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Chamfering operation on k-orbit maps*

María del Río Francos
Institute of Mathematics Physics and Mechanics,
University of Ljubljana, Slovenia,
Jadranska 19, Ljubljana 1000, Slovenia

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Abstract

A map, as a 2-cell embedding of a graph on a closed surface, is called a k-orbit map if the group of automorphisms (or symmetries) of the map partitions its set of flags into k orbits. Orbanić, Pellicer and Weiss studied the effects of operations as medial and truncation on k-orbit maps. In this paper we study the possible symmetry types of maps that result from other maps after applying the chamfering operation and we give the number of possible flag-orbits that has the chamfering map of a k-orbit map, even if we repeat this operation t times.

Keywords: Map, flag graph, symmetry type graph, chamfering operation.

Math. Subj. Class.: 52B15, 05C10, 57M15, 51M20, 52B10

1 Introduction

Topologically, a *map* $\mathcal{M}$ is a cellular embedding of a connected graph on a closed surface, with no boundary. While combinatorially, we define a map by an edge coloured cubic graph $\mathcal{G}_{\mathcal{M}}$, to which we refer as the *flag graph* of the map $\mathcal{M}$, as Lins and Vince (1982-83) define it in [18] and [25], respectively. The vertex set of $\mathcal{G}_{\mathcal{M}}$ is the set of flags of the map, and the edges define the connectivity between pairs of flags. Flags are a very important tool in describing combinatorially the structure of a map. They have been used not only for maps but also for hypermaps [9, 23], maps on the surfaces with boundary [1], abstract polytopes [22] or maniplexes [28].

A map $\mathcal{M}$ is called a k-orbit map if its group of automorphisms, or symmetries, partitions the set of flags into exactly k orbits. The most symmetric maps are well known as *regular* (or *reflexible*) maps, those for which its automorphism group acts transitively on their set of flags, i.e. they have exactly one flag-orbit. Other highly symmetric type of maps

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E-mail address: maria.delrio@fmf.uni-lj.si (María del Río Francos)

are the so called *chiral* maps, which flags are partitioned into two orbits in such way that any two adjacent flags belong to different orbits, [12, 13, 22]. In other words, the flag graph of a chiral map is a bipartite graph and each part is an orbit.

In [19] the question of possible symmetry types of maps resulting from other maps after applying various operations was raised. In particular, the medial and truncation operations on k-orbit maps were considered, for $k \leq 4$. In this paper we use the chamfering operation on k-orbit maps and determine, in terms of k, the number of possible flag-orbits that has the chamfering map of a k-orbit map. Table 1 depicts all possible cases.

The operation of *chamfering* an object is related to the idea of beveling ("to file down") the edges of a solid object. Given a map $\mathcal{M}$, the chamfering operation replaces the edges of $\mathcal{M}$ with hexagonal faces while keeping the faces of $\mathcal{M}$. This operation divides each flag of the map $\mathcal{M}$ into four different flags in the chamfering map. This operation is also used on the study of fullerenes (see [6]), for instance, which also leads to chemical applications as in [17]. Theorem 5.3 summarizes all the results presented in this paper.

To solve our problem, we define another graph to which we refer as the *symmetry type graph* of a map, this is, the quotient graph of the flag graph of a map under the action of its automorphism group. A strategy of how to generate symmetry type graphs is shown in [2]. Dress and Huson (1987) refer to such graphs as the Delaney-Dress symbol, [7]. Dress and Brinkmann (1996), as well as Balaban and Pisanski (2012), give applications to mathematical chemistry in [8] and [1], respectively.

In [20], Orbanić, Pellicer, Pisanski and Tucker (2011), show the 14 symmetry type graphs of edge-transitive maps. Later, in [4] and [5], the complete list of possible symmetry type graphs with at most 6 vertices is determined. In particular, in [4] are described some properties of the symmetry type graphs, and also, the advantages of symmetry type graphs were applied to completely solve the problem of symmetry types of medial maps. While, in [5] is given an extension of the results in [19] of all possible symmetry type graphs of a map and its truncated map might have, for up to 7 and 9 vertices.

The paper is organized in the following way. In Section 2, we formally define a map and its flag graph. In Section 3, we define the symmetry type graph of a k-orbit map and give some of its properties, also studied in [4]. In Section 4, we define the chamfering map and find some conditions for the original map as for its chamfering map in manner to determine whether the chamfering map of a k-orbit map has $4k$ flag-orbits or not. Finally, we conclude with Theorems 5.1 and 5.3 where we obtain the number of flag-orbits that the chamfering map has if we repeat this operation t times on the same map.

2 Maps

A *map* $\mathcal{M}$ is defined as a cellular embedding of a connected graph on a surface. Let $\mathcal{BS}$ be the barycentric subdivision of $\mathcal{M}$ and let Φ be a triangle in $\mathcal{BS}$. Label the vertices of Φ by Φ_0, Φ_1 and Φ_2 according to whether they represent a vertex, an edge or a face (mutually incident) in the map $\mathcal{M}$. Note that every triangle of $\mathcal{BS}$ is adjacent to other three triangles, see Figure 1. If two triangles Φ and Ψ of $\mathcal{BS}$ are adjacent by the edge with vertices Φ_j and Φ_k, with $j, k \in \{0, 1, 2\}$ and $j \neq k$, then we say that Φ and Ψ are i-adjacent, for $i \in \{0, 1, 2\}$ and $i \neq j, k$. In this case we shall denote Ψ by Φ^i (likewise Φ by Ψ^i) and note that for every triangle Φ and $i \in \{0, 1, 2\}$, $(\Phi^i)^i = \Phi$.

Combinatorially, a map can be seen as a set $\mathcal{F}(\mathcal{M})$ of *flags*, and the relation between pairs of elements in $\mathcal{F}(\mathcal{M})$ in the following way. To each flag in $\mathcal{F}(\mathcal{M})$, we assign a

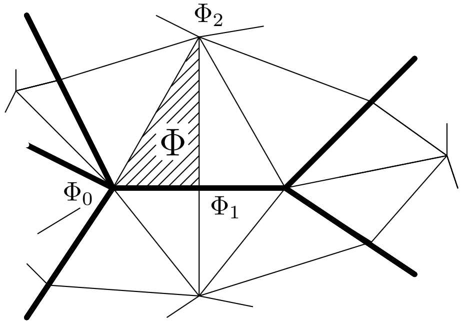

Figure 1: Barycentric subdivision of $\mathcal{M}$ and the flag $\Phi = (\Phi_0, \Phi_1, \Phi_2) \in \mathcal{F}(\mathcal{M})$.

triangle Φ in $\mathcal{BS}$ described by the ordered triple (Φ_0, Φ_1, Φ_2), represented by the vertices of Φ in $\mathcal{BS}$, and denote by Φ^i the corresponding i-adjacent flag of Φ in $\mathcal{M}$, with $i \in \{0, 1, 2\}$. Note that as it happens for the degenerated cases: as a map with a single vertex and edge but two faces, or a map with two vertices, one edge and a face; two adjacent flags can be represented by the same triple, however they are assigned to different triangles. We shall say that a map $\mathcal{M}$ is a non-degenerated map if the triples (Φ_0, Φ_1, Φ_2) are in one to one correspondence to the flags of $\mathcal{M}$.

Let s_0, s_1 and s_2 be the three permutations in the symmetric group $Sym(\mathcal{F}(\mathcal{M}))$ such that, for every flag Φ,

$$\Phi^{s_i} = \Phi \cdot s_i = \Phi^i,$$

with $i = 0, 1, 2$. Note that s_0, s_1, s_2 and $s_0 s_2$ are fixed point free involutions. Furthermore, by the connectivity of the map the action of the subgroup of $Sym(\mathcal{F}(\mathcal{M}))$ generated by these three distinguished involutions, denoted by $\mathrm{Mon}(\mathcal{M}) := \langle s_0, s_1, s_2 \rangle$, is transitive on the set of flags $\mathcal{F}(\mathcal{M})$. The group $\mathrm{Mon}(\mathcal{M})$ is known as the *monodromy (or connection) group* of the map $\mathcal{M}$, [10].

