

Two-arc-transitive two-valent digraphs of certain orders

Katja Berčič

IAM, University of Primorska, Muzejski trg 2, SI-6000 Koper, Slovenia

Primož Potočnik *

*Faculty of Mathematics and Physics, University of Ljubljana,
Jadranska 19, SI-1000 Ljubljana, Slovenia,*

*IAM, University of Primorska, Muzejski trg 2, SI-6000 Koper, Slovenia
IMFM, Jadranska 19, SI-1000 Ljubljana, Slovenia*

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Abstract

The topic of this paper is digraphs of in-valence and out-valence 2 that admit a 2-arc-transitive group of automorphisms. We classify such digraphs that satisfy certain additional conditions on their order. In particular, a classification of those with order kp or kp^2 where $k \leq 14$ and p is a prime can be deduced from the results of this paper.

Keywords: Graph, digraph, arc-transitive, order.

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1 Introduction

This paper is about finite connected arc-transitive digraphs of in- and out-valence 2 the order of which has a specific prime factorisation. We refer the reader to Section 2.1 for exact definitions of notions such as *digraph*, *arc-transitive*, *valence* etc. To simplify exposition, we tacitly assume throughout the paper (even where not stated explicitly) that all digraphs are finite and connected.

Studying arc-transitive graphs and digraphs of orders with a specific prime factorisation has a long history and has become increasingly popular in the last decade or two. For example, arc-transitive graphs and digraphs of order p or $2p$, where p is a prime, were classified in [3] and [4], respectively; later, using the classification of finite simple groups, all arc-transitive graphs and digraphs of order a product of two distinct primes were characterised

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E-mail addresses: katja.bercic@upr.si (Katja Berčič), primoz.potocnik@fmf.uni-lj.si (Primož Potočnik)

in [36], and independently in [26], and those that are 2-arc-transitive were determined in [23].

Once the prime factorisation of the order becomes more complex, results of this type become considerably more complicated (see [38] for an illustration of the difficulties that can arise when the order is a product of three distinct primes). However, when one fixes the valence (and perhaps imposes some further restrictions), further analysis becomes possible (see for example [5, 8, 10]).

Since every connected digraph of valence 1 is isomorphic to a directed cycle, valence 2 is the smallest interesting valence in the context of arc-transitive digraphs. In the literature, arc-transitive 2-valent digraphs often arise in disguise as undirected 4-valent graphs admitting a group of automorphisms acting transitively on the edges, vertices, but not on the arcs of the graph; such group actions are usually called $\frac{1}{2}$ -arc-transitive. Namely, if Γ is a G -arc-transitive 2-valent digraph, then its underlying (undirected) graph Γ' admits a $\frac{1}{2}$ -arc-transitive action of the group G ; and conversely, if the automorphism group of an undirected 4-valent graph Γ' contains a subgroup G acting $\frac{1}{2}$ -arc-transitively on Γ' , then there exists an orientation of the edges of Γ' that gives rise to a G -arc-transitive 2-valent digraph whose underlying graph is Γ' (in fact, there are precisely two such orientations giving rise to a pair of opposite digraphs). In this sense, the study of G -arc-transitive 2-valent digraphs is equivalent to the study of $(G, \frac{1}{2})$ -arc-transitive graphs of valence 4. There is a substantial literature about the latter class of graphs (see for example [6, 18, 19, 20, 21, 25, 39, 40]).

If Γ is an arc-transitive 2-valent digraph, then, for some positive integer s , the automorphism group $\text{Aut}(\Gamma)$ acts regularly on the set of all s -arcs of the digraph. If $s = 1$, then the automorphism group acts regularly on the arc-set, and if the order of the digraph has a simple prime factorisation, one is usually able to classify all possible automorphism groups and use this information to determine all digraphs upon which such groups can act. An instructive example of how this can be done (in the case of undirected 4-valent graphs) can be found in [10]. Here, we will avoid this case and restrict ourselves to the case $s \geq 2$; that is, we will assume that our digraphs are all 2-arc-transitive.

The two main results of the paper are Theorems 1.1 and 1.2, stated below and proved in Section 3. The digraphs $\overrightarrow{PX}(t, s)$ appearing in the statements are defined in Section 2.5.

Theorem 1.1. *Let p and q be distinct odd primes, and let a, b, c be integers satisfying $a \in \{0, 1, 2, 3\}$, $b, c \in \{0, 1, 2\}$, and $(b, c) \neq (2, 2)$. If Γ is a connected $(G, 2)$ -arc-transitive 2-valent digraph of order $2^a q^b p^c$ and G is non-solvable, then the order of Γ is at most 1224 and Γ is isomorphic to one of the sixty-seven digraphs in Table 1.*

Remark. Exact descriptions of the sixty-seven exceptional digraphs of Theorem 1.1 are available in [29] (for the digraphs of order up to 1000) and [1] (for digraphs of larger order). The digraphs are given there in a form readable by Magma [2].

Theorem 1.2. *Let Γ be a connected $(G, 2)$ -arc-transitive 2-valent digraph and suppose that G is solvable. Let n be the order of Γ , and suppose that one of the following holds:*

- (i) n is odd and cube-free;
- (ii) $n = 2^a m$, where $a \in \{1, 2, 3\}$ and m is an odd, square-free integer;
- (iii) $n = 2^a q^b p^2$, where $a \in \{1, 2, 3\}$, $b \in \{0, 1\}$ and p, q are distinct odd primes.

Then one of the following conclusions holds:

Order	Name	$ \text{Aut}_v $	$ S $	$\text{soc}(\text{Aut})$
$2 \cdot 3 \cdot 5$	ATD[30;6]	4	1	$\text{Alt}(5)$
$2 \cdot 3 \cdot 7$	ATD[42;3]	8	1	$\text{PSL}(2, 7)$
$2^2 \cdot 3 \cdot 5$	ATD[60;16]	4	1	$\text{Alt}(5) \times C_2$
$2^2 \cdot 3 \cdot 7$	ATD[84;20]	8	1	$\text{PSL}(2, 7) \times C_2$
$2^2 \cdot 3 \cdot 7$	ATD[84;23]	4	1	$\text{PSL}(2, 7)$
$2^2 \cdot 3 \cdot 7$	ATD[84;24]	4	1	$\text{PSL}(2, 7)$
$2 \cdot 3^2 \cdot 5$	ATD[90;12]	4	1	$\text{Alt}(5) \times C_3$
$2 \cdot 3^2 \cdot 5$	ATD[90;13]	16	3	$\text{Alt}(6)$
$2^3 \cdot 3 \cdot 5$	ATD[120;11]	4	1	$\text{Alt}(5) \times C_2$
$2^3 \cdot 3 \cdot 5$	ATD[120;54]	4	1	$\text{Alt}(5) \times C_2$
$2^3 \cdot 3 \cdot 5$	ATD[120;56]	4	1	$\text{Alt}(5) \times C_2$
$2 \cdot 3^2 \cdot 7$	ATD[126;15]	8	1	$\text{PSL}(2, 7) \times C_3$
$2 \cdot 3 \cdot 5^2$	ATD[150;16]	4	1	$\text{Alt}(5) \times C_5$
$2^3 \cdot 3 \cdot 7$	ATD[168;53]	8	1	$\text{PSL}(2, 7) \times C_2$
$2^3 \cdot 3 \cdot 7$	ATD[168;64]	4	1	$\text{PSL}(2, 7) \times C_2$
$2^3 \cdot 3 \cdot 7$	ATD[168;65]	4	1	$\text{PSL}(2, 7) \times C_2$
$2^3 \cdot 3 \cdot 7$	ATD[168;81]	4	1	$\text{PSL}(2, 7) \times C_2$
$2^3 \cdot 3 \cdot 7$	ATD[168;82]	4	1	$\text{PSL}(2, 7) \times C_2$
$2^2 \cdot 3^2 \cdot 5$	ATD[180;42]	4	1	$\text{Alt}(6)$
$2^2 \cdot 3^2 \cdot 5$	ATD[180;45]	4	1	$\text{Alt}(5) \times C_2 \times C_3$
$2^2 \cdot 3^2 \cdot 5$	ATD[180;57]	8	3	$\text{Alt}(6)$
$2^2 \cdot 3^2 \cdot 5$	ATD[180;58]	16	3	$\text{Alt}(6) \times C_2$
$2^2 \cdot 3^2 \cdot 7$	ATD[252;59]	8	1	$\text{PSL}(2, 7) \times C_2 \times C_3$
$2^2 \cdot 3^2 \cdot 7$	ATD[252;69]	4	1	$\text{PSL}(2, 7) \times C_3$
$2^2 \cdot 3^2 \cdot 7$	ATD[252;70]	4	1	$\text{PSL}(2, 7) \times C_3$
$2 \cdot 3 \cdot 7^2$	ATD[294;19]	8	1	$\text{PSL}(2, 7) \times C_7$
$2^2 \cdot 3 \cdot 5^2$	ATD[300;66]	4	1	$\text{Alt}(5) \times C_2 \times C_5$
$2 \cdot 3^2 \cdot 17$	ATD[306;11]	8	1	$\text{PSL}(2, 17)$
$2^3 \cdot 3^2 \cdot 5$	ATD[360;146]	4	1	$\text{Alt}(6) \times C_2$
$2^3 \cdot 3^2 \cdot 5$	ATD[360;148]	4	1	$\text{Alt}(6)$
$2^3 \cdot 3^2 \cdot 5$	ATD[360;150]	8	3	$\text{Alt}(6) \times C_2$
$2^3 \cdot 3^2 \cdot 5$	ATD[360;153]	8	3	$\text{Alt}(6) \times C_2$
$2^3 \cdot 3^2 \cdot 5$	ATD[360;154]	4	1	$\text{Alt}(6)$
$2^3 \cdot 3^2 \cdot 5$	ATD[360;158]	4	1	$\text{Alt}(5) \times C_2^2$
$2^3 \cdot 3^2 \cdot 5$	ATD[360;163]	4	1	$\text{Alt}(5) \times C_2 \times C_3$
$2^3 \cdot 3^2 \cdot 5$	ATD[360;172]	4	1	$\text{Alt}(5) \times C_2 \times C_3$
$2^3 \cdot 3^2 \cdot 5$	ATD[360;174]	4	1	$\text{Alt}(5) \times C_2 \times C_3$
$2^3 \cdot 3^2 \cdot 5$	ATD[360;201]	8	3	$\text{Alt}(6) \times C_2$
$2^3 \cdot 3^2 \cdot 5$	ATD[360;202]	16	3	$\text{Alt}(6) \times C_2$
$2^3 \cdot 3^2 \cdot 7$	ATD[504;162]	8	1	$\text{PSL}(2, 7) \times C_2 \times C_3$
$2^3 \cdot 3^2 \cdot 7$	ATD[504;180]	4	1	$\text{PSL}(2, 7) \times C_2 \times C_3$
$2^3 \cdot 3^2 \cdot 7$	ATD[504;182]	4	1	$\text{PSL}(2, 7) \times C_2 \times C_3$
$2^3 \cdot 3^2 \cdot 7$	ATD[504;232]	4	1	$\text{PSL}(2, 7) \times C_2 \times C_3$
$2^3 \cdot 3^2 \cdot 7$	ATD[504;233]	4	1	$\text{PSL}(2, 7) \times C_2 \times C_3$
$2^2 \cdot 3 \cdot 7^2$	ATD[588;87]	8	1	$\text{PSL}(2, 7) \times C_2 \times C_7$
$2^2 \cdot 3 \cdot 7^2$	ATD[588;90]	4	1	$\text{PSL}(2, 7) \times C_7$
$2^2 \cdot 3 \cdot 7^2$	ATD[588;91]	4	1	$\text{PSL}(2, 7) \times C_7$
$2^3 \cdot 3 \cdot 5^2$	ATD[600;199]	4	1	$\text{Alt}(5) \times C_2 \times C_5$
$2^3 \cdot 3 \cdot 5^2$	ATD[600;201]	4	1	$\text{Alt}(5) \times C_2 \times C_5$
$2^3 \cdot 3 \cdot 5^2$	ATD[600;204]	4	1	$\text{Alt}(5) \times C_2 \times C_5$
$2^2 \cdot 3^2 \cdot 17$	ATD[612;48]	4	1	$\text{PSL}(2, 17)$
$2^2 \cdot 3^2 \cdot 17$	ATD[612;49]	8	1	$\text{PSL}(2, 17)$
$2^3 \cdot 3 \cdot 7^2$	X_1	4	1	$\text{PSL}(2, 7) \times C_2 \times C_7$
$2^3 \cdot 3 \cdot 7^2$	X_2	4	1	$\text{PSL}(2, 7) \times C_2 \times C_7$
$2^3 \cdot 3 \cdot 7^2$	X_3	4	1	$\text{PSL}(2, 7) \times C_2 \times C_7$
$2^3 \cdot 3 \cdot 7^2$	X_4	4	1	$\text{PSL}(2, 7) \times C_2 \times C_7$
$2^3 \cdot 3 \cdot 7^2$	X_5	8	1	$\text{PSL}(2, 7) \times C_2 \times C_7$
$2^3 \cdot 3^2 \cdot 17$	X_6	4	1	$\text{PSL}(2, 17) \times C_2$
$2^3 \cdot 3^2 \cdot 17$	X_7	8	1	$\text{PSL}(2, 17) \times C_2$
$2^3 \cdot 3^2 \cdot 17$	X_8	4	1	$\text{PSL}(2, 17) \times C_2$
$2^3 \cdot 3^2 \cdot 17$	X_9	4	1	$\text{PSL}(2, 17) \times C_2$
$2^2 \cdot 3^2 \cdot 17$	X_{10}	4	1	$\text{PSL}(2, 17)$
$2^2 \cdot 3^2 \cdot 17$	X_{11}	4	1	$\text{PSL}(2, 17)$
$2^2 \cdot 3^2 \cdot 17$	X_{12}	4	1	$\text{PSL}(2, 17)$
$2^2 \cdot 3^2 \cdot 17$	X_{13}	4	1	$\text{PSL}(2, 17)$
$2^2 \cdot 3^2 \cdot 17$	X_{14}	4	1	$\text{PSL}(2, 17)$
$2^2 \cdot 3^2 \cdot 17$	X_{15}	4	1	$\text{PSL}(2, 17)$

