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The Petra Šparl Award 2020

The Petra Šparl Award was established in 2017 to recognise in each even-numbered year the best paper published in the previous five years by a young woman mathematician in one of the two journals *Ars Mathematica Contemporanea* (AMC) and *The Art of Discrete and Applied Mathematics* (ADAM). It was named after Dr Petra Šparl, a talented woman mathematician who died mid-career in 2016 after a battle with cancer. The first award was made in 2018 to Dr Monika Pilśniak (AGH University, Poland) for a paper she published in AMC on the distinguishing index of graphs.

Nominations for the 2020 award were invited in 2019, and all cases were considered by a committee (consisting of the three of us, listed below) appointed by the Editors-in-Chief of AMC and ADAM. As judges we were impressed by the large number of papers in these journals over the five years 2015–2019 having a woman author or co-author, with many of these being women in the early stages of their career. With helpful commentaries from co-authors and/or nominators, we drew up a long list of candidates for the 2020 award, sought reports from referees on those, and also considered the papers themselves, before making a decision.

In fact it seemed very appropriate to us to celebrate these awards in a year in which Slovenia was to be hosting the European Congress of Mathematicians, by recommending two awards. Sadly the ECM is being postponed to 2021, but we are proceeding with an announcement of the 2020 awards.

The two winners of the Petra Šparl Award for 2020 are as follows:

- Dr **Simona Bonvicini** (of the Università di Modena e Reggio Emilia, Italy), for her contributions to the paper 'Octahedral, dicyclic and special linear solutions of some Hamilton-Waterloo problems', co-authored with Marco Buratti and published in *Ars Mathematica Contemporanea* **14** (2018), 1–14. This paper provides a solution for each of the nine Hamilton-Waterloo problems with cycles of length three and four, left open by Danziger, Quattrocchi and Stevens in their paper in 2009. Its value lies not only in dealing with the missing cases, but also in the elegant approach taken, and the novel techniques used. Solutions with high degree of symmetry are constructed by taking appropriate groups of automorphisms with sharply vertex-transitive action (which was not an easy task, as only one group for each case could be used). Simona's contribution was described by her co-author Marco Buratti as significant at all stages, showing deep understanding and great skills.

- Dr **Klavdija Kutnar** (of the University of Primorska, Slovenia), for her contributions to the paper on 'Odd automorphisms in vertex-transitive graphs', co-authored with Ademir Hujdurović and Dragan Marušič, and published in *Ars Mathematica Contemporanea* **10** (2016), 427–437. This paper addresses the question of which vertex-transitive graphs admit automorphisms that act as an odd permutation of the vertices. It reports on work that was initiated by Klavdija Kutnar, and is the first paper to consider such a question. The paper provides background motivation, makes some very interesting observations, and poses open questions for further study. Klavdija's two co-authors attest that her contributions to the paper were of utmost importance, and that she provided key ideas in solving various problems. Also the number of citations the paper has received on MathSciNet shows that it has opened up a new line of study, attracting several other researchers in the field.

We heartily congratulate the two awardees.

Also we encourage nominations for the next Petra Šparl Award in 2022, as well as submissions of high quality new papers that will be worthy of consideration for future awards.

Marston Conder, Asia Ivić Weiss and Aleksander Malnič
Members of the *2020 Petra Šparl Award Committee*

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Classification of cubic vertex-transitive tricirculants*

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Abstract

A finite graph is called a tricirculant if admits a cyclic group of automorphism which has precisely three orbits on the vertex-set of the graph, all of equal size. We classify all finite connected cubic vertex-transitive tricirculants. We show that except for some small exceptions of order less than 54, each of these graphs is either a prism of order $6k$ with k odd, a Möbius ladder, or it falls into one of two infinite families, each family containing one graph for every order of the form $6k$ with k odd.

Keywords: Graph, cubic, semiregular automorphism, tricirculant, vertex-transitive.

Math. Subj. Class. (2020): 05E18, 05C25

1 Introduction

All graphs in this paper are finite. A connected graph Γ admitting a cyclic group of automorphisms H having k orbits of vertices of equal size larger than 1 is called a *k-multicirculant* and a generator of H is then called a *k-multicirculant automorphism* of Γ. In particular, 1-, 2- and 3-multicirculants are generally called circulants, bicirculants and tricirculants, respectively. A graph is called cubic if it is connected and regular of valence 3.

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A famous and longstanding *polycirculant conjecture* claims that every vertex-transitive graph and digraph is a k-multicirculant for some k (see [3, 13, 14, 15]). It is particularly interesting to study conditions under which vertex-transitive graphs admit k-multicirculant automorphisms of large order (and thus relatively small k); see, for example, [2, 20]. On the other hand, existence of a k-multicirculant automorphism with small k often restricts the structure of a vertex-transitive graph to the extent that allows a complete classification. There is a series of classification results about cubic arc-transitive k-multicirculant for $k \leq 5$ (see [7, 10]). In particular, cubic arc-transitive tricirculants where completely classified in [10], where it was shown that only 4 such graphs exist: $K_{3,3}$, the Pappus graph, Tutte's 8-cage and a graph on 54 vertices. This work culminated in a beautiful paper [9], where it was shown that for all square-free values of k coprime to 6 there exist only finitely many cubic arc-transitive k-multicirculants.

In this paper we widen the focus to a much wider and structurally richer class of cubic vertex-transitive graphs that are not necessarily arc-transitive. It is know that every cubic vertex-transitive graph has a k-multicirculant automorphisms [16] and that no fixed k exists such that every cubic vertex-transitive graph is a k-multicirculant [20]. As for the classification results, it is clear that a cubic graph is a vertex-transitive 1-circulant if and only if it is a cubic Cayley graph on a cyclic group. Furthermore, vertex-transitive cubic bicirculants were classified in [17]. The goal of this paper is to provide a complete classification of cubic vertex-transitive tricirculants.

The following theorem and remarks are a brief summary of the contents of this paper.

Theorem 1.1. *A cubic graph Γ is a vertex-transitive tricirculant if and only if the order of Γ is $6k$ for some positive integer k and one of the following holds:*

1. *$k > 1$, k is odd and Γ is isomorphic to one of the graphs $\mathrm{X}(k)$ or $\mathrm{Y}(k)$, described in Section 4 (Definition 4.1) and Section 5 (Definition 5.1), respectively.*

2. *Γ is a prism and k is odd, or Γ is a Möbius ladder (defined in Section 6).*

3. *Γ is the truncated tetrahedron or Tutte's 8-cage.*

Remark 1.2. All Möbius ladders are circulants. A prism P_k is a circulant whenever k is odd. Meanwhile $\mathrm{X}(k)$ and $\mathrm{Y}(k)$ are bicirculants if and only if k is odd and $3 \nmid k$. In this case, $\mathrm{X}(k)$ is isomorphic to the generalized Petersen graph $\mathrm{GP}(3k, k + (-1)^\alpha)$ while $\mathrm{Y}(k)$ is isomorphic to the graph $\mathrm{H}(3k, 1, \alpha k + 2)$, where $\alpha \in \{1, 2\}$ and $\alpha \equiv k \pmod 3$ (see [17] for definitions pertaining to bicirculant graphs).

Remark 1.3. $\mathrm{Y}(k)$ can be embedded on the torus yielding the toroidal map $\{6, 3\}_{\alpha,3}$ where $\alpha = \frac{1}{2}(k-1)$ (see [5] for the definitions pertaining to the maps on the torus).

Remark 1.4. For $k \geq 7$, the graphs $\mathrm{X}(k)$ and $\mathrm{Y}(k)$ have girth 8 and 6, respectively. Meanwhile, prisms and Möbius ladders have girth 4. The graph $\mathrm{X}(k)$ is not bipartite for any integer k, while $\mathrm{Y}(k)$ is bipartite for all odd $k > 1$.

Remark 1.5. There are exactly four cubic arc-transitive tricirculants: $K_{3,3}$, the Pappus graph, Tutte's 8-cage and the graph with 54 vertices denoted F054A in [1]. All other cubic vertex-transitive tricirculants have two edge orbits and their arc-type is $2 + 1$ (see [4] for details). There are no semi-symmetric cubic tricirculants.

2 Diagrams

Even though this paper is about simple graphs, it will be convenient in the course of the analysis to have a slightly more general definition, which allows the graphs to have loops, parallel edges and semi-edges. To distinguish between a simple graph (which we will refer to simply as a graph) and these more general objects, that we shall call *pregraphs*. In what follows, we briefly introduce the concept of a pregraph and refer the reader to [11, 12] for more detailed explanation.

A *pregraph* is an ordered 4-tuple $(V, D; \mathrm{beg}, \mathrm{inv})$ where D and $V \neq \emptyset$ are disjoint finite sets of *darts* and *vertices*, respectively, $\mathrm{beg}\colon D \to V$ is a mapping which assigns to each dart x its *initial vertex* $\mathrm{beg}\ x$, and $\mathrm{inv}\colon D \to D$ is an involution which interchanges every dart x with its *inverse dart*, also denoted by x^{-1}. The *neighbourhood* of a vertex v is defined as the set of darts that have v for its initial vertex and the *valence* of v is the cardinality of the neighbourhood. Note that a simple graph Γ can be represented as a pregraph by letting $V = \mathrm{V}(\Gamma)$ be the vertex-set of Γ, letting $D = \{(u, v) : u, v \in \mathrm{V}(\Gamma), u \sim_\Gamma v\}$ be the arc-set of Γ, and by letting $\mathrm{beg}(u, v) = u$ and $\mathrm{inv}(u, v) = (v, u)$. Notions such as morphism, isomorphism and automorphism of pregraphs are obvious generalisations of those for (simple) graphs and precise definitions can be found in [11, 12].

The orbits of inv are called *edges*. The edge containing a dart x is called a *semi-edge* if $\mathrm{inv}\ x = x$, a *loop* if $\mathrm{inv}\ x \neq x$ while $\mathrm{beg}\,(x^{-1}) = \mathrm{beg}\ x$, and is called a *link* otherwise. The *endvertices of an edge* are the initial vertices of the darts contained in the edge. Two links are *parallel* if they have the same endvertices. When we present a pregraph as a drawing, the links are drawn in the usual way as a line between the points representing its endvertices, a loop is drawn as a closed curve at its unique endvertex and a semi-edge is drawn as a segment attached to its unique endvertex. The following lemma, which will serve as the starting point of our analysis of cubic tricirculants, is an easy exercise illustrating the definition of a pregraph.

Lemma 2.1. *Up to isomorphism there are precisely four non-isomorphic cubic pregraphs on three vertices such that no vertex is the initial vertex of more than one semi-edge, namely the pregraphs $\Delta_1, \Delta_2, \Delta_3$ and Δ_4 depicted in Figure 1.*

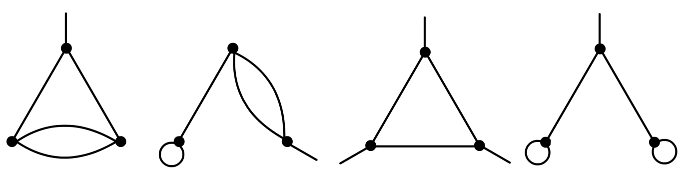


Figure 1: Cubic pregraphs on three vertices without parallel semi-edges.

Given a pregraph Γ and a group $H \leq \mathrm{Aut}(\Gamma)$ we define the quotient Γ/H to be the pregraph $(V', D'; \mathrm{beg}', \mathrm{inv}')$ where V' and D' are the sets $\mathrm{V}(\Gamma)/H$ and $\mathrm{D}(\Gamma)/H$ of orbits of H on V and D, respectively, and beg' and inv' are defined by $\mathrm{beg}'(x^H) = (\mathrm{beg}\,x)^H$ and $\mathrm{inv}'(x^H) = (\mathrm{inv}\,x)^H$ for every $x \in \mathrm{D}(\Gamma)$. The mapping $\wp_H\colon\ \mathrm{V}(\Gamma) \cup \mathrm{D}(\Gamma) \to V' \cup D'$ that maps a vertex or a dart x to its H-orbit x^H is then an epimorphism of pregraphs, called *the quotient projection with respect to H*. If H acts semiregularly on $\mathrm{V}(\Gamma)$ (that is, if the vertex-stabiliser H_v is trivial for every vertex $v \in \mathrm{V}(\Gamma)$), then the quotient projection $\wp_H$

is a *regular covering projection* and in particular, it preserves the valence of the vertices. Moreover, in this case the graph Γ is isomorphic to the *derived covering graph* of Γ/H, a notion which we now define.

Let $\Gamma' = (V', D', \mathrm{beg}', \mathrm{inv}')$ be an arbitrary connected pregraph, let N be a group and let $\zeta\colon D' \to N$ be a mapping (called a *voltage assignment*) satisfying the condition $\zeta(x) = \zeta(\mathrm{inv}' x)^{-1}$ for every $x \in D'$. Then $\mathrm{Cov}(\Gamma', \zeta)$ is the graph with $D' \times N$ and $V' \times N$ as the sets of darts and vertices, respectively, and the functions beg and inv defined by $\mathrm{beg}(x, a) = (\mathrm{beg}' x, a)$ and $\mathrm{inv}(x, a) = (\mathrm{inv}' x, a\zeta(x))$. If, for a voltage assignment $\zeta\colon \mathrm{D}(\Gamma') \to N$, there exists a a spanning tree T in Γ' such that $\zeta(x)$ is the trivial element of N for every dart x in T, then we say that ζ is *normalised*; note that in this case $\mathrm{Cov}(\Gamma', \zeta)$ is connected if and only if the images of ζ generate N. The following lemma is a well-known fact in the theory of covers:

Lemma 2.2. *Let H be a group of automorphisms acting semiregularly on the vertex-set of a connected graph Γ, let $\Gamma' = \Gamma/H$ and let T be a spanning tree in Γ'. Then there exists a voltage assignment $\zeta\colon \mathrm{D}(\Gamma') \to N$ which is normalised with respect to T and such that $\Gamma \cong \mathrm{Cov}(\Gamma', \zeta)$.*

Let us now move our attention back to cubic tricirculants. Consider the voltage assignments $\zeta_i\colon \Delta_i \to \mathbb{Z}_{2k}$ given in Figure 2 where the value $\zeta_i(x)$ is written next to the drawing of each dart $x \in \mathrm{D}(\Delta_i)$.

Lemma 2.3. *Let Γ be a cubic tricirculant, let ρ be a corresponding tricirculant automorphism and let n be the order of ρ. Then $n = 2k$ for some positive integer k and there exist elements $r, s \in \mathbb{Z}_n$ and $i \in \{1, 2, 3, 4\}$ such that $\Gamma \cong \mathrm{Cov}(\Delta_i, \zeta_i)$ where $\zeta_i\colon \mathrm{D}(\Delta_i) \to \mathbb{Z}_n$ is the voltage assignment defined by Figure 2.*

Proof. Note first that the quotient $\Gamma/\langle\rho\rangle$ is a pregraph with three vertices of valency 3, one for each orbit under the action of $\langle\rho\rangle$. Since ρ is a semiregular automorphism of Γ of order n, the group $\langle\rho\rangle$ is isomorphic to $\mathbb{Z}_n$ and acts semiregularly on $\mathrm{V}(\Gamma)$. By Lemma 2.2, $\Gamma \cong \mathrm{Cov}(\Gamma/\langle\rho\rangle, \zeta)$ for some voltage assignment $\zeta\colon\ \mathrm{D}(\Gamma/\langle\rho\rangle, \mathbb{Z}_n)$. Note that by definition of voltage assignments, it follows that $\zeta(x) = -\zeta(x^{-1})$ for every $x \in \mathrm{D}(\Gamma/\langle\rho\rangle)$, implying that if x is a semi-edge, then $\zeta(x)$ is an element of order at most 2 in $\mathbb{Z}_n$, and since Γ has no semi-edges, $\zeta(x)$ must in fact have order 2 in $\mathbb{Z}_n$. Since every pregraph with three vertices in which every vertex has valence 3 contains at least one semi-edge, it follows that $n = 2k$ for some positive integer k, and moreover, $\zeta(x) = k$ for every semi-edge x of $\Gamma/\langle\rho\rangle$. Since Γ has no parallel edges, this also implies that every vertex of $\Gamma/\langle\rho\rangle$ is the initial vertex of at most one semi-edge. By Lemma 2.1, $\Gamma/\langle\rho\rangle \cong \Delta_i$ for some $i \in \{1, 2, 3, 4\}$ and we may thus assume that $\Gamma \cong \mathrm{Cov}(\Delta_i, \zeta_i)$ for some $i \in \{1, 2, 3, 4\}$ and some voltage assignment $\zeta_i\colon \Delta_i \to \mathbb{Z}_n$. Finally, in view of Lemma 2.2, we may assume that $\zeta_i(x) = 0$ for every edge x belonging to a chosen spanning tree of Δ_i. In particular, ζ_i can be chosen as shown in Figure 2. $\square$

A cubic tricirculant isomorphic to $\mathrm{Cov}(\Delta_i, \zeta_i)$, $i \in \{1, 2, 3, 4\}$, is said to be of *Type i* and is denoted by $T_i(k, r, s)$, if $i \neq 3$, or $T_3(k, r)$. From Lemma 2.3 we get the following characterization of connected cubic tricirculants.

Theorem 2.4. *A cubic graph Γ is a tricirculant if and only if $n = 6k$ and it is isomorphic to $T_i(k, r, s)$ or $T_3(k, r)$, for some $i \in \{1, 2, 4\}$ and $r, s \in \mathbb{Z}_{2k}$. Furthermore, $T_i(k, r, s)$*

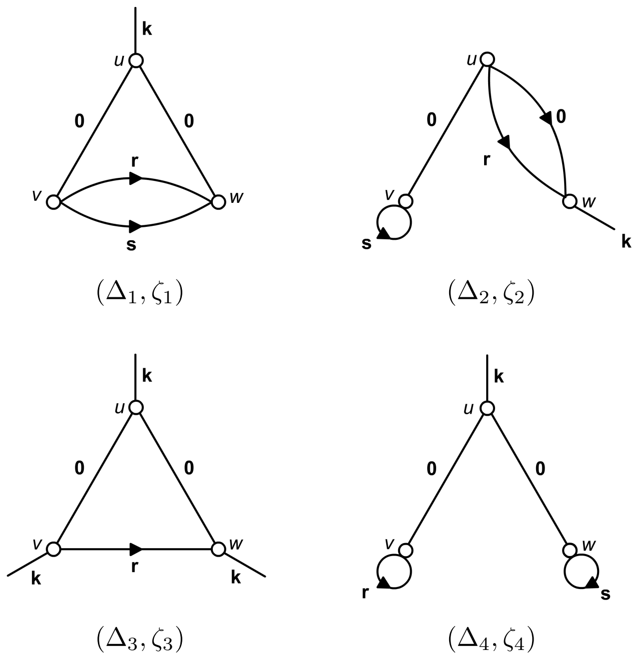


Figure 2: The voltage assignments giving rise to cubic tricirculants.

is connected if and only if $\gcd(2k, k, r, s) = 1$ *while* $T_3(k, r)$ *is connected if and only if* $\gcd(2k, k, r) = 1$.

Note that in principle, a cubic tricirculant could be of more than one type. However, each vertex-transitive cubic tricirculant is of Type i for exactly one $i \in \{1, 2, 3, 4\}$, with the exception of the triangular prism P_3 which is both of Type 1 as well as of Type 3.

The following sections are devoted to the analysis of the tricirculants graphs arising from the voltage assignments ζ_i, and in particular, to determining sufficient and necessary conditions for vertex-transitivity. We will consider the graphs of order at most 48 separately in Section 3 and as for the graphs of larger order, we will show that no graph of Type 4 is vertex-transitive, that Type 3 yields prisms and Möbius ladders, and that Types 1 and 2 each yield one infinite family of vertex-transitive graphs, namely the graphs $\mathrm{X}(k)$ defined in Definition 4.1 and the graphs $\mathrm{Y}(k)$ defined in Definition 5.1.

Finally, to facilitate discussion in the forthcoming sections, we specify the notion of a walk. A walk ω in a (pre)graph is a sequence of darts $(x_1, x_2, \ldots, x_n)$ such that $\mathrm{beg}(x_{i+1}) = \mathrm{beg}(x_i^{-1})$ for all $i \in \{1, \ldots, n-1\}$; that is, the initial vertex of x_{i+1} is the end vertex of x_i. We say that ω is closed if the initial vertex of x_1 is the endvertex of x_n. If additionally $\mathrm{beg}(x_i) \neq \mathrm{beg}(x_j)$ for all $i \neq j$, then we say that ω is a cycle. We define the inverse of ω as the walk $\omega^{-1} = (x_n^{-1}, x_{n-1}^{-1}, \ldots, x_1^{-1})$. We say that a walk ω is a *reduced walk* if no two consecutive arcs are inverse to one another (if ω is closed, we will consider the first arc and the last arc of ω to be consecutive). It is pertinent to point out that, from this definition, all walks and cycles are directed and have an initial vertex. Therefore, no reduced walk is equal to its inverse.

3 Graphs of small order

To make this classification as general and as neat as possible, we will treat cubic tricirculant graphs of small order separately, as if we allow graphs that are "too small", special cases and exceptions will inevitably occur. Since a cubic tricirculant must have order $6k$, for some positive k, we present, in Table 1, the complete list of all cubic vertex-transitive tricirculants of order at most 48 obtained from the census [18] of cubic vertex-transitive graphs. Notice that with the exception of the truncated tetrahedron, $\mathrm{Tr}(K_4)$, of order 12 and Tutte's 8-cage, T8, of order 30 (see Figure 3), all vertex-transitive cubic tricirculants of order at most 48 are isomorphic to either $\mathrm{X}(k)$, $\mathrm{Y}(k)$, P_k (the prism of order $2k$) or M_k (the Möbius ladder of order $2k$) for some integer k.

Table 1: Graphs of small order.

Order	Type	Graph	Order	Type	Graph	Order	Type	Graph
6	$1,3$	P_3	18	3	M_9	36	3	M_{18}
6	3	M_3	24	3	M_{12}	42	1	$\mathrm{X}(7)$
12	1	$\mathrm{Tr}(K_4)$	30	1	$\mathrm{X}(5)$	42	2	$\mathrm{Y}(7)$
12	3	M_6	30	2	$\mathrm{Y}(5)$	42	3	P_{21}
18	1	$\mathrm{X}(3)$	30	3	P_{15}	42	3	M_{21}
18	2	$\mathrm{Y}(3)$	30	3	M_{15}	48	3	M_{24}
18	3	P_9	30	4	T8			

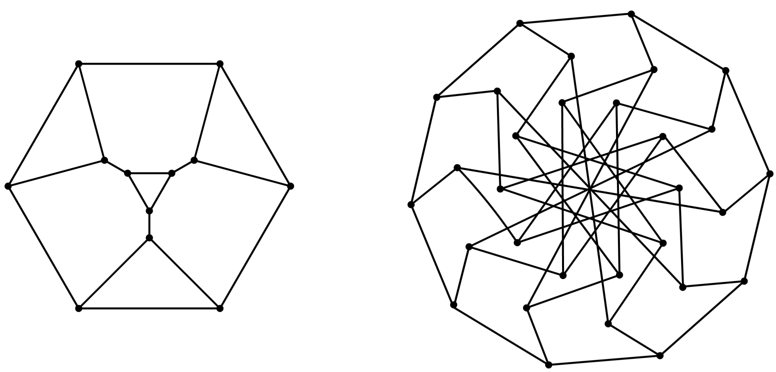


Figure 3: The truncated tetrahedron $\mathrm{Tr}(K_4)$ and Tutte's 8-cage.

The graphs $K_{3,3}$, $\mathrm{Y}(3)$ (the Pappus graph) and Tutte's 8-cage are arc-transitive. The fourth arc-transitive cubic tricirculant, denoted F054A in [1], does not appear in Table 1, as it has 54 vertices; it corresponds to the graph $\mathrm{Y}(9)$ defined in Section 5. The four aforementioned graphs are the only edge-transitive cubic tricirculants. Indeed, a cubic graph that is both vertex- and edge-transitive is automatically arc-transitive [22]. A graph that is edge-transitive but not vertex-transitive is called a semi-symmetric graph, and must have two orbits of vertices under its full automorphism group [6]. It follows that no tricirculant can be semi-symmetric, as its tricirculant automorphism would mix the two vertex orbits, making the graph vertex-transitive.

For the remainder of this paper, all cubic tricirculants will have order at least 54.

4 Type 1

Let $k \geq 9$ be an integer, and let r and s be two distinct elements of $\mathbb{Z}_{2k}$. Recall that $T_1(k, r, s)$ is the covering graph arising from (Δ_1, ζ_1) shown in Figure 2; for convenience, we repeat the drawing here (see Figure 4).

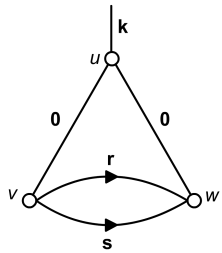


Figure 4: The voltage assignment giving rise to the graph $T_1(k, r, s)$.

By the definition of a derived cover, the vertices of $T_1(k, r, s)$ are then the pairs (x, i) where $x \in \{u, v, w\}$ is a vertex of Δ_1 and $i \in \mathbb{Z}_{2k}$. Note that $T_1(k, r, s)$ is connected if and only if k, r and s generate $\mathbb{Z}_{2k}$, that is, if and only if $\gcd(k, r, s) = 1$.

To simplify notation, we write u_i instead of (u, i) and similarly for v_i and w_i. Then $U = \{u_0, u_1, \ldots, u_{2k-1}\}$, $V = \{v_0, v_1, \ldots, v_{2k-1}\}$ and $W = \{w_0, w_1, \ldots, w_{2k-1}\}$ are the respective fibres of vertices u, v and w, and the edge-set of $T_1(k, r, s)$ is then the union of the sets

$$\begin{aligned} E_K &= \{u_i u_{i+k} : i \in \mathbb{Z}_{2k}\}, \\ E_0 &= \{u_i v_i : i \in \mathbb{Z}_{2k}\} \cup \{u_i w_i : i \in \mathbb{Z}_{2k}\}, \\ E_R &= \{v_i w_{i+r} : i \in \mathbb{Z}_{2k}\}, \\ E_S &= \{v_i w_{i+s} : i \in \mathbb{Z}_{2k}\}. \end{aligned}$$

Definition 4.1. For an odd integer $k > 1$, let

$$r^* = \begin{cases} \frac{k+3}{2}, & \text{if } k \equiv 1 \pmod 4 \\ \frac{k+3}{2} + k, & \text{if } k \equiv 3 \pmod 4 \end{cases}$$

and let $\mathrm{X}(k) = T_1(k, r^*, 1)$.

We can now state the main theorem of this section.

Theorem 4.2. *Let $k \geq 9$ and let Γ be a connected cubic tricirculant of Type 1 with $6k$ vertices. Then the following holds:*

(1) Γ is vertex-transitive if and only if it is isomorphic to $\mathrm{X}(k)$ with k odd.

(2) If Γ is not vertex-transitive, then it has two vertex orbits under its full automorphism group, one twice the size of the other.

The next theorem gives more information about the graph $\mathrm{X}(k)$.

Theorem 4.3. *Let k be an odd integer, $k > 1$. Then the following holds:*

(1) $\mathrm{X}(k)$ *is a bicirculant if and only if* $3 \nmid k$, *in which case it is isomorphic to the generalized Petersen graph* $\mathrm{GP}(3k, k + (-1)^{\alpha})$ *where* $\alpha \in \{1, 2\}$ *and* $\alpha \equiv k \pmod 3$.

(2) $\mathrm{X}(k)$ *has arc-type* $2 + 1$.

The rest of the section is devoted to the proof of Theorems 4.2 and 4.3. Clearly, as a tricirculant of Type 1, Γ is isomorphic to $T_1(k, r, s)$ for some integer k and elements $r, s \in \mathbb{Z}_{2k}$ (see Lemma 2.3). We shall henceforth assume that $\Gamma = T_1(k, r, s)$ for some $k \geq 9$ and that Γ is connected. For a symbol X from the set of symbols $\{0, R, S, K\}$, edges in E_X will be called edges of type X, or simply X-edges.

Define ρ as the permutation given by $\rho(u_i) = u_{i+1}$, $\rho(v_i) = v_{i+1}$ and $\rho(w_i) = w_{i+1}$, and note that ρ is a tricirculant automorphism of $T_1(k, r, s)$. Further, observe that

$$T_1(k, r, s) \cong T_1(k, s, r), \tag{4.1}$$

and if $a \in \mathbb{Z}$ is such that $\gcd(2k, a) = 1$, then

$$T_1(k, ar, as) \cong T_1(k, r, s).$$

Lemma 4.4. *Let* k *be an odd integer,* $k > 1$, *and let* r^* *and* $\mathrm{X}(k)$ *be as in Definition 4.1. Then the graph* $\mathrm{X}(k)$ *is vertex-transitive.*

Proof. Recall that $\mathrm{X}(k) \cong T_1(k, r^*, 1)$. Define ϕ as the mapping given by:

$$\begin{array}{llll} u_i \mapsto w_{i-r^*+2}, & w_i \mapsto v_{i-2r^*+2}, & v_i \mapsto u_{i-r^*+2} & \text{if } i \text{ is even;} \\ u_i \mapsto v_{i+r^*-2}, & v_i \mapsto w_{i+2r^*-2}, & w_i \mapsto u_{i+r^*-2} & \text{if } i \text{ is odd.} \end{array}$$

That ϕ is indeed an automorphism follows from the fact that k is odd, r^* is even and the congruence $k + 3 - 2r^* \equiv 0$ holds. Since ϕ transitively permutes the three $\langle\rho\rangle$-orbits U, V and W, the group $\langle\rho, \phi\rangle$ acts transitively on the vertices of $T_1(k, r^*, 1)$. □

Lemma 4.5. *For* $k > 1$, $\mathrm{X}(k)$ *has arc-type* $2 + 1$.

Proof. Let G be the full automorphism group of $\mathrm{X}(k)$ and consider the automorphism φ interchanging u_i with u_{-i} and v_i with w_{-i}, $i \in \mathbb{Z}_{2k}$. Note that φ is an element of G_{u_0}, the stabiliser of u_0. Moreover, φ fixes u_k but interchanges v_0 with w_0. This is, φ fixes one of the neighbours of u_0 while interchanging the remaining two. This implies that the arc-type of $\mathrm{X}(k)$ is either 3, when $\mathrm{X}(k)$ is arc-transitive, or $2 + 1$. However, there are no arc-transitive graphs of Type 1. The result follows. □

Remark 4.6. The automorphism given in the proof above is an automorphism of $\Gamma \cong T_1(k, r, s)$, from which we see that if Γ is not vertex-transitive, then $V \cup W$ is a vertex orbit under its full automorphism group $\mathrm{Aut}(\Gamma)$. Then Γ must have two vertex orbits: U and $V \cup W$. This shows that item (2) of Theorem 4.2 holds.

Now, recall that Γ is connected and equals $T_1(k, r, s)$, for some $k \geq 9$, $r, s \in \mathbb{Z}_{2k}$. We may also assume that Γ is vertex-transitive, however, for some of the results in this section vertex-transitivity is not needed. When possible, we will not assume the graph to be vertex-transitive, but rather only to have some weaker form of symmetry, that we will define in the following paragraphs.

A simple graph Γ is said to be *c-vertex-regular*, for some $c \geq 3$, if there are the same number of c-cycles through every vertex of Γ.

For an edge e of Γ and a positive integer c denote by $\epsilon_c(e)$ the number of c-cycles that pass through e. For a vertex v of Γ, let $\{e_1, e_2, e_3\}$ be the set of edges incident to v ordered in such a way that $\epsilon_c(e_1) \leq \epsilon_c(e_2) \leq \epsilon_c(e_3)$. The triplet $(\epsilon_c(e_1), \epsilon_c(e_2), \epsilon_c(e_3))$ is then called the *c-signature* of v. If for $c \geq 3$ all vertices in Γ have the same c-signature, then we say Γ is *c-cycle-regular* and the signature of Γ is the signature of any of its vertices. Note that for every $c \geq 3$, a vertex-transitive graph is necessarily c-cycle-regular, and every c-cycle-regular graph is c-vertex-regular.

If c equals the girth of the graph, then following [19], a c-cycle-regular graph will be called *girth-regular*.

Lemma 4.7. *If Γ is 4-vertex-regular, then neither r nor s equals k.*

Proof. Suppose $r = k$. Since Γ has no parallel edges, we see that $s \neq k$. Observe that (u_0, v_0, w_r, u_r) and $(u_0, w_0, v_{-r}, u_{-r})$ are the only 4 cycles of Γ through u_0.

Meanwhile, there exists a unique 4-cycle through v_0, namely (v_0, w_r, u_r, u_0), which contradicts Γ being 4-vertex-regular. Hence $r \neq k$, and since $T_1(k, s, r) \cong T_1(k, r, s)$ (see Equation (4.1)), this also shows that $s \neq k$. □

For the sake of simplicity denote a dart in Δ_1 starting at vertex a, pointing to vertex b and having voltage x by $(ab)_x$. Recall that a walk in Δ_1 is a sequence of darts $(x_1, x_2, \ldots, x_n)$, for instance $((vw)_r, (wu)_0, (uu)_k)$ is a walk of length 3. However, since Γ is a simple graph a walk in Γ will be denoted as a sequence of vertices, as it is normally done.

Lemma 4.8. *If Γ is 8-cycle-regular, then neither r nor s equals 0.*

Proof. Suppose $r = 0$. Since Γ has no parallel edges, we see that $s \neq 0$. Suppose there is an 8-cycle C in Γ. Such a cycle, when projected to Δ_1, yields a reduced closed walk ω in Δ_1 whose ζ_1-voltage is 0. Note that ω cannot trace three darts with voltage 0 consecutively, as this would lift into a 3-cycle contained in C. This implies ω necessarily visits the dart $(vw)_s$ or its inverse at least once. Furthermore, since $\gcd(k, s) = 1$ and $8 < 9 \leq k$, ω must trace $(wv)_{-s}$ as many times as it does $(vw)_s$. By observing Figure 4, the reader can see that if ω traces the dart $(vw)_s$, then it must also trace the semi-edge $(uu)_k$ before tracing $(wv)_{-s}$, as ω cannot trace three consecutive darts with voltage 0. Since ω has net voltage 0, it necessarily traces $(uu)_k$ an even number of times. Moreover, ω must trace a dart in $\{(uv)_0, (vu)_0, (uw)_0, (wu)_0\}$ immediately before and immediately after tracing $(uu)_k$. Hence, ω must visit the set $\{(vw)_s, (wv)_{-s}\}$ at least twice; the semi-edge $(uu)_k$ at least twice; and the set $\{(uv)_0, (vu)_0, (uw)_0, (wu)_0\}$ at least 4 times. This already amounts to 8 darts, none of which is $(vw)_r$ or its inverse. Therefore, no 8-cycle in Γ visits an R-edge. However, for $X \neq R$ there is at least one 8-cycle through every X-edge, as $(u_0, v_0, w_s, u_s, u_{s+k}, w_{s+k}, v_k, u_k)$ and $(u_0, w_0, v_{-s}, u_{-s}, u_{k-s}, v_{k-s}, w_k, u_k)$ are 8-cycles in Γ. This contradicts our hypothesis of Γ being 8-cycle-regular. The proof when $s = 0$ is analogous. □

Now, observe that $(u_0, u_k, v_k, w_{k+r}, u_{k+r}, u_r, w_r, v_0, u_0)$ is a cycle of length 8 in Γ starting at u_0. In what follows we will study the 8-cycle structure of Γ to determine conditions for vertex-transitivity. Recall that each 8-cycle in Γ quotients down into a closed walk of length 8 having net voltage 0 in Δ_1. We can thus, provided we are careful, determine how many 8-cycles pass through any given vertex of Γ by counting closed walks of length 8 and net voltage 0 in the quotient Δ_1. Note that only closed walks that are reduced will lift into cycles of Γ. Therefore, we may safely ignore non-reduced walks and focus our attention exclusively on reduced ones. Define $\mathcal{W}_8$ as the set of all reduced closed walks of length 8 in Δ_1.

Every element of $\mathcal{W}_8$ having net voltage 0 lifts into a closed walk of length 8 in Γ. In principle, the latter walk might not be a cycle of Γ. However, we will show that this never happens in our particular case. For this, it suffices to show that Γ does not admit cycles of length 4 or smaller, as any closed walk of length 8 that is not a cycle is a union of smaller cycles, one of which will have length smaller or equal to 4.

Lemma 4.9. Γ *has girth at least* 5.

Proof. Suppose Γ has a 3-cycle. Since Γ is vertex-transitive, this would mean that there is a reduced closed walk with net voltage 0 of length 3 in Δ_1 that visits u. From Figure 4 we get that such a closed walk must be either $((uv)_0, (vw)_r, (wu)_0)$, $((uv)_0, (vw)_s, (wu)_0)$ or one of their inverses. This would imply that either $r = 0$ or $s = 0$, a contradiction. Similarly, the existence of 4-cycles in Γ would imply there is a closed walk with net voltage 0 of length 4 visiting u. The only such walks are

$$((uv)_0, (vw)_r, (wu)_0, (uu)_k)), \qquad ((uu)_k, (uv)_0, (vw)_r, (wu)_0),$$
$$((uv)_0, (vw)_s, (wu)_0, (uu)_k)), \qquad ((uv)_0, (vw)_s, (wu)_0, (uu)_k))$$

and their inverses. In all eight cases, we have that $r = k$ or $s = k$, contradicting Lemma 4.7. □

Now, let N be the set of the net voltages of walks in $\mathcal{W}_8$, expressed as linear combinations of k, r and s, where we view r and s as indeterminants over $\mathbb{Z}_{2k}$. For instance, the closed walk $((uv)_0, (vw)_r, (wv)_{-s}, (vw)_r, (wu)_0, (uv)_0, (vw)_r, (wu)_0)$ has net voltage $3r - s$. For an element $\nu \in N$, let $\mathcal{W}(\nu)$ be the set of walks in $\mathcal{W}$ with net voltage ν and for a vertex x of Δ_1, let $\mathcal{W}_8^x(\nu)$ be the set of walks in $\mathcal{W}(\nu)$ starting at x.

For each $\nu \in N$, we have computed the number of elements in $\mathcal{W}_8^u(\nu)$ and in $\mathcal{W}_8^v(\nu)$. The result is displayed in Table 2. Notice that, if $\omega \in \mathcal{W}_8(\nu)$, then $\omega^{-1} \in \mathcal{W}_8^x(-\nu)$. This means ω has voltage 0 if and only if ω^{-1} does too. For this reason, and since we are interested in walks with net voltage 0, we have grouped walks with net voltage ν along with their inverses having net voltage $-\nu$. This computation is straightforward but somewhat lengthy. For this reason and to avoid human error, we have done it with the help of a computer programme written in SageMath [21].

From the first Row of Table 2 we know that there are always 12 walks starting at u in $\mathcal{W}_8$ that have net voltage 0, regardless of the values of r and s. Similarly, there are always 10 such walks starting at v. Since Γ is vertex-transitive, the number of walks in $\mathcal{W}_8$ having net voltage 0 (after evaluating r and s in $\mathbb{Z}_{2k}$) starting at u and those starting at v must be the same. It follows that, for some $\nu \in N$ with $|\mathcal{W}_8^v(\nu)| > |\mathcal{W}_8^u(\nu)|$, the equation $\nu \equiv 0 \pmod{2k}$ holds when we evaluate r and s in $\mathbb{Z}_{2k}$. We thus see that at least one expression in VIII – XIII is congruent to 0 modulo $2k$. In fact, if Γ is vertex-transitive, then at most one of these expressions can be congruent to 0.

Table 2: Net voltages of closed walks of length 8 in Δ_1.

Label	Net voltage	Starting at u	Starting at v
I	0	12	10
II	$\pm(2r)$	8	8
III	$\pm(2s)$	8	8
IV	$\pm(r+s)$	8	4
V	$\pm(r-s)$	8	4
VI	$\pm(k+2r-s)$	12	10
VII	$\pm(k+2s-r)$	12	10
VIII	$\pm(k+3r-2s)$	4	6
IX	$\pm(k+3s-2r)$	4	6
X	$\pm(3r-s)$	4	6
XI	$\pm(3s-r)$	4	6
XII	$\pm(2r-2s)$	4	6
XIII	$\pm(4r-4s)$	0	2

Lemma 4.10. *If Γ is 4-vertex-regular and 8-cycle-regular, then exactly one of the following equations modulo $2k$ holds:*

$$3s-2r+k \equiv 0, \tag{4.2}$$

$$3r-2s+k \equiv 0, \tag{4.3}$$

$$3r-s \equiv 0, \tag{4.4}$$

$$3s-r \equiv 0, \tag{4.5}$$

$$4r-4s \equiv 0. \tag{4.6}$$

Proof. We will slightly abuse notation and, for a label $x \in \{\mathrm{I}, \mathrm{II}, \ldots, \mathrm{XIII}\}$, we will refer by x to the congruence equation modulo $2k$ obtained by making the expression labelled x in Table 2 congruent to 0. For instance, the equation $2r \equiv 0$ will be referred to as II.

First note that if II or III holds then $r = k$ or $s = k$, contradicting Lemma 4.7. If V holds then (v_0, w_r, v_0) is a 2-cycle in Γ, which is not possible. Similarly, XII implies the existence of a 4-cycle in Γ, contradicting Lemma 4.9. Now, if XIII holds, then either $2r-2s \equiv 0$, which is not possible, or $2r-2s+k \equiv 0$, which gives Equation (4.6).

If VI holds, then neither X nor XI can hold as this would imply $2r-2s \equiv 0$ (subtracting IV from X or from XI, respectively). Therefore, IV excludes X and XI. If IV and I are the only equations to hold, we would have 6 more elements of $\mathcal{W}_8$ through u than through v. This means that if IV holds then necessarily VII, IX and XII also hold. However, XII implies $2r-2s+k \equiv 0$ and subtracting this from VIII we get $r = 0$, in contradiction with Lemma 4.8. Hence, VII, IX and XII cannot all hold at the same time and so IV can never hold.

Suppose VI holds, then one of the equations in $\{\mathrm{VIII}, \mathrm{IX}, \mathrm{X}, \mathrm{XI}, \mathrm{XIII}\}$ must also hold. Note that VI and VIII imply III; VI and IX imply V; VI and X imply $r = k$; VI and XIII imply $s = k$. Therefore, by Lemma 4.7, only XI can hold. But VI and XI imply VII, and so we would have 40 elements of $\mathcal{W}_8$ starting at u but only 36 starting at v. This would contradict Γ being 8-cycle-regular. Thus VI cannot hold. An analogous reasoning, where the roles of r and s are interchanged, shows that VII cannot hold. We have shown that the only equations that can hold are in $\{\text{VIII}, \text{IX}, \text{X}, \text{XI}, \text{XIII}\}$. However, if two or more of these equations hold, then we would have more elements of $\mathcal{W}_8$ through v than we would have through u. It follows that exactly one equation in $\{\text{VIII}, \text{IX}, \text{X}, \text{XI}, \text{XIII}\}$ holds. $\square$

Lemma 4.10 tells us, for each of the 5 possible equations, exactly which walks in $\mathcal{W}_8$ will lift to 8-cycles and so we can count exactly how many 8-cycles pass through a given vertex or edge of Γ. For instance, if $3s - 2r + k \equiv 0$, then there are 16 walks in $\mathcal{W}_8$ through u that will lift to an 8-cycle (see Table 2). Since a closed walk in $\mathcal{W}_8$ and its inverse lift to the same cycle in Γ, we see that there are 8 cycles of length 8 through every vertex in Γ. Moreover, there are exactly 5 cycles of length 8 through every edge of type R or $\mathit{0}$ while there are 6 such cycles through each edge of type K or S. It follows that the 8-signature of a vertex x of Γ is $(5,5,6)$ whenever $3s-2r+k \equiv 0$ and thus, in this case, Γ is 8-cycle-regular with 8-signature $(5,5,6)$. Table 3 shows the number of 8-cycles through each vertex and through each edge, depending on its type, for each of the 5 cases described in Lemma 4.10.

Table 3: Number of 8-cycles through each edge-type of $T_1(k,r,s)$.

Congruence	$\mathit{0}$-edge	R-edge	S-edge	K-edge	8-signature
$3s-2r+k$	5	5	6	6	$(5,5,6)$
$3r-rs+k$	5	6	5	6	$(5,5,6)$
$3r-s$	6	6	4	4	$(4,6,6)$
$3s-r$	6	4	6	4	$(4,6,6)$
$4r-4s$	4	4	4	4	$(4,4,4)$

We will now show that Equations (4.4), (4.5) and (4.6) cannot hold when Γ is vertex-transitive. This will be proved in Lemmas 4.12, 4.13 and 4.14. But first we need to show a result about the subgraph induced by the $\mathit{0}$- and R-edges.

Recall that that $\Gamma = T_1(k,r,s)$ with $k \geq 9$ and that it is connected. Henceforth we also assume that Γ is vertex-transitive. Denote by $\Gamma_{\mathit{0},R}$ the subgraph that results from deleting the edges of type S and K from Γ. Equivalently, $\Gamma_{\mathit{0},R}$ is the subgraph induced by $\mathit{0}$- and R-edges. We see that $\Gamma_{\mathit{0},R}$ is 2-valent and that it has $\gcd(2k,r)$ connected components, each of which is a cycle of length $6k/\gcd(2k,r)$.

Since $\Gamma_{\mathit{0},R}$ only has edges of type $\mathit{0}$ and R, any two vertices in the same connected components must have indices that differ in a multiple of r. It is not hard to see that, indeed, each connected component consist precisely of all those vertices whose indices are congruent modulo $\gcd(2k,r)$. We now prove an auxiliary result that will be used both in the case where Equation (4.4) as well as in the case where Equation (4.2) or Equation (4.3) holds.

Suppose that no automorphism of Γ maps an S-edge to a $\mathit{0}$-edge. Since Γ is vertex-transitive, there is an automorphism that maps a vertex from V to a vertex in U. Such an automorphism then maps an S-edge to a K-edge. Since all S-edges as well as all the K-edges are in the same orbit, this then implies that $E_S \cup E_K$ forms a single edge-orbit of $\mathrm{Aut}(\Gamma)$.

Lemma 4.11. *Suppose that no automorphism of* Γ *maps an* S*-edge to a* $\mathit{0}$*-edge, then* $\Gamma_{\mathit{0},R}$ *is disconnected, and the set of connected components induce a block system for the vertex-set of* Γ.

Proof. By the discussion above the lemma, $E_S \cup E_K$ forms a single edge-orbit in Γ. Now note that $\mathrm{Aut}(\Gamma)$ also acts as a group of automorphism on $\Gamma_{\mathit{0},R}$ and that this action is transitive, since the edges removed consist of an edge-orbit of Γ. It follows that connected components of $\Gamma_{\mathit{0},R}$ form a block system for the vertex set of Γ. Now, suppose $\Gamma_{\mathit{0},R}$ consists of a single cycle of length $6k$, $(w_0, u_0, v_0, w_r, u_r, v_r, \dots, v_{(2k-1)r})$. Notice that the vertex antipodal to u_0 in this cycle is u_{kr} which is the same as u_k, as r is odd. We see that edges of type K in Γ join vertices that are antipodal in $\Gamma_{\mathit{0},R}$, while S-edges do not. Since this $6k$-cycle is a block of imprimitivity, K-edges can only be mapped into K-edges and thus U is an orbit of $\mathrm{Aut}(\Gamma)$, contradicting the assumption that Γ is vertex-transitive. Hence, $\Gamma_{\mathit{0},R}$ is disconnected. □

Lemma 4.12. *If* $3r - s \equiv 0 \pmod{2k}$, $k \geq 9$, *then* $T_1(k, r, s)$ *is not vertex-transitive.*

Proof. Let $\Gamma = T_1(k, r, s)$ where $3r - s \equiv 0 \pmod{2k}$. A computer assisted counting shows that S-edges and K-edge have exactly 4 distinct 8-cycles passing through them, while *0*- and R-edges have 6 such cycles (see Table 3). It follows that any automorphism sending a vertex $u \in U$ to a vertex $v \in V$ must necessarily map the K-edge incident to u into the S-edge incident to v. Thus $E_S \cup E_K$ is an orbit of edges under the action of $\mathrm{Aut}(\Gamma)$.

Since we are assuming that $3r - s \equiv 0$ we get that any number dividing k and r must also divide s, but since $\gcd(k, r, s) = 1$, we have that $\gcd(k, r) = 1$ and then $\gcd(2k, r) \in \{1, 2\}$. In light of Lemma 4.11, $\Gamma_{\mathit{0},R}$ is disconnected, implying $\gcd(2k, r) \neq 1$ and therefore $\gcd(2k, r) = 2$. From the congruence $3r - s \equiv 0$ we also get that r and s must have the same parity thus making s even and k odd. Recall that each connected component of $\Gamma_{\mathit{0},R}$ consists of all the vertices whose indices are congruent modulo $\gcd(2k, r)$, so $\Gamma_{\mathit{0},R}$ has two connected components: one containing all the vertices with even index, and the other containing those with odd index. Since s is even and k is odd, S-edges in Γ join vertices with same parity, while K-edges join vertices with distinct parity. We know that each of the two connected components of $\Gamma_{\mathit{0},R}$ is a block of imprimitivity of Γ, an so any automorphism of Γ must either preserve the parity of all indices, or of none at all. It follows that no automorphism can send S-edges into K-edges, and therefore no automorphism can send a vertex in V to a vertex in U. We conclude Γ cannot be vertex-transitive. □

Lemma 4.13. *If* $3s - r \equiv 0 \pmod{2k}$, $k \geq 9$, *then* $T_1(k, r, s)$ *is not vertex-transitive.*

Proof. Set $\Gamma = T_1(k, r, s)$ and $\Gamma' = T_1(k, r', s')$, where $r' = s$ and $s' = r$. Notice that the triplet (k, r', s') satisfies Equation (4.4) and so, by Lemma 4.12, Γ' cannot be vertex-transitive. By observation (4.1), $\Gamma \cong \Gamma'$. Therefore Γ is not vertex-transitive. □

Lemma 4.14. *If* $4r - 4s \equiv 0 \pmod{2k}$, $k \geq 9$, *then* $T_1(k, r, s)$ *is not vertex-transitive.*

Proof. Let $\Gamma = T_1(k, r, s)$, with $4r - 4s \equiv 0 \pmod{2k}$ and $k \geq 9$. From the congruence $4r - 4s \equiv 0$, we see that either $2r - 2s \equiv 0$ or $2r - 2s + k \equiv 0$, but the former contradicts Lemma 4.9. Hence $2r - 2s + k \equiv 0$ and the closed walk $((uv)_0, (vw)_r, (wv)_{-s}, (vw)_r, (wv)_{-s}, (vu)_0, (uu)_k)$ in Δ_1 lifts to a 7-cycle in Γ. We will show that Γ cannot be 7-vertex-regular. Following a similar procedure as the one used to obtain Table 2, we have counted all closed walks of length 7 in Δ_1 starting at u and those starting at v, and we have computed their net voltage. The result is displayed in Table 4.

It is plain to see that if, in addition to I, any one of the voltages in Rows III – VI is congruent to 0, then either $r = 0$, $r = k$, $s = 0$ or $s = k$, contradicting Lemmas 4.7 or 4.8.

Table 4: Net voltages of closed walks of length 7 in Δ_1.

Label	Net voltage	Starting at u	Starting at v
I	$\pm(k+2r-2s)$	8	10
II	$\pm(k+r+s)$	12	8
III	$\pm(k+2r)$	6	4
IV	$\pm(k+2s)$	6	4
V	$\pm(3r-2s)$	2	6
VI	$\pm(3s-2r)$	2	6

We see that only the voltages in Row I or II can hold, and thus Γ is not 7-vertex-regular and hence cannot be vertex-transitive. $\square$

In what follow we deal with the case where Equation (4.2) holds, that is, when $3s - 2r + k \equiv 0 \pmod{2k}$ and $k \geq 9$. From Table 3 we see that S-edges and K-edge have exactly 6 distinct 8-cycles passing through them, while 0- and R-edges have only 5. In particular, no automorphism of Γ maps an S-edge to a 0-edge, implying that $E_S \cup E_K$ is an edge-orbit of Γ (see the discussion above Lemma 4.11).

Lemma 4.15. *Suppose that $3s-2r+k \equiv 0 \pmod{2k}$, $k \geq 9$. If e is an edge in $E_K \cup E_S$, then the endpoints of e belong to different connected components of $\Gamma_{0,R}$.*

Proof. Suppose a K-edge e has both of its endpoints in the same connected component, C, of $\Gamma_{0,R}$. Then, because Γ is vertex-transitive and C is a block of imprimitivity, both of the endpoints of any K-edge must belong to the same connected component of $\Gamma_{0,R}$. This means that the subgraph induced by 0-edges, R-edges and K-edges is disconnected. Recall that K-edges and S-edges belong to a single edge-orbit in Γ and thus, for every S-edge, e', there exists $\phi \in \mathrm{Aut}(\Gamma)$ such $\phi(e) = e'$. Since ϕ also acts as an automorphism on $\Gamma_{0,R}$, it follows that the endpoints of $\phi(e)$ are contained in the component $\phi(C)$, thus making Γ a disconnected graph, which is a contradiction. $\square$

Lemma 4.16. *Suppose that $3s - 2r + k \equiv 0 \pmod{2k}$, $k \geq 9$. Then the subgraph $\Gamma_{0,R}$ has an even number of connected components.*

Proof. Let e be a K-edge whose endpoints are in two different connected components C_1 and C_2. Let $\rho^r \in \mathrm{Aut}(\Gamma)$ be an "r-fold" rotation. That is, ρ^r adds r to the index of every vertex. It is plain to see that ρ^r fixes the connected components of $\Gamma_{0,R}$ set-wise and that it send K-edges into K-edges. Moreover, since it is transitive on $U \cap C_1$, we have that all K-edges having an endpoint in C_1 will have the other endpoint in C_2. Hence K-edges "pair up" connected components of $\Gamma_{0,R}$. The result follows. $\square$

Proposition 4.17. *If $T_1(k, r, s)$ is connected and vertex-transitive, then the following statements (or the analogous three statements obtained by interchagning r and s) hold:*

(1) $3s - 2r + k \equiv 0 \pmod{2k}$

(2) *k and s are odd and* $\gcd(k, s) = 1$

(3) *r is even and* $\gcd(k, r) = \{1, 3\}$.

Proof. First, note that item (1) follows from the combination of Lemma 4.10 and Lemmas 4.12, 4.13 and 4.14.

We will continue by proving (3). Set $d = \gcd(k, r)$, so that $r = dr'$ and $k = dk'$ for some integers k' and r'. Since $3s - 2r + k \equiv 0 \pmod{2k}$, we see that, for some odd integer a:

$$3s = d(ak' + 2r').$$

By connectedness, $\gcd(k, r, s) = 1$, and hence $\gcd(d, s) = 1$ implying that d divides 3. Hence $d = \gcd(k, r) \in \{1, 3\}$. Moreover, since $\Gamma_{0,R}$ has $\gcd(2k, r)$ connected components and this number must be even by Lemma 4.16, we see that $\gcd(2k, r) \in \{2, 6\}$, and therefore r is even.

Now, from the congruence $3s - 2r + k \equiv 0 \pmod{2k}$ we see that s and k are either both even, or both odd. Since $\gcd(2k, r, s) = 1$ and r is even, s must necessarily be odd and then k is also odd. Set $d' := \gcd(k, s)$ so that $s = d's'$ and $k = d'k'$ for some integers s' and k'. Again, from Equation (4.2) we obtain

$$2r = d'(ak' + 3s')$$

for some odd integer a. Thus $d' \in \{1, 2\}$, but since s is odd, we see that $\gcd(k, s) = 1$. This proves item (2) and completes the proof. □

Remark 4.18. Conditions (1) – (3) (or their analogues) in Proposition 4.17 are also sufficient to prove vertex-transitivity of a connected graph $T_1(k, r, s)$. This will be shown after the proof on Lemma 4.20. Moreover, observe that if k and s satisfy condition (2) of Lemma 4.17, then $\gcd(2k, 1) = 1$ and $T_1(k, r, s)$ is isomorphic to $T_1(k, rs^{-1}, 1)$ where s^{-1} is the multiplicative inverse of s in $\mathbb{Z}_{2k}$. Hence, in what follows we may assume that $s = 1$.

Lemma 4.19. *Let k be an odd number and set $s = 1$. There exists a unique element $r \in \mathbb{Z}_{2k}$ such that conditions (1) and (3) of Proposition 4.17 are satisfied. This unique r equals r^* from Definition 4.1.*

Proof. Suppose that such r exists. From condition (1) of Lemma 4.19, we get $r = (3 - ak)/2$ for some odd integer a. Since r must be a positive integer smaller than $2k$, it follows that $a \in \{-1, -3\}$. Hence $(3 + k)/2$ and $(3 + 3k)/2 = (3 + k)/2 + k$ are the only possible candidates for r. Since k is odd, the integers $(3 + k)/2$ and $(3 + k)/2 + k$ have different parity. Set r to be whichever one of these two expressions is even. Note that any odd number that divides both r and k must divide 3 and that this is true whether r equals $(3 + k)/2$ or $(3 + k)/2 + k$. Thus condition (3) of Lemma 4.19 is satisfied and r equals r^* from Definition 4.1. □

We have just proved that once an odd $k \geq 9$ is prescribed, then there is at most one graph $T_1(k, r, 1)$ that is connected and vertex-transitive. The following lemma generalizes this to an arbitrary value of s.

Lemma 4.20. *Let $k \geq 9$ be an odd integer and let $T_1(k, r, s)$ be vertex-transitive. Then $T_1(k, r, s) \cong \mathrm{X}(k)$.*

Proof. Recall that $\mathrm{X}(k) \cong T_1(k, r^*, 1)$, where r^* is as in Definition 4.1. Since $T_1(k, r, s)$ is connected and vertex-transitive, conditions (1) – (3) of Proposition 4.17 hold. In particular,

s is relatively prime to $2k$. Then we have $T_1(k, r, s) \cong T_1(k, s^{-1}r, 1)$, where s^{-1} is the multiplicative inverse of s in $\mathbb{Z}_{2k}$. It follows by Lemma 4.19 that $T_1(k, s^{-1}r, 1) = T_1(k, r^*, 1)$. □

Notice that in the previous proof vertex-transitivity is only used to ensure that the graph $T_1(k, r, s)$ satisfies conditions (1) – (3) of Proposition 4.17, and from there $T_1(k, r, s)$ is shown to be isomorphic to $T_1(k, r^*, 1)$. Since we know from Lemma 4.4 that $T_1(k, r^*, 1)$ is vertex-transitive, we have that conditions (1) – (3) of Proposition 4.17 are not only necessary, but also sufficient for vertex-transitivity, and thus Proposition 4.17 may be regarded a characterization theorem.

Remark 4.21. For $k \geq 9$, $\mathrm{X}(k)$ has girth 8. The proof of this fact is straightforward but lengthy and for this reason we decide to omit it. Note as well that $\mathrm{X}(k)$ is not a bipartite graph as every connected component of $\mathrm{X}(k)_{0,R}$ is a cycle of odd length.

We have now completed the proof of Theorem 4.2. Item (1) follows from Lemmas 4.4 and 4.20 while item (2) follows from Remark 4.6. Now that we have a characterization for vertex-transitivity, we would like to know when a cubic vertex-transitive tricirculant of Type 1 is also a bicirculant. Note that the only cubic vertex-transitive circulants are prisms and Möbius ladders, which have girth 4. It follows from Lemma 4.9 that no cubic vertex-transitive tricirculant of Type 1 is a circulant.

Lemma 4.22. *Let* $\Gamma = T_1(k, r, s)$ *be connected and vertex-transitive. If* $3 \nmid k$*, then* Γ *is a bicirculant.*

Proof. Let $\varphi \colon V(\Gamma) \to V(\Gamma)$ be the mapping given by:

$$\begin{array}{llll} u_i \mapsto v_i, & v_i \mapsto w_{i+r}, & w_i \mapsto u_i & \text{if } i \text{ is even;} \\ u_i \mapsto w_{i+k+s}, & v_i \mapsto u_{i+k+s}, & w_i \mapsto v_{i+k+s-r} & \text{if } i \text{ is odd.} \end{array}$$

It can be readily seen that φ is indeed a graph automorphism. Moreover, since r is even and both k and s are odd, φ preserves the parity of the index of all vertices. In fact, it is plain to see that φ is the product of the following two disjoint permutation cycles ϕ_1 and ϕ_2 of length $3k$:

$$\begin{aligned} \phi_0 &= (u_0, v_0, w_r, u_r, v_r, w_{2r}, u_{2r}, v_{2r}, \ldots, u_{(k-1)r}, v_{(k-1)r}, w_0); \\ \phi_1 &= (u_k, w_s, v_{k+2s-r}, u_{k+r}, w_{r+s}, v_{k+2s}, u_{k+2r} \ldots v_{k+2s+(k-2)r}). \end{aligned}$$

Hence, Γ is a bicirculant. □

Lemma 4.23. *If* $3 \mid k$*, then* $\mathrm{X(k)}$ *is not a bicirculant.*

Proof. Suppose $\mathrm{X(k)}$ admits a semiregular automorphism φ having two orbits, O_1 and O_2 of size $3k$. Following the notation in [17], $\mathrm{X(k)}$ must be isomorphic to $\mathrm{H}(3k, i, j)$, $\mathrm{I}(3k, i, j)$ or $\mathrm{F}(3k, i)$ for some $i, j \in \mathbb{N}$. Since $\mathrm{X(k)}$ is not bipartite, it cannot be isomorphic to $\mathrm{H}(3k, i, j)$. Further, if $\mathrm{X(k)} \cong \mathrm{F(3k, i)}$ then each of its orbits under φ would admit a matching, which is not possible because $3k$ is odd. Suppose $\mathrm{X(k)} \cong \mathrm{I(3k, i, j)}$ for some $i, j \in \mathbb{N}$. It is known that vertex-transitive I-graphs are generalized Petersen graphs and as such, one of the two orbits, say O_1, consists of a single cycle having, in this case, length $3k$. Recall that the connected components of $\mathrm{X(k)}_{0,\mathrm{R}}$ are blocks of imprimitivity. Since

$\gcd(2k, r) = 6$, there are 6 such blocks, each of size k. It follows that no two adjacent vertices of O_1 are in the same block. On the other hand, it is plain to see that any 2-path in $\mathrm{X(k)}$ has at least two vertices in the same block (see Figure 5). We conclude that $\mathrm{X(k)}$ cannot be a bicirculant. □

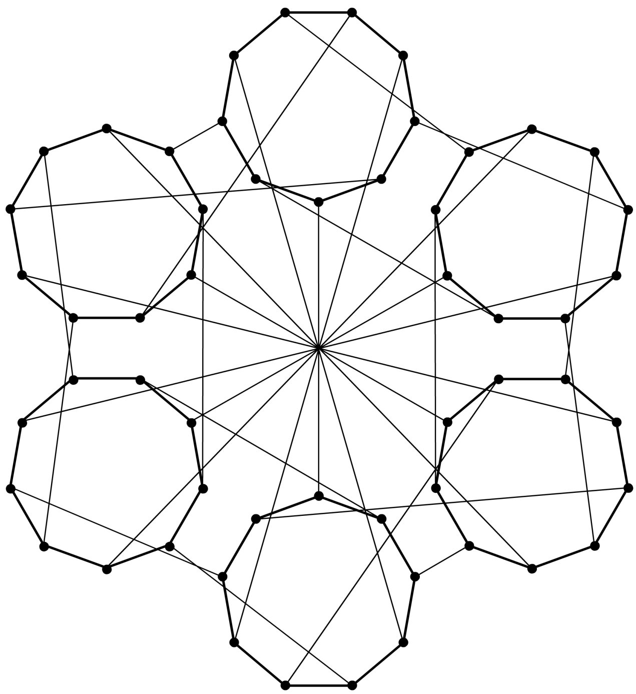


Figure 5: $T_1(9, 6, 1)$. Bold edges correspond to *0*- and R-edges.