An *automorphism* of the map $\mathcal{M}$ is a bijection between vertices, edges and faces preserving their adjacency on the map. Thus, an automorphism of $\mathcal{M}$ induces a permutation of the flags in $\mathcal{F}(\mathcal{M})$ such that its action commutes with the elements of $\mathrm{Mon}(\mathcal{M})$. In other words, for every automorphism α of $\mathcal{M}$, every flag $\Phi \in \mathcal{F}(\mathcal{M})$ and every $i \in \{0, 1, 2\}$ it follows that

$$\Phi^{s_i}\alpha = (\Phi\alpha)^{s_i},$$

[14]. The connectivity of the map implies that the only automorphism that fixes a flag is the identity one. That is, the action of the automorphism group $\mathrm{Aut}(\mathcal{M})$ over the set $\mathcal{F}(\mathcal{M})$ is semi-regular, and hence divides $\mathcal{F}(\mathcal{M})$ into k orbits of the same size; in such case $\mathcal{M}$ is called a k-orbit map. If the action of $\mathrm{Aut}(\mathcal{M})$ over the set $\mathcal{F}(\mathcal{M})$ is transitive we say that the map is *regular* (or *reflexible*). The 2-orbit maps were widely studied and classified (in different contexts) in [9] and [15]. The most studied and understood type of 2-orbit maps is the *chiral* one, which has two orbits on its flags where any two adjacent flags belong to different orbits.

2.1 Flag graph

Given a map $\mathcal{M}$, we can construct a graph $\mathcal{G}_{\mathcal{M}}$ in the following way. The set of flags $\mathcal{F}(\mathcal{M})$ of the map $\mathcal{M}$ corresponds to the vertex set of the graph $\mathcal{G}_{\mathcal{M}}$, and two vertices Φ and Ψ in $V(\mathcal{G}_{\mathcal{M}})$ are adjacent by an edge of colour $i = 0, 1, 2$ if and only if the corresponding flags are i-adjacent in $\mathcal{M}$ (see Figure 2). We shall refer to the graph $\mathcal{G}_{\mathcal{M}}$ as the *flag graph* of the map $\mathcal{M}$.

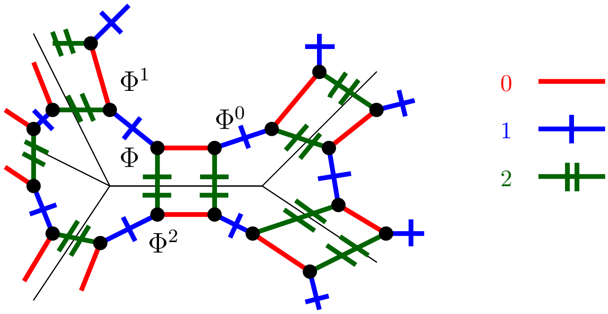

Figure 2: Local representation of the flag graph $\mathcal{G}_{\mathcal{M}}$ of a map $\mathcal{M}$.

Observe that the distinguished generators s_0, s_1 and s_2 of the monodromy group $\mathrm{Mon}(\mathcal{M})$ of the map $\mathcal{M}$ label the coloured edges of its flag graph $\mathcal{G}_{\mathcal{M}}$ in a natural way. Hence, for each flag $\Phi \in \mathcal{F}(\mathcal{M})$, a word $w = s_{i_0}s_{i_1}\cdots s_{i_n} \in \mathrm{Mon}(\mathcal{M})$ describes a path along the edges in $\mathcal{G}_{\mathcal{M}}$, coloured by $i_0, i_1, \ldots, i_n$, starting at the vertex Φ and ending at the vertex Φ^w, with

$$\Phi^w = (\Phi^{i_0}) \cdot s_{i_1} \cdots s_{i_n} =: \Phi^{i_0,i_1,\ldots,i_n}.$$

Since in general the action of $\mathrm{Mon}(\mathcal{M})$ is not semi-regular on $\mathcal{F}(\mathcal{M})$, this implies that one can have differently “coloured” walks in $\mathcal{G}_{\mathcal{M}}$ going from Φ to another flag Ψ that induce different words of $\mathrm{Mon}(\mathcal{M})$ that act on the flag Φ in the same way.

Note that in $\mathcal{G}_{\mathcal{M}}$ the edges of a given colour form a perfect matching (an independent set of edges containing all the vertices of the graph), and the union of two sets of edges of different colour is a subgraph whose components are even cycles; such subgraph is known as a *2-factor* of $\mathcal{G}_{\mathcal{M}}$. In particular, note that since $(s_0s_2)^2 = 1$ and s_0s_2 is fixed-point free, the cycles with edges of alternating colours 0 and 2 are all of length four.

Note that the connected components of the 2-factor of colours 0 and 2 in $\mathcal{G}_{\mathcal{M}}$, define the set of edges of $\mathcal{M}$. In other words, the edges of $\mathcal{M}$ can be identified with the orbits of $\mathcal{F}(\mathcal{M})$ under the action of the subgroup generated by the involutions s_0 and s_2; that is, $E(\mathcal{M}) = \{\Phi^{\langle s_0,s_2\rangle} \mid \Phi \in \mathcal{F}(\mathcal{M})\}$. Similarly, we find that the vertices and faces of $\mathcal{M}$ are identified with the respective orbits of the subgroups $\langle s_1, s_2\rangle$ and $\langle s_0, s_1\rangle$ on $\mathcal{F}(\mathcal{M})$. That is, $V(\mathcal{M}) = \{\Phi^{\langle s_1,s_2\rangle} \mid \Phi \in \mathcal{F}(\mathcal{M})\}$ and $F(\mathcal{M}) = \{\Phi^{\langle s_0,s_1\rangle} \mid \Phi \in \mathcal{F}(\mathcal{M})\}$. Thus, the group $\langle s_0, s_1, s_2\rangle$ acts transitively on the sets of vertices, edges and faces of $\mathcal{M}$.

The automorphism group $\mathrm{Aut}(\mathcal{M})$ of $\mathcal{M}$ induces a bijection between the flags of $\mathcal{M}$ preserving their adjacencies, and an edge-coloured preserving automorphism of the graph $\mathcal{G}_{\mathcal{M}}$ is a bijection between the vertices of $\mathcal{G}_{\mathcal{M}}$, preserving the adjacencies on the elements of $\mathcal{F}(\mathcal{M})$. Consequently, $\mathrm{Aut}(\mathcal{M})$ is isomorphic to the edge-coloured preserving automorphism group of the flag graph $\mathcal{G}_{\mathcal{M}}$.

Recall that $\mathrm{Aut}(\mathcal{M})$ partitions the set $\mathcal{F}(\mathcal{M})$ into k orbits of the same size. Let $\mathrm{Orb}(\mathcal{M}) := \{\mathcal{O}_\Phi | \Phi \in \mathcal{F}(\mathcal{M})\}$ be the set of all flag-orbits of $\mathcal{M}$. By the connectivity of $\mathcal{G}_\mathcal{M}$ we have the following lemma.

Lemma 2.1. *Let $\mathcal{O}_1, \mathcal{O}_2 \in \mathrm{Orb}(\mathcal{M})$, $\Phi \in \mathcal{F}(\mathcal{M})$, and $w \in \mathrm{Mon}(\mathcal{M})$. If $\Phi \in \mathcal{O}_1$ and $\Phi^w \in \mathcal{O}_2$, then $\Psi \in \mathcal{O}_1$ if and only if $\Psi^w \in \mathcal{O}_2$, for any $\Psi \in \mathcal{F}(\mathcal{M})$.*

3 Symmetry type graph of a map

We define a graph $T(\mathcal{M})$ (fairly, a pre-graph as in [24]), that we call the *symmetry type graph* of $\mathcal{M}$, as the quotient of the flag graph $\mathcal{G}_\mathcal{M}$, under the action of the automorphism group $\mathrm{Aut}(\mathcal{M})$ of the map. Hence, the vertices of $T(\mathcal{M})$ correspond to the elements in $\mathrm{Orb}(\mathcal{M})$, where two vertices $\mathcal{O}_\Phi, \mathcal{O}_\Psi \in \mathrm{Orb}(\mathcal{M})$ are adjacent by an edge of colour $i = 0, 1, 2$ if and only if there are flags $\Phi' \in \mathcal{O}_\Phi$ and $\Psi' \in \mathcal{O}_\Psi$ that are i-adjacent in $\mathcal{G}_\mathcal{M}$ (if the two i-adjacent flags Φ' and Ψ' belong to the same flag-orbit, then the edge of colour i is projected into a semi-edge in $T(\mathcal{M})$). By Lemma 2.1 the symmetry type graph $T(\mathcal{M})$ is a 3-valent (pre-)graph of chromatic index 3.