Table 1: Exceptional digraphs for Theorem 1.1. The column “Name” refers to the digraph names as given in [28] (up to order 1000) or [1] (for orders greater than 1000). The number of non-solvable 2-arc-transitive subgroups of $\text{Aut}(\Gamma)$ (up to conjugacy) is given in the column called $|S|$.

- (a) $\Gamma \cong \overrightarrow{PX}(t, s)$ for some $t \geq 1$ and $s \geq 0$;
- (b) condition (iii) holds, G has a normal Sylow p -subgroup P , which is elementary abelian of order p^2 , and $\Gamma/P \cong \overrightarrow{PX}(t, s)$ for some $t \geq 1$ and $s \geq 0$.

Remark. Let us spend a few words on the seemingly unfinished case (b) of Theorem 1.2. The digraphs appearing in this case arise from regular covering projections onto the digraphs $\overrightarrow{PX}(t, s)$ of order $2^a q^b$ where the groups of covering transformations are elementary abelian of order p^2 , along which a 2-arc-transitive group of automorphisms of $\overrightarrow{PX}(t, s)$ lifts. The theory of lifting groups along elementary abelian covering projections was developed in [14] and illustrated in several papers (see for example [15, 31]). If desired, one could use this theory to determine all the resulting covering digraphs for fixed (a, q, b) . In particular, we could easily obtain a complete classification in the case of order kp or kp^2 for every $k \leq 14$ and prime p .

Recently, numerous papers have been written in which authors classified arc-transitive graphs and digraphs of fixed valence and orders with a simple prime factorisation (usually kp or kp^2 for a fixed small k and variable prime p). Unlike in many of the above mentioned papers, we have tried to prove our results in as general a form as our approach allowed. Slight improvements are certainly possible (for example, using the classification of finite simple groups whose order is divisible by four primes only [13], one could extend Theorem 1.1 to orders divisible by a third odd prime). However, it seems that major improvements would require new ideas.

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2 Preliminaries

2.1 On graphs and digraphs

Even though we are mainly interested in simple digraphs, it will be convenient in the proofs to allow digraphs to be non-simple. We therefore define a *digraph* Γ as a quadruple $(V, A, \text{head}, \text{tail})$ where V and A are finite non-empty sets and head and tail are functions mapping from A to V ; we call the sets V and A the *vertex-set* and the *arc-set* of Γ and denote them by $V(\Gamma)$ and $A(\Gamma)$, respectively. We then think of an arc *point* from its *tail* to its *head*. The cardinality of $V(\Gamma)$ is called the *order* of Γ .

Similarly, a *graph* Γ is determined by a vertex-set $V(\Gamma)$, edge-set $E(\Gamma)$ and a function $\text{end}: E(\Gamma) \rightarrow \{X \subseteq V(\Gamma) : |X| \in \{1, 2\}\}$, assigning a pair of *endvertices* to each edge of Γ . An edge e of a graph Γ is a *loop* provided that $|\text{end}(e)| = 1$, and two edges y and x are *parallel* if $\text{end}(x) = \text{end}(y)$. A graph Γ without loops and parallel edges is *simple* and is uniquely determined by $V(\Gamma)$ and the set $\{\text{end}(e) : e \in E(\Gamma)\}$.

If Γ is a digraph, then the *underlying graph* of Γ is the graph with vertex-set $V(\Gamma)$, edge-set $A(\Gamma)$ and the end-function defined by $\text{end}(x) = \{\text{tail}(x), \text{head}(x)\}$. A digraph is *simple* provided that its underlying graph is simple.

A sequence (x_1, \dots, x_s) of arcs of a digraph Γ is called an *s-arc* of Γ provided that $\text{head}(x_i) = \text{tail}(x_{i+1})$ for every $i \in \{1, \dots, s-1\}$. The set of all *s*-arcs of Γ is denoted by $A_s(\Gamma)$.

An automorphism of a digraph Γ is a permutation of $V(\Gamma) \cup A(\Gamma)$ that preserves $V(\Gamma)$ set-wise and commutes with the functions head and tail. If G is a subgroup of the automorphism group $\text{Aut}(\Gamma)$, then Γ is said to be G -arc-transitive (or (G, s) -arc-transitive) provided that G acts transitively on $A(\Gamma)$ (or $A_s(\Gamma)$, respectively). When $G = \text{Aut}(\Gamma)$, the symbol G can be omitted from this notation.

If v is the tail and u the head of some arc x , then we say that u is an *out-neighbour* of v and v an *in-neighbour* of u . For a vertex $v \in V(\Gamma)$, we let $\Gamma^+(v) = \{x \in A(\Gamma) : \text{tail}(x) = v\}$ and $\Gamma^-(v) = \{x \in A(\Gamma) : \text{head}(x) = v\}$, and call the sizes of these two sets the *out-valence* and the *in-valence* of v in Γ , respectively. (Note that when the digraph is not simple the out-valence does not necessarily equal the number of out-neighbours of v , and similarly for the in-valence). If for some integer k , the in-valence (out-valence) of every vertex equals k , then we say that the digraph has in-valence (out-valence, respectively) k . A digraph is called k -valent if it is of out-valence and in-valence k .

Observe that every arc-transitive digraph without vertices of out-valence 0 (in particular, every connected arc-transitive digraph) is vertex-transitive.

2.2 Non-simple arc-transitive 2-valent digraphs

In this section, we characterise arc-transitive 2-valent digraphs that are not simple. To formulate the characterisation (Lemma 2.1), we first need to introduce the digraphs $\vec{C}_n^{(2)}$ and \overleftarrow{C}_n for $n \geq 1$. Both digraphs arise from an undirected cycle with each edge doubled, and their vertex-sets and arc-sets can be taken to be \mathbb{Z}_n and $\mathbb{Z}_n \times \mathbb{Z}_2$, respectively. In $\vec{C}_n^{(2)}$ the functions head and tail are defined with $\text{tail}(i, \epsilon) = i$ and $\text{head}(i, \epsilon) = i + 1$ for every arc $(i, \epsilon) \in \mathbb{Z}_n \times \mathbb{Z}_2$. Similarly, in \overleftarrow{C}_n , the functions head and tail are defined with $\text{tail}(i, 0) = i$, $\text{head}(i, 0) = i + 1$, $\text{tail}(i, 1) = i + 1$, and $\text{head}(i, 1) = i$. Note that $\vec{C}_1^{(2)}$ and \overleftarrow{C}_1 are both isomorphic to a digraph with a single vertex and two directed loops attached to it, while $\vec{C}_2^{(2)}$ and \overleftarrow{C}_2 consist of two vertices and four arcs between them, two pointing in each of the two possible directions. The proof of the following lemma is straightforward and is left to the reader.