Corollary 4.24. *If $T_1(k, r, s)$ is vertex-transitive with $3 \nmid k$, then $T_1(k, r, s) \cong \mathrm{GP}(3k,$ $k + (-1)^\alpha)$ where $\alpha \in \{1, 2\}$ and $\alpha \equiv k \pmod 3$.*

Proof. Let $T_1(k, r, s)$ be vertex-transitive and suppose $k \equiv 1 \pmod 3$. Let $\varphi = \phi_0\phi_1$ be the automorphism described in the proof of Lemma 4.22. We will show that u_k is adjacent to $\varphi^{(k+1)}(u_k)$. Notice that $\varphi^{(k+1)}(u_k) = v_{k+2s-r+\frac{k-1}{3}r}$, but $k + 2s - r \equiv r - s$ so we may write $\varphi^{(k+1)}(u_k) = v_{r-s+\frac{k-1}{3}r}$. Recall that $\gcd(2k, 3) = 1$ and r is even. From the congruence $k + 2r - 3s \equiv 0$ we see that:

$$\begin{aligned} 2r - 3s &\equiv 3k, \\ 3r - 3s + r(k-1) &\equiv 3k, \\ r - s + r(k-1)/3 &\equiv k. \end{aligned}$$

So that $\varphi^{(k+1)}(u_k) = v_k$, which is adjacent to u_k. The case when $k \equiv 2 \pmod 3$ can be proved with the use of a similar argument. □

This completes the proof of Theorem 4.3. Item (1) follows from Lemma 4.22 and Corollary 4.24, while item (2) is proved in Lemma 4.5.

5 Type 2

Let $k \geq 9$ be an integer, and let $r, s \in \mathbb{Z}_{2k}$. Recall that $T_2(k, r, s)$ is the derived graph of Δ_2 with the normalized voltage assignment for $\mathbb{Z}_{2k}$ shown in Figure 6.

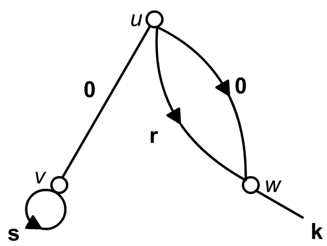


Figure 6: Type 2.

Then, with the same notation as in Section 4,

$$U = \{u_0, u_1, \dots, u_{2k-1}\},\ V = \{v_0, v_1, \dots, v_{2k-1}\} \text{ and } W = \{w_0, w_1, \dots, w_{2k-1}\}$$

are the respective fibres of vertices u, v and w in Δ_2. The set of edges of $T_2(k, r, s)$ can be expressed as the union $E_K \cup E_R \cup E_S \cup E_0$ where:

$$\begin{aligned} E_K &= \{w_i u_{i+k} : i \in \mathbb{Z}_{2k}\}, \\ E_0 &= \{u_0 v_0 : i \in \mathbb{Z}_{2k}\} \cup \{u_0 w_0 : i \in \mathbb{Z}_{2k}\}, \\ E_R &= \{u_i w_{i+r} : i \in \mathbb{Z}_{2k}\}, \\ E_S &= \{v_i v_{i+s} : i \in \mathbb{Z}_{2k}\}. \end{aligned}$$

Similarly as with Type 1 cubic tricirculants, we see that every cubic tricirculant of Type 2 is isomorphic to $T_2(k, r, s)$ for an appropriate choice of k and $r, s \in \mathbb{Z}_{2k}$.

Definition 5.1. For an odd integer $k > 1$, let $\mathrm{Y}(k) = T_2(k, 2, 1)$.

Theorem 5.2. *Let $k \geq 9$ and let Γ be a connected cubic tricirculant of Type 2 with $6k$ vertices. Then the following holds:*

1. *Γ is vertex-transitive if and only if it is isomorphic to $\mathrm{Y}(k)$ for some odd integer k.*

2. *If Γ is not vertex-transitive, then it has three distinct vertex orbits under its full automorphism group.*

The next theorem gives more information about the graph $\mathrm{Y}(k)$.

Theorem 5.3. *For an odd integer $k > 1$ the following holds:*

(1) $\mathrm{Y}(k)$ *a bicirculant if and only if* $3 \nmid k$*, in which case it is isomorphic to the graph* $H(6k, 1, \alpha k + 2)$ *where* $\alpha \in \{1, 2\}$ *and* $\alpha \equiv k \pmod 3$ *(see [17]).*

(2) $\mathrm{Y}(k)$ *is the underlying graph of the toroidal map* $\{6, 3\}_{\alpha,3}$ *where* $\alpha = \frac{1}{2}(k - 3)$*.*

(3) $\mathrm{Y}(k)$ *has arc-type* $2 + 1$*.*

We devote the remainder of this section to proving Theorems 5.2 and 5.3.

Remark 5.4. Note that $T_2(k, r, s)$ and $T_2(k, r, -s)$ are in fact the exact same graph. Then, we can safely assume $s < k$.

Lemma 5.5. *For an odd integer* $k > 1$*,* $\mathrm{Y}(k)$ *is vertex-transitive.*

Proof. Since $\mathrm{Y}(k) = T_2(k, r, s)$, it suffices to give an automorphism of $T_2(k, 2, 1)$ that mixes the sets U, V and W. Let φ be the mapping given by:

$$
\begin{array}{llll}
u_i \mapsto v_{i+1}, & v_i \mapsto u_{i+1}, & w_i \mapsto v_i & \text{if } i \text{ is even;} \\
u_i \mapsto w_{i+2+k}, & v_i \mapsto w_{i+2}, & w_i \mapsto u_{i+k} & \text{if } i \text{ is odd.}
\end{array}
$$

Now consider a vertex $u_i \in U$ with i even. Observe that φ maps u_i to v_{i+1} and that the neighbourhood $N(u_i) := \{v_i, w_i, w_{i+2}\}$ is mapped to $\{u_{i+1}, v_i, v_{i+2}\}$, which is precisely the neighbourhood of v_{1+i}. That is, φ maps the neighbourhood of any vertex u_i into the neighbourhood of its image, when i even. The reader can verify the remaining cases and see that φ is indeed a graph automorphism. □

Lemma 5.6. *For an odd integer* $k > 9$*,* $\mathrm{Y}(k)$ *has arc-type* $2 + 1$*.*

Proof. Let G be the full automorphism group of $\mathrm{Y}(k)$, $k > 9$, and consider the automorphism φ interchanging w_i with w_{-i}, u_i with u_{-i-2} and v_i with v_{-i-2}, $i \in \mathbb{Z}_{2k}$. Note that $\varphi \in G_{w_0}$ and that it fixes w_k while interchanging u_0 with u_{-2}. Since no $\mathrm{Y}(k)$ with $k > 9$ is arc-transitive, it follows that its arc-type is $2 + 1$. □

For the rest of this section, let $k \geq 9$ be an integer, let $r, s \in \mathbb{Z}_{2k}$ and suppose $\Gamma = T_2(k, r, s)$ is vertex-transitive. In what follows we will show that Γ is isomorphic to $\mathrm{Y}(k)$. As with cubic tricirculants of Type 1, the strategy will be to count reduced closed walks in the quotient Δ_2. Observe that the walk $((wu)_0, (uw)_r, (ww)_k, (wu)_{-r}, (uw)_0, (ww)_k)$ starting at w is a closed walk of length 6 having net voltage 0, regardless of what r and s evaluate to in $\mathbb{Z}_{2k}$. As was done with graphs of Type 1, we computed all closed walks in Δ_2 along with their net voltages. Again, it would be convenient if closed walks of length 6 with net voltage 0 always lift into 6-cycles. For this it suffices to show that the girth of Γ is at least 4.

Lemma 5.7. Γ *has no cycles of length* 3*.*

Proof. Suppose to the contrary, that Γ contains a triangle T. By observing Figure 6 we can see that all three vertices of T belong to V or they belong to $U \cup W$. Since Γ is vertex-transitive, there is at least one triangle through every vertex in W. We can thus assume without loss of generality that all three vertices of T are in $U \cup W$. This implies that $r = k$ and further, that there are 2 distinct triangles through every vertex in W: the lifts of $((wu)_0, (uw)_r, (ww)_k)$ and $((wu)_{-r}, (uw)_0, (ww)_k)$. However, since the subgraph induced by V is 2-valent, there is at most one triangle through every vertex in V, contradicting the vertex-transitivity of Γ. □

Note that since Γ is simple, $s \neq k$ and if $r = k$, Γ would contain a triangle. Thus the following corollary holds.

Corollary 5.8. *Neither r nor s equals k.*

Now, let $\mathcal{W}_6$ be the set of all walks of length 6 in Δ_2 and let N be the set of net voltages of elements of $\mathcal{W}_6$, expressed in terms of k, r and s. It follows from Lemma 5.7 that walks in $\mathcal{W}_6$ with net voltage 0 will lift into 6-cycles, and not just closed walks. Table 5 shows, for each element n of N, how many closed walks with net voltage n start at each vertex of Δ_2.

Table 5: Net voltages of closed walks of length 6 in Δ_2.

Label	Net voltage	Starting at u	Starting at v	Starting at w
I	0	2	0	4
II	$\pm(k-s)$	8	8	8
III	$\pm(k-r-s)$	4	4	4
IV	$\pm(k+r-s)$	4	4	4
V	$\pm(3r)$	2	0	2
VI	$\pm(2r)$	2	0	4
VII	$\pm(6s)$	0	2	0
VII	$\pm(r-2s)$	4	6	2
IX	$\pm(r+2s)$	4	6	2

From Row I of Table 5 we see that there are at least 4 walks $\omega \in \mathcal{W}_6$ starting at w. Therefore, at least one expression from Rows II – IX must be congruent to $0 \pmod{2k}$. However, a careful inspection of Table 5 shows that if neither of the expressions in Rows VIII and IX are congruent to $0 \pmod{2k}$, then there are at least 2 more walks in $\mathcal{W}_6$ with net voltage 0 starting at w than there are those starting at v. This means that if Γ is vertex-transitive, then one of the following two equations modulo $2k$ hold:

$$\begin{aligned} r &\equiv 2s, \qquad (5.1)\\ r &\equiv -2s \end{aligned}$$

Suppose Equation (5.1) holds. By Remark 5.4 we can in fact write $r = 2s$. Then the mapping $\phi\colon T_2(k, 2s, s) \to T_2(k, -2s, -s)$ given by $x_i \mapsto x_{-i}$, for all $x \in U \cup V \cup W$, is a graph isomorphism. But, again by Remark 5.4, $T_2(k, -2s, -s) = T_2(k, -2s, s)$ so that any graph $T_2(k, r, s)$ with $r \equiv 2s$ is isomorphic to a graph $T_2(k, r', s)$ satisfying $r' \equiv -2s$. We can therefore limit our analysis to the case when Equation (5.1) holds.

Let $\Gamma = T_2(k, r, s)$ be connected and vertex-transitive, and suppose $r \equiv 2s$. Note that any number dividing both k and s must also divide r, and since by connectedness of Γ $\gcd(k, r, s) = 1$ we see that $\gcd(k, s) = 1$ and $\gcd(2k, s) \in \{1, 2\}$. We will show that $\gcd(2k, s)$ cannot be 2.

Lemma 5.9. *If Γ is vertex-transitive,* $\gcd(2k, s) = 1$ *and* $\gcd(2k, r) = 2$.

Proof. In order to get a contradiction, suppose $\gcd(2k, s) = 2$. Then s is even and k is odd, as $\gcd(k, s) = 1$. Furthermore, the 2-valent subgraph of Γ induced by the set V is the union of two k-cycles. We will show that these are in fact the only two k-cycles in

Γ which will imply that Γ cannot be vertex-transitive. Suppose there is a k-cycle C' that visits a vertex not in V. Define $\Gamma[U \cup W]$ as the subgraph of Γ induced by the set $U \cup W$. Since r is even and k is odd, it is not difficult to see that $\Gamma[U \cup W]$ is bipartite, with sets $\{w_i : i \text{ is even}\} \cup \{u_i : i \text{ is odd}\}$ and $\{w_i : i \text{ is odd}\} \cup \{u_i : i \text{ is even}\}$. Therefore no k-cycle of Γ is contained in $\Gamma[U \cup W]$. This means that C' visits at least one vertex in each of sets U, V and W.

Now, C' projects onto a closed walk C in Δ_2 that has net voltage 0. Define x_S as the number of times C traces the dart $(vv)_s$ minus the number of times it traces $(vv)_{-s}$, so that $x_S s$ is the total voltage contributed to C by darts in $\{(vv)_{\pm s}\}$. Define x_R similarly, and define x_K as the number of times C traces the semi-edge $(ww)_k$. We obtain the following congruence modulo $2k$:

$$x_R r + x_S s + x_K k \equiv 0.$$

Recall that both r and s are even, making both $x_R r$ and $x_S s$ even. It follows that $x_K k$ is even, but since k is odd, x_K must be even, and hence $x_K k \equiv 0 \pmod{2k}$. We thus have

$$x_R r + x_S s \equiv 0$$

and since $r \equiv 2s \pmod{2k}$,

$$x_R 2s + x_S s \equiv 0,$$
$$s(2x_R + x_S) \equiv 0.$$

From this, and because $\gcd(k, s) = 1$, we see that $2x_R + x_S = 0$ or $2x_R + x_S \geq k$. To see that $2x_R + x_S$ cannot equal 0, define A as the set of darts in Δ_2 with voltage 0 or r and observe that C must visit A an even number of times. Since C has odd length, C visits $\{(vv)_{\pm s}\} \cup \{(ww)_k\}$ an odd number of times. Then, since x_K is even, x_S must be odd. But $2x_R$ is even so $2x_R + x_S \neq 0$. We thus have

$$2x_R + x_S \geq k.$$

Now, notice that if C traces the dart $(uw)_r$, then it must immediately trace a dart in $\{(wu)_0, (ww)_k\}$. This means that C traces a dart in the subgraph of Δ_2 induced by $U \cup W$ at least $2x_R$ times. Then $2x_R + x_S < k$, since C has length k and it must visit $(vu)_0$ at least once. We thus have $k > 2x_R + x_S \geq k$, a contradiction. The result follows. $\square$

Lemma 5.10. *If $T_2(k, r, s)$ is vertex-transitive, k is odd.*

Proof. Suppose to the contrary that k is even. Since s is odd, we have $sk \equiv k \pmod{2k}$, and thus $s(2 \cdot \frac{1}{2})k \equiv k$. But k is even, so we can rewrite that expression as $2s \cdot \frac{k}{2} \equiv k$. Recall that $r \equiv 2s$ and then $r\frac{k}{2} \equiv k$.

Now, consider the walk of length $k+1$ in Δ_2 that starts in w, traces the semi-edge $(ww)_k$ once and then traces the 2-path $((wu)_r, (uw)_0)$ exactly $\frac{k}{2}$ times. Note that this walk is closed and has net voltage $\frac{k}{2}r + k \equiv 0$. Furthermore, it is easy to see that it lifts into a $(k+1)$-cycle in Γ and that it does not visit any vertex in V.

Since Γ is vertex-transitive, there must be a $(k+1)$-cycle C' through each vertex in V. Note that the subgraph of Γ induced by V is a single $2k$-cycle, and thus C' must visit all three set U, V and W. It follows that C' projects into a walk C of length $k+1$ having net voltage 0 and that it visits all three vertices of Δ_2. Furthermore, since $k+1$ is odd, the number of times C visits $\{(vv)_{\pm s}\}$ must be of different parity than the number of times it traces $(ww)_k$. Define x_S, x_R and x_K as in the proof of Lemma 4.16. Hence

$$x_R r + x_S s + x_K k \equiv 0.$$

Now, r and k are even, and so $x_R r$ and $x_K k$ are also even. It follows that $x_S s$ is even, but since s is odd, s_S is necessarily even. This means C visits $\{(vv)_{\pm s}\}$ an even number of times. It follows that C traces $(ww)_k$ and odd number of times, that is, x_K is odd. We thus have

$$\begin{aligned} x_R r + x_S s + k &\equiv 0, \\ x_R r + x_S s &\equiv k, \end{aligned}$$

and since $r \equiv 2s$,

$$s(2x_R + x_S) \equiv k.$$

This implies $2x_R + x_S \geq k$, as s and k are relatively prime. However, $2x_R + x_R \leq k - 1$ since C has length $k + 1$ and C visits the set $\{(vu)_0, (uv)_0\}$ at least twice. This contradiction arises from the assumption that k is even. We conclude that k is odd. □

Lemma 5.11. *For an odd $k > 9$, if $T_2(k, r, s)$ is vertex-transitive then we have*

$$T_2(k, r, s) \cong \mathrm{Y}(k).$$

Proof. Let $T_2(k, r, s)$ be vertex-transitive. Then, by Lemmas 5.10 and 5.9 k is an odd integer, $\gcd(2k, s) = 1$ and $r \equiv 2s$. Denote by s^{-1} the multiplicative inverse of s in $\mathbb{Z}_{2k}$. Now, define φ as the mapping between $T_2(k, 2, 1)$ and $T_2(k, r, s)$ given by $x_i \mapsto x_{s^{-1}i}$, for $x \in U \cup V \cup W$. It is plain to see that φ is the desired isomorphism. Therefore $T_2(k, r, s) \cong T_2(k, 2, 1) = \mathrm{Y}(k)$. □

This, together with Lemma 5.5, proves the first claim of Theorem 5.2. We now proceed to show that a tricirculant of Type 2 that is not vertex-transitive must have 3 distinct orbits on vertices.

Lemma 5.12. *Let Γ be a tricirculant of Type 2. If Γ is not vertex-transitive, then it has 3 distinct vertex orbits.*

Proof. We will slightly abuse notation and say $\Gamma = T_2(k, r, s)$ for some $k \in \mathbb{Z}$ and $r, s \in \mathbb{Z}_{2k}$. Let $\varphi \in \mathrm{Aut}(\Gamma)$ and let $u_i \in U$. Suppose that φ maps u_i to some vertex $v_j \in V$. Then φ must map the neighbourhood of u_i, $\{v_i, w_i, w_{i+r}\}$ to $\{u_j, v_{j-s}, v_{j+s}\}$, the neighbourhood of v_j. It is straightforward to see that φ must map a vertex in W to a vertex in V. This is, φ "mixes" the set V with the sets U and W implying that Γ is vertex-transitive, a contradiction. Similarly, if φ maps u_i to some $w_j \in W$ then it must map $\{v_i, w_i, w_{i+r}\}$ to $\{w_{j+k}, u_j, u_{j-r}\}$. Thus φ maps v_i into either U or W contradicting again Γ not being vertex-transitive. This shows that the set U is a single vertex orbit of Γ under $\mathrm{Aut}(\Gamma)$. An analogous argument shows that V is a single vertex orbit and therefore Γ has three distinct orbits on vertices: U, V and W. □

The proof of Theorem 5.2 is now complete. In the following paragraphs we will give further details about the structure of a vertex-transitive tricirculant of Type 2. In particular, we will show that $\mathrm{Y}(k)$ can be seen as a map on the torus and we will give sufficient and necessary conditions for $\mathrm{Y}(k)$ to be a bicirculant. This will prove items (1) and (2) of Theorem 5.3.

Proposition 5.13. *Let $k \geq 3$ be an odd integer, then* $\mathrm{Y}(k)$ *admits an embedding on the torus with hexagonal faces, yielding a map of type* $\{6, 3\}_{\alpha,3}$*, where* $\alpha = \frac{1}{2}(k-3)$ *(see [5] for details about the notation).*

Proof. First, notice that vertices in $U \cup W$ with even index form a cycle of length $2k$, $C_1 = (w_0, u_0, w_2, u_2, \ldots, w_{2k-2}, u_{2k-2})$. Likewise, vertices in $U \cup W$ with odd index form a $2k$-cycle, C_2. The vertices in V form a third cycle of length $2k$, C_3.

Now, $w_i w_{i+k} \in E$ for all $i \in \mathbb{Z}_{2k}$, and $i+k$ has different parity than i. This means each vertex of C_1 of the form w_i is adjacent, through a K-edge, to a vertex of C_2. Further, each vertex of the form u_i in C_1 is adjacent to a vertex in C_3, namely v_i. Note that the vertices of C_2 that are not adjacent to a vertex of C_1 are precisely those of the form u_i (with i odd), and that $u_i v_i \in E$ for all odd i. This is, every other vertex in C_2 has a neighbour in C_3.

Therefore, we can think of $\mathrm{Y}(k)$ as three stacked $2k$-cycles C_1, C_2 and C_3 where every other vertex of C_i has a neighbour in C_{i-1} while each of the remaining vertices has a neighbour in C_{i+1}, where $i \pm 1$ is computed modulo 3 (see Figure 7 for a detailed example). From here it is clear that $T_2(k, 2, 1)$ can be embedded on a torus, tessellating it with $3k$ hexagons. It can be readily verified that this yields the map $\{6, 3\}_{\alpha,3}$, where $\alpha = \frac{1}{2}(k-3)$, following the notation in [5]. $\square$

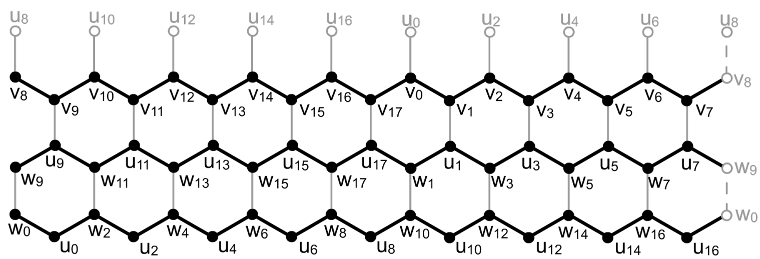


Figure 7: The graph $\mathrm{Y}(9)$. The subgraph induced by bold edges has three connected components that correspond, from bottom to top, to the cycles C_1, C_2 and C_3 described in the proof of Proposition 5.13.

Corollary 5.14. *For $k \geq 3$,* $\mathrm{Y}(k)$ *has girth* 6 *and is bipartite.*

Observe that $\mathrm{Y}(9)$ corresponds to the map $\{6, 3\}_{3,3}$, which is a regular map (see Chapter 8.4 of [5]). It follows that $T_2(9, 2, 1)$ is arc-transitive, making it one of the four possible cubic arc-transitive tricirculant graphs (called F054A in [10], following Foster's notation).

Lemma 5.15. *If $k > 1$ is an odd integer such that $3 \nmid k$, then* $\mathrm{Y}(k)$ *is a bicirculant.*

Proof. Consider the mapping φ defined by:

$$\begin{array}{llll} u_i \mapsto w_{i+2+k}, & v_i \mapsto w_{i+2}, & w_i \mapsto u_{i+k} & \text{if } i \text{ is even;} \\ u_i \mapsto v_{i+1}, & v_i \mapsto u_{i+1}, & w_i \mapsto v_i & \text{if } i \text{ is odd.} \end{array}$$

For a vertex $u_i \in U$, we have $\varphi^l(u_i) \in U$ if and only if $3 \mid l$. This is, if we start at U, every third iteration of φ lands us in U again. Moreover, we have $\varphi^3(u_i) = u_{i+3+k}$. In general $\varphi^{3l}(u_i) = u_{i+3l+lk}$. Hence $\varphi^{3l}(u_i) = u_i$ if and only if $3l + lk \equiv 0$. If $3 \nmid k$, then

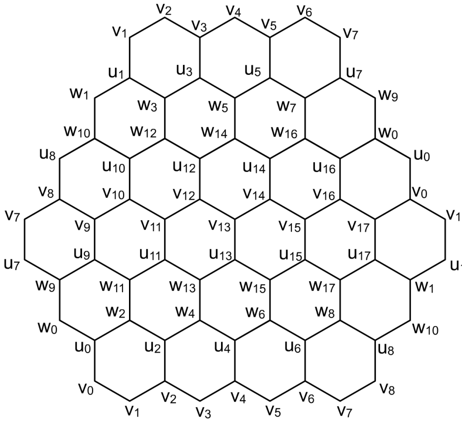


Figure 8: A drawing of $\mathrm{Y}(9)$ as the map on the torus $\{6,3\}_{3,3}$.

$l = k$ is the smallest value for l that satisfies $3l + lk \equiv 0$. It follows that the orbit of u_i has size $3k$. It is plain to see that the vertices not in this orbit form an orbit of size $3k$ on their own, namely, the orbit of u_{i+1}. Hence $\mathrm{Y}(k)$ is a bicirculant and is isomorphic to the graph $\mathrm{H}(3k, 1, \alpha k + 2)$, where $\alpha \in \{1, 2\}$ and $\alpha \equiv k \pmod 3$ (see [17] for details). It is worthwhile to mention that if 3 does divide k, then φ has 6 orbits of size k. $\square$

Lemma 5.16. *If $k > 1$ is an odd integer such that $3 \mid k$, then $\mathrm{Y}(k)$ is not a bicirculant.*

Proof. Let k be an odd integer divisible by 3. To make this proof more succinct, we will assume that $\mathrm{Y}(k)$ is not arc-transitive, this is, that $k > 9$. The graphs $\mathrm{Y}(3)$ and $\mathrm{Y}(9)$ can be verified to not be bicirculants individually. By Lemma 5.6, $\mathrm{Y}(K)$ has arc-type $2+1$ and the set $E' = \{u_i v_i \mid i \in \mathbb{Z}_{2k}\} \cup \{w_i w_{k+1} \mid i \in \mathbb{Z}_{2k}\}$ is an orbit of edges under the full automorphism group of $\mathrm{Y}(k)$. Then each connected component of $\mathrm{Y}(k) - E'$ is a block of imprimitivity for $\mathrm{Aut}(\mathrm{Y}(k))$. Furthermore $\mathrm{Y}(k) - E'$ has three connected components that correspond precisely to the three $2k$-cycles C_i, $i \in \{1, 2, 3\}$, in the proof of Proposition 5.13. For convenience, we will relabel the vertices of $\mathrm{Y}(k)$ as follows: for each $i \in \{1, 2, 3\}$, let $\mathrm{V}(C_i) = \{v_{i,j} \mid j \in \mathbb{Z}_k\}$, where $v_{i,j}$ is adjacent to $v_{i,j-1}$, $v_{i,j+1}$ and $v_{i+1,j-1}$. It is straightforward to see that this labelling is always possible.

Now, suppose $\mathrm{Y}(k)$ admits a bicirculant automorphism ρ. Since a ρ-orbit has size $3k$, and every $2k$-cycle C_i is a block of imprimitivity, ρ must cyclically permute them. Without loss of generality $\rho(C_i) = C_{i+1}$. Then, ρ is determined, up to two possibilities, by its action on a single vertex. Indeed, if $\rho(v_{i,0}) = v_{i+1,a}$ then the two neighbours of $v_{i,0}$ in C_i must be mapped to the neighbours of $\rho(v_{i,0})$ in C_{i+1}. In other words, the set $\{v_{i,-1}, v_{i,1}\}$ is bijectively mapped to $\{v_{i,a-1}, v_{i,a+1}\}$ in one of two possible ways, and this completely determines the action of ρ. In short, for all $i \in \{1, 2, 3\}$ and $j \in \mathbb{Z}_{2k}$ one of the following holds:

(1) $\rho(v_{i,j}) = v_{i+1,j+a}$,

(2) $\rho(v_{i,j}) = v_{i+1,-j+a}$.

If (2) holds, then $\rho^3(v_{i,j}) = v_{i,-j+a}$ and so $\rho^6(v_{i,j}) = v_{i,j}$. This is, ρ has order 6, which implies that $\mathrm{Y}(k)$ has order 12 and hence $k = 2$, which contradicts k being odd and divisible by 3. If (1) holds, then $\rho^3(v_{i,j}) = v_{i,j+3a}$ and since $3 \mid k$, a vertex $v_{1,j}$ can only be mapped to a vertex $v_{1,b}$ by an element of $\langle\rho\rangle$ if $j \equiv b \pmod 3$. This implies that ρ has at least 3 orbits on vertices, contradicting that it is a bicirculant automorphism. The result follows. □

Theorem 5.3 now follows from Proposition 5.13 and Lemmas 5.6, 5.15 and 5.16.

6 Type 3

Let us begin this section by giving the definition of two well-known families of cubic graphs. For a positive integer k, the *prism* P_k is the graph of order $2k$ with vertex set $\{u_0, u_1, \ldots, u_{k-1}\} \cup \{v_0, v_1, \ldots, v_{k-1}\}$ and edges of the form u_iu_{i+1}, v_iv_{i+1} and u_iv_i, where the indices are taken modulo k. The *Möbius ladder* M_k is the graph with vertex set $\{u_0, u_1, \ldots, u_{2k-1}\}$ and edges of the form u_iu_{i+1} and u_iu_{i+k}, where indices are taken modulo $2k$.

Let $k \geq 9$ be an integer, and let $r \in \mathbb{Z}_{2k}$. Recall that $T_3(k, r)$ is the derived graph of Δ_3 with the normalized voltage assignment for $\mathbb{Z}_{2k}$ shown in Figure 9.

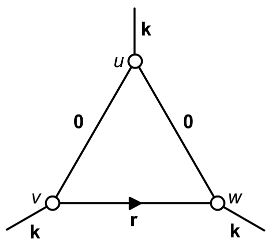


Figure 9: Type 3.

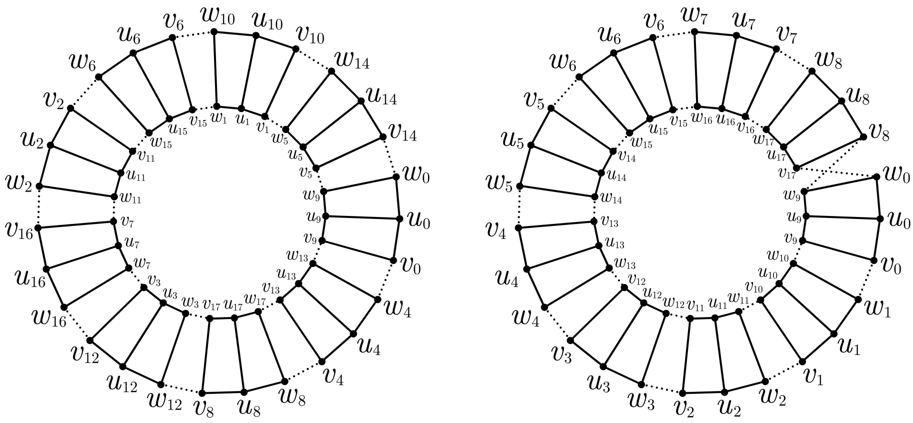
Let $U = \{u_0, u_1, \ldots, u_{2k-1}\}$, $V = \{v_0, v_1, \ldots, v_{2k-1}\}$ and $W = \{w_0, w_1, \ldots, w_{2k-1}\}$ be the respective fibers of vertices u, v and w in Δ_3. It is clear that any cubic tricirculant of Type 3 is isomorphic to $T_3(k, r)$ for some k and r.

Theorem 6.1. *If $T_3(k, r)$ is connected, then it is isomorphic to either a prism or a Möbius ladder.*

Proof. First, recall that $T_3(k, r)$ is connected if and only if $\gcd(k, r) = 1$. Then $\gcd(2k, r) \in \{1, 2\}$, depending on whether r is even or odd.

If r is odd, then the subgraph induced by *0*- and R-edges is a single cycle of length $6k$, $(w_0, u_0, v_0, w_r, u_r, v_r, \ldots, w_{2k-r}, u_{2k-r}, v_{2k-r}, w_0)$. It is straighforward to see that K-edges join antipodal vertices in this cycle. Hence, in this case $T_3(k, r)$ is a Möbius ladder.

If r is even, then the graph induced by *0*- and R-edges is the union of two disjoint cycles of length $3k$: one consisting of all the vertices with even index, and the other consisting on those with odd index. Observe that K-edges connect these two cycles creating a prism. □



(a) $T_3(9,4)$ (b) $T_3(9,1)$

Figure 10: A prism and a Möbius ladder on 54 vertices.

7 Type 4

Let k be a positive integer, and let r and s be two distinct integers in $\mathbb{Z}_{2k}$. Recall that $T_4(k,r,s)$ is the derived graph of Δ_4 with the normalized voltage assignment for $\mathbb{Z}_{2k}$ shown in Figure 11. Let $U = \{u_0, u_1, \ldots, u_{2k-1}\}$, $V = \{v_0, v_1, \ldots, v_{2k-1}\}$ and

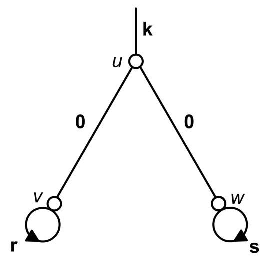


Figure 11: Type 4.

$W = \{w_0, w_1, \ldots, w_{2k-1}\}$ be the respective fibers of vertices u, v and w in Δ_4. Then, the set of edges of $T_4(k,r,s)$ can be expressed as the union $E_K \cup E_R \cup E_S \cup E_0$ where:

$$\begin{aligned}
E_K &= \{u_i u_{i+k} : i \in \mathbb{Z}_{2k}\},\\
E_0 &= \{u_i v_i : i \in \mathbb{Z}_{2k}\} \cup \{u_i w_i : i \in \mathbb{Z}_{2k}\},\\
E_R &= \{v_i v_{i+r} : i \in \mathbb{Z}_{2k}\},\\
E_S &= \{w_i w_{i+s} : i \in \mathbb{Z}_{2k}\}.
\end{aligned}$$

For $X \in \{0, R, S, K\}$, edges in E_X will be called edges of type X, or simply X-edges. Similarly as with tricirculants of Types 1, 2 and 3, every cubic tricirculant of Type 4 with $6k$ vertices is isomorphic to $T_4(k,r,s)$ for an appropriate choice of r and s.

Remark 7.1. Note that for any k, r and s the following isomorphism holds:

$$T_4(k,r,s) \cong T_4(k,-r,s) \cong T_4(k,r,-s) \cong T_4(k,s,r).$$

Theorem 7.2. *There are no cubic vertex-transitive tricirculants of Type* 4 *of order greater or equal to* 54.

The rest of this section is devoted to prove Theorem 7.2 as well as to characterise Type 4 tricirculants with 2 and 3 vertex orbits. We will assume henceforth that $k \geq 9$ and thus the order of $T_4(k,r,s)$ is at least 54.

Lemma 7.3. *If* $T_4(k,r,s)$ *is vertex-transitive, then* $r \neq s$ *and* $r \neq -s$.

Proof. Suppose that $r = s$ and consider the graph $\Gamma := T_4(k,r,r)$, with $k \geq 9$ and $1 \leq r \leq k-1$. Observe that $(v_0, v_r, u_r, w_r, w_0, u_0, v_0)$ is a 6-cycle of Γ that does not contain any K-edge but does contain edges of all other types. For Γ to be vertex-transitive, there must be at least one 6-cycle through a K-edge; otherwise K-edges conform a single edge-orbit and thus U is a single vertex-orbit. However, such a cycle would quotient down to a closed walk of length 6 in Δ_4 having voltage 0 and tracing the semi-edge $(uu)_k$. By observing Figure 11 we see that any closed walk of length 6 visiting $(uu)_k$ has net voltage $k \pm 3r$. That is, if Γ is vertex-transitive, then $3r \equiv k \pmod{2k}$. Observe that under these conditions, Γ is connected if and only if $r = 1$ and thus making $k = 3$, contradicting that $k \geq 9$. This shows that $r \neq s$ and in view of Remark 7.1, $r \neq -s$. □

Lemma 7.4. *Let* e *be a* K*-edge of* $T_4(k,r,s)$ *and let* a *be an integer greater than* 4. *If* C *is an cycle of length* a *containing* e *then there exists another cycle of length* a, C', *such that the intersection* $C \cap C'$ *contains a* 3*-path whose middle edge is* e.

Proof. Without loss of generality, we can assume $e = u_0 u_k$. Define H as the subgraph of Γ induced by vertices at distance at most one from e, and let C be a k-cycle through e (see Figure 12). Observe that C intersects H in a 3-path, P, whose middle edge is e. Now, let ϕ be the mapping that acts by multiplying the index of each vertex by -1; that is ϕ maps u_i into u_{-i}, v_i into v_{-i}, and w_i to w_{-i}. Observe that ϕ is in fact an automorphism of Γ. Moreover, ϕ fixes each vertex and each edge of H. In particular, it fixes P. Therefore, $\phi(C)$ is a k-cycle through e and P lies in the intersection of C and $\phi(C)$. To see that $\phi(C)$ is different from C, observe that ϕ interchanges the two vertices in each of the following sets: $\{v_{k+r}, v_{k-r}\}$, $\{v_r, v_{-r}\}$,$\{w_{k+s}, w_{k-s}\}$ and $\{w_s, w_{-s}\}$ (white vertices in Figure 12). It is clear that C can visit at most one vertex in each of these four sets, and if C visits one vertex in one of these sets, then $\phi(C)$ must visit the other. □

Lemma 7.5. *If* $T_4(k,r,s)$ *is vertex-transitive, then in* $\mathbb{Z}_{2k}$ *one of the equalities* (A1) – (A4) *below and one of the equalities* (B1) – (B4) *below hold:*

(A1)	$k + 2r + s = 0,$	(B1)	$k + 2s + r = 0,$
(A2)	$k - 2r + s = 0,$	(B2)	$k - 2s + r = 0,$
(A3)	$r + 3s = 0,$	(B3)	$s + 3r = 0,$
(A4)	$r - 3s = 0,$	(B4)	$s - 3r = 0.$

Proof. Suppose $T_4(k,r,s)$ is vertex-transitive. We will first show that one of the equalities (A1) – (A4) holds in $\mathbb{Z}_{2k}$. The rest of the lemma will follow from the fact that $T_4(k,r,s) \cong$

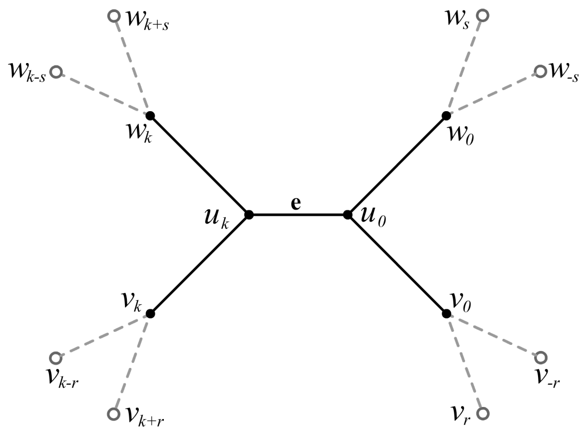


Figure 12: Neighbourhood of u_0u_k. The subgraph H shown in solid edges and vertices.

$T_4(k, s, r)$. Since $T_4(k, r, s)$ is vertex-transitive, the edge-neighbourhood of any vertex in U can be mapped by an automorphism to the edge-neighbourhood of any vertex in V. In particular, either there exists an automorphism mapping the K-edge u_0u_k to the *0*-edge u_0v_0 or there is an automorphism mapping u_0u_k to the R-edge v_0v_r. Therefore, the property described in Lemma 7.4 should also hold for u_0v_0 or for v_0v_r (or possibly both). We will see what this means in terms of k, r and s.

Suppose that the property described in Lemma 7.4 holds for u_0v_0. Observe that there is an 8-cycle $C = (u_0, v_0, v_{-r}, u_{-r}, u_{k-r}, v_{k-r}, v_k, u_k, u_0)$ through u_0v_0 (see Figure 13). Then there must be another 8-cycle C' whose intersection with C contains the 3-path (v_{-r}, v_0, u_0, u_k), but no 4-path containing this 3-path. This in turns implies that some vertex in $\{v_{-3r}, u_{-2r}\}$ is adjacent to a vertex in $\{w_{k-s}, w_{k+s}\}$. Since no vertex in V is adjacent to a vertex in W, we have that either $u_{-2r}w_{k-s} \in E$ or $u_{-2r}w_{k+s} \in E$. This implies $2r \equiv k + s$ or $2r \equiv k - s$, and so (A1) or (A2) holds.

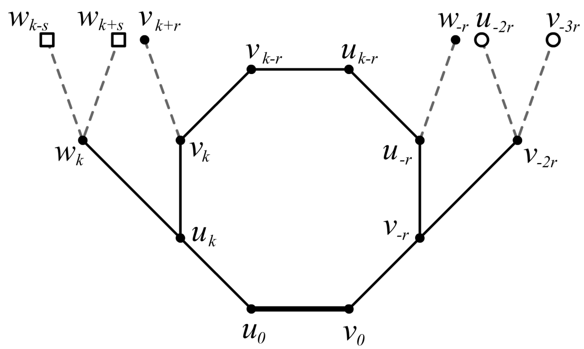


Figure 13: A part of the graph $T_4(k, r, s)$ containing u_0v_0.

If, on the other hand, the property described in Lemma 7.4 holds for v_0v_r, then one vertex in $\{w_{r+s}, w_{r-s}\}$ must be adjacent to a vertex in $\{w_s, w_{-s}\}$ (see Figure 14), implying that one of the following expressions must be equal to 0 in $\mathbb{Z}_{2k}$: $r+3s, r+s, r-3s, r-s$. However, in view of Lemma 7.3, $r+s \neq 0$ and $r-s \neq 0$. Hence (A3) or (A4) holds. $\square$

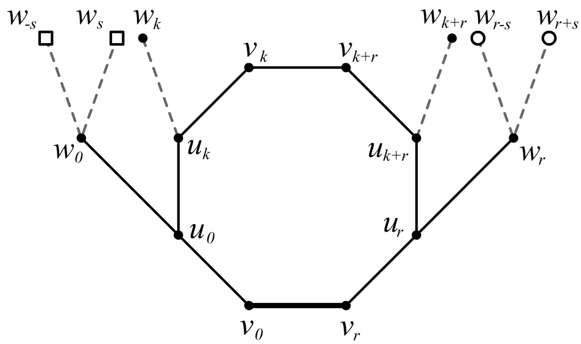


Figure 14: A subgraph containing v_0v_r.

We are now ready to prove Theorem 7.2. Let $k \geq 9$ be an integer and suppose $T_4(k,r,s)$ is vertex-transitive. Then, by Lemma 7.5, one of the equalities (A1) – (A4) and one of the equalities (B1) – (B4) must hold. If, for instance both (A1) and (B1) hold, then by adding them we see that $r = -s$, which contradicts Lemma 7.3. Similarly, we get that $r \pm s$ if both equalities in any of the following pair hold: (A1) and (B2), (A2) and (B2), (A3) and (B3). If (A1) and (B4), or (A2) and (B4) hold, then $k = r$, which contradicts the simplicity of Γ. If (A1) and (B3), (A2) and (B3), or (A3) and (B4) hold, then either $k = 5$ or $\gcd(k,r,s) \neq 1$. It is readily seen that we get a contradiction in each of the 16 possible cases that arise from Lemma 7.5. We conclude that a connected $T_4(k,r,s)$ cannot be vertex-transitive if $k \geq 9$.

Lemma 7.6. *Let $k \geq 9$ and $T_4(k,r,s)$ be connected. Let $\alpha = \gcd(2k,r)$, $R = r/\alpha$, $S = s/\alpha$. Then $T_4(k,r,s)$ has two orbits on vertices if and only if $\alpha \in \{1,2\}$ and $(SR^{-1})^2 \equiv \pm 1 \pmod{2k/\alpha}$, where R^{-1} is the multiplicative inverse of R modulo $2k/\alpha$ (or the equivalent statement obtained from interchaging r and s).*

Proof. For a graph Γ of Type 4, let Γ^* be the graph obtained from Γ by deleting all vertices in U (along with the edges incident to them) and making v_i adjacent to w_i, $i \in \mathbb{Z}_{2k}$. Then Γ^* is isomorphic to the bicirculant I-graph $I(2k,r,s)$ of order $4k$. Moreover, each automorphism $\varphi \in \mathrm{Aut}(\Gamma)$ induces an automorphism $\varphi^* \in \mathrm{Aut}(\Gamma^*)$ in a natural way. Denote by $\mathrm{Aut}^*(\Gamma^*)$ the group of all automorphism of Γ^* induced by an automorphism of Γ.

Now suppose Γ has two orbits on vertices. Since K-edges can only be mapped to K-edges, these two orbits must necessarily be U and $V \cup W$. This is, $\mathrm{Aut}(\Gamma)$ acts transitively on $V \cup W$ and so $\mathrm{Aut}^*(\Gamma^*)$ is transitive on the vertices of Γ^*.

If $\Gamma^* = I(2k,r,s)$ is connected then it must be a vertex-transitive generalised Petersen graph [17]. This is, without loss of generality, the graph induced by V is a $2k$-cycle, and thus $\gcd(2k,r) = 1$. Let r^{-1} be the multiplicative inverse of r modulo $2k$ and let $\Lambda = T_4(k,1,sr^{-1})$. Then $\Lambda \cong \Gamma = T_4(k,r,s)$. Moreover, Λ^* is isomorphic to the generalised Petersen graph GP$(2k,sr^{-1})$, and since it is vertex-transitive we have $(sr^{-1})^2 \equiv \pm 1 \pmod{2k}$ [8].

If $\Gamma^* = I(2k, r, s)$ is disconnected, then it must have two connected components. Indeed, since Γ is connected we have $\gcd(2k, k, r, s) = 1$, but $I(2k, r, s)$ is disconnected if and only if $\gcd(2k, r, s) \neq 1$. Then necessarily $\gcd(2k, r, s) = 2$ and both r and s are even while k is odd. Each connected component is isomorphic to $I(k, r/2, s/2)$. An analogous argument as the one used in the connected case shows that $\gcd(k, r/2) = 1$, (and since k is odd, $\gcd(k, r) = 1$ and $\gcd(2k, r) = 2$) and $(SR^{-1})^2 = ((s/2)(r/2)^{-1})^2 \equiv \pm 1 \pmod{k}$.

For the reverse implication let $k \geq 9$ and $r, s \in \mathbb{Z}_{2k}$. Suppose that $\alpha = \gcd(2k, r) \in \{1, 2\}$ and $(SR^{-1})^2 \equiv \pm 1 \pmod{2k/\alpha}$, where R^{-1} is the multiplicative inverse of R modulo $2k/\alpha$. Let $\Gamma = T_4(k, r, s)$. Define $\varphi\colon \Gamma \to \Gamma$ as follows:

$$u_i \mapsto u_{(SR^{-1})i}, \qquad v_i \mapsto w_{(SR^{-1})i}, \qquad w_i \mapsto v_{(SR^{-1})i} \qquad \text{for all } i \in \mathbb{Z}_{2k}.$$

Observe that $SR^{-1}r \equiv s \pmod{2k}$ and $sSR^{-1} \equiv \pm r \pmod{2k}$. From here, it is routine to check that φ is an automorphism of Γ. We conclude that $V \cup W$ is a vertex orbit of Γ. □

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Clustering via the modified Petford-Welsh algorithm

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Abstract

Detecting meaningful communities has become crucial to advance our knowledge in diverse research areas that deal with datasets which can be naturally represented as networks. By regarding vertex clustering as the opposite problem of vertex colouring, we were able to leverage the Petford-Welsh colouring algorithm to develop a fast, efficient, and highly-scalable decentralised clustering algorithm. Its greatest potential lies in outperforming conventional methods when the community structure is fairly clear-cut.

Keywords: Graph algorithm, community detection, the Petford-Welsh algorithm, simulated annealing, complex networks.

Math. Subj. Class. (2020): 05C85, 05C15, 91C20

1 Introduction

Over the past few decades, deploying complex networks to represent and analyse vast amounts of data has become prevalent across various disciplines [4, 21, 30]. Being able to uncover a network's community structure has turned out to be an invaluable tool when trying to shed light on its intricate organisation and functioning [11].

However, to date we have not even gained an insight into what constitutes a satisfactory clustering solution, let alone developed an approach that would have proven to be universally applicable and scalable to an ever-increasing abundance of data. As a matter of fact, according to the *"No Free Lunch Theorem"* for community detection, *"there can be no algorithm that is optimal for all possible community detection tasks"* [22]. As a consequence,

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this has led to a number of clustering algorithms, stemming from different aspects, a variety of objective functions, stand-alone quality metrics, and information recovery metrics [10].

In this paper, we introduce a clustering algorithm motivated by a heuristic approach to vertex colouring proposed by Petford and Welsh [23, 37]. Our method was founded on the observation that finding an appropriate vertex clustering is, in essence, dual to the problem of finding a proper vertex colouring. Indeed, local proliferation of the colour of a vertex should, in principle, lead to densely interconnected monochromatic neighbourhoods. By associating colours with clusters one thus attains a clustering solution that captures the very fundamental intuition behind community detection – a division into tightly-knit groups, loosely connected to one another. Hence, the terms *colour(ing)* and *cluster(ing)* will be used interchangeably.

Due to its heuristic nature, the resulting algorithm typically runs in linear time $\mathcal{O}(|E|)$ in the number of edges, which makes it computationally less demanding compared to other widely used methods, and, consequently, highly scalable. Moreover, as it only leverages local knowledge about the network, it does not require specifying any objective function to be optimised. Nevertheless, as there is a general lack of consensus regarding both validity and quality of clustering solutions, it may be beneficial to define an application-specific quality measure in order to filter the resulting solutions based on desirable properties. These can also be accommodated by varying the only prerequisite parameter of the algorithm, the weight parameter ω that enables tuning the degree of randomness, and thus renders the algorithm more widely applicable.

2 Method

It should be noted that our algorithm can be applied to any kind of graphs – be it directed or undirected, connected or disconnected, weighted or unweighted. Regardless, for simplicity, let us assume that we are only dealing with simple connected undirected graphs with possibly weighted edges.

Thus, let $G = (V, E)$ be a simple connected undirected graph with possibly weighted edges. Furthermore, let $A = [a_{uv}]_{u,v \in V}$ denote its (weighted) adjacency matrix with the entry a_{uv} representing the weight of the edge $uv \in E$ ($a_{uv} = 0$ if $uv \notin E$) and assume that, in the case of an unweighted graph, $a_{uv} = 1$ if and only if $uv \in E$.

For the purpose of the algorithm, we say that a vertex $v \in V$ is *bad* if $c(u) \neq c(v)$ for the colouring $c\colon V \to \{1, 2, \ldots, k\}$ currently constructed by the algorithm and some vertex $u \in V$ such that $uv \in E$. Analogously, an edge $uv \in E$ that satisfies $c(u) \neq c(v)$, i.e., a bichromatic edge, is said to be a *bad edge*. It will also prove convenient to adopt the notation $\mathcal{W}(v, i) = \sum_{u \in V : c(u) = i} a_{uv}$ for the sum of the weights of the edges incident to vertices of colour i adjacent to vertex v. As a side note, in the case of an unweighted graph, $a_{uv} \in \{0, 1\}$ for all $u, v \in V$, and the expression $\mathcal{W}(v, i)$ reduces to the number of neighbours of vertex v of colour i.

The main idea of the algorithm goes as follows. During initialisation, each vertex gets assigned a random colour. Afterwards, in an iterative procedure, a vertex with at least one neighbour of a different colour – dubbed a bad vertex – is chosen uniformly at random and is reassigned colour i chosen proportionally to $e^{\mathcal{W}(v,i)/T}$ where $\mathcal{W}(v, i)$ denotes the sum of the weights of the edges incident to v with an endpoint of colour i, and T denotes the temperature parameter. Hence, from a local point of view, the more abundant a particular colour is, the more likely it will spread out – even more so, when the corresponding vertices form strong bonds with the rest of the graph. This weighting function is motivated by the simulated annealing heuristics [15] – we will be using an equivalent expression, i.e., $\omega^{\mathcal{W}(v,i)}$ for appropriately chosen weight $\omega > 1$.

The recolouring process repeats until there are no more bad vertices or the stopping criterion, a sufficiently low variance `Var` in the number of bad edges calculated over a sliding window of length $l \in \mathbb{N}$, is met. To facilitate the computation and subsequent updating of the sliding-window variance, at each step of the iteration, the algorithm records the number of bad edges, stores it in the list `bad_edges`, and re-evaluates `Var` only by adding and subtracting local contributions.

To be more specific, let $b_{\texttt{step}} = \texttt{bad\_edges}\,[\texttt{step}]$ be a shorthand notation for the number of bad edges at step `step` of the iteration, and let $\mu_{\texttt{step}}$ and $\mathrm{Var}_{\texttt{step}}$ denote the sample mean and the variance of the number of bad edges at step `step` over the sliding window of length l, respectively, i.e.,

$$\mu_{\texttt{step}} = \frac{1}{l} \sum_{s=\texttt{step}-l+1}^{\texttt{step}} b_s,$$

$$\mathrm{Var}_{\texttt{step}} = \mathrm{Var}\left([b_s]_{s=\texttt{step}-l+1}^{\texttt{step}}\right) = \frac{1}{l-1} \sum_{s=\texttt{step}-l+1}^{\texttt{step}} b_s^2 - \frac{l}{l-1}\mu_{\texttt{step}}^2.$$

The variance at step $\texttt{step}+1$ is then calculated by first updating the sample mean according to

$$\mu_{\texttt{step}+1} = \mu_{\texttt{step}} + \frac{1}{l}\left(b_{\texttt{step}+1} - b_{\texttt{step}-l+1}\right),$$

which is immediately followed by

$$\mathrm{Var}_{\texttt{step}+1} = \mathrm{Var}_{\texttt{step}} + \frac{1}{l-1}\left(b_{\texttt{step}+1} - b_{\texttt{step}-l+1}\right) \cdot \\ \cdot \left(b_{\texttt{step}+1} + b_{\texttt{step}-l+1} - \mu_{\texttt{step}+1} - \mu_{\texttt{step}}\right).$$

All in all, the algorithm thus takes as an input a given graph G, a suitably chosen weight $\omega > 1$, and an initial number of clusters $k \in \mathbb{N}$. Optionally, two additional parameters that serve as a stopping condition may be specified – a tolerance $\texttt{tol} > 0$ on the variance `Var` in the number of edges spanning distinct clusters and a length $l \in \mathbb{N}$ of the sliding window over which this variance is computed. Alternatively – or additionally, a pre-given maximum number of steps can be used as a termination condition. Schematically, the algorithm can be outlined as detailed below (refer to Algorithm 1).

Observe that for the initial number of colours k any upper bound on the number of clusters may be taken. E.g., one may even use the cardinality of the vertex set. Regardless of the choice, the number of colours present in the graph will drop to its natural level as the iterations proceed. Indeed, as soon as a colour $1 \le i \le k$ disappears, $i \notin \{c(v) \mid v \in V\}$, it cannot reappear ever again, since the prerequisite $\mathcal{W}(v,i) \neq 0$ is not met by any vertex $v \in V$ (cf. line 7 in Algorithm 1). Note that permanently discarding absent colours also turned out to enhance numerical stability and consistency of the clustering solutions generated by the algorithm.

More crucially, the weight parameter ω should be chosen more carefully as it may have a profound affect on the algorithm's performance. A sufficiently small ω yields a uniform

Algorithm 1 The modified Petford-Welsh algorithm (mPW)

Input: $G, \omega, k, \texttt{tol}, l$
Output: c

generate an initial k-colouring c
$\texttt{bad\_edges}\,[0] \leftarrow \big|\{uv \in E \mid c(u) \neq c(v)\}\big|$
$\texttt{step} \leftarrow 1$
$\texttt{Var} \leftarrow \texttt{tol}$
while $(\texttt{bad\_edges}\,[\texttt{step} - 1] > 0)$ **and** $(\texttt{Var} \geq \texttt{tol})$ **do**
  choose a bad vertex v uniformly at random
  choose a new colour $1 \leq i \leq k$ for v with probability proportional to $\omega^{\mathcal{W}(v,i)}$ if $\mathcal{W}(v,i) \neq 0$
  $c(v) \leftarrow i$
  $\texttt{bad\_edges}\,[\texttt{step}] \leftarrow \big|\{uv \in E \mid c(u) \neq c(v)\}\big|$
  if $(\texttt{step} \geq l - 1)$ **then**
    $\texttt{Var} \leftarrow \mathrm{Var}\left([\texttt{bad\_edges}\,[s]]_{s=\texttt{step}-l+1}^{\texttt{step}}\right)$
  end if
  $\texttt{step} \leftarrow \texttt{step} + 1$
end while
return c

distribution over the set of all possible outcomes that result from recolouring a vertex at each iteration step, leaving the diffusion of colours to chance. At the other end of the spectrum, a large ω puts higher weight on more frequent colours – the more neighbours of a certain colour a vertex has, the higher the probability of it being assigned the same colour. The final decision whether to choose an intermediate value of ω or a value closer to the extremes should be based on the specifics of the application. Moreover, to achieve better results, the model could be extended by implementing a self-tuning mechanism that iteratively optimises ω in a way analogous to [29].

Nevertheless, ω was set to the constant value of 6 for all test runs, the length l of the sliding window was fixed at the number of vertices $|V|$, and the tolerance $\texttt{tol}$ was chosen by trial and error but was kept within the range of $[10^{-4}, 10^{-2}]$. A key insight that led to the deployment of $\texttt{tol}$ as a stopping criterion in the first place was a typical steady decline in the number of bad edges followed by a flatter region observed across all experiments.

It is worth noting that optimising the initial parameters ω and $\texttt{tol}$ through, e.g., dynamic adaptation to the specifics of the problem under consideration should lead to an improvement in overall performance. Undertaking this task lies beyond the scope of this work but nonetheless deserves more attention in an independent study. All data and the source code for the modified Petford-Welsh algorithm are available at [14].

2.1 Fine-tuning

Essentially, the vertex colouring that is returned as the output of the algorithm lends itself to a natural interpretation in terms of clusters – each vertex colour class corresponds to a cluster. However, there are a few hurdles to overcome.

Initial random colouring may inadvertently lead to disconnected clusters, which invalidates one of the main premises underlying clustering in general. This is resolved in the final part of the algorithm by extracting subgraphs corresponding to different colour classes and assigning new colours to their connected components, thus obtaining a subdivision of the original clustering solution. Besides, an additional optional tweak is implemented to enable a further refinement of the clustering solution by discarding singleton clusters that are potentially left over after the iterations terminate. This is achieved by recolouring every such cluster with the most frequently represented colour in its neighbourhood.

Although our algorithm already typically runs in linear time $\mathcal{O}(|E|)$, there is still room for improvement. In the era of ever-expanding data, developing fast and efficient tools is of utmost importance, as the sheer amount and complexity of data can make many existing algorithms computationally unfeasible. A further significant reduction in runtime could be achieved through parallelism, as the very design of our method makes it easily implementable. In fact, as vertex updates are computed locally and independently of one another (that is, neglecting the unlikely event of simultaneously updating both endpoints of an edge), they can be performed synchronously. This approach has already proved efficient in practice for the Petford-Welsh algorithm [38].

3 Numerical experiments

In this section we compare the performance of our algorithm against several state-of-the-art clustering algorithms. In particular, we restrict our analysis to algorithms that, similar to ours, do not require specifying any additional parameters beforehand. In order to make experiments more comparable, we use the implementations of these algorithms that are readily available in the `python-igraph` (version 0.7.1) software package that comprises of the `igraph` high performance `C` library for network analysis and a `Python` interface [8]. These are *Edge betweenness* relying on iterative removal of edges with high betweenness scores, *Fastgreedy*, *Multilevel*, and *Leading eigenvector* optimising modularity in different ways, *Infomap* analysing the information flow of random walks, *Label propagation* based on an iterative process in which each vertex adopts a label that the majority of its neighbours has, *Spinglass* utilising a spin-glass model and simulated annealing, and *Walktrap* performing random walks. For a thorough review of the methods, their computational complexity and performance, we refer to survey [35] and the references therein.

As far as our algorithm is concerned, it is implemented in `Python`. However, in order to make it comparable to `igraph`'s implementations in `C`, the part of the code that corresponds to the iterative procedure is compiled using `Cython` [5] that converts it into `C` code. The initialisation and the fine-tuning processes are compiled in `Python`. The experiments were performed on a computer equipped with an Intel Core i7-8550U 1.80 GHz processor with 16.0 GB of physical memory (RAM).

3.1 Datasets

We investigated the performance of our proposed method on a class of artificially generated networks as well as on several real-world networks, some of which were given a ground-truth division into clusters. However, it should be noted that it can be misleading to compare the ground-truth clusters, possibly derived from latent network information, with those detected by clustering algorithms operating solely with structural information about the network.

All synthetic networks with embedded community structure were generated by means of the Lancichinetti-Fortunato-Radicchi (LFR) benchmark [17], a commonly used benchmark for testing clustering algorithms. It tries to mimic observations found in many real-world networks by producing networks whose vertex degrees and cluster sizes follow power-law distributions with exponents $2 \leq \gamma \leq 3$ and $1 \leq \beta \leq 2$, respectively. Moreover, its tunable mixing parameter $\mu \in [0, 1]$, the fraction of a vertex's neighbours that belong to different clusters, enables varying how well-defined the clusters are. For $\mu > 0.5$, clusters tend to blend in and become increasingly indistinguishable; as noted in [35], clusters in the strong sense disappear.