It can be seen that the action of $\mathrm{Mon}(\mathcal{M})$ on the set $\mathrm{Orb}(\mathcal{M})$ is defined as $\mathcal{O}_\Phi \cdot w = \mathcal{O}_{\Phi^w}$, for any $w \in \mathrm{Mon}(\mathcal{M})$ and $\Phi \in \mathcal{F}(\mathcal{M})$. This action is transitive, as is the action of $\mathrm{Mon}(\mathcal{M})$ on $\mathcal{F}(\mathcal{M})$. Since $\mathcal{G}_\mathcal{M}$ is a connected graph, then its corresponding symmetry type graph $T(\mathcal{M})$ is connected as well.

The symmetry type graph of regular maps is a graph with a single vertex and three semi-edges of colours 0, 1 and 2. Moreover, the symmetry type graph of chiral maps is a graph with two vertices and three parallel edges coloured by 0 ,1 and 2, connecting both vertices. In fact, chiral maps are commonly said to be of symmetry type 2.

The number of symmetry types of k-orbit maps is bounded by the number of connected cubic graphs with k vertices, properly three edge-coloured, where the colours 0 and 2 are as in the Figure 3. The reader can refer to [4] and [5] for all possible symmetry type graphs with at most 6 vertices.

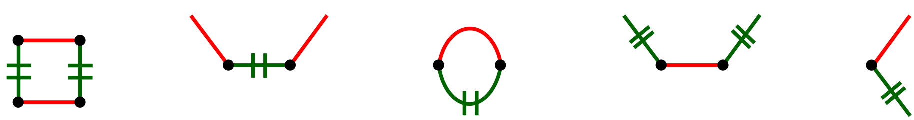

Figure 3: Possible quotients of 0-2 coloured 4-cycles of $\mathcal{G}_\mathcal{M}$.

4 Chamfering map

The chamfering map $\mathrm{Cham}(\mathcal{M})$ of any (non-degenerated) map $\mathcal{M}$ is produced, as its name says: by chamfering the edges in $\mathcal{M}$. More precisely, the edges of a map $\mathcal{M}$ are replaced by hexagonal faces, surrounding the faces of $\mathcal{M}$, in $\mathrm{Cham}(\mathcal{M})$ (see Figure 4). Hence, the set of faces of $\mathrm{Cham}(\mathcal{M})$ is in correspondence with the set of faces $F(\mathcal{M})$ and the set of edges $E(\mathcal{M})$ of $\mathcal{M}$. That is, the set of faces of $\mathrm{Cham}(\mathcal{M})$ is

$$F(\mathrm{Cham}(\mathcal{M})) = F(\mathcal{M}) \cup E(\mathcal{M}).$$

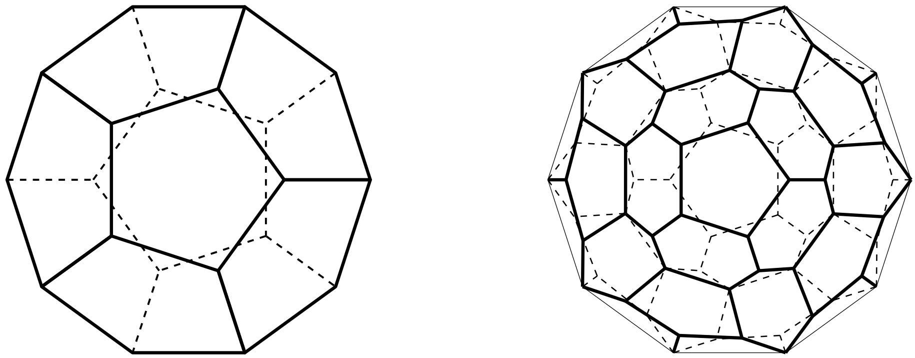

Figure 4: Dodecahedron (left) and the chamfering of the dodecahedron (right).

It is straightforward to see that the map $\mathrm{Cham}(\mathcal{M})$ has two types of edges: those between hexagonal faces and those between a face Φ_2 in $F(\mathcal{M})$ and its adjacent hexagonal faces (corresponding to the incident edges on the face Φ_2 in $\mathcal{M}$). This is, the set of edges of $\mathrm{Cham}(\mathcal{M})$ is

$$E(\mathrm{Cham}(\mathcal{M})) = \{\{\Phi_0, \{\Phi_0, \Phi_2\}\} | \Phi \in \mathcal{F}(\mathcal{M})\} \cup \{\{\Phi_1, \Phi_2\} | \Phi \in \mathcal{F}(\mathcal{M})\}.$$

In fact, $\mathrm{Cham}(\mathcal{M})$ has exactly $4|E(\mathcal{M})|$ edges. Finally, the set of vertices of $\mathcal{M}$ is a proper subset of the vertices of $\mathrm{Cham}(\mathcal{M})$, and the remaining $2|E(\mathcal{M})|$ vertices in $V(\mathrm{Cham}(\mathcal{M})) \setminus V(\mathcal{M})$ (each of these vertices are adjacent to exactly one vertex in $V(\mathcal{M})$), all have degree 3. Thus, the set of vertices of $\mathrm{Cham}(\mathcal{M})$ is

$$V(\mathrm{Cham}(\mathcal{M})) = V(\mathcal{M}) \cup \{\{\Phi_0, \Phi_2\} | \Phi \in \mathcal{F}(\mathcal{M})\}.$$

For an alternative definition of chamfering we refer the reader to [6].

Observe that the map on the left (dodecahedron) in Figure 4 is regular, while the map on the right is a 4-orbit map with symmetry type 4_{D_p} (Figure 5). There is a single orbit of

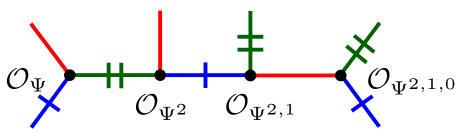

Figure 5: Symmetry type graph 4_{D_p}.

flags, $\mathcal{O}_\Psi$, on a pentagon and three different flags on a hexagon. Note that by chamfering a non-degenerated map $\mathcal{M}$, every flag $\Phi := (\Phi_0, \Phi_1, \Phi_2)$ in $\mathcal{F}(\mathcal{M})$ is divided into four flags of $\mathrm{Cham}(\mathcal{M})$, as is depicted in Figure 6, and the corresponding four flags to $\Phi \in \mathcal{F}(\mathcal{M})$ in $\mathrm{Cham}(\mathcal{M})$ can be written as

$$\begin{aligned}
(\Phi, 0) &:= (\Phi_0, \{\Phi_0, \{\Phi_0, \Phi_2\}\}, \Phi_1), \qquad & (\Phi, 1) &:= (\{\Phi_0, \Phi_2\}, \{\Phi_0, \{\Phi_0, \Phi_2\}\}, \Phi_1), \\
(\Phi, 2) &:= (\{\Phi_0, \Phi_2\}, \{\Phi_1, \Phi_2\}, \Phi_1), \qquad & (\Phi, 3) &:= (\{\Phi_0, \Phi_2\}, \{\Phi_1, \Phi_2\}, \Phi_2).
\end{aligned}$$

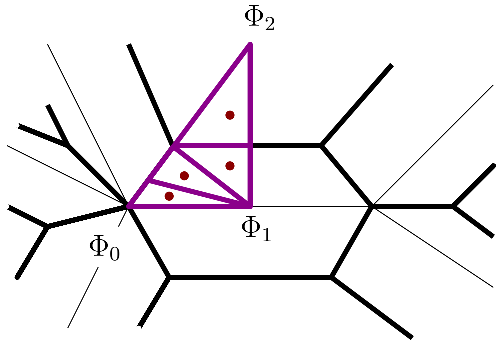

Figure 6: The four respective flags of $\mathcal{F}(Cham(\mathcal{M}))$ to the flag $\Phi = \{\Phi_0, \Phi_1, \Phi_2\} \in \mathcal{F}(\mathcal{M})$.