Lemma 2.1. *If Γ is a connected non-simple arc-transitive 2-valent digraph of order n , then $\Gamma \cong \vec{C}_n^{(2)}$ or $\Gamma \cong \overleftarrow{C}_n$, and if in addition Γ is 2-arc-transitive, then $\Gamma \cong \vec{C}_n^{(2)}$ for some $n \geq 2$.*

The following result will be needed in the proof of Theorem 1.2.

Lemma 2.2. *Let G be a subgroup of $\text{Aut}(\vec{C}_n^{(2)})$ acting transitively on the s -arcs but not on the $(s+1)$ -arcs of $\vec{C}_n^{(2)}$ and let v be a vertex of $\vec{C}_n^{(2)}$. Then G_v is an elementary abelian 2-group of order 2^s and is normal in G . If G_v has order 4 and contains a non-trivial central element of G , then n is even.*

Proof. Observe that every automorphism of $\vec{C}_n^{(2)}$ that fixes v fixes every vertex of $\vec{C}_n^{(2)}$, implying that G_v is the kernel of the action of G on the vertex-set of $\vec{C}_n^{(2)}$, and is therefore normal in G . Furthermore, G_v preserves set-wise each pair of arcs with the same tail (and thus the same head). In particular, G_v is an elementary abelian 2-group. Since G is transitive on the s -arcs but not on the $(s+1)$ -arcs, it is an easy exercise to show that G_v acts regularly on the s -arcs starting at v , and since there are 2^s of them, it follows that $|G_v| = 2^s$.

Suppose now that n is odd, that $|G_v| = 4$, and that τ is a non-trivial central element of G contained in G_v . Without loss of generality, we may assume that τ acts non-trivially on the pair of arcs pointing out of v . Furthermore, since the index of G_v in G is n , it follows that G_v is the unique Sylow 2-subgroup of G , and thus $G = G_v \rtimes H$, where H is a group of order n . Moreover, since G_v is the kernel of the action of G on the vertices of $\vec{C}_n^{(2)}$, it follows that H acts regularly on the vertices of $\vec{C}_n^{(2)}$; in particular, $H = \langle g \rangle$ where g is an automorphism of order n that maps every vertex to its unique out-neighbour.

Since $\tau = \tau^g$, the element τ acts non-trivially on every pair of arcs sharing the same tail. In particular, τ is the unique non-trivial central element of G contained in G_v . Since $G = G_v H$ and since G_v is abelian, this shows that H centralises no element of $G_v \setminus \{1, \tau\}$. However, this is impossible since H has odd order and $|G_v \setminus \{1, \tau\}| = 2$. This contradiction completes the proof of the lemma. \square

2.3 Alter-relations, alter-exponent, radius and perimeter

In this section, we present a very useful tool for studying digraphs, based on the orientation of arcs in the walks of a digraph. The concepts presented in this section were first introduced in [24] (for a generalisation to infinite digraphs, see [16]). All the facts stated below were proved in [24] for simple digraphs and extend without any change to digraphs with loops and multiple arcs.

A walk from a vertex v_0 to a vertex v_s of length s in a digraph Γ is a sequence $(v_0, x_1, v_1, \dots, v_{s-1}, x_s, v_s)$ of arcs $x_i \in A(\Gamma)$ and vertices $v_j \in V(\Gamma)$ such that for any $i \in \{1, \dots, s\}$ the pair $(\text{tail}(x_i), \text{head}(x_i))$ equals either (v_{i-1}, v_i) or (v_i, v_{i-1}) . In the former case, we say that x_i is *positively oriented*, while in the latter case we say that x_i is *negatively oriented* in the walk. A walk is *directed* if all of its arcs are positively oriented and is *alternating* if the orientation of the arcs in the walk alternates. A digraph Γ is (*strongly*) *connected* provided that for any two vertices $u, v \in \Gamma$ there exists a (directed) walk from u to v . A vertex-transitive digraph is strongly connected if and only if it is connected (see, for example, [27, Lemma 2]). A walk is closed provided that it begins and ends in the same vertex.

Let $W = (v_0, x_1, v_1, x_2, \dots, x_n, v_n)$ be a walk in a digraph Γ . The *sum* $s(W)$ is the difference between the number of positively oriented arcs in W and the number of negatively oriented arcs in W . The k -th partial sum $s_k(W)$ is defined as the sum of the initial walk $(v_0, x_1, v_1, \dots, v_k)$ of length k . The set $\{s_k(W), 0 \leq k \leq n\}$ is the *tolerance* of W and vertices u and v are *alter-equivalent* with tolerance \mathcal{J} (written $uA_{\mathcal{J}}v$) if there exists a walk from u to v with sum 0 and tolerance contained in \mathcal{J} . It transpires that $A_{\mathcal{J}}$ is an equivalence relation (called an *alter-relation*) for every interval \mathcal{J} containing 0 and that it is invariant under every automorphism of Γ . We will denote the equivalence class containing a vertex v with $A_{\mathcal{J}}(v)$ and use the shorthand $A_i(v)$ to mean $A_{[0,i]}(v)$ (when $i \geq 0$) or $A_{[i,0]}(v)$ (when $i < 0$). Note that since Γ is a finite digraph, there exists a non-negative integer e such that $A_e = A_{e+1}$ and (by induction) $A_e = A_{\infty}$. The smallest such integer e is called the *alter-exponent* of Γ and denoted $\exp(\Gamma)$. It can be shown that $\exp(\Gamma)$ also equals the smallest non-negative integer i for which $A_{-i} = A_{-i-1}$ as well as the smallest i such that $A_{[-i,i]} = A_{[-i-1,i+1]}$. When we consider alter-relations in several different digraphs, we shall use the symbol $A_{\mathcal{J}}^{\Gamma}$ (instead of $A_{\mathcal{J}}$) to denote the one in the digraph Γ .

The number of equivalence classes of the alter-relation A_{∞} is called the *perimeter* of

Γ and denoted $\text{perim}(\Gamma)$. If the in-valence and the out-valence of each vertex is positive, then the equivalence classes B_i of A_∞ can be indexed by \mathbb{Z}_p (where $p = \text{perim}(\Gamma)$) in such a way that every arc of Γ having its tail in B_i , has its head in B_{i+1} .

We will be particularly interested in the sets $A_1(v)$ and $A_{-1}(v)$. Note that these sets consists of precisely those vertices that can be reached from v by alternating walks of even length starting with a positively (negatively, respectively) oriented arc. The intersection $A_1(v) \cap A_{-1}(v)$ will be denoted $\text{Att}(v)$ and called the *attachment set* (at vertex v).

Suppose henceforth that Γ is a G -arc-transitive digraph. Then the sets $A_{\mathcal{J}}(v)$ (as well as $\text{Att}(v)$) are all blocks for the action of G on $V(\Gamma)$ and their size depends only on \mathcal{J} (but not on v). One can thus define the *radius* of Γ (denoted $\text{rad}(\Gamma)$) to be the cardinality of $|A_1(v)|$ for any $v \in \Gamma$, and the *attachment number* of Γ (denoted $\text{att}(\Gamma)$) to be the cardinality of $\text{Att}(v)$ for any $v \in V(\Gamma)$. Since $\text{Att}(v) \subseteq A_1(v) \subseteq A_2(v) \subseteq \dots$, we see that $\text{att}(\Gamma)$ divides $\text{rad}(\Gamma)$, and that $|A_i(v)|$ divides $|A_{i+1}(v)|$ for every $i \geq 1$.

Suppose now that Γ is a 2-valent arc-transitive digraph. Then the sub-digraph of Γ induced by a closed alternating walk of sum 0 that traverses every arc of Γ at most once is called an *alternating cycle*. The *length* of an alternating cycle is defined to be the length of the closed alternating walk that induces it. (Alternating cycles were introduced in [19] in the context of simple $(G, \frac{1}{2})$ -arc-transitive 4-valent graphs.)

Note that an alternating cycle is uniquely determined by any of its arcs, implying that the set of alternating cycles induces a decomposition of the arc-set of Γ . Furthermore, this decomposition is preserved by every automorphism of Γ , implying that all alternating cycles in Γ have the same length.

In addition to the assumption that Γ is a 2-valent arc-transitive digraph, assume for the rest of the section that Γ is not isomorphic to any \overleftrightarrow{C}_n with n odd. Then an alternating cycle is indeed a cycle (in the sense that the walk that generates it traverses every vertex of the digraph at most once), and Γ contains at least two alternating cycles.

Furthermore, observe that $A_1(v)$ consists of every second vertex of an alternating cycle starting with a positively oriented arc with its tail in v , and similarly, $A_{-1}(v)$ consist of every second vertex of an alternating cycle starting with a negatively oriented arc with its head in v . In particular, $|A_1(v)| = |A_{-1}(v)|$ and the length of each alternating cycle is twice the radius of Γ . Note also that there are precisely two alternating cycles meeting in a given vertex v and the set of vertices that are contained in both of these alternating cycles is precisely $\text{Att}(v)$. Two alternating cycles therefore meet in either 0 or $\text{att}(\Gamma)$ vertices.

Suppose now that $\text{att}(\Gamma) \geq 3$ and let $g \in \text{Aut}(\Gamma)$ fix an arc x of Γ . Then g fixes point-wise the alternating cycle C containing x . Since $\text{att}(\Gamma) \geq 3$, g fixes also at least three vertices of each alternating cycle intersecting C , and therefore fixes each of these cycles point-wise. But then by connectivity, g fixes each alternating cycles of Γ point-wise. In particular, g is trivial. This proves the following easy, but very useful result.