Further analysis was carried out on diverse real-world networks ranging from social to infrastructural networks. With the exception of one, all of them have an observationally or experimentally pre-defined community structure. Their basic properties are summarised in Table 1.

Table 1: Real-world networks used in our experiments.

Graph	Vertices	Edges	Ground truth
Zachary's karate club	34	78	Yes
Dolphins	62	159	Yes
UK faculty	79	552	Yes
Political books	105	441	Yes
American college football	115	613	Yes
Political blogs	1222	16714	Yes
Cora citation network	23166	89157	Yes
International E-road network	1040	1305	No

For simplicity, all networks are treated as unweighted and undirected, and we only consider their largest connected components.

3.2 Quality metrics

Most commonly, clustering results are evaluated using internal and external quantitative approaches [34]. The former are derived on the basis of distinct intrinsic statistical characteristics of the clustering solution obtained, whereas the latter compare it to a given ground-truth clustering, provided such a clustering exists.

With regard to internal quality measures, we used the now-standard *modularity* Q [21], *conductance* ϕ [10], and *coverage* γ [10], defined as

$$Q(\mathcal{C}) = \frac{1}{2|E|} \sum_{v,w \in V} \left(a_{vw} - \frac{k_v k_w}{2|E|} \right) \delta(c_v, c_w),$$

$$\phi(\mathcal{C}) = 1 - \frac{1}{|\mathcal{C}|} \sum_{C_i \in \mathcal{C}} \phi(C_i) \quad \text{with}$$

$$\phi(C_i) = \frac{\sum_{v \in C_i, w \notin C_i} a_{vw}}{\min\left\{ \sum_{v \in C_i, w \in V} a_{vw}, \sum_{v \notin C_i, w \in V} a_{vw} \right\}},$$

$$\gamma(\mathcal{C}) = \frac{\sum_{v,w \in V} a_{vw} \delta(c_v, c_w)}{\sum_{v,w \in V} a_{vw}}.$$

Here, $\mathcal{C} = \{C_i\}_i$ denotes a partition into clusters, $A = [a_{vw}]_{v,w \in V}$ the network's adjacency matrix, k_v the degree of vertex v and c_v the community it has been assigned to with respect to $\mathcal{C}$.

One should bear in mind that despite the general prevalence of these indices, they come with a number of caveats. For example, modularity suffers from the resolution limit that displays a bias towards detecting larger clusters, as its value increases by merging clusters smaller than the characteristic size [12]. In spite of there being no clear winner in terms of general applicability, the study published in [10] suggests that conductance correlates best with information recovery metrics, which tend to be more reliable than internal indices when ground truth is supplied.

To validate against externally given ground truth, we made use of two well-established information-theoretic measures, *normalised mutual information* NMI [9] and the *adjusted Rand index* ARI [34]. Given two partitions $\mathcal{C}_1$ and $\mathcal{C}_2$, they can be calculated as follows

$$\begin{aligned}
\mathrm{NMI}(\mathcal{C}_1,\mathcal{C}_2) &= \frac{\mathrm{MI}(\mathcal{C}_1,\mathcal{C}_2)}{\sqrt{\mathrm{H}(\mathcal{C}_1)\mathrm{H}(\mathcal{C}_2)}} \quad \text{with} \quad \mathrm{MI}(\mathcal{C}_1,\mathcal{C}_2) = \mathrm{H}(\mathcal{C}_1) + \mathrm{H}(\mathcal{C}_2) - \mathrm{H}(\mathcal{C}_1,\mathcal{C}_2),\\
\mathrm{ARI}(\mathcal{C}_1,\mathcal{C}_2) &= \frac{\mathrm{RI}(\mathcal{C}_1,\mathcal{C}_2) - \mathrm{E}[\mathrm{RI}(\mathcal{C}_1,\mathcal{C}_2)]}{\max(\mathrm{RI}(\mathcal{C}_1,\mathcal{C}_2)) - \mathrm{E}[\mathrm{RI}(\mathcal{C}_1,\mathcal{C}_2)]}\\
&= \frac{2(n_{00}n_{11} - n_{01}n_{10})}{(n_{00}+n_{01})(n_{01}+n_{11}) + (n_{00}+n_{10})(n_{10}+n_{11})}.
\end{aligned}$$

In the formulas above, H denotes the Shannon entropy, i.e., $\mathrm{H}(\mathcal{C}_i) = -\sum_{C\in\mathcal{C}_i} \frac{|C|}{|V|}\log\frac{|C|}{|V|}$ and $\mathrm{H}(\mathcal{C}_1,\mathcal{C}_2) = -\sum_{C_i^1\in\mathcal{C}_1, C_j^2\in\mathcal{C}_2} \frac{|C_i^1\cap C_j^2|}{|V|}\log\frac{|C_i^1\cap C_j^2|}{|V|}$, and n_{ij} denotes the number of pairs of vertices that fall in the same cluster (different clusters) of $\mathcal{C}_1$ if $i=1$ ($i=0$) and in the same cluster (different clusters) of $\mathcal{C}_2$ if $j=1$ ($j=0$). Both NMI and ARI were computed using `Python` library `Scikit-learn`.

As a final remark, all metrics under consideration assume values in the range of 0-1, and combined higher values across all quality measures indicate better performance.

3.3 Results

Adequately equipped with all the necessary tools, we are now able to present our experimental results. First of all, let us see how well our algorithm performs on the set of real-world networks which was briefly described in Table 1.

As evident from Tables 2 to 8 provided in Section A.1 of the Supplementary materials, the modified Petford-Welsh algorithm, if ran a sufficiently large number of times, either recovered the underlying ground-truth division into clusters or constructed a clustering closest to the ground truth in terms of both NMI and ARI. Notice that modularity may indeed not be the best indicator of the quality of clustering solutions. On the other hand, conductance tends to achieve highest values for the mPW, which is consistent with findings in [10].

What is more, although the International E-road network does not come with a predetermined community structure, solutions generated by the mPW – besides achieving the highest conductance (see Table 9 in Section A.1 of the Supplementary materials) – also seem to be in line with a natural partition of Europe and Central Asia into countries, as shown on Figure 1 below. On top of that, relaxing the tolerance `tol` on the variance in the number of bichromatic edges leads to a coarser partition into geopolitical regions within both continents.

In all experiments, the mPW turned out to be the fastest-performing method, although it should be pointed out that the time elapsed during the initialisation phase and the fine-tuning procedures was not measured in order to be able to benchmark the code against `python-igraph` C implementations.

On a different note, results on the LFR benchmark (here, we refer to Section A.2 in the Supplementary materials and the figures within) indicate that the mPW is among the best-performing methods in terms of NMI and ARI when the community structure is well-defined, that is, in the range $\mu \in [0.1, 0.4]$. As the clusters become more and more intertwined, its performance drops, but achieves best results in the region of $\mu \gtrapprox 0.7$, where all methods, as expected, struggle the most. Here, the difference between the mPW and Label propagation becomes the most apparent – although both methods use only local information, are non-deterministic (and thus need to be executed repeatedly in order to achieve best results), with almost linear complexity in the number of edges, Label propagation, as opposed to the mPW, can either easily get trapped into local minima or inevitably lead to a single giant cluster.

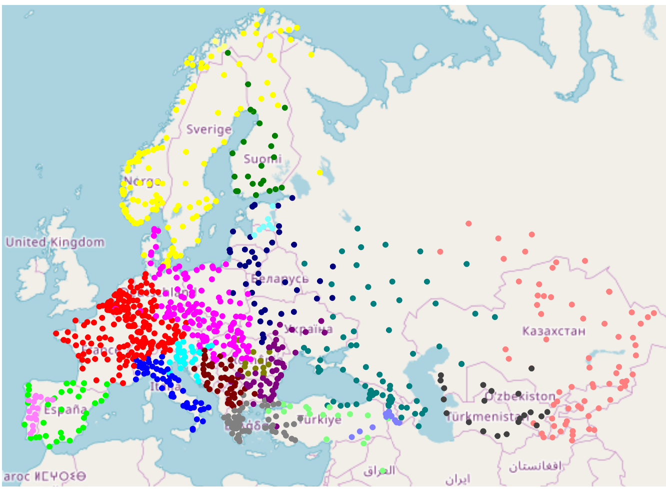

Figure 1: A clustering solution generated by the modified Petford-Welsh algorithm. It consists of 20 clusters; the corresponding quality metrics are equal to $Q = 0.830$, $\phi = 0.847$, and $\gamma = 0.933$.

It should be taken into account that optimising the choice of the initial parameters ω and `tol` lies beyond the scope of this study; implementing it should lead to better results. Indeed, dynamically adapting `tol` to account for the mixing parameter μ already yielded a significant improvement.

All in all, predominantly, the mPW is a fast algorithm that scales efficiently to large networks and produces high-quality clustering solutions when the community structure is reasonably well-defined. Due to its non-deterministic nature, it can be best used in order to find a division into clusters that optimises an application-dependant pre-given quality measure.

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Appendix A Supplementary materials

A.1 Real-world networks

Our analysis on real-world networks is supported by the numerical results presented in the tables below. In columns NMI, ARI, ϕ, γ, and Q, maximum values attained over r runs (as specified below) are reported. Column *Clusters* shows the median number of clusters returned and column $t[s]$ the average running time in seconds over the same set of runs. For the modified Petford–Welsh algorithm (mPW), only running times of the code compiled to C are measured (excluding the initialisation phase and the fine-tuning procedures). In all cases except for the Spinglass method applied to the *Political blogs* and the *International E-road network*, the number of runs r was set to 100. Due to a relatively high complexity of Spinglass, we needed to resort to a smaller value of $r = 10$ when running it on networks of order greater than ≈ 1000 (although this was still too much of a hurdle for the *Cora citation network*). Edge betweenness was not even run on these networks noting its prohibitively high computational complexity, namely $\mathcal{O}(|E|^2|V|)$ [35]. Bold font indicates maximum values attained column-wise (except for the column corresponding to $t[s]$, where the minimum values are indicated).

Table 2: Zachary's karate club [36].

Method	NMI	ARI	ϕ	γ	Q	Clusters	$t[s]$
Edge betweenness [13]	0.517	0.392	0.424	0.692	0.401	5	2.083e−03
Fastgreedy [7]	0.576	0.568	0.574	0.756	0.381	3	1.630e−04
Multilevel [6]	0.516	0.392	0.558	0.731	0.419	4	1.512e−04
Leading eigenvector [20]	0.612	0.435	0.487	0.667	0.393	4	3.191e−03
Infomap [28, 27]	0.578	0.591	0.668	0.821	0.402	3	5.579e−03
Label propagation [25]	0.865	0.882	**0.773**	**0.949**	0.415	3	7.300e−05
Spinglass [26, 33]	0.627	0.509	0.563	0.756	**0.420**	4	2.922e−01
Walktrap [24]	0.531	0.321	0.434	0.590	0.353	5	1.538e−04
mPW	**1.000**	**1.000**	**0.773**	**0.949**	0.403	2	**3.080e−07**

Table 3: Dolphins [18].

Method	NMI	ARI	ϕ	γ	Q	Clusters	$t[s]$
Edge betweenness	0.600	0.395	0.570	0.799	0.519	5	1.792e−02
Fastgreedy	0.602	0.451	0.575	0.824	0.495	4	8.213e−04
Infomap	0.649	0.390	0.571	0.774	0.528	5	8.299e−02
Label propagation	**1.000**	**1.000**	**0.880**	**0.962**	0.526	4	3.263e−04
Leading eigenvector	0.497	0.283	0.544	0.711	0.491	5	7.615e−03
Multilevel	0.564	0.327	0.585	0.755	0.519	5	1.217e−03
Spinglass	0.690	0.452	0.654	0.805	**0.529**	5	4.357e−01
Walktrap	0.565	0.417	0.613	0.824	0.489	4	2.257e−03
mPW	**1.000**	**1.000**	**0.880**	**0.962**	0.528	3	**4.210e−07**

Table 4: UK faculty [19].

Method	NMI	ARI	ϕ	γ	Q	Clusters	$t[s]$
Edge betweenness	0.875	0.898	0.540	0.841	0.441	4	1.826e−01
Fastgreedy	0.882	0.854	0.564	0.783	0.457	4	5.425e−04
Infomap	0.838	0.817	0.583	0.763	**0.461**	4	5.510e−03
Label propagation	0.898	0.920	0.722	**0.953**	0.443	3	1.763e−04
Leading eigenvector	0.898	0.913	0.495	0.772	0.408	4	4.711e−03
Multilevel	0.948	0.957	0.683	0.826	0.446	3	6.230e−04
Spinglass	0.894	0.842	0.583	0.764	**0.461**	4	5.776e−01
Walktrap	0.838	0.817	0.583	0.763	**0.461**	4	1.544e−03
mPW	**0.951**	**0.968**	**0.743**	**0.953**	0.443	3	**4.250e−07**

Table 5: Political books [16].

Method	NMI	ARI	ϕ	γ	Q	Clusters	$t[s]$
Edge betweenness	0.562	0.682	0.626	0.905	0.517	5	1.153e−01
Fastgreedy	0.531	0.638	0.648	0.918	0.502	4	4.967e−04
Infomap	0.503	0.536	0.584	0.855	0.523	6	1.142e−02
Label propagation	0.607	0.702	**0.917**	**0.957**	0.523	3	2.345e−04
Leading eigenvector	0.525	0.547	0.555	0.778	0.467	4	7.717e−03
Multilevel	0.516	0.558	0.675	0.853	0.520	4	4.436e−04
Spinglass	0.566	0.657	0.644	0.889	**0.527**	6	8.279e−01
Walktrap	0.544	0.653	0.687	0.914	0.507	4	9.237e−04
mPW	**0.645**	**0.727**	**0.917**	**0.957**	0.511	3	**5.230e−07**

Table 6: American college football [13].

Method	NMI	ARI	ϕ	γ	Q	Clusters	$t[s]$
Edge betweenness	0.880	0.778	0.533	0.710	0.600	10	2.486e−01
Fastgreedy	0.708	0.474	0.567	0.731	0.550	6	8.467e−04
Multilevel	0.891	0.807	0.547	0.708	**0.605**	10	4.085e−04
Leading eigenvector	0.703	0.464	0.456	0.641	0.493	8	9.658e−03
Infomap	0.924	0.897	0.505	0.690	0.601	12	9.109e−03
Label propagation	0.927	0.889	0.568	0.741	**0.605**	11	2.301e−04
Spinglass	0.929	**0.900**	0.563	0.728	**0.605**	11	3.580e−01
Walktrap	0.888	0.815	0.547	0.705	0.603	10	1.383e−03
mPW	**0.936**	**0.900**	**0.600**	**0.780**	0.603	9	**6.200e−07**

Table 7: Political blogs [3].

Method	NMI	ARI	ϕ	γ	Q	Clusters	$t[s]$
Edge betweenness	–	–	–	–	–	–	–
Fastgreedy	0.659	0.785	0.451	0.923	**0.427**	10	6.923e−01
Infomap	0.523	0.651	0.250	0.899	0.423	41	3.500e+00
Label propagation	0.723	0.813	**0.857**	**1.000**	0.426	3	6.523e−03
Leading eigenvector	0.693	0.781	0.854	0.926	0.424	2	8.246e−02
Multilevel	0.651	0.774	0.476	0.920	**0.427**	9	3.876e−02
Spinglass	0.649	0.783	0.315	0.922	**0.427**	15	5.976e+01
Walktrap	0.646	0.760	0.484	0.925	0.425	11	4.427e−01
mPW	**0.732**	**0.820**	**0.857**	0.927	0.426	4	**2.000e−06**

Table 8: Cora citation network [1, 32].

Method	NMI	ARI	ϕ	γ	Q	Clusters	$t[s]$
Edge betweenness	–	–	–	–	–	–	–
Fastgreedy	0.373	0.085	0.701	**0.906**	0.693	159	7.465e+00
Infomap	**0.574**	0.122	0.505	0.678	0.670	1162	1.450e+02
Label propagation	0.546	0.170	0.580	0.793	0.721	722	5.967e−01
Leading eigenvector	0.157	0.008	0.252	0.853	0.311	11	6.480e+00
Multilevel	0.450	**0.190**	**0.792**	0.860	**0.790**	42	2.600e−01
Walktrap	0.522	0.127	0.523	0.797	0.710	1204	3.347e+01
Spinglass	–	–	–	–	–	–	–
mPW	0.537	0.184	0.602	0.771	0.729	517	**7.100e−05**

Table 9: International E-road network [2, 31].

Method	ϕ	γ	Q	Clusters	$t[s]$
Edge betweenness	–	–	–	–	–
Fastgreedy	0.860	0.917	0.861	24	3.741e−03
Infomap	0.663	0.787	0.777	126	4.615e−01
Label propagation	0.731	0.856	0.828	82	7.130e−03
Leading eigenvector	0.794	0.887	0.835	26	3.492e−01
Multilevel	0.873	0.921	0.867	24	4.546e−03
Spinglass	0.866	0.924	**0.872**	25	1.210e+01
Walktrap	0.757	0.886	0.828	67	8.510e−03
mPW	**0.945**	**0.979**	0.845	17	**2.000e−06**

A.2 LFR benchmark

Experiments were also conducted on networks generated by the LFR benchmark model [17]. To this end, we constructed three qualitatively different families of networks on 1000 vertices and, for each of the regimes separately, studied how varying the mixing parameter μ in the range of 0.1 to 0.9 affects the overall performance.

First, the power-law exponents were set to $\gamma = 2$ and $\beta = 1$, the average and the maximum degree to 15 and 100, respectively, and the sizes of the embedded ground-truth clusters were restricted to the interval $[50, 100]$. Further, the second set of trials was carried out on networks with the average degree and the maximum degree equal to 25 and 150, respectively, whereas the power-law exponents were kept the same, i.e., $\gamma = 2$ and $\beta = 1$, and no constraints were imposed on the sizes of the clusters. Lastly, the networks in the third regime followed power-law distributions at the other extreme, with $\gamma = 3$ and $\beta = 2$. For the average degree and the maximum degree we chose 15 and 50, respectively. Again, we left out the optional parameters controlling the permitted cluster sizes.

The plots below display how maximum values of NMI, ARI, and Q attained over a series of runs vary as a function of the mixing parameter μ. For each particular value of μ, 10 networks were generated and each of the methods was run 100 times on top of them. Edge betweenness and Spinglass were omitted from the analysis due to their rather slow execution speed. Moreover, in order for Leading eigenvector to converge, new networks had to be generated occasionally.

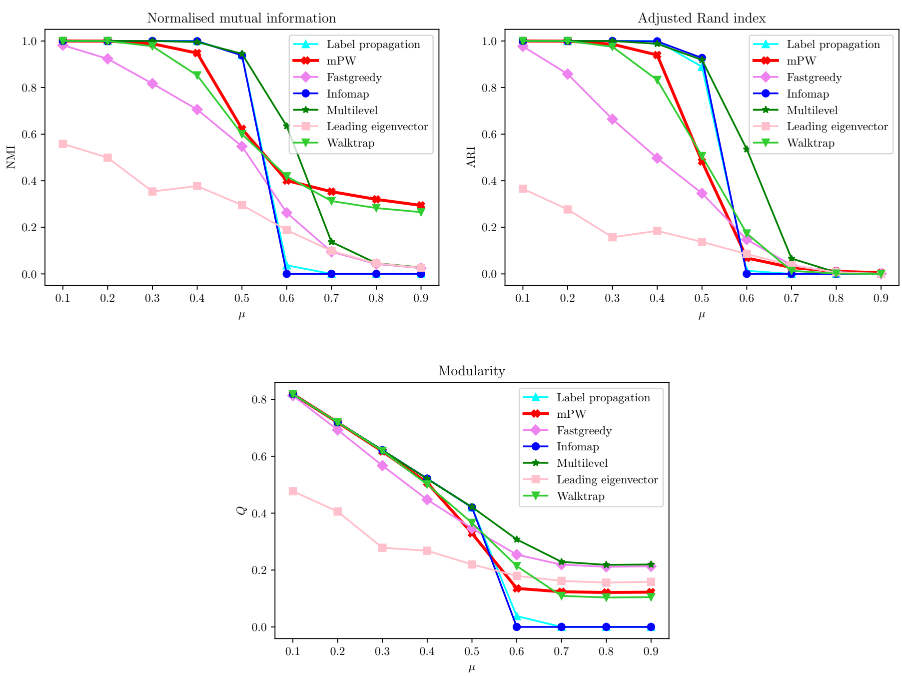

Figure 2: NMI, ARI, and Q plotted as functions of $\mu \in [0.1, 0.9]$, evaluated on networks generated by the LFR benchmark model using $|V| = 1000, \gamma = 2, \beta = 1, \texttt{k\_avg} = 15, \texttt{k\_max} = 100, \texttt{c\_min} = 50$ and $\texttt{c\_max} = 100$.

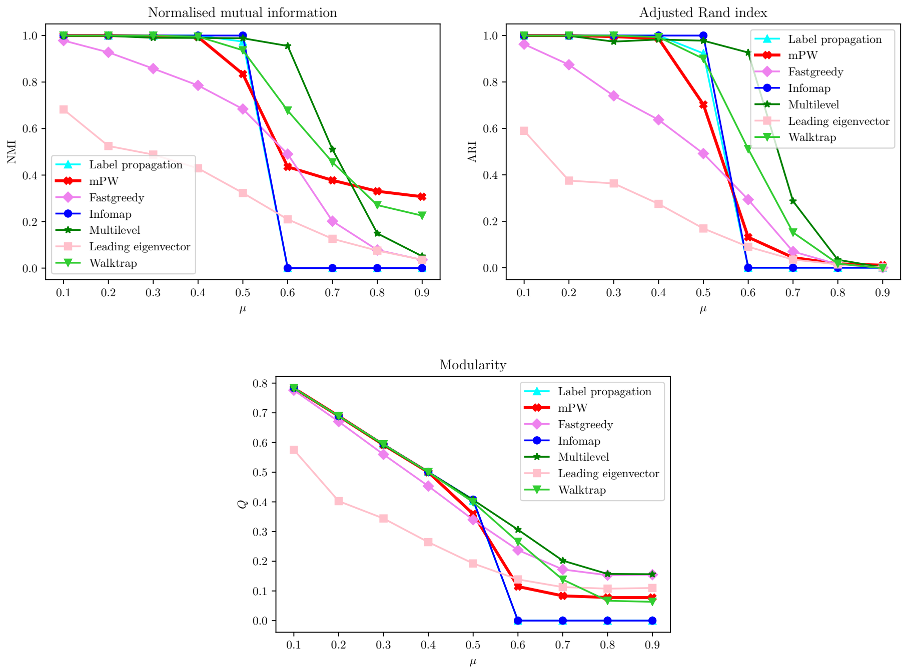

Figure 3: NMI, ARI, and Q plotted as functions of $\mu \in [0.1, 0.9]$, evaluated on networks generated by the LFR benchmark model using $|V| = 1000, \gamma = 2, \beta = 1$, $\texttt{k\_avg} = 25$ and $\texttt{k\_max} = 150$.

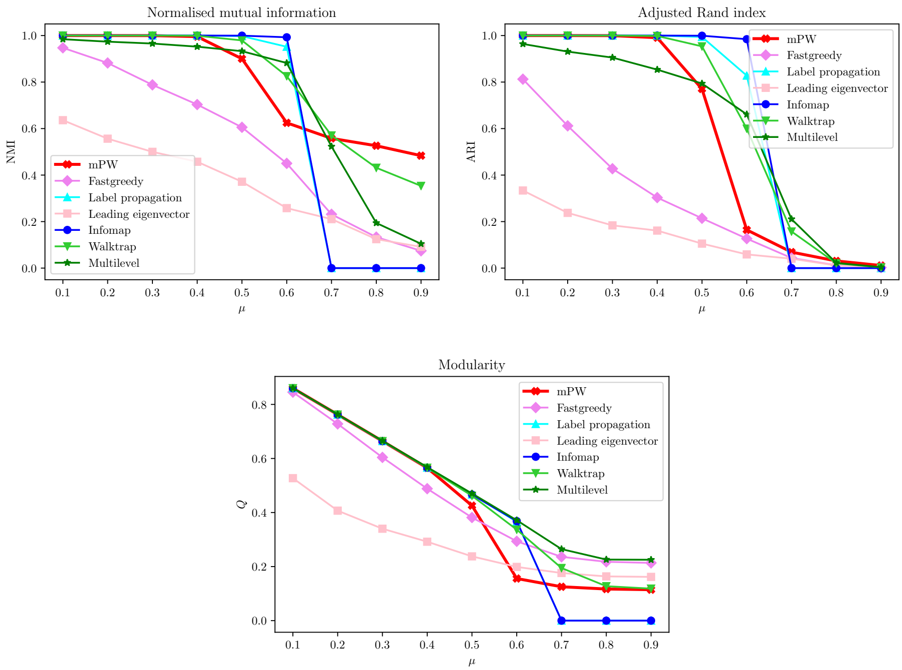

Figure 4: NMI, ARI, and Q plotted as functions of $\mu \in [0.1, 0.9]$, evaluated on networks generated by the LFR benchmark model using $|V| = 1000, \gamma = 3, \beta = 2,$ `k_avg` $= 15$ and `k_max` $= 50$.

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4-edge-connected 4-regular maps on the projective plane*

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Abstract

In this paper rooted (near-)4-regular maps on the projective plane are investigated with respect to the root-valency, the number of edges, the number of inner faces, the number of nonroot-vertex-loops and the number of separating cycles. In particular, 4-edge connected 4-regular maps (which are related to the 3-flow conjecture by Tutte) are handled. Formulae of several types of rooted 4-edge-connected 4-regular maps on the projective plane are presented. Several known results on the number of 4-regular maps on the projective plane are also derived. Finally, using Darboux's method, a nice asymptotic formula for the numbers of this type of maps is given which implies that almost every (loopless) 4-regular map on the projective plane has a separating cycle.

Keywords: (Rooted) near-4-regular map, separating cycle, Lagrangian inversion, enumerating function, asymptotic.

Math. Subj. Class. (2020): 05C10, 05C30, 05C45

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1 Introduction

Graphs here are connected with loops or multi-edges permitted. Terms mentioned without definition may be found in [8, 28, 48]. A graph (map) is k*-(edge-)connected* if it takes the removal of at least k (edges) vertices to separate the graph (map) [8]. Notice that a 2-connected graph (map) may have loops which have been excluded by Tutte [44]. A pair of edges $\{e_1, e_2\}$ is called a 2*-edge-cut* if removing of them will result in a disconnected graph (map). A cycle is *separating* if removing its edges will leave a disconnected graph (map). It is clear that in a 4-regular graph (map) on the projective plane any 2-edge-cut is contained in a separating cycle.

A *planar map* (*projective planar map*) is a graph G drawn on the sphere S_0 (the projective plane N_1) such that no two edges cross and each component of $S_0 - G$ ($N_1 - G$) is homeomorphic to an open disc called a *face* whose boundary is the *facial walk*. In the same way, we may define a map on higher surfaces. A circuit $\mathcal{C}$ on a surface Σ is *essential* (or *noncontractible* as some people named it) if $\Sigma - \mathcal{C}$ has no component which is homeomorphic to an open disc. Otherwise it is called *planar* (or *trivial*).

A map is *rooted* if an edge, a vertex incident to the edge together with a direction along one side of the edge are all distinguished. The essence of rooting a map is to trivialize the *automorphism group* and set up equation(s) of the enumerating functions which makes it possible to determine the solution in an algebraical or analytic way. Furthermore, many results on rooted maps are valid for unrooted maps in terms of asymptotic evaluation since there are at most $4n$ ways to root an n*-edged* map. There has been a trend to evaluate the distributions of a certain type of maps in random map theory [6, 16, 18, 41]. Following the path of Tutte [46] and Brown [9], people have investigated many quadratic equations (or equations much related to them) and got much information about the number of maps. Although some pioneers such as Walsh et al. [50, 51], D. Arquès [1], E. A. Bender et al. [4, 5] and Gao et al. [17] did some influential works on nonplanar maps in an exact way, elegant formulae are very difficult to obtain for maps on higher surfaces. People such as E. A. Bender et al. started systematically working on asymptotic evaluations of rooted maps. Many scholars such as E. A. Bender et al. [3], G. Chapuy et al. [12], Gao [14, 15], A. Mednykh et al. [33, 34] and T. R. S. Walsh et al. [49] have investigated many types of maps on general surfaces and gotten asymptotic evaluations of nonplanar maps up to now. For a survey one may see [7] or [25].

4-regular maps are very important for their applications in many fields such as *rectilinear embedding* in VLSI, the *Gaussian crossing problem* in graph theory, the *knot problem* in topology and the enumerations of some other types of maps [26, 27, 28, 29]. For instance, Bender et al. showed [6] that there exists a close relation between the *face-width* (a concept due to Hutchinson [21]) and the *edge-width* of a graph on a surface through an extended bijection of Brown [10] between n-edged maps and n-faced bipartite quadrangulations; a result of Tutte [43], *every* $2k$*-edge-connected graph has* k *edge-disjoint spanning trees*, together with a formula of Xuong [52] shows that a 4-edge-connected graph is *up-embeddable* (i.e., may be embedded into an orientable surface with at most two faces). Let $k \geq 1$ be an integer and $G = (V, E)$ a multigraph. A Z-flow f on G such that $0 < |f(\vec{e})| < k$ for all $\vec{e} \in \vec{E}$ is called a k*-flow*. Tutte's 3-flow conjecture [42], which is still open, states that *every* 4*-edge-connected* 4*-regular graph has a nowhere-zero* 3*-flow*. Although Grötzsch [20] showed its validity for planar graphs, Tutte conjectured that this statement should still be valid without the requirement that the graph be planar. Jaeger proved [22] that *every* 4*-edge-connected graph has a nowhere-zero* 4*-flow*. However, the difficulty to give a positive answer for the existence of a nowhere-zero 3-flow shows that there is still a long way to go. Instead, Jaeger raised the weak 3-flow conjecture: *There is a fixed integer* k *such that every* k*-edge-connected graph has a nowhere-zero* 3*-flow*.

A *(rooted) near-*4*-regular map* is a map having all the vertices 4-valent except possibly the rooted one. A map is called *Eulerian* if all the valencies of its vertices are even. It is clear that a near-4-regular map is *Eulerian* and 2-edge-connected. Rooted (near-)4-regular maps (or their duals: *quadrangulations*) have been investigated by many scholars. We list them (as far as we know) as follows (among which some are not 4-regular):

(1) rooted bicubic maps [46];

(2) rooted trees [45];

(3) rooted quadrangulations [10];

(4) rooted c-nets via quadrangulations [35];

(5) rooted one-faced maps on surfaces [50, pp. 212–213];

(6) rooted 4-regular planar maps [28, pp. 159–166];

(7) rooted near-4-regular planar Eulerian trials [38];

(8) rooted loopless 4-regular maps on the projective plane, the torus and the Klein bottle [36, 39, 38, 37];

(9) rooted 2-connected 4-regular maps on the plane and the projective plane [30, 40];

(10) rooted 4-edge-connected 4-regular maps on the plane [31].

We expect that several more classes of 4-regular maps could be added into this list. In fact, Z. C. Gao showed us his interest in enumeration of *simple Eulerian maps* through the conversations at 2000's *Workshop of Graph Theory with Applications* at Nanjing Normal University. As an approach to this open problem, we investigate the number of rooted 4-edge-connected 4-regular maps on the projective plane and give a formula, through which an exact formula may be derived, for this type of maps. Furthermore, by using the Darboux's method, we present a nice asymptotic formula for them. Although we did some work on this as shown in the list, the main result(s) of this paper is much closer to the answer since a 4-edged-connected 4-regular map is 2-connected without loops and there are at most two multi-edges between a pair of vertices and there are infinitely many 2-connected 4-regular maps with three multi-edges connecting two vertices. Finally, as an application of those facts, we conclude that among rooted 4-regular maps on the sphere (or the projective plane) almost all of such maps have at least one separating cycle.

Remark 1.1.

(1) Since Tutte's 3-flow conjecture is still open for general graphs, our results may be viewed as an approach to it.

(2) Here we regard planar trees or generally maps with one face on surfaces, which some people also called *monopoles*, as a special kind of near-4-regular maps.

2 Maps on the projective plane

In this section we shall set up a general equation with up to five more parameters for rooted near-4-regular maps on the projective plane. We first give some definitions on maps.

Let $\mathcal{U}$ and $\mathcal{U}_p$, respectively, denote the set of the rooted near-4-regular maps on the plane and the projective plane. Let their *enumerating functions* be, respectively, as

$$f(x,y,z,t,w) = \sum_{M\in\mathcal{U}} x^{2m(M)}y^{s(M)}z^{n(M)}t^{\alpha(M)}w^{\beta(M)},$$
$$f_p(x,y,z,t,w) = \sum_{M\in\mathcal{U}_p} x^{2m(M)}y^{s(M)}z^{n(M)}t^{\alpha(M)}w^{\beta(M)},$$

where the variables x, y, z, t and w mark, respectively, the root-valency, the number of edges, the number of inner faces, the number of nonroot-vertex-loops and the number of separating cycles of M. The set $\mathcal{U}_p$ may be partitioned into three parts as

$$\mathcal{U}_p = \mathcal{U}_{p1} + \mathcal{U}_{p2} + \mathcal{U}_{p3},$$

where

$$\mathcal{U}_{p1} = \{M \mid e_r(M) \text{ is a planar loop}\},$$
$$\mathcal{U}_{p2} = \{M \mid e_r(M) \text{ is an essential loop}\},$$
$$\mathcal{U}_{p3} = \{M \mid e_r(M) \text{ is a link}\},$$

in which $e_r(M)$ is the rooted edge of M; when $e_r(M)$ is not a loop, it is called a *link*.

Lemma 2.1. *Let* $\mathcal{U}_{\langle p1\rangle} = \{M - e_r(M) \mid M \in \mathcal{U}_{p1}\}$. *Then*

$$\mathcal{U}_{\langle p1\rangle} = \mathcal{U} \odot \mathcal{U}_p + \mathcal{U}_p \odot \mathcal{U},$$

where "$\odot$" is the $1v$*-addition of the sets of maps defined in [28, pp. 88–89].*

Here the operation $\odot$ of two maps is defined as follows: For two maps M_1 and M_2 with their respective roots $r_1 = r(M_1)$ and $r_2 = r(M_2)$, the map

$$M = M_1 \cup M_2$$

with $M_1 \cap M_2 = \{v\}$ such that

$$v = v_{r_1} = v_{r_2}$$

is defined such that its root, root-vertex and root-edge are as the same as those of M_1 but the root-face is the composition of $f_r(M_1)$ and $f_r(M_2)$. The operation for getting M from M_1 and M_2 is called the $1v$*-addition* and the map is written as

$$M = M_1 \odot M_2.$$

Further, for two sets of rooted maps $\mathcal{M}_1$ and $\mathcal{M}_2$, the set of maps

$$\mathcal{M}_1 \odot \mathcal{M}_2 = \{M_1 \odot M_2 \mid M_1 \in \mathcal{M}_1, M_2 \in \mathcal{M}_2\}$$

is said to be the $1v$*-production* of $\mathcal{M}_1$ and $\mathcal{M}_2$.

If $\mathcal{M}_1 = \mathcal{M}_2 = \mathcal{M}$, then we are allowed to write

$$\mathcal{M} \odot \mathcal{M} = \mathcal{M}^{\odot 2},$$

and further,

$$\mathcal{M}^{\odot k} = \mathcal{M}^{\odot(k-1)} \odot \mathcal{M}.$$

Proof. For a map $M \in \mathcal{U}_p$, the root-edge $e_r(M)$ is a planar loop. The inner and outer regions determined by $e_r(M)$ are, respectively, two elements of $\mathcal{U}$ and $\mathcal{U}_p$. Since this procedure is reversible, the lemma follows. □

By the above lemma and the reasoning used in [36, Lemma 3.1] the enumerating functions of $\mathcal{U}_{p1}$ and $\mathcal{U}_{p2}$ are, respectively,

$$f_{p1} = 2x^2yzff_p \tag{2.1}$$

and

$$f_{p2} = x^2y\frac{\partial(xf)}{\partial x}. \tag{2.2}$$

The following result is easy to obtain from the definition.

Lemma 2.2. *Let* $\mathcal{U}_{(p3)} = \{M \bullet e_r(M) \mid M \in \mathcal{U}_{p3}\}$*. Then* $\mathcal{U}_{(p3)} = \mathcal{U}_p - \mathcal{U}_p(2)$*, where* $\mathcal{U}_p(2)$ *is the set of maps in* $\mathcal{U}_p$ *whose root-valencies are all* 2*.*

Here, $M \bullet e_r(M)$ is the map obtained by contracting (or shrinking) the root-edge $e_r(M)$ of M into a vertex (i.e., identifying the two ends of $e_r(M)$ along it). The reverse operation is the so-called *splitting the root-vertex*. It is clear that splitting the root-vertex of a near-4-regular map M may create at most one near-4-regular map.

By Lemma 2.2, the enumerating function of $\mathcal{U}_{(p3)}$ is

$$f_{(p3)} = f_p - x^2F_p(2), \tag{2.3}$$

where $F_p(2)$ is the enumerating function of $\mathcal{U}_p(2)$.

If the splitting operation results in a nonroot-vertex loop, the root-edge of the resultant map is contained in a 2-edge-cut. Since splitting the root-vertex may create nonroot-vertex loops, among which some are planar ones while others are essential, the set $\mathcal{U}_{(p3)}$ has to be divided into several more parts as $\mathcal{U}_{(p3)} = \sum_{i=1}^{6} \mathcal{U}^i_{(p3)}$, where maps in $\mathcal{U}^i_{(p3)}$ $(1 \le i \le 5)$ have the structures as depicted in Figure 1 (which defines several types of possibilities of a nonroot-vertex loop after splitting the root-vertex).

Remark 2.3.

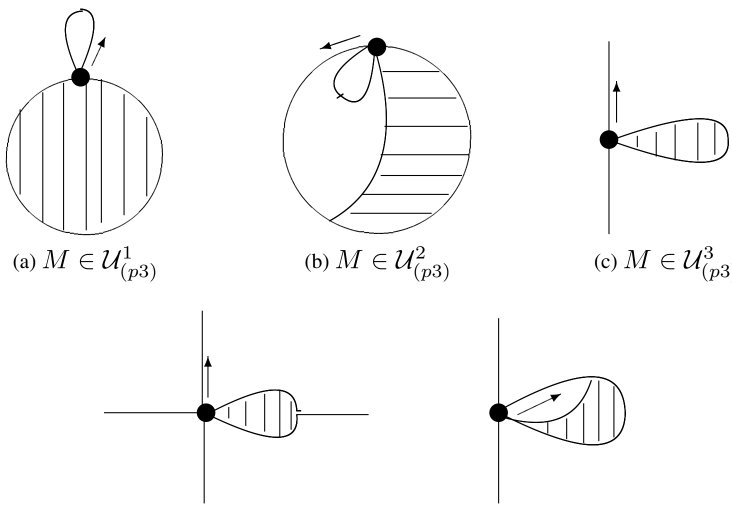
(1) Maps of type 1 (or 2) have their edge $e_r(M)$ (or $e_{P_r}(M)$) the planar loop. Here P is the permutation of the edges of M incident to the root-vertex.

(2) Maps of the above types may create new separating cycles after splitting the root-vertex.

In order to set up recursive relations of maps, we should first state some basic facts for embeddings of graphs on surfaces.

Fact 2.4. *Let* $G(M)$ *be a 2-edge-connected graph (map) with* $\{e_1, e_2\}$ *a* 2*-edge-cut. Then* e_1 *and* e_2 *are on the same separating cycle (which is a part of facial walk of* M*).*

Proof. One may verify this from the fact that for a 2-edge-cut as defined above, each cycle passing through e_1 will also contain e_2 and each facial walk containing e_1 will also has a shortest simple subwalk passing through e_1 and e_2. □



(a) $M \in \mathcal{U}^1_{(p3)}$ (b) $M \in \mathcal{U}^2_{(p3)}$ (c) $M \in \mathcal{U}^3_{(p3)}$

(d) $M \in \mathcal{U}^4_{(p3)}$ (e) $M \in \mathcal{U}^5_{(p3)}$

Figure 1: Five types of maps which will induce a nonroot-vertex loop after splitting the root-vertex.

Fact 2.5. *Let M be a map as defined in Figure 1. Then after splitting the root-vertex, the root-edge of the resulting map will be on a separating cycle.*

Proof. This may be concluded from the definition of the vertex-splitting procedure. □

An edge in a map is called *singular* if it lies on the boundary of exactly one face. Otherwise, it is *nonsingular*. The following fact shows that a projective planar map with a singular edge is determined by performing an *edge-twisting* operation on an edge in planar maps.

Here the so-called *edge-twisting* operation is defined to replace an edge e with a Möbius strip.

Fact 2.6. *Removing of a singular edge on a cycle in a projective planar map will result in a planar map. Conversely, by twisting a nonsingular edge e in a planar map will re-embed its underlying graph into the projective plane.*

Since splitting the root-vertex may lead to a new nonroot-vertex loop or separating cycle, there are several more cases which must be handled.

Case 1. *The maps of $\mathcal{U}^1_{(p3)}$.*

Notice that for a map of $\mathcal{U}^1_{(p3)}$, one thing will happen: both the number of nonroot-vertex loops and the number of separating cycles will increase after splitting the root-vertex. Let $e_1 = e_{P^2r}$, where P is the permutation of the edges incident to the root-vertex. Then the set $\mathcal{U}^1_{(p3)}$ can be partitioned into two parts as

$$\mathcal{U}^1_{(p3)} = \mathcal{U}^{11}_{(p3)} + \mathcal{U}^{12}_{(p3)},$$

where

$$\mathcal{U}_{(p3)}^{11} = \{M \mid e_1 \text{ is contained in a 2-edge-cut}\},$$
$$\mathcal{U}_{(p3)}^{12} = \{M \mid e_1 \text{ is not contained in any 2-edge-cut}\}.$$

Hence, by Facts 2.4 and 2.5, a map M in $\mathcal{U}_{(p3)}^{11}$ will have a 2-edge-cut containing e_1 together with another edge, say e_2. Let e_2 be chosen such that size of the component of $M-\{e_1, e_2\}$ containing the root-vertex will be as small as possible which is as shown in the left side of Figure 2, where the shaded region is either a planar map or a nonplanar map and f represents the possible (inner) face containing e_1 and e_2. Notice that f may be identical to the root-face which will happen only in the case of e_2 being singular.

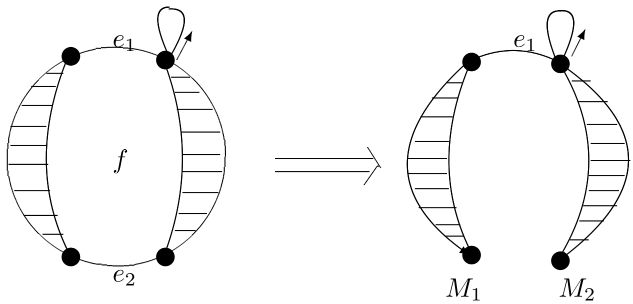


Figure 2: One type of maps containing a 2-edge-cut $\{e_1, e_2\}$.

In general, people use ∂f to denote the boundary of a face f. Here we use this to denote the shortest closed subwalk of f which contains the 2-edge-cut $\{e_1, e_2\}$. One may see that those two concepts coincide in the planar case.

Subcase 1.1. *e_2 is a singular edge.*

By Fact 2.6 and the discussion used in the planar case in [31], the contribution of such type of maps to $\mathcal{U}_p$ is

$$\frac{x^2y(F(2) - yz)}{F(2)}(f - 1), \tag{2.4}$$

where $F(2)$ is the enumerating function of those maps in $\mathcal{U}$ whose root-valency are all 2.

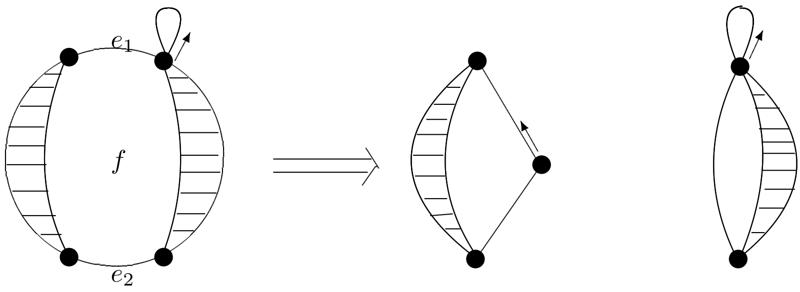
In fact, in this situation the shortest closed subwalk of ∂f in a map M (as shown in the left part of Figure 2) containing $\{e_1, e_2\}$ is an essential cycle and the operation in Figure 2 results in two planar maps M_1 and M_2 where $M_1 \in \mathcal{U}(2) - L$, the set of non-loop maps in $\mathcal{U}$ whose root-valencies are all 2 while M_2 is a map which is the $1v$-addition of a loop map and a planar one. We denote the enumerating functions of the above three types of maps M, M_1 and M_2 are, respectively, P, Q, R. It is clear that the reverse operation from M_1 and M_2 to M in Figure 2 does not increase the number of inner faces, nonroot-vertex loops and separating cycles. Therefore, $P = \frac{F(2)-yz}{yz}R$. Since $R = \frac{x^2y^2z}{F(2)}(f-1)$, we have (2.4).

Subcase 1.2. *e_2 is a nonsingular edge.*

In this case, removing the edge e_2 from M will leave a projective planar map, i.e., $M - e_2$ is a map on N_1 with e_1 a *bridge* (i.e., a separating singular edge) connecting two types of maps M_1 and M_2 as shown in the right side of Figure 2.

Subcase 1.2.1. *M_1 is nonplanar and M_2 is planar.*

Now M corresponds to two types of maps. One is in $\mathcal{U}_p^1(2)$, a subset of $\mathcal{U}_p$ containing the maps with their root-valencies being 2 while the other is a planar one with the edge e_{P^2r} $(= e_1)$ which is not on any separating cycle. This correspondence has been depicted in Figure 3.



A map in $\mathcal{U}_p^1(2)$

Figure 3: Decomposition of a map into two types of maps.

Let $\mathcal{U}_p(2)$ be the set of those maps in $\mathcal{U}_p$ such that their root-valencies are all 2. Since rooted edges of maps in $\mathcal{U}_p^1(2)$ are all nonsingular, its contribution to $\mathcal{U}_p$ is

$$F_p^1(2) = F_p(2) - \frac{1}{z}F(2),$$

where $F_p(2)$ and $F(2)$ are, respectively, the enumerating functions of $\mathcal{U}_p(2)$ and $\mathcal{U}(2)$ (which is the set of maps in $\mathcal{U}$ such that their root-valencies are all 2). As we have reasoned in the proof of (2.4), this combined with (2.4), implies that the total contribution of the maps in this subcase is

$$\frac{F_p(2) - \frac{1}{z}F(2)}{yz} \times \frac{x^2y^2z^2(f-1)}{F(2)}. \tag{2.5}$$

Subcase 1.2.2. *M_1 is planar and M_2 is nonplanar.*

This time we have a similar situation as we got in Subcase 1.2.1, i.e., maps in this case may be decomposed into two types of maps such that the map in rightmost part of Figure 3 is a nonplanar map. Notice that a map of the type in the rightmost side of Figure 3 is in $\mathcal{U}_{(p3)}^{12}$ such that its edge e_{P^2r} is nonsingular. We denote the set of such type of maps as $\mathcal{U}_{(p3)}^{121}$. Then

$$\mathcal{U}_{(p3)}^{12} = \mathcal{U}_{(p3)}^{121} + \mathcal{U}_{(p3)}^{122},$$

where $\mathcal{U}_{(p3)}^{122}$ is the subset of $\mathcal{U}_{(p3)}^{12}$ whose maps have their edge e_{p^2r} being singular. Also by the same reason as above, the enumerating function of $\mathcal{U}_{(p3)}^{122}$ is

$$\frac{1}{z}\frac{x^2y^2z^2(f-1)}{F(2)}.$$

Thus, we have the contribution of the maps in this case as

$$\frac{F(2)-yz}{yz}\left(f^{12}_{(p3)}-\frac{x^2y^2z^2(f-1)}{F(2)}\right), \tag{2.6}$$

where $f^{12}_{(p3)}$ is the enumerating function of $\mathcal{U}^{12}_{(p3)}$. By (2.4), (2.5) and (2.6) and the fact $f^{1}_{(p3)}=x^2yzf_p$, we see that the total contributions of the maps in those cases satisfy

$$\begin{aligned} x^2yzf_p = {} & \frac{x^2y(F(2)-yz)}{F(2)}(f-1)+\frac{x^2yz(F_p(2)-\frac{1}{z}F(2))}{F(2)}(f-1) \\ & +\frac{F(2)-yz}{yz}\left(f^{12}_{(p3)}-\frac{x^2y^2z^2(f-1)}{F(2)}\right)+f^{12}_{(p3)}. \end{aligned}$$

After simplification we find that

$$f^{12}_{(p3)}=\frac{x^2y^2z^2}{F(2)}\left\{f_p-\frac{F_p(2)-\frac{1}{z}F(2)}{F(2)}(f-1)\right\}.$$

Since

$$f^1_{p3}=\frac{yt}{x^2}f^{11}_{(p3)}+\frac{ytw}{x^2}f^{12}_{(p3)},$$

where f^1_{p3} is the enumerating function of $\mathcal{U}^1_{p3}$, we have that

$$f^1_{p3}=\left\{\frac{y^3zt(w-1)}{F(2)}+y^2t\right\}f_p-\frac{y^3zt(w-1)(F_p(2)-\frac{1}{z}F(2))}{F^2(2)}(f-1).$$

Hence,

$$\begin{aligned} f^1_{p3}-\frac{y}{x^2}f^1_{(p3)}= {} & \left\{\frac{y^3zt(w-1)}{F(2)}+y^2(t-1)\right\}f_p \\ & -\frac{y^3zt(w-1)(F_p(2)-\frac{1}{z}F(2))}{F^2(2)}(f-1). \end{aligned} \tag{2.7}$$

Case 2. *The maps of* $\mathcal{U}^2_{(p3)}$.

Since maps in this case have a very similar structure as those in Case 1 except for the ways of putting the planar loop e_{P^2r}, so we have

$$f^2_{p3}-\frac{y}{x^2}f^2_{(p3)}=f^1_{p3}-\frac{y}{x^2}f^1_{(p3)}. \tag{2.8}$$

Case 3. *The maps of* $\mathcal{U}^3_{(p3)}$.

Since the set of maps of this type satisfies

$$\mathcal{U}^3_{p3}=\mathcal{L}_p\odot\mathcal{U}_p,$$

where $\mathcal{L}_p$ is the loop map on N_1, the projective plane. By the result obtained in [31] (or a similar procedure as the proof of (2.4)) we conclude that

$$f^3_{p3}-\frac{y}{x^2}f^3_{(p3)}=\frac{y^3zt(w-1)}{F(2)}(f-1)+y^2(t-1)(f-1). \tag{2.9}$$

By rearranging the root-edges of those in $\mathcal{U}^3_{(p3)}$ which are singular we get all of those in $\mathcal{U}^3_{(p3)} + \mathcal{U}^4_{(p3)}$. After an analogous procedure as we did in Case 3 we obtain

$$f^4_{p3} - \frac{y}{x^2} f^4_{(p3)} = f^5_{p3} - \frac{y}{x^2} f^5_{(p3)} = f^3_{p3} - \frac{y}{x^2} f^3_{(p3)}, \tag{2.10}$$

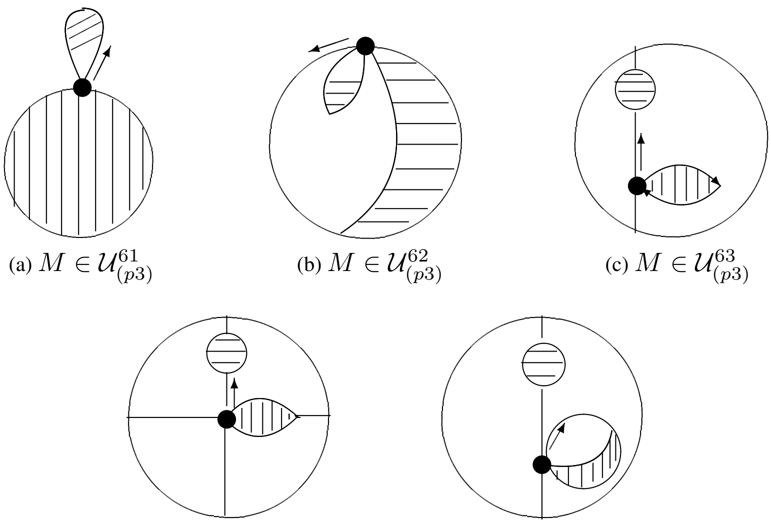
where f^i_{p3} ($f^i_{(p3)}$) is the enumerating function of $\mathcal{U}^i_{p3}$ ($\mathcal{U}^i_{(p3)}$), $1 \le i \le 6$.

Case 4. *The maps of* $\mathcal{U}^6_{(p3)}$.

Since splitting the root-vertex of a map of this type will not create a nonroot-vertex loop but may result in a 2-edge-cut containing the root-edge, we have to partition the set $\mathcal{U}^6_{(p3)}$ into several more parts as

$$\mathcal{U}^6_{(p3)} = \sum_{i=1}^{6} \mathcal{U}^{6i}_{(p3)},$$

where maps in $\mathcal{U}^{6i}_{(p3)}$ $(1 \le i \le 5)$ have the structure shown in Figure 4 where splitting the root-vertex will lead to a 2-edge-cut containing their root-edges.



(a) $M \in \mathcal{U}^{61}_{(p3)}$ (b) $M \in \mathcal{U}^{62}_{(p3)}$ (c) $M \in \mathcal{U}^{63}_{(p3)}$

(d) $M \in \mathcal{U}^{64}_{(p3)}$ (e) $M \in \mathcal{U}^{65}_{(p3)}$

Figure 4: Maps which are composed of $\mathcal{U} - L$ (or $\mathcal{U}_p(2) - L_p$) and a subset of $\mathcal{U}_p$ (or $\mathcal{U}$).

Here we use $\mathcal{L}$ and $\mathcal{L}_p$ to denote, respectively, the loop-map on the sphere and the projective plane. Further, we use f^{6i}_{p3} ($f^{6i}_{(p3)}$) to denote the enumerating function of $\mathcal{U}^{6i}_{p3}$ ($\mathcal{U}^{6i}_{(p3)}$), $1 \le i \le 6$.

One may see that the structures for maps M in $\mathcal{U}^i_{(p3)}$ and $\mathcal{U}^i_{(63)}$ $(1 \le i \le 6)$ are the same except for whether a new nonroot-vertex loop appears or not after splitting the root-vertex of M. Therefore,

$$f^{61}_{(p3)} = \frac{x^2yz(F(2) - yz)}{F(2)} \left\{ f_p - \frac{F_p(2) - \frac{1}{z}F(2)}{F(2)}(f - 1) \right\}$$

and further

$$f_{p3}^{61} - \frac{y}{x^2} f_{(p3)}^{61} = \frac{y^2 z(w-1)(F(2)-yz)}{F(2)} \left\{ f_p - \frac{F_p(2) - \frac{1}{z}F(2)}{F(2)}(f-1) \right\}. \quad (2.11)$$

Since correspondences like this exist between $\mathcal{U}_{p3}^{6i}$ and $\mathcal{U}_{p3}^{i}$ for $2 \le i \le 5$, repeating similar procedures we may establish that

$$\begin{aligned} f_{p3}^{62} - \frac{y}{x^2} f_{(p3)}^{62} &= \frac{y^2 z(w-1)(F(2)-yz)}{F(2)} \left\{ f_p - \frac{F_p(2) - \frac{1}{z}F(2)}{F(2)}(f-1) \right\}, \\ f_{p3}^{6i} - \frac{y}{x^2} f_{(p3)}^{6i} &= \frac{y^2 (w-1)(F(2)-yz)}{F(2)}(f-1), \quad 3 \le i \le 5. \end{aligned} \quad (2.12)$$

We now begin to handle the maps in $\mathcal{U}_{p3}^{66}$. By the definition, a map in $\mathcal{U}_{p3}^{66}$ is not separable at the root-vertex. Given that the possibility of creating new separating cycles, the set $\mathcal{U}_{p3}^{66}$ has to be further partitioned into two parts as

$$\mathcal{U}_{p3}^{66} = \mathcal{U}_{p3}^{661} + \mathcal{U}_{p3}^{662},$$

where splitting the root-vertex of the maps in $\mathcal{U}_{p3}^{661}$ will create a new separating cycle with a structure shown in the left side of Figure 5.

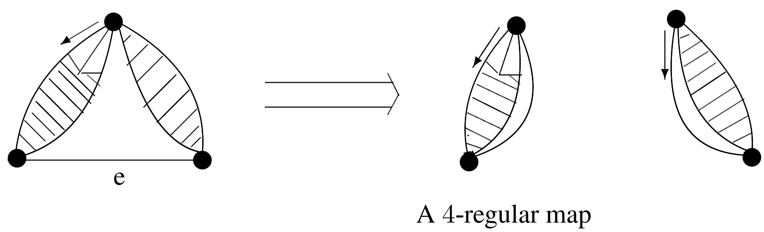


Figure 5: A map in $\mathcal{U}_{(p3)}^{661}$ with an unique edge e lying on a new separating cycle after splitting the root-vertex and may be decomposed into two types of maps.

Let M be a map in $\mathcal{U}_{p3}^{661}$. Then it is not separable at the root-vertex and contains an unique edge, say e, such that removal of e will make it separable at the root-vertex. Further, $M - e = M_1 \odot M_2$ and the root-valency of M_1 is 3. Since M_1 and M_2 may be of distinct types of maps, there are several cases that must be handled.

Subcase 4.1. *e is singular.*

By Fact 2.6 this is in fact a planar case we have handled in [31] and maps of this type contribute

$$\frac{x^2 F(4)}{zF(2)}(f-1) \quad (2.13)$$

to $\mathcal{U}_p$, where $F(4)$ is the enumerating function of those 4-regular maps in $\mathcal{U}$ which are not separable at the root-vertex.

Subcase 4.2. *e is nonsingular.*

For a map M of this case, $M - e = M_1 \odot M_2$ and there is only one nonplanar map among M_1 and M_2. Notice that this will not hold in the general case for the maps which are composed this way. But this possibility has been excluded now. Thus, two subcases will be introduced.

Subcase 4.2.1. *M_1 is nonplanar and M_2 is planar.*

As we have reasoned in the Subcase 1.2.1 of the maps in $\mathcal{U}^1_{(p3)}$, M may be decomposed into two maps as shown in Figure 5. This determined by M_1 is a 4-regular map in $\mathcal{U}_p(4)$ such that the edge $e_{P^{-1}r}$ is nonsingular. Further, we let $\mathcal{U}^1_p(4)$ be the set of those maps. On the other hand, the one determined by M_2 is a planar map having its root-edge outside of any separating cycle. Through a very similar procedure as we used in the Subcase 1.2.1 of the maps in $\mathcal{U}^1_{(p3)}$ we see that the contribution of this case to $\mathcal{U}_p$ is

$$\frac{x^2 F^1_p(4)}{F(2)}(f-1),$$

where $F^1_p(4)$ is the enumerating function of $\mathcal{U}^1_p(4)$. By subtracting the contribution of those determined by the maps in $\mathcal{U}(4)$ we conclude that the enumerating function for maps of this case is

$$\frac{x^2(F_p(4) - \frac{1}{z}F(4))}{F(2)}(f-1), \tag{2.14}$$

where $F_p(4)$ counts the maps in $\mathcal{U}_p(4)$.