It is then straightforward to see that the adjacencies of the flags of $\mathrm{Cham}(\mathcal{M})$ are closely related to those of the flags of $\mathcal{M}$. In fact, we have that,

$$
\begin{array}{lll}
(\Phi,0)^0 = (\Phi,1), & (\Phi,0)^1 = (\Phi^{s_2},0), & (\Phi,0)^2 = (\Phi^{s_1},0), \\
(\Phi,1)^0 = (\Phi,0), & (\Phi,1)^1 = (\Phi,2), & (\Phi,1)^2 = (\Phi^{s_1},1), \\
(\Phi,2)^0 = (\Phi^{s_0},2), & (\Phi,2)^1 = (\Phi,1), & (\Phi,2)^2 = (\Phi,3), \\
(\Phi,3)^0 = (\Phi^{s_0},3), & (\Phi,3)^1 = (\Phi^{s_1},3), & (\Phi,3)^2 = (\Phi,2).
\end{array}
$$

Thus, we define the algorithm in Figure 7 to construct the flag graph of $\mathrm{Cham}(\mathcal{M})$ out of $\mathcal{G}_{\mathcal{M}}$.

Proposition 4.1. *The flag graph $\mathcal{G}_{\mathrm{Cham}(\mathcal{M})}$, of the chamfering map $\mathrm{Cham}(\mathcal{M})$ of any map $\mathcal{M}$, can be quotient into a graph as the symmetry type graph 4_{D_p}.*

Proof. Let $\mathcal{A}_i = \{(\Phi,i) | \Phi \in \mathcal{F}(\mathcal{M})\}$ be the subset of $\mathcal{F}(\mathrm{Cham}(\mathcal{M}))$ containing all flags of $\mathrm{Cham}(\mathcal{M})$ of the form (Φ,i), with $i = 0,1,2,3$. Then, $\mathcal{F}(\mathrm{Cham}(\mathcal{M})) = \mathcal{A}_0 \cup \mathcal{A}_1 \cup \mathcal{A}_2 \cup \mathcal{A}_3$ and $\mathcal{A}_i \cap \mathcal{A}_j = \emptyset$ whenever $i \neq j$. Hence, $(\mathcal{A}_0, \mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3)$ is a partition of the set of flags $\mathcal{F}(\mathrm{Cham}(\mathcal{M}))$. Based on Figure 7, it is straightforward to see that the quotient of $\mathcal{G}_{\mathrm{Cham}(\mathcal{M})}$ over such partition, is isomorphic to the symmetry type graph of a map with symmetry type 4_{D_p} (see Figure 5). □

Note that for any flags $\Upsilon \in \mathcal{A}_3$, $\Upsilon^2 \in \mathcal{A}_2$, $\Upsilon^{2,1} \in \mathcal{A}_1$ and $\Upsilon^{2,1,0} \in \mathcal{A}_0$, we can define a flag $\Phi_\Upsilon \in \mathcal{F}(\mathcal{M})$, by assembling these four flags in $\mathrm{Cham}(\mathcal{M})$. Observe that an automorphism $\bar{\alpha} \in \mathrm{Aut}(\mathrm{Cham}(\mathcal{M}))$ that sends a flag $\Upsilon' \in \mathcal{A}_i$ to another flag also contained in $\mathcal{A}_i$, with $i = 0,1,2,3$, is induced by an automorphism $\alpha \in \mathrm{Aut}(\mathcal{M})$ that sends $\Phi_{\Upsilon'}$ to the assembled flag $\Phi_{\Upsilon'\bar{\alpha}}$ in $\mathcal{M}$. Say this in other way, for each automorphism $\alpha \in \mathrm{Aut}(\mathcal{M})$, there is an automorphism $\bar{\alpha} \in \mathrm{Aut}(\mathrm{Cham}(\mathcal{M}))$ such that $(\Phi,i)\bar{\alpha} = (\Phi\alpha, i)$, with $\Phi \in \mathcal{F}(\mathcal{M})$ and $i = 0,1,2,3$. Then, it follows that

$$|\mathrm{Orb}(\mathrm{Cham}(\mathcal{M}))| \leq 4|\mathrm{Orb}(\mathcal{M})|.$$

Motivated by Proposition 4.3 of [19], we are interested in studying the number of possible flag-orbits of the chamfering map $\mathrm{Cham}(\mathcal{M})$.

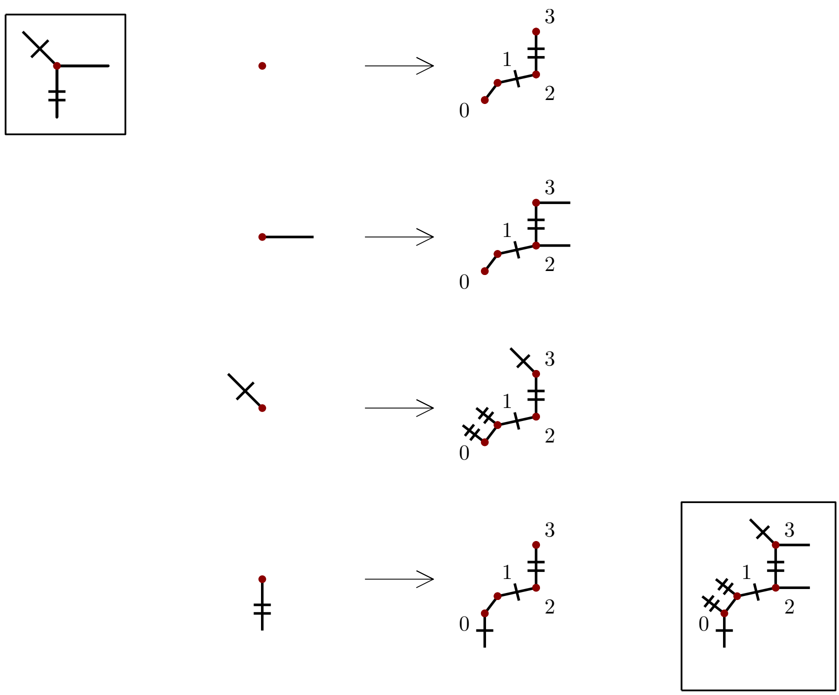

Figure 7: Local representation of a flag in $\mathcal{G}_{\mathcal{M}}$, in the left. The image under the chamfering operation, locally obtained, in the right.

Certainly, the chamfering map $\mathrm{Cham}(\mathcal{M})$, of a k-orbit map $\mathcal{M}$, has $4k$ orbits on the set of flags $\mathcal{F}(\mathrm{Cham}(\mathcal{M}))$, if for any $\Phi, \Psi \in \mathcal{F}(\mathcal{M})$ there is no flag of the form (Φ, i) in the same orbit as a flag of the form (Ψ, j), with $i, j \in \{0,1,2,3\}$ and $i \neq j$. In fact, if the chamfering map $\mathrm{Cham}(\mathcal{M})$ of a k-orbit map $\mathcal{M}$ is a $4k$-orbit map then, the algorithm presented in Figure 7 works as an algorithm on the vertices of $T(\mathcal{M})$ to obtain the symmetry type graph $T(\mathrm{Cham}(\mathcal{M}))$ with $4k$ vertices, of the chamfering map of $\mathcal{M}$.

We denote by r_0, r_1 and r_2 the distinguished generators of $\mathrm{Mon}(\mathrm{Cham}(\mathcal{M}))$. Observe that, in particular, $(\Phi^{s_0}, 3) = (\Phi, 3) \cdot r_0$, $(\Phi^{s_1}, 3) = (\Phi, 3) \cdot r_1$, and $(\Phi^{s_2}, 3) = (\Phi, 3) \cdot r_2r_1r_0r_1r_0r_1r_2$, for any $\Phi \in \mathcal{F}(\mathcal{M})$. This is, the action of the subgroup

$$D = \langle r_0, r_1, r_2r_1r_0r_1r_0r_1r_2 \rangle \leq \mathrm{Mon}(\mathrm{Cham}(\mathcal{M}))$$

over the subset of flags $\mathcal{F}(\mathcal{M}) \times \{3\}$ in $\mathrm{Cham}(\mathcal{M})$ is isomorphic to the action of the monodromy group $\mathrm{Mon}(\mathcal{M})$ over the set $\mathcal{F}(\mathcal{M})$, inducing the following action isomorphism.