Lemma 2.3. *If Γ is a connected 2-valent 2-arc-transitive digraph, then $\text{att}(\Gamma) \leq 2$.*

We finish this section with another useful result.

Lemma 2.4. *If Γ is a connected 2-valent 2-arc-transitive digraph and $\exp(\Gamma) = 1$, then $\Gamma \cong \overrightarrow{\text{PX}}(m, 1)$ for some integer m .*

Proof. If Γ is not simple, then by Lemma 2.1, $\Gamma \cong \overrightarrow{C}_n^{(2)}$ for some $n \geq 2$, implying that $\exp(\Gamma) = 0$; a contradiction. Hence Γ is simple, and we can apply [30, Theorem 7.1] to

conclude that $\text{rad}(\Gamma) = 2$. Since $\exp(\Gamma) = 1$, it is then easy to see that $\text{att}(\Gamma) = 2$, and also that $\Gamma \cong \overrightarrow{PX}(m, 1)$ for some m (see, for example, [19, Proposition 3.1]). \square

2.4 Covers and quotients

The second tool that we will use extensively is the concept of (di)graph coverings. This tool is usually defined in the setting of undirected graphs, but extends naturally to digraphs. In this section, we present a few basic facts and results and refer the reader to [14, 17] for more details.

Let Γ and Λ be two digraphs. A *morphism* from Γ onto Λ is a function $f : V(\Gamma) \cup A(\Gamma) \rightarrow V(\Lambda) \cup A(\Lambda)$ mapping $V(\Gamma)$ to $V(\Lambda)$ and $A(\Gamma)$ to $A(\Lambda)$ such that $f(\text{tail}(x)) = \text{tail}(f(x))$ and $f(\text{head}(x)) = \text{head}(f(x))$ for every $x \in A(\Gamma)$. A morphism is an *epimorphism* or *isomorphism* if it is *surjective* or *bijective*, respectively. (Note that an automorphism of a digraph is precisely an isomorphism from the digraph onto itself.)

An epimorphism $\wp : \Gamma \rightarrow \Lambda$ is a *covering projection* provided that for every $v \in V(\Gamma)$ the restrictions $\wp_v^+ : \Gamma^+(v) \rightarrow \Lambda^+(\wp(v))$ and $\wp_v^- : \Gamma^-(v) \rightarrow \Lambda^-(\wp(v))$ of \wp to the out- and in-neighbourhoods of v are bijective. For simplicity, we shall also require both Γ and Λ to be connected. The preimage $\wp^{-1}(x)$ of a vertex or an arc x of Λ is called a *fibre* of the covering projection \wp and the group of all automorphisms of Γ that preserve each fibre set-wise is called the *group of covering transformations*. If the latter is transitive on each fibre, then the covering projection is *regular*.

Normal quotients of simple graphs were introduced in [33, 34] and have now become a standard tool in studying symmetric graphs. Here we adapt this concept slightly to fit into the setting of digraphs admitting loops and multiple arcs. This adaptation will prove most useful in the proofs of our main results.

Let Γ be a digraph and let $N \leq \text{Aut}(\Gamma)$. Let $A_N = \{x^N : x \in A(\Gamma)\}$ and $V_N = \{v^N : v \in V(\Gamma)\}$ denote the sets of N -orbits on the arcs and vertices of Γ , respectively. Further, let $\text{tail}_N : A_N \rightarrow V_N$ and $\text{head}_N : A_N \rightarrow V_N$ be defined by $\text{tail}_N(x^N) = \text{tail}(x)^N$ and $\text{head}_N(x^N) = \text{head}(x)^N$. This defines the *quotient digraph* $\Gamma/N = (V_N, A_N, \text{head}_N, \text{tail}_N)$, together with the obvious epimorphism $\wp_N : \Gamma \rightarrow \Gamma/N$ satisfying $\wp_N(x) = x^N$ for every $x \in V(\Gamma) \cup A(\Gamma)$, called the *normal quotient projection relative to N* . If $N \trianglelefteq G \leq \text{Aut}(\Gamma)$, then there is an obvious, but not necessarily faithful action of the quotient group G/N on the digraph Γ/N . Note also that if G acts transitively on vertices, arcs or s -arcs of Γ , then so does G/N on Γ/N . If the quotient projection \wp_N is a covering projection, then the situation is particularly nice; for example:

Lemma 2.5. *Let Γ be a digraph, let $G \leq \text{Aut}(\Gamma)$ and let N be a normal subgroup of G . If the quotient projection $\wp : \Gamma \rightarrow \Gamma/N$ is a covering projection, then the action of G/N on $V(\Gamma/N) \cup A(\Gamma/N)$ is faithful, and moreover, the stabilisers G_v and $(G/N)_{v^N}$ are isomorphic for every $v \in V(\Gamma)$.*

We say in this case that the group G/N *lifts along* \wp . More precisely, a group $H \leq \text{Aut}(\Gamma/N)$ *lifts along* \wp if there exists some $G \leq \text{Aut}(\Gamma)$, containing N as a normal subgroup, such that $G/N = H$.

We now state two very useful sufficient and necessary conditions for a normal quotient projection to be a regular covering projection. (We shall call a group N of automorphisms of Γ *semiregular* provided that the stabiliser N_v is trivial for every $v \in V(\Gamma)$.)

Lemma 2.6. *Let Γ be a connected digraph, let $N \leq \text{Aut}(\Gamma)$ and let $\wp: \Gamma \rightarrow \Gamma/N$ be the corresponding quotient projection. Then the following statements are equivalent:*

- (a) *N is semiregular;*
- (b) *the in-valence as well as the out-valence of v and $\wp(v)$ coincide for every $v \in V(\Gamma)$;*
- (c) *\wp is a regular covering projection.*

The rest of the section is devoted to the interplay between the concepts of alter-relations and covering projections.

Lemma 2.7. *Let $\wp: \Gamma \rightarrow \Lambda$ be a covering projection, let v be a vertex of Γ and let \mathcal{J} be an interval of integers containing 0. Then $\wp(A_{\mathcal{J}}^{\Gamma}(v)) = A_{\mathcal{J}}^{\Lambda}(\wp(v))$.*

Proof. Suppose that $\tilde{u} \in \wp(A_{\mathcal{J}}^{\Gamma}(v))$. Then there exists $u \in V(\Gamma)$ such that $\wp(u) = \tilde{u}$ and a walk $(v, x_1, v_1, \dots, x_n, u)$ in Γ of sum 0 and tolerance within \mathcal{J} . But then the projected walk $(\wp(v), \wp(x_1), \wp(v_1), \dots, \wp(x_n), \tilde{u})$ is also a walk of sum 0 and tolerance within \mathcal{J} , implying that $\tilde{u} \in A_{\mathcal{J}}^{\Lambda}(\wp(v))$.

Conversely, suppose that $\tilde{u} \in A_{\mathcal{J}}^{\Lambda}(\wp(v))$. Then there exists a walk $(\wp(v), \tilde{x}_1, \tilde{v}_1, \dots, \tilde{x}_n, \tilde{u})$ of sum 0 and tolerance within \mathcal{J} . Since \wp is a local bijection, one can then construct a lift $(v, x_1, v_1, \dots, x_n, u)$ such that $\wp(x_i) = \tilde{x}_i$, $\wp(v_i) = \tilde{v}_i$, and $\wp(u) = \tilde{u}$. Note that this lift will also have sum 0 and tolerance within \mathcal{J} , implying that $u \in A_{\mathcal{J}}^{\Gamma}(v)$, and therefore $\tilde{u} \in \wp(A_{\mathcal{J}}^{\Gamma}(v))$. \square

Lemma 2.8. *Let Γ be a G -vertex-transitive digraph, let N be a semiregular normal subgroup of G , let $\Lambda = \Gamma/N$ and let $\wp: \Gamma \rightarrow \Lambda$ be the corresponding covering projection. Further, let v be a vertex of Γ , and let \mathcal{J} be an interval of integers containing 0. Then $|A_{\mathcal{J}}^{\Gamma}(v)|$ divides $|N||A_{\mathcal{J}}^{\Lambda}(\wp(v))|$.*

Proof. In view of Lemma 2.7, we see that $A_{\mathcal{J}}^{\Gamma}(v) \subseteq \wp^{-1}(\wp(A_{\mathcal{J}}^{\Gamma}(v))) = \wp^{-1}(A_{\mathcal{J}}^{\Lambda}(\wp(v)))$. Since $A_{\mathcal{J}}^{\Lambda}(\wp(v))$ is a block for the action of G/N on Λ , it follows easily that $\wp^{-1}(A_{\mathcal{J}}^{\Lambda}(\wp(v)))$ is a block for the action of G on Γ . Since $A_{\mathcal{J}}^{\Gamma}(v)$ is also a block for G , it follows that $|A_{\mathcal{J}}^{\Gamma}(v)|$ divides $|\wp^{-1}(A_{\mathcal{J}}^{\Lambda}(\wp(v)))|$. However, since the \wp -preimage of a vertex in Λ is an N -orbit on Γ , it follows that the latter equals $|N||A_{\mathcal{J}}^{\Lambda}(\wp(v))|$. \square

Lemma 2.9. *Let Γ be a connected, $(G, 2)$ -arc-transitive 2-valent digraph and let N be a normal subgroup of G . If N has odd prime order, then $\text{rad}(\Gamma/N) = \text{rad}(\Gamma)$.*

Proof. Let q be the order of N , let $\Lambda = \Gamma/N$ and let $\wp: \Gamma \rightarrow \Lambda$ be the corresponding quotient projection. Suppose that the conclusion of the lemma is false, that is, $\text{rad}(\Gamma/N) \neq \text{rad}(\Gamma)$.