Subcase 4.2.2. *M_1 is planar and M_2 is nonplanar.*

Similar to what we have analyzed in the corresponding situation of the maps in $\mathcal{U}^1_{(p3)}$, maps of this case may be decomposed into two types of maps, among which one is determined by twisting the edge which is one edge ahead of the root-edge along the root-face of a map in $\mathcal{U}(4)$ and the other is in $\mathcal{U}^{662}_{(p3)}$ whose root-edge is nonsingular. By repeating an analogous but much more complicated manipulation we find that the enumerating function for maps of this type is

$$\frac{x^2 F^1_p(4)}{F(2)}\left\{f_p - \frac{F_p(2) - \frac{1}{z}F(2)}{F(2)} - \frac{1}{z}(f-1)\right\}.$$

Combining all the three cases shows that the enumerating function of $\mathcal{U}^{661}_{(p3)}$ is

$$\begin{aligned} f^{661}_{(p3)} = \frac{x^2 F(4)}{zF(2)}(f-1) &+ \frac{x^2(F_p(4) - \frac{1}{z}F(4))}{F(2)}(f-1) \\ &+ \frac{x^2 F^1_p(4)}{F(2)}\left\{f_p - \frac{F_p(2) - \frac{1}{z}F(2)}{F(2)} - \frac{1}{z}(f-1)\right\}. \end{aligned}$$

This implies that

$$\begin{aligned} f^{66}_{p3} - \frac{y}{x^2} f^{66}_{(p3)} = \frac{y(w-1)}{F(2)}\Bigg\{ & F_p(4)(f-1) \\ &+ F^1(4)\left(f_p - \frac{F_p(2) - \frac{1}{z}F(2)}{F(2)} - \frac{1}{z}(f-1)\right)\Bigg\}. \end{aligned} \tag{2.15}$$

Now we are in a position to obtain our first main result. Equations (2.1) – (2.15) yield

$$\begin{aligned}
f_p &= 2x^2yzff_p + x^2y\frac{\partial xf}{\partial x} + \frac{y}{x^2}\left\{f_p - x^2F_p(2)\right\} \\
&\quad + \sum_{i=1}^{5}\left(f_{p3}^{i} - \frac{y}{x^2}f_{(p3)}^{i}\right) + \sum_{i=1}^{5}\left(f_{p3}^{6i} - \frac{y}{x^2}f_{(p3)}^{6i}\right) + f_{p3}^{661} - \frac{y}{x^2}f_{(p3)}^{661} \\
&= 2x^2yzff_p + x^2y\frac{\partial xf}{\partial x} + \frac{y}{x^2}\left\{f_p - x^2F_p(2)\right\} \\
&\quad + 2\left\{\frac{y^3zt(w-1)}{F(2)} + y^2(t-1)\right\}f_p - 2\frac{y^3zt(w-1)(F_p(2)-\frac{1}{z}F(2))}{F^2(2)}(f-1) \\
&\quad + 3\frac{y^3zt(w-1)}{F(2)}(f-1) + 3y^2(t-1)(f-1) \\
&\quad + 2\frac{y^2z(w-1)(F(2)-yz)}{F(2)}\left\{f_p - \frac{F_p(2)-\frac{1}{z}F(2)}{F(2)}(f-1)\right\} \\
&\quad + 3\frac{y^2(w-1)(F(2)-yz)}{F(2)}(f-1) \\
&\quad + \frac{y(w-1)}{F(2)}\left\{F_p(4)(f-1) + F^1(4)\left(f_p - \frac{F_p(2)-\frac{1}{z}F(2)}{F(2)} - \frac{1}{z}(f-1)\right)\right\}.
\end{aligned}$$

By considering the coefficient of f_p we have that

$$\begin{aligned}
x^2 - y - 2x^4yzf - 2x^2\left\{\frac{y^3zt(w-1)}{F(2)} + y^2z(t-1)\right\}& \\
- \frac{2x^2y^2z(w-1)(F(2)-yz)}{F(2)} - \frac{x^2y(w-1)F(4)}{F(2)} &= -\sqrt{\triangle},
\end{aligned}$$

where $\triangle$ is the discriminant of the following quadratic equation for planar maps obtained in [31]:

$$\begin{aligned}
&x^4yzf^2 + (y - x^2G)f + x^2G - y - x^2yF(2) = 0, \\
&G = 1 - \frac{2y^3z^2t(w-1)}{F(2)} - 2y^2z(t-1) \\
&\qquad - y^2z(w-1)\frac{F(2)-yz}{F(2)} - y(w-1)\frac{F(4)}{F(2)}.
\end{aligned} \tag{2.16}$$

Theorem A. *The enumerating functions f_p and f satisfy the following equation:*

$$\begin{aligned}
-\sqrt{\triangle}f_p &= x^4y\frac{\partial xf}{\partial x} - x^2yF_p(2) - 2\frac{x^2y^3zt(w-1)(F_p(2)-\frac{1}{z}F(2))}{F^2(2)}(f-1) \\
&\quad + 3\frac{x^2y^3zt(w-1)}{F(2)}(f-1) + 3y^2(t-1)(f-1) + 3x^2y^2(t-1)(f-1) \\
&\quad - 2\frac{x^2y^2z(w-1)(F(2)-yz)}{F(2)}\left\{\frac{F_p(2)-\frac{1}{z}F(2)}{F(2)}(f-1)\right\} \\
&\quad + 3\frac{x^2y^2(w-1)(F(2)-yz)}{F(2)}(f-1) \\
&\quad + \frac{x^2y(w-1)}{F(2)}\left\{F_p(4)(f-1) - \frac{F^1(4)}{F(2)}(F_p(2) - \frac{1}{z}F(2)) - \frac{F^1(4)}{z}(f-1)\right\}.
\end{aligned}$$

Remark 2.7. Since the above equation contains 5 or more parameters and much information, many known results such as for monopoles on N_1 [36], 4-regular maps on N_1 [36] (the case of $t = w = 1$) and loopless 4-regular maps on N_1 [39] (the case of $t = 0, w = 1$) may be derived. If more parameters are introduced, more results will be obtained. But here we are only interested in the case of 4-edge-connected 4-regular maps on N_1 since they are more closer to the simple Eulerian maps which is what we shall handle in the next section.

3 Calculations

In this section we shall concentrate on the calculation of the function in Theorem A and all the arguments are under the restriction $t = w = 0, y = 1$ since this is what we are really interested in and may be handled by the *double-root method* developed by Brown [11]. Since $t = w = 0$ will destroy all the nonroot-vertex loops and all the separating cycles, we have that $F(2) = z, F_p(2) = 1$. Now the equation in Theorem A becomes

$$-\sqrt{\triangle} f_p = x^2 \left\{ x^2 \frac{\partial (xf)}{\partial x} - 1 - 3(f-1) - \frac{F_p(4)}{z}(f-1) + \frac{F(4)}{z^2}(f-1) \right\},$$

where

$$-\sqrt{\triangle} = Hx^2 - 1 - 2x^4 zf, \qquad H = 1 + 2z + \frac{F(4)}{F(2)}. \tag{3.1}$$

Let $x^2 = \eta$ denote a double root of $\triangle$. Then after simplifications we obtain

Theorem B. *The enumerating function of rooted* 4*-edge-connected* 4*-regular maps on the projective plane is*

$$F_p(4) = \frac{z}{1-f}\left(\eta f - \frac{1 - \sqrt{1 - 4z\eta^2 f}}{z\eta} \right) + \frac{z}{1-f} - 3z + \frac{F(4)}{z},$$

where

$$f\eta(2-f) = 1, \qquad \eta = 1 + \frac{z^2\eta^3}{1 - 4z\eta^2}. \tag{3.2}$$

Remark 3.1. By applying Lagrangian inversion (for a reference one may see [19]) for (3.2) and its other version

$$t = 1 + \frac{z^2}{4z - t^2}, \qquad t\eta = 1,$$

we may expand $F_p(4)$ into a power series of z, i.e.,

$$F_p(4) = \sum_{m \ge 0} z(\eta + 1) f^{m+1} - \sum_{\substack{m \ge 1 \\ n \ge 0}} \frac{2}{m} \binom{2m-2}{m-1} z^m \eta^{2m-1} f^{m+n} - 4z - 1 + 2z\eta f + t,$$

where

$$f^s = \sum_{k \ge s} \frac{s}{k 2^{2k-s}} \binom{2k-s-1}{k-1} t^k,$$

$$\eta^s = \frac{z^2}{1-4z} + \sum_{\substack{m\geq 1\\ n\geq 0}} \frac{4^n s}{m!}\binom{m+n-1}{m-1} D_{\eta=1}^{(m-1)}(\eta^{2m+2n+s-1})z^{2m+n},$$

$$t^s = \frac{z^2}{4z-1} + \sum_{\substack{m\geq 1\\ n\geq 0}} \frac{s}{4^{m+n}m!}\binom{m+n-1}{m-1} D_{t=1}^{(m-1)}(t^{2n+s-1})z^{m-n}.$$

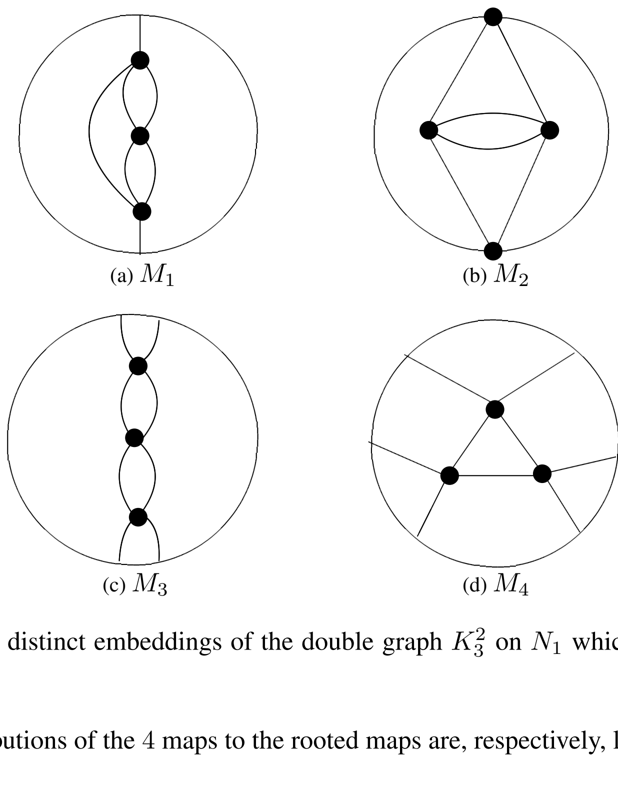
Here

$$D_{x=a}^{(m)} f(x) = \left.\frac{d^m f(x)}{dx^m}\right|_{x=a}.$$

By using an algebraic symbolic system such as Maple [32], the first few coefficients of $F_p(4)$ may be calculated:

$$\begin{aligned} F_p(4) = 6z^2 &+ 21z^3 + 94z^4 + 471z^5 + 2516z^6 + 14006z^7 + 80246z^8 + 469635z^9 \\ &+ 2793758z^{10} + 16835979z^{11} + 102530832z^{12} + 629875252z^{13} \\ &+ 3898000312z^{14} + 24274540180z^{15} + 151988898790z^{16} \\ &+ 956148311939z^{17} + 6040124414714z^{18} + \cdots \end{aligned}$$

One may compare the initial values of the maps with the number presented by the formula above and see that they coincide with each other. For instance, there are 6 rooted 4-edge-connected 4-regular maps with 3 faces on N_1 and the following group of unrooted maps (as shown in Figure 6) will induce 21 distinct rooted maps with 4 faces on N_1.



(a) M_1 (b) M_2

(c) M_3 (d) M_4

Figure 6: Four distinct embeddings of the double graph K_3^2 on N_1 which will induce 21 rooted maps.

The contributions of the 4 maps to the rooted maps are, respectively, listed in Table 1.

Table 1: Initial values for 3-vertex rooted maps on N_1.

M_1	M_2	M_3	M_4
12	6	2	1

4 Asymptotic evaluations

In this section we concentrate on the approximate values of the number of the maps obtained in the previous section. We first state some basic facts.

Fact 4.1. *Let M be a class of infinite rooted maps with M_1 as its subset. Suppose that their enumerating functions may be expanded into power series and their respective convergence radiuses are R and R_1. Then R and R_1 are, respectively, the singularities of them. Furthermore, $R \leq R_1$.*

Fact 4.2 (Long [30])**.** *The convergence radius of the power series expansion with the number of inner faces as the parameter of the enumerating function for the rooted 2-connected 4-regular maps without loops on the projective plane is $\frac{27}{196}$. Consequently, rooted 4-edge-connected 4-regular maps on the projective plane must have its convergence radius, say r, of enumerating function satisfying $\frac{27}{196} \leq r \leq 1$.*

As we have reasoned in [31], the function $H = G|_{y=1}$ (defined in Theorem A) and the parameter $\eta = x^2 = x$ may be rewritten as

$$Hx = \frac{1 - 2z^2x^3}{1 - 2zx^2}, \qquad x = 1 + \frac{z^2x^3}{1 - 4zx^2}. \tag{4.1}$$

Remark 4.3. Here we use $x^2 = x = \eta$ to denote a double root of $\triangle$ in convenience.

Since Darboux's Theorem [2, Theorem 4] shows that the *dominating singularity* (i.e., that corresponds to the convergence radius) of the power series expansion of an enumerating function of a type of maps contains a lot of information about the number of maps considered, we have to locate the wanted singularity. By the definition of $F_p(4)$ we have the following

Fact 4.4. *The singularities of $F_p(4)$ satisfy either*

(1) $f - 1 = 0$ *or*

(2) $x = 0$ *or*

(3) $vx = 1$ *or*

(4) Equations (4.1).

From the above fact we may conclude that the dominating singularity of $F_p(4)$ must satisfy (4.1). We now investigate the singularities of (4.1). The equation on the right hand side of (4.1) may be rewritten as

$$S(x, z) = (z^2 + 4z)x^3 - 4zx^2 - x + 1 = 0. \tag{4.2}$$

By the implicit function theorem [24, Theorem 1.3.1], (4.2) will determine a function $x = x(z)$ with its singularities also satisfying $\frac{\partial S(x,z)}{\partial x} = 0$. From this we may obtain a group of equations as

$$\begin{aligned} (z^2+4z)x^3 - 4zx^2 - x + 1 = 0, \\ 3(z^2+4z)x^2 - 8zx - 1 = 0. \end{aligned} \tag{4.3}$$

The variable z may be extracted from (4.3) such that

$$z = \frac{3-2x}{4x^2}.$$

Now we substitute it into an equation of (4.3) and obtain another high order equation of x:

$$3(3-2x)^2 + 48x^2(3-2x) - 32x(3-2x) - 16x^2 = 0. \tag{4.4}$$

Factoring the left hand side of (4.4) yields

$$(-8x+9)(2x-1)^2 = 0.$$

One may see that the only possible singularity satisfying Facts 4.1 and 4.2 is $z = \frac{4}{27}$ (which corresponds to $x = \frac{9}{8}$).

In order to apply Darboux's Theorem we also need to investigate the behavior of $x = x(z)$ near the dominating singularity $m = \frac{4}{27}$. Since both $\frac{\partial^2 S(x,z)}{\partial x^2}$ and $\frac{\partial S(x,z)}{\partial x}$ are not equal to zero (where the function $S(x,z)$ is defined by the Lagrangian inversion formula in (4.2)) at m, $x = x(z)$ may be expanded into a power series of $(1-\frac{z}{m})^{\frac{1}{2}}$ near m. Let

$$\begin{aligned} x &= a + b\Big(1-\frac{z}{m}\Big)^{\frac{1}{2}} + c\Big(1-\frac{z}{m}\Big) + d\Big(1-\frac{z}{m}\Big)^{\frac{3}{2}}, \\ z &= m - m\Big(1-\frac{z}{m}\Big). \end{aligned} \tag{4.5}$$

Substitute those into the equation system (4.3) and set the coefficients of $(1-\frac{z}{m})$ and $(1-\frac{z}{m})^{\frac{3}{2}}$ to zero we obtain a group of relations such as

$$\begin{aligned} a &= \frac{4}{27}, \\ b^2 &= \frac{27}{256}, \\ c &= \frac{6ma^2 - 3mb^2 + 12a^2 - 12b^2 - 8a}{2(3ma+12a-4)}, \\ d &= \frac{203\sqrt{3}}{16}. \end{aligned}$$

We now begin to study the asymptotic behavior of $F_p(4)$ near m. By Theorem B and Darboux's Theorem, we have the following

Fact 4.5. *The coefficients of $F_p(4)$ will be dominated by those of $\frac{\sqrt{1-4zx^2f}}{x(1-f)}$, when the number of faces is sufficiently large.*

Remark 4.6. Although there are several other terms in the expression of $F_p(4)$ in Theorem B which will contribute several corresponding coefficients to the power series of $F_p(4)$ by the famous *transfer theorem* discussed by Flajolet and Odlyzko [13], their effects may be ignored when the number of faces is large enough.

Let $u = \sqrt{1 - \frac{z}{m}}$. Then the coefficient of u^0 in $1 - 4zx^2 f$ is

$$1 - 4ma^2 \frac{1 - am}{1 - 2ma^2} = 0, \qquad a = \frac{9}{8}, \qquad m = \frac{4}{27}.$$

Now we concentrate on the evaluation of the coefficient of u in $1 - 4zx^2 f$. After many manipulations we see that this number is

$$-\frac{b(3 - 2ma^2)}{a(1 - 2ma^2)},$$

which implies that

$$\sqrt{1 - 4zx^2 f} \approx \sqrt{-\frac{b(3 - 2ma^2)}{a(1 - 2ma^2)}} \left(1 - \frac{z}{m}\right)^{\frac{1}{4}}.$$

Substitute this into (4.3) we see that

$$\frac{\sqrt{1 - 4zx^2 f}}{x(1 - f)} \approx \frac{\sqrt{-\frac{b(3-2ma^2)}{a(1-2ma^2)}}}{\frac{ma^2(1-2a)}{1-2ma^2}} \left(1 - \frac{z}{m}\right)^{\frac{1}{4}}.$$

This together with the Darboux's Theorem concludes the following

Theorem C. *The number of rooted 4-edge-connected 4-regular maps on the projective plane with n inner faces is asymptotic to*

$$-\frac{C}{n^{\frac{5}{4}} m^n \Gamma(-\frac{1}{4})},$$

where $\Gamma(x)$ is the gamma function at x and

$$C = \frac{\sqrt{-\frac{b(3-2ma^2)}{a(1-2ma^2)}}}{\frac{ma^2(1-2a)}{1-2ma^2}},$$

and

$$a = \frac{9}{8}, \qquad m = \frac{4}{27}, \qquad |b| = \frac{3\sqrt{3}}{16}.$$

Here the sign of the number b is chosen to guarantee that the number C is a real number.

Remark 4.7. Although $\frac{4}{27}$ is very close to $\frac{27}{196}$, the dominating singularity corresponding to the rooted 2-connected loopless 4-regular maps on N_1, almost all the maps of such type are not 4-edge-connected by Theorem C, that is, nearly all of such maps must contain at least one separating cycle.

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Mapification of n-dimensional abstract polytopes and hypertopes

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Abstract

The n-dimensional abstract polytopes and hypertopes, particularly the regular ones, have gained great popularity over recent years. The main focus of research has been their symmetries and regularity. The planification of a polyhedron helps its spatial construction, yet it destroys symmetries. No "planification" of n-dimensional polytopes do exist, however it is possible to make a "mapification" of an n-dimensional polytope; in other words it is possible to construct a restrictedly-marked map representation of an abstract polytope on some surface that describes its combinatorial structures as well as all of its symmetries. There are infinitely many ways to do this, yet there is one that is more natural that describes reflections on the sides of $(n-1)$-simplices (flags or n-flags) with reflections on the sides of n-gons. The restrictedly-marked map representation of an abstract polytope is a cellular embedding of the flag graph of a polytope. We illustrate this construction with the 4-cube, a regular 4-polytope with automorphism group of size 384. This paper pays a tribute to Lynne James' last work on map representations.

Keywords: Maps, hypermaps, representation, polytopes, hypertopes.

Math. Subj. Class. (2020): 05C10, 05C25, 05C65, 05E18, 20F65

1 Introduction

This paper stands to be a tribute to Lynne James' last, and unfinished, work [9], where she outlines a method of representing topological categories, such as the categories of cell decompositions of n-manifolds, by other categories, for example the category of cell decompositions of oriented surfaces.

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A map is a cellular embedding of a graph (possibly with multiple edges, loops and/or free edges) on connected surfaces with or without boundary. Algebraically a map $\mathcal{M} = (\Omega; r_0, r_1, r_2)$ is a set Ω of triangular pieces of surface called *flags*, and 3 involutory permutations r_0, r_1, r_2 on Ω satisfying $(r_0 r_2)^2 = 1$ and generating a transitive group on Ω called the *monodromy group* of the map.

An abstract n-polytope $\mathcal{P}$ is a partially ordered set (poset) of faces with a strictly monotone rank function of range $\{-1, 0, \dots, n\}$, represented by a Hasse diagram with $n + 1$ layers, where the poset obey the diamond condition and the set of flags are strongly flag-connected. *Flags* are maximal chains of faces, that is, vectors consisting of $n + 2$ faces of rank $-1, 0, 1, \dots, n$ respectively. There is a unique least face, the (-1)-face F_{-1} and a unique greatest face the n-face F_n. Faces of rank $0, 1$ and $n - 1$ are called *vertices*, *edges* and *facets*, respectively. Two flags are *adjacent* if they differ only by one face (entry). Flags are *strongly flag-connected* means that any two flags Ψ_1, Ψ_2 are connected by a sequence of flags $\Phi_0 = \Psi_1, \Phi_1, \dots, \Phi_m = \Psi_2$ such that two successive flags Φ_i, Φ_{i+1} are adjacent and for any i, j, $\Phi_i \cap \Phi_j \supseteq \Psi_1 \cap \Psi_2$. The *diamond condition* says that whenever F_{i-1} and F_{i+1} are faces of ranks $i-1$ and $i+1$ for some i, with $F_{i-1} < F_{i+1}$, then there are exactly two faces F_i of rank i containing F_{i-1} and contained in F_{i+1}, that is, $F_{i-1} < F_i < F_{i+1}$. In other words, the poset of the section $F_{i+1}/F_{i-1} = \{F \in \mathcal{P} \mid F_{i-1} \leq F \leq F_{i+1}\}$ is like a diamond.

An abstract 2-polytope is just a polygon while a 3-polytope is a non-degenerate map (cellular embedding of a loopless graph on some compact connected (i.e. closed) surface), with the property that every edge is incident with exactly two faces, and every vertex on a face is incident with two edges of that face.

A n-hypertope is an extension of an n-polytope by eliminating the partial order set condition [6]. All n-polytopes are finite in this paper and $n > 2$ everywhere. For a further reading on polytopes we address the reader to the classical book by McMullen and Schulte [13].

2 Restrictedly-marked maps

Lynne James in [9] introduced maps representations and associate it to a non-commutative multiplication operation between map type objects. Although restrictedly-marked map representations [4] lie in a different category, they represent the same topological objects with a different perspective and semantics.

Consider the "right triangle" group $\Gamma = \langle R_0, R_2 \rangle * \langle R_1 \rangle \cong (C_2 \times C_2) * C_2$ generated by the three reflections R_0, R_1, R_2 in the sides of a hyperbolic right triangle with two zero internal angles.

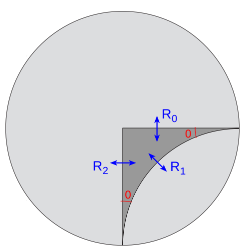


Figure 1: Hyperbolic right triangle on the Poincaré disc.

Every finite index subgroup $M < \Gamma$ determines a finite map $\mathcal{M} = (\Gamma/_{r}M; M^*R_0, M^*R_1, M^*R_2)$, where M^* is the core of M in Γ and each M^*R_i acts as a permutation on the right cosets $\Gamma/_{r}M$ of M in Γ by right multiplication. M is called the *fundamental map subgroup* of $\mathcal{M}$ (or just "map subgroup"). Let Θ be a normal subgroup of Γ with finite index n. A map is Θ-*conservative* if M is a subgroup of Θ. In this case the flags of $\mathcal{M}$ are n coloured under the action of Θ, each colour determined by an orbit (the Θ-orbit) under the action of Θ. By the Kurosh Subgroup Theorem [11, Proposition 3.6, p. 120], Θ freely decomposes into a free product $C_2 * \cdots * C_2 * D_2 * \cdots * D_2 * C_\infty * \cdots * C_\infty = \langle Z_1, \ldots, Z_m \rangle$ for some finite number (possibly zero) of factors $C_2, D_2 = C_2 \times C_2$ and C_∞. This decomposition is unique up to a permutation of the factors [12, p. 245]. A Θ-conservative map can then be represented by a Θ-*marked map* $\mathcal{Q} = (\Omega; z_1, \ldots, z_m)$, where Ω is the set of right cosets $\Theta/_{r}M$ of M in Θ, and each $z_i = M_\Theta Z_i \in \Theta/M_\Theta$ (where M_Θ is the core of M in Θ). The geometric construction described in [2], which can be adapted to Γ [4], uses Θ-slices, polygonal regions determined by a Schreier transversal for Θ in Γ. Θ-slices represent the elements of Ω. For example, a Γ-slice is a "flag" and a Γ^+-slice is a "dart", where Γ^+ is the normal subgroup of index 2 in Γ consisting of the words of even length on R_0, R_1, R_2. The group generated by $z_1, \ldots, z_m$, called the *monodromy group* of $\mathcal{Q}$ (denoted $\mathrm{Mon}(\mathcal{Q})$), or the Θ-monodromy group of $\mathcal{M}$, acts transitively on the set of the Θ-slices Ω.

A covering, or morphism, ψ from a Θ-marked map $\mathcal{Q}_1 = (\Omega_1; z_1, \ldots, z_m)$ to another Θ-marked map $\mathcal{Q}_2 = (\Omega_2; z'_1, \ldots, z'_m)$ is a function $\psi\colon \Omega_1 \longrightarrow \Omega_2$ that commutes the diagram

$$\begin{tikzcd}
\Omega_1 \times \mathrm{Mon}(\mathcal{Q}_1) \arrow[r] \arrow[d, "\psi"', "\iota\colon z_i \mapsto z'_i"] & \Omega_1 \arrow[d, "\psi"] \\
\Omega_2 \times \mathrm{Mon}(\mathcal{Q}_2) \arrow[r] & \Omega_2
\end{tikzcd}$$

An automorphism of $\mathcal{Q}$ is just a bijective covering from $\mathcal{Q}$ to $\mathcal{Q}$. A Θ-marked map $\mathcal{Q}$ is *regular*, or the Γ-marked map $\mathcal{M}$ is Θ-regular, if M is a normal subgroup of Θ. In this case the automorphism group of $\mathcal{Q}$, which is the automorphism group of $\mathcal{M}$ preserving each Θ-orbit, coincides with the monodromy group $\mathrm{Mon}(\mathcal{Q})$, but with different action on Ω. For a more detailed exposition see [2]; though the focus here has been hypermaps the results are easily adapted to maps (see [4]).

By a *restrictedly-regular* (or resctrictly-regular) map we mean a map that is Θ-regular for some (finite index) normal subgroup $\Theta \triangleleft \Gamma$. In a similar way as done in [2], not all maps are restrictedly-regular. However, any group G is the monodromy group (and hence the automorphism group) of a restrictedly-regular map ([3, Lemma 2.2] easily adapted to Γ).

3 Algebraic representation of finite n-polytopes

A Coxeter group is a group with presentation $\langle s_0, s_1, \ldots, s_{n-1} \mid s_i^2 = (s_i s_j)^{p_{ij}} = 1 \rangle$ where $p_{ij} \geq 2$ is a positive integer possibly ∞. If $p_{ij} = \infty$ then the relation $(s_i s_j)^{p_{ij}}$ is vacuous and is not considered in the above presentation. Let $\mathcal{P}$ be an abstract n-polytope, and denote by $\Omega_{\mathcal{P}}$ the set of flags of $\mathcal{P}$. As an immediate consequence of the diamond condition, for any flag $\Phi \in \Omega_{\mathcal{P}}$ and for any $0 \leq i \leq n-1$, the set $\{\Phi' \in \Omega_{\mathcal{P}} \mid F_j(\Phi') = F_j(\Phi), \forall j \neq i\}$, where $F_j(\Phi)$ is the face of rank j of Φ, contains exactly two elements, being Φ one of them. Denote by $\Phi r_i = \Phi'$ the other flag of this set. Then we have n permutations $r_i = \prod_{\Phi \in \Omega_{\mathcal{P}}} (\Phi, \Phi r_i)$ for $i \in \{0, 1, \dots, n-1\}$, giving rise to a transitive permutation group $G(\mathcal{P}) = \langle r_0, r_1, \dots, r_{n-1} \rangle$ on $\Omega_{\mathcal{P}}$, called the *connection group* (or *monodromy group*) of $\mathcal{P}$, that describes the polytope $\mathcal{P}$: each rank i face F_i for $i \in \{0, 1, \dots, n-1\}$, corresponds to an orbit of $\langle r_0, \dots, \hat{r_i}, \dots, r_{n-1} \rangle$ on $\Omega_{\mathcal{P}}$, where $\hat{r_i}$ denotes the absence of r_i. In fact, if F_i is a face of rank i and Φ and Ψ are two flags containing F_i, then by strong connectedness

$$\Phi \langle r_0, \dots, \hat{r_i}, \dots, r_{n-1} \rangle = \Psi \langle r_0, \dots, \hat{r_i}, \dots, r_{n-1} \rangle.$$

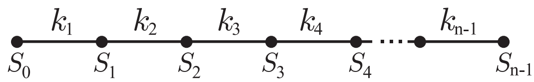
So $\Phi \langle r_0, \dots, \hat{r_i}, \dots, r_{n-1} \rangle$ is the set of all flags containing the common i-face F_i. An i-face $\Phi \langle r_0, \dots, \hat{r_i}, \dots, r_{n-1} \rangle$ is incident to a j-face $\Psi \langle r_0, \dots, \hat{r_j}, \dots, r_{n-1} \rangle$ ($i \neq j$) if and only if $\Phi \langle r_0, \dots, \hat{r_i}, \dots, r_{n-1} \rangle \cap \Psi \langle r_0, \dots, \hat{r_j}, \dots, r_{n-1} \rangle \neq \emptyset$; so incidence corresponds to non-empty intersection.

Hence the polytope $\mathcal{P}$ can be identified with the $n+1$ tuple $(\Omega_{\mathcal{P}}; r_0, r_1, \dots, r_{n-1})$. Two such $n+1$ tuples $(\Omega_1; r_0, r_1, \dots, r_{n-1})$ and $(\Omega_2; s_0, s_1, \dots, s_{n-1})$ are *isomorphic* if there is a bijection f from Ω_1 to Ω_2 that satisfy $\omega r_i f = \omega f s_i$ for every $\omega \in \Omega_1$ and $i \in \{0, 1, \dots, n-1\}$.

Denote by Δ_{n-1} the Coxeter group $\langle S_0, S_1, \dots, S_{n-1} \mid S_i^2 = 1 \rangle$. Then we have a natural epimorphism $\pi \colon \Delta_{n-1} \to G(\mathcal{P})$, mapping each S_i to r_i, inducing an action $\Phi d := \Phi d \pi$ of Δ_{n-1} on $\Omega_{\mathcal{P}}$. Similarly to [2, §1.2], fixing a flag $\Phi \in \Omega_{\mathcal{P}}$ and letting P be the stabiliser of Φ in Δ_{n-1}, then Δ_{n-1} acts on $\Delta_{n-1}/_{r}P$ by right multiplication, inducing a bijective function $\pi_\Phi \colon \Delta_{n-1}/_{r}P \to \Omega_{\mathcal{P}}$, $Pd \mapsto \Phi d \pi$. The kernel of π is the core P^* of P in Δ_{n-1}[1] and the group Δ_{n-1}/P^* acts transitively on $\Delta_{n-1}/_{r}P$ by right multiplication in a similar way as $G(\mathcal{P})$ acts on $\Omega_{\mathcal{P}}$. Hence the polytope $(\Omega_{\mathcal{P}}; r_0, r_1, \dots, r_{n-1})$ is isomorphic to $(\Delta_{n-1}/_{r}P; P^*S_0, P^*S_1, \dots, P^*S_{n-1})$. Every polytope $\mathcal{P}$ is described by such $(n+1)$-tuples; the converse is false. The set of all such $(n+1)$-tuples will be called for the moment the set of $(n-1)$-*hypermaps* (see also Section 7). So both n-polytopes and n-hypertopes are $(n-1)$-hypermaps, the converse is false. The subgroup P will be called a *fundamental subgroup* of $\mathcal{P}$. This is unique up to a conjugacy in Δ_{n-1}.

A $(n-1)$-hypermap $\mathcal{H} = (\Omega_{\mathcal{P}}; r_0, r_1, \dots, r_{n-1})$ is regular if the connection group acts regularly on Ω; this is equivalent to say that the fundamental subgroup P is normal in Δ_{n-1}. In such case $P^* = P$ and, up to an $(n-1)$-hypermap isomorphism, $\Omega = \Delta_{n-1}/_{r}P = G$ is the connection group, which coincides with the automorphism group of $\mathcal{H}$. The action of G on $\Omega = G$ as a connection group is done by right multiplication, while as automorphism group is done by left multiplication.

A string Coxeter group of type $[k_1, k_2, \dots, k_{n-1}]$ ($k_1 > 2$, $k_2 > 2$, $\dots$, $k_{n-1} > 2$) is a Coxeter group $\langle S_0, S_1, \dots, S_{n-1} \mid S_i^2 = 1 \rangle$ satisfying the Dynking diagram (or string diagram) of type $[k_1, k_2, \dots, k_{n-1}]$:



A regular polytope $\mathcal{P}$ is of type $[k_1, k_2, \dots, k_{n-1}]$ if its automorphism group (or connection group) is a smooth quotient of a string Coxeter group Δ_{n-1} of type $[k_1, k_2, \dots, k_{n-1}]$.

For additional details on polytopes and related subjects we address the reader to [14].

[1]Let $\alpha \colon G(\mathcal{P}) \to S_\Omega$ be the action homomorphism of $G(\mathcal{P})$ on Ω, then $\pi\alpha \colon \Delta_{n-1} \to S_\Omega$ (right action notation) is the action homomorphism of Δ_{n-1} on Ω. As $G(\mathcal{P})$ acts faithfully on Ω, then $P^* = \mathrm{Ker}(\pi\alpha) = \pi^{-1}(\mathrm{Ker}(\alpha)) = \pi^{-1}(1) = \mathrm{Ker}(\pi)$.

4 Topological approach to n-polytopes and n-hypertopes

Polytopes appear in the literature both as abstract and geometric. Hypertopes have been essentially introduced as abstract constructions. There is a topological construction that is relevant for what follows later in Section 8. From the example below (Section 6) one see that a flag of the hypercube can be associated to a tetrahedron, that is, a 3-simplex. Replacing faces in the poset of a polytope (hypertope) by simplices and the rank function by the dimension function, under the same conditions, we get an abstract simplicial complex model of a polytope (hypertope).

Towards a more topological approach, let a n-*flag* be an $(n-1)$-simplex with its n vertices labelled $0, 1, \dots, n-1$ and its n facets labelled by the opposite vertex label. Let also Ω be a set of n-flags.

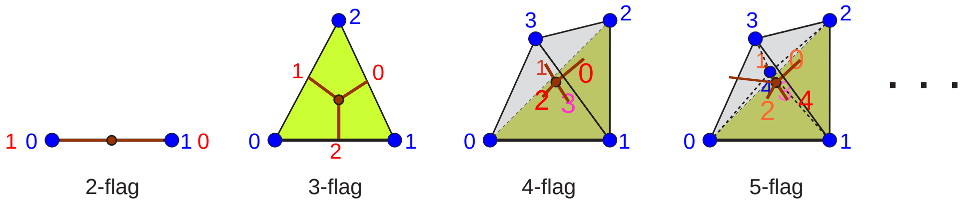


Figure 2: Example of n-flags.

For each $i \in \{0, 1, \dots, n-1\}$ we denote by the transposition $\tau_i = (a, b)$ the joining of two n-flags $a, b \in \Omega$ along their facet labelled i so that its facet's vertex numbers match up, and call it an i-transposition. Denote by r_i the product of i-transpositions recording those pairs of n-flags that are joined by their i-labelled facets; $r_i = 1$ just means that no pairs of n-flags are joined by their facets labelled i. The group $G = \langle r_0, r_1, \dots, r_{n-1} \rangle$ records all the existing joining between the n-flags. We call it a connection group (or monodromy group). If this group acts transitively on Ω then $(\Omega; r_0, r_1, \dots, r_{n-1})$ describes a topological/algebraic object isomorphic[2] to an $(n-1)$-hypermap. Call it n-*hyperplex*[3]. Since n-flags are only connected by their facets, no k-face, for $k < n-1$, can occur as the intersection of two consecutive n-flags. Moreover, if an n-flag is fixed by some r_i this means that the facet labelled i of the flag is on the boundary. Thus boundary cannot be made of k-faces for $k < n-1$. Thus if two n-flags have in common a k-face ($k < n-1$) then necessarily there must be a sequence of n-flags that intersect two by two on a facet containing the k-face. Hence by construction, if this connection group acts transitively on Ω it does so strongly transitively.

This bring polytopes (hypertopes) close to Piecewise Linear Manifolds (PL-manifolds).

5 Regular representation of n-polytopes by restrictedly-marked maps

Following Lynne's ideas [9], and more explicitly the notations and definitions expressed in [4], a regular representation of $(n-1)$-hypermaps by restrictedly-marked maps is a $(m+1)$-tuple $(\Theta; X_0, X_1, \dots, X_{m-1})$, consisting of a normal subgroup Θ of Γ freely generated by $X_0, X_1, \dots, X_{m-1}$ for some $m \geq n$, together with an epimorphism ρ from Θ to Δ_{n-1}.

[2]Isomorphisms taken in the same sense as isomorphisms between Θ-marked maps defined in Section 3.
[3]Terminology introduced by Steve Wilson in BIRS Workshop 17w5162, Canada, 2017.

Such representation gives rise to a bijection between the set of $(n-1)$-hypermaps $\mathcal{P}$ with fundamental subgroup H to the set of regular Θ-marked maps with fundamental subgroup $H\rho^{-1}$, henceforth a representation of n-polytopes (or n-hypertopes).

$$\begin{array}{ccc}
 & \Gamma = C_2 * D_2 = \langle R_1\rangle * \langle R_0, R_2\rangle & \\
 & \| & \\
\underset{\substack{\Theta\text{-marked}\\ \text{map}}}{Q}\left\{\begin{array}{c} \Theta \\ | \\ H\rho^{-1} \end{array}\right. & \begin{array}{c} \xrightarrow{\rho\ \ } \hspace{-1.2em}\twoheadrightarrow \\ \\ \twoheadrightarrow \end{array} & \left.\begin{array}{c} \Delta_{n-1} = \langle S_0, S_1, \dots, S_{n-1}\rangle \\ | \\ H \end{array}\right\} P
\end{array}$$

Theorem 5.1. *There is a regular restrictedly-marked representation of n-polytopes such that:*

(1) n-flags ($(n-1)$-simplices for n-polytopes and -hypertopes) correspond to n-gons;

(2) local reflections about facets of an n-flag corresponds to local reflections on the sides of the n-gon;

(3) the (full) automorphism group of the n-polytope (-hypertope) is the (full) automorphism group of the restrictedly marked map;

(4) the n-polytope (-hypertope) is orientable if and only if the restrictedly marked map is orientable.

Proof. Lynne James's first example [9], more specifically the example given by the alternative construction, gives an answer to this question for $n = 4$. The proof could be resumed to find a normal subgroup Θ of Γ which is freely generated by reflections. However there are only four subgroups that are freely generated by, and only by, reflections, namely

$$\begin{aligned}
&\Gamma_{2.1} = \langle R_0, R_1, R_2R_1R_2\rangle = C_2 * C_2 * C_2,\\
&\Gamma_{2.4} = \langle R_1, R_2, R_0R_1R_0\rangle = C_2 * C_2 * C_2,\\
&\Gamma_{2.5} = \langle R_1, R_2R_0, R_0R_1R_0\rangle = C_2 * C_2 * C_2, \text{ and}\\
&\Gamma_{4.2} = \langle R_1, R_0R_1R_0, R_2R_1R_2, R_0R_2R_1R_2R_0\rangle = C_2 * C_2 * C_2 * C_2.
\end{aligned}$$

These solve the problem for $n = 3$ and 4. The following approach gives a general construction for all $n \geq 3$.

Denote by $\prod_k(R_i, R_j)$ the product $R_iR_jR_iR_jR_i\dots$ of R_i and R_j in alternate form, starting from R_i and counting k total factors. If $k = 0$ then put $\prod_0(R_i, R_j) = 1$. Now take the normal subgroup[4]

$$\Gamma_n = \langle R_0, R_0^{R_1}, R_0^{R_1R_2}, \dots, R_0^{\prod_{n-1}(R_1,R_2)}, (R_1R_2)^n\rangle$$

of rank $n+1$ and index $2n$ in Γ (Γ/Γ_n is a dihedral group of order $2n$). By the Kurosh's Subgroup Theorem [11, Proposition 3.6], these generators decompose Γ_n as a free product

[4]There is another subgroup generated by reflections and one rotation with the same decomposition as a free product $C_2 * C_2 * C_2 * \cdots * C_\infty$, it is the dual resulting from swapping R_0 with R_2. Another subgroup also appears with such free product decomposition $C_2 * C_2 * C_2 * \cdots * C_\infty$, however one of the C_2 is generated by the rotation R_0R_2, instead of a reflection.

$C_2 * C_2 * \cdots * C_2 * C_\infty$. We take the epimorphism $\rho\colon \Gamma_n \to \Delta_{n-1}$ by mapping each $R_0^{\prod_k(R_1,R_2)}$ to S_k, for $k = 0, 1, \ldots, n-1$, and $(R_1R_2)^n$ to 1. Then the regular map with dihedral automorphism group of size $2n$ corresponding to the quotient Γ/Γ_n, called *trivial* Γ_n*-map*, is a star graph cellular embedded in the disk, thus a boundary map with one vertex and n edges. We need to cut open this disk to its centre to create a Γ_n-slice (see [4] for the constructing example of such a Γ_n-slice) for the restricted Γ_n-marked map, however we need to join it back to accomplish $(R_1R_2)^n = 1$, satisfied by the epimorphism ρ, to create a Γ_n-slice for this representation ρ. Each $(n-1)$-hypermap $\mathcal{P}$, and hence each n-polytope (and each n-hypertope), corresponding to a fundamental subgroup P, is isomorphic to a Γ_n-marked map $\mathcal{Q}$ with fundamental subgroup the inverse image $Q = P\rho^{-1}$. The rooted Γ_n-slice for $\mathcal{Q}$ is the above n-gon with a distinguished flag (in black) as shown in Figure 3.

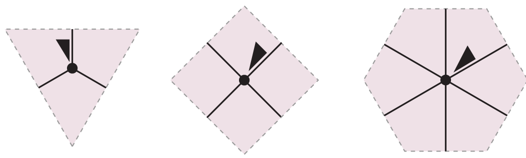


Figure 3: Rooted Γ_3-slice, Γ_4-slice and Γ_6-slice.

The monodromy group (which corresponds to the connection group of the $(n-1)$-hypermap, n-polytope or n-hypertope) is generated by the reflections on the sides of this n-gon. The isomorphism $\bar{\rho}$ between the restricted Γ_n-marked map $\mathcal{Q}$ and $\mathcal{P}$ establishes the third statement. A $(n-1)$-hypermap is orientable (and so is an n-polytope and an n-hypertope) if and only if any word on $r_0, r_1, \ldots, r_{n-1}$ that turns to be the identity has even length, that is, it can be expressed as a word on the rotations $r_1r_2, r_2r_3, \ldots, r_{n-2}r_{n-1}$. Given that the isomorphism $\bar{\rho}$ sends each odd length word $R_0^{\prod_i(R_1,R_2)}$ to r_i, that is also true in the restrictedly marked map representation, that is, the representation word will also be a word of even length on $R_0, R_1, \ldots, R_n$. This establishes the last statement. $\square$

As it turns out from the rooted Γ_n-slice, a restrictedly-marked map representation of an abstract polytope is a cellular embedding of the flag graph of the polytope on a connected surface without boundary. In [4] we dealt only with *clean* restrictedly-marked representations (of hypermaps), that is, regular restricted Θ-marked map representations of n-ranked objects where the generators of Θ give rise to free product decompositions of Θ of equal rank n; this translates, for instance, to 3 generators for representations of hypermaps (rank 3 polytopes). As $(n-1)$-hypermaps (n-polytopes, n-hypertopes) have rank n and Γ_n has rank $n+1$, the above restricted Γ_n-marked map representations are not clean. As a consequence of this fact we have,

Proposition 5.2. *There are infinitely many regular restricted Γ_n-marked map representations of $(n-1)$-hypermaps. Thus in general, there are infinitely many regular restrictedly-marked representations of $(n-1)$-hypermaps, and so of n-polytopes and n-hypertopes.*

Proof. In order to get distinct epimorphisms $\rho\colon \Gamma_n \to \Delta_{n-1} = \langle S_0, \ldots, S_{n-1}\rangle = C_2 * C_2 * \cdots * C_2$ we just assign $R_0^{\prod_k(R_1,R_2)}$ to S_k as before, for $k = 0, 1, \ldots, n-1$, and $(R_1R_2)^n$ to different, and non conjugate, elements in Δ_{n-1}. In this way we get infinitely many distinct epimorphism and hence infinitely many regular restricted Γ_n-marked map representations of $(n-1)$-hypermaps. $\square$

6 Example: The hypercube

As an illustration we take the hypercube, an orientable and regular 4-polytope with 384 flags. The rooted Γ_4-slice of the restricted Γ_4-marked map representation is illustrated in the picture above (Figure 3). To construct the regular restricted Γ_4-map $\mathcal{Q}$ that represents the hypercube, we need to join the 384 rooted Γ_4-slices through their four sides according to the rule dictated by the side reflections $r_0 = R_0, r_1 = R_0^{R_1}, r_2 = R_0^{R_1 R_2}$ and $r_3 = R_0^{R_1 R_2 R_1}$.

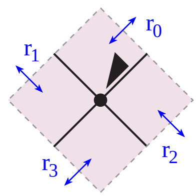


Figure 4: Identification sides on the rooted Γ_4-slice.

The automorphism group G of the hypercube is a Coxeter group of type $[4, 3, 3]$ with presentation

$$\langle r_0, r_1, r_2, r_3 \mid r_0^2, r_1^2, r_2^2, r_3^2, (r_0r_2)^2, (r_0r_3)^2, (r_1r_3)^2, (r_0r_1)^4, (r_1r_2)^3, (r_2r_3)^3 \rangle.$$

Since it is regular, its connection group coincide with its automorphism group (only its action on the flags is different), and since the automorphism group acts regularly on the set of flags its size is the number of flags. So we may replace the set of flags by the automorphism group, in which case the action of the automorphism group on the flags is done by left multiplication while the action of the connection group is done by right multiplication. For the constructing we use the group as a connection group and automatically label its elements $1, 2, 3, \ldots,$ being the identity element the element labelled 1, followed by the elements $2 = r_0 = R_0, 3 = r_1 = R_0^{R_1}, 4 = r_0 r_1, 5 = r_0 r_1 r_0$ etc, so that the first 8 label all the elements of the dihedral subgroup $\langle r_0, r_1 \rangle$ (the central 8-gon). As the hypercube is symmetric around the central 8-gon, we only need to construct one sector, being the rest of the 7 sectors obtained by reflections and rotations about this central 8-gon. So we only need to figure out how to arrange the 48 Γ_4-slices and the final labelling of the outside border of this sector. This is done (with the help of GAP [15]) in the figure below (Figure 5). GAP was used twice:

(i) to ensure that each rooted Γ_4-slice placed inside the sector does not appear when reflecting or rotating around the central 8-gon,

(ii) to get the side-pairings between the labelled sides of the sector with the sides of the rest of the picture.

Bold numbers and letters label the sides of this sector; the red labels signalize identifications inside the same sector, while the black ones label indentifications outside this sector.

Now copy reflecting this sector about the central 8-gon we get the final picture of the hypercube (Figure 6) which reflects a Γ_4-restrictedly regular map on an orientable surface of genus 41. Not all the sides were labelled. To complete the labelling we use the reflections and rotations about the central polygonal region. For example, the central bottom

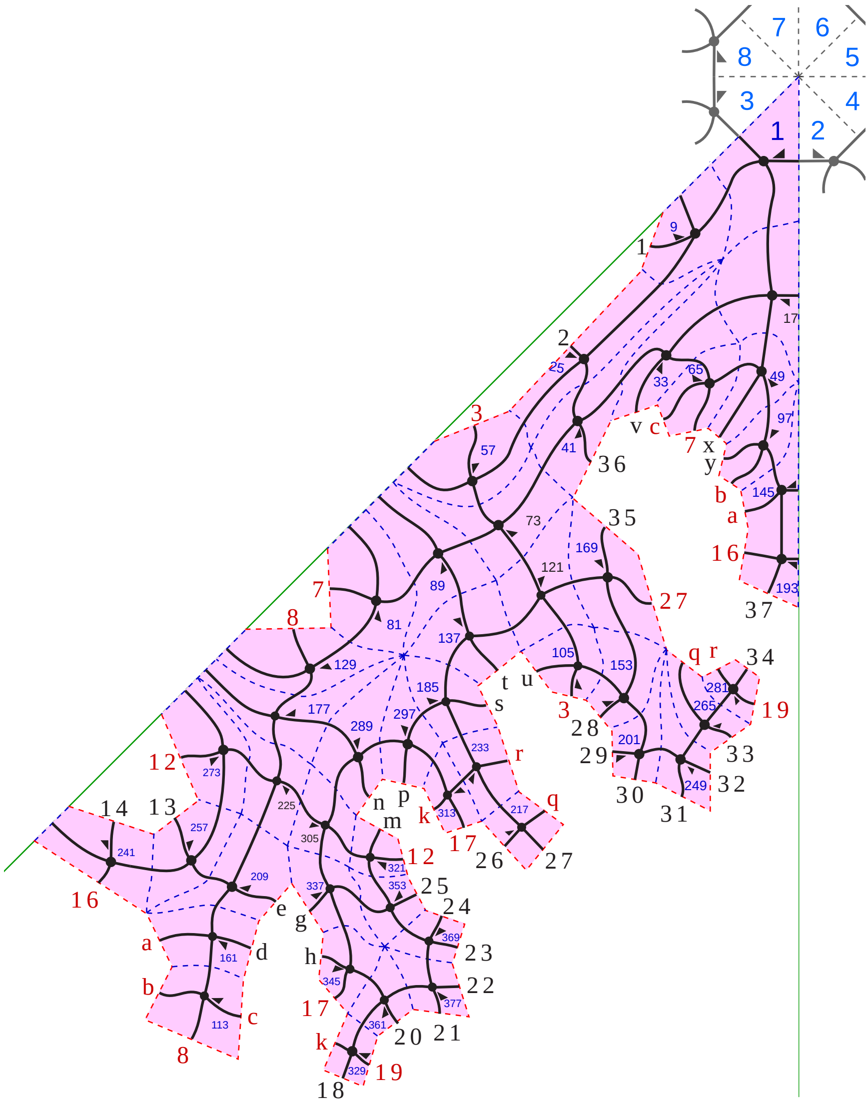


Figure 5: The first sector of the hypercube.

side labelled 37 has its right side unlabelled; label it x for a while. This x vertically mirror reflects to 37, so the x side should be identified to the side y that is the vertical mirror reflection of the identification pair of 37. There is also no arrows to instruct the identification side pairing; this is unnecessary as well since the identification is done similarly to the matching of the internal sides, which was done by following the words R_0, $R_0^{R_1}$, $R_0^{R_1R_2}$ and $R_0^{R_1R_2R_1}$ corresponding to the sides (Figure 4); any of these words will take a rooted Γ_4-slice to a neighbouring rooted Γ_4-slice.

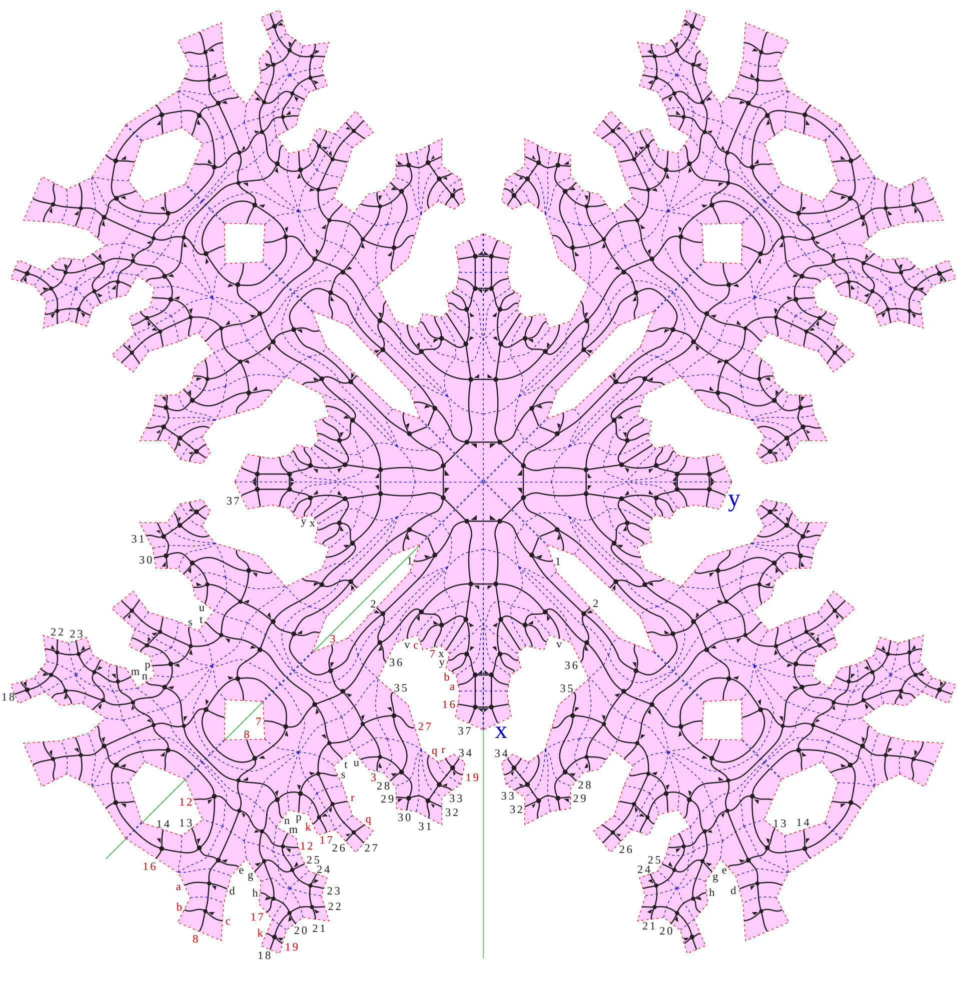


Figure 6: The hypercube.

7 Genus of a regular orientable n-polytope and n-hypertope

The genus $\mathfrak{g}$ of an orientable n-polytope (resp. orientable n-hypertope) can be defined to be the genus of the orientable $(n-1)$-hypermap it corresponds to, which is the genus of the regular Γ_n-marked map representation $\mathcal{Q}$ without boundary. Recall that $\mathfrak{g} = \frac{2-\chi}{z}$, where $z = 2$ if $\mathcal{Q}$ is orientable and 1 otherwise. The formula of the characteristic χ (i.e. the Euler characteristic of the underlying surface of the regular Γ_n-marked map representation) presented below is derived from the characteristic formula in [2] taking into account that the trivial Γ_n-map is a boundary map with edges and faces on the boundary. A direct calculation can go as follows: we see from a rooted Γ_n-slice (Figure 7) that the embedding of the n-coloured graph (flag graph in the case of polytopes and hypertopes) produces n type of faces $f_1, f_2, \ldots, f_n$ determined by

$$\begin{aligned}
\rho_1 &= r_0 r_1 = (R_0 R_1)^2, \\
\rho_2 &= r_0 r_2 = (R_0 R_1)^{R_2}, \\
\rho_3 &= r_1 r_3 = ((R_0 R_1)^2)^{R_2 R_1}, \\
&\vdots \\
\rho_{n-1} &= r_{n-3} r_{n-1} = ((R_0 R_1)^2)^{\prod_{n-2}(R_2, R_1)}, \text{ and} \\
\rho_n &= r_{n-2} r_{n-1} = ((R_0 R_1)^2)^{\prod_{n-1}(R_2, R_1)}.
\end{aligned}$$

Then $F_i = \frac{|G|}{2m_i}$ is the number of faces of type i, where $m_i = |\rho_i|$ and $G = \langle r_0, \ldots, r_{n-1} \rangle$ is the set of n-flags. It also produces n types of edges, one for each r_i, being the number E_i of edges of type i given by $E_i = \frac{|G|}{2}$.

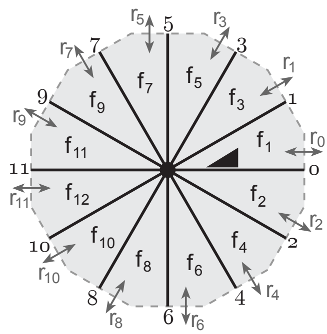


Figure 7: Γ_{12}-slice showing the i-labelled edges and the i-labelled faces.

Finally, as the number V of vertices is $|G|$, the number of edges is $E = E_0 + E_1 + \cdots + E_{n-1}$ and the number of faces is $F = F_1 + F_2 + \cdots + F_n$, then the characteristic $\chi = V - E + F$ of a regular $(n-1)$-hypermap (or a regular n-hypertope) $\mathcal{H} = (G; r_0, r_1, \ldots, r_{n-1})$ is given by

$$\chi = \frac{|G|}{2}\left(\frac{1}{m_1} + \frac{1}{m_2} + \cdots + \frac{1}{m_{n-1}} + \frac{1}{m_n} + 2 - n\right).$$

In regular $(n-1)$-hypermaps we may have $m_i = 1$, so writing $N = -\chi$, we have

$$|G| = \frac{2N}{n - 2 - (\frac{1}{m_1} + \frac{1}{m_2} + \cdots + \frac{1}{m_n})} \leq \frac{2N}{n - 2 - (\frac{1}{1} + \cdots + \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{7})} = 84N,$$

the usual Hurwitz bound.

However, for regular n-polytopes, of type $[m_1, k_2, k_3, \dots, k_{n-1}, m_n]$, we have $m_2 = m_3 = \cdots = m_{n-1} = 2$, which gives the formula

$$\chi = \frac{|G|}{2}\left(\frac{1}{m_1} + \frac{1}{m_n} + \frac{2-n}{2}\right).$$

We have also $m_1 \geq 3$ and $m_n \geq 3$, so that

$$|G| = \frac{2N}{\frac{n-2}{2} - (\frac{1}{m_1} + \frac{1}{m_n})} \leq \frac{12N}{3n-10}.$$

In particular if $n > 3$, then

$$|G| < \frac{4N}{n-4} \quad \text{and} \quad N \geq \frac{|G|}{12}(3n-10) > \frac{|G|}{4}(n-4).$$

For $n > 8$ the minimum size of a regular polytope is $|G| = 2.4^{n-1}$ (Conder [5]), which gives $N > 2.4^{n-2}(n-4)$. A better refinement for n in $\{3, 4, 5, 6, 7, 8\}$ can be made by taking the Propositions 4.1, 4.2, 4.3, 4.4, 4.5 and 4.6 of [5] into account:

Table 1: Leasted values for $|G|$ and negative characteristic N for regular n-polytopes.

n	$\min \lvert G\rvert$ (from [5])	$\min N = -\chi$	inf
3	$\lvert G\rvert \geq 24$	$N \geq -2$	from $\lvert G\rvert = 24$, type $[3,3]$
4	$\lvert G\rvert \geq 96$	$N \geq 18$	from $\lvert G\rvert = 108$, type $[3,6,3]$
5	$\lvert G\rvert \geq 432$	$N \geq 198$	from $\lvert G\rvert = 432$, type $[3,6,3,4]$
6	$\lvert G\rvert \geq 1728$	$N \geq 1296$	from $\lvert G\rvert = 1728$, type $[4,3,6,3,4]$
7	$\lvert G\rvert \geq 7776$	$N \geq 7452$	from $\lvert G\rvert = 7776$, type $[3,6,3,6,3,4]$
8	$\lvert G\rvert \geq 31104$	$N \geq 38234$	estimated with $\lvert G\rvert = 32772$, type $[3,\dots,3]$
n	$\lvert G\rvert \geq 2.4^{n-1}$	$N > 2.4^{n-2}(n-4)$	

Regular n-polytopes and their duals have the same genus. The restricted Γ_4-map pictured in Figure 6 that represents the hypercube has genus 41 ($N = 80$). If we have done the same for the hypertetrahedron, an orientable and regular 4-polytope with 120 flags and automorphism group the Coxeter group of type $[3,3,3]$, we would end up with a regular restricted Γ_4-map of genus 11 ($N = 20$).

The above formulae do not take into account the smallest dimension that a n-polytope (or n-hypertope) might be realised as a complex simplicial manifold. For this we have the genus $g(M)$ of a piecewise linear manifold M introduced by Gagliardi [8] as being the minimum genus of any colour-graph that induces the same piecewise linear manifold M, where the genus of a coloured-graph is the minimum genus of its strongly-regular embeddings. Despite $g(M)$ be a topological invariant, $g(M) = 0$ if and only if M is the $(n-1)$-sphere, and $g(M)$ coincides with usual surface genus if $\dim(M) = 2$ and with Heegaard genus if $\dim(M) = 3$, to calculate $g(M)$ one needs to apply dipoles operations of addition and/or subtraction consecutively on the n-graph in order to transform it into a minimal coloured-graph embedding for M, called a *crystallisation* of M [7]. The genus of a crystallisation is a topological invariant and coincides with $g(M)$. However it has been shown to be difficult to get a transition from a crystallisation to another one [16].

8 n-hypermaps as generalisation of hypermaps

Hypermap are generalisations of maps by allowing edges to join more than two vertices. They are accomplished by cellular embeddings of hypergraphs (bipartite graphs) on connected surfaces. Now n-hypermaps can also be seen as a further generalisation of hypermaps. A map is a cellular embedding of a (1-partite) graph on a connected surface (so to speak a 1-partite map). A *n-hypermap*, $n > 1$, is a cellular embedding of an n-partite graph on a connected surface (that is an n-partite map or n-coloured map) such that each vertex coloured k with $1 < k < n$, is alternately surrounded by vertices coloured $k-1$ and $k+1$, while vertices coloured 1 (resp. n) are surrounded by vertices coloured 2 (resp. $n-1$). They arise as quotients of "$(n+1)$-gonal" groups. Take a hyperbolic $(n+1)$-gon with zero internal angles (Figure 8 left). The dual is a cellular embedding tree (Figure 8 right) and so the Coxeter group generated by the reflections on the sides of this hyperbolic $(n+1)$-gon is a free product $C_2 * C_2 * \cdots * C_2$. Each conjugacy class of a subgroup H in Δ_n determines, up to isomorphism, a n-hypermap $\mathcal{H} = (\Delta_n/_r H; H^*r_0, H^*r_1, \ldots, H^*r_n)$ with monodromy group $\mathrm{Mon}(\mathcal{H}) = \Delta_n/H^*$, where H^* is the core of H in Δ_n, and

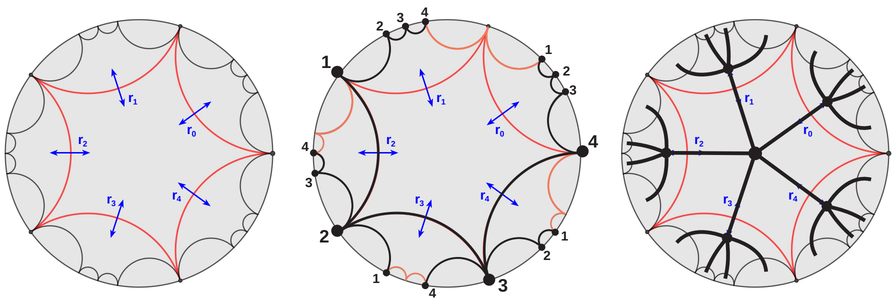


Figure 8: Hyperbolic 5-gonal tesselation (left), 4-hypermap (centre), 5-valency tree (right).

$\mathrm{Aut}(\mathcal{H}) = N_{\Delta_n}(H)/H$. It occurs as an orbifold of the universal hyperbolic n-hypermap illustrated in Figure 8 centre (for $n = 4$). A map is a 1-hypermap and a hypermap is a 2-hypermap.

These $(n+1)$-gonal hyperbolic tessellations on the Poincaré disc are maps. Their duals are maps whose edges are $(n+1)$ coloured (Figure 8 right) representing hyperbolic cellular embeddings of universal $(n+1)$-coloured graphs [1] or $(n+1)$-GEMs ($(n+1)$-graph encoding manifolds) [10].