$$(f, g) : (\mathcal{F}(\mathcal{M}), \langle s_0, s_1, s_2 \rangle) \to (\mathcal{F}(\mathcal{M}) \times \{3\}, \langle r_0, r_1, r_2r_1r_0r_1r_0r_1r_2 \rangle),$$

where $f : \Phi \mapsto (\Phi, 3)$ is a bijective function, and $g : (s_0, s_1, s_2) \mapsto (r_0, r_1, r_2r_1r_0r_1r_0r_1r_2)$ is a group isomorphism, as that defined in [14]. Then, the action of D is transitive on the set of flags $\mathcal{F}(\mathcal{M}) \times \{3\}$. Moreover, the action of D on $\mathcal{F}(\mathrm{Cham}(\mathcal{M}))$ fixes the set $\mathcal{A}_3$ and permutes the sets $\mathcal{A}_0$, $\mathcal{A}_1$ and $\mathcal{A}_2$. Further on, because

$$(\Phi, 3) \cdot r_2 = (\Phi, 2), \ (\Phi, 3) \cdot r_2r_1 = (\Phi, 1) \ \text{ and } \ (\Phi, 3) \cdot r_2r_1r_0 = (\Phi, 0),$$

conjugating D by the elements r_2, r_2r_1 and $r_2r_1r_0$ in $\mathrm{Mon}(\mathrm{Cham}(\mathcal{M}))$, we obtain three different subgroups of $\mathrm{Mon}(\mathrm{Cham}(\mathcal{M}))$, that act transitively on the set of flags $\mathcal{F}(\mathcal{M}) \times \{2\}$, $\mathcal{F}(\mathcal{M}) \times \{1\}$ and $\mathcal{F}(\mathcal{M}) \times \{0\}$, respectively. Therefore, we say that the conjugate subgroup $D^{a_i} \leq \mathrm{Mon}(\mathrm{Cham}(\mathcal{M}))$ fixes the set $\mathcal{A}_i$, for each $i = 0, 1, 2, 3$, and permutes the sets $\mathcal{A}_{j_1}$, $\mathcal{A}_{j_2}$ and $\mathcal{A}_{j_3}$, with $j_1, j_2, j_3 \in \{0,1,2,3\} \setminus \{i\}$, where $a_0 = r_2r_1r_0$, $a_1 = r_2r_1$, $a_2 = r_2$, and $a_3 = id$

With the following lemma we see that the chamfering map of a k-orbit map $\mathcal{M}$, not necessarily has $4k$ flag-orbits. By an *equivelar* map with Schläfli type $\{6, 3\}$ we mean a map that all its faces are 6-gons, and all its vertices have degree 3.

Lemma 4.2. *Let* $\mathrm{Cham}(\mathcal{M})$ *be the chamfering map of a map* $\mathcal{M}$*. If there is an automorphism* $\alpha \in \mathrm{Aut}(\mathrm{Cham}(\mathcal{M}))$ *such that* $(\Phi, i)\alpha = (\Psi, j)$ *for some* $\Phi, \Psi \in \mathcal{F}(\mathcal{M})$ *and* $i \neq j$*, with* $i, j \in \{0,1,2,3\}$*. Then,* $\mathcal{M}$ *is an equivelar map with Schläfli type* $\{6, 3\}$*.*

Proof. Consider the partition $(\mathcal{A}_0, \mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3)$ of the set $\mathcal{F}(\mathrm{Cham}(\mathcal{M}))$, where $\mathcal{A}_i = \{(\Phi, i) | \Phi \in \mathcal{F}(\mathcal{M})\}$, $i = 0, 1, 2, 3$, and recall that if we assemble the flags $\Upsilon \in \mathcal{A}_3$, $\Upsilon^2 \in \mathcal{A}_2$, $\Upsilon^{2,1} \in \mathcal{A}_1$ and $\Upsilon^{2,1,0} \in \mathcal{A}_0$, we can define a flag $\Phi_\Upsilon \in \mathcal{F}(\mathcal{M})$.

Suppose that there is an automorphism $\alpha \in \mathrm{Aut}(\mathrm{Cham}(\mathcal{M}))$ such that $(\Phi, i)\alpha = (\Psi, j)$ for some $\Phi, \Psi \in \mathcal{F}(\mathcal{M})$ and $i \neq j$, with $i, j \in \{0,1,2,3\}$. We shall verify the image, under α, of the assembled flags $(\Phi, 0), (\Phi, 1), (\Phi, 2), (\Phi, 3)$, corresponding to $\Phi \in \mathcal{F}(\mathcal{M})$, in terms of the adjacent flags of (Ψ, j). Note that $\Phi_0 \in (\Phi, 0)$ and $\Phi_2 \in (\Phi, 3)$, but they are neither in $(\Phi, 1)$ nor in $(\Phi, 2)$. Then, we have the following cases.

0) For $i = 0$.

- If $(\Phi,0)\alpha = (\Psi,1)$, then $\Phi_0\alpha = \{\Psi_0,\Psi_2\}$ and $\Phi_2\alpha = (\Psi^{2,1})_1$, since $(\Phi,0)\alpha = (\Psi,1) := (\{\Psi_0,\Psi_2\},\{\Psi_0\{\Psi_0,\Psi_2\}\},\Psi_1)$ and $(\Phi,3)\alpha = (\Phi,0)^{0,1,2}\alpha = ((\Phi,0)\alpha)^{0,1,2} = (\Psi,1)^{0,1,2} = (\Psi^{2,1},0) := (\Psi_0,\{\Psi_0,\{\Psi_0,(\Psi^2)_2\}\},(\Psi^{2,1})_1)$.
- If $(\Phi,0)\alpha = (\Psi,2)$, then $\Phi_0\alpha = \{\Psi_0,\Psi_2\}$ and $\Phi_2\alpha = (\Psi^{0,1})_1$, since $(\Phi,0)\alpha = (\Psi,2) := (\{\Psi_0,\Psi_2\},\{\Psi_1,\Psi_2\},\Psi_1)$ and $(\Phi,3)\alpha = (\Phi,0)^{0,1,2}\alpha = ((\Phi,0)\alpha)^{0,1,2} = (\Psi,2)^{0,1,2} = (\Psi^{0,1},1) := (\{(\Psi^0)_0,\Psi_2\},\{(\Psi^0)_0,\{(\Psi^0)_0,\Psi_2\}\},(\Psi^{0,1})_1)$.
- If $(\Phi,0)\alpha = (\Psi,3)$, then $\Phi_0\alpha = \{\Psi_0,\Psi_2\}$ and $\Phi_2\alpha = (\Psi^{0,1})_1$, since $(\Phi,0)\alpha = (\Psi,3) := (\{\Psi_0,\Psi_2\},\{\Psi_1,\Psi_2\},\Psi_2)$ and $(\Phi,3)\alpha = (\Phi,0)^{0,1,2}\alpha = ((\Phi,0)\alpha)^{0,1,2} = (\Psi,3)^{0,1,2} = (\Psi^{0,1},2) := (\{(\Psi^0)_0,\Psi_2\},\{(\Psi^{0,1})_1,\Psi_2\},(\Psi^{0,1})_1)$.

Similarly, we follow the same analysis in the next cases.

1) For $i = 1$.

- If $(\Phi,1)\alpha = (\Psi,0)$, then $\Phi_0\alpha = \{\Psi_0,\Psi_2\}$ and $\Phi_2\alpha = (\Psi^{2,1})_1$.
- If $(\Phi,1)\alpha = (\Psi,2)$, then $\Phi_0\alpha = \{(\Psi^0)_0,\Psi_2\}$ and $\Phi_2\alpha = (\Psi^1)_1$.
- If $(\Phi,1)\alpha = (\Psi,3)$, then $\Phi_0\alpha = \{(\Psi^0)_0,\Psi_2\}$ and $\Phi_2\alpha = (\Psi^1)_1$.

2) For $i = 2$.

- If $(\Phi,2)\alpha = (\Psi,0)$, then $\Phi_0\alpha = \{\Psi_0,(\Psi^2)_2\}$ and $\Phi_2\alpha = (\Psi^1)_1$.
- If $(\Phi,2)\alpha = (\Psi,1)$, then $\Phi_0\alpha = \{(\Psi^0)_0,\Psi_2\}$ and $\Phi_2\alpha = (\Psi^1)_1$.
- If $(\Phi,2)\alpha = (\Psi,3)$, then $\Phi_0\alpha = \{(\Psi^{1,0})_0,\Psi_2\}$ and $\Phi_2\alpha = \Psi_1$.

3) For $i = 3$.