Since G_v is a 2-group (see Lemma 3.1) and N is of odd order, N acts semiregularly on $V(\Gamma)$. By Lemma 2.6, the quotient projection \wp is then a regular covering projection. Choose a vertex v of Γ and $\epsilon \in \{-1, 1\}$, and consider the set $T = A_{\epsilon}^{\Gamma}(v)$. Recall that $|T| = \text{rad}(\Gamma)$. By Lemma 2.7, $\wp(T) = A_{\epsilon}^{\Lambda}(\wp(v))$. Since the size of the latter is $\text{rad}(\Lambda)$, it follows by our initial assumption that $|\wp(T)| \neq |T|$, implying that T contains at least two elements of the orbit v^N . Since both T and v^N are blocks for the action of G on $V(\Gamma)$, so is their intersection. However, v^N is of prime size, implying that $v^N = v^N \cap T$, and thus $v^N \subseteq T$. Since this is true for any choice of ϵ , it follows that $v^N \subseteq A_1^{\Gamma}(v) \cap A_{-1}^{\Gamma}(v) = \text{Att}(v)$. But then by Lemma 2.3 it follows that Γ is not 2-arc-transitive, a contradiction. \square

2.5 Partial line graphs and digraphs of Praeger and Xu

In this section, we give a brief overview of the very useful concept of *partial line graph construction*, which was invented in [21] to analyse G -arc-transitive 2-valent digraphs of radius 2, and was further developed in [30].

For a digraph Γ and a positive integer s , the s -th *partial line graph* $\text{Pl}_s(\Gamma)$ of Γ is the digraph with vertex-set being the set of s -arcs $A_s(\Gamma)$, the arc-set being $A_{s+1}(\Gamma)$, and the functions tail and head defined by the rules $\text{tail}(x_1, \dots, x_{s+1}) = (x_1, \dots, x_s)$ and $\text{head}(x_1, \dots, x_{s+1}) = (x_2, \dots, x_{s+1})$ for every $(s+1)$ -arc (x_1, \dots, x_{s+1}) of Γ . Moreover, we let $\text{Pl}_0(\Gamma) = \Gamma$ and write Pl instead of Pl_1 . Note that if Γ is a 2-valent digraph, then so is $\text{Pl}_s(\Gamma)$ for every $s \geq 0$. The following formula (which appeared as [30, Lemma 3.2(i)] in the context of simple digraphs), provides an alternative, recursive definition of the Pl_s operator:

$$\text{Pl}_s(\Gamma) \cong \text{Pl}(\text{Pl}_{s-1}(\Gamma)) \quad \text{for } s \geq 1. \quad (2.1)$$

The lemma below follows from [30, Lemma 3.1(iv)] and [30, Lemma 3.2(ii)] in the context of simple digraphs. The proof remains unchanged in the case of non-simple digraphs.

Lemma 2.10. *If Γ is a vertex-transitive digraph, then $\exp(\text{Pl}(\Gamma)) = \exp(\Gamma) + 1$.*

The following result appeared as [30, Lemma 5.1] in the context of simple digraphs, and extends to general digraphs via Lemma 2.1.

Lemma 2.11. *If Γ is a 2-valent $(G, 2)$ -arc-transitive digraph such that $\text{rad}(\Gamma) = 2$, then $\Gamma \cong \text{Pl}(\Lambda)$, where Λ is a 2-valent $(G, 3)$ -arc-transitive digraph of order half that of Γ .*

The Pl operator can be used to define a very important class of digraphs, first studied by Praeger and Xu [37] in the context of simple graphs, and by Praeger [35] in the context of simple digraphs. For integers n and s , $n \geq 1$, $s \geq 0$, let

$$\overrightarrow{\text{PX}}(n, s) = \begin{cases} \overrightarrow{C}_n^{(2)} & \text{if } s = 0 \\ \text{Pl}(\overrightarrow{\text{PX}}(n, s-1)) & \text{if } s \geq 1 \end{cases} \quad (2.2)$$

We shall call a graph isomorphic to some $\overrightarrow{\text{PX}}(n, s)$ simply a $\overrightarrow{\text{PX}}$ -digraph. Note that, in view of (2.1), we have

$$\overrightarrow{\text{PX}}(n, s) \cong \text{Pl}_s(\overrightarrow{C}_n^{(2)}). \quad (2.3)$$

The automorphism group of $\overrightarrow{C}_n^{(2)}$ acts naturally as a group of automorphisms on each $\overrightarrow{\text{PX}}(n, s)$ for $s \geq 1$. The following surprising characterisation of $\overrightarrow{\text{PX}}$ -digraphs was proved in [35, Theorem 2.9] in the context of simple digraphs. In view of Lemma 2.1, the result extends to non-simple digraphs.

Lemma 2.12. *Let Γ be a connected 2-valent G -arc-transitive digraph and let $v \in V(\Gamma)$. If G contains an abelian normal subgroup N that is not semiregular, then Γ is a $\overrightarrow{\text{PX}}$ -digraph.*

The following lemma is an analogue of a similar result for the undirected graphs (see [9, Lemma 3.1]). Our proof is just a slight modification of the proof given there.

Lemma 2.13. *Let Γ be a connected 2-valent, G -arc-transitive digraph and let N be a minimal normal subgroup of G . Suppose that N is a 2-group and that $\Gamma/N \cong \vec{C}_n^{(2)}$ for some $n \geq 1$. Then Γ is a \vec{PX} -digraph.*

Proof. Since N is a minimal normal subgroup of G and a 2-group, it is elementary abelian. Let K be the kernel of the action of G on the set of N -orbits on $V(\Gamma)$, and observe that G/K acts faithfully on $V(\Gamma/N)$. Let C be the centraliser of N in K . Then $N \leq C \leq K$. Since N and K are normal in G , so is C . Since N and K have the same orbits on $V(\Gamma)$, so does C , implying that $K = NK_v$ and $C = NC_v$ for any vertex v .

Since the quotient Γ/N is 2-valent, Lemma 2.6 implies that the quotient projection $\Gamma \rightarrow \Gamma/N$ is a covering projection, and also that N is semiregular (for otherwise the valence of the quotient Γ/N would be less than that of Γ). Therefore, $N \cap C_v \leq N_v = 1$, and since C_v centralises N , we see that $C = N \times C_v$. Since the quotient projection $\Gamma \rightarrow \Gamma/N$ is a covering projection, Lemma 2.5 implies that G_v embeds into a vertex-stabiliser in $\text{Aut}(\vec{C}_n^{(2)})$. In particular, G_v (and thus C_v) is an elementary abelian 2-group, implying that C is an abelian normal subgroup of G .

Let us now show that $C_v \neq 1$. By way of contradiction, assume that $C_v = 1$, and thus that $C = NC_v = N$. Now recall that $K = NK_v$ and $N \cap K_v = 1$. Since both N and K_v are 2-groups, so is K . In particular, the centre $Z(K)$ is non-trivial. On the other hand, since $Z(K) \leq C$ and since $C = N$, we see that $Z(K) \leq N$. Since N is a minimal normal subgroup of G , this implies that $N = Z(K)$. But then $K = NK_v = N \times K_v$, and thus K is an elementary abelian 2-group. In particular, N , being the centre of K , equals K . Now recall that G/K acts faithfully on $V(\Gamma/N)$. On the other hand, G/K equals G/N , which is clearly unfaithful on $V(\Gamma/N)$. This contradiction shows that $C_v \neq 1$, and by Lemma 2.12, Γ is a \vec{PX} -digraph, as claimed. \square

Lemma 2.14. *Let n and s be integers, $n \geq 1$, $s \geq 0$, let $\Lambda = \vec{PX}(n, s)$ and let v be a vertex of Λ . Then $\exp(\Lambda) = s$, $|A_\infty^\Lambda(v)| = 2^s$ and $\text{perim}(\Lambda) = n$. Suppose G is a group acting transitively on the arcs of Λ and let $K = \langle G_u : u \in V(\Lambda) \rangle$, that is, the group generated by all the vertex-stabilisers in G . Then K is the kernel of the action of G on the partition $\{A_\infty^\Lambda(u) : u \in V(\Lambda)\}$ and $v^K = A_\infty^\Lambda(v)$; in particular, K is normal in G . Furthermore, the group K is elementary abelian of order $2^s |G_v|$, the quotient digraph Λ/K is isomorphic to a directed cycle of length n , and G/K is a cyclic group of order n .*

Proof. Observe first that $\exp(\vec{PX}(n, 0)) = \exp(\vec{C}_n^{(2)}) = 0$. On the other hand, by formula (2.2), $\Lambda = \text{Pl}(\vec{PX}(n, s-1))$, and thus by induction and Lemma 2.10, $\exp(\Lambda) = s$, as claimed.

By formula (2.3), a vertex of Λ is an s -arc of $\vec{C}_n^{(2)}$. Now recall that $V(\vec{C}_n^{(2)}) = \mathbb{Z}_n$ and that there is an arc pointing from i to j if and only if $j - i = 1$. It is now clear that if v is an s -arc of $\vec{C}_n^{(2)}$ starting in a vertex i of $\vec{C}_n^{(2)}$, and W is a walk in Λ of sum k starting in v , then the end-point of W will be an s -arc of $\vec{C}_n^{(2)}$ starting in $i + k$; in particular, every member of $A_\infty^\Lambda(v)$ is one of the 2^s s -arcs of $\vec{C}_n^{(2)}$ starting in i . On the other hand, if w and u are arbitrary s -arcs of $\vec{C}_n^{(2)}$ starting in i and $i + s$, respectively, then there clearly exists a directed walk in Λ of length s from v to u . By combining two such walks from v to u and from an arbitrary w to u , one gets a walk from v to w of sum 0. This shows that $A_\infty^\Lambda(v)$ is precisely the set of all s -arcs of $\vec{C}_n^{(2)}$ starting in i . In particular, $|A_\infty^\Lambda(v)| = 2^s$,

as claimed. Since $|\mathbf{V}(\Lambda)| = 2^s n$ and since $\text{perim}(\Lambda) = |\mathbf{V}(\Lambda)|/|A_\infty^\Lambda(v)|$, it follows that $\text{perim}(\Lambda) = n$.

The equality $v^K = A_\infty^\Lambda(v)$ follows directly from [30, Lemma 4.1] and [30, Corollary 4.2]. In particular, K fixes every class $A_\infty^\Lambda(u)$, $u \in \Lambda$, set-wise, implying that K is contained in the kernel (call it M) of the action of G on the partition $\{A_\infty^\Lambda(u) : u \in \mathbf{V}(\Lambda)\}$. Moreover, $v^K = v^M$, and since $K_v = G_v = M_v$, it follows that $K = M$. In particular, $|K| = |v^K| |K_v| = 2^s |G_v|$, as claimed.