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An equivalent formulation of the Fan-Raspaud Conjecture and related problems

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Abstract

In 1994, it was conjectured by Fan and Raspaud that every simple bridgeless cubic graph has three perfect matchings whose intersection is empty. In this paper we answer a question recently proposed by Mkrtchyan and Vardanyan, by giving an equivalent formulation of the Fan-Raspaud Conjecture. We also study a possibly weaker conjecture originally proposed by the first author, which states that in every simple bridgeless cubic graph there exist two perfect matchings such that the complement of their union is a bipartite graph. Here, we show that this conjecture can be equivalently stated using a variant of Petersen-colourings, we prove it for graphs having oddness at most four and we give a natural extension to bridgeless cubic multigraphs and to certain cubic graphs having bridges.

Keywords: Cubic graph, perfect matching, oddness, Fan-Raspaud Conjecture, Berge-Fulkerson Conjecture, Petersen-colouring.

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1 Introduction and terminology

Many interesting problems in graph theory are about the behaviour of perfect matchings in cubic graphs. One of the early classical results was made by Petersen [28] and states that every bridgeless cubic graph has at least one perfect matching. Some years ago, one of the most prominent conjectures in this area was completely solved by Esperet et al. in [5]: the conjecture, proposed by Lovász and Plummer in the 1970s, stated that the number

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of perfect matchings in a bridgeless cubic graph grows exponentially with its order (see [20]). However, many others are still open, such as Conjecture 2.1 proposed independently by Berge and Fulkerson in the 1970s as well, and Conjecture 2.2 by Fan and Raspaud (see [10] and [7], respectively). These two conjectures are related to the behaviour of the union and intersection of sets of perfect matchings, and properties of this kind are already largely studied: see, amongst others, [1, 2, 15, 16, 17, 19, 22, 23, 25, 30, 31]. In this paper we prove that a seemingly stronger version of the Fan-Raspaud Conjecture is equivalent to the classical formulation (Theorem 3.3), and so to another interesting formulation proposed in [21] (see also [18]). In the second part of the paper (Section 4 and Section 5), we study a weaker conjecture proposed by the first author in [24]: we show how we can state it in terms of a variant of Petersen-colourings (Proposition 4.1) and we prove it for cubic graphs of oddness four (Theorem 5.4). Although all mentioned conjectures are about simple cubic graphs without bridges, we extend our study of the union of two perfect matchings to bridgeless cubic multigraphs and to particular cubic graphs having bridges (Section 6.1 and Section 6.2).

Graphs considered in the sequel, unless otherwise stated, are simple connected bridgeless cubic graphs and so do not contain loops and parallel edges. Graphs that may contain parallel edges will be referred to as *multigraphs*. For a graph G, let $V(G)$ and $E(G)$ be the set of vertices and the set of edges of G, respectively. A *matching* of G is a subset of $E(G)$ such that any two of its edges do not share a common vertex. For an integer $k \geq 0$, a *k-factor* of G is a spanning subgraph of G (not necessarily connected) such that the degree of every vertex is k. The edge-set of a 1-factor is said to be a *perfect matching*. The least number of odd cycles amongst all 2-factors of G, denoted by $\omega(G)$, is called the *oddness* of G, and is clearly even for a cubic graph since G has an even number of vertices. For $M \subseteq E(G)$, we denote the graph $G \setminus M$ by $\overline{M}$. In particular, when M is a perfect matching of G, then $\overline{M}$ is a 2-factor of G. In this case, following the terminology used for instance in [8], if $\overline{M}$ has $\omega(G)$ odd cycles, then M is said to be a *minimal perfect matching*.

A *cut* in G is any set $X \subseteq E(G)$ such that $\overline{X}$ has more components than G, and no proper subset of X has this property, i.e. for any $X' \subset X$, $\overline{X'}$ does not have more components than G. The set of edges with precisely one end in $W \subseteq V(G)$ is denoted by $\partial_G W$, or just ∂W when it is obvious to which graph we are referring. Moreover, a cut X is said to be *odd* if there exists a subset W of $V(G)$ having odd cardinality such that $X = \partial W$.

We next define some standard operations on graphs that will be useful in the sequel. Let G_1 and G_2 be two bridgeless graphs (not necessarily cubic), and let e_1 and e_2 be two edges such that $e_1 = u_1v_1 \in E(G_1)$ and $e_2 = u_2v_2 \in E(G_2)$. A *2-cut connection* on u_1v_1 and u_2v_2 is a graph operation that consists of constructing the new graph

$$[G_1 - e_1] \cup [G_2 - e_2] \cup \{u_1u_2, v_1v_2\},$$

and denoted by $G_1(u_1v_1) * G_2(u_2v_2)$. It is clear that another possible graph obtained by a 2-cut connection on e_1 and e_2 is $G_1(u_1v_1) * G_2(v_2u_2)$. Clearly, the two graphs obtained are bridgeless, and, unless otherwise stated, if it is not important which of these two graphs is obtained, we use the notation $G_1(e_1) * G_2(e_2)$ and we say that it is a graph obtained by a 2-cut connection on e_1 and e_2.

Now, let G_1 and G_2 be two bridgeless cubic graphs, $v_1 \in V(G_1)$ and $v_2 \in V(G_2)$ such that the vertices adjacent to v_1 are x_1, y_1 and z_1, and those adjacent to v_2 are x_2, y_2 and z_2. A 3-*cut connection* (sometimes also known as the star product, see for instance [11]) on v_1

and v_2 is a graph operation that consists of constructing the new graph

$$[G_1 - v_1] \cup [G_2 - v_2] \cup \{x_1x_2, y_1y_2, z_1z_2\},$$

and denoted by $G_1(x_1y_1z_1) * G_2(x_2y_2z_2)$. The 3-edge-cut $\{x_1x_2, y_1y_2, z_1z_2\}$ is referred to as the *principal* 3-*edge cut* (see for instance [9]). As in the case of 2-cut connections, other graphs can be obtained by a 3-cut connection on v_1 and v_2, and, unless otherwise stated, if it is not important how the adjacencies in the principal 3-edge cut look like, we use the notation $G_1(v_1) * G_2(v_2)$ and we say that it is a graph obtained by a 3-cut connection on v_1 and v_2. It is clear that any resulting graph is also bridgeless and cubic.

2 A list of relevant conjectures

One of the aims of this paper is to study the behaviour of perfect matchings in cubic graphs, more specifically the union of two perfect matchings (see Section 4 and Section 5). We relate this to well-known conjectures stated here below, in particular: the Berge-Fulkerson Conjecture and the Fan-Raspaud Conjecture.

Conjecture 2.1 (Berge-Fulkerson [10]). *Every bridgeless cubic graph G admits six perfect matchings $M_1, \ldots, M_6$ such that any edge of G belongs to exactly two of them.*

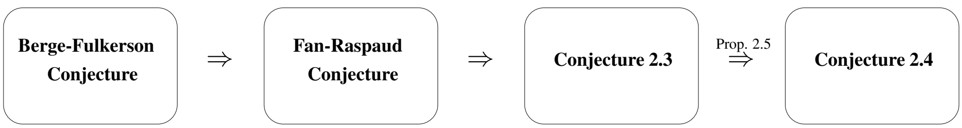

Figure 1: Conjectures mentioned and how they are related.

We also state here other (possibly weaker) conjectures implied by the above conjecture.

Conjecture 2.2 (Fan-Raspaud [7]). *Every bridgeless cubic graph admits three perfect matchings M_1, M_2, and M_3 such that $M_1 \cap M_2 \cap M_3 = \emptyset$.*

In the sequel we will refer to three perfect matchings satisfying Conjecture 2.2 as an *FR-triple*. We can see that Conjecture 2.2 is immediately implied by the Berge-Fulkerson Conjecture, since we can take any three perfect matchings out of the six which satisfy Conjecture 2.1. A still weaker statement implied by the Fan-Raspaud Conjecture is the following:

Conjecture 2.3 ([21]). *For each bridgeless cubic graph G, there exist two perfect matchings M_1 and M_2 such that $M_1 \cap M_2$ contains no odd-cut of G.*

We claim that any two perfect matchings out of the three in an FR-triple have no odd-cut in their intersection, in other words that Conjecture 2.2 implies Conjecture 2.3. For, suppose not. Then, without loss of generality, suppose that $M_2 \cap M_3$ contains an odd-cut X. Hence, since every perfect matching has to intersect an odd-cut at least once, $|M_1 \cap (M_2 \cap M_3)| \geq |M_1 \cap X| \geq 1$, a contradiction, since we assumed that $M_1 \cap M_2 \cap M_3$ is empty. In relation to the above, the first author proposed the following conjecture:

Conjecture 2.4 (S_4-Conjecture [24]). *For any bridgeless cubic graph G, there exist two perfect matchings such that the deletion of their union leaves a bipartite subgraph of G.*

For reasons which shall be obvious in Section 4 we let such a pair of perfect matchings be called an S_4*-pair* of G and shall refer to Conjecture 2.4 as the S_4-Conjecture. We will first proceed by showing that this conjecture is implied by Conjecture 2.3, and so, by what we have said so far, is a consequence of the Berge-Fulkerson Conjecture. In particular, we can see the S_4-Conjecture as Conjecture 2.3 restricted to odd-cuts $\partial V(C)$, where C is an odd cycle of G.

Proposition 2.5. *Conjecture 2.3 implies the* S_4*-Conjecture.*

Proof. Let M_1 and M_2 be two perfect matchings such that their intersection does not contain any odd-cut. Consider $\overline{M_1 \cup M_2}$, and suppose that it contains an odd cycle C. Then, all the edges of $\partial V(C)$ belong to $M_1 \cap M_2$. If $\overline{\partial V(C)}$ has exactly two components, then $\partial V(C)$ is an odd-cut belonging to $M_1 \cap M_2$, a contradiction. Therefore, $\overline{\partial V(C)}$ must have more than two components, say k, denoted by $C_1, C_2, \ldots, C_k$, where the first component C_1 is the cycle C. Let $[C_1, C_j]$ denote the set of edges between C_1 and C_j, for $j \in \{2, \ldots, k\}$. Since $\sum_{j=2}^{k} |[C_1, C_j]| = |\partial V(C)| \equiv 1 \pmod 2$, there exists $j' \in \{2, \ldots, k\}$, such that $|[C_1, C_{j'}]| \equiv 1 \pmod 2$. However, $[C_1, C_{j'}]$ is an odd-cut which belongs to $M_1 \cap M_2$, a contradiction. □

3 Statements equivalent to the Fan-Raspaud Conjecture

Let $M_1, \ldots, M_t$ be a list of perfect matchings of G, and let $a \in E(G)$. We denote the number of times a occurs in this list by $\nu_G[a : M_1, \ldots, M_t]$. When it is obvious which list of perfect matchings or which graph we are referring to, we will denote this as $\nu(a)$ and refer to it as the *frequency of* a. We will sometimes need to refer to the frequency of an ordered list of edges, say (a, b, c), and we will do this by saying that the frequency of (a, b, c) is (i, j, k), for some integers i, j and k. Mkrtchyan et al. [27] showed that the Fan-Raspaud Conjecture, i.e. Conjecture 2.2, is equivalent to the following:

Conjecture 3.1 ([27]). *For each bridgeless cubic graph* G*, any edge* $a \in E(G)$ *and any* $i \in \{0, 1, 2\}$*, there exist three perfect matchings* $M_1, M_2,$ *and* M_3 *such that* $M_1 \cap M_2 \cap M_3 = \emptyset$ *and* $\nu_G[a : M_1, M_2, M_3] = i$*.*

In other words they show that if a graph has an FR-triple then, for every i in $\{0, 1, 2\}$, there exists an FR-triple in which the frequency of a pre-chosen edge is exactly i. In the same paper, Mkrtchyan et al. state the following seemingly stronger version of the Fan-Raspaud Conjecture:

Conjecture 3.2 ([27]). *Let* G *be a bridgeless cubic graph,* w *a vertex of* G *and* i, j *and* k *three integers in* $\{0, 1, 2\}$ *such that* $i + j + k = 3$*. Then,* G *has an FR-triple in which the edges incident to* w *in a given order have frequencies* (i, j, k)*.*

This means that we can prescribe the frequencies to the three edges incident to a given vertex. At the end of [27], the authors remark that it would be interesting to show that Conjecture 3.2 is equivalent to the Fan-Raspaud Conjecture. We prove here that this is actually the case.

Theorem 3.3. *Conjecture 3.2 is equivalent to the Fan-Raspaud Conjecture.*

Proof. Since the Fan-Raspaud Conjecture is equivalent to Conjecture 3.1, it suffices to show the equivalence of Conjectures 3.1 and 3.2. The latter clearly implies the former, so assume Conjecture 3.1 is true and let a, b and c be the edges incident to w such that the frequencies (i, j, k) are to be assigned to (a, b, c). It is sufficient to show that there exist two FR-triples in which the frequencies of (a, b, c) are $(2, 1, 0)$ in one FR-triple (Case 1 below) and $(1, 1, 1)$ in the other FR-triple (Case 2 below).

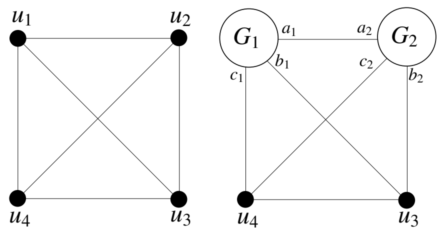

Figure 2: The graphs K_4 and K_4^* in Case 1 of the proof of Theorem 3.3.

Case 1. Let u_1, u_2, u_3 and u_4 be the vertices of the complete graph K_4 as in Figure 2. Consider two copies of G, and let the vertex w in the i^{th} copy of G be denoted by w_i, for each $i \in \{1, 2\}$. We apply a 3-cut connection between u_i and w_i, for each $i \in \{1, 2\}$. With reference to this resulting graph, denoted by K_4^*, we refer to the copy of the graph $G - w$ at u_1 as G_1, and to the corresponding edges a, b and c as a_1, b_1 and c_1, respectively. The graph G_2 and the edges a_2, b_2 and c_2 are defined in a similar way, and the 3-cut connection is done in such a way that b_1 and b_2 are adjacent, and also c_1 and c_2, as Figure 2 shows. Note also that a_1 and a_2 coincide in K_4^*. By our assumption, there exists an FR-triple M_1, M_2 and M_3 of K_4^* in which the edge u_3u_4 has frequency 2. Without loss of generality, let $u_3u_4 \in M_1 \cap M_2$. Then, a_1 (and so a_2) must belong to $M_1 \cap M_2$. Clearly, a_1 (and so a_2) cannot belong to M_3, and so the principal 3-edge-cuts with respect to G_1 and G_2 do not belong to M_3. If $b_1 \in M_3$, then we are done, as then M_1, M_2 and M_3 restricted to G_1, together with a and b having the same frequencies as a_1 and b_1, induce an FR-triple of G such that the frequencies of (a, b, c) are $(2, 1, 0)$. So suppose $c_1 \in M_3$. Then, $b_2 \in M_3$, and so by a similar argument applied to G_2 and the corresponding edges, M_1, M_2 and M_3 induce an FR-triple in G such that the frequencies of (a, b, c) are $(2, 1, 0)$.

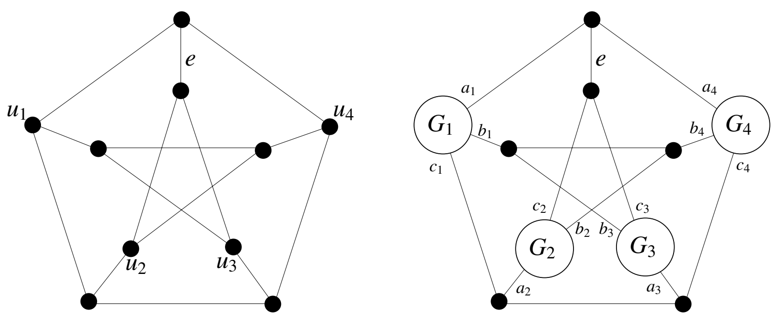

Figure 3: The graphs P and P^* in Case 2 of the proof of Theorem 3.3.

Case 2. Let P be the Petersen graph and $\{u_1, u_2, u_3, u_4\}$ be a maximum independent set of vertices in P as in Figure 3. Consider four copies of G. Let the vertex w in the i^{th} copy of G be denoted by w_i, for each $i \in \{1, \ldots, 4\}$. Let P^* be the graph obtained by applying a 3-cut connection between each u_i and w_i, as shown in Figure 3. Similar to Case 1 we refer to the copy of $G - w$ at u_i as G_i and to the corresponding edges a, b and c as a_i, b_i and c_i, respectively. Since we are assuming that Conjecture 3.1 is true, we can consider an FR-triple M_1, M_2 and M_3 of P^* in which the edge e incident to both a_1 and a_4 has frequency 2. Without loss of generality, let the two perfect matchings containing e be M_1 and M_2. The edges a_1, c_2, c_3 and a_4 are not contained in M_1 and neither M_2, since they are all incident to e, and so no principal 3-edge-cut leaving G_i belongs to M_1 or M_2. Then, M_1 and M_2 induce perfect matchings of P (clearly distinct), and since there are exactly two perfect matchings of P containing e, we can assume that M_1 contains $\{e, b_1, a_2, a_3, b_4\}$, and M_2 contains $\{e, c_1, b_2, b_3, c_4\}$.

If the third perfect matching M_3 induces a perfect matching of the Petersen graph then the induced perfect matching cannot be one of the perfect matchings induced by M_1 and M_2 in P. Hence, since every two distinct perfect matchings of P intersect in exactly one edge of P, there exists $i \in \{1, 2, 3, 4\}$ such that the frequencies of (a_i, b_i, c_i) are $(1, 1, 1)$ and so, M_1, M_2 and M_3 restricted to G_i, together with a, b and c having the same frequencies as a_i, b_i and c_i, induce an FR-triple in G with the needed property.

Therefore, suppose M_3 contains the principal 3-edge-cut of one of the G_is, say G_1 by symmetry of P^*. Thus, a_1, b_1 and c_1 belong to M_3. The perfect matching M_3 can intersect the principal 3-edge-cut at G_2 either in b_2 or c_2 (not both). If $c_2 \in M_3$ we are done by the same reasoning above applied to G_2 and the corresponding edges. So suppose $b_2 \in M_2 \cap M_3$. Then, $c_4 \in M_3$, and M_3 can only intersect the principal 3-edge-cut at G_3 in c_3, implying that the frequencies of (a_3, b_3, c_3) are $(1, 1, 1)$ in P^* and that M_1, M_2 and M_3 restricted to G_3, together with a, b and c having the same frequencies as a_3, b_3 and c_3, induce an FR-triple in G with the needed property. □

In [27] it is also shown that a minimal counterexample to Conjecture 3.2 is cyclically 4-edge-connected. It remains unknown whether a smallest counterexample to the original formulation of the Fan-Raspaud Conjecture has the same property. Indeed, we only prove that the two assertions are equivalent, but we cannot say whether a possible counterexample to Conjecture 3.2 is itself a counterexample to the original formulation.

4 Statements equivalent to the S_4-Conjecture

All conjectures presented in Section 2 are implied by a conjecture made by Jaeger in the late 1980s. In order to state it we need the following definitions. Let G and H be two graphs. An *H-colouring* of G is a proper edge-colouring f of G with edges of H, such that for each vertex $u \in V(G)$, there exists a vertex $v \in V(H)$ with $f(\partial_G\{u\}) \subseteq \partial_H\{v\}$. If G admits an H-colouring, then we will write $H \prec G$. In this paper we consider S_4-colourings of bridgeless cubic graphs, where S_4 is the multigraph shown in Figure 4.

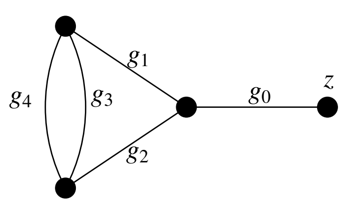

Figure 4: The multigraph S_4.

The importance of H-colourings is mainly due to Jaeger's Conjecture [14] which states that each bridgeless cubic graph G admits a P-colouring (where P is again the Petersen graph). For recent results on P-colourings, known as Petersen-colourings, see for instance [12, 13, 26, 29]. The following proposition shows why we choose to refer to a pair of perfect matchings whose deletion leaves a bipartite subgraph as an S_4-pair.

Proposition 4.1. *Let G be a bridgeless cubic graph, then $S_4 \prec G$ if and only if G has an S_4-pair.*

Proof. Along the entire proof we denote the edges of S_4 by using the same labelling as in Figure 4. Let M_1 and M_2 be an S_4-pair of G. The graph induced by $M_1 \cup M_2$, denoted by $G[M_1 \cup M_2]$, is made up of even cycles and isolated edges, whilst the bipartite graph $\overline{M_1 \cup M_2}$ is made up of even cycles and paths. We obtain an S_4-colouring of G as follows:

- the isolated edges in $M_1 \cup M_2$ are given colour g_0,
- the edges of the even cycles in $M_1 \cup M_2$ are properly edge-coloured with g_3 and g_4, and
- the edges of the paths and even cycles in $\overline{M_1 \cup M_2}$ are properly edge-coloured with g_1 and g_2.

One can clearly see that this gives an S_4-colouring of G. Conversely, assume that $S_4 \prec G$. We are required to show that there exists an S_4-pair of G. Let M_1 be the set of edges of G coloured g_3 and g_0, and let M_2 be the set of edges of G coloured g_4 and g_0. If e and f are edges of G coloured g_3 (or g_4) and g_0, respectively, then e and f cannot be adjacent, otherwise we contradict the S_4-colouring of G. Thus, M_1 and M_2 are matchings. Next, we show that they are in fact perfect matchings. This follows since for every vertex v of G, $f(\partial_G\{v\})$ is equal to $\{g_1, g_3, g_4\}$, or $\{g_2, g_3, g_4\}$, or $\{g_0, g_1, g_2\}$. Thus, $\overline{M_1 \cup M_2}$ is the graph induced by the edges coloured g_1 and g_2, which clearly cannot induce an odd cycle. $\square$

Hence, by the previous proof, Conjecture 2.4 can be stated in terms of S_4-colourings, which clearly shows why we choose to refer to it as the S_4-Conjecture. In analogy to what we did for FR-triples, here we prove that for S_4-pairs we can prescribe the frequency of an edge and the frequencies of the edges leaving a vertex (the proof of the latter also implies that we can prescribe the frequencies of the edges of each 3-cut). Consider the following conjecture, analogous to Conjecture 3.1:

Conjecture 4.2. *For any bridgeless cubic graph G, any edge $a \in E(G)$ and any $i \in \{0, 1, 2\}$, there exists an S_4-pair, say M_1 and M_2, such that $\nu_G[a : M_1, M_2] = i$.*

In Theorem 4.3 we show that the latter conjecture is actually equivalent to the S_4-Conjecture. The proof given in [27] to show the equivalence of the Fan-Raspaud Conjecture and Conjecture 3.1 is very similar to the proof we give here for the analogous case for the S_4-Conjecture, however, we need a slightly more complicated tool in our context.

Theorem 4.3. *Conjecture 4.2 is equivalent to the S_4-Conjecture.*

Proof. Clearly, Conjecture 4.2 implies the S_4-Conjecture so it suffices to show the converse. Assume the S_4-Conjecture to be true and let f_1, f_2, f_3 be three consecutive edges of K_4 inducing a path. Consider two copies of G. Let the edge a in the i^{th} copy of G

be denoted by a_i, for each $i \in \{1,2\}$. Let K_4' be the graph obtained by applying a 2-cut connection between f_i and a_i for each $i \in \{1,2\}$. We refer to the copy of the graph $G - a$ on f_i as G_i.

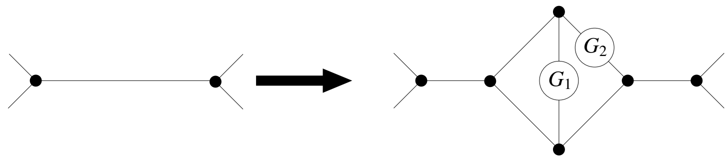

Figure 5: An edge in P transformed into the corresponding structure in H.

Let $\{e_1, \ldots, e_{15}\}$ be the edges of the Petersen graph and let $T_1, \ldots, T_{15}$ be fifteen copies of K_4'. For every $i \in \{1, \ldots, 15\}$, apply a 2-cut connection on e_i and the edge f_3 of T_i. Consequently, every edge e_i of the Petersen graph is transformed into the structure E_i as in Figure 5, and we refer to G_1 and G_2 on E_i as G_1^i and G_2^i, respectively. Let H be the resulting graph. By our assumption, there exists an S_4-pair of H, say M_1 and M_2, which induces a pair of two distinct perfect matchings in P, say N_1 and N_2, respectively. There exists an edge of P, say e_j, for some $j \in \{1, \ldots, 15\}$, such that $\nu_P[e_j : N_1, N_2] = 1$, since every two distinct perfect matchings of P have exactly one edge of P in common. Hence, the restriction of M_1 and M_2 to the edge set of G_1^j, together with the edge a having the same frequency as e_j, gives rise to an S_4-pair of G in which the frequency of a is 1.

Moreover, there exists an edge of P, say e_k, for some $k \in \{1, \ldots, 15\}$, such that $\nu_P[e_k : N_1, N_2] = 2$. Restricting M_1 and M_2 to the edge set of G_1^k, together with the edge a having the same frequency as e_k, gives rise to an S_4-pair of G, in which the frequency of a is 2. Also, the restriction of M_1 and M_2 to the edge set of G_2^k gives rise to an S_4-pair of G (G_2^k together with a), in which the frequency of a is 0, because if not, then there exists an odd cycle in G, say of length α, passing through a and having all its edges with frequency 0. However, this would mean that there is an odd cycle of length $\alpha + 4$ on E_k in $\overline{M_1 \cup M_2}$ (in H), a contradiction. □

As in Section 3, we state an analogous conjecture to Conjecture 3.2, but for S_4-pairs:

Conjecture 4.4. *Let G be a bridgeless cubic graph, w a vertex of G and i, j and k three integers in $\{0,1,2\}$ such that $i + j + k = 2$. Then, G has an S_4-pair in which the edges incident to w in a given order have frequencies (i, j, k).*

The following theorem shows that this conjecture is actually equivalent to Conjecture 4.2, and so to the S_4-Conjecture by Theorem 4.3.

Theorem 4.5. *Conjecture 4.4 is equivalent to the S_4-Conjecture.*

Proof. Since the S_4-Conjecture is equivalent to Conjecture 4.2, it suffices to show the equivalence of Conjectures 4.2 and 4.4. Clearly, Conjecture 4.4 implies Conjecture 4.2 and so we only need to show the converse. Let a, b and c be the edges incident to w such that the frequencies (i, j, k) are to be assigned to (a, b, c). We only need to prove the case when (i, j, k) is equal to $(1, 1, 0)$, as all other cases follow from Conjecture 4.2.

Consider the graph $G(w) * P(v)$, where P is the Petersen graph and v is any vertex of P. We refer to the edges corresponding to a, b and c in $G(w) * P(v)$, as a_w, b_w and c_w.

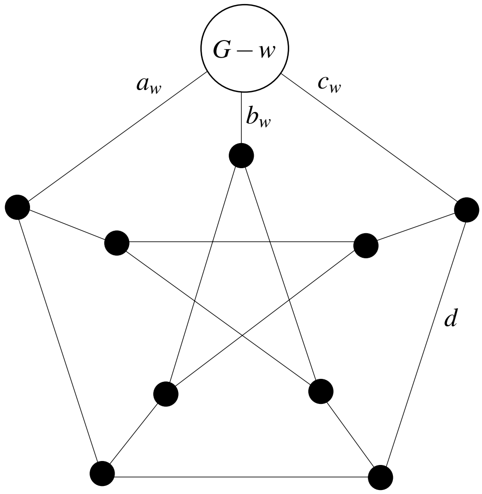

Figure 6: The graph $G(w) * P(v)$ from Theorem 4.5.

Let d be an edge originally belonging to P and adjacent to c_w in $G(w) * P(v)$. Since we are assuming Conjecture 4.2 to be true, there exists an S_4-pair in $G(w) * P(v)$ in which d has frequency 2. If the frequencies of (a_w, b_w, c_w) are $(1, 1, 0)$, then we are done, because the S_4-pair for $G(w) * P(v)$ restricted to the edges in $G - w$, together with a and b having the same frequencies as a_w and b_w, give an S_4-pair for G with the desired property. We claim that this must be the case. For, suppose not. Then, without loss of generality, the frequencies of (a_w, b_w, c_w) are $(2, 0, 0)$. This implies that all the edges of $G(w) * P(v)$ originally in P have either frequency 0 or 2, since the two perfect matchings in the S_4-pair induce the same perfect matching in P. However, this implies that P has a perfect matching whose complement is bipartite, a contradiction since P is not 3-edge-colourable. $\square$

As in [27], a minimal counterexample to Conjecture 4.4 (but not necessarily to the S_4-Conjecture) is cyclically 4-edge-connected. We omit the proof of this result as it is very similar to the proof of Theorem 2 in [27].

5 Further results on the S_4-Conjecture

Little progress has been made on the Fan-Raspaud Conjecture so far. Bridgeless cubic graphs which trivially satisfy this conjecture are those which can be edge-covered by four perfect matchings. In this case, every three perfect matchings from a cover of this type form an FR-triple since every edge has frequency one or two with respect to this cover. Therefore, a possible counterexample to the Fan-Raspaud Conjecture should be searched for in the class of bridgeless cubic graphs whose edge-set cannot be covered by four perfect matchings, see for instance [6]. In 2009, Máčajová and Škoviera [22] shed some light on the Fan-Raspaud Conjecture by proving it for bridgeless cubic graphs having oddness two. One of the aims of this paper is to show that even if the S_4-Conjecture is still open, some results are easier to extend than the corresponding ones for the Fan-Raspaud Conjecture. Clearly, the result by Máčajová and Škoviera in [22] implies the following result:

Theorem 5.1. *Let G be a bridgeless cubic graph of oddness two. Then, G has an S_4-pair.*

We first give a proof of Theorem 5.1 in the same spirit of that used in [22], however much shorter since we are proving a weaker result.

Proof 1 of Theorem 5.1. Let M_1 be a minimal perfect matching of G, and let C_1 and C_2 be the two odd cycles in $\overline{M_1}$. Colour the even cycles in $\overline{M_1}$ using two colours, say 1 and 2. For each $i \in \{1, 2\}$, let E_i be the set of edges belonging to the even cycles in $\overline{M_1}$ and having colour i. In G, there must exist a path Q whose edges alternate in M_1 and E_1 and whose end-vertices belong to C_1 and C_2, respectively, since C_1 and C_2 are odd cycles. Note that since the edges of C_1 and C_2 are not edges in $M_1 \cup E_1$, every other vertex on Q which is not an end-vertex does not belong to C_1 and C_2.

For each $i \in \{1, 2\}$, let v_i be the end-vertex of Q belonging to C_i, and let M_{C_i} be the unique perfect matching of $C_i - v_i$. Let $M_2 := (M_1 \cap Q) \cup (E_1 \setminus Q) \cup M_{C_1} \cup M_{C_2}$. Clearly, M_2 is a perfect matching of G which intersects C_1 and C_2, and so $\overline{M_1 \cup M_2}$ is bipartite. □

We now give a second alternative proof of the same theorem using fractional perfect matchings, which we will show to be easier to use for graphs having larger oddness. Let w be a vector in $\mathbb{R}^{|E(G)|}$. The entry of w corresponding to $e \in E(G)$ is denoted by $w(e)$, and for $A \subseteq E(G)$, we let the weight of A, denoted by $w(A)$, to be equal to $\sum_{e \in A} w(e)$. The vector w is said to be a *fractional perfect matching* of G if:

1. $w(e) \in [0, 1]$ for each $e \in E(G)$,
2. $w(\partial\{v\}) = 1$ for each $v \in V(G)$, and
3. $w(\partial W) \geq 1$ for each $W \subseteq V(G)$ of odd cardinality.

The following lemma is presented in [16] and it is a consequence of Edmonds' characterisation of perfect matching polytopes in [3].

Lemma 5.2. *If w is a fractional perfect matching in a graph G, and $c \in \mathbb{R}^{|E(G)|}$, then G has a perfect matching N such that*

$$c \cdot \chi^N \geq c \cdot w,$$

where $\cdot$ denotes the inner product. Moreover, there exists a perfect matching satisfying the above inequality and which contains exactly one edge of each odd-cut X with $w(X) = 1$.

Remark 5.3. If we let $w(e) = 1/3$ for all $e \in E(G)$, for some graph G, then we know that w is a fractional perfect matching of G. Also, since the weight of every 3-cut is one, then by Lemma 5.2 there exists a perfect matching of G containing exactly one edge of each 3-cut of G.

Proof 2 of Theorem 5.1. Let M_1 be a minimal perfect matching of G, and let C_1 and C_2 be the two odd cycles in $\overline{M_1}$. For each $i \in \{1, 2\}$, let e_1^i and e_2^i be two adjacent edges belonging to C_i. We define the vector $c \in \mathbb{R}^{|E(G)|}$ such that

$$c(e) = \begin{cases} 1 & \text{if } e \in \cup_{i=1}^{2}\{e_1^i, e_2^i\}, \\ 0 & \text{otherwise.} \end{cases}$$

By Remark 5.3, we also know that if we let $w(e) = 1/3$ for all $e \in E(G)$, then w is a fractional perfect matching of G. Hence, by Lemma 5.2, there exists a perfect matching M_2 such that $c \cdot \chi^{M_2} \geq c \cdot w$, which implies that

$$|\cup_{i=1}^{2} \{e_1^i, e_2^i\} \cap M_2| \geq 1/3 \times 2 \times 2 = 4/3 > 1.$$

Therefore, for each $i \in \{1, 2\}$, there exists exactly one $j \in \{1, 2\}$ such that $e_j^i \in M_2$. Hence, M_2 intersects C_1 and C_2 and so $\overline{M_1 \cup M_2}$ is bipartite. □

Using the same idea as in Proof 2 of Theorem 5.1, we also prove that the S_4-Conjecture is true for graphs having oddness four.

Theorem 5.4. *Let G be a bridgeless cubic graph of oddness four. Then, G has an S_4-pair.*

Proof. Let M_1 be a minimal perfect matching of G, and let C_1, C_2, C_3 and C_4 be the four odd cycles in $\overline{M_1}$. By Remark 5.3, there exists a perfect matching N of G such that if G has any 3-cuts, then N intersects every 3-cut of G in one edge. Moreover, for every $i \in \{1, \ldots, 4\}$, there exists at least a pair of adjacent edges e_1^i and e_2^i belonging to $C_i \cap \overline{N}$. We define the vector $c \in \mathbb{R}^{|E(G)|}$ such that

$$c(e) = \begin{cases} 1 & \text{if } e \in \cup_{i=1}^{4} \{e_1^i, e_2^i\}, \\ 0 & \text{otherwise.} \end{cases}$$

We also define the vector $w \in \mathbb{R}^{|E(G)|}$ as follows:

$$w(e) = \begin{cases} 1/5 & \text{if } e \in N, \\ 2/5 & \text{otherwise.} \end{cases}$$

The vector w is clearly a fractional perfect matching of G because, in particular, N intersects every 3-cut in one edge and so $w(X) \geq 1$ for each odd-cut X of G. Hence, by Lemma 5.2, there exists a perfect matching M_2 such that $c \cdot \chi^{M_2} \geq c \cdot w$, which implies that

$$|\cup_{i=1}^{4} \{e_1^i, e_2^i\} \cap M_2| \geq 2/5 \times 2 \times 4 = 16/5 > 3.$$

Therefore, for each $i \in \{1, \ldots, 4\}$, there exists exactly one $j \in \{1, 2\}$ such that $e_j^i \in M_2$. Hence, M_2 intersects C_1, C_2, C_3 and C_4 and so $\overline{M_1 \cup M_2}$ is bipartite. □

As the above proofs show us, extending results with respect to the S_4-Conjecture is easier than in the case of the Fan-Raspaud Conjecture and this is why we believe that a proof of the S_4-conjecture could be a first feasible step towards a solution of the Fan-Raspaud Conjecture. For graphs having oddness at least six we are not able to prove the existence of an S_4-pair and we wonder how many perfect matchings we need such that the complement of their union is bipartite. In the next proposition we use the technique used in Theorem 5.4 and show that given a bridgeless cubic graph G, if $\omega(G) \leq 5^{k-1} - 1$ for some positive integer k, then there exist k perfect matchings such that the complement of their union is bipartite. Note that for $k = 2$ we obtain $\omega(G) \leq 4$.

Proposition 5.5. *Let G be a bridgeless cubic graph and let $\mathcal{C}$ be a collection of disjoint odd cycles in G such that $|\mathcal{C}| \leq 5^{k-1} - 1$ for some positive integer k. Then, there exist $k - 1$ perfect matchings of G, say $M_1, \ldots, M_{k-1}$, such that for every $C \in \mathcal{C}$, there exists $j \in \{1, \ldots, k-1\}$ for which $C \cap M_j \neq \emptyset$. Moreover, if $\omega(G) \leq 5^{k-1} - 1$, then there exist k perfect matchings such that the complement of their union is bipartite.*

Proof. We proceed by induction on k. For $k = 1$, the assertion trivially holds since $\mathcal{C}$ is the empty set. Assume the result is true for some $k \geq 1$ and consider $k+1$. Let $C_1, C_2, \ldots, C_t$, with $t \leq 5^k - 1$, be the odd cycles of G in $\mathcal{C}$. Let N be a perfect matching of G which intersects every 3-cut of G once. For every $i \in \{1, \ldots, t\}$, there exists at least a pair of adjacent edges e_1^i and e_2^i belonging to $C_i \cap \overline{N}$. We define the vector $c \in \mathbb{R}^{|E(G)|}$ such that

$$c(e) = \begin{cases} 1 & \text{if } e \in \cup_{i=1}^t \{e_1^i, e_2^i\}, \\ 0 & \text{otherwise.} \end{cases}$$

We also define the vector $w \in \mathbb{R}^{|E(G)|}$ as follows:

$$w(e) = \begin{cases} 1/5 & \text{if } e \in N, \\ 2/5 & \text{otherwise.} \end{cases}$$

As in the proof of Theorem 5.4, w is a fractional perfect matching of G and by Lemma 5.2 there exists a perfect matching M_k such that $c \cdot \chi^{M_k} \geq c \cdot w$. This implies that

$$| \cup_{i=1}^t \{e_1^i, e_2^i\} \cap M_k| \geq 2 \times 2/5 \times t.$$

Let $\mathcal{C}'$ be the subset of $\mathcal{C}$ which contains the odd cycles of $\mathcal{C}$ with no edge of M_k. Then, $|\mathcal{C}'| \leq |\mathcal{C}| - \frac{4}{5}t = t - \frac{4}{5}t = \frac{t}{5} \leq 5^{k-1} - \frac{1}{5}$, and so $|\mathcal{C}'| \leq 5^{k-1} - 1$. By induction, there exist $k-1$ perfect matchings of G, say $M_1, \ldots, M_{k-1}$, having the required property with respect to $\mathcal{C}'$. Therefore, $M_1, \ldots, M_k$ intersect all odd cycles in $\mathcal{C}$. The second part of the statement easily follows by considering $\mathcal{C}$ to be the set of odd cycles in the complement of a minimal perfect matching M of G, since the union of M with the $k-1$ perfect matchings which intersect all the odd cycles in $\mathcal{C}$ has a bipartite complement. □

Remark 5.6. We note that with every step made in the proof of Proposition 5.5, one could update the weight w of the edges using the methods presented in [16, 23] which gives a slightly better upper bound for $\omega(G)$. For reasons of simplicity and brevity, we prefer the present weaker version of Proposition 5.5.

6 Extension of the S_4-Conjecture to larger classes of cubic graphs

6.1 Multigraphs

In this section we discuss natural extensions of some previous conjectures to bridgeless cubic multigraphs. We note that bridgeless cubic multigraphs cannot contain any loops. We will make use of the following standard operation on parallel edges, referred to as *smoothing*. Let G' be a bridgeless cubic multigraph. Let u and v be two vertices in G' such that there are exactly two parallel edges between them.

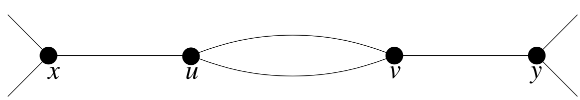

Figure 7: Vertices x, u, v and y in G'.

Let x and y be the vertices adjacent to u and v, respectively (see Figure 7). We say that we *smooth* uv if we delete the vertices u and v from G' and add an edge between x

and y (even if x and y are already adjacent in G'). One can easily see that the resulting multigraph, say G, after smoothing uv is again bridgeless and cubic.

In what follows, we will say that a perfect matching M of G and a perfect matching M' of G' are *corresponding* perfect matchings if the following holds:

$$M = \begin{cases} M' \cup xy - \{xu, vy\} & \text{if } xu \in M', \\ M' - uv & \text{otherwise.} \end{cases}$$

The following theorem can be easily proved by using smoothing operations.

Theorem 6.1. *The S_4-Conjecture is true if and only if every bridgeless cubic multigraph has an S_4-pair.*

Now we show that Conjecture 4.4 can also be extended to multigraphs.

Theorem 6.2. *Let i, j and k be three integers in $\{0, 1, 2\}$ such that $i + j + k = 2$ and let w be a vertex in a bridgeless cubic multigraph G'. Then, the S_4-Conjecture is true if and only if G' has an S_4-pair in which the edges incident to w in a given order have frequencies (i, j, k).*

Proof. It suffices to assume that the S_4-Conjecture is true and only show the forward direction, by Theorem 6.1. Let G' be a minimal counterexample and suppose it has some parallel edges. If $G' = C_{2,3}$ then the result clearly follows. So assume $G' \neq C_{2,3}$. Let a, b and c be the edges incident to w such that the frequencies (i, j, k) are to be assigned to (a, b, c). We proceed by considering two cases: when w has two parallel edges incident to it (Figure 8) and otherwise (Figure 9).

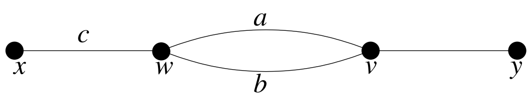

Figure 8: Case 1 in the proof of Theorem 6.2.

Case 1. Let G be the resulting multigraph after smoothing wv. By minimality of G', G has an S_4-pair (say M_1 and M_2) in which $\nu(xy) = k$. It is easy to see that a pair of corresponding perfect matchings in G' give $\nu_{G'}(c) = \nu_{G'}(vy) = k$ and can be chosen in such a way such that $\nu_{G'}(a) = i$ and $\nu_{G'}(b) = j$, a contradiction to our initial assumption. Therefore, we must have Case 2.

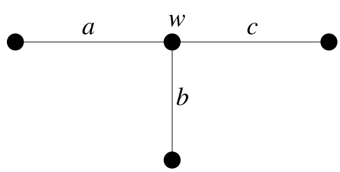

Figure 9: Case 2 in the proof of Theorem 6.2.

Case 2. Let G be the resulting multigraph after smoothing some parallel edge in G' and let a_w, b_w and c_w be the corresponding edges incident to w in G after smoothing is done.

In G, there exists an S_4-pair such that the frequencies of (a_w, b_w, c_w) are equal to (i, j, k). Clearly, the corresponding perfect matchings in G' form an S_4-pair in which the frequencies of (a, b, c) are (i, j, k), a contradiction, proving Theorem 6.2. □

Using the same ideas as in Theorem 6.1 and Theorem 6.2 one can also state analogous results for the Fan-Raspaud Conjecture in terms of multigraphs.

6.2 Graphs having bridges

Since every perfect matching must intersect every bridge of a cubic graph, then the Fan-Raspaud Conjecture cannot be extended to cubic graphs containing bridges. The situation is quite different for the S_4-Conjecture as Theorem 6.3 shows. By Errera's Theorem [4] we know that if all the bridges of a connected cubic graph lie on a single path, then the graph has a perfect matching. We use this idea to show that there can be graphs with bridges that can have an S_4-pair.

Theorem 6.3. *Let G be a connected cubic graph having k bridges, all of which lie on a single path, for some positive integer k. If the S_4-Conjecture is true, then G admits an S_4-pair.*

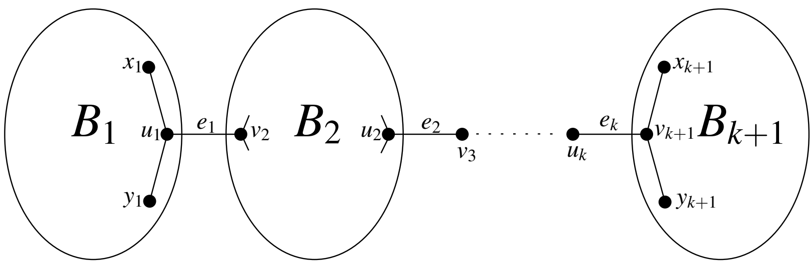

Figure 10: G with k bridges lying all on the same path.

Proof. Let $B_1, B_2, \dots, B_{k+1}$ be the 2-connected components of G and let $e_1, \dots, e_k$ be the bridges of G such that $e_i = u_i v_{i+1}$ for each $i \in \{1, \dots, k\}$, where $u_i \in V(B_i)$ and $v_{i+1} \in V(B_{i+1})$. Let x_1 and y_1 be the two vertices adjacent to u_1 in B_1, and let x_{k+1} and y_{k+1} be the two vertices adjacent to v_{k+1} in B_{k+1}. Let $B_1' = (B_1 - u_1) \cup x_1 y_1$ and $B_{k+1}' = (B_{k+1} - v_{k+1}) \cup x_{k+1} y_{k+1}$. Also, let $B_i' = B_i \cup v_i u_i$ for every $i \in \{2, \dots, k\}$. Clearly, $B_1', \dots, B_{k+1}'$ are bridgeless cubic multigraphs. Since we are assuming that the S_4-Conjecture holds, then, by Theorem 6.1, for every $i \in \{1, \dots, k+1\}$, B_i' has an S_4-pair, say M_1^i and M_2^i. Using Theorem 6.2, we choose the S_4-pair in:

- B_1', such that the two edges originally incident to x_1 (not $x_1 u_1$) both have frequency 1,
- B_i', for each $i \in \{2, \dots, k\}$, such that $\nu_{B_i'}(v_i u_i) = 2$, and
- B_{k+1}', such that the two edges originally incident to x_{k+1} (not $x_{k+1} v_{k+1}$) both have frequency 1.

Let $M_1 := (\cup_{i=1}^{k+1} M_1^i) \cup (\cup_{j=1}^{k} e_j) - \cup_{l=2}^{k} v_l u_l$, and let $M_2 := (\cup_{i=1}^{k+1} M_2^i) \cup (\cup_{j=1}^{k} e_j) - \cup_{l=2}^{k} v_l u_l$. Then, M_1 and M_2 are an S_4-pair of G. □

Finally, we remark that there exist cubic graphs which admit a perfect matching however do not have an S_4-pair. For example, since the edges u_iv_i in Figure 11 are bridges, then they must be in any perfect matching. Consequently, every pair of perfect matchings do not intersect the edges of the odd cycle T. This shows that it is not possible to extend the S_4-Conjecture to the entire class of cubic graphs.

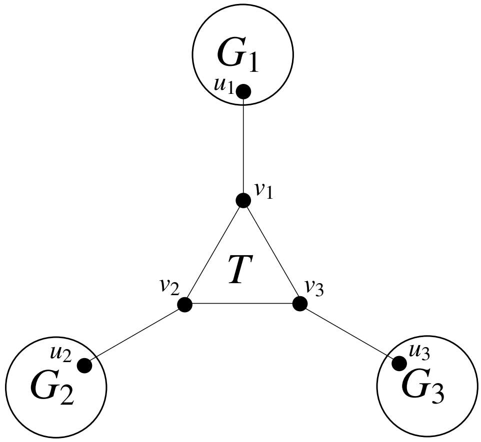

Figure 11: A cubic graph with bridges having no S_4-pair.

7 Remarks and problems

Many problems about the topics presented above remain unsolved: apart from asking if we can solve the Fan-Raspaud Conjecture and the S_4-Conjecture completely, or at least partially for higher oddness, we do not know which are those graphs containing bridges which admit an S_4-pair and we do not know either if the S_4-Conjecture is equivalent to Conjecture 2.3. Here we would like to add some other specific open problems.

For a positive integer k, we define ω_k to be the largest integer such that any graph with oddness at most ω_k, admits k perfect matchings with a bipartite complement. Clearly, for $k = 1$, we have $\omega_1 = 0$, since the existence of a perfect matching of G with a bipartite complement is equivalent to the 3-edge-colourability of G. Moreover, the S_4-Conjecture is equivalent to $\omega_k = \infty$, for $k \geq 2$, but a complete result to this is still elusive. Proposition 5.5 (see also Remark 5.6) gives a lower bound for ω_k and it would be interesting if this lower bound can be significantly improved. We believe that the following problem, weaker than the S_4-Conjecture, is another possible step forward.

Problem 7.1. Prove the existence of a constant k such that every bridgeless cubic graph admits k perfect matchings whose union has a bipartite complement.

It is also known that not every perfect matching can be extended to an FR-triple and neither to a Berge-Fulkerson cover, where the latter is a collection of six perfect matchings which cover the edge set exactly twice. We do not see a way how to produce a similar argument for S_4-pairs and so we also suggest the following problem.

Problem 7.2. Establish whether any perfect matching of a bridgeless cubic graph can be extended to an S_4-pair.

It can be shown that Problem 7.2 is equivalent to saying that given any collection of disjoint odd cycles in a bridgeless cubic graph, then there exists a perfect matching which intersects all the odd cycles in this collection.

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Notes on exceptional signed graphs

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Abstract

A connected signed graph is called exceptional if it has a representation in the root system E_8, but has not in any D_k. In this study we obtain some properties of these signed graphs, mostly expressed in terms of those that are maximal with a fixed number of eigenvalues distinct from -2. As an application, we characterize exceptional signed graphs with exactly 2 eigenvalues. In some particular cases, we prove the (non-)existence of such signed graphs.

Keywords: Adjacency matrix, least eigenvalue, root system, signed line graph, exceptional signed graph, signed graph decomposition.

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1 Introduction

A *signed graph* $\dot{G}$ is a pair (G, σ), where $G = (V, E)$ is a finite unsigned graph, called the *underlying graph*, and $\sigma\colon E \longrightarrow \{-1, 1\}$ is the *signature*. The edge set of a signed graph is composed of subsets of positive and negative edges. Throughout the paper we interpret a graph as a signed graph with all the edges being positive.

The $n \times n$ adjacency matrix $A_{\dot{G}}$ of $\dot{G}$ is obtained from the $(0, 1)$-adjacency matrix of G by reversing the sign of all 1s which correspond to negative edges. The eigenvalues of $A_{\dot{G}}$ are real and form the spectrum of $\dot{G}$.

We establish some properties of exceptional signed graphs. Precisely, we extend some results concerning the particular case of exceptional graphs, which can be found in [3]. It occurs that the 'signed' generalizations of some known theorems have a nice form and give an interesting insight into the nature of signed graphs. We also give a characterization

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of exceptional signed graphs with 2 (distinct) eigenvalues in terms of certain decompositions which are specified in the corresponding section. Some particular exceptional signed graphs with 2 eigenvalues are determined.

Sections 2 and 3 are preparatory. In the former we fix some terminology and notation, and in the latter we give a brief review of the root systems with a special emphasis on D_k and E_8. Our main contribution is reported in Sections 4 and 5.

2 Preliminaries

If the vertices i and j of a signed graph are adjacent, then we write $i \sim j$; in particular, the existence of a positive (resp. negative) edge between these vertices is designated by $i \overset{+}{\sim} j$ (resp. $i \overset{-}{\sim} j$).

The *degree* of a vertex in a signed graph is equal to the number of edges incident with it. A signed graph is said to be r*-regular* if the degree of all its vertices is r.

The signed graphs $\dot{G}$ and $\dot{H}$ are said to be *switching equivalent* if there exists a set $U \subseteq V(\dot{G})$, such that $\dot{H}$ is obtained from $\dot{G}$ by reversing the sign of every edge with exactly one end in U. Switching equivalent signed graphs share the same underlying graph and the same spectrum. Equivalently, if the vertex labelling of $\dot{G}$ and $\dot{H}$ is transferred from their common underlying graph, then they are switching equivalent if there exists a diagonal matrix D of ± 1s, such that $A_{\dot{H}} = D^{-1}A_{\dot{G}}D$.

A walk in a signed graph is *positive* if the number of its negative edges (counted with their multiplicity if there are repeated edges) is not odd. Otherwise, it is *negative*. In the same way we decide whether a cycle, considered as a subgraph of a signed graph, is positive or negative.

A *signed doubled graph* $\ddot{G}$ is obtained by doubling every edge of a graph G (i.e., a signed graph with positive signature) with a negative edge. In fact, this is a signed multigraph.

We proceed by a concept of signed line graphs which is frequently used in spectral considerations, see [2, 5, 8]; a slightly different approach can be found in [10]. For a signed graph $\dot{G}$, we introduce the vertex-edge *orientation* $\eta\colon V(\dot{G}) \times E(\dot{G}) \longrightarrow \{1, 0, -1\}$ formed by obeying the following rules:

(1) $\eta(i, jk) = 0$ if $i \notin \{j, k\}$,

(2) $\eta(i, ij) = 1$ or $\eta(i, ij) = -1$ and

(3) $\eta(i, ij)\eta(j, ij) = -\sigma(ij)$.

In fact, each edge has two orientations, and thus η is also called a *bi-orientation*. The vertex-edge *incidence matrix* B_η is the matrix whose rows and columns are indexed by $V(\dot{G})$ and $E(\dot{G})$, respectively, in such a way that its (i, e)-entry is equal to $\eta(i, e)$. The following relation, valid also if multiple edges exist, is well-known:

$$B_\eta^\intercal B_\eta = 2I + A_{L(\dot{G})},$$

where $L(\dot{G})$ is taken to be the *signed line graph* of $\dot{G}$. In fact, a signed line graph is a switching class, not a single signed graph, since $L(\dot{G})$ depends on orientation.

We say that a signed graph is *maximal* with a fixed property, if it is not an induced subgraph of a signed graph having the same property.

3 Root systems

A *representation* of a signed graph $\dot{G}$ with least eigenvalue ≥ -2 is a mapping $f\colon V(\dot{G}) \longrightarrow \mathbb{R}^k$, $k \geq 1$, such that

$$f(i)\cdot f(j) = \begin{cases} 2 & \text{if } i=j,\\ 1 & \text{if } i \overset{+}{\sim} j,\\ -1 & \text{if } i \overset{-}{\sim} j,\\ 0 & \text{if } i \nsim j.\end{cases}$$

If $S \subset \mathbb{R}^k$ is the image of f, then we say that $\dot{G}$ is *represented* by S. Clearly, $A_{\dot{G}} + 2I$ is the Gram matrix of S, so it is positive semidefinite and the least eigenvalue of $\dot{G}$ is, indeed, ≥ -2. We do not make a difference between the set S and the matrix (also denoted by S) whose columns are the vectors of S.

If $\mathbf{e_1}, \mathbf{e_2}, \ldots, \mathbf{e_k}$ are the vectors of the canonical basis of $\mathbb{R}^k$, $k \geq 4$, then the root system D_k consists of the vectors (in the context of root systems, they are also known as the *roots*) $\pm\mathbf{e_i} \pm \mathbf{e_j}$, for $1 \leq i < j \leq k$. The root system E_8 consists of the 112 roots that also belong to D_8 and the additional 128 roots of the form $\frac{1}{2}\sum_{i=1}^{8} \pm\mathbf{e_i}$, where the number of positive summands is not odd. The root system E_7 consists of the 126 roots of E_8 orthogonal to a fixed root $\mathbf{x} \in E_8$, and E_6 consists of the 72 roots of E_8 orthogonal to a fixed pair of roots $\mathbf{x}, \mathbf{y} \in E_8$, such that $\mathbf{x} \cdot \mathbf{y} = -1$.

A root system determines a line system, i.e., a system of one-dimensional subspaces generated by the corresponding roots. The definitions of E_7 and E_6 do not essentially depend on the choice of roots, as different choices produce isometric line systems.

For a root system R, a system of positive roots R^+ is its subset such that:

(1) for $\pm\mathbf{x} \in R$, exactly one of $\mathbf{x}$, $-\mathbf{x}$ belongs to R^+ and

(2) for $\mathbf{x}, \mathbf{y} \in R^+$, if $\mathbf{x}+\mathbf{y} \in R$ then $\mathbf{x}+\mathbf{y} \in R^+$.

Observe that R and R^+ produce the same line system. In particular, we shall frequently deal with the systems of positive roots E_k^+ (of E_k, for $6 \leq k \leq 8$).

It is known that every signed line graph has a representation in some D_k [2, 5]. Accordingly, the least eigenvalue of every signed line graph is ≥ -2. A connected signed graph is said to be *exceptional* if it has a representation in E_8, but has not in any D_k. It follows that a connected signed graph is exceptional if and only if it is not a signed line graph, but its least eigenvalue is ≥ -2.

For a signed graph $\dot{G}$ with n vertices and least eigenvalue greater than -2, an *extension* (or a (-2)-*extension*) of $\dot{G}$ is a signed graph with the following properties:

(1) its least eigenvalue is equal to -2,

(2) it contains $\dot{G}$ as an induced subgraph and

(3) exactly n of its eigenvalues (with possible repetitions) are greater than -2.

4 Properties of exceptional signed graphs

We start with the following lemma.

Lemma 4.1. *If a signed graph is represented by $S \subset \mathbb{R}^k$, then the set S' obtained by reversing the sign of some elements of S represents a switching equivalent signed graph.*

In particular, signed graphs represented by all the possible sets of positive roots R^+ of R (where R stands for any of D_k, E_6, E_7, E_8) are switching equivalent.

Proof. Given $S = \{\mathbf{x_1}, \mathbf{x_2}, \dots, \mathbf{x_n}\}$ and $N \subseteq \{1, 2, \dots, n\}$, let S' be obtained by reversing the sign of vectors whose indices belong to N. If S represents a signed graph $\dot{G}$, then S' obviously represents some signed graph, say $\dot{H}$. Moreover, if D is the diagonal matrix of ± 1s with -1 in the ith position if and only if $i \in N$, then

$$A_{\dot{G}} + 2I = S^{\intercal}S = D^{-1}S'^{\intercal}S'D = D^{-1}(A_{\dot{H}} + 2I)D = D^{-1}A_{\dot{H}}D + 2I,$$

giving the first part of the statement.

Let R^+ be a fixed set of positive roots. If $\mathbf{x}, \mathbf{y} \in R^+$, then $\mathbf{x} \cdot \mathbf{y} \in \{-1, 0, 1\}$, and so R^+ indeed represents a signed graph. In addition, any other set of positive roots is obtained by reversing the sign of some roots of R^+, and the result follows by the previous part of the proof. □

The previous result can also be proved on the basis of Vijayakumar's [9, Lemma 2.3].

Throughout the remainder of the paper, we do not make a difference between switching equivalent signed graphs; in particular, when we say that some signed graph is unique (for certain property), we always mean that it is unique up to the switching equivalence.

The following 3 signed graphs play a significant role in our study: $\dot{M}_6$ represented by E_6^+, $\dot{M}_7$ represented by E_7^+ and $\dot{M}_8$ represented by E_8^+. Each of them is unique (by Lemma 4.1). Their and spectra of the corresponding underlying graphs are given in Table 1. Note that the corresponding underlying graphs are known from literature; for example, they can be met in [3].

Table 1: Spectra of signed graphs $\dot{M}_6$, $\dot{M}_7$ and $\dot{M}_8$.

Signed graph	Spectrum	Spectrum of the underlying graph
$\dot{M}_6$	$\left[10^6, (-2)^{30}\right]$	$\left[20, 2^{20}, (-4)^{15}\right]$
$\dot{M}_7$	$\left[16^7, (-2)^{56}\right]$	$\left[32, 4^{27}, (-4)^{35}\right]$
$\dot{M}_8$	$\left[28^8, (-2)^{112}\right]$	$\left[56, 8^{35}, (-4)^{84}\right]$

We proceed with the following theorem.

Theorem 4.2. *$\dot{M}_8$ is the unique maximal exceptional signed graph.*

Proof. If $\dot{G}$ is a maximal exceptional signed graph, then it has a representation in E_8, and consequently it has a representation in some E_8^+. The maximality of $\dot{G}$ implies that it is represented by the entire set of positive roots. The uniqueness follows by Lemma 4.1. □

In what follows we prove a sequence of results that give a closer insight into exceptional signed graphs. We start with the following lemma and conclude the section with the forthcoming Theorem 4.8 which, together with Theorem 4.2, gives an interesting generalization of certain results concerning exceptional graphs.

Lemma 4.3. *Given* $\mathbf{x}, \mathbf{x}', \mathbf{y}, \mathbf{y}' \in E_8$*, such that* $\mathbf{x} \cdot \mathbf{y} = -1$ *and* $\mathbf{x}' \cdot \mathbf{y}' = -1$*, let* E_7 *and* E_7' *stand for the sets of roots of* E_8 *orthogonal to* $\mathbf{x}$ *and* $\mathbf{y}$*, respectively, and let* E_6 *and* E_6' *stand for the sets of roots of* E_8 *orthogonal to the pair* $\mathbf{x}, \mathbf{y}$ *and* $\mathbf{x}', \mathbf{y}'$*, respectively. The set of signed graphs with a representation in* E_7 (*resp.* E_6) *is equal to the set of those with a representation in* E_7' (*resp.* E_6').

Proof. Let E stand for one of E_7 or E_6 and E' for a corresponding E_7' or E_6'.

If a signed graph $\dot{G}$ has a representation in E, then $A_{\dot{G}} + 2I = S^\intercal S$, for some $S \subseteq E$. Observe that an entry of $S^\intercal S$ is twice the cosine of the angle between the corresponding vectors. Since E and E' determine isometric line systems, there exists $S' \subseteq E'$ such that $S'^\intercal S' = S^\intercal S$, that is $\dot{G}$ has a representation in E'. □

At this point we need a result of [5]. We restate it in a slightly modified form by putting focus on relations between signed graphs in question.

Lemma 4.4. *An exceptional signed graph whose least eigenvalue is greater than* -2 *is*

(i) one of 32 *signed graphs with* 6 *vertices,*

(ii) one of 233 *signed graphs with* 7 *vertices or*

(iii) one of 1242 *signed graphs with* 8 *vertices*

listed in [5]. Every signed graph of (ii) (*resp. (iii)*) *is a one-vertex extension of some of (i)* (*resp. (ii)*).