- If $(\Phi,3)\alpha = (\Psi,0)$, then $\Phi_0\alpha = \{\Psi_0,(\Psi^{1,2})_2\}$ and $\Phi_2\alpha = \Psi_1$.
- If $(\Phi,3)\alpha = (\Psi,1)$, then $\Phi_0\alpha = \{(\Psi^{1,2})_0,(\Psi^{1,2})_2\}$ and $\Phi_2\alpha = \Psi_1$.
- If $(\Phi,3)\alpha = (\Psi,2)$, then $\Phi_0\alpha = \{(\Psi^{1,2})_0,\Psi_2\}$ and $\Phi_2\alpha = (\Psi^{0,1})_1$.

Observe from the cases above, that all the vertices $\{\Psi_0,\Psi_2\}$, $\{(\Psi^0)_0,\Psi_2\}$, $\{\Psi_0,(\Psi^2)_2\}$, $\{(\Psi^{1,0})_0,\Psi_2\}$, $\{\Psi_0,(\Psi^{1,2})_2\}$, $\{(\Psi^{1,2})_0,(\Psi^{1,2})_2\}$ and $\{(\Psi^{1,2})_0,\Psi_2\}$ are vertices with degree 3 in $\mathrm{Cham}(\mathcal{M})$. So as all the faces $(\Psi^{2,1})_1$, $(\Psi^{0,1})_1$, $(\Psi^1)_1$ and Ψ_1, correspond to 6-gons in $\mathrm{Cham}(\mathcal{M})$. Thus, the vertex Φ_0 has degree 3 and the face Φ_2 is a 6-gon in $\mathcal{M}$, with $\Phi \in \mathcal{F}(\mathcal{M})$. Furthermore, let $\Phi^w = \Delta \in \mathcal{F}(\mathcal{M})$, with $w \in \mathrm{Mon}(\mathcal{M})$. Then we have that

$$(\Delta,i)\alpha = (\Phi^w,i)\alpha = (\Phi,i)^{\bar{w}}\alpha = ((\Phi,i)\alpha)^{\bar{w}} = (\Psi,j)^{\bar{w}},$$

with $\bar{w} \in \mathrm{Mon}(\mathrm{Cham}(\mathcal{M}))$. Recall that the conjugated subgroup D^{a_i} of $\mathrm{Mon}(\mathrm{Cham}(\mathcal{M}))$ fixes the set $\mathcal{A}_i$ and permutes the sets $\mathcal{A}_{j_1}$, $\mathcal{A}_{j_2}$ and $\mathcal{A}_{j_3}$, with $j_1, j_2, j_3 \in \{0,1,2,3\} \setminus \{i\}$, where $a_0 = r_2r_1r_0$, $a_1 = r_2r_1$, $a_2 = r_2$, and $a_3 = id$. Since $(\Delta,i) = (\Phi,i)^{\bar{w}}$, it follows that $\bar{w} \in D^{a_i}$, and henceforth $(\Delta,i)\alpha = (\Psi,j)^{\bar{w}} \in \mathcal{A}_{j_k}$, with $j, j_k \in \{0,1,2,3\} \setminus \{i\}$.

Thus, we follow with a similar analysis as the previous one for $(\Delta,i)\alpha = (\Psi,j)^{\bar{w}}$, and we conclude that the vertex Δ_0 has degree 3 and the face Δ_2 is a 6-gon in $\mathcal{M}$. This latter was for arbitrary $\Delta \in \mathcal{F}(\mathcal{M})$ and $w \in \mathrm{Mon}(\mathcal{M})$. Therefore, we have that each vertex in $V(\mathcal{M})$ has degree 3 and every face $F(\mathcal{M})$ is a 6-gon. Consequently, the map $\mathcal{M}$ is an equivelar map with Schäfli type $\{6, 3\}$. □

By the Euler characteristic of a map, the surface of an equivelar map with Schläfli type $\{6, 3\}$ is either the torus or Klein bottle. In the following subsection we find the number of flag-orbits of the chamfering of an equivelar map of type $\{6, 3\}$.

4.1 Chamfering of equivelar maps of type $\{6, 3\}$

In [16] Hubard, Orbanić, Pellicer and Weiss studied the symmetry types of equivelar maps in the torus, described as $\{6, 3\}_{v_1,v_2}$, where v_1 and v_2 are two linearly independent vectors. In [27] Wilson shows that there are two kinds of maps of type $\{6, 3\}$ in the Klein bottle, and denotes them by $\{6, 3\}_{|m,n|}$ and $\{6, 3\}_{\backslash m,n\backslash}$ respectively, where the two glide reflections of these maps are on axes that are at distance a multiple of n and have length a multiple of m.

Regarding equivelar toroidal maps of type $\{6, 3\}$, from Theorem 8 in [16], we obtain the following proposition.

Proposition 4.3. *Equivelar toroids with Schläfli type* $\{6, 3\}$ *are either regular, chiral, or have symmetry type* 3^{02} *or* 6_{H_p}.

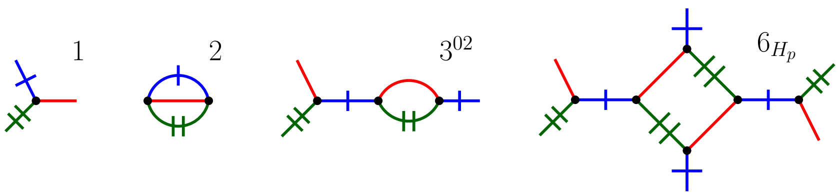

Figure 8: Symmetry type graphs of regular, chiral, 3^{02} and 6_{H_p} maps.

An equivelar toroidal map of type $\{6, 3\}$ is described as $\{6, 3\}_{v_1,v_2}$, where the linearly independent vectors v_1 and v_2 are a linear combination of the basis $\{\sqrt{3}e_1, \frac{\sqrt{3}}{2}e_1 + \frac{3}{2}e_2\}$, with the origin in the centre of an hexagon in the $\{6, 3\}$-tessellation of the plane.

In Figures 9–12 examples of equivelar toroids and their corresponding chamfering maps are depicted. Note that by chamfering a toroidal map $\mathcal{M} := \{6, 3\}_{v_1,v_2}$ we replace the edges of $\mathcal{M}$ by the corresponding hexagonal faces in $\mathrm{Cham}(\mathcal{M})$. Thus, the centres of adjacent faces of $\mathrm{Cham}(\mathcal{M})$ are at half distance than in the centres of adjacent hexagons of $\mathcal{M}$. This implies that the chamfering map $\mathrm{Cham}(\mathcal{M})$ is the equivelar toroidal map $\{6, 3\}_{2v_1,2v_2}$. Thus, we have the following lemma.

Lemma 4.4. *Let* $\mathcal{M}$ *be an equivelar toroidal map of type* $\{6, 3\}$. *Then the symmetry type graph* $T(\mathrm{Cham}(\mathcal{M}))$ *is isomorphic to* $T(\mathcal{M})$.

As what it concerns to equivelar maps of type $\{6, 3\}$ in the Klein bottle. Following [27], the two kinds of maps of type $\{6, 3\}$ in the Klein bottle are denoted by $\{6, 3\}_{|m,n|}$ and $\{6, 3\}_{\backslash m,n\backslash}$ respectively, where m and n are measured in
respect to the centres of the hexagons. The map in the Klein bottle, described as $\{6, 3\}_{|m,n|}$,

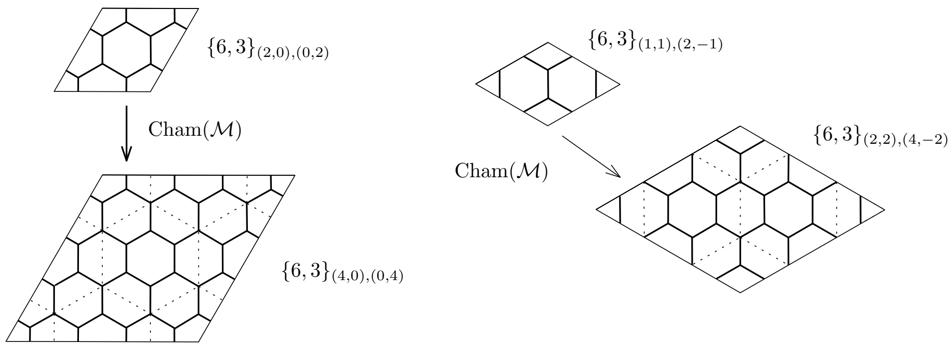

Figure 9: Chamfering of regular toroids of type $\{6, 3\}$.