The fact that Λ/K is isomorphic to the directed cycle of length $\text{perim}(\Lambda)$ and that G/K is a cyclic group of order $\text{perim}(\Lambda)$ is now a direct consequence of either [24, Propositions 3.2 and 3.5] or [35, Proposition 2.1].

Finally, to see that K is elementary abelian, recall that a vertex of Λ is an s -arc in $\vec{C}_n^{(2)}$, and thus the stabiliser of a vertex in $\text{Aut}(\Lambda)$ equals the stabiliser of an s -arc in $\text{Aut}(\vec{C}_n^{(2)})$. However, each stabiliser of an s -arc in $\text{Aut}(\vec{C}_n^{(2)})$ is contained in the kernel of the action of $\text{Aut}(\vec{C}_n^{(2)})$ on $\mathbf{V}(\vec{C}_n^{(2)})$, which is elementary abelian of order 2^n . Since K is generated by the vertex-stabilisers G_u , $u \in \mathbf{V}(\Lambda)$, and thus by the stabilisers of the s -arcs of $\vec{C}_n^{(2)}$ in G , it follows that K is also elementary abelian. \square

3 Proofs of the main results

3.1 Auxilliary results

We start this section by a folklore fact about the vertex-stabilisers in arc-transitive 2-valent digraphs (see for example [22, Theorem 1.1] or [30, Theorem 1.2]).

Lemma 3.1. *If Γ be a connected 2-valent (G, s) -arc-transitive but not $(G, s+1)$ -arc-transitive digraph and $v \in \mathbf{V}(\Gamma)$, then G_v is a group of order 2^s , generated by s involutions, and acts regularly upon the set of all s -arcs starting in v .*

The following is a well-known fact about the general linear groups $\text{GL}(2, q)$.

Lemma 3.2. *If q is a power of an odd prime, and H an elementary abelian 2-subgroup of $\text{GL}(2, q)$, then $|H| \leq 4$, and if $|H| = 4$, then H contains the central involution of $\text{GL}(2, q)$, namely the minus identity matrix.*

Proof. Recall that the group $\text{SL}(2, q)$ contains a unique involution, namely the minus identity matrix. This implies that the intersection $H \cap \text{SL}(2, q)$ is of order at most 2. On the other hand, the quotient $\text{GL}(2, q)/\text{SL}(2, q)$ is cyclic (of order $q-1$), implying that $H\text{SL}(2, q)/\text{SL}(2, q)$ is cyclic; but this cyclic group is isomorphic to $H/(\text{SL}(2, q) \cap H)$, and so of order at most 2. Hence the order of H is at most 4; and if it is of order 4, then $\text{SL}(2, q) \cap H$ is non-trivial and thus contains the minus identity matrix. \square

The following situation will occur several times in the proofs of the main results of the paper. To avoid repetition, we formulate it as a lemma.

Lemma 3.3. *Let Z be a group containing normal subgroups X and Y , such that X is abelian and contained in Y . Let C be the centraliser of X in Y . If the order of X is coprime to its index in C , then $C = X \times T$ for some normal subgroup T of Z . Moreover, T is isomorphic to a normal subgroup of Y/X .*

Proof. Since both X and Y are normal in Z , so is C . Moreover, since X is abelian, it is contained in C . Since, by assumption, the order of X is coprime to its index in C , the Schur-Zassenhaus theorem implies that X has a complement, say T . However, since X is centralised by C , it follows that $C = X \times T$. Now observe that T consists of all the elements of C of order coprime to $|X|$, implying that T is characteristic in C . Since C is normal in Z , so is T . Furthermore, since $X \leq C \trianglelefteq Y$, the quotient Y/X contains a normal subgroup isomorphic to C/X , which is isomorphic to T . \square

Lemma 3.4. *Let m be an odd positive integer and G an arc-transitive group of automorphisms of the digraph $\overrightarrow{PX}(m, s)$. Then G contains a normal cyclic subgroup, the order of which divides m and is at least $m/(2^s|G_v| - 1)$.*

Proof. Let $\Lambda = \overrightarrow{PX}(m, s)$. By Lemma 3.1, the vertex-stabiliser G_v has order 2^r for some positive integer r . Let $K = \langle G_v : v \in V(\Lambda) \rangle$, and recall that by Lemma 2.14, K is an elementary abelian normal subgroup of G of order 2^{s+r} and G/K is a cyclic group of order m . In particular, K is a normal Sylow 2-subgroup of G .

Let C be the centraliser of K in G . By applying Lemma 3.3 with $Z = Y = G$ and $X = K$, we can conclude that $C = K \times T$ for some normal subgroup T of G , isomorphic to a subgroup of the quotient G/K . Since G/K is cyclic of order m , T is cyclic of order dividing m .

Further, the quotient G/C is isomorphic to the quotient of G/K by C/K , and since $C/K \cong T$ and G/K is cyclic of order m , the quotient G/C is a cyclic group of order $m/|T|$. However, G/C embeds into $\text{Aut}(K)$, which is isomorphic to $\text{GL}(r+s, 2)$. It is well known that every cyclic subgroup of $\text{GL}(r+s, 2)$ is of order at most $2^{r+s} - 1$ (see for example [11, Corollary 2.7]), implying that $m/(2^{s+r} - 1) \leq |T|$. \square

3.2 Proof of Theorem 1.1

As in Theorem 1.1, let Γ be a connected $(G, 2)$ -arc-transitive 2-valent digraph of order $2^a q^b p^c$, where p and q are distinct odd primes, $a \in \{0, 1, 2, 3\}$, $b, c \in \{0, 1, 2\}$, $(b, c) \neq (2, 2)$, and G is non-solvable. We need to show that Γ is isomorphic to one of the digraphs in Table 1.

All such digraphs of order up to 1000 can be found by inspecting the census [28] of arc-transitive digraphs of valence 2. It transpires that there are precisely fifty-two of them, and they are all listed in Table 1 as digraphs labelled ATD. We may thus assume throughout the proof that $|V(\Gamma)| > 1000$.

Suppose that G acts transitively on the s -arcs but not on the $(s+1)$ -arcs of Γ . Then $|G_v| = 2^s$ (see Lemma 3.1) and therefore $|G| = 2^{a+s} q^b p^c$. Now consider a composition series $1 = G_0 \triangleleft G_1 \triangleleft \dots \triangleleft G_k = G$ of G , and the corresponding set of composition factors $F_i = G_i/G_{i-1}$ for $i \in \{1, \dots, k\}$. Recall that F_i are simple groups. Since G is non-solvable, there exists $j \in \{1, \dots, k\}$ such that F_j is non-abelian. Let $T = F_j$ and note that $|T|$ divides $|G|$, which equals $2^{a+s} q^b p^c$.

It is known that there are precisely eight non-abelian simple groups whose orders are divisible by at most three distinct primes (see, for example, [12]); these are $\text{Alt}(5)$, $\text{PSL}(2, 7)$, $\text{Alt}(6)$, $\text{PSL}(2, 8)$, $\text{PSL}(2, 17)$, $\text{PSL}(3, 3)$, $\text{PSU}(3, 3)$, and $\text{PSU}(4, 2)$. Out of these, only the first five are such that the odd primes appear with multiplicity at most 2; these five groups, together with their orders and the orders of their automorphism groups are listed in Table 2.

T	$ T $	$ \text{Aut}(T) $
Alt(5)	$2^2 \cdot 3 \cdot 5$	$2^2 \cdot 3 \cdot 5 = 120$
PSL(2, 7)	$2^3 \cdot 3 \cdot 7$	$2^4 \cdot 3 \cdot 7 = 336$
Alt(6)	$2^3 \cdot 3^2 \cdot 5$	$2^5 \cdot 3^2 \cdot 5 = 1440$
PSL(2, 8)	$2^3 \cdot 3^2 \cdot 7$	$2^3 \cdot 3^3 \cdot 7 = 1512$
PSL(2, 17)	$2^4 \cdot 3^2 \cdot 17$	$2^5 \cdot 3^2 \cdot 17 = 4896$

Table 2: Simple groups of orders divisible by three primes only, with odd part cube-free

Observe that the order of each of these groups is divisible by 3 and that the other odd prime divisor is 5, 7, or 17. We may thus assume without loss of generality that $q = 3$ and $p \in \{5, 7, 17\}$.

If $p = 5$, then $|V(\Gamma)| \leq 8 \cdot 3 \cdot 5^2 = 600$, contradicting our initial assumption. This rules out the groups Alt(5) and Alt(6) as possibilities for T .

If $p = 7$, then the order $2^a 3^b 7^c$ of Γ is larger than 1000 only when $a = 3$, $b = 1$ and $c = 2$. Since 9 divides the order of PSL(2, 8), this implies that $T \not\cong \text{PSL}(2, 8)$, and therefore $T \cong \text{PSL}(2, 7)$ and $|V(\Gamma)| = 8 \cdot 3 \cdot 7^2 = 1176$.

Finally, if $p = 17$, then $T \cong \text{PSL}(2, 17)$, and since 3^2 divides the order of PSL(2, 17), it follows that the order of Γ is $8 \cdot 3^2 \cdot 17 = 1224$.

We shall now distinguish two cases, depending on whether G contains a non-trivial abelian normal subgroup or not.

Case I. Suppose that G contains a non-trivial abelian normal subgroup. Then G contains a minimal normal subgroup N that is abelian. Since G is non-solvable, Γ is not isomorphic to a $\overrightarrow{\text{PX}}$ -digraph. In view of Lemma 2.12, N is then semiregular, and thus $\wp: \Gamma \rightarrow \Gamma/N$ is a regular covering projection.

If N is a 2-group, then, since the 2-part of $|V(\Gamma)|$ is 8, we see that $|N| \in \{2, 4, 8\}$. The possible orders of Γ/N are then 147 and 153 (when $|N| = 8$, and $T \cong \text{PSL}(2, 7)$ and PSL(2, 17), respectively), 294 and 306 (when $|N| = 4$), and 588 and 612 (when $|N| = 2$).