In [9], Vijayakumar characterized signed graphs represented in D_k, for all k. In particular, the set of connected minimal forbidden subgraphs for signed line graphs has been described. It occurs that they have at most 6 vertices, and all with least eigenvalue greater than -2 are given in the previous lemma. In addition, we claim that none of them has -2 as the least eigenvalue. One may confirm this by constructing all the minimal forbidden subgraphs (by following the particular procedure described in [9]) and checking their least eigenvalue. This would be an extensive work, as just in case of graphs there are the 31 minimal forbidden subgraphs [3, p. 35]. We applied a different approach based on the computer search. Namely, we first obtained all the connected signed graphs with at most 6 vertices that are represented in D_6 and whose least eigenvalue is -2. This procedure was performed very quickly by computer, as there are just 30 roots in D_6^+. Then we have just checked whether each connected signed graph with at most 6 vertices and least eigenvalue -2 (they can be found at the web page [6]) appears in the list found by computer. We record this as the following lemma.

Lemma 4.5. *Every signed graph with at most* 6 *vertices and least eigenvalue* -2 *has a representation in* D_6.

Here is an immediate consequence.

Corollary 4.6. *Every exceptional signed graph* $\dot{G}$ *with least eigenvalue* -2 *has at least* 7 *vertices and contains an exceptional induced subgraph with* 6 *vertices and least eigenvalue greater than* -2.

Proof. Since $\dot{G}$ is exceptional, by the mentioned result of [9], it contains a connected induced subgraph, say $\dot{H}$, with at most 6 vertices and with no representation in any D_k; in other words, $\dot{H}$ is an exceptional signed graph. By Lemma 4.5, the least eigenvalue of $\dot{H}$ is greater than -2. By Lemma 4.4, $\dot{H}$ has exactly 6 vertices and it is some of signed graphs mentioned in that lemma. □

Similarly to a result concerning graphs (cf. [2]), we have the following.

Lemma 4.7. *An exceptional signed graph $\dot{G}$ has between 6 and 8 eigenvalues (with possible repetitions) greater than -2.*

Proof. By Corollary 4.6, $\dot{G}$ contains an induced exceptional subgraph with 6 vertices and least eigenvalue greater than -2, which together with the interlacing argument, gives the lower bound.

It holds $A_{\dot{G}} + 2I = S^{\intercal}S$, where S is obtained by arranging some roots of E_8 as its columns. Since the dimension of the linear span of E_8 is 8, the rank of $A_{\dot{G}} + 2I$ is at most 8, giving the upper bound. □

Theorem 4.8. *For $6 \leq k \leq 8$, $\dot{M}_k$ is the unique maximal extension of exceptional signed graphs with k vertices and least eigenvalue greater than -2.*

Proof. Every exceptional signed graph with k $(6 \leq k \leq 8)$ vertices and least eigenvalue greater than -2 has a representation in E_k, and also in E_k^+. Clearly, such a signed graph is an induced subgraph of a signed graph represented by the entire set of positive roots. For $k = 8$, the result follows by Theorem 4.2.

For $k = 7$, assuming to the contrary, we arrive at the existence of an extension, say $\dot{G}$, such that $A_{\dot{G}} + 2I = S^{\intercal}S$, where S is obtained by arranging some roots of E_7^+ and at least one root, say $\mathbf{x}$, of $E_8^+ \setminus E_7^+$ as its columns. Since $\mathbf{x} \cdot \mathbf{y} = 0$, for all $\mathbf{y} \in E_7^+$, it follows that the rank of $A_{\dot{G}} + 2I$ is 8, meaning that 8 eigenvalues (with repetitions) of $\dot{G}$ are greater than -2, and so $\dot{G}$ cannot be an extension – a contradiction.

For $k = 6$, the proof is essentially the same. □

In particular case of graphs, there are exactly 473 maximal exceptional graphs, along with a large number of maximal extensions of exceptional graphs with 6 or 7 vertices, as follows by [3, Chapter 6]. On the contrary, the 'signed' generalization reported in Theorems 4.2 and 4.8 has a very simple formulation.

5 Exceptional signed graphs with 2 eigenvalues

In [8], we determined all the signed line graphs with 2 eigenvalues. This result is also needed here.

Theorem 5.1. *A connected signed line graph has 2 eigenvalues if and only if the corresponding signed root graph is switching equivalent to*

(i) the star $K_{1,n-1}$,

(ii) the negative quadrangle or the signed multigraph obtained by inserting the (parallel) negative edge between the vertices of degree 2 of the path P_4,

(iii) the complete graph K_n or

(iv) a signed doubled regular graph with n vertices,

in all cases, for $n \geq 3$.

The next natural step is a determination of exceptional signed graphs with 2 eigenvalues. In the forthcoming Theorem 5.4, we give a characterization of such signed graphs; this result relies on Theorems 5.2 and 5.3.

If A is the adjacency matrix of a signed graph with 2 eigenvalues, λ and μ, then

$$A^2 - (\lambda + \mu)A + \lambda\mu I = O. \tag{5.1}$$

It follows that such a signed graph is regular with vertex degree $-\lambda\mu$.

Let m_λ and m_μ denote the multiplicity of λ and μ, respectively, and let t (resp. q) denote the difference between the numbers of positive and negative triangles (resp. quadrangles) contained.

Theorem 5.2. *If $\dot{G}$ is an exceptional signed graph with 2 eigenvalues, then either $\dot{G}$ is the 3-dimensional cube with negative quadrangles or its parameters $(n, r, \lambda, m_\lambda, \mu, m_\mu, t, q)$ have the form*

$$\left(\frac{k}{2}(\lambda+2),\ 2\lambda,\ \lambda,\ k,\ -2,\ \frac{k}{2}\lambda,\ \frac{k}{6}\lambda(\lambda^2-4),\ \frac{k}{8}\lambda(\lambda+2)(\lambda-1)(\lambda-5)\right),$$

where λ is a positive integer and either

(a) $k = 6,\ 2 \leq \lambda \leq 10,$

(b) $k = 7,\ 2 \leq \lambda \leq 16$ and λ is even or

(c) $k = 8,\ 2 \leq \lambda \leq 28.$

Proof. Let $\mu > -2$. Then either $\mu = -1$ or $\lambda = -\mu$ (as follows by considering (5.1)). In the former case, we arrive at the complete signed graph (which is non-exceptional). In the latter case, we have $r = -\lambda\mu < 4$, and so $r = 3$, as $r < 3$ leads to non-exceptional solutions. It is known from [8] that the mentioned cube is the unique 3-regular signed graph with 2 eigenvalues ($\pm\sqrt{3}$); in addition, this is an exceptional signed graph. The last can be verified either by hand or by identifying our cube among 1242 signed graphs of Lemma 4.4(iii) (it is enumerated by 641 in the original reference).

Let $\mu = -2$. We immediately get $r = 2\lambda$, while, by virtue of Theorem 4.8, we have $m_\lambda = k$, for $6 \leq k \leq 8$. By using $m_\lambda + m_{-2} = n$ and $\lambda m_\lambda - 2m_{-2} = 0$, we arrive at $n = \frac{k}{2}(\lambda + 2)$ and $m_{-2} = \frac{k}{2}\lambda$. Similarly, the parameters t and q are obtained by

$$\begin{aligned} m_\lambda\lambda^3 - 8m_{-2} &= 6t, \\ m_\lambda\lambda^4 + 16m_{-2} &= (2r-1)rn + 8q, \end{aligned}$$

where the right-hand sides count the difference between the numbers of positive and negative closed walks of length 3 and 4, respectively.

Since $\dot{G}$ is an induced subgraph of $\dot{M}_k$, we get the upper bounds for λ. Taking into account that all the parameters are integral if and only if either k is even or k is odd and λ is even, we conclude the proof. □

Here is another result giving a specified decomposition of $\dot{M}_k$.

Theorem 5.3. *Let $\dot{G}$ be a signed graph with a representation in E_k ($6 \leq k \leq 8$) having 2 eigenvalues, λ and -2, and $\frac{k}{2}(\lambda+2)$ vertices. If n denotes the number of vertices in $\dot{M}_k$ then, unless $\dot{G}$ coincides with $\dot{M}_k$, there exists a signed graph $\dot{H}$ with eigenvalues $\frac{2n}{k}-\lambda-4$ (of multiplicity k) and -2 (of multiplicity $n-\frac{k}{2}(\lambda+4)$), such that*

$$A_{\dot{M}_k} = \begin{pmatrix} A_{\dot{G}} & B^{\intercal} \\ B & A_{\dot{H}} \end{pmatrix}. \tag{5.2}$$

Proof. Observe that, under given assumptions, the multiplicity of λ (in $\dot{G}$) is k. Clearly, the matrix $A_{\dot{M}_k}$ can be written in the form (5.2), where $A_{\dot{H}}$ is the adjacency matrix of some signed graph $\dot{H}$. If so, then by (5.1) we have $A_{\dot{H}}B = -BA_{\dot{G}} + \big(\frac{2(n-k)}{k}-2\big)B$, as the positive eigenvalue of $\dot{M}_k$ is $\frac{2(n-k)}{k}$. If $\mathbf{x}$ is an eigenvector associated with an eigenvalue of $\dot{G}$, then $A_{\dot{H}}B\mathbf{x} = -BA_{\dot{G}}\mathbf{x} + \big(\frac{2(n-k)}{k}-2\big)B\mathbf{x}$, giving

$$A_{\dot{H}}B\mathbf{x} = \left(-\nu + \frac{2(n-k)}{k} - 2\right)B\mathbf{x}, \tag{5.3}$$

where ν stands for either λ or -2.

If $B\mathbf{x} = \mathbf{0}$, then (again, by (5.1)) we have

$$A_{\dot{M}_k}\begin{pmatrix}\mathbf{x}\\ \mathbf{0}\end{pmatrix} = \begin{pmatrix}A_{\dot{G}}\mathbf{x}\\ \mathbf{0}\end{pmatrix} = \nu\begin{pmatrix}\mathbf{x}\\ \mathbf{0}\end{pmatrix}. \tag{5.4}$$

The case $\nu = \lambda$ cannot occur in (5.4), since $\dot{M}_k$ and $\dot{G}$ cannot share the same eigenvalues because they differ in vertex degree. Therefore, $\frac{2n}{k}-\lambda-4$ is an eigenvalue of $\dot{H}$, with multiplicity at least k. Moreover, its multiplicity must be exactly k, as follows by interchanging the roles of $\dot{G}$ and $\dot{H}$ and considering (5.3) (with $A_{\dot{G}}$ instead of $A_{\dot{H}}$ and $B^{\intercal}$ instead of B); a larger multiplicity would imply either $\lambda = \frac{2(n-k)}{k}$ or $\lambda = -2$, both impossible.

Since $\dot{H}$ has $n-\frac{k}{2}(\lambda+2)$ vertices, there are exactly $n-\frac{k}{2}(\lambda+4)$ remaining eigenvalues (with repetitions). If $n-\frac{k}{2}(\lambda+4) = 0$, then $\dot{H}$ is totally disconnected (with k vertices), and we are done; this is a degenerate case where $\dot{H}$ has only one eigenvalue. Otherwise, if $\overline{d}$ is the average value of the remaining eigenvalues, then by

$$k\left(\frac{2n}{k}-\lambda-4\right) + \left(n-\frac{k}{2}(\lambda+4)\right)\overline{d} = 0,$$

we get $\overline{d} = -2$, which (since $\dot{H}$ is an induced subgraph of $\dot{M}_k$) implies that -2 is the unique remaining eigenvalue with the desired multiplicity. □

In the light of the previous result, we say that $\dot{M}_k$ is decomposed into $\dot{G}$ and $\dot{H}$. We are now in position to give the following characterization.

Theorem 5.4. *If $\dot{G}$ is an exceptional signed graph with 2 eigenvalues, then either*

(i) $\dot{G}$ is the 3-dimensional cube with negative quadrangles, $\dot{M}_6, \dot{M}_7, \dot{M}_8$ or

(ii) the parameters of $\dot{G}$ are determined by Theorem 5.2 and $\dot{G}$ is obtained from a decomposition of $\dot{M}_k$ in sense of Theorem 5.3.

The proof follows directly. Observe that if $\dot{M}_k$ is decomposed as in Theorem 5.3, then a special case where $\dot{G}$ and $\dot{H}$ are isomorphic (possibly, after an appropriate switching) may occur. Here are some particular cases:

- Besides the cube described in Theorem 5.2, in [8] (but see also [4, 7]) the existence of the 2 additional exceptional signed graphs with 2 eigenvalues is confirmed; they correspond to data of Theorem 5.2 obtained for $(k, \lambda) = (7, 2)$ and $(k, \lambda) = (8, 2)$, respectively.
- Setting $(k, \lambda) = (7, 14)$ and $(k, \lambda) = (8, 26)$ in Theorem 5.2, we get that $\dot{G}$ is paired (in sense of Theorem 5.3) with a totally disconnected graph with 7 and 8 vertices, respectively. Since $\dot{M}_k$ contains a co-clique with the corresponding number of vertices, we may confirm the existence of $\dot{G}$; clearly, it is exceptional.

Note that every signed graph represented by all the non-integral roots of E_k^+, for $6 \leq k \leq 8$, is exceptional. Therefore, every signed graph represented by these and possibly some other roots of E_k^+ is also exceptional.

- Up to the switching equivalence, $L(K_8)$ is represented by the roots $\mathbf{e_i} - \mathbf{e_j}$ $(1 \leq i < j \leq 8)$. Thus, it has a representation in E_8^+; moreover, since all the corresponding roots are orthogonal to $\frac{1}{2}\mathbf{j} \in E_8^+$, $L(K_8)$ has a representation in E_7^+, as well. It is paired with the signed graph with $(k, \lambda) = (7, 8)$, which is exceptional by the previous observation.

To get another example we need the following theorem.

Theorem 5.5. *If F is a regular graph with 8 vertices, then $L(\ddot{F})$ is represented by integral roots of E_8^+.*

Proof. Since each column of the vertex-edge incidence matrix B_η of $\ddot{F}$ has length 8 and contains exactly 2 non-zero entries, it follows that $L(\ddot{F})$ has a representation in D_8^+, which is sufficient to conclude the proof. □

- If F is r-regular with k vertices, then by (5.1) the spectrum of $L(\ddot{F})$ is
$$\Big[2(r-1)^k,\ (-2)^{k(r-1)}\Big], \tag{5.5}$$
and so, for $k = 8$, $L(\ddot{F})$ is paired with an exceptional signed graph with spectrum $\big[2(14-r)^8,\ (-2)^{8(14-r)}\big]$. Its existence covers many particular cases of Theorem 5.2.

We continue by some non-existences.

Theorem 5.6. *There is no exceptional signed graph neither for $k = 6$ and $\lambda \notin \{4, 5, 10\}$ nor $(k, \lambda) = (8, 27)$, where the parameters k and λ are determined in the formulation of Theorem 5.2.*

Proof. Denote the putative signed graph by $\dot{G}$, and let $\dot{H}$ be the other signed graph of a decomposition described in Theorem 5.3.

The case $(k, \lambda) = (6, 2)$ cannot occur, since the only connected signed graph with the corresponding spectrum is $L(\ddot{C}_6)$ [8], so non-exceptional.

The case $(k, \lambda) = (6, 3)$ is eliminated by the computer search.

For $(k, \lambda) = (6, 6)$ and $(k, \lambda) = (6, 7)$, $\dot{H}$ would be the mentioned $L(\ddot{C}_6)$ and the disconnected signed graph consisting of the 3 negative triangles, respectively, but this is impossible since these signed graphs have no representation in E_6.

For $(k, \lambda) = (6, 8)$, $\dot{H}$ would be totally disconnected with 6 vertices, but this is impossible since the size of the largest co-clique in $\dot{M}_6$ is 4.

The case $(k, \lambda) \in \{(6, 9), (8, 27)\}$ is resolved easily on the basis of Theorem 5.3 ($\dot{H}$ does not exist).

Therefore, $\dot{G}$ does not exist. □

We conclude the section with the 'line' counterparts to Theorem 4.2 and Lemma 4.4.

Theorem 5.7. *If $\dot{G}$ is a signed line graph with a representation in D_k (for some fixed k), then $\dot{G}$ is an induced subgraph of $L(\ddot{K}_k)$.*

Proof. This follows since $L(\ddot{K}_k)$ is represented by all the roots of D_k^+. □

We believe that the reader is familiar with the concept of star complements in signed graphs, where an induced subgraph $\dot{H}$ is said to be a star complement in its k-vertex extension $\dot{G}$ for an eigenvalue ν if ν is not an eigenvalue of $\dot{H}$, but is an eigenvalue of $\dot{G}$ with multiplicity k. For more details, see [3]. A connected signed multigraph with equal number of vertices and edges is called *unicyclic*.

Lemma 5.8. *For every connected r-regular graph F with vertex degree at least 2, $\ddot{F}$ contains a unicyclic subgraph whose signed line graph is a star complement for -2 in $L(\ddot{F})$.*

Proof. By (5.5), the multiplicity of -2 in $L(\ddot{F})$ is $t(r-1)$ (t being the number of vertices in F), and thus there exists a star complement, say $\dot{H}$, for -2, where $\dot{H}$ has $tr - t(r-1) = t$ vertices. By [1], it can be chosen to be connected. Then, its signed root multigraph is also connected and has t edges, so it is unicyclic. □

A cycle in $\ddot{F}$ can be formed by 2 parallel edges, as well.

By Lemmas 4.4 and 4.7, if $\dot{H}$ is an exceptional star complement for -2 (in some exceptional signed graph), then $\dot{H}$ is one of signed graphs listed in the former lemma. By following the proof of Lemma 5.8, one can obtain star complements in signed line graphs of Theorem 5.1(i)–(iii); in fact, this is a matter of routine. In this way, we have described the basis of all star complements for -2 in signed graphs with 2 eigenvalues.

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The distinguishing index of connected graphs without pendant edges*

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Abstract

We consider edge colourings, not necessarily proper. The distinguishing index $D'(G)$ of a graph G is the least number of colours in an edge colouring that is preserved only by the identity automorphism. It is known that $D'(G) \leq \Delta$ for every countable, connected graph G with finite maximum degree Δ except for three small cycles. We prove that $D'(G) \leq \lceil\sqrt{\Delta}\rceil + 1$ if additionally G does not have pendant edges.

Keywords: Symmetry breaking, distinguishing index of a graph.

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1 Introduction and main result

We use standard graph theory terminology and notation [3]. In particular, the *circumference* of a graph G, denoted by $c(G)$, is the length of a longest cycle of G. A graph that contains a Hamiltonian path, i.e. a path that visits each vertex of the graph, is called *traceable*. A finite tree is called *symmetric* (respectively, *bisymmetric*) if it contains a central vertex v_0 (resp. a central edge e_0), all leaves are at the same distance from v_0 (resp. e_0) and all vertices that are not leaves have the same degree. An *infinite path* P_∞ is an infinite connected 2-regular graph.

We consider edge colourings, not necessarily proper. Such a colouring *breaks an automorphism* $\varphi \in \operatorname{Aut}(G)$ if there exists an edge that is mapped into an edge with a different

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colour. A colouring is called *distinguishing*, if it breaks all non-trivial automorphisms. The minimum number of colours in a distinguishing colouring of a graph G is called the *distinguishing index* of G and is denoted by $D'(G)$. Obviously, the distinguishing index is defined only for graphs without K_2 as a component and with at most one isolated vertex. In this paper, we exclusively consider connected graphs, hence in the sequel, we assume that each graph in question is of order at least 3.

The following general upper bound for the distinguishing index of finite connected graphs was proved in [5].

Theorem 1.1 ([5]). *If G is a finite, connected graph of order $n \geq 3$, then*

$$D'(G) \leq \Delta(G)$$

unless G is C_3, C_4 or C_5.

This result was improved by the third author who characterized all connected graphs with the distinguishing index equal to the maximum degree.

Theorem 1.2 ([9]). *Let G be a finite, connected graph of order $n \geq 3$. Then*

$$D'(G) \leq \Delta(G) - 1$$

unless G is a cycle, a symmetric or a bisymmetric tree, K_4 or $K_{3,3}$.

For infinite graphs, a sharp upper bound was proved by Pilśniak and Stawiski in [10].

Theorem 1.3 ([10]). *Let G be a connected, infinite graph with finite maximum degree $\Delta(G)$. Then*

$$D'(G) \leq \Delta(G) - 1$$

unless G is an infinite path P_∞ with $D'(P_\infty) = \Delta(P_\infty) = 2$.

However, $D'(G) \leq 2$ if G belongs to some classes of graphs, e.g. G is a traceable graph of order at least 7 ([9]), G is any Cartesian power of a graph unless $G = K_2^2$ ([4]), or G is a countable graph every non-identity automorphism of which moves infinitely many vertices ([6]).

Pilśniak [9] formulated the following conjecture.

Conjecture 1.4 ([9]). *If G is a 2-connected graph, then*

$$D'(G) \leq \left\lceil \sqrt{\Delta(G)} \right\rceil + 1.$$

We prove this conjecture in a bit stronger form.

Theorem 1.5 (Main result). *If G is a connected, countable graph with minimum degree $\delta(G) \geq 2$, then*

$$D'(G) \leq \left\lceil \sqrt{\Delta(G)} \right\rceil + 1.$$

In view of Theorem 1.3, the claim obviously holds for countable graphs with infinite $\Delta(G)$. Hence, from now on, we assume that the maximum degree of any graph G in question is finite.

This result is tight for complete bipartite graphs K_{2,r^2} since $D'(K_{2,r^2}) = r+1$ for any integer $r \geq 2$. Indeed, there are r^2 internally disjoint paths of length two between the two vertices u, v of maximum degree in K_{2,r^2}, and they have to be coloured with distinct ordered pairs of colours. With r colours, we have r^2 distinct pairs, but if all of them are used, we can transpose u and v and permute the other vertices to obtain an automorphism preserving this colouring. There are also some small graphs G with $D'(G) = \lceil \sqrt{\Delta(G)} \rceil + 1$, e.g. K_n, C_n for $n \in \{3,4,5\}$, and $K_{3,3}$. We conjecture that the only connected graphs of order at least seven satisfying this equality are complete bipartite graphs K_{2,r^2}.

The proof of the Main result for 2-connected graphs is given in Section 2 (Theorem 2.2). In Section 3, we prove it for graphs of connectivity 1 (Theorem 3.5).

We believe that the following much stronger claim is true[1].

Conjecture 1.6. *If G is a connected, countable graph with finite minimum degree $\delta \geq 2$, then*

$$D'(G) \leq \left\lceil \sqrt[\delta]{\Delta(G)} \right\rceil + 1.$$

Moreover, for graphs of order at least seven, the equality holds if and only if $G = K_{\delta, r^\delta}$, for some positive integer r.

If true, this would imply that $D'(G) \leq 2$ for every regular graph G of order at least seven. Observe that for vertex-transitive graphs this result would be also implied by the famous Lovász Conjecture [8] that every vertex-transitive graph is traceable since, as it was already mentioned, $D'(G) \leq 2$ for every traceable graph of order at least 7. Let us add that recently Lehner, Pilśniak and Stawiski [7] proved that $D'(G) \leq 3$ for every connected, regular graph G, finite or infinite.

2 2-connected graphs

In the proof of the Main result, we colour the edges of a graph G with colours from the set $Z = \{0, 1, \ldots, \lceil \sqrt{\Delta} \rceil\}$. Observe that we always have at least three colours at our disposal. Given a vertex a of a graph H, by $\mathrm{Aut}(H)_a$ we denote the stabilizer of a vertex a, i.e. $\mathrm{Aut}(H)_a = \{\varphi \in \mathrm{Aut}(H) : \varphi(a) = a\}$. For two vertices a, b, we denote $\mathrm{Aut}(H)_{a,b} = \mathrm{Aut}(H)_a \cap \mathrm{Aut}(H)_b$.

In this section, we prove the Main result for 2-connected, countable graphs. The following lemma plays a key role in the proof.

Lemma 2.1. *Let a, b be two vertices of a graph H of finite maximum degree at most Δ, such that*

$$\mathrm{dist}(a, v) + \mathrm{dist}(v, b) = \mathrm{dist}\,(a, b)$$

for every vertex $v \in V(H)$. Then H admits an edge colouring with $\lceil \sqrt{\Delta} \rceil$ colours breaking every automorphism of $\mathrm{Aut}(H)_{a,b}$.

Proof. For $r \in \{0, 1, \ldots, \mathrm{dist}(a, b)\}$, let

$$S_r(a) = \{v \in V(H) \ : \ \mathrm{dist}(a, v) = r\}$$

[1] Actually, this conjecture was earlier posed by Pilśniak and Woźniak, and the first part of it was independently formulated by Alikhani and Soltani in [1]. The latter authors claimed a proof therein but we found two errors and a few gaps that we cannot fix.

be the r-th sphere centered at the vertex a. Thus, for $r < \operatorname{dist}(a, b)$, every vertex $v \in S_r(a)$ has at least one and at most $\Delta - 1$ neighbours in $S_{r+1}(a)$. For $r \geq 1$, denote $H_r = H\left[S_0(a) \cup \dots \cup S_r(a)\right]$. We recursively colour the edges between $S_r(a)$ and $S_{r+1}(a)$ with $\lceil\sqrt{\Delta}\rceil$ colours such that for each r the following two conditions are satisfied:

(1) $S_r(a)$ is fixed pointwise by every automorphism $\varphi \in \operatorname{Aut}(H)_{a,b}$ preserving the colouring of H_{r+1}, whenever $S_{r+1}(a)$ is fixed so;

(2) if $A \subseteq S_{r+1}(a)$ is a set of vertices such that there exists a cyclic permutation of A that can be extended to an automorphism $\varphi \in \operatorname{Aut}(H)_{a,b}$ preserving the colouring of H_{r+1}, then $|A| \leq \lceil\sqrt{\Delta}\rceil$.

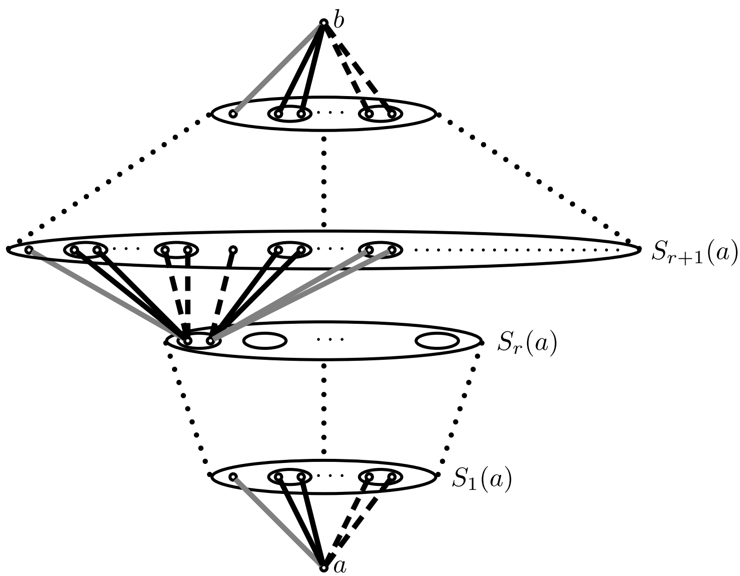

Figure 1: Colouring of edges between subsequent spheres centered at the vertex a breaking all automorphisms of $\operatorname{Aut}_{a,b}(H)$.

First we partition the set of edges incident to a into at most $\lceil\sqrt{\Delta}\rceil$ subsets of cardinality at most $\lceil\sqrt{\Delta}\rceil$, and we colour edges in each subset with the same colour.

Suppose that we have already defined an edge colouring f of H_r satisfying the above two conditions for $r-1$ instead of r. Let $A = \{v_1, \dots, v_p\}$, with $p \geq 2$, be a set of vertices of $S_r(a)$ such that there exists a cyclic permutation of A that can be extended to an automorphism $\varphi \in \operatorname{Aut}(H)_{a,b}$ preserving the colouring f. By assumption, $1 \leq |A| \leq \lceil\sqrt{\Delta}\rceil$. Each vertex v_i of A has the same number $k \leq \Delta - 1$ of incident edges joining it to $S_{r+1}(a)$, and let a_j denote the number of those of them that will obtain the colour j, for $j \in Z \setminus \{0\}$.

Every vertex $v_i \in A$, for $i \in \{1, \dots, p\}$, can be assigned a distinct sequence $(a_1, \dots, a_{\lceil\sqrt{\Delta}\rceil})$, where $a_j \leq \lceil\sqrt{\Delta}\rceil$ for every j. Indeed, take a vertex $v_i \in A$ for some $i \in \{1, \dots, p\}$. If $k \leq \lceil\sqrt{\Delta}\rceil$, then we put $a_i = k$ and $a_j = 0$ for $j \in Z \setminus \{0, i\}$. If $k > (\lceil\sqrt{\Delta}\rceil - 1)\lceil\sqrt{\Delta}\rceil$, then we put $a_i = k - (\lceil\sqrt{\Delta}\rceil - 1)\lceil\sqrt{\Delta}\rceil$, and $a_j = \lceil\sqrt{\Delta}\rceil$ for

$j \in Z \setminus \{0, i\}$. Otherwise, $\lceil\sqrt{\Delta}\rceil + 1 \le k \le (\lceil\sqrt{\Delta}\rceil - 1)\lceil\sqrt{\Delta}\rceil$. If this case, $a_i = 0$, and for $j \neq i$ we put $a_j > 0$ such that $|a_j - \frac{k}{\lceil\sqrt{\Delta}\rceil-1}| \le 1$. Thus, $a_i \neq a_j$ for $j \neq i$, whence the vertices of A are distinguished.

Moreover, this way we produce at most $\lceil\sqrt{\Delta}\rceil$ vertices in $S_{r+1}(a)$ that can be interchanged by an automorphism preserving the colouring of H_{r+1} because at most $\lceil\sqrt{\Delta}\rceil$ edges joining any vertex of A to $S_{r+1}(a)$ have the same colour.

The second last sphere $S_r(a)$, i.e. for $r = \mathrm{dist}(a, b) - 1$, has at most Δ vertices and, due to our construction, any of its subsets A that can be permuted, has at most $\lceil\sqrt{\Delta}\rceil$ vertices. We colour the edges between b and the vertices of A with distinct colours. The unique vertex b of the last sphere is fixed by assumption, hence all spheres are fixed pointwise with respect to any automorphism $\varphi \in \mathrm{Aut}(H)_{a,b}$ preserving the edge colouring f of H.

Finally, we colour edges within each sphere with an arbitrary colour. □

Given a cycle C of a 2-connected graph $G \neq C_4$ and two distinct colours $\alpha, \beta \in Z \setminus \{0\}$, by $C_0(\alpha, \beta)$ we denote this cycle coloured such that three consecutive edges are coloured with $\alpha, 0, \beta$ in that order, and all other edges of the cycle are coloured with 0. Exceptionally, in case $G = C_4$, by a colouring $C_0(\alpha, \beta)$ of C_4 we mean the colouring $\alpha, \beta, 0, 0$ of its consecutive edges.

Theorem 2.2. *If G is a 2-connected, countable graph with finite maximum degree Δ, then*

$$D'(G) \le \left\lceil\sqrt{\Delta}\right\rceil + 1.$$

Proof. If the circumference of G equals 3, then $G = C_3$ and the claim holds.

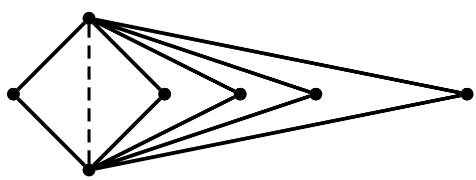

Figure 2: For a given $n \ge 5$, there are exactly two non-isomorphic graphs of order n and circumference 4 (the dashed edge may exist or not).

Let $c(G) = 4$. If $G \in \{C_4, K_4, K_4 - e\}$, then $D'(G) \le 3$, and the claim is true. Suppose $|V(G)| \ge 5$. Then either $G = K_{2,\Delta}$ or $G = K_{2,\Delta-1} + e$, where e is an edge between the two vertices of maximum degree $\Delta \ge 3$ (see Figure 2). Indeed, let $V(G) = \{u_1, \ldots, u_n\}$ with $n \ge 5$, and suppose that u_1, u_2, u_3, u_4 are consecutive vertices of a cycle C of length 4. As G is 2-connected and $c(G) = 4$, all other vertices $u_5, \ldots, u_n$ have to be adjacent to the same pair of non-consecutive vertices of C, say u_1, u_3. Thus, we get a complete bipartite graph $K_{2,n-2}$. The only possible additional edge not violating the condition $c(G) = 4$, is $e = u_1u_3$. As it was mentioned in the Introduction, in such a graph $G \in \{K_{2,\Delta}, K_{2,\Delta-1} + e\}$, we have enough colours for a distinguishing colouring of G. It is also easy to see that we can require that colour 0 appears only in one cycle of length four, coloured as $C_0(\alpha, \beta)$.

Suppose then that G has a cycle C of length at least 5. We colour it as $C_0(\alpha, \beta)$ for some distinct $\alpha, \beta \in Z \setminus \{0\}$. Moreover, we colour all chords of C with α. We can require that the colour 0 appears only in this cycle C. Thus each vertex of C is fixed by every automorphism of G preserving the colouring. To see this it is enough to observe that both endpoints of the unique edge coloured with β are fixed.

We define a colouring of the remaining edges of G recursively, as follows. Suppose that we have already coloured the edges of an induced 2-connected subgraph F of G such that the only automorphism of G preserving this colouring is the identity, regardless of a colouring of the edges outside F. Thus $F = G[V(C)]$ in the initial step. If $F \neq G$, then we choose a shortest path P between any two vertices of F containing at least one vertex outside F. Let a, b be the end-vertices of P, and let p be the length of P. As P is shortest possible, each path of length p between a and b in G is either uncoloured or fully coloured. Consider the subgraph H induced by the vertices of P and of all uncoloured paths between a and b of length p. Thus, all edges of H are uncoloured yet. We colour the edges of H with $\lceil\sqrt{\Delta}\rceil$ colours according to Lemma 2.1. Let $F' = G[V(F) \cup V(H)]$. We thus obtained a colouring of all edges of F' that breaks every non-trivial automorphism of G, because the colouring of F already break them.

If $F' \neq G$, the we repeat this procedure with F' instead of F until all edges of G are coloured. □

Actually, we have proved the following result which we use in the next section.

Corollary 2.3. *Let G be a 2-connected, countable graph with finite maximum degree Δ and let C be a longest cycle or any cycle of length at least five in G. Then for every two distinct colours $\alpha, \beta \in Z \setminus \{0\}$, any colouring $C_0(\alpha, \beta)$ of its edges can be extended to a distinguishing colouring of G with colours from the set Z such that all edges coloured with 0 belong to the cycle C.*

Proof. The claim follows directly from the proof of Theorem 2.2 unless $|G| = 4$. In the latter case, $Z = \{0, 1, 2\}$, and G contains a cycle C_4, which we colour as $C_0(1, 2)$ (recall that this means a colouring $\alpha, \beta, 0, 0$ of consecutive edges if $G = C_4$, and $\alpha, 0, \beta, 0$ otherwise). This is clearly a distinguishing colouring of $G = C_4$. If $G = K_4$, then the other two edges get distinct colours $1, 2$. Otherwise, $G = K_4 - e$, and the chord of C_4 gets colour 1. □

3 Graphs of connectivity 1

Observe that it follows from Theorem 1.2 and Theorem 1.3 that Theorem 1.5 is true for graphs of maximum degree $\Delta \leq 5$ satisfying the conditions of the Theorem. This fact did not matter in the proof for 2-connected graphs, but it facilitates a bit our proof for graphs with cut vertices. From now on, assume that G is a countable graph of finite maximum degree $\Delta \geq 6$ and connectivity 1. Hence, we have at least four colours $0, 1, 2, 3$ at our disposal.

Two edge colourings f_1, f_2 of a graph H are called *isomorphic with respect to a group* $\Gamma \subseteq \mathrm{Aut}(H)$ if there exists an automorphism $\varphi \in \Gamma$ such that $f_1(uv) = f_2(\varphi(u)\varphi(v))$ for every edge $uv \in E(H)$. Furthermore, f_1, f_2 are called *isomorphic* if they are isomorphic with respect to $\mathrm{Aut}(H)$.

Lemma 3.1. *Let u_0 be a vertex of a 2-connected, countable block H_0 of finite maximum degree at most Δ, where $\Delta \geq 6$. Then H_0 admits at least $\lceil\sqrt{\Delta}\rceil$ edge colourings with colours from the set Z, breaking all non-trivial automorphisms of* $\mathrm{Aut}(H_0)_{u_0}$*, which are pairwise non-isomorphic with respect to* $\mathrm{Aut}(H_0)_{u_0}$*, and do not contain* $C_0(1, 2)$.

Proof. First observe that if H_0 is a 2-connected, countable graph and $c(H_0) \geq 5$, then every vertex of H_0 belongs to a cycle of length at least five. This easily follows from the fact that every vertex outside a given cycle C of length at least five is the origin of two internally disjoint paths to two vertices of C, which partition the cycle C into two paths, one of which is of length at least three.

We choose a longest cycle C in B passing through the vertex u_0 if B is finite, or a cycle of length at least five if B is infinite. It is easy to see that C is a longest cycle in G whenever $|C| \leq 4$. We colour C as $C_0(\alpha, \beta)$, for $\{\alpha, \beta\} \neq \{1, 2\}$. By Corollary 2.3, regardless of location of colours α and β on C, this colouring can be extended to a distinguishing colouring of H_0 using colours from $Z \setminus \{0\}$. We have $\binom{\lceil\sqrt{\Delta}\rceil}{2} - 1$ choices for the set of two colours $\{\alpha, \beta\} \neq \{1, 2\}$. For a given colouring $C_0(\alpha, \beta)$, there are at least three possible placements for u_0 (actually, there are more possibilities if $|H_0| > 3$). Hence, there are at least $L = 3(\binom{\lceil\sqrt{\Delta}\rceil}{2} - 1)$ edge colourings of H_0 that are pairwise non-isomorphic with respect to $\mathrm{Aut}(H_0)_{u_0}$. The inequality $L \geq \lceil\sqrt{\Delta}\rceil$, equivalent to $3(\lceil\sqrt{\Delta}\rceil)^2 - 5\lceil\sqrt{\Delta}\rceil - 6 \geq 0$, holds whenever $\lceil\sqrt{\Delta}\rceil) \geq 3$. □

Lemma 3.2. *Let H_0 be a graph of finite maximum degree at most Δ, with $\Delta \geq 6$, consisting of $s \geq 2$ copies of a 2-connected block B sharing a common cut vertex u_0. Then H_0 admits at least $\lceil\sqrt{\Delta}\rceil$ pairwise non-isomorphic distinguishing edge colourings with colours from the set Z, such that $C_0(1, 2)$ does not appear in any of these colourings.*

Proof. Let H_0 be a graph satisfying the assumptions. Observe that $\mathrm{Aut}(H_0) = \mathrm{Aut}(H_0)_{u_0}$ for $s \geq 2$. Clearly, $s \leq \lfloor\frac{\Delta}{2}\rfloor$.

As we have shown above, the block B admits at least $L = 3(\binom{\lceil\sqrt{\Delta}\rceil}{2} - 1)$ colourings which are pairwise non-isomorphic with respect to $\mathrm{Aut}(H_0)_{u_0}$. To colour the graph H_0, we select s of them. Hence, we have at least $\binom{L}{s}$ pairwise non-isomorphic colourings of H_0. Clearly, $\binom{L}{s} \geq L$, when $1 \leq s \leq \lfloor\frac{\Delta}{2}\rfloor \leq L-1$, which is the case for $\Delta \geq 6$. We know that $L \geq \lceil\sqrt{\Delta}\rceil$, hence there are at least $\lceil\sqrt{\Delta}\rceil$ non-isomorphic distinguishing colourings of H_0. □

Lemma 3.3. *Let H_0 be a graph satisfying the assumptions of Lemma 3.1 or of Lemma 3.2. Let T be a symmetric tree of order at least 3 with a central vertex v_0. A graph H is obtained by attaching a copy of the graph H_0 to every leaf of T in such a way that each pendant edge of T is incident to the same vertex u_0 of H_0. If the maximum degree of H is at most Δ and $\Delta \geq 6$, then there exists a distinguishing colouring of H with colours from the set Z, and without $C_0(1, 2)$.*

Proof. We colour the edges of T with $\lceil\sqrt{\Delta}\rceil$ colours similarly as in the proof of Lemma 2.1. That is, at most $\lceil\sqrt{\Delta}\rceil$ edges incident to the central vertex v_0 of T get the same colour. Then recursively, at each level of T, any set A of vertices that can be cyclically permuted by an automorphism of H preserving the hitherto defined colouring, has at most $\lceil\sqrt{\Delta}\rceil$ elements. For each vertex v of A, we choose a distinct colour $i \in Z$ such that the number of edges joining v to the next level of T and coloured with i, is different from the number of such edges coloured with any other colour. We arrive at the leaves of T with a colouring such that every set A of leaves that can be cyclically permuted by an automorphism of T preserving the colouring of T has at most $\lceil\sqrt{\Delta}\rceil$ elements. It follows from Lemma 3.1 or Lemma 3.2 (depending on the structure of H_0) that each copy of H_0 attached at a leaf

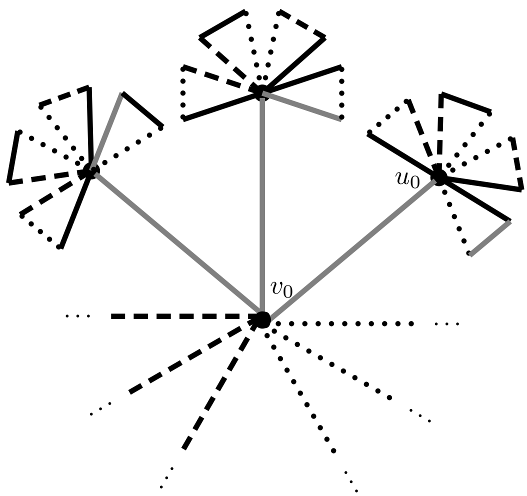

Figure 3: Example of a 4-colouring of a graph H of Lemma 3.3 with $\Delta = 9$, where a graph H_0 consisting of four triangles sharing a cut vertex u_0 is attached to every leaf of a tree $T = K_{1,9}$.

belonging to the same set A has a distinct colouring satisfying our assumptions. Thus we obtain a desired colouring of the whole graph H. □

The following result of Broere and Pilśniak [2] will be useful in the next proof.

Theorem 3.4 ([2]). *If T is an infinite tree without leaves, then $D'(T) \leq 2$.*

We are now ready to prove Theorem 1.5 for graphs with cut vertices.

Theorem 3.5. *Let G be a countable graph of connectivity 1 and without pendant edges. If G has finite maximum degree Δ, then*

$$D'(G) \leq \left\lceil \sqrt{\Delta} \right\rceil + 1.$$

Proof. Recall that we may assume that $\Delta \geq 6$, whence the set Z contains at least four colours. If G is a tree, then it has no leaves, so $D'(G) \leq 2$ by Theorem 3.4.

Suppose then that G contains a 2-connected block B_0. Let C be a longest cycle of B_0, if it exists, or any cycle of length at least five of B_0. Colour it with $C_0(1, 2)$. By Corollary 2.3, there is a distinguishing edge colouring of B_0, where colour 0 is used only on the cycle C. In our colouring of the whole graph G, we ensure that $C_0(1, 2)$ does not appear again, therefore each vertex of B_0 is fixed by every automorphism of G preserving the colouring of B_0. We recursively extend this colouring in such a way that every vertex with at least one coloured incident edge is fixed as well.

Now, we set any ordering $v_1, v_2, \ldots$ of all cut vertices of G. We recursively consider the first vertex v_i in this ordering that belongs to an already coloured block, as well as to an uncoloured one. Hence, v_i is already fixed by every $\varphi \in \mathrm{Aut}(G)$ preserving the existing partial colouring. We proceed with v_i in four stages.

1. Let $B_1, \dots, B_l$ be pairwise non-isomorphic, uncoloured 2-connected blocks containing v_i. For each $j \in \{1, \dots, l\}$, we consider a maximal subgraph H_0 of G that consists of $s \geq 1$ copies of B_j sharing the cut vertex v_i. Taking $u_0 = v_i$, by Lemma 3.1 if $s = 1$, or by Lemma 3.2 if $s \geq 2$, there is a distinguishing colouring of H_0 with colours from the set Z, not containing $C_0(1, 2)$.

2. We consider every uncoloured maximal subgraph H satisfying the assumptions of Lemma 3.3. We colour H distinguishingly according to the conclusion of Lemma 3.3. Consequently, all 2-connected blocks containing v_i are now coloured.

3. If some components of $G - v_i$ are uncoloured infinite trees, then we consider the infinite tree T containing all those components and the vertex v_i, and we colour the edges of T with two colours, by Theorem 3.4.

4. We colour every yet uncoloured edge e incident to v_i arbitrarily. Note that both endpoints of e are already fixed by any $\varphi \in \mathrm{Aut}(G)$ preserving the existing partial colouring, because uncoloured components of $G - v_i$ are pairwise non-isomorphic.

This recursive procedure yields a distinguishing colouring of G with $\lceil \sqrt{\Delta} \rceil + 1$ colours, which completes the proof of Theorem 3.5, and thus of Theorem 1.5. □

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On the upper embedding of Steiner triple systems and Latin squares*

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Abstract

It is proved that for any prescribed orientation of the triples of either a Steiner triple system or a Latin square of odd order, there exists an embedding in an orientable surface with the triples forming triangular faces and one extra large face.

Keywords: Upper embedding, Steiner triple system, Latin square.

Math. Subj. Class. (2020): 05B07, 05B15, 05C10

1 Introduction

The motivation for the work described herein comes from two previous papers, respectively on the upper embedding of Steiner triple systems [1] and the upper embedding of Latin squares [2]. First we recall the relevant definitions. Let $\mathcal{X} = (V, \mathcal{B})$ be a (partial) triple system on a point set V, that is, a collection $\mathcal{B}$ of 3-element subsets of V, called blocks

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or triples, such that every 2-element subset of V is contained in at most one triple in $\mathcal{B}$. Equivalently, such a triple system $\mathcal{X}$ may be viewed as a pair $(K, \mathcal{B})$, where K is a graph with vertex set V and edge set E consisting of all pairs uv for distinct points $u, v \in V$ such that $\{u, v\}$ is a subset of some block in $\mathcal{B}$. In other words, in the graph setting, $\mathcal{B}$ is regarded as a decomposition of the edge set E of K into triangles. Of course, such a graph $K = (V, E)$ may admit many decompositions into triangles and so the set of blocks $\mathcal{B}$ needs to be specified. We will refer to $K = (V, E)$ with the specified decomposition $\mathcal{B}$ of E as the graph *associated* with the triple system $\mathcal{X} = (V, \mathcal{B})$. We will be assuming throughout that the triple systems considered here are *connected*, meaning that their associated graphs are connected. In the case of a Steiner triple system $\mathcal{S} = \mathrm{STS}(n)$ where $n = |V|$, the associated graph is the complete graph K_n. Such systems exist if and only if $n \equiv 1$ or 3 $(\mathrm{mod}\ 6)$ [4]. For a Latin square $L = \mathrm{LS}(n)$, the associated graph is the complete tripartite graph $K_{n,n,n}$ where the three parts of the tripartition are the rows, the columns and the entries of the Latin square.

By an embedding of a triple system $\mathcal{X} = (V, \mathcal{B})$ we will understand a cellular embedding $\vartheta\colon K \to \Sigma$ of the associated graph $K = (V, E)$ of $\mathcal{X}$ in an orientable surface Σ, such that every triangle in $\mathcal{B}$ bounds a face of ϑ. Such faces will be called *block faces*, and the remaining faces of the embedding ϑ will be called *outer faces*. By the properties of the set $\mathcal{B}$, in the embedding ϑ every edge of E lies on the boundary of exactly one block face, so that there is at least one outer face in ϑ. The extreme case occurs if such an embedding has exactly one outer face; we then speak about an *upper embedding* and call the triple system $\mathcal{X} = (V, \mathcal{B})$ *upper embeddable*.

A necessary and sufficient condition for upper embeddability of triple systems follows from available knowledge about upper embeddings of graphs in general. To make use of this we will represent triple systems by their point-block incidence graphs as usual in design theory. For a triple system $\mathcal{X} = (V, \mathcal{B})$ its *point-block incidence graph* is the bipartite graph $G(\mathcal{X})$ with vertex set $V \cup \mathcal{B}$ and edge set consisting of pairs $\{v, B\}$ for $v \in V$ and $B \in \mathcal{B}$ such that $v \in B$. The pair $(V, \mathcal{B})$ forms the bi-partition of the vertex set of $G(\mathcal{X})$; vertices in V and $\mathcal{B}$ will be referred to as *point vertices* and *block vertices* respectively. By our convention regarding triple systems, the graph $G(\mathcal{X})$ is assumed to be connected, and note that every block vertex has valency 3 in $G(\mathcal{X})$.

It is a folklore fact in topological design theory that there is a one-to-one correspondence between orientable embeddings of a triple system $\mathcal{X}$ and its point-block incidence graph $G(\mathcal{X})$ in orientable surfaces; the correspondence is illustrated in Figure 1. As is obvious from this figure, an embedding of $G(\mathcal{X})$ arises from an embedding of $\mathcal{X}$ naturally. On the other hand, given an embedding of $G(\mathcal{X})$ one obtains the corresponding embedding of $\mathcal{X}$ by 'inflating' every block vertex into a triangle on the surface. In particular, a triple system $\mathcal{X}$ is upper embeddable if and only if its point-block incidence graph is embeddable with exactly one face; such graphs are also called upper-embeddable. By a classical result of Jungerman [3] and Xuong [5], a graph (in particular, the point-block incidence graph of a triple system) is upper-embeddable if and only if the graph contains a spanning tree such that each of its co-tree components has an even number of edges.

Let us now look more closely at an upper embedding of a triple system $\mathcal{X} = (V, \mathcal{B})$ in an orientable surface Σ, or, equivalently, at an upper embedding $\vartheta\colon G(\mathcal{X}) \to \Sigma$ of the point-block intersection graph of $\mathcal{X}$ in Σ. For any triple $B = \{u, v, w\} \in \mathcal{B}$ the (given) orientation of Σ induces one of the two cyclic permutations (u, v, w) or (u, w, v) when looking at the three points from the 'centre' of the triangular face representing B. Equivalently

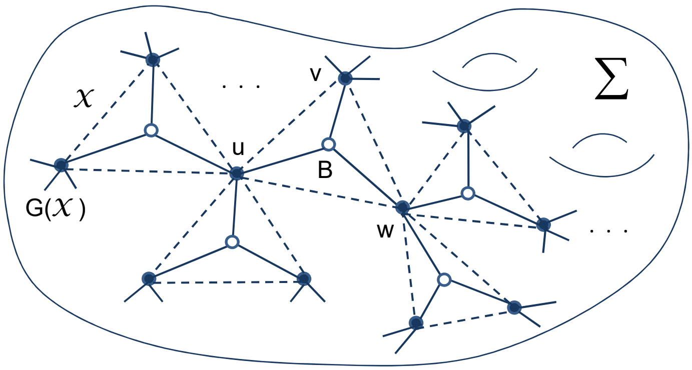

Figure 1: Correspondence between embeddings of $\mathcal{X}$ and $G(\mathcal{X})$.

for any block vertex B of $G(\mathcal{T})$ with point vertex neighbours u, v, w, the orientation of Σ induces one of the cyclic permutations (u, v, w) or (u, w, v) of the three points 'around' the block vertex B. We refer to any of the two permutations as an *orientation of the triple*, or equivalently, an *orientation of the neighbourhood*. In this terminology, the orientation of Σ induces one of the two possible orientations of every triple (or, of the neighbourhood of every block vertex) in the embedding. It was proved in [1] that every Steiner triple system $\mathrm{STS}(n)$ and in [2] that every Latin square $\mathrm{LS}(n)$ where n is odd has an upper embedding in an orientable surface. In the Latin square case the restriction that n must be odd is determined by Euler's formula ($V + F - E = 2 - 2g$). However in both [1] and [2] no attention was paid to the orientation of the triples in the upper embeddings.

The driving question of our research is the following question. Which triple systems $\mathcal{X} = (V, \mathcal{B})$ have the property that, for every choice of the orientation of all triples $B \in \mathcal{B}$, the system $\mathcal{X}$ admits an upper embedding in an orientable surface such that the orientation of the surface induces the preassigned orientation of B for *every* triple $B \in \mathcal{B}$? We will refer to this property simply as *upper embeddability in every orientation of triples*. The main results of this paper are the two theorems below which substantially extend the results in the two papers [1] and [2].

Theorem 1.1. *Every Steiner triple system admits an upper embedding in every orientation of triples.*

Theorem 1.2. *Every Latin square of odd order admits an upper embedding in every orientation of triples.*

2 Spanning tree

In what follows we prove a sufficient condition for a triple system to admit upper embeddability in every orientation of triples. The result will be stated in terms of the point-block incidence graph of a triple system and, as one expects by Jungerman and Xuong's Theorem [3] and [5], the statement will involve existence of a spanning tree with particular properties. We recall the connectivity assumption of our triple systems.

Theorem 2.1. *Let $\mathcal{X}$ be a triple system and let $G = G(\mathcal{X})$ be its point-block incidence graph. If G admits a spanning tree such that every point vertex has even valency in the corresponding co-tree, then $\mathcal{X}$ admits an upper embedding in every orientation of triples.*

Proof. Let $\mathcal{X} = (V, \mathcal{B})$ and suppose that every triple $B = \{u, v, w\} \in \mathcal{B}$ has been assigned an orientation, i.e., one of the two cyclic permutations (u, v, w), (u, w, v). Using the ideas of Jungerman and Xuong we will show (by induction) how to build an upper embedding of the point-block incidence graph $G = G(\mathcal{X})$ of $\mathcal{X}$ in an oriented surface in such a way that its orientation will induce the preassigned orientation on every $B \in \mathcal{B}$.

Let T be a spanning tree of G as in the assumption of our theorem, that is, such that every point vertex of the subgraph of G induced by the set $E(G) \setminus E(T)$ of co-tree edges has even valency. Since G is bipartite, this assumption implies that the set $E(G) \setminus E(T)$ has a decomposition into paths of length two that have a point vertex in the centre; in particular, the number of co-tree edges here is even. We note that the quantity $\beta = |E(G) \setminus E(T)|$ is known as the Betti number of G. The set of co-tree edges thus decomposes into $\beta/2$ paths P of the form $P = AuB$, where A, B are block vertices and u is a point vertex.

To proceed, we prove quite a general auxiliary statement on upper embeddability of extensions of spanning subgraphs of our point-block incidence graph by paths as above. Let H be a connected spanning subgraph of G and let A and B be block vertices and u be a point vertex of H such that $P = AuB$ is a path in G but the edges Au and Bu are not in H. Further, let $\mathcal{C}$ be the set of block vertices of G that have valency 3 in H. We will say that *H upper embeds in every orientation at $\mathcal{C}$* if, for every vertex $C \in \mathcal{C}$ and every orientation of the neighbourhood of C, there is an embedding of H in an orientable surface such that its orientation induces the preassigned orientation around every vertex $C \in \mathcal{C}$. Extending this terminology in a natural way to the graph $H' = H \cup P$ and the set $\mathcal{C}'$ of block vertices of valency 3 in H', we prove the following.

Claim. *If H upper embeds in every orientation at $\mathcal{C}$, then H' upper embeds in every orientation at $\mathcal{C}'$.*

To prove our Claim, let $H \to \Sigma$ be an upper embedding of H in an orientable surface Σ such that its orientation induces the preassigned orientations of neighbourhoods of block vertices in $\mathcal{C}$. Let $P = AuB$ be a path as above. The boundary of the single face F of this embedding is, without loss of generality, a closed walk in H of the form $(uXAYBZ)$, where X, Y, Z are $u \to A$, $A \to B$ and $B \to u$ walks of H traversed in the direction induced by the (clockwise) orientation of Σ, as one can see in Figure 2 when disregarding the arcs (edges with direction) a, b, c, d. We point out that in our considerations the order of appearance of the vertices A and B on the boundary of F will be immaterial.

Ignoring the condition on the set $\mathcal{C}'$, the embedding of H can be extended to an upper embedding of H'. Namely, letting $a = uA$ and $b = uB$ denote the arcs from u to A and u to B respectively, one 'adds' the path $P = AuB$ to the single face F of ϑ in such a way that all the local cyclic orderings of arcs emanating from vertices distinct from A, u, B are kept intact and the local cyclic ordering of neighbours of u is extended from $(\ldots, c, d, \ldots)$ to $(\ldots, c, b, a, d, \ldots)$ as in Figure 2; the local cyclic order of arcs at A and B is obvious from Figure 2.

The single face F' of the new embedding of H' in an oriented surface Σ' is obtained by tracing down its boundary, which is the closed walk $(uXa^{-}bZaYb^{-})$, where a^{-} and b^{-} indicate traversals along a and b in the opposite direction. We again emphasize that, for

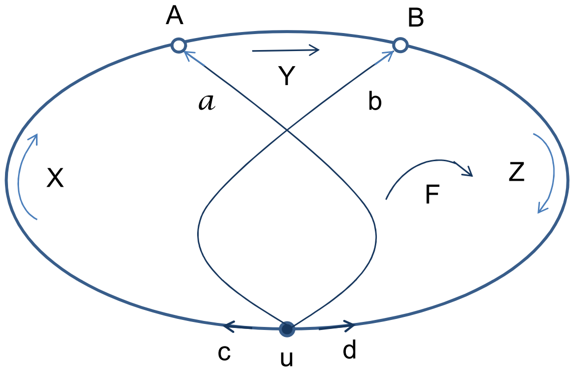

Figure 2: Extending F by adding the path $P = AuB$.

this construction, the position of A and B on the boundary of F is irrelevant. Note that the genus of Σ' is equal to the genus of Σ increased by 1.

We now show that the above construction can be carried out in such a way that the orientation of Σ' induces the preassigned orientations of neighbours of vertices in $\mathcal{C}'$. This orientation is obviously maintained for block vertices in the subset $\mathcal{C} \subset \mathcal{C}'$ and so one only needs to consider the block vertices in $\mathcal{C}'\backslash\mathcal{C} \subset \{A, B\}$; note that $A, B \notin \mathcal{C}$. If neither A nor B are in $\mathcal{C}'$, we simply use the above construction. If one or both of A, B are in $\mathcal{C}'\backslash\mathcal{C}$, then we proceed as follows.

Say, without loss of generality, that $A \in \mathcal{C}'$, which means that the valency of A in H must have been equal to 2. We know that $u \in A$; let $A = \{u, v, w\}$ with $\{v, w\}$ being the point vertices constituting the neighbourhood of A in H. Since the boundary of the single face F in H must contain every edge twice (and traversed in each direction once) and A has valency 2 in H, it follows that the boundary of F must have the form $(u \ldots vAw \ldots wAv \ldots)$, as displayed in Figure 3; the relative position of the vAw and wAv paths is without loss of generality.

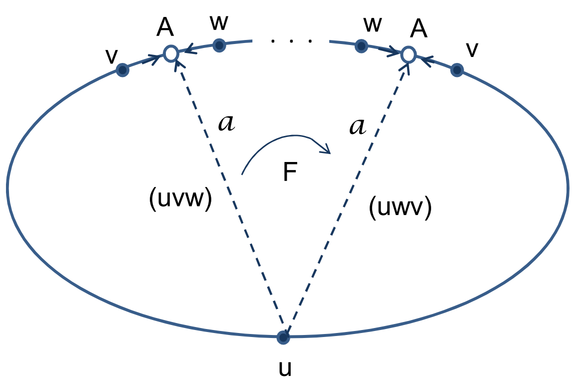

Figure 3: Adding the edge uA to control the orientation of the triple A.

Now, if the preassigned orientation of the block $A = \{u, v, w\}$ is given by the cyclic permutation (uvw), we will add the arc $a = uA$ pointing to the position of A appearing on the left-hand side of Figure 3; if the preassigned orientation is (uwv) we use the arc a pointing to the occurrence of A on the right-hand side in Figure 3. If the vertex B also has valency 2 in H, we make a similar choice for the position of B for addition of the arc $b = uB$. We then complete the construction of the single face embedding of H' with the required properties as in the previous paragraph.

Having proved our Claim, the rest of the proof is straightforward. We begin by embedding the spanning tree T in a sphere in such a way that its orientation induces a preassigned orientation at every block vertex that has valency 3 in T. We then apply the construction of our Claim $\beta/2$ times for every path in the decomposition of the set $E(G)\backslash E(T)$ of co-tree edges into $\beta/2$ paths whose middle vertex is a point vertex. As a result we obtain the upper embeddability of G in every orientation at its set $\mathcal{B}$ of block vertices. □

3 Steiner triple systems

We deal first with the case of Steiner triple systems. In view of the previous section, Theorem 1.1 follows immediately from the following Proposition.

Proposition 3.1. *Let $S = (V, \mathcal{B})$ be a Steiner triple system of order n and let $G = G(S)$ be its point-block incidence graph. Then G admits a spanning tree such that every point vertex has even valency in the corresponding co-tree.*

Proof. If $n = 3$, the result is true trivially. If $n = 7$, then S is unique up to isomorphism and G is the Heawood graph. A breadth first search starting from any block gives a tree in which each point has valency 1 or 3 and hence valency 2 or 0 in the co-tree. Now assume that $n \geq 9$.

Let $V = \{0, 1, \ldots, n-1\}$. Construct a spanning tree T of the point-block incidence graph G as follows. For convenience, we will refer to point vertices simply as points and block vertices as blocks. Let the root (Level 0) of the tree be the point 0. Connect the point to all $(n-1)/2$ blocks containing it, which will be at Level 1. At Level 2 put all the $n-1$ points $x \in V \setminus \{0\}$ and connect these to the blocks at Level 1 which contain them. The remaining $n(n-1)/6 - (n-1)/2 = (n-1)(n-3)/6$ blocks are at Level 3. Connect each one of these to a point at Level 2 which is contained in the block. The structure of the spanning tree is shown below.

In the co-tree the point 0 has zero valency and the points at Level 2 have either odd or even valency including zero. We will refer to such points as either *odd points* or *even points* respectively. The number of edges in the co-tree is $3n(n-1)/6 - n(n-1)/6 - n + 1 = (n-1)(n-3)/3$ which is even. Thus the number of odd points is even. The aim is to construct a spanning tree with no odd points. The proof is in the form of an iterative algorithm. Beginning with a spanning tree constructed as above, at any stage, if there are odd points at Level 2, modify how the blocks at Level 3 are connected to the points at Level 2 in order to reduce the number of odd points until they are absent. Let $\mathcal{B}_3$ denote the set of blocks at Level 3, i.e. the blocks that do not contain 0. Each of these blocks is always adjacent in T to exactly one point. Let P be the number of odd points. For any distinct $a, b \in V$ let ab be the point so that $\{a, b, ab\} \in \mathcal{B}$.

