), \\
& (0,0,0,1,1), (0,1,1,0,0)\}).
\end{aligned}
$$

3 Eigenspace bases of gcd-graphs

The *eigenvalues* and *eigenspaces* of an undirected graph G are the eigenvalues and eigenspaces, respectively, of any adjacency matrix of G. The multiset of all eigenvalues of a graph is called its *spectrum*. According to HARARY and SCHWENK [14], a graph G is defined to be *integral* if all of its eigenvalues are integers. Integral graphs have been a focus of research for some time; see [4] for a survey.

In particular, many notable results on integrality of Cayley graphs have been obtained. Integral cubic and quartic Cayley graphs on abelian groups have been characterized in [1] and [2], respectively. Circulant graphs are the Cayley graphs over Z_n, $n \geq 1$. So [26] showed that the integral circulant graphs with n vertices are exactly the gcd-graphs over Z_n. This result was extended in [18] to groups of the form $Z_2 \oplus \ldots \oplus Z_2 \oplus Z_n, n \geq 2$. A complete characterization of integral Cayley graphs over abelian groups has recently been achieved by ALPERIN and PETERSON [3].

The eigenvalues of $G = \mathrm{NEPS}(G_1, \ldots, G_n; B)$ are certain sums of products of the eigenvalues of the G_i, cf. [10]:

Theorem 3.1. *Let $G_1, \ldots, G_n$ be graphs with $n_1, \ldots, n_r$ vertices, respectively. Further, for $i = 1, \ldots, r$ let $\lambda_{i1}, \ldots, \lambda_{in_i}$ be the eigenvalues of G_i. Then, the spectrum of the graph $G = \mathrm{NEPS}(G_1, \ldots, G_n; B)$ with respect to basis B consist of all possible values*

$$
\mu_{i_1,\ldots,i_n} = \sum_{(\beta_1,\ldots,\beta_n) \in B} \lambda_{1i_1}^{\beta_1} \cdot \ldots \cdot \lambda_{ni_n}^{\beta_n}
$$

with $1 \leq i_k \leq n_k$ for $1 \leq k \leq n$.

A first consequence is that every NEPS of integral graphs is integral. It is easily checked that the complete graph K_n on $n \geq 2$ vertices has the simple eigenvalue $n - 1$ and the eigenvalue -1 with multiplicity $n - 1$. Hence NEPS of complete graphs are integral. Using Theorem 2.5, we now readily confirm the following result of [18]:

Proposition 3.2. *Every gcd-graph is integral.*

An interesting property of a graph is the ability to choose an eigenspace basis such that its vectors have entries from a very small set only. This may be possible only for certain or for all of its eigenvalues. For example, in [9] a construction is given for a basis of the eigenspace of eigenvalue -2 of a generalized line graph whose vectors contain only entries from $\{0, \pm 1, \pm 2\}$.

Imposing an even greater restriction on the admissible entries, we call an eigenspace basis *simply structured* if it consists of vectors containing only entries from $\{0, 1, -1\}$. Accordingly, an eigenspace is considered as simply structured if it has a simply structured basis. Observe that the eigenvalue belonging to a simply structured eigenspace is necessarily integral.

For a trivial example of a simply structured eigenspace basis, consider a connected r-regular graph. Here the all ones vector constitutes a basis of the eigenspace corresponding to the eigenvalue r. Moreover, for several graph classes, the eigenspaces corresponding to the eigenvalues 0 or -1 are simply structured, cf. [5],[20],[24].

It is somewhat remarkable if *all* of the eigenspaces of a graph are simply structured. In [25] is has been shown that Sudoku graphs are NEPS of complete graphs (recall Example 2.9) and admit simply structured eigenspace bases for all eigenvalues. As we shall see, this is true for any NEPS of complete graphs. For this we require the following theorem [11]:

Theorem 3.3. *If X and Y are graphs of orders n and m with linearly independent eigenvectors $x^{(1)}, \ldots, x^{(n)}$ and $y^{(1)}, \ldots, y^{(m)}$, respectively, then the nm tensor products*

$$x^{(i)} \otimes y^{(j)} \qquad (i = 1, \ldots, n; j = 1, \ldots, m)$$

form a set of linearly independent eigenvectors of any NEPS of X and Y. This fact readily extends to NEPS with more factors.

Corollary 3.4. *Any NEPS of graphs for which all eigenspaces are simply structured inherits that very property.*

Proof. Using the notation of the previous theorem, it is obvious that $x^{(i)} \otimes y^{(j)}$ has only entries from $\{0, 1, -1\}$ if the same holds for $x^{(i)}$ and $y^{(j)}$. This remains true for an arbitrary number of factors. $\square$

We can now prove the following result:

Proposition 3.5. *All eigenspaces of a gcd-graph are simply structured.*

Proof. Consider the complete graph K_n, $n \geq 2$. The all-ones vector $(1, 1, \ldots, 1)$ forms a basis of the eigenspace of eigenvalue $n - 1$. A basis of the eigenspace of eigenvalue -1 is formed by the vectors

$$
\begin{aligned}
x^{(1)} &= (-1, 1, 0, 0, \ldots, 0, 0), \\
x^{(2)} &= (-1, 0, 1, 0, \ldots, 0, 0), \\
&\vdots \\
x^{(n-1)} &= (-1, 0, 0, 0, \ldots, 0, 1).
\end{aligned}
$$

Thus the result follows from Corollary 3.4 and Theorem 2.5. $\square$

4   Open problems

Let us conclude with a number of open problems we think are worth investigating in the future:

1. Does every integral Cayley graph over a finite abelian group have a simply structured eigenspace basis for every eigenvalue?

2. Find a small class of integral graphs such that every integral Cayley graph over an abelian group is a NEPS of some graphs of this class.

3. It has been shown by So [26] that integral Cayley graphs over Z_{p^α}, p prime, are uniquely determined by their spectrum. Find more groups Γ such that cospectral integral Cayley graphs $\mathrm{Cay}(\Gamma, S_1)$, $\mathrm{Cay}(\Gamma, S_2)$ are necessarily isomorphic.

4. Try to determine or estimate the number $\varrho(n)$ of nonisomorphic gcd-graphs on n vertices. In [18] we showed that for a prime $p \geq 5$ we have $\varrho(p^2) = 6$. Observe that $\varrho(2^\alpha)$ is the number of nonisomorphic cubelike graphs on 2^α vertices, cf. [19].

5. Determine graph invariants for gcd-graphs such as connectivity, clique number, and chromatic number, cf. [6], [7].

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