Now suppose that N is of odd order. Since N is solvable, T is a composition factor of G/N and thus $|T|$ divides $|G|/|N| = 2^{a+s} q^b p^c / |N|$. Since $|N|$ is odd and $b + c \leq 3$, it follows that the odd part of $|T|$ is of the form $q^{b'} p^{c'}$ where $b' + c' \leq 2$; in particular, $T \not\cong \text{PSL}(2, 17)$, and therefore $T \cong \text{PSL}(2, 7)$, $|N| = 3$ or $|N| = 7$, and $|V(\Gamma/N)| = 2^a \cdot 3 \cdot 7 \leq 168$. In fact, since we have already established that $|V(\Gamma)| = 8 \cdot 3 \cdot 7^2$ when $T \cong \text{PSL}(2, 7)$, it follows that $|N| = 7$ and $|V(\Gamma/N)| = 168$.

We have thus shown that in Case I, we have $|V(\Gamma/N)| \in \{147, 153, 168, 294, 306, 588, 612\}$ and therefore the quotient digraph Γ/N appears in the census [28]. By searching the census for 2-arc-transitive digraphs of these orders with a non-solvable automorphism group, one sees that the triple $(T, |N|, \Gamma/N)$ is as one given in Table 3 (here the data in the last column corresponds to the names of digraphs as given in [28]).

Using the methods described in, say, [14, 32], for each of the digraphs Γ/N from Table 3, all the corresponding N -regular covers were computed for which a 2-arc-transitive subgroup of $\text{Aut}(\Gamma/N)$ lifts, and the resulting nine covering digraphs were included in Table 1 under the names X_1, X_2, \dots, X_9 .

Case II. Suppose now that G contains no non-trivial abelian normal subgroups. Let us now consider the group generated by all minimal normal subgroups of G , called the *socle* of G and denoted $\text{soc}(G)$. Since G contains no non-trivial abelian normal subgroups, it

T	$ N $	Γ/N
$\text{PSL}(2, 7)$	2	$\text{ATD}[588;87], \text{ATD}[588;90], \text{ATD}[588;91]$
$\text{PSL}(2, 7)$	4	$\text{ATD}[294;19]$
$\text{PSL}(2, 7)$	8	order 147; none
$\text{PSL}(2, 7)$	7	$\text{ATD}[168;53], \text{ATD}[168;64], \text{ATD}[168;65], \text{ATD}[168;81], \text{ATD}[168;82]$
$\text{PSL}(2, 17)$	2	$\text{ATD}[612;48], \text{ATD}[612;49]$
$\text{PSL}(2, 17)$	4	$\text{ATD}[306;11]$
$\text{PSL}(2, 17)$	8	order 153; none

Table 3: Possible quotients of Γ by a minimal abelian normal subgroup

follows that $\text{soc}(G)$ is a direct product of non-abelian simple groups (see, for example, [7, Theorem 4.3A]). Since the order of every non-abelian simple group is divisible by at least three distinct primes, and since not both b and c are 2, $\text{soc}(G)$ is a simple normal subgroup of G and is therefore isomorphic to the non-abelian composition factor T of G .

Moreover, G acts faithfully by conjugation on $\text{soc}(G)$ and thus embeds into its automorphism group. Since $\text{soc}(G)$ is isomorphic to either $\text{PSL}(2, 7)$ or $\text{PSL}(2, 17)$, we see that G is isomorphic to one of $\text{PSL}(2, 7)$, $\text{PGL}(2, 7)$, $\text{PSL}(2, 17)$ or $\text{PGL}(2, 17)$. On the other hand, recall that $|G| = 2^{a+s}q^bp^a$ and that $a = 3$ and $s \geq 2$, implying that $|G|$ is divisible by 2^5 . This rules out all but the last possibility, that is $G \cong \text{PGL}(2, 17)$. Since, in this case, $|V(\Gamma)| = 2^3 \cdot 3^2 \cdot 17$ and $|G| = 2^5 \cdot 3^2 \cdot 17$, it follows that $|G_v| = 4$. By Lemma 3.1, G_v is elementary abelian. In particular, Γ is a coset digraph of G with respect to an elementary abelian subgroup of order 4 and a non-self-paired suborbit of length 2. A direct inspection of the appropriate subgroups of $\text{PGL}(2, 17)$ and their coset digraphs reveals that there are six pairwise non-isomorphic digraphs arising in this way. They are listed in Table 1 as digraphs $X_{10}, X_{11}, \dots, X_{15}$. This concludes the proof of Theorem 1.1.

3.3 Proof of Theorem 1.2

We shall say that a positive integer n satisfies condition (i), (ii) or (iii), respectively, if the following holds:

- (i) n is odd and cube-free;
- (ii) $n = 2^am$, where $a \in \{1, 2, 3\}$ and m is an odd, square-free integer;
- (iii) $n = 2^aq^bp^2$, where $a \in \{1, 2, 3\}$, $b \in \{0, 1\}$ and p, q are distinct odd primes.

As in the statement of Theorem 1.2, we assume that Γ is a connected 2-valent $(G, 2)$ -arc-transitive digraph with G solvable, and that one of the conditions (i), (ii) or (iii) holds for $n = |V(\Gamma)|$. We need to show that either:

- (a) Γ is a $\overrightarrow{\text{PX}}$ -digraph; or that
- (b) n satisfies the condition (iii) and G contains a normal Sylow p -subgroup P , which is elementary abelian of order p^2 and such that Γ/P is a $\overrightarrow{\text{PX}}$ -digraph.

Suppose that the theorem is false and let Γ be a minimal counter-example (in terms of n). In particular, Γ is not a $\overrightarrow{\text{PX}}$ -digraph. By Lemma 2.1, Γ is then simple. Since G acts transitively on the vertex-set of Γ and since the vertex-stabiliser G_v is of order 2^s for some $s \geq 2$ (see Lemma 3.1), it follows that $|G| = |G_v|n = 2^sn$.

We shall now prove a few facts about Γ and G , finally resulting in a contradiction.

Fact 0: *If N is a semiregular normal subgroup of G , then Γ/N is a \overrightarrow{PX} -digraph, or $n/|N|$ (and thus also n) satisfies the condition (iii) and the Sylow p -subgroup of G/N is elementary abelian of order p^2 and normal in G/N .*

Proof: Since N is semiregular, by Lemma 2.6, $\Gamma \rightarrow \Gamma/N$ is a covering projection, and by Lemma 2.5, Γ/N is a connected 2-valent $(G/N, 2)$ -arc-transitive digraph. Moreover, since every divisor of an integer satisfying one of the conditions (i), (ii), or (iii) also satisfies one of these conditions, the minimality of the counterexample Γ implies that either Γ/N is a \overrightarrow{PX} -digraph or that $n/|N|$ satisfies the condition (iii) and the Sylow p -subgroup of G/N is indeed as claimed.

Fact 1: *n does not satisfy the condition (i); in particular, n is even.*

Proof: Assume the contrary (that is, n is odd and cube-free). Since n is odd, the vertex-stabiliser in G is a Sylow 2-subgroup of G , and every 2-subgroup of G is contained in some vertex-stabiliser in G .

Let N be a minimal normal subgroup of G . Since G is solvable, N is elementary abelian. If N is a 2-group, then $N \leq G_v$ for some vertex v , and thus the action of G on the vertices of Γ is not faithful, implying that Γ is not simple, a contradiction.

Hence N is an elementary abelian group of odd order, and thus acts semiregularly on the vertices of Γ . By Fact 0, Γ/N is a \overrightarrow{PX} -digraph, and since its order is odd, it must be isomorphic to $\overrightarrow{PX}(n', 0)$ where $n' = n/|N|$. Further, by Lemma 2.9 (note that $\text{rad}(\Gamma/N) = 1 \neq \text{rad}(\Gamma)$), we see that N is not of prime order. Since the order of Γ is cube-free, it follows that N is elementary abelian of order p^2 for some odd prime p .

Let us now consider the group G/N acting on Γ/N . Since $\Gamma/N \cong \overrightarrow{PX}(n', 0)$, by Lemma 2.2, the stabiliser $(G/N)_{v^N}$ of a vertex v^N of Γ/N is elementary abelian and normal in G/N . Note also that $(G/N)_{v^N} = G_v N/N$, implying that $G_v N$ is normal in G .

Let C be the centraliser of N in $G_v N$. If we apply Lemma 3.3 with $X = N$, $Y = G_v N$ and $Z = G$, we see that $C = N \times T$ for some normal subgroup T of G , isomorphic to a subgroup of $Y/N \cong G_v$. In particular, T is a 2-group. Since the order of Γ is odd and T is a 2-group, T fixes a vertex of Γ , and being normal in G , it acts trivially on the vertex-set of Γ . Since Γ is a simple digraph, it follows that $T = 1$, and thus $C = N$.

Since $G_v N$ is normal in G and contains G_v , it contains G_u for every vertex $u \in V(\Gamma)$. In particular, $G_v N$ contains every involution of G . Together with the fact that N is self-centralising in $G_v N$ this implies that no involution of G centralises N .

Now consider the centraliser D of N in G . We have just shown that D has odd order, implying that D is semiregular, and thus, $\Gamma/D \cong \overrightarrow{PX}(n'', 0)$ for some odd integer n'' . Moreover, since G/D acts 2-arc-transitively on Γ/D , the Sylow 2-subgroup S of G/D is the vertex-stabiliser of every vertex of Γ/D , and is thus normal in Γ/D , elementary abelian, and of order at least 4. On the other hand G/D embeds into $\text{Aut}(N) \cong \text{GL}(2, p)$. By Lemma 3.2, it follows that S is of order 4 and contains an involution that is central in G/D . However, by Lemma 2.2, this implies that n'' is even. This contradiction concludes the proof of Fact 1.

Fact 2: *The group G does not contain a normal elementary abelian subgroup of order p^2 for any odd prime p .*

Proof: Assume the contrary and note that in view of Fact 1, n then satisfies the condition (iii); that is, $n = 2^a q^b p^2$ for some $a \leq 3$ and $b \leq 1$. Moreover, G contains a normal

elementary abelian subgroup P of order p^2 . Since p is odd, P is semiregular, and by Fact 0, either Γ/P is a $\overrightarrow{\text{PX}}$ -digraph, or $n/|P| = 2^a q^b$ satisfies the condition (iii). The latter is clearly false, while the former implies that the conclusion (b) of Theorem 1.2 holds for Γ , a contradiction.