Claim 1. *If $\{a, b, c\} \in \mathcal{B}_3$, $\{a, \{a, b, c\}\} \in T$, a is odd, and at least one of b or c is odd, then we can reduce P.*

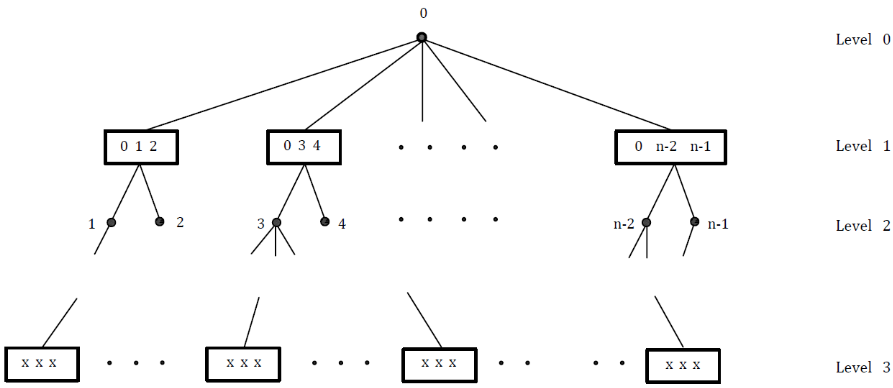

Figure 4: Spanning tree of $G(S)$.

Proof. Suppose that b is odd. Replace $\{a, \{a, b, c\}\}$ by $\{b, \{a, b, c\}\}$. $\square$

Claim 2. *If $P > 0$ then we may assume that there is $\{p, q, r\} \in \mathcal{B}_3$ with p and q odd, r even, and $\{r, \{p, q, r\}\} \in T$; otherwise we can reduce P.*

Proof. If $P > 0$ then there are at least two odd points: say a and b are odd.

Suppose that $w = ab \neq 0$, so $\{a, b, w\} \in \mathcal{B}_3$. If w is odd or $\{w, \{a, b, w\}\} \notin T$, then we can reduce P by Claim 1. So we may assume that $\{a, b, w\}$ is the required block, with $p = a$, $q = b$, $r = w$.

Suppose now that $ab = 0$, so that $\{a, \{a, b, 0\}\} \in T$. Since a does not have valency 0 in the co-tree, there is some block $\{a, c, x\} \in \mathcal{B}_\ni$ with $\{a, \{a, c, x\}\} \notin T$. Then $c, x \in V \setminus \{0, a, b\}$ and we may assume that $\{c, \{a, c, x\}\} \in T$. If c is odd, then we can reduce P by Claim 1, so assume that c is even. Replace $\{a, \{a, c, x\}\}$ in T by $\{c, \{a, c, x\}\}$, so that a becomes even and c becomes odd. There is also some $\{b, c, y\} \in \mathcal{B}_3$. If y is odd or $\{y, \{b, c, y\}\} \notin T$ then we can reduce P by Claim 1. So we may assume that $\{b, c, y\}$ is the required block, with $p = b$, $q = c$, $r = y$. $\square$

Claim 3. *Given a block $\{p, q, r\}$ as in Claim 2, we can reduce P.*

Proof. Let S_p be the set of points z such that $\{p, z, pz\} \in \mathcal{B}_3$, z is even, and either $\{p, \{p, z, pz\}\} \in T$ or $\{z, \{p, z, pz\}\} \in T$. Define S_q and S_r similarly.

There are $N = (n-5)/2$ blocks $\{p, z, s\} \in \mathcal{B} \setminus \{\{p, q, r\}, \{p, 0, p0\}\}$, which all belong to $\mathcal{B}_3$. Consider such a block $\{p, z, s\}$. If $\{p, \{p, z, s\}\} \in T$ then we can reduce P by Claim 1 unless z and s are both even, so $z \in S_p$. Otherwise we may assume that $\{z, \{p, z, s\}\} \in T$, and again we may reduce P by Claim 1 unless z is even, so again $z \in S_p$. Since each of the N blocks $\{p, z, s\}$ contains an element of S_p, $|S_p| \geq N$.

Similarly, $|S_q| \geq N$. Temporarily replacing $\{r, \{p, q, r\}\}$ in T by $\{p, \{p, q, r\}\}$, which does not change S_r, we also obtain $|S_r| \geq N$. Then $S_p, S_q, S_r \subseteq V \setminus \{0, p, q, r\}$ and since $n \geq 9$, we have $|S_p| + |S_q| + |S_r| \geq 3N = 3(n-5)/2 > n - 4$. Hence by the pigeonhole principle there is some z belonging to at least two of S_p, S_q, S_r.

If $z \in S_p \cap S_q$, replace $\{p, \{p, z, pz\}\}$ in T by $\{z, \{p, z, pz\}\}$ or vice versa, and replace $\{q, \{q, z, qz\}\}$ in T by $\{z, \{q, z, qz\}\}$ or vice versa. This makes p and q even, and the parity of z is changed twice, so remains even; hence P is reduced. If $z \in S_p \cap S_r$ or $z \in S_q \cap S_r$ apply a similar argument after replacing $\{r, \{p, q, r\}\}$ in T by $\{q, \{p, q, r\}\}$ or $\{r, \{p, q, r\}\}$, respectively. □

By applying Claims 2 and 3 repeatedly we can reduce P to 0, as required. □

4 Latin squares

Now we turn our attention to the case of Latin squares of odd order. Let $\mathrm{LS}(n)$ be a Latin square of odd order. Denote the sets of row points, column points and entry points by $\mathcal{R}$, $\mathcal{C}$ and $\mathcal{E}$ respectively. Let $\mathcal{B}$ be the set of triples, which we will also for convenience refer to as blocks, $\{i_r, j_c, k_e\}$ where $i_r \in \mathcal{R}$, $j_c \in \mathcal{C}$, $k_e \in \mathcal{E}$ and $k = L(i, j)$. The *point-block incidence graph* is the bipartite graph with vertex set $\mathcal{R} \cup \mathcal{C} \cup \mathcal{E} \cup \mathcal{B}$ and edge set

$$\{(v_r, B) : v_r \in \mathcal{R}, B \in \mathcal{B}, v_r \in B\} \cup \{(v_c, B) : v_c \in \mathcal{C}, B \in \mathcal{B}, v_c \in B\} \\ \cup \{(v_e, B) : v_e \in \mathcal{E}, B \in \mathcal{B}, v_e \in B\}.$$

The following result is an exact analogy of Proposition 3.1 above for Steiner triple systems and establishes Theorem 1.2. However the proof is much simpler. We are able to construct the spanning tree with the appropriate property directly rather than by an iterative procedure.

Proposition 4.1. *Let $L = \mathrm{LS}(n)$ be a Latin square of odd order n and let $G = G(L)$ be its point-block incidence graph. Then G admits a spanning tree such that every point vertex has even valency in the corresponding co-tree.*

Proof. Let $\mathcal{R} = \{0_r, 1_r, \ldots, (n-1)_r\}$, $\mathcal{C} = \{0_c, 1_c, \ldots, (n-1)_c\}$ and $\mathcal{E} = \{0_e, 1_e, \ldots, (n-1)_e\}$. Construct a spanning tree T of the point-block incidence graph G as follows.

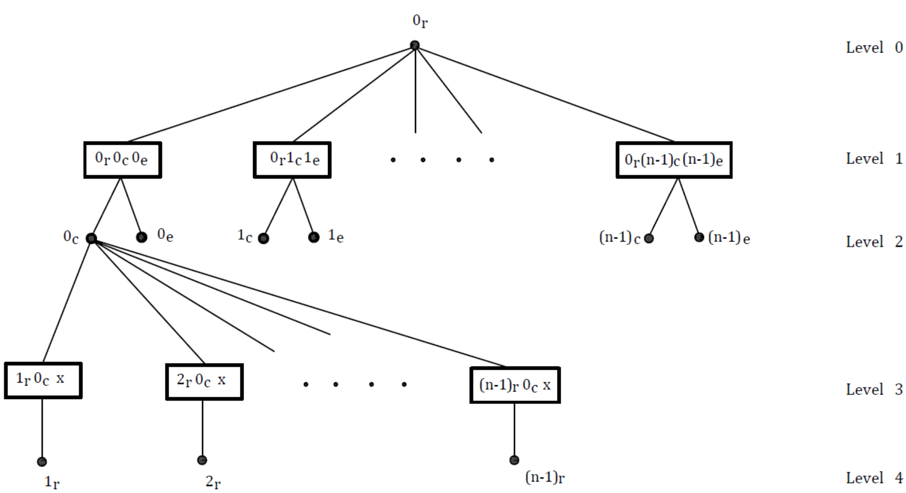

Figure 5: Spanning tree of $G(L)$.

For convenience, we will again refer to point vertices simply as points and block vertices as blocks. Let the root (Level 0) of the tree be the point 0_r. Connect the point to all n blocks containing it, which will be at Level 1. At Level 2 put all the $2n$ points $x \in \mathcal{C} \cup \mathcal{E}$ and connect each of these to the unique block at Level 1 which contains it. At Level 3 put the remaining $n-1$ blocks which contain the point 0_c and connect these to 0_c. Then at Level 4 put the $n-1$ points $\mathcal{R} \setminus \{0_r\}$ and connect each of these to the unique block at Level 3 which contains it. In standard form, i.e. $L(0,j) = j$, the structure of the tree is shown in Figure 5.

At this stage we have a tree which contains all the points but only the blocks which contain the points 0_r or 0_c. Now consider the point 1_c. There are $n-1$ blocks containing the point which as yet are not in the tree. Connect all of these to 1_c. Repeat this procedure for all of the points in the set $\mathcal{C} \setminus \{0_c, 1_c\}$. We now have a spanning tree of the point-block incidence graph in which the valency of the points 0_r and $x \in \mathcal{C}$ are n and all other points have valency 1. Thus in the co-tree all points have even valency. □

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A universality theorem for stressable graphs in the plane

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Abstract

Universality theorems (in the sense of N. Mnëv) claim that the realization space of a combinatorial object (a point configuration, a hyperplane arrangement, a convex polytope, etc.) can be arbitrarily complicated. In the paper, we prove a universality theorem for a graph in the plane with a prescribed *oriented matroid of stresses*, that is the collection of signs of all possible equilibrium stresses of the graph.

This research is motivated by the Grassmanian stratification (Gelfand, Goresky, MacPherson, Serganova) by thin Schubert cells, and by a recent series of papers on stratifications of configuration spaces of tensegrities (Doray, Karpenkov, Schepers, Servatius).

Keywords: Maxwell-Cremona correspondence, Grassmanian stratification, oriented matroid, equilibrium stress.

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1 Preliminaries and the main theorem

Let $\Gamma = (V, E)$ be a graph without loops and multiple edges, where $V = \{v_1, \dots, v_m\}$ is the set of vertices, and E is the set of edges. A *realization* of Γ is a map $p\colon V \to \mathbb{R}^2$ such that $(ij) \in E$ implies $p(v_i) \neq p(v_j)$. We abbreviate $p(v_i)$ as p_i.

That is, we have a planar drawing of Γ with possible intersections of edges and possible coinciding vertices. However, each edge is mapped to a non-degenerate line segment.

A *stress* $\mathfrak{s}$ on a realization (Γ, p) is an assignment of real scalars $\mathfrak{s}(i, j)$ to the edges. One imagines that each edge is represented by a (either compressed or extended) spring. Each spring produces some forces at its endpoints.

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A stress $\mathfrak{s}$ is called a *self-stress*, or an *equilibrium stress*, if at every vertex p_i, the sum of the forces produced by the springs vanishes:

$$\sum_{(ij)\in E} \mathfrak{s}(i,j)\mathbf{u}_{ij} = 0.$$

Here $\mathbf{u}_{ij} = \frac{p_i - p_j}{|p_i - p_j|}$ is the unit vector pointing from p_j to p_i.

A self-stress is *non-trivial* if it is not identically zero.

The set of all self-stresses $\mathfrak{S}(\Gamma, p)$ is a linear space which naturally embeds in $\mathbb{R}^e$, where $e = |E|$; the space $\mathfrak{S}$ depends on p.

A realization (Γ, p) is *stressable* if $\dim \mathfrak{S}(\Gamma, p) > 0$.

Given (Γ, p), define an oriented matroid $\mathcal{M}(\Gamma, p) := \mathrm{SIGN}(\mathfrak{S}(\Gamma, p))$.

In simple words, to obtain the matroid, enumerate somehow the edges of the graph, and for each non-trivial stress, list the signs of $\mathfrak{s}(i,j)$. We obtain a collection of strings (elements of $(+,-,0)^{\sharp(E)}$), which is an oriented matroid.[1] For the purpose of the present paper, it is sufficient to imagine an oriented matroid as a collection of strings. For a general theory of oriented matroids see [1].

Example 1.1. Let (Γ, p) be a planar realization of the graph K_4 such that p_4 lies inside the triangle $p_1p_2p_3$. Assume that the edges are enumerated in a way such that first come the edges of the triangle. Then $\mathcal{M}(\Gamma, p) = \{(+++---), (---+++)\}$.

The *realization space* of a graph Γ is the space of all realizations of Γ factorized by the action of the affine group:

$$\mathcal{R}(\Gamma) = \{p : p \text{ is a realization of } \Gamma\}/\operatorname{Aff}(\mathbb{R}^2).$$

Given a graph Γ and an oriented matroid $\mathcal{M}$, the *realization space* of $(\Gamma, \mathcal{M})$ is the space of all realizations of Γ that yield the oriented matroid $\mathcal{M}$:

$$\mathcal{R}(\Gamma, \mathcal{M}) = \{p \in \mathcal{R}(\Gamma) : \mathcal{M}(\Gamma, p) = \mathcal{M}\}.$$

For a fixed graph Γ, the realization spaces $\mathcal{R}(\Gamma, \mathcal{M})$ stratify $\mathcal{R}(\Gamma)$. Each of $\mathcal{R}(\Gamma, \mathcal{M})$ becomes a *stratum*.

In general, semialgebraic sets are subsets of some Euclidean space $\mathbb{R}^N$ defined by polynomial equations and inequalities. A semialgebraic set is called a *open basic primary semialgebraic set* (*OBP semialgebraic set*) if there are no defining equations, all the defining inequalities are strict, and the coefficients of all the defining polynomials are rational.

We borrow the notion of stable equivalency from traditional papers on universality, e.g. from [9]: *stable equivalence* is an equivalence relation on OBP semialgebraic sets generated by rational equivalence and stable projections.

The main result of the paper is:

Theorem 1.2. *For each open basic primary semialgebraic set $\mathfrak{A}$, there exists a graph Γ and an oriented matroid $\mathcal{M}$ such that the realization space $\mathcal{R}(\Gamma, \mathcal{M})$ is stably equivalent to $\mathfrak{A}$.*

[1] This is some realizable oriented matroid indeed, since it represents the set of vectors of some easy-to-build vector configuration related to the *rigidity matrix*. For rigidity matrices see [10].

Our first motivation comes from the complex Grassmanian stratifications [4], where strata are labeled by realizable matroids, and each stratum equals the realization space of a matroid. The stratification has a version over the field $\mathbb{R}$, where strata are labeled by realizable oriented matroids.

The other motivation is a series of papers [3, 5, 6] on stratifications of configuration spaces of tensegrities. Although the setup of the present paper might look different from the setup of [3, 5, 6], there is very much in common, see Section 3. In particular, we have the following analogue of Theorem 1.2:

Theorem 1.3. *For each connected open basic primary semialgebraic set $\mathfrak{A}$, there exists a graph Γ and a stratum (in the sense of [3]) R in the realization space of Γ such that R is stably equivalent to $\mathfrak{A}$.*

2 Proof of Theorem 1.2

Combinatorial equivalence of planar point configurations and line configurations see [8] is a classical subject and a starting point of our research. Given a combinatorial type, the equivalence class of point configurations (or line configurations) having this type is called the *realization space*. We shall use the same letter $\mathcal{R}$ for realization spaces of line configurations. In particular, for a line configuration L we denote by $\mathcal{R}(L)$ the realization space of all line configurations that are combinatorially equivalent to L.

We shall use the following version (chronologically, one of the first ones) of the celebrated Universality Theorem [7]: *generic planar point configurations are universal*. More precisely, for each OBP semialgebraic set $\mathfrak{A}$, there exists a planar point configuration with points in generic position[2] such that the realization space of the configuration is stably equivalent to $\mathfrak{A}$. An immediate consequence of the theorem is: generic planar line arrangements are universal.

Assume that a OBP semialgebraic set $\mathfrak{A}$ is fixed. For the set $\mathfrak{A}$, we shall construct a graph Γ together with its realization p, depicted in Figure 1. Thus, we get the associated oriented matroid $\mathcal{M} = \mathcal{M}(\Gamma, p)$. Our final aim is to show that $\mathcal{R}(\Gamma, \mathcal{M})$ is stably equivalent to $\mathfrak{A}$.

Here is the construction.

(1) Take a generic line configuration $L = \{l_i\}_{i=1}^n$ whose realization space is stably equivalent to $\mathfrak{A}$. Since the configuration is generic, there are no triple intersections.

(2) Take a rhombus $ABCD$ such that all mutual intersections $T_{ij} = T_{ji} = l_i \cap l_j$ lie strictly inside the rhombus, and each line $l_i \in L$ intersects the interiors of the segments AB and AD. Denote the intersection points points by A_i and D_i respectively. We may assume that the points $A, A_1, A_2, \ldots, A_n, B$ appear on the segment AB in this very order. Therefore, the order of the points D_i (from left to right) is reverse.

(3) Add to our construction the diagonals of the rhombus AC and BD.

(4) Add to our construction the points $B_i \in BC$, $C_i \in CD$, and the segments A_iB_i, B_iC_i, C_iD_i, and D_iA_i such that A_iD_i is symmetric to B_iC_i with respect to the diagonal BD for all i.

(5) Finally, add the intersection points $T_{ij} = T_{ji} = A_iD_i \cap A_jD_j$.

[2]No three points are collinear.

(6) Now let us specify edges of the graph. The points A_i split the segment AB into edges. The points B_i split BC into edges, etc. Besides, the points T_{ij} split A_iD_i into edges. The segments A_iB_i, B_iC_i, C_iD_i, AC, and BD are edges as well. All the edges are depicted in Figure 1.

We obtain a realization of a graph, whose vertices are $\{A, B, C, D, A_i, B_i, C_i, D_i, T_{ij}\}_{i,j}$.

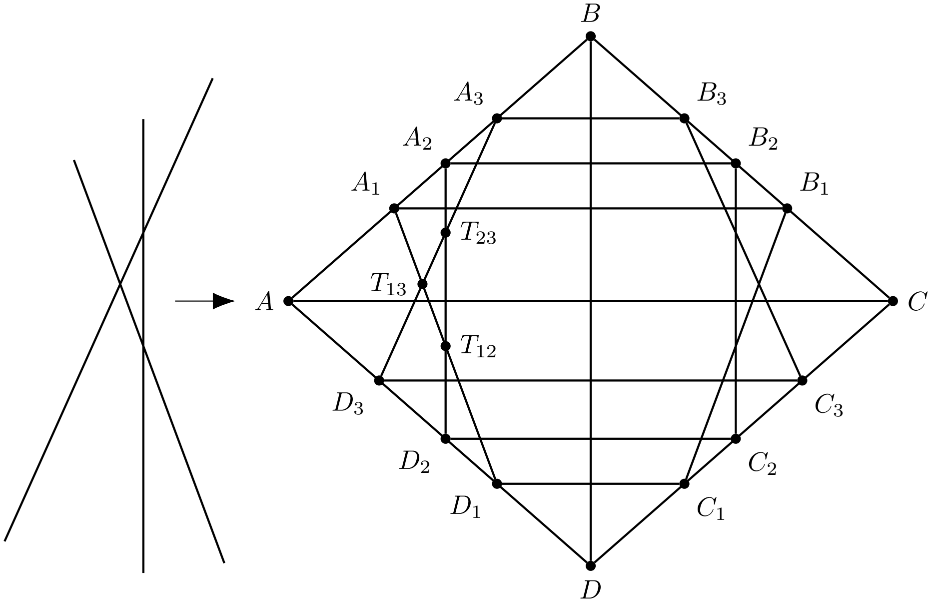

Figure 1: The graph Γ with its realization p.

Lemma 2.1. *Assume that we have a self-stressed realization of an arbitrary graph, and a vertex of the graph looks as is depicted in Figure 2. Then, in notation of the figure, the stresses satisfy:*

(a) (left) $\operatorname{Sign} s_1 = \operatorname{Sign} s_3 = -\operatorname{Sign} s_2$,

(b) (right) $s_1 = s_2$, $s_3 = s_4$.

(c) If a self-stress $\mathfrak{s}$ *of the above constructed* (Γ, p) *(see Figure 1) vanishes at one of the segments of the quadrilateral* $A_iB_iC_iD_i$*, then it vanishes on each of the segments of the quadrilateral.*

(d) If a self-stress $\mathfrak{s}$ *of* (Γ, p) *from Figure 1 vanishes at all the segments lying on* AB*, then it vanishes everywhere.* □

Let us look at some particular elements of $\mathfrak{S}(\Gamma, p)$ (in matroid terminology, they all are *circuits* of the oriented matroid $\mathcal{M}$). At most of the edges, these stresses vanish, so we depict them as subgraphs of (Γ, p). That is, we leave stressed edges only, and indicate the signs of the stress.

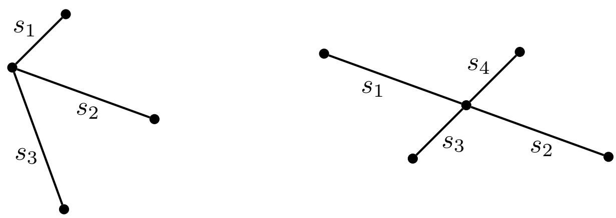

Figure 2: Local stresses.

Lemma 2.2.

(1) The subgraphs depicted in Figure 3 are stressable. The signs of the associated stresses are indicated. (Clearly, simultaneous inversion of signs also represents some self-stress.)

(2) The stresses (a) and (d) from Figure 3 (for all $i = 1, \dots, n$) form a basis of the linear space $\mathfrak{S}(\Gamma, p)$.

(3) The stresses (a) and (c) (for all $i = 1, \dots, n$) also form a basis of $\mathfrak{S}(\Gamma, p)$.

(4) For any stress of (Γ, p), the ratio of stresses on edges A_iA_{i+1} and B_iB_{i+1} does not depend on i (provided that the stresses are non-zero). The ratio of stresses on edges C_iC_{i+1} and D_iD_{i+1} does not depend on i either.

Proof. (1): (a) is known to be stressable.[3] (c) and (d) are stressable since these are *Desargues configurations*. This means that the three lines A_iD_i, BC and B_iC_i meet at a point; the three lines A_iB_i, C_iD_i and AC are parallel, that is, meet at a point at infinity. (b) is the difference of two different stresses of type (c), therefore stressable. The signs in all the cases follow from Lemma 2.1.

(2): Assume we have a stress $\mathfrak{s} \in \mathfrak{S}(\Gamma, p)$. Adding an appropriate stress of type (a), we kill the value of the stress on the edge AA_1, and therefore, on all the edges emanating from A. Next, adding appropriate stresses of type (d) kills the stresses on all the edges of AB. By Lemma 2.1, the result is identically zero.

(3): It follows from (2).

(4): It is true for (a) and (d), therefore, it is true for all self-stresses. □

Now we analyze the realization space of matroid $(\Gamma, \mathcal{M})$.

Proposition 2.3. *Assume that $\mathcal{M}(\Gamma, p') = \mathcal{M}(\Gamma, p)$, that is, $(\Gamma, p') \in \mathcal{R}(\Gamma, p)$. Denote the vertices of the realization by the same letters with primes (that is, by A'_i, B'_i, etc.). Then:*

(1) All collinearities of vertices that are present in p survive in p'. Besides, the order of collinear vertices is maintained.

(2) Points $A'B'C'D'$ lie in the convex position.

(3) p' yields an arrangement of lines $L' = \{l'_i\}$ with the same combinatorics as the initial arrangement L.

[3](a) can be viewed as a projection of a tetrahedron, therefore (a) is *liftable*. By Maxwell-Cremona correspondence, it is stressable.

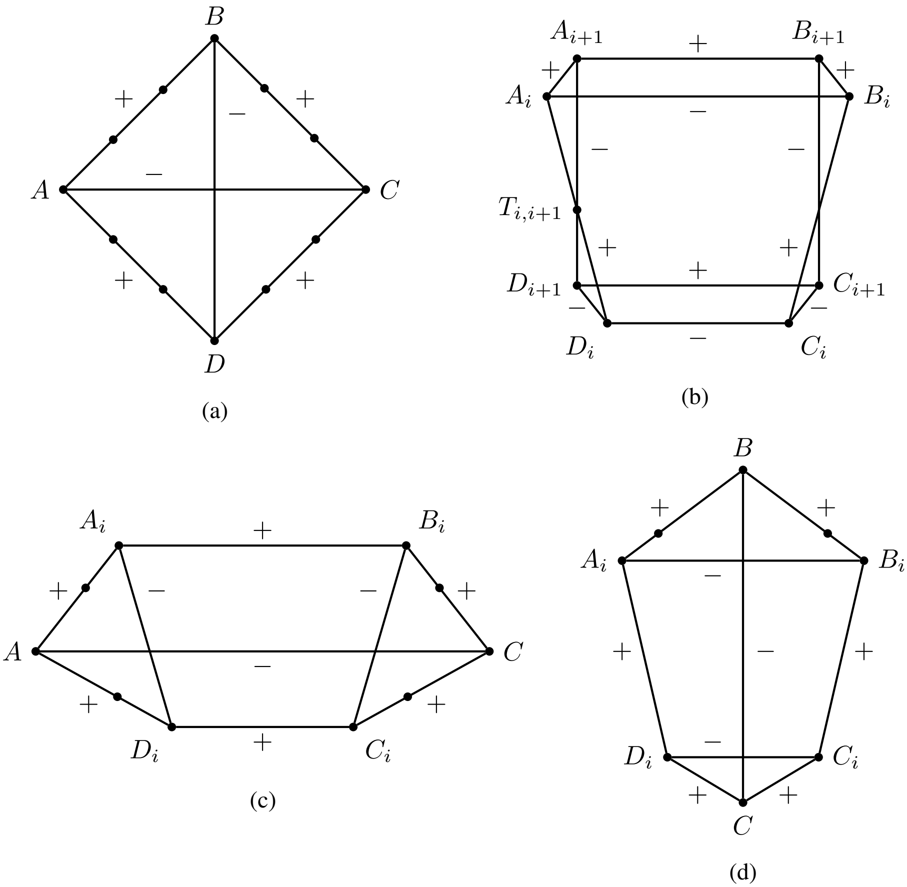

Figure 3: Some particular stresses.

Proof. The stressed graph (a) from Figure 3 should be stressed with the same signs (and with no other signs) for p' as well. Therefore $A'B'C'D'$ is a convex quadrilateral, and all the edges belonging to $A'B'$, are collinear (all the edges belonging to $C'B'$, etc. are collinear as well). The graphs of type (b) from Figure 3 remain stressed for p', so the segments of l_i stay collinear. Therefore we have an arrangement of lines $L' = \{l_i'\}$.

The graph Γ records the combinatorics of L, therefore the L and L' are combinatorially equivalent, that is, $L' \in \mathcal{R}(L)$. □

Corollary 2.4. *There exists a natural mapping between the realization space of* $(\Gamma, \mathcal{M})$ *and the realization space of the arrangements of lines* L*:*

$$\pi\colon \mathcal{R}(\Gamma, \mathcal{M}) \to \mathcal{R}(L).$$

The mapping π *extracts the arrangement* L' *from* (Γ, p') *and forgets the rest.* □

Proposition 2.5. *Let* $A'B'C'D'$ *be a convex quadrilateral. Assume that the points* $\{A_i', B_i', C_i', D_i'\}_{i=1}^n$ *are such that:*

(i) The points A'_i lie on the segment $A'B'$ and come in the same order as A_i. The points B'_i lie on the segment $B'C'$ and come in the same order as B_i; and the same condition for C'_i and D'_i.

(ii) The affine hulls of $A'_iD'_i$ form an arrangement of lines combinatorially equivalent to L.

(iii) All the associated subgraphs of type (c) from Figure 3 are Desargues ones.

Then:

(1) All the associated subgraphs of type (d) from Figure 3 are Desargues ones, and therefore, stressable.

(2) The realization of the graph Γ with these vertices has the same oriented matroid as $\mathcal{M} = \mathcal{M}(\Gamma, p)$.

Proof. (1): Follows from Desargues' theorem. The conditions imply that we have some realization p' of Γ, and that all circuits depicted in Figure 3 are circuits relative p'.

Before we proceed with the claim (2), let us observe the following:

Lemma 2.6. *Assume that $\mathfrak{s}$ is a self-stress of (Γ, p'), whose values on the edges $A'_{i-1}A'_i$ and $A'_{i+1}A'_i$ we denote by s_1 and s_2. Then the signs of the stresses of the other two edges E_1 and E_2 (each of them equals $A'_iT'_{ij}$ for some j), emanating from A'_i are:*

$$\mathrm{SIGN}(\mathfrak{s}(E_1)) = \mathrm{SIGN}(s_2 - s_1) = -\mathrm{SIGN}(\mathfrak{s}(E_2)),$$

assuming that E_1 lies to the left of E_2, see Figure 4. Similar statements are valid for edges emanating from B'_i, C'_i, and D'_i. □

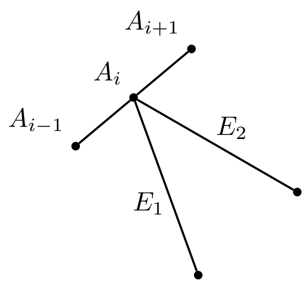

Figure 4: Illustration for the proof of Proposition 2.5.

Now let us prove the statement (2) of Proposition 2.5. First observe that Lemma 2.2 stays valid for (Γ, p'). Let $\mathfrak{d}_i$ (respectively, $\mathfrak{d}'_i$) be a stress of (Γ, p) (respectively, (Γ, p')) depicted in Figure 3(d), such that its value on AA_1 (respectively, $A'A'_1$) equals 1.

Let $\mathfrak{a}$ be a stress of (Γ, p) depicted in Figure 3(a), such that its value on AA_1 equals 1, let $\mathfrak{a}'$ be defined analogously for (Γ, p').

Assume that $\mathfrak{s}$ is a stress of (Γ, p). By Proposition 2.3, $\mathfrak{s} = \lambda\mathfrak{a} + \sum \lambda_i \mathfrak{d}_i$ for some real coefficients. Consider the stress $\mathfrak{s}' = \lambda\mathfrak{a}' + \sum \lambda_i \mathfrak{d}'_i$ of (Γ, p'). By Lemma 2.2(4) and Lemma 2.6, we have $\mathrm{SIGN}(\mathfrak{s}) = \mathrm{SIGN}(\mathfrak{s}')$. Conversely, each stress $\mathfrak{s}'$ of (Γ, p'), has a similar counterpart for (Γ, p). □

Proposition 2.7. *Mapping π from Corollary 2.4 is a stable projection.*

Proof. Assume that a line arrangement L' belongs to the realization space $\mathcal{R}(L)$. Let us look at the preimage $\pi^{-1}(L')$. Specification of $A'B'C'D'$ is stable since the positions of the lines $A'B'$, $A'D'$, etc. are defined by a number of strict inequalities depending on the lines L.

Now we may choose arbitrary distinct points $A'_1, \dots, A'_n$ on the segment $A'B'$ that come in the same order as $A_1, A_2, \dots, A_n$. The same happens with $B'_1, \dots, B'_n$: here we care about their order only. Now let us choose C'_1 as an arbitrary point on the segment $B'C'$. Once C' is specified, the position of D'_1 is uniquely determined, since Desargues condition implies that the lines $A'C'$, $A'_1B'_1$, and $C'_1D'_1$ meet at a point.

The point C'_2 should be chosen to the left of C'_1 but in such a way that D'_2 lies to the right of D'_1. This is always possible but dictates some extra condition, still in the framework of stable equivalence. The rest of the points C'_i and D'_i are treated analogously. $\square$

Corollary 2.4 and Proposition 2.7 imply that $\mathcal{R}(\Gamma, \mathcal{M})$ is stably equivalent to $\mathfrak{A}$. Theorem 1.2 is proven.

3 Relations between different settings. Proof of Theorem 1.3

In this section we show the equivalence of the settings of the present paper and that of [3].

Let us start with the definition of equilibrium stress. The paper [3] puts no restrictions on a realization of a graph p, that is, the endpoints of an edge might be mapped to one and the same point. Besides, [3] presents a more usual setting of the equilibrium stress (as in [2]): the equilibrium condition reads as

$$\sum_{(ij)\in E} s(i,j)(p_i - p_j) = 0.$$

Let us denote by $S(\Gamma, p)$ the linear space of stresses and by $M(\Gamma, p) = \mathrm{SIGN}(S(\Gamma, p))$ the associated matroid.

Clearly, if no edge is degenerate (that is, $p_i - p_j \neq 0$), a stress s in this setting gives a stress in the setting of the present paper $\mathfrak{s}(i,j) = s(i,j)|p_i - p_j|$ and vice versa. Therefore, $M(\Gamma, p) = \mathcal{M}(\Gamma, p)$. The only subtlety may arise if a realization p produces degenerate edges.

Lemma 3.1. *The matroid $M(\Gamma, p)$ "knows" all the degenerate edges. In particular, if there exists $p \in R(\Gamma, M)$ with no degenerate edges, then each $p' \in R(\Gamma, M)$ has no degenerate edges.*

Proof. Degenerate edges are detected by almost everywhere zero stresses: an edge number i is degenerate for (Γ, p) iff $(0, \dots, 0, +, 0, 0, \dots, 0) \in M(\Gamma, p)$. $\square$

3.1 Strong equivalence vs weak equivalence

Assume that a realization p of a graph Γ has no degenerate edges.

Repeating [6], let us say that two realizations of one and the same graph (Γ, p) and (Γ, p') are *strongly equivalent*, if there exists a sign preserving homeomorphism between the stress spaces $\mathfrak{S}(\Gamma, p)$ and $\mathfrak{S}(\Gamma, p')$.

Two realizations of one and the same graph (Γ, p) and (Γ, p') are *weakly equivalent*, if the associated matroids coincide: $\mathcal{M}(\Gamma, p) = \mathcal{M}(\Gamma, p')$.

Classes of weak equivalence are realization spaces, defined in the introduction (Section 1). Classes of strong equivalence are strata considered in [6].[4]

Proposition 3.2. *Strong equivalence equals weak equivalence.*

Proof. Clearly, strong equivalence implies weak equivalence. Let us prove the converse. The linear space $\mathfrak{S}(\Gamma, p)$ is tiled by convex cones, each cone corresponds to some string of signs from $\mathcal{M}(\Gamma, p)$. Let us intersect this tiling with the unit sphere centered at the origin. This gives a tiling of the sphere where each tile is a spherically convex polytope. The matroid $\mathcal{M}(\Gamma, p)$ "knows" the incidence relation of the tiles: a tile labeled by $(\varepsilon_1, \ldots, \varepsilon_e)$, $\varepsilon_i \in \{+, -, 0\}$, belongs to the closure of the tile $(\varepsilon'_1, \ldots, \varepsilon'_e)$ iff either $\varepsilon_i = 0$, or $\varepsilon_i = \varepsilon'_i$.

Besides, the matroid $\mathcal{M}(\Gamma, p)$ "knows" the dimension of each tile. Indeed, the matroid records the face poset of each tile. Since each tile is some pointed cone, its dimension is determined by the length of a longest chain in the poset.

Now it becomes possible to inductively build a sign-preserving homeomorphism between two spaces $\mathfrak{S}(\Gamma, p)$ and $\mathfrak{S}(\Gamma, p')$ with equal matroids. One should start with zero-dimensional spherical tiles, then extend the homeomorphism to one-dimensional tiles, etc. $\square$

Now let us prove Theorem 1.3. Given $\mathfrak{A}$, take the pair (Γ, p) as in the proof of Theorem 1.2. By Lemma 3.1, $R(\Gamma, p) = \mathcal{R}(\Gamma, p)$. By Proposition 3.2, $\mathcal{R}(\Gamma, p)$ is a strong equivalence class, that is, a stratum in the sense of [3, 5, 6]. Finally, by Theorem 1.2 the stratum is stably equivalent to $\mathfrak{A}$.

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A Appendix

A.1 One more example

A simpler (but in a sense, more "degenerate") example of (Γ', p) with the same realization space as in the previous section can be obtained if one takes (Γ, p) from Figure 1, removes all the edges lying on $AB, BC, CD, DA, AC, BD, A_iB_i, B_iC_i, C_iD_i$, and adds the new edges A_iD_i. That is, all the edges of the new graph lie on the lines l_i.

A.2 "No parallel edges" condition

The graph from Figure 1 has parallel edges emanating from one and the same vertex (in fact, almost all the vertices have parallel emanating edges). If parallel edges emanating of one and the same vertex are forbidden we still have a universality-type theorem:

Theorem A.1. *For each OBP semialgebraic set $\mathfrak{A}$, there exists a graph Γ, an oriented matroid $\mathcal{M}$, and a number N such that*

(1) $(\Gamma, \mathcal{M})$ has a realization with no parallel edges at all, and

(2) the realization space $\mathcal{R}(\Gamma, \mathcal{M})$ is stably equivalent to 2^N disjoint copies of $\mathfrak{A}$.

Proof. The idea is depicted in Figure 5: take the graph from Figure 1 and for each edge (ij), add two new vertices and replace (ij) by five new edges. One imagines a stressed realization of K_4 added to a stressed realization of Γ in such a way that the stresses on (ij) cancel. Denote the realization of the new graph by $(\hat{\Gamma}, \hat{p})$, and set $\hat{M} = M(\hat{\Gamma}, \hat{p})$. There exists a natural mapping

$$R(\hat{\Gamma}, \hat{M}) \to R(\Gamma, M).$$

The preimage of each point has $2^{e(\Gamma)}$ connected components since each stressed copy of K_4 can be attached both on the righthand side and on the lefthand side of (ij), but never degenerates. $\square$

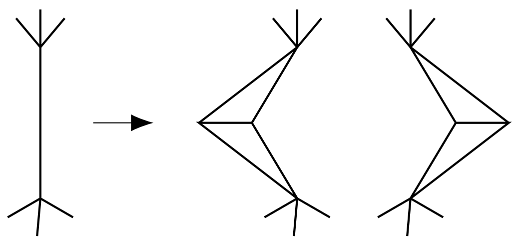

Figure 5: Adding a stressed copy of K_4.

A.3 Intersection of closures of two strata is not necessarily the closure of a stratum

This phenomenon was observed in [4] for Grassmanian stratifications. Let us adjust an example from [4] to show the same for stressed graphs.

Take the point configuration from Figure 6 and associate to it a graph (Γ, p) by the following rule: for each three collinear points i, j, k add the edges (ij), (jk), and (ik). So each three collinear points yield a stressable subgraph K_3. We conclude that all the

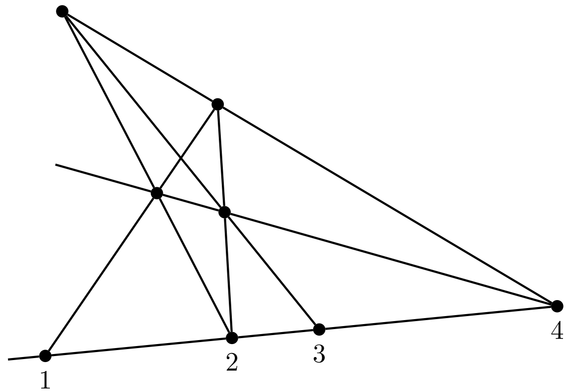

Figure 6: A graph which forces harmonic relation.

collinearities of vertices persist for all the elements of the realization space $\mathcal{R}(\Gamma, \mathcal{M}(\Gamma, p))$. However, all these collinearities imply that the four points $1, 2, 3, 4$ are harmonic, that is, their cross ratio equals -1.

Let us take a realization p' of Γ with all the vertices lying on a line. The corresponding matroid depends on the order of the vertices only and "does not see" the cross ratio. Therefore the intersection of the closures of the strata $\mathcal{R}(\Gamma, \mathcal{M}(\Gamma, p))$ and $\mathcal{R}(\Gamma, \mathcal{M}(\Gamma, p'))$ is not a closure of a stratum.

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An algorithm for constructing all supercharacter theories of a finite group*

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Abstract

In 2008, Diaconis and Isaacs introduced the notion of a supercharacter theory of a finite group in which supercharacters replace with irreducible characters and superclasses by conjugacy classes. In this paper, we introduce an algorithm for constructing supercharacter theories of a finite group by which all supercharacter theories of groups containing up to 14 conjugacy classes are calculated.

Keywords: Supercharacter theory, superclass, conjugacy class, irreducible character.

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1 Introduction

Suppose $\mathrm{UT}_n(q)$ denotes the set of all $n \times n$ unipotent upper-triangular matrices over the finite field $\mathrm{GF}(q)$. While working on the complex characters of this group, André constructed something nowadays called a supercharacter theory [1, 2, 3]. Diaconis and Isaacs in their seminal paper [9], axiomatized the notion of supercharacter theories of finite groups. To define, we assume that G is a finite group, $\mathrm{Irr}(G)$ denotes the set of all ordinary irreducible characters of G and $\mathrm{Con}(G)$ is the set of all conjugacy classes of G. A pair $(\mathcal{X}, \mathcal{K})$ is a supercharacter theory of G if the following conditions hold:

1. $\mathcal{X}$ and $\mathcal{K}$ are set partitions of $\mathrm{Irr}(G)$ and $\mathrm{Con}(G)$, respectively;
2. $\mathcal{K}$ contains $\{e\}$, where e denotes the identity element of G;
3. $|\mathcal{X}| = |\mathcal{K}|$;
4. For every $X \in \mathcal{X}$, characters $\sigma_X = \sum_{\chi \in X} \chi(e)\chi$ are constant on each $K \in \mathcal{K}$.

Characters σ_X are called *supercharacters*, and the members of $\mathcal{K}$ are *superclasses* of G [9]. Throughout this paper, $\mathrm{Sup}(G)$ denotes the set of all supercharacter theories of G. Assume $\mathcal{X} = \{\{1_G\}, \mathrm{Irr}(G) \setminus \{1_G\}\}$ and $\mathcal{K} = \{\{e\}, \mathrm{Con}(G) \setminus \{e\}\}$. Then $m(G) = (\mathrm{Irr}(G), \mathrm{Con}(G))$ and $M(G) = (\mathcal{X}, \mathcal{K})$ are the trivial supercharacter theories of G.

We now review some constructive results on supercharacter theories of finite groups. Hendrickson [13] provided several constructions which are used to classify all supercharacter theories of cyclic groups and obtained an exact formula for the number of supercharacter theories of a finite cyclic p-group. By studying partitions of the set of irreducible characters, Clifford theory and some well-known results regarding the structure of simple rational groups, Burkett et al. [6] proved that there are only three groups with exactly two supercharacter theories: the cyclic group Z_3, the symmetric group S_3 which is solvable, and the non-abelian simple group $\mathrm{Sp}(6, 2)$. Furthermore, Wynn [22] described all supercharacter theories of extraspecial and Frobenius groups. The number of supercharacter theories of dihedral groups of order $2p$, p is a Mersenne prime, was also calculated. In particular, he proved that if G is a Frobenius group of order pq, where p, q are primes and $p > q$, then G has exactly $1 + \tau(\frac{p-1}{q})\tau(q-1)$ supercharacter theories in which $\tau(n)$ denotes the number of positive divisors of n. In [5] the authors continued these works by providing some constructive methods in order to find new supercharacter theories. Then, they applied these methods towards a classification of finite simple groups with exactly three or four supercharacter theories.

The aim of this paper is to present an algorithm for constructing all supercharacter theories of finite groups. To explain and then evaluate our algorithm, we need some concepts in computer science. The time complexity of a program with a given input data of size n is defined as the number of elementary instructions that this program executes as a function of n. Moreover, the space complexity of a program with a given input data of size n is defined as the number of elementary objects that this program needs to store during its execution with respect to n. For two matrices A and B with the same number of rows, the augmented matrix $C = [A|B]$ is formed by appending the columns of B to A.

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Throughout this paper, our calculations are done with the aid of GAP [12]. Our group theory notations and terminologies can be found in [14, 18]. Moreover, we refer the interested readers to the book [8] for more information on algorithms.

2 Algorithm

Set $[n] = \{1, 2, \ldots, n\}$, G is a finite group, $\mathrm{Irr}(G) = \{\chi_1, \ldots, \chi_n\}$ and $\mathrm{Con}(G) = \{K_1, \ldots, K_n\}$. Choose $A \subseteq [n]$. Define $A_I(G) = \{\chi_i \mid i \in A\}$ and $A_C(G) = \{K_i \mid i \in A\}$. Then the mappings $\xi_1 \colon \mathcal{P}([n]) \longrightarrow \mathcal{P}(\mathrm{Irr}(G))$ and $\xi_2 \colon \mathcal{P}([n]) \longrightarrow \mathcal{P}(\mathrm{Con}(G))$ are given by $\xi_1(A) = A_I(G)$ and $\xi_2(A) = A_C(G)$, where $\mathcal{P}(Y)$ denotes the power set of a given set Y. Conversely, we assume that $B \subseteq \mathrm{Irr}(G)$, $C \subseteq \mathrm{Con}(G)$ and define $[n]_B = \{i \mid \chi_i \in B\}$ and $[n]^C = \{j \mid K_j \in C\}$. We now define $\gamma_1 \colon \mathcal{P}(\mathrm{Irr}(G)) \longrightarrow \mathcal{P}([n])$ and $\gamma_2 \colon \mathcal{P}(\mathrm{Con}(G)) \longrightarrow \mathcal{P}([n])$ by $\gamma_1(B) = [n]_B$ and $\gamma_2(C) = [n]^C$. Then the mapping ξ_t and γ_t, $t = 1, 2$, are mutually inverse and so we can use $\mathcal{P}([n])$ instead of both $\mathcal{P}(\mathrm{Con}(G))$ and $\mathcal{P}(\mathrm{Irr}(G))$ in our algorithms.

Let $X = \{\chi_1, \ldots, \chi_u\}$ and $K = \{K_1, \ldots, K_s\}$ be parts of set partitions $\mathcal{X}$ of $\mathrm{Irr}(G)$ and $\mathcal{K}$ of $\mathrm{Con}(G)$, respectively. If $\sigma_X(K_1) = \cdots = \sigma_X(K_s)$, then we say that X and K are *consistent*. If all parts of $\mathcal{X}$ are mutually consistent with all parts of $\mathcal{K}$, then the set partitions $\mathcal{X}$ and $\mathcal{K}$ are said to be consistent. In a part of the proof of [9, Theorem 2.2(c)], the following equivalence relation on G is given.

$$u \sim v \Longleftrightarrow \forall\, X \in \mathcal{X},\ \sigma_X(u) = \sigma_X(v).$$

Note that if u and v are conjugate in G, then $u \sim v$. As a result, it is enough to compute σ_X on all conjugacy classes of G. Suppose $I = \{X_1, \ldots, X_r\}$ is a set partition of $\mathrm{Irr}(G)$ and define $\sigma_i = \sigma_{X_i}$, $1 \le i \le r$. Set $K_x = \{y \mid \forall i, 1 \le i \le r; \sigma_i(x) = \sigma_i(y)\}$, $x \in G$, and let J be the set of all such subsets. If all members of J are non-empty, then J is a set partition of $\mathrm{Con}(G)$ consistent with I and so $(I, J) \in \mathrm{Sup}(G)$. It is far from true that for each set partition $\mathcal{X}$ of the irreducible characters of G there exists a set partition $\mathcal{K}$ of G-conjugacy classes such that $\mathcal{X}$ and $\mathcal{K}$ are consistent. Hence the problem of computing supercharacter theories of G is reduced to the problem of computing all consistent pairs for G.

For the sake of completeness, we introduce here some notations in order to work with supercharacter theories of a group G in GAP. The notation $\sigma_X(i)$ denotes the image of σ_X in the i-th conjugacy class of G. We also use the notation $[n, i] = \mathrm{SmallGroup}(n, i)$ to denote the i-th group of order n in the small group library of GAP.

The aim of this section is to present an algorithm for constructing all supercharacter theories of a finite group G. We partition this algorithm into three sub-algorithms. These sub-algorithms are presented in three sub-sections. In the first sub-section, a sub-algorithm for finding the set of all bad parts is provided. The second sub-section devotes to calculating all set partitions of an n-element set such that these set partitions do not have any bad part. In the last sub-section, a sub-algorithm for computing a consistent set partition of the conjugacy classes of a group G with respect to a set partition of $\mathrm{Irr}(G)$ is given.

2.1 Bad parts and bad set partitions

A part X of a set partition $\mathcal{X}$ of $\mathrm{Irr}(G)$ is said to be *bad* if X is consistent with only singleton subsets of $\mathrm{Con}(G)$. A set partition containing a bad part is called a *bad set* *partition*. It is easy to see that a bad set partition $\mathcal{X}$ of $\mathrm{Irr}(G)$ does not have a mate $\mathcal{K}$ such that $(\mathcal{X}, \mathcal{K}) \in \mathrm{Sup}(G)$.

We recall that in computing supercharacter theories of a finite group G, $\{e\}$ and $\{1_G\}$ are always parts of $\mathcal{K}$ and $\mathcal{X}$, respectively. As a result, it is enough to work with the set partitions of $[n]^\star = \{2, \ldots, n\}$.

Lemma 2.1. *Suppose $X \in \mathcal{P}([n]^\star)$ is a bad part. Then all values of $\sigma_X(i)$, $2 \le i \le n$, are distinct.*

Proof. Choose $2 \le j \ne k \le n$ such that $\sigma_X(j) = \sigma_X(k)$. Then the part X is consistent with $\{j\}$, $\{k\}$ and $\{j, k\}$. This is a contradiction to the definition of a bad part. □

By Lemma 2.1, to check whether a part $X \in \mathcal{P}([n]^\star)$ is bad or not, it is enough to compute $\sigma_X(i)$ for $i \in [n]^\star$. If all values are different, then X is a bad part.

2.2 Create set partitions

The computer algebra system GAP does not have the potential to construct set partitions for large enough $[n]$. In literature, there are two algorithms by Semba [17] and Er [11] for computing set partitions of $[n]$. The Semba's algorithm is based on the backtrack technique [17, Theorem 1] and has the time complexity $\Theta(4B(n))$, where the Bell number $B(n)$ is defined as the number of set partitions of an n-element set. The Er's algorithm is recursive. He claimed (without proof) that $\sum_{i=1}^{n} B(i) < 1.6B(n)$. This is while Nayak and Stojmenović [16, p. 12] proved that $\sum_{i=2}^{n} B(i) < 2B(n)$. In an exact phrase, Er claimed that the time complexity of his algorithm for generating all set partitions of $[n]$ is $\Theta(1.6B(n))$.

There exists a limitation in the Er's algorithm: Each set partition is determined at the end step and so we cannot identify whether a given set partition is bad or not, unless it is done completely. As a consequence, we design an algorithm which generates set partitions part by part. When a bad part occurs, calculations of all set partitions containing that part are pruned.

To explain our algorithm, we set $S = \{s_1, s_2, \ldots, s_n\}$. The $2^n - 1$ non-empty subsets of S are used as the parts of the set partitions of S. Note that when n is enough large, it is not possible to save all set partitions on the memory. In order to reduce memory usage, we use the integers of the closed interval $I = I(S) = [1, 2^{|S|} - 1]$. In fact, we define a one-to-one correspondence $\alpha_S \colon I \longrightarrow \mathcal{P}(S) \setminus \{\emptyset\}$ by $\alpha_S(k) = \{s_i \mid a_{n+1-i} = 1\}$, where $a_1 a_2 \ldots a_n$ is the n-bit binary form of k. For example, if $S = \{s_1, s_2, s_3, s_4, s_5\}$ and $k = 13$ then $(13)_2 = 1101$ and since $|S| = 5$, the 5-bit binary form of 13 is 01101. Thus, $\alpha_S(13) = \{s_1, s_3, s_4\}$. Let $I_O = I_O(S)$ be the set of all odd integers in the closed interval I. Then $\alpha_S(I_O)$ is the set of all subsets of S containing s_1. On the other hand, if F is a non-empty subset of S, then we conclude that $\alpha_S^{-1}(F) = \sum_{i \in F} 2^{\mathrm{Position}(S,i)-1}$, where $\mathrm{Position}(S, i)$, $i \in F$, is the position of i in S. In this example, if $F = \{s_1, s_3, s_4\}$ then $\alpha_S^{-1}(F) = 2^{1-1} + 2^{3-1} + 2^{4-1} = 13$.

Now, we are ready to present our algorithm for constructing all set partitions with no bad part. We use two lists *SPs* and RE in order to keep the set partitions and remaining elements, respectively. At the first step, we have $SPs = \emptyset$ and $RE = [n]^\star$. We fill *SPs* by parts constructed from the elements of RE which are not bad. Thus, $RE := RE \setminus \alpha_{RE}(k)$ and $SPs := SPs \cup \{\alpha_{RE}(k)\}$, where $k \in I_O$ and $\sigma_{RE}(k)$ is not a bad part. This algorithm will be returned to the previous step, when $RE = \emptyset$. Since our algorithm is recursive, its generating tree is constructed by DFS strategy. In Sub-algorithm 2, all set partitions of $[n]^\star$

that do not contain any bad part are generated. In Figure 1, an example of a generating tree for set partitions of $[4]^\star$ is presented.

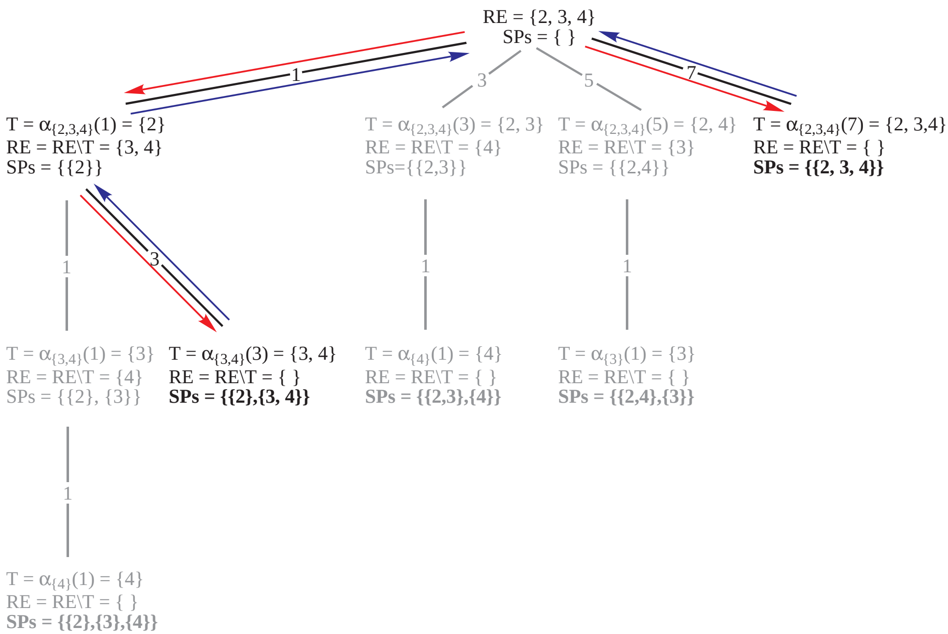

Figure 1: A schematic diagram of Sub-algorithm 2 for the case that $RE = [4]^\star$ and badparts $= \{\{2,3\},\{2,4\},\{3\},\{4\}\}$. The red and blue arrows represent forward and backward directions in the generating tree traversal, respectively. The number on each edge is an odd integer in $I_O(RE)$ of the parent node of the edge. The gray part of the tree is pruned due to the occurrence of a bad part in the process of creating corresponding set partition.

Note that after creating a set partition for $\mathrm{Irr}(G)$, we invoke the Sub-algorithm 3 for it, to check whether there exists a consistent set partition of $\mathrm{Con}(G)$ or not. This function is explained in details in the next subsection.

2.3 Create the consistent set partition $\mathcal{K}$ with respect to a given set partition of $\mathbf{Irr}(G)$

We recall that finding all supercharacter theories of a group G with n conjugacy classes is equivalent to constructing all consistent pairs of set partitions. Suppose *Irrp* is a given set partition of the irreducible characters of G. To find a consistent set partition of *Irrp*, we first define the matrix A as follows, see Table 1.

- The rows of A are the parts of *Irrp* and so A has exactly $|Irrp|$ rows.
- The columns of A are the conjugacy classes of G.
- If $A = (a_{ij})$, then $a_{ij} = \sigma_{X_i}(K_j)$ where $1 \le i \le |Irrp|$, $1 \le j \le n$ and K_j are the conjugacy classes of G.

Suppose $C_1, C_2, \dots, C_n$ are all columns of A. We construct a matrix ST and a list $Kappa$ as follows. Since $\{e\}$ is a part of each consistent set partition with *Irrp*, we conclude that $\{1\} \in Kappa$. We start our algorithm by defining $Kappa = \{\{1\}, \{2\}\}$ and the submatrix $ST = [C_1, C_2]$. For each j, $3 \leq j \leq n$, we compare C_j with all constructed columns of ST other than its first column. If C_j is different from such columns of ST, then we add C_j to ST as a new column and add j to $Kappa$ as a singleton part. Hence $Kappa := Kappa \cup \{\{j\}\}$ and $ST := [ST|C_j]$. If C_j is equal to the r-th column of ST, then we add j to the r-th part of $Kappa$, i.e.

$$Kappa = \{\{1\}, \dots, \{\underbrace{\dots, j}_{\text{part } r}\}, \dots, \{\dots\}\}.$$

Table 1: Matrix A.

	K_1	K_2	K_3	$\cdots$	K_n
χ_1	1	1	1	$\cdots$	1
$\vdots$	$*$	$*$	$*$	$\dots$	$*$
$X_i \begin{cases} \chi_{i_1} \\ \vdots \\ \chi_{i_t} \end{cases}$	$\sigma_{X_i}(K_1)$	$\sigma_{X_i}(K_2)$	$\sigma_{X_i}(K_3)$	$\cdots$	$\sigma_{X_i}(K_n)$
$\vdots$	$*$	$*$	$*$	$\dots$	$*$

If in the process of constructing $Kappa$ and ST the inequality $|Kappa| > |Irrp|$ occurs, then we stop calculations without any result. It is because there is no consistent set partition with the same size as *Irrp*. If at the end of our calculations, $|Kappa| = |Irrp|$, then we conclude that $(Irrp, Kappa)$ is a supercharacter theory and ST is the supercharacter table of G.

To construct all supercharacter theories of a group G, it is possible to combine the Er's algorithm which creates all set partitions of $\mathrm{Irr}(G)$ with the algorithm based on the proof of Theorem 2.2(c) in [9]. This is our *first algorithm*. Our main algorithm is a combination of the Sub-algorithms 1, 2 and 3. In [4], Sub-algorithm 3 for computing bad parts of the cyclic group Z_{13} is analyzed.

We end this section by noticing that:

1. The result of our main algorithm is supercharacter theories of a given group G. In fact, conditions of being a supercharacter theory are checked by the main algorithm in each case.

2. All supercharacter theories are generated by our algorithm. This is guaranteed by the proof of [9, Theorem 2.2(c)].

3 Analysis of algorithms

In Section 2, three sub-algorithms for computing bad parts, set partitions and supercharacter theories were presented. The aim of this section is to calculate the running time and the space complexity of these sub-algorithms and also our main algorithm.

Theorem 3.1. *Let $T_1(n)$, $S_1(n)$ and BP be the time complexity function, the space complexity function and the list of bad parts for a given group, respectively. Then, $T_1(n) \in O((n^2-n)\cdot 2^{n-1})$ and $S_1(n) \in O(n) + O(|BP|)$, where $n = |\operatorname{Con}(G)|$.*

Proof. To compute the running time of the Sub-algorithm 2, we should know the values of $\sigma_X(j)$, $2 \le j \le n$, where $X = \{x_1, \ldots, x_i\} \in \mathcal{P}([n]^\star)$. For this purpose, $i(n-1)$ multiplications and $(i-1)(n-1)$ additions are needed. Therefore, we have $\frac{(n-1)(n-2)}{2}$ comparisons for investigating the property that σ_X's are distinct. Since there are $\binom{n-1}{i}$ i-subsets, the complexity of this sub-algorithm can be computed by the following formula:

$$T_1(n) = \sum_{i=1}^{n-1}\left[(n-1)(2i-1)\binom{n-1}{i} + \frac{(n-1)(n-2)}{2}\right].$$

Therefore,

$$\begin{aligned}
T_1(n) &= \sum_{i=1}^{n-1}\left[(n-1)(2i-1)\binom{n-1}{i} + \frac{(n-1)(n-2)}{2}\right] \\
&= (n-1)\sum_{i=1}^{n-1}(2i-1)\binom{n-1}{i} + \frac{(n-1)^2(n-2)}{2} \\
&< n\cdot\sum_{i=1}^{n-1} 2i\cdot\binom{n-1}{i} + n^3 = 2n\cdot\sum_{i=1}^{n-1} i\cdot\binom{n-1}{i} + n^3 \\
&= 2n\cdot(n-1)\cdot 2^{n-2} + n^3 = (n^2-n)\cdot 2^{n-1} + n^3.
\end{aligned}$$

Hence $T_1(n) \in O((n^2-n)\cdot 2^{n-1})$.

To compute the space complexity of this sub-algorithm, we note that all parts are generated one by one. If a generated part is bad, then we add it to BP. To keep each part, our calculations need an array of size $n-1$. Moreover, an $(n-1)$-length array is needed in order to save the values $\sigma_X(i)$. Therefore, this sub-algorithm needs a memory of size $O(n)$ to keep each part. For saving all bad parts, we need another array such that its size depends only on the number of bad parts of irreducible characters of a given group. Consequently, $S_1(n) \in O(n) + O(|BP|)$. □

Theorem 3.2. *Suppose $T_2(n)$ and $S_2(n)$ are the time and space complexity functions of the Sub-algorithm 2, respectively. Then,*

(1) $T_2(n) \in O(2B(n))$;

(2) The space complexity $S_2(n)$ belongs to $\Theta(n)$.