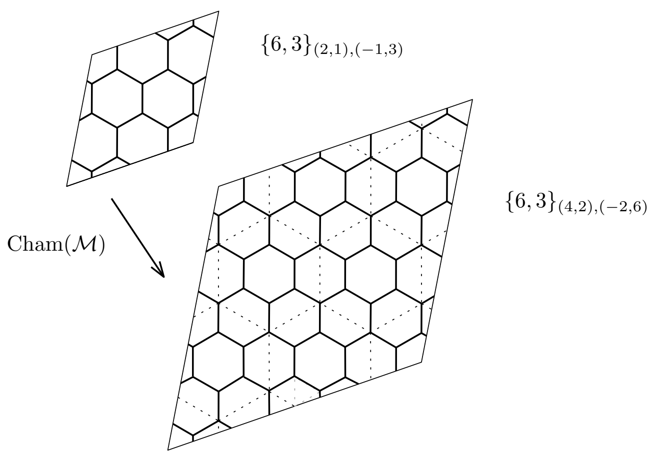

Figure 10: Chamfering of chiral toroids of type $\{6, 3\}$.

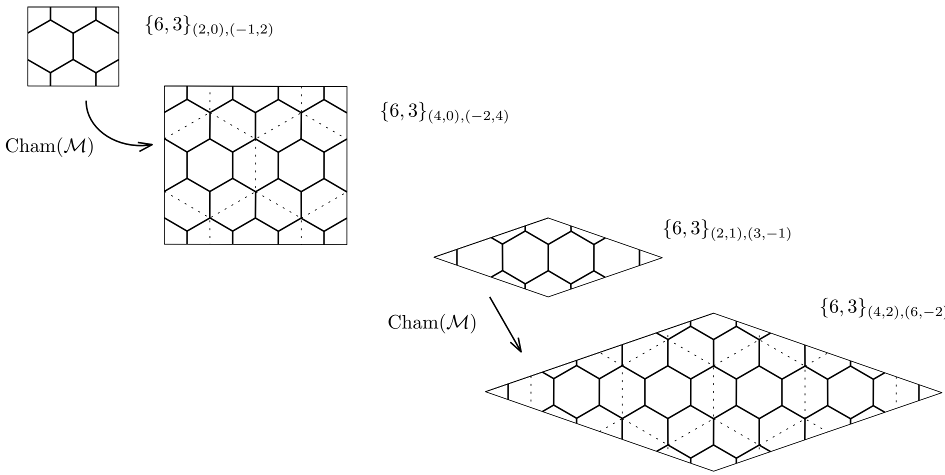

Figure 11: Chamfering of 3-orbit toroids of type $\{6, 3\}$.

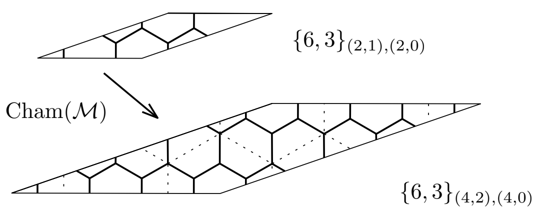

Figure 12: Chamfering of 3-orbit toroids of type $\{6, 3\}$.

results by using two glide reflections of length $\frac{m}{2}$ on axes of type (a) or (b), as in Figure 13, that are $n\frac{\sqrt{3}}{2}$ apart. And, the map in the Klein bottle, described as $\{6,3\}_{\backslash m,n\backslash}$, results by using two glide reflections of length $m\frac{\sqrt{3}}{2}$ on axes of type (c) or (d) as in Figure 13, that are $\frac{n}{2}$ apart. In both cases, the generating glide reflections are symmetries of the regular hexagonal tessellation of the plane. Since the glide reflection axes (a), (b), (c) and (d) are either

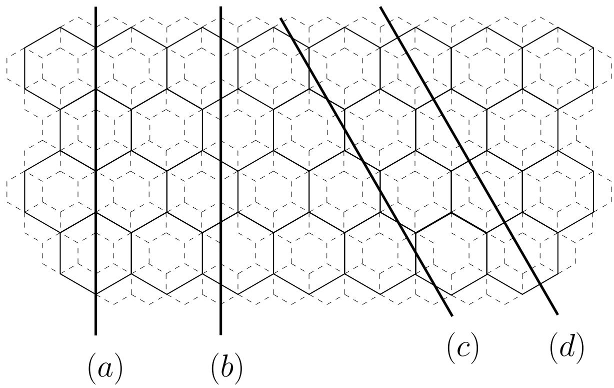

Figure 13: Possible glide reflection axes in $\{6,3\}$.

parallel to the edges of the hexagons or cross the edges in their midpoint, by chamfering an equivelar map $\mathcal{M}$ in the Klein bottle, of type either $\{6,3\}_{|m,n|}$ or $\{6,3\}_{\backslash m,n\backslash}$, the distance between both glide reflection axes and their length are the half than for those in $\mathcal{M}$. This is, in $\mathrm{Cham}(\mathcal{M})$, the values of m and n are the half as those for $\mathcal{M}$. Therefore, the chamfering map $\mathrm{Cham}(\mathcal{M})$ is an equivelar map in the Klein bottle described as $\{6,3\}_{|2m,2n|}$, or as $\{6,3\}_{\backslash 2m,2n\backslash}$, with glide reflection axes of type (a) or (d), respectively.

Hence, we obtain the following lemma.

Lemma 4.5. *If $\mathcal{M}$ is the toroidal map $\{6,3\}_{v_1,v_2}$ or a map in the Klein bottle of type either $\{6,3\}_{|m,n|}$, or $\{6,3\}_{\backslash m,n\backslash}$, then $\mathrm{Cham}(\mathcal{M})$ is a map on the same surface of type $\{6,3\}_{2v_1,2v_2}$, $\{6,3\}_{|2m,2n|}$, or $\{6,3\}_{\backslash 2m,2n\backslash}$, respectively.*

Following [27] we can see that maps $\{6,3\}_{|m,n|}$ and $\{6,3\}_{\backslash m,n\backslash}$ have $3mn$ edges and thereby $12mn$ flags. Moreover, the automorphism group of these maps have $4m$ elements. Thus, the maps $\{6,3\}_{|m,n|}$ and $\{6,3\}_{\backslash m,n\backslash}$ are $3n$-orbit maps. Hence, $\mathrm{Cham}(\{6,3\}_{|m,n|}) = \{6,3\}_{|2m,2n|}$ and $\mathrm{Cham}(\{6,3\}_{\backslash m,n\backslash}) = \{6,3\}_{\backslash 2m,2n\backslash}$ have $48mn$ flags and their respective automorphism group have $8m$ elements. Therefore, $\{6,3\}_{|2m,2n|}$ and $\{6,3\}_{\backslash 2m,2n\backslash}$ are $6n$-orbit maps. In Figures 14 and 15 are depicted examples of maps of type $\{6,3\}_{|m,1|}$ and $\{6,3\}_{\backslash m,1\backslash}$, with m even and odd, and its chamfering maps. Note that both maps of type $\{6,3\}_{|m,1|}$ and $\{6,3\}_{\backslash m,1\backslash}$ have symmetry type 3^{02}, while their chamfering maps $\{6,3\}_{|2m,2|}$ and $\{6,3\}_{\backslash 2m,2\backslash}$ have symmetry type 6_{H_p}.

Corollary 4.6. *If $\mathcal{M}$ is a k-orbit toroidal equivelar map of Schläfli type $\{6,3\}$, then $\mathrm{Cham}(\mathcal{M})$ is a k-orbit map, with $k = 1,2,3,6$. If $\mathcal{M}$ is a k-orbit equivelar map of Schläfli type $\{6,3\}$ in the Klein bottle, then $3|k$ and $\mathrm{Cham}(\mathcal{M})$ is a $2k$-orbit map.*

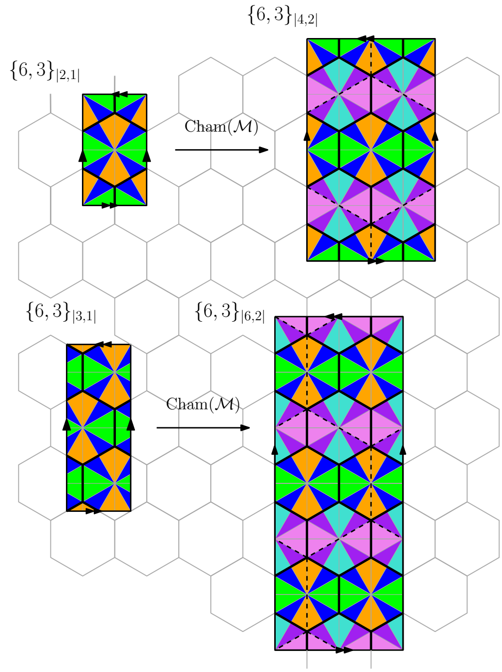

Figure 14: Chamfering of a 3-orbit map of type $\{6,3\}_{|m,1|}$ in the Klein bottle.