Fact 3: $\text{rad}(\Gamma) \geq 3$; that is, the alternating cycles of Γ are of length at least 6.

Proof: Assume the contrary; that is, $\text{rad}(\Gamma) < 3$. Since Γ is simple, we have $\text{rad}(\Gamma) \neq 1$. Hence $\text{rad}(\Gamma) = 2$, and by Lemma 2.11, it follows that $\Gamma \cong \text{Pl}(\Lambda)$ for some connected 2-valent $(G, 3)$ -arc-transitive digraph Λ of order $\frac{1}{2}n$. If Λ is a $\overrightarrow{\text{PX}}$ -digraph, then by formula (2.2), so is Γ , a contradiction. By the minimality of the counterexample Γ , this implies that conclusion (b) holds for the pair (Λ, G) in place of (Γ, G) , and in particular, that G contains a normal elementary abelian subgroup of order p^2 for some odd prime p . However, the latter contradicts Fact 2.

Fact 4: The group G contains no normal subgroup of odd prime order.

Proof: Suppose the contrary and let N be a normal subgroup of G of odd prime order q . Since G_v is a 2-group, N is semiregular. By Fact 3 and Lemma 2.9, Γ/N is not a $\overrightarrow{\text{PX}}$ -digraph. But then Fact 0 implies that $n = 2^a q p^2$ and the Sylow p -subgroup \tilde{P} of G/N is normal in G/N and isomorphic to \mathbb{Z}_p^2 .

Let Q be the preimage of \tilde{P} with respect to the quotient projection $G \rightarrow G/N$. Then Q is a normal subgroup of G of order $q p^2$, containing the normal subgroup N of order q . Let C be the centraliser of N in Q . Since N is abelian and since N has order coprime to its index in Q , we may apply Lemma 3.3 with $Z = G$, $Y = Q$ and $Z = N$, to conclude that $C = N \times P$ for some normal subgroup P of G , isomorphic to a normal subgroup of Q/N . Since the latter is isomorphic to \tilde{P} , we see that P is either trivial, cyclic of order p , or isomorphic to \mathbb{Z}_p^2 .

If P is trivial, then $C = N$, and Q/N embeds into $\text{Aut}(N)$, implying that Q/N is cyclic. However, Q/N is isomorphic to \tilde{P} , which is isomorphic to \mathbb{Z}_p^2 , a contradiction. Further, by Fact 2, the order of P is not p^2 . This leaves us with the possibility that $|P| = p$.

Now consider the quotient Γ/P . Since the order of Γ/P is $2^a q p$, Fact 0 implies that Γ/P is a $\overrightarrow{\text{PX}}$ -digraph. But then, by Lemma 2.9, $\text{rad}(\Gamma) = \text{rad}(\Gamma/P)$, which is at most 2, since Γ/P is a $\overrightarrow{\text{PX}}$ -digraph, contradicting Fact 3.

Fact 5: If N is a minimal normal subgroup of G , then N is semiregular and of order 2 or 4. If $|N| = 2$, then $\Gamma/N \cong \overrightarrow{\text{PX}}(m, 2)$, and if $|N| = 4$, then $\Gamma/N \cong \overrightarrow{\text{PX}}(m, 1)$ for some odd integer m . Moreover, $\exp(\Gamma) = 2$.

Proof: Let m be the odd part of n . By Fact 1, $n = 2^a m$ where $a \geq 1$ and m is cube-free. Let N be a minimal normal subgroup of G . Since G is solvable, N is elementary abelian, and since $|G| = 2^{a+s} m$, Facts 2 and 4 imply that N is a 2-group. If N is not semiregular, then by Lemma 2.12, Γ is a $\overrightarrow{\text{PX}}$ -digraph, contradicting our assumptions. Hence N is semiregular, and thus $|N|$ divides n , and therefore $|N| = 2^t$ for some integer t satisfying $1 \leq t \leq a$.

By Fact 0, either Γ/N is a $\overrightarrow{\text{PX}}$ -digraph, or $n/|N|$ satisfies the condition (iii) and the group G/N contains a normal elementary abelian subgroup \tilde{P} of order p^2 .

Suppose first that the latter case occurs. Then $n/|N| = 2^{a-t} q^b p^2$ where $a - t \geq 1$. Since $a \leq 3$, this implies that $t \in \{1, 2\}$. As in the proof of Fact 4, let Q be the preimage

of \tilde{P} with respect to the quotient projection $G \rightarrow G/N$. Then Q is a normal subgroup of G of order $2^t p^2$, containing the normal subgroup N of order 2^t . Now consider the centraliser C of N in Q , apply Lemma 3.3 with $Z = G$, $Y = Q$ and $X = N$, and conclude that $C = N \times P$ for some (possibly trivial) p -group P which is normal in G . If P is trivial, then $Q/N \cong \tilde{P} \cong \mathbb{Z}_p^2$ embeds into $\text{Aut}(N) \cong \text{GL}(t, 2)$. Since $t \leq 2$, this is clearly not the case. Hence P is non-trivial, contradicting either Fact 2 or Fact 4.

This contradiction shows that the former case occurs, that is $\Gamma/N \cong \overrightarrow{\text{PX}}(2^{a-t-r}m, r)$ for some integer r such that $0 \leq r \leq a - t$. Let $\Lambda = \Gamma/N$ and let $\wp: \Gamma \rightarrow \Lambda$ be the corresponding quotient projection. Since $a \leq 3$ and $t \geq 1$, we see that $r \leq 2$.

If $r = 0$, then Lemma 2.13 implies that Γ is a $\overrightarrow{\text{PX}}$ -digraph, a contradiction.

If $r = 1$, either $a = 2$ and $t = 1$, or $a = 3$ and $t \in \{1, 2\}$. Let $v \in V(\Gamma)$ and let $v' = \wp(v)$. Observe that $\exp(\Lambda) = 1$ (see Lemma 2.14) and $|A_i^\Lambda(v')| = 2$ for every $i \geq 1$. By Lemma 2.8, it follows that $|A_i^\Gamma(v)|$ divides $2^t |A_i^\Lambda(v')| = 2^{t+1} \leq 8$ for every $i \geq 1$. Since $|A_1^\Gamma(v)| = \text{rad}(\Gamma) \geq 3$, it follows that $|A_1^\Gamma(v)| \in \{4, 8\}$. If $|A_1^\Gamma(v)| = |A_2^\Gamma(v)|$, then $\exp(\Gamma) = 1$, and by Lemma 2.4, Γ is a $\overrightarrow{\text{PX}}$ -digraph, a contradiction. Hence $|A_1^\Gamma(v)| < |A_2^\Gamma(v)|$, implying that $|A_1^\Gamma(v)| = 4$ and $|A_i^\Gamma(v)| = 8$ for every $i \geq 2$ (hence $\exp(\Gamma) = 2$). Moreover, since $8 = |A_2^\Gamma(v)|$ divides 2^{t+1} , we see that $t = 2$ and $a = 3$, implying that $\Lambda = \overrightarrow{\text{PX}}(m, 1)$, as claimed.

Similarly, if $r = 2$, then $a = 3$, $t = 1$ and $\Lambda = \overrightarrow{\text{PX}}(m, 2)$. Hence $\exp(\Lambda) = 2$, $|A_1^\Lambda(v')| = 2$ and $|A_i^\Lambda(v')| = 4$ for every $i \geq 2$. Moreover, as above, $|A_1^\Gamma(v)| \geq 3$ and $|A_1^\Gamma(v)| < |A_2^\Gamma(v)|$. In view of Lemma 2.8, it thus follows that $|A_1^\Gamma(v)| = 2|A_1^\Lambda(v')| = 4$, and $|A_2^\Gamma(v)| = |A_\infty^\Gamma(v)| = 8$. In particular, $\exp(\Gamma) = 2$, as claimed. This concludes the proof of Fact 5.

Fact 6: *The order n of Γ is at most 744.*

Proof: Let N be a minimal normal subgroup of Γ and recall Fact 5. Since $\exp(\Gamma) = 2$, [30, Theorem 7.1] implies that $|G_v| \leq 2^4$. By Lemma 2.5, also the stabiliser $(G/N)_{vN}$ has order at most 2^4 . By Lemma 3.4, this implies that G/N contains a normal cyclic group \tilde{Y} whose order is ℓ for some odd integer ℓ satisfying

$$\ell \geq m/(2^{\alpha+4} - 1), \quad (*)$$

where α is either 1 or 2, depending on whether $|N| = 4$ or $|N| = 2$, respectively. Let $Y \leq G$ be the preimage of \tilde{Y} with respect to the quotient projection $G \rightarrow G/N$, let C be the centraliser of N in Y , and apply Lemma 3.3 to deduce that $C = N \times T$ for some cyclic group $T \trianglelefteq G$ of order dividing ℓ . Since T is cyclic, every subgroup of T is characteristic in T and thus normal in G . If T is non-trivial, this implies that G contains a normal subgroup of odd prime order, contradicting Fact 4. Hence $T = 1$, and $C = N$.

If $|N| = 2$, then $\alpha = 2$, N is central in Y , and $Y = C = N$. However, $\ell = |Y|/|N|$, and thus $\ell = 1$. In view of (*), we see that $m \leq 2^{2+4} - 1 \leq 63$, and therefore $n = |N||V(\Gamma/N)| = 2|V(\overrightarrow{\text{PX}}(m, 2))| = 8m \leq 504$.

If $|N| = 4$, then $\alpha = 1$, and by (*), we see that $m \leq 31\ell$, and thus $n = 4|V(\overrightarrow{\text{PX}}(m, 1))| = 8m \leq 248\ell$. On the other hand, since $C = N$, the cyclic group Y/N of order ℓ embeds into $\text{Aut}(N) \cong \text{GL}(2, 2) \cong \text{Sym}(3)$, and thus $\ell \leq 3$. But then $n \leq 3 \cdot 248 = 744$. This concludes the proof of Fact 6.

Since a census of all simple arc-transitive digraphs of valence 2 is available in [28], we can easily see that no counter-example to the theorem of order at most 1000 exists. This,

however, contradicts Fact 6, and thus proves the theorem.

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