Proof. To prove (1), let $T_2(n)$ denote the number of calculations needed to obtain all set partitions which do not contain bad parts and are from the $(n-1)$-element set RE. For computing the time complexity in the worst case $|BP| = 0$, we have to count the number of edges in the generating tree of this sub-algorithm in general, see Figure 1. Then,

$$T_2(n) = \sum_{i=0}^{n-1}\binom{n-1}{i}(T_2(i)+1).$$

In OEIS [10], the sequence $\{a(n)\}_{n\geq 0}$ with code A060719 exists which is defined as follows.

$$a(n+1) = a(n) + \sum_{i=0}^{n} \binom{n}{i}(a(i)+1); \quad a(0) = 1$$

By [15], $a(n) = 2B(n+1) - 1$ and since $T_2(n) = a(n-1)$, we conclude that $T_2(n) = 2B(n) - 1$. Hence, the time complexity of this algorithm is $O(2B(n))$.

The space complexity depends on the sizes of *SPs* and RE. Since the union of RE with the members of *SPs* is $[n]$, $S_2(n) \in \Theta(n)$ which proves (2). □

In the next theorem, we calculate the complexity of our Sub-algorithm 3 which computes the supercharacter theories of G.

Theorem 3.3. *Suppose G has exactly n conjugacy classes. For a given set partition Irrp, the time and space complexities of the Sub-algorithm 3 are $O(n^3)$ and $O(n^2)$, respectively.*

Proof. Suppose *Irrp* is a set partition for the set of all irreducible characters of a group G and $|Irrp| = k$. To calculate the matrix A, we first obtain all values $\chi_i(1)\chi_i$, $1 \leq i \leq n$. Since χ_i has n values, there are n^2 different products $\chi_i(1)\chi_i$. To compute σ_X and in the worst case $X = \{\chi_1, \ldots, \chi_n\}$, we need $n(n-2)$ products. As a result, the calculations for obtaining the matrix A is of the time complexity $O(n^2)$.

Now, we count all the operations that we need to construct ST and the list $Kappa$. The first and second columns of ST are the same as the first and second columns of A, respectively. Therefore, we have nothing to count for these columns. For the third column of ST, we have to compare the third column of A with the second column of ST and so there are k comparisons. For the fourth column of ST, the fourth column of A should be compared with the second and third columns of ST, and so there are at most $2k$ comparisons, and so on. Suppose that from the column C_j to the next, $|Kappa| = |Irrp|$. In this case, the remaining conjugacy classes of the group should be distributed among the other parts of $Kappa$ and hence we do not have a new part in $Kappa$. Thus from C_j to C_n, the number of comparisons is equal to $k(k-1)$. Therefore, the total number of comparisons for constructing ST and $Kappa$ is:

$$T_3(n) := k(1) + k(2) + \cdots + \underbrace{k(k-1) + k(k-1) + \cdots + k(k-1)}_{n-j+1}.$$

Since $j \geq 3$, $n - j + 1 \leq n - 2$. Thus $T_3(n) \leq k(\frac{n^2-5n+6}{2}) \leq n(\frac{n^2-5n+6}{2})$, and so, $T_3(n) \in O(n^3)$.

Suppose $S_3(n)$ is the space complexity of the Sub-algorithm 3. We have two matrices A and ST of sizes $k \times n$ and $k \times k$, respectively. Since in the worst case $k = n$, we conclude that $S_3(n) \in O(n^2)$. □

We are now ready to compute the time and space complexity of the first and main algorithms. In the first algorithm, the generated set partitions are used as an input to compute all supercharacter theories of a finite group. In what follows, we assume that our group has exactly n conjugacy classes. The running time of our first algorithm is $T_{Er}(n) = O(n^3) \cdot \Theta(2B(n))$ and so $T_{Er}(n) \in O(n^3 B(n))$.

In the main algorithm, we do not need to create consistent set partition $Kappa$ for bad set partitions. As a result, the Sub-algorithm 3 should be called $B(n) - |BP_s|$ times, where

$|BP_s|$ is the number of bad set partitions. Therefore, the time complexity of the main algorithm is $O((n^2 - n)2^{n-1} + (B(n) - |BP_s|)n^3) = O(n^3(B(n) - |BP_s|))$ and so the following result holds.

Theorem 3.4. *The time complexity of our first and main algorithms are $O(n^3B(n))$ and $O(n^3(B(n) - |BP_s|))$, respectively.*

4 Performance evaluation

To evaluate the performance of the main algorithm and then compare it with the first one, both algorithms have been implemented in the computer algebra system GAP under Windows 10 Home Single Language. The average running times for both algorithms after three runs on a computer with processor Intel(R) Core(TM) m7-6Y75 CPU @ 1.20 GHz 1.51 GHz, installed memory (RAM) 8.00 GB (7.90 GB usable), system type 64-bit operating system and x64-based processor are summarized in Table 2. In this table, we have chosen groups which have the maximum or the minimum number of supercharacter theories with different number of conjugacy classes. Let $BP(G)$ be the set of all bad parts in a group G and $\alpha(G) = \frac{|BP(G)|}{2^{\kappa(G)-1}-1}$ in which $\kappa(G)$ denotes the number of distinct conjugacy classes of G. We have the following two cases in general.

1. **There is not any bad part in $\mathcal{P}(\mathrm{Irr}(G))$.** In this case, the algorithm for computing supercharacter theories based on the Er's algorithm has a faster running time. Note that we have a pre-process for finding bad parts but such an overhead is very small with respect to the total running time.

2. **There are some bad parts in $\mathcal{P}(\mathrm{Irr}(G))$.** In this case, by removing these parts from our calculations, the main algorithm will have a faster running time. For example, in the cyclic group Z_{13} in which $\%98.17$ of all parts are bad, our main algorithm takes less than one second to run while the other algorithm takes more than 548 seconds. In rare cases such as the Mathieu group M_{22} in which a few percentage of existence of parts are bad, the running time of the first algorithm is a bit faster.

To compare the first algorithm with the main one, we use Figure 2 in which the running time of the groups with exactly 13 conjugacy classes with respect to these algorithms are shown. As it can bee seen, groups numbered 29–53 have faster running times with the main algorithm. The result shows that the main algorithm is better for some classes of groups.

5 Concluding remarks

In this paper, two algorithms for computing all supercharacter theories of a finite group G have been presented. The first algorithm is based on the Er's algorithm. In the main algorithm, we have introduced the new feature "bad part" for the parts of $\mathrm{Irr}(G)$. Since none of the supercharacter theories contains these bad parts, by filtering and detecting the set partitions of $\mathrm{Irr}(G)$ which have at least one bad part, the running time of this algorithm decreases significantly.

Suppose $BP(G)$ denotes the set of all bad parts in a group G. Let $\alpha(G) = \frac{|BP(G)|}{2^{\kappa(G)-1}-1}$ and assume $|\operatorname{Sup}(G)|$ is the number of all supercharacter theories of G. In Table 4, the percentage of existence of bad parts for some cyclic and dihedral groups are given.

Table 2: Comparing the running times for some groups.

$\kappa(G)$	G	$\lvert \mathrm{Sup}(G)\rvert$	$\lvert BP(G)\rvert$	$\alpha(G)$	Main algorithm (MA)(second)	First algorithm (FA)(second)	FA/MA
10	$[100, 11]$	623	0	0	1.4	1.1	0.8
11	$[32, 43]$	376	0	0	7.6	6.7	0.9
11	$[32, 44]$	376	0	0	8.1	7.5	0.9
12	$[1296, 3523]$	1058	0	0	70.1	62	0.9
13	$[64, 32]$	325	0	0	464.6	429.8	0.9
12	D_{36}	51	168	8.2	65.4	68.2	1.04
12	M_{22}	5	288	14.1	65.8	61.8	0.9
10	M_{11}	5	112	21.9	0.8	1.1	1.4
10	D_{28}	23	144	28.9	0.8	1.7	2.1
10	$[120, 35]$	10	152	29.7	0.6	1.1	1.8
13	$[93, 1]$	9	1980	48.4	169.7	662.3	3.9
13	$[253, 1]$	9	1980	48.4	127.8	521.3	4.08
10	Z_{10}	10	376	73.6	0.06	1.2	20
10	D_{34}	5	480	93.9	0.04	1.3	32.5
11	Z_{11}	4	990	96.8	0.1	7.9	79
13	Z_{13}	6	4020	98.1	1.2	548.2	456.8
11	D_{38}	4	1008	98.5	0.1	8.5	85
13	D_{46}	3	4092	99.9	1.2	610.6	508.8

Table 3: The GAP id of all groups with exactly 13 conjugacy classes.

1	$[162, 21]$	12	$[162, 20]$	23	$[960, 11359]$	34	$[100, 10]$	45	$[328, 12]$
2	$[96, 191]$	13	$[1944, 2290]$	24	$[64, 33]$	35	$[162, 15]$	46	$[148, 3]$
3	$[400, 206]$	14	$[64, 37]$	25	$[1000, 86]$	36	$[40, 6]$	47	$[333, 3]$
4	$[162, 22]$	15	$[64, 32]$	26	$[720, 409]$	37	$[1053, 51]$	48	$[156, 7]$
5	$[192, 1494]$	16	$[96, 190]$	27	$[576, 8652]$	38	$[120, 38]$	49	$[301, 1]$
6	$[96, 193]$	17	$[192, 1491]$	28	$[40, 4]$	39	$[600, 148]$	50	$[205, 1]$
7	$[1944, 2289]$	18	$[64, 36]$	29	$[\mathbf{162}, \mathbf{11}]$	40	$[216, 86]$	51	$[150, 5]$
8	$[192, 1493]$	19	$[192, 1492]$	30	$[160, 199]$	41	$[258, 1]$	52	$[13, 1]$
9	$[1440, 5841]$	20	$[64, 35]$	31	$[324, 160]$	42	$[310, 1]$	53	$[46, 1]$
10	$[40, 8]$	21	$[162, 19]$	32	$[162, 13]$	43	$[253, 1]$		
11	$[64, 34]$	22	$[216, 87]$	33	$[1320, 133]$	44	$[93, 1]$		

Our calculations given in Table 4 suggests the following conjecture.

Conjecture 5.1. *If $\beta_n = \alpha(Z_{p_n})$ and $\gamma_n = \alpha(D_{2p_n})$, then*

$$\lim_{n\to\infty} \beta_n = \lim_{n\to\infty} \gamma_n = 1,$$

where p_n is the n-th prime number.

In Table 5, the percentage of existence of bad parts in some groups of order $3p$, $3 \mid p-1$ is presented. The calculations given in this table show that the Conjecture 5.1 is not valid for groups of order $3p$. Suppose p and q are primes such that $q < p$ and $q \mid p-1$. Let

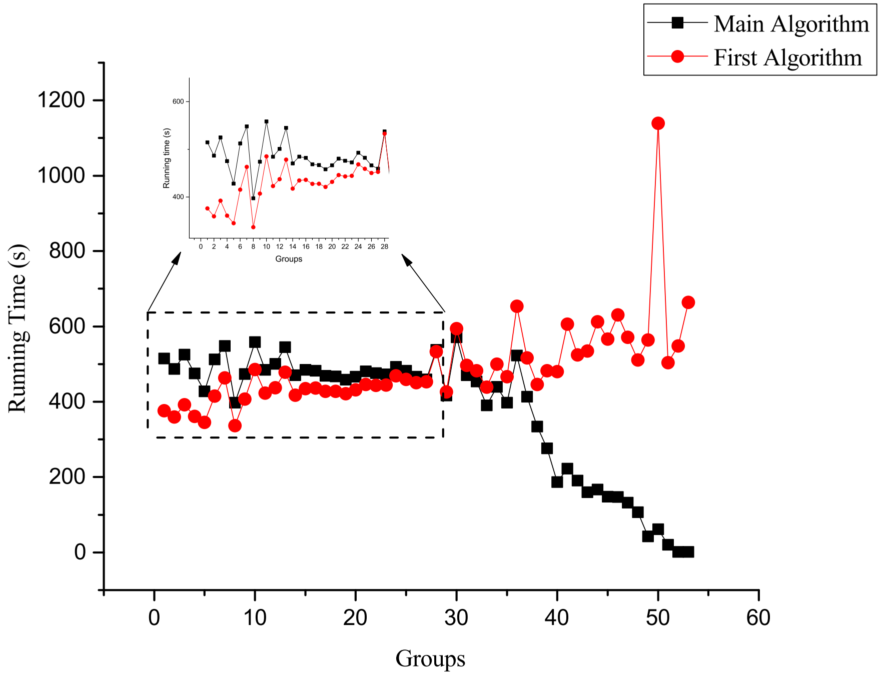

Figure 2: A diagram for the running time of all 53 groups which are listed in Table 3.

Table 4: Percentage of existence of bad parts for some cyclic and dihedral groups.

$\kappa(G)$	Group	$\alpha(G)$	$\lvert \mathrm{Sup}(G) \rvert$	$\kappa(G)$	Group	$\alpha(G)$	$\lvert \mathrm{Sup}(G) \rvert$
4	D_4	$\%\ 0$	5	2	Z_2	$\%\ 100$	1
3	D_6	$\%\,66.67$	2	3	Z_3	$\%\ 66.67$	2
4	D_{10}	$\%\,57.14$	3	5	Z_5	$\%\ 80$	3
5	D_{14}	$\%\,80$	3	7	Z_7	$\%\ 85.7$	4
7	D_{22}	$\%\,95$	3	11	Z_{11}	$\%\ 96.77$	4
8	D_{26}	$\%\,84.3$	5	13	Z_{13}	$\%\ 98.16$	6
10	D_{34}	$\%\,93.75$	5	17	Z_{17}	$\%\ 99.6$	5
11	D_{38}	$\%\,98.53$	4	19	Z_{19}	$\%\ 99.78$	6
13	D_{46}	$\%\,99.92$	3				
16	D_{58}	$\%\,99.2$	5				
17	D_{62}	$\%\,99.88$	5				

$T_{p,q}$ denote the non-abelian group of order pq and q_n denote the n-th prime number with the property $3 \mid q_n - 1$.

Table 5: Percentage of existence of bad parts in $T_{p,3}$.

p	$\alpha(G)$	p	$\alpha(G)$	p	$\alpha(G)$
7	$\%\,26$	13	$\%\,38$	19	$\%\,42$
31	$\%\,48$	37	$\%\,49$	43	$\%\,49.6$

By the results of Table 5, we offer the following conjecture:

Conjecture 5.2. *If* $\delta_n = \alpha(T_{q_n,3})$, *then* $\lim_{n\to\infty} \delta_n = 0.5$.

Table 6: Supercharacter theory form of groups with $\kappa \leq 14$ conjugacy classes.

κ	Supercharacter Theory Form
3	2^2
4	$3^3\ 5^1$
5	$3^3\ 5^3\ 9^2$
6	$4^1\ 5^1\ 7^1\ 8^1\ 9^1\ (15)^1\ (18)^1\ (20)^1$
7	$3^1\ 4^1\ 5^1\ 7^3\ 8^2\ (11)^1\ (20)^3$
8	$5^2\ 7^1\ (10)^1\ (11)^1\ (12)^3\ (13)^1\ (14)^1\ (15)^1\ (16)^1\ (18)^1\ (19)^1\ (22)^1\ (23)^1\ (25)^1\ (28)^1\ (54)^1\ (100)^1\ (110)^1$
9	$3^1\ 7^3\ 9^1\ (10)^1\ (12)^1\ (13)^1\ (15)^1\ (18)^1\ (19)^1\ (21)^1\ (22)^2\ (32)^2\ (36)^1\ (43)^1\ (40)^1\ (45)^3\ (49)^1\ (65)^1\ (128)^2$
10	$5^2\ (10)^2\ (11)^1\ (13)^1\ (14)^1\ (15)^1\ (16)^1\ (23)^1\ (25)^2\ (23)^1\ (24)^1\ (28)^1\ (32)^2\ (34)^1\ (35)^2\ (44)^1\ (51)^1\ (52)^1\ (57)^2\ (58)^3\ (64)^2\ (80)^1\ (83)^2\ (165)^1\ (215)^2\ (623)^1$
11	$4^2\ 8^2\ (11)^4\ (13)^4\ (15)^1\ (17)^1\ (18)^2\ (25)^1\ (26)^1\ (31)^1\ (47)^3\ (53)^2\ (55)^1\ (81)^1\ (89)^2\ (124)^1\ (144)^3\ (232)^1\ (376)^2$
12	$5^1\ 7^1\ (13)^2\ (16)^1\ (18)^2\ (19)^2\ (22)^3\ (23)^2\ (32)^2\ (34)^1\ (35)^1\ (36)^1\ (46)^1\ (49)^1\ (51)^1\ (65)^1\ (68)^1\ (69)^1\ (76)^3\ (81)^2\ (88)^2\ (94)^1\ (99)^2\ (100)^1\ (105)^1\ (133)^4\ (144)^1\ (152)^1\ (197)^1\ (205)^1\ (212)^1\ (233)^1\ (255)^1\ (360)^1\ (484)^1\ (1058)^1$
13	$3^1\ 6^1\ 9^2\ (11)^1\ (13)^2\ (17)^1\ (18)^1\ (24)^2\ (25)^2\ (35)^1\ (38)^2\ (40)^2\ (42)^1\ (43)^1\ (46)^2\ (50)^1\ (53)^2\ (71)^3\ (72)^2\ (81)^2\ (89)^1\ (102)^1\ (110)^4\ (129)^4\ (132)^3\ (138)^1\ (175)^1\ (313)^3\ (325)^3$
14	$5^2\ 9^1\ (10)^1\ (12)^1\ (13)^1\ (14)^1\ (15)^2\ (21)^1\ (22)^2\ (23)^1\ (29)^1\ (35)^2\ (38)^1\ (39)^1\ (41)^1\ (43)^1\ (45)^3\ (47)^1\ (49)^1\ (51)^1\ (53)^1\ (57)^1\ (63)^2\ (71)^1\ (76)^1\ (78)^2\ (79)^1\ (81)^2\ (85)^1\ (105)^1\ (110)^2\ (119)^1\ (123)^1\ (125)^1\ (130)^1\ (138)^1\ (139)^1\ (140)^2\ (145)^1\ (157)^1\ (172)^2\ (186)^2\ (206)^1\ (213)^3\ (222)^1\ (244)^2\ (270)^1\ (272)^2\ (304)^2\ (308)^2\ (320)^1\ (482)^1\ (601)^2\ (613)^2\ (620)^3\ (627)^1\ (645)^3\ (904)^1\ (940)^3\ (1048)^2\ (1324)^3\ (2093)^1\ (29016)^1$

Suppose $\mathrm{Irr}(Z_7) = \{1_{Z_7}, \chi_2, \ldots, \chi_7\}$ and $\mathrm{Con}(Z_7) = \{e, x_2^{Z_7}, \ldots, x_7^{Z_7}\}$. The cyclic group Z_7 has exactly four supercharacter theories $m(Z_7)$, $M(Z_7)$, $\mathcal{C}_1 = (\mathcal{X}_1, \mathcal{K}_1)$ and

$\mathcal{C}_2 = (\mathcal{X}_2, \mathcal{K}_2)$ such that

$$\mathcal{X}_1 := \{\{1_{Z_7}\}, \{\chi_2, \chi_3, \chi_5\}, \{\chi_4, \chi_6, \chi_7\}\},$$
$$\mathcal{K}_1 := \{\{e\}, \{{x_2}^{Z_7}, {x_3}^{Z_7}, {x_5}^{Z_7}\}, \{{x_4}^{Z_7}, {x_6}^{Z_7}, {x_7}^{Z_7}\}\},$$
$$\mathcal{X}_2 := \{\{1_{Z_7}\}, \{\chi_2, \chi_7\}, \{\chi_3, \chi_6\}, \{\chi_4, \chi_5\}\},$$
$$\mathcal{K}_2 := \{\{e\}, \{{x_2}^{Z_7}, {x_7}^{Z_7}\}, \{{x_3}^{Z_7}, {x_6}^{Z_7}\}, \{{x_4}^{Z_7}, {x_5}^{Z_7}\}\}.$$

This shows that each part in $\mathcal{X}$ and $\mathcal{K}$ in a supercharacter theory $(\mathcal{X}, \mathcal{K})$ has size $1, 2, 3$ or 6. On the other hand, if p is prime and d is the number of divisors of $p-1$, then by [13, Table 1], the cyclic group Z_p has exactly d supercharacter theories. As a result, the following conjecture is suggested.

Conjecture 5.3. *For each divisor r of $p-1$, there exists only one supercharacter theory $(\mathcal{X}, \mathcal{K})$ of Z_p such that the sizes of all non-trivial parts of $\mathcal{X}$ and $\mathcal{K}$ are equal to r. Moreover, if we sort the conjugacy classes and irreducible characters of Z_p by ATLAS notations [7], then $\gamma_1(\mathcal{X}) = \gamma_2(\mathcal{K})$.*

It is a well-known result in group theory that for any positive integer k, there are finitely many number of non-isomorphic finite groups with exactly k conjugacy classes. This number is denoted by $f(k)$. The supercharacter theory form of $f(k)$ is defined as $n_1^{\alpha_1} n_2^{\alpha_2} \cdots n_s^{\alpha_s}$ where α_i, $1 \leq i \leq s$, denote the number of groups with exactly k conjugacy classes containing n_i supercharacter theories and note that $f(k) = \sum_{i=1}^{s} \alpha_i$. The supercharacter theory form of groups with at most 14 conjugacy classes are recorded in Table 6 and the following conjecture has been suggested based on this table.

Conjecture 5.4. *The number of supercharacter theories of all members of $\Gamma(k)$ are distinct if and only if $k = 6$. In this case, all groups are Z_5, D_{14}, A_5, $Z_5 : Z_4$, $Z_7 : Z_3$, S_4, D_8 and Q_8.*

Vera-López and his co-authors [19, 20, 21] classified all finite groups containing up to 14 conjugacy classes. We apply these classification theorems and our main algorithm to find all supercharacter theories of groups containing up to 14 conjugacy classes. These calculations are presented in [4].

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Hypergeometric degenerate Bernoulli polynomials and numbers

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Abstract

Carlitz defined the degenerate Bernoulli polynomials $\beta_n(\lambda, x)$ by means of the generating function $t\big((1+\lambda t)^{1/\lambda}-1\big)^{-1}(1+\lambda t)^{x/\lambda}$. In 1875, Glaisher gave several interesting determinant expressions of numbers, including Bernoulli, Cauchy and Euler numbers. In this paper, we show some expressions and properties of hypergeometric degenerate Bernoulli polynomials $\beta_{N,n}(\lambda, x)$ and numbers, in particular, in terms of determinants.

The coefficients of the polynomial $\beta_n(\lambda, 0)$ were completely determined by Howard in 1996. We determine the coefficients of the polynomial $\beta_{N,n}(\lambda, 0)$. Hypergeometric Bernoulli numbers and hypergeometric Cauchy numbers appear in the coefficients.

Keywords: Bernoulli numbers, hypergeometric Bernoulli numbers, hypergeometric Cauchy numbers, hypergeometric functions, degenerate Bernoulli numbers, determinants, recurrence relations.

Math. Subj. Class. (2020): 11B68, 11B37, 11C20, 15A15, 33C15

1 Introduction

Carlitz [7, 8] defined the *degenerate Bernoulli polynomials* $\beta_n(\lambda, x)$ by means of the generating function

$$\left(\frac{t}{(1+\lambda t)^{1/\lambda}-1}\right)(1+\lambda t)^{x/\lambda} = \sum_{n=0}^{\infty} \beta_n(\lambda, x)\frac{t^n}{n!}\,. \tag{1.1}$$

When $\lambda \to 0$ in (1.1), $B_n(x) = \beta_n(0, x)$ are the ordinary Bernoulli polynomials because

$$\lim_{\lambda\to 0}\left(\frac{t}{(1+\lambda t)^{1/\lambda}-1}\right)(1+\lambda t)^{x/\lambda} = \frac{te^{xt}}{e^t-1} = \sum_{n=0}^{\infty} B_n(x)\frac{t^n}{n!}\,.$$

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When $\lambda \to 0$ and $x = 0$ in (1.1), $B_n = \beta_n(0,0)$ are the classical Bernoulli numbers defined by

$$\frac{t}{e^t - 1} = \sum_{n=0}^{\infty} B_n \frac{t^n}{n!}\,. \tag{1.2}$$

The degenerate Bernoulli polynomials in λ and x have rational coefficients. When $x = 0$, $\beta_n(\lambda) = \beta_n(\lambda, 0)$ are called *degenerate Bernoulli numbers*. In [16], explicit formulas for the coefficients of the polynomial $\beta_n(\lambda)$ are found. In [26], a general symmetric identity involving the degenerate Bernoulli polynomials and the sums of generalized falling factorials are proved.

In another direction, *hypergeometric Bernoulli polynomials* $B_{N,n}(x)$ (see, e.g., [17]) are defined by the generating function

$$\frac{e^{tx}}{{}_1F_1(1; N+1; t)} = \sum_{n=0}^{\infty} B_{N,n}(x) \frac{t^n}{n!}\,, \tag{1.3}$$

where ${}_1F_1(a; b; z)$ is the confluent hypergeometric function defined by

$${}_1F_1(a; b; z) = \sum_{n=0}^{\infty} \frac{(a)^{(n)}}{(b)^{(n)}} \frac{z^n}{n!}$$

with the rising factorial $(x)^{(n)} = x(x+1)\cdots(x+n-1)$ $(n \ge 1)$ and $(x)^{(0)} = 1$. When $x = 0$ in (1.3), $B_{N,n} = B_{N,n}(0)$ are the hypergeometric Bernoulli numbers ([12, 13, 14, 15, 19]). When $N = 1$ in (1.3), $B_n(x) = B_{1,n}(x)$ are the ordinary Bernoulli polynomials. When $x = 0$ and $N = 1$ in (1.3), $B_n = B_{1,n}(0)$ are the classical Bernoulli numbers.

Many kinds of generalizations of the Bernoulli numbers have been considered by many authors. For example, such generalizations include poly-Bernoulli numbers, Apostol Bernoulli numbers, various types of q-Bernoulli numbers, Bernoulli Carlitz numbers. One of the advantages of hypergeometric numbers is the natural extension of determinant expressions of the numbers.

A determinant expression of hypergeometric Bernoulli numbers ([2, 18]) is given by

$$B_{N,n} = (-1)^n n! \begin{vmatrix} \frac{N!}{(N+1)!} & 1 & 0 & & \\ \frac{N!}{(N+2)!} & \frac{N!}{(N+1)!} & & & \\ \vdots & \vdots & \ddots & 1 & 0 \\ \frac{N!}{(N+n-1)!} & \frac{N!}{(N+n-2)!} & \cdots & \frac{N!}{(N+1)!} & 1 \\ \frac{N!}{(N+n)!} & \frac{N!}{(N+n-1)!} & \cdots & \frac{N!}{(N+2)!} & \frac{N!}{(N+1)!} \end{vmatrix}\,. \tag{1.4}$$

The determinant expression for the classical Bernoulli numbers $B_n = B_{1,n}$ was discovered by Glaisher ([11, p. 53]).

In this paper, we introduce and study the hypergeometric degenerate Bernoulli numbers as total generalizations of degenerate Bernoulli numbers and hypergeometric Bernoulli numbers in the aspects of determinants. By applying Trudi's formula and the inversion formula, we show several arithmetical and combinatorial identities. The coefficients of the polynomial $\beta_n(\lambda)$ were completely determined by Howard in 1996. We determine the coefficients of the polynomial $\beta_{N,n}(\lambda)$. The constant term and the leading coefficient are exactly equal to Hypergeometric Bernoulli numbers and hypergeometric Cauchy numbers, respectively.

2 Definition and preliminary results

Denote the generalized falling factorial by for $n \ge 1$

$$(x|\alpha)_n = x(x-\alpha)(x-2\alpha)\cdots\bigl(x-(n-1)\alpha\bigr)$$

with $(x|\alpha)_0 = 1$. When $\alpha = 1$, $(x)_n = (x|1)_n$ is the ordinary falling factorial. Define *hypergeometric degenerate Bernoulli polynomials* $\beta_{N,n}(\lambda, x)$ by

$$\left({}_2F_1\left(1, N-\frac{1}{\lambda}; N+1; -\lambda t\right)\right)^{-1}(1+\lambda t)^{x/\lambda} = \sum_{n=0}^\infty \beta_{N,n}(\lambda, x)\frac{t^n}{n!}\,, \tag{2.1}$$

where ${}_2F_1(a, b; c; z)$ is the Gauss hypergeometric function defined by

$${}_2F_1(a;b;z) = \sum_{n=0}^\infty \frac{(a)^{(n)}(b)^{(n)}}{(c)^{(n)}}\frac{z^n}{n!}\,.$$

When $x = 0$ in (2.1), $\beta_{N,n}(\lambda) = \beta_{N,n}(\lambda, 0)$ are the *hypergeometric degenerate Bernoulli numbers*. When $\lambda \to 0$, $B_{N,n}(x) = \lim_{\lambda\to 0}\beta_{N,n}(\lambda, x)$ are the hypergeometric Bernoulli polynomials in (1.3). Since

$$\frac{t}{(1+\lambda t)^{1/\lambda}-1} = t\left(\sum_{n=1}^\infty \frac{(1-\lambda|\lambda)_{n-1}}{n!}t^n\right)^{-1}$$

in (1.1), we can write

$$\begin{aligned}
&{}_2F_1\left(1, N-\frac{1}{\lambda}; N+1; -\lambda t\right)\\
&\qquad = \left(\frac{(1-\lambda|\lambda)_{N-1}}{N!}t^N\right)\left(\sum_{n=N}^\infty \frac{(1-\lambda|\lambda)_{n-1}}{n!}t^n\right)^{-1}\\
&\qquad = 1 + \sum_{n=1}^\infty \frac{(1-\lambda|\lambda)_{N+n-1}N!}{(1-\lambda|\lambda)_{N-1}(N+n)!}t^n\\
&\qquad = 1 + \sum_{n=1}^\infty \frac{(1-N\lambda|\lambda)_n}{(N+n)_n}t^n\,.
\end{aligned} \tag{2.2}$$

When $N = 1$, $\beta_n(\lambda, x) = \beta_{N,1}(\lambda, x)$ are degenerate Bernoulli polynomials, defined by

$$\left(1 + \sum_{n=1}^\infty \frac{(1-\lambda|\lambda)_n}{(n+1)!}t^n\right)^{-1}(1+\lambda t)^{x/\lambda} = \sum_{n=0}^\infty \beta_n(\lambda, x)\frac{t^n}{n!}\,.$$

When $N = 1$ and $\lambda \to 0$, $B_n(x) = \lim_{\lambda\to 0}\beta_{N,1}(\lambda, x)$ are the classical Bernoulli polynomials, defined by

$$\left(1 + \sum_{n=1}^\infty \frac{t^n}{(n+1)!}\right)^{-1}e^{xt} = \sum_{n=0}^\infty B_n(x)\frac{t^n}{n!}\,.$$

The definition (2.1) may be obvious or artificial for the readers with different backgrounds. One of our motivations is mentioned in Section 5.

We have the following recurrence relation of hypergeometric degenerate Bernoulli numbers $\beta_{N,n}(\lambda)$.

Proposition 2.1. *For $N, n \geq 1$, we have*

$$\beta_{N,n}(\lambda) = -\sum_{k=0}^{n-1} \frac{n!(1-N\lambda|\lambda)_{n-k} N!}{(N+n-k)!k!} \beta_{N,k}(\lambda)$$

with $\beta_{N,0}(\lambda) = 1$.

Proof. By (2.1) with (2.2), we get

$$\begin{aligned} 1 &= \left(1 + \sum_{l=1}^{\infty} \frac{(1-N\lambda|\lambda)_l N!}{(N+l)!} t^l\right) \left(\sum_{n=0}^{\infty} \beta_{N,n}(\lambda) \frac{t^n}{n!}\right) \\ &= \sum_{n=0}^{\infty} \beta_{N,n}(\lambda) \frac{t^n}{n!} + \sum_{n=1}^{\infty} \sum_{k=0}^{n-1} \frac{(1-N\lambda|\lambda)_{n-k} N!}{(N+n-k)!} \frac{\beta_{N,k}(\lambda)}{k!} t^n . \end{aligned}$$

Comparing the coefficients on both sides, we obtain for $n \geq 1$

$$\frac{\beta_{N,n}(\lambda)}{n!} + \sum_{k=0}^{n-1} \frac{(1-N\lambda|\lambda)_{n-k} N!}{(N+n-k)!} \frac{\beta_{N,k}(\lambda)}{k!} = 0 . \qquad \square$$

We can use Proposition 2.1 to give an explicit expression for $\beta_{N,n}(\lambda)$.

Theorem 2.2. *For $N, n \geq 1$,*

$$\beta_{N,n}(\lambda) = n! \sum_{k=1}^{n} (-N!)^k \sum_{\substack{i_1+\cdots+i_k=n \\ i_1,\ldots,i_k \geq 1}} \frac{(1-N\lambda|\lambda)_{i_1}}{(N+i_1)!} \cdots \frac{(1-N\lambda|\lambda)_{i_k}}{(N+i_k)!} . \tag{2.3}$$

Remark 2.3. When $\lambda \to 0$, Theorem 2.2 is reduced to

$$B_{N,n} = n! \sum_{k=1}^{n} \sum_{\substack{i_1+\cdots+i_k=n \\ i_1,\ldots,i_k \geq 1}} \frac{(-N!)^k}{(N+i_1)! \cdots (N+i_k)!} , \tag{2.4}$$

as seen in [2, 18]. When $\lambda \to 0$ and $N = 1$, there is a combinatorial interpretation of Bernoulli numbers in terms of the cardinality of $\mathbb{Z}_2$-graded groupoids [4, Corollary 45].

Proof of Theorem 2.2. The proof is by induction on n. From Proposition 2.1 with $n = 1$,

$$\beta_{N,1}(\lambda) = -\frac{(1-N\lambda)N!}{(N+1)!} \beta_{N,0}(\lambda) = -\frac{N!(1-N\lambda)}{(N+1)!} .$$

This matches the expression (2.1) when $n = 1$. Assume that the result is valid up to $n-1$. For simplicity, put

$$S_k(n) = \sum_{\substack{i_1+\cdots+i_k=n \\ i_1,\ldots,i_k \geq 1}} \frac{(1-N\lambda|\lambda)_{i_1}}{(N+i_1)!} \cdots \frac{(1-N\lambda|\lambda)_{i_k}}{(N+i_k)!} . \tag{2.5}$$

Then by Proposition 2.1

$$
\begin{aligned}
\frac{\beta_{N,n}(\lambda)}{n!} &= -\sum_{l=0}^{n-1}\frac{(1-N\lambda|\lambda)_{n-l}N!}{(N+n-l)!}\frac{\beta_{N,l}(\lambda)}{l!}\\
&= -\frac{(1-N\lambda|\lambda)_n N!}{(N+n)!}-\sum_{l=1}^{n-1}\frac{(1-N\lambda|\lambda)_{n-l}N!}{(N+n-l)!}\sum_{k=1}^{l}(-N!)^k S_k(l)\\
&= -\frac{(1-N\lambda|\lambda)_n N!}{(N+n)!}-\sum_{k=1}^{n-1}(-N!)^k\sum_{l=k}^{n-1}\frac{(1-N\lambda|\lambda)_{n-l}N!}{(N+n-l)!}S_k(l)\\
&= -\frac{(1-N\lambda|\lambda)_n N!}{(N+n)!}-\sum_{k=2}^{n}(-N!)^{k-1}\sum_{l=k-1}^{n-1}\frac{(1-N\lambda|\lambda)_{n-l}N!}{(N+n-l)!}S_{k-1}(l)\\
&= -\frac{(1-N\lambda|\lambda)_n N!}{(N+n)!}+\sum_{k=2}^{n}(-N!)^k S_k(n)\\
&= \sum_{k=1}^{n}(-N!)^k S_k(n)\,.
\end{aligned}
$$

Here, we put $n-l=i_k$ in the second last equation. □

There is an alternative form of $\beta_{N,n}(\lambda)$ using binomial coefficients. The proof may be similar to that of Theorem 2.2, but a different proof is given.

Theorem 2.4. *For $N,n\ge 1$,*

$$
\beta_{N,n}(\lambda)=n!\sum_{k=1}^{n}(-N!)^k\binom{n+1}{k+1}\sum_{\substack{i_1+\cdots+i_k=n\\ i_1,\dots,i_k\ge 0}}\frac{(1-N\lambda|\lambda)_{i_1}}{(N+i_1)!}\cdots\frac{(1-N\lambda|\lambda)_{i_k}}{(N+i_k)!}\,.
$$

Proof. Put

$$
1+w={}_2F_1\left(1,N-\frac{1}{\lambda};N+1;-\lambda t\right)\,.
$$

By the definition (2.1) with $x=0$, we have

$$
\begin{aligned}
\beta_{N,n}(\lambda) &= \left.\frac{d^n}{dt^n}(1+w)^{-1}\right|_{t=0}=\left.\frac{d^n}{dt^n}\left(\sum_{l=0}^{\infty}(-w)^l\right)\right|_{t=0}=\sum_{l=0}^{n}\left.\frac{d^n}{dt^n}(-w)^l\right|_{t=0}\\
&=\sum_{l=0}^{n}\sum_{k=0}^{l}(-1)^k\binom{l}{k}\left.\frac{d^n}{dt^n}\left({}_2F_1\left(1,N-\frac{1}{\lambda};N+1;-\lambda t\right)\right)^k\right|_{t=0}\,.
\end{aligned}
$$

By (2.2), we get

$$
\begin{aligned}
\left.\frac{d^n}{dt^n}\left({}_2F_1\left(1,N-\frac{1}{\lambda};N+1;-\lambda t\right)\right)^k\right|_{t=0} &= \left.\frac{d^n}{dt^n}\left(\sum_{l=0}^{\infty}\frac{(1-N\lambda|\lambda)_l}{(N+l)_l}t^l\right)^k\right|_{t=0}\\
&= n!R_k(n)\,,
\end{aligned}
$$

where

$$R_k(n)=\sum_{\substack{i_1+\cdots+i_k=n\\ i_1,\dots,i_k\ge 0}}\frac{(1-N\lambda|\lambda)_{i_1}}{(N+i_1)_{i_1}}\cdots\frac{(1-N\lambda|\lambda)_{i_k}}{(N+i_k)_{i_k}}\,.$$

Thus, we have

$$\begin{aligned}\beta_{N,n}(\lambda)&=\sum_{l=0}^{n}\sum_{k=0}^{l}(-1)^k\binom{l}{k}n!R_k(n)\\&=n!\sum_{k=0}^{n}(-1)^kR_k(n)\sum_{l=k}^{n}\binom{l}{k}\\&=n!\sum_{k=0}^{n}(-1)^kR_k(n)\binom{n+1}{k+1}\\&=n!\sum_{k=1}^{n}(-1)^k\binom{n+1}{k+1}R_k(n)\\&=n!\sum_{k=1}^{n}(-N!)^k\binom{n+1}{k+1}\sum_{\substack{i_1+\cdots+i_k=n\\ i_1,\dots,i_k\ge 0}}\frac{(1-N\lambda|\lambda)_{i_1}}{(N+i_1)!}\cdots\frac{(1-N\lambda|\lambda)_{i_k}}{(N+i_k)!}\,.\end{aligned}$$

□

3 Hypergeometric degenerate Bernoulli polynomials

In this section, a relation between hypergeometric degenerate Bernoulli polynomials and numbers and some more related properties are shown.

Theorem 3.1. *For $N\ge 1$ and $n\ge 0$,*

$$\beta_{N,n}(\lambda,x+y)=\sum_{k=0}^{n}\binom{n}{k}(y|\lambda)_{n-k}\beta_{N,k}(\lambda,x)\,.$$

Proof. By the definition in (2.1),

$$\begin{aligned}&\sum_{n=0}^{\infty}\beta_{N,n}(\lambda,x+y)\frac{t^n}{n!}\\&\qquad=\left({}_2F_1\left(1,N-\frac{1}{\lambda};N+1;-\lambda t\right)\right)^{-1}(1+\lambda t)^{(x+y)/\lambda}\\&\qquad=\left(\sum_{n=0}^{\infty}\beta_{N,n}(\lambda,x+y)\frac{t^n}{n!}\right)\left(\sum_{l=0}^{\infty}\binom{y/\lambda}{l}(\lambda t)^l\right)\\&\qquad=\left(\sum_{n=0}^{\infty}\beta_{N,n}(\lambda,x+y)\frac{t^n}{n!}\right)\left(\sum_{l=0}^{\infty}(y|\lambda)_l\frac{t^l}{l!}\right)\\&\qquad=\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\binom{n}{k}(y|\lambda)_{n-k}\beta_{N,k}(\lambda,x)\right)\frac{t^n}{n!}\,.\end{aligned}$$

Comparing the coefficients on both sides, we get the desired result. □

By specializing $y = 0$ in Theorem 3.1, we have a relation between the hypergeometric degenerate Bernoulli polynomials and numbers.

Corollary 3.2. *For $N \ge 1$ and $n \ge 0$,*

$$\beta_{N,n}(\lambda, x) = \sum_{k=0}^{n} \binom{n}{k} (x|\lambda)_{n-k} \beta_{N,k}(\lambda)\,.$$

Theorem 3.3. *For $N \ge 1$ and $n \ge 0$,*

$$\frac{d}{dx}\beta_{N,n}(\lambda, x) = \sum_{k=0}^{n-1} \frac{(-\lambda)^{n-k-1} n!}{(n-k)k!} \beta_{N,k}(\lambda, x)\,.$$

Proof. By the definition in (2.1),

$$\begin{aligned}
&\frac{d}{dx}\sum_{n=0}^{\infty} \beta_{N,n}(\lambda, x)\frac{t^n}{n!}\\
&= \left({}_2F_1\left(1, N - \frac{1}{\lambda}; N+1; -\lambda t\right)\right)^{-1} \frac{d}{dx}(1+\lambda t)^{(x)/\lambda}\\
&= \log(1+\lambda t)^{1/\lambda} \sum_{n=0}^{\infty} \beta_{N,n}(\lambda, x)\frac{t^n}{n!}\\
&= \left(\frac{1}{\lambda}\sum_{l=1}^{\infty}(-1)^{l-1}\frac{(\lambda t)^l}{l}\right)\left(\sum_{n=0}^{\infty} \beta_{N,n}(\lambda, x)\frac{t^n}{n!}\right)\\
&= \sum_{n=1}^{\infty}\left(\sum_{k=0}^{n-1}\frac{(-\lambda)^{n-k-1}}{(n-k)k!}\beta_{N,k}(\lambda, x)\right) t^n\,.
\end{aligned}$$

Comparing the coefficients on both sides, we get the desired result. □

4 A determinant expression of hypergeometric degenerate Bernoulli numbers

Theorem 4.1. *For $N, n \ge 1$, we have*

$$\beta_{N,n}(\lambda)$$
$$= (-1)^n n! \begin{vmatrix}
\frac{(1-N\lambda)N!}{(N+1)!} & 1 & 0 & & \\
\frac{(1-N\lambda|\lambda)_2 N!}{(N+2)!} & \frac{(1-N\lambda)N!}{(N+1)!} & & & \\
\vdots & \vdots & \ddots & 1 & 0 \\
\frac{(1-N\lambda|\lambda)_{n-1} N!}{(N+n-1)!} & \frac{(1-N\lambda|\lambda)_{n-2} N!}{(N+n-2)!} & \cdots & \frac{(1-N\lambda)N!}{(N+1)!} & 1 \\
\frac{(1-N\lambda|\lambda)_n N!}{(N+n)!} & \frac{(1-N\lambda|\lambda)_{n-1} N!}{(N+n-1)!} & \cdots & \frac{(1-N\lambda|\lambda)_2 N!}{(N+2)!} & \frac{(1-N\lambda)N!}{(N+1)!}
\end{vmatrix}\,.$$

Remark 4.2. When $\lambda \to 0$ in Theorem 4.1, we get a determinant expression of hypergeometric Bernoulli numbers $B_{N,n}$ in (1.4). If $\lambda \to 0$ and $N = 1$ in Theorem 4.1, we recover the classical determinant expression of the Bernoulli numbers B_n ([11, p. 53]).

Proof of Theorem 4.1. For simplicity, we put $\tilde{\beta}_{N,n} = (-1)^n \beta_{N,n}(\lambda)/n!$ and

$$f(i,j) = \begin{cases} \dfrac{(1-N\lambda|\lambda)_{i-j+1} N!}{(N+i-j+1)!} & \text{if } i \ge j; \\ 1 & \text{if } i = j-1; \\ 0 & \text{otherwise} \end{cases}$$

and shall prove that

$$\tilde{\beta}_{N,n} = |f(i,j)|_{1 \le i,j \le n} \,. \tag{4.1}$$

From Proposition 2.1, we have

$$\begin{aligned} \tilde{\beta}_{N,n} &= \sum_{m=0}^{n-1} \frac{(-1)^{n-m-1}(1-N\lambda|\lambda)_{n-m} N!}{(N+n-m)!} \tilde{\beta}_{N,m} \\ &= \sum_{m=0}^{n-1} (-1)^{n-m-1} f(n-m,1) \tilde{\beta}_{N,m} \,. \end{aligned} \tag{4.2}$$

When $n = 1$, it is trivial because by Theorem 2.2

$$\tilde{\beta}_{N,1} = -\frac{(1-N\lambda)N!}{(N+1)!} \,.$$

Assume that (4.1) is valid up to $n-1$. By expanding along the first row, the right-hand side of (4.1) is equal to

$$\begin{aligned} & f(1,1)\tilde{\beta}_{N,n-1} - \begin{vmatrix} f(2,1) & 1 & 0 & & \\ f(3,1) & 1 & & & \\ \vdots & \vdots & \ddots & 1 & 0 \\ f(n-1,1) & f(n-1,3) & \cdots & f(n-1,n-1) & 1 \\ f(n,1) & f(n,3) & \cdots & f(n,n-1) & f(n,n) \end{vmatrix} \\ &= f(1,1)\tilde{\beta}_{N,n-1} - f(2,1)\tilde{\beta}_{N,n-2} + \cdots + (-1)^{n-2} \begin{vmatrix} f(n-1,1) & 1 \\ f(n,1) & f(n,n) \end{vmatrix} \\ &= \sum_{m=0}^{n-1} (-1)^{n-m-1} f(n-m,1)\tilde{\beta}_{N,m} = \tilde{\beta}_{N,n} \,. \end{aligned}$$

Here, we used the relation (4.2) with $\tilde{\beta}_{N,0} = 1$. □

5 Applications of Trudi's formula and inversion relations

One motivation of this paper comes from a 1989 paper of Cameron [6], in which he considered the operator A defined on the set of sequences of non-negative integers as follows: for $\mathbf{x} = \{x_n\}_{n \ge 1}$ and $\mathbf{z} = \{z_n\}_{n \ge 1}$, set $A\mathbf{x} = \mathbf{z}$, where

$$1 + \sum_{n=1}^{\infty} z_n t^n = \left(1 - \sum_{n=1}^{\infty} x_n t^n\right)^{-1} \,. \tag{5.1}$$

Suppose that x enumerates a class C. Then Ax enumerates the class of disjoint unions of members of C, where the order of the "component" members of C is significant. The operator A also plays an important role for free associative (non-commutative) algebras. More motivations and background together with many concrete examples (in particular, in the aspects of graph theory) by this operator can be seen in [6].

Though only nonnegative numbers in the sequence are treated with combinatorial interpretations in [6], the transformation in (5.1) can be extended to negative or rational numbers too. Some combinatorial interpretations for rational numbers can be found in [3, 4], where a categorical setting is proposed. In the sense of Cameron's operator A, we have the following relations.

$$
\begin{aligned}
A\left\{-\frac{1}{(n+1)!}\right\}&=\left\{\frac{B_n}{n!}\right\}\\
A\left\{-\frac{1}{(N+n)_n}\right\}&=\left\{\frac{B_{N,n}}{n!}\right\}\\
A\left\{-\frac{(1-\lambda|\lambda)_n}{(n+1)!}\right\}&=\left\{\frac{\beta_n(\lambda)}{n!}\right\}\\
A\left\{-\frac{(1-N\lambda|\lambda)_n}{(N+n)_n}\right\}&=\left\{\frac{\beta_{N,n}(\lambda)}{n!}\right\}
\end{aligned}
$$

These relations are interchangeable in the sense of determinants too.

We shall use Trudi's formula to obtain different explicit expressions and inversion relations for the numbers $\beta_{N,n}(j)$.

Lemma 5.1. *For $n\ge 1$, we have*

$$
\begin{vmatrix}
a_1 & a_0 & 0 & \cdots & \\
a_2 & a_1 & \ddots & & \vdots\\
\vdots & \vdots & \ddots & \ddots & 0\\
a_{n-1} & & \cdots & a_1 & a_0\\
a_n & a_{n-1} & \cdots & a_2 & a_1
\end{vmatrix}
=\sum_{t_1+2t_2+\cdots+nt_n=n}\binom{t_1+\cdots+t_n}{t_1,\dots,t_n}(-a_0)^{n-t_1-\cdots-t_n}a_1^{t_1}a_2^{t_2}\cdots a_n^{t_n},
$$

where $\binom{t_1+\cdots+t_n}{t_1,\dots,t_n}=\frac{(t_1+\cdots+t_n)!}{t_1!\cdots t_n!}$ are the multinomial coefficients.

This relation is known as Trudi's formula [24, Vol. 3, p. 214], [25] and the case $a_0=1$ of this formula is known as Brioschi's formula [5], [24, Vol. 3, pp. 208–209].

In addition, there exists an inversion formula (see, e.g. [22]). From Cameron's operator $A\mathbf{x}=\mathbf{z}$ in (5.1),

$$
\sum_{n=0}^\infty\sum_{k=0}^n(-1)^{n-k}x_{n-k}z_k=1\,.
$$

Hence, for $n\ge 1$

$$
\sum_{k=0}^n(-1)^{n-k}x_{n-k}z_k=0\,.
$$

When $x_0=z_0=1$, we have the following inversion formula.

Lemma 5.2.

$$\textit{If } x_n = \begin{vmatrix} z_1 & 1 & & \\ z_2 & \ddots & \ddots & \\ \vdots & \ddots & \ddots & 1 \\ z_n & \cdots & z_2 & z_1 \end{vmatrix}, \textit{ then } z_n = \begin{vmatrix} x_1 & 1 & & \\ x_2 & \ddots & \ddots & \\ \vdots & \ddots & \ddots & 1 \\ x_n & \cdots & x_2 & x_1 \end{vmatrix}.$$

From Trudi's formula, it is possible to give the combinatorial expression

$$x_n = \sum_{t_1+2t_2+\cdots+nt_n=n} \binom{t_1+\cdots+t_n}{t_1,\ldots,t_n} (-1)^{n-t_1-\cdots-t_n} z_1^{t_1} z_2^{t_2} \cdots z_n^{t_n}.$$

By applying these lemmas to Theorem 4.1, we obtain an explicit expression for the hypergeometric degenerate Bernoulli numbers.

Theorem 5.3. *For* $N, n \ge 1$,

$$\beta_{N,n}(\lambda) = n! \sum_{t_1+2t_2+\cdots+nt_n=n} \binom{t_1+\cdots+t_n}{t_1,\ldots,t_n} (-1)^{t_1+\cdots+t_n} \\ \times \left(\frac{(1-N\lambda)N!}{(N+1)!}\right)^{t_1} \left(\frac{(1-N\lambda|\lambda)_2 N!}{(N+2)!}\right)^{t_2} \cdots \left(\frac{(1-N\lambda|\lambda)_n N!}{(N+n)!}\right)^{t_n}.$$

Theorem 5.4. *For* $N, n \ge 1$,

$$\frac{(-1)^n (1-N\lambda|\lambda)_n N!}{(N+n)!} = \begin{vmatrix} \beta_{N,1}(\lambda) & 1 & 0 & & \\ \frac{\beta_{N,2}(\lambda)}{2!} & \beta_{N,1}(\lambda) & & & \\ \vdots & \vdots & \ddots & 1 & 0 \\ \frac{\beta_{N,n-1}(\lambda)}{(n-1)!} & \frac{\beta_{N,n-2}(\lambda)}{(n-2)!} & \cdots & \beta_{N,1}(\lambda) & 1 \\ \frac{\beta_{N,n}(\lambda)}{n!} & \frac{\beta_{N,n-1}(\lambda)}{(n-1)!} & \cdots & \frac{\beta_{N,2}(\lambda)}{2!} & \beta_{N,1}(\lambda) \end{vmatrix}.$$

Applying the Trudi's formula in Lemma 5.1 to Theorem 5.4, we get the inversion relation of Theorem 5.3.

Theorem 5.5. *For* $N, n \ge 1$,

$$\frac{(1-N\lambda|\lambda)_n N!}{(N+n)!} = \sum_{t_1+2t_2+\cdots+nt_n=n} \binom{t_1+\cdots+t_n}{t_1,\ldots,t_n} (-1)^{t_1+\cdots+t_n} \\ \times (\beta_{N,1}(\lambda))^{t_1} \left(\frac{\beta_{N,2}(\lambda)}{2!}\right)^{t_2} \cdots \left(\frac{\beta_{N,n}(\lambda)}{n!}\right)^{t_n}.$$

6 Coefficients of hypergeometric degenerate Bernoulli numbers

Hypergeometric Cauchy polynomials $c_{N,n}(x)$ ([20]) have similar properties. The generating function is given by

$$\frac{1}{(1+t)^x {}_2F_1(1,N;N+1;-t)}=\sum_{n=0}^\infty c_{N,n}(x)\frac{t^n}{n!}\,. \tag{6.1}$$

When $x=0$ in (6.1), $c_{N,n}=c_{N,n}(0)$ are the hypergeometric Cauchy numbers ([12, 13, 14, 15, 19]). When $N=1$ in (6.1), $c_n(x)=c_{1,n}(x)$ are the ordinary Cauchy polynomials (e.g., [9]). When $x=0$ and $N=1$ in (6.1), $c_n=c_{1,n}(0)$ are the classical Cauchy numbers (see, e.g., [10, Chapter VII]), defined by

$$\frac{t}{\log(1+t)}=\sum_{n=0}^\infty c_n\frac{t^n}{n!}\,. \tag{6.2}$$

The number $c_n/n!$ is sometimes referred to as the Bernoulli number of the second kind (see, e.g. [16]). A determinant expression of hypergeometric Cauchy numbers ([1, 23]) is given by

$$c_{N,n}=n!\begin{vmatrix}\frac{N}{N+1} & 1 & 0 & & \\ \frac{N}{N+2} & \frac{N}{N+1} & & & \\ \vdots & \vdots & \ddots & 1 & 0 \\ \frac{N}{N+n-1} & \frac{N}{N+n-2} & \cdots & \frac{N}{N+1} & 1 \\ \frac{N}{N+n} & \frac{N}{N+n-1} & \cdots & \frac{N}{N+2} & \frac{N}{N+1}\end{vmatrix}\,. \tag{6.3}$$

The determinant expression for the classical Cauchy numbers $c_n=c_{1,n}$ was discovered by Glaisher ([11, p. 50]). A more general case is considered in [21].

From the expression in Theorem 5.3, the hypergeometric degenerate Bernoulli number $\beta_{N,n}$ is a polynomial in λ with rational coefficients and degree at most n. Thus, we can write

$$\beta_{N,n}(\lambda)=d_{n,n}\lambda^n+d_{n,n-1}\lambda^{n-1}+\cdots+d_{n,1}\lambda+d_{n,0}\,. \tag{6.4}$$

In this section, we give some coefficients explicitly. By this theorem, we can see that hypergeometric degenerate Bernoulli numbers are closely related with both hypergeometric Bernoulli numbers and hypergeometric Cauchy numbers.

Theorem 6.1. *For $N\ge 1$ and $n\ge 0$, we have*

$$d_{n,n}=c_{N,n}\quad\textit{and}\quad d_{n,0}=B_{N,n}\,.$$

Remark 6.2. When $N=1$, Theorem 6.1 is reduced to [16, Theorem 3.1]. This implies that the leading coefficient of $\beta_n(\lambda)$ is equal to the n-th Cauchy number c_n and the constant term is equal to the n-th Bernoulli number B_n.

Proof of Theorem 6.1. Since

$$\begin{aligned}(1-N\lambda|\lambda)_{n-k}&=\lambda^{n-k}\sum_{l=0}^{n-k}(-1)^{n-k-l}\begin{bmatrix}n-k\\ l\end{bmatrix}\left(\frac{1}{\lambda}-N\right)^l\\ &=\sum_{l=0}^{n-k}\begin{bmatrix}n-k\\ l\end{bmatrix}\sum_{i=0}^{l}(-1)^{n-k-i}\binom{l}{i}\lambda^{n-k-i}N^{l-i}\,,\end{aligned}$$

by Proposition 2.1 we obtain for $n \ge 1$

$$\begin{aligned} &\frac{\beta_{N,n}(\lambda)}{n!} \\ &\quad = -\sum_{k=0}^{n-1} \frac{(1-N\lambda|\lambda)_{n-k} N!}{(N+n-k)!k!} \beta_{N,k}(\lambda) \\ &\quad = -\sum_{k=0}^{n-1} \frac{N!\beta_{N,k}(\lambda)}{(N+n-k)!k!} \sum_{l=0}^{n-k} \begin{bmatrix} n-k \\ l \end{bmatrix} \sum_{i=0}^{l} (-1)^{n-k-i} \binom{l}{i} \lambda^{n-k-i} N^{l-i} . \end{aligned} \tag{6.5}$$

Note that $\beta_{N,0}(\lambda) = 1$.

For constant term of the polynomial in λ, as $i = n - k$ in (6.5)

$$\begin{aligned} \frac{d_{0,0}}{n!} &= -\sum_{k=0}^{n-1} \frac{(1-N\lambda|\lambda)_{n-k} N!}{(N+n-k)!k!} d_{k,0}(\lambda) \\ &= -\sum_{k=0}^{n-1} \frac{N! d_{k,0}}{(N+n-k)!k!} \sum_{l=0}^{n-k} \begin{bmatrix} n-k \\ l \end{bmatrix} \binom{l}{n-k} N^{l-n+k} \\ &= -\sum_{k=0}^{n-1} \frac{N! d_{k,0}}{(N+n-k)!k!} . \end{aligned}$$

Hence,

$$\sum_{k=0}^{n} \binom{N+n}{k} d_{k,0} = 0$$

with $d_{0,0} = 1$. Since the hypergeometric Bernoulli numbers $B_{N,n}$ satisfies the same recurrence relation, namely,

$$\sum_{k=0}^{n} \binom{N+n}{k} B_{N,k} = 0$$

with $B_{N,0} = 1$ ([2, Proposition 1], [18, (6)]), we can conclude that

$$d_{n,0} = B_{N,n} .$$

For the leading coefficient, that is, the coefficient of λ^n of the polynomial in λ, as $i = 0$ in (6.5)

$$\begin{aligned} \frac{d_{n,n}}{n!} &= -\sum_{k=0}^{n-1} \frac{(-1)^{n-k} N! d_{k,k}}{(N+n-k)!k!} \sum_{l=0}^{n-k} \begin{bmatrix} n-k \\ l \end{bmatrix} N^{l} \\ &= -\sum_{k=0}^{n-1} \frac{(-1)^{n-k} N! (N)^{(n-k)}}{(N+n-k)!k!} d_{k,k} \\ &= -\sum_{k=0}^{n-1} \frac{(-1)^{n-k} N}{(N+n-k)k!} d_{k,k} . \end{aligned}$$

Thus, for $n \ge 1$

$$\sum_{k=0}^{n} \frac{(-1)^{n-k} N}{(N+n-k)k!} d_{k,k} = 0$$

or

$$\sum_{k=0}^n \frac{(-1)^k}{(N+n-k)k!} d_{k,k} = 0$$

with $d_{0,0} = 1$. Since the hypergeometric Cauchy numbers $c_{N,n}$ satisfies the same recurrence relation, namely,

$$\sum_{k=0}^n \frac{(-1)^k}{(N+n-k)k!} c_{N,k} = 0$$

with $c_{N,0} = 1$ ([20, Proposition 1]), we can conclude that

$$d_{n,n} = c_{N,n}\,. \qquad \square$$

6.1 Another method

Howard [16] found explicit formulas for all the coefficients by proving the following. For $n \ge 2$

$$\beta_n(\lambda) = c_n\lambda^n + \sum_{j=1}^n (-1)^{n-j}\frac{n}{j} B_j \begin{bmatrix} n-1 \\ j-1 \end{bmatrix} \lambda^{n-j}\,, \tag{6.6}$$

where $\left[{n \atop k}\right]$ are the Stirling numbers of the first kind, determined by

$$x(x-1)\cdots(x-n+1) = \sum_{k=0}^n (-1)^{n-k} \begin{bmatrix} n \\ k \end{bmatrix} x^k\,. \tag{6.7}$$

and c_n are the Cauchy numbers (of the firs kind), defined by the generating function

$$\frac{t}{\log(1+t)} = \sum_{n=0}^\infty c_n \frac{t^n}{n!}\,. \tag{6.8}$$

We have another expression of the coefficients of $\beta_{N,n}(\lambda)$ directly from Theorem 2.2. Since by (6.7)

$$\begin{aligned}(1-N\lambda|\lambda)_i &= \sum_{j=0}^i (-1)^{i-j} \sum_{l=1}^i \begin{bmatrix} i \\ l \end{bmatrix} \binom{l}{j} N^{l-j}\cdot\lambda^{i-j} \\ &= \sum_{j=0}^i (-1)^{i-j} \sum_{l=j}^i \begin{bmatrix} i \\ l \end{bmatrix} \binom{l}{j} N^{l-j}\cdot\lambda^{i-j}\,,\end{aligned}$$

we have for $j = 0, 1, \dots, n$

$$\begin{aligned} d_{n,n-j} = n! \sum_{k=1}^n \frac{(-N!)^k(-1)^{n-j}}{j!} &\sum_{\substack{i_1+\cdots+i_k=n \\ i_1,\dots,i_k\ge 1}} \frac{1}{(N+i_1)!\cdots(N+i_k)!} \\ \times \frac{d^j}{dx^j} &\left(\left(\sum_{l_1=1}^{i_1}\begin{bmatrix} i_1 \\ l_1