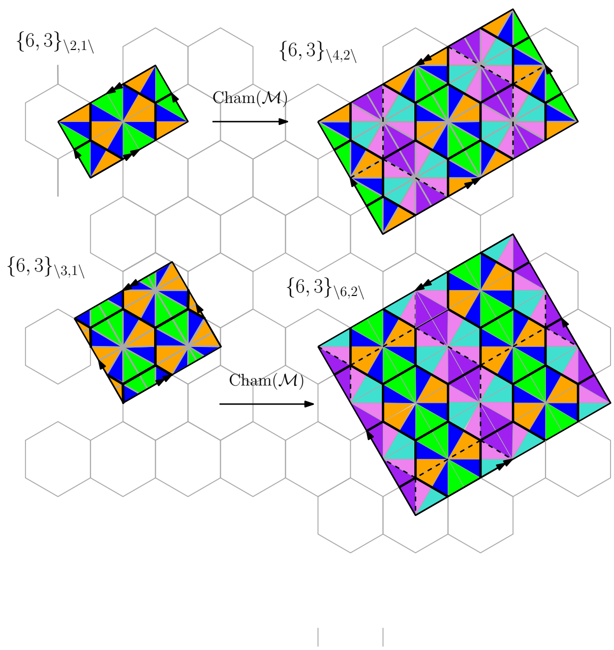

Figure 15: Chamfering of a 3-orbit map of type $\{6,3\}_{\backslash m,1\backslash}$ in the Klein bottle.

5 Conclusion

Putting our results together we see that lemma 4.2 implies that if $\mathcal{M}$ is a k-orbit map such that $\mathrm{Cham}(\mathcal{M})$ is not a $4k$-orbit map, then it is of type $\{6,3\}$. Hence, Corollary 4.6 implies the following theorem.

Theorem 5.1. *Let $\mathcal{M}$ be a k-orbit map. Then,* $\mathrm{Cham}(\mathcal{M})$ *has either k, $2k$ or $4k$ flag-orbits.*

We denote as $T(\mathrm{Cham}(T'))$ the chamfering symmetry type graph with $4k$ vertices that results from applying the algorithm in Figure 7 to the symmetry type graph T' of a k-orbit map. (See for instance Figures 16). As a consequence of the above discussion we have the following corollary.

Corollary 5.2. *Let $\mathcal{M}$ be a k-orbit map with symmetry type either 1, 2, 3^{02} or 6_{H_p}, and* $\mathrm{Cham}(\mathcal{M})$ *its chamfering map. Then the following holds.*

(1) If $\mathcal{M}$ is a regular map, then $\mathrm{Cham}(\mathcal{M})$ *is either regular of type $\{6,3\}$ (and hence toroidal), or has symmetry type 4_{D_p}.*

(2) If $\mathcal{M}$ is a chiral map, then $\mathrm{Cham}(\mathcal{M})$ *is either chiral of type $\{6,3\}$ (and hence toroidal), or has symmetry type graph* $T(\mathrm{Cham}(2))$ *with 8 vertices. (See Figure 16.)*

(3) If $\mathcal{M}$ has symmetry type 3^{02}, then $\mathrm{Cham}(\mathcal{M})$ *is either a toroidal map of type $\{6,3\}$ with symmetry type graph 3^{02}, or* $\mathrm{Cham}(\mathcal{M})$ *is a 6-orbit map in the Klein bottle and has symmetry type graph 6_{H_p}, or it has symmetry type graph* $T(\mathrm{Cham}(3^{02}))$ *with 12 vertices. (See Figure 16.)*

(4) If $\mathcal{M}$ has symmetry type 6_{H_p}, then $\mathrm{Cham}(\mathcal{M})$ *is either a toroidal map of type $\{6,3\}$ and has symmetry type graph 6_{H_p}, or* $\mathrm{Cham}(\mathcal{M})$ *is a 12-orbit map in the Klein bottle, or it has symmetry type graph* $T(\mathrm{Cham}(6_{H_p}))$ *with 24 vertices. (See Figure 16.)*

In [6] A. Deza, M. Deza and V. Grishukhin denote by $\mathrm{Cham}_t(\mathcal{M})$ the t-times chamfering of $\mathcal{M}$. It is straightforward to see that $\mathrm{Cham}_t(\mathcal{M})$ of a k-orbit equivelar map $\mathcal{M}$ on the torus is a k-orbit map described as $\{6,3\}_{2^t v_1, 2^t v_2}$. Similarly, $\mathrm{Cham}_t(\mathcal{M})$ of a k-orbit equivelar map $\mathcal{M}$ on the Klein bottle is a $2k$-orbit map denoted either $\{6,3\}_{|2^t m, 2^t n|}$ or $\{6,3\}_{\backslash 2^t m, 2^t n \backslash}$.

Finally, based on the results obtained in the previous section, we conclude with the following theorem.

Theorem 5.3. *Let $\mathcal{M}$ be a k-orbit map and* $\mathrm{Cham}_t(\mathcal{M})$ *the t-times chamfering map of $\mathcal{M}$ having s flag-orbits. Then at least one of the following holds.*

1. *$s = 4^t k, 2^t k$ or k.*
2. *If $s \neq 4^t k$, then $\chi(\mathcal{M}) = 0$ ($\mathcal{M}$ is on the torus or on the Klein bottle) and $\mathcal{M}$ is of type $\{6,3\}$.*
3. *If $\mathcal{M}$ is on the torus of type $\{6,3\}$ then $s = k$ and $k = 1,2,3,4$.*
4. *If $\mathcal{M}$ is on the Klein bottle of type $\{6,3\}$ then $s = 2^t k$ and $3|k$.*

Furthermore, joining the results obtained for the medial and truncation operations on k-orbit maps, in [19], that motivated the work done for this paper, with the results in obtained for the chamfering operation on k-orbit maps, we obtain the following table.

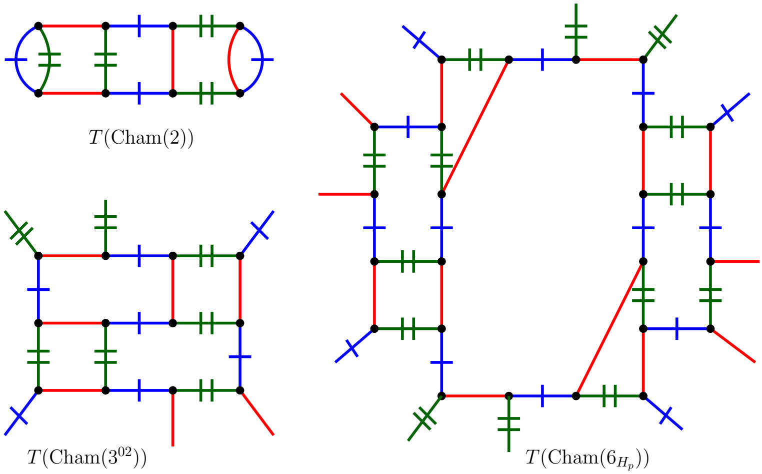

Figure 16: Symmetry type graphs of $\mathrm{Cham}(\mathcal{M})$, with $\mathcal{M}$ of type 2, 3^{02} and 6_{H_p}.

$\mathcal{M}'$	$\mathrm{Me}(\mathcal{M})$	$\mathrm{Tr}(\mathcal{M})$	$\mathrm{Cham}(\mathcal{M})$		
$\lvert\mathrm{Orb}(\mathcal{M}')\rvert$	$2k$ or k	$3k$, $\frac{3k}{2}$ or k	$4k$,	$2k$ $k\|3$	or k $k = 1, 2, 3, 6$

Table 1: Possible number of possible flag-orbits of a map $\mathcal{M}'$ with regard to $k = \lvert\mathrm{Orb}(\mathcal{M})\rvert$, where $\mathcal{M}'$ is the medial, truncation or chamfering map of $\mathcal{M}$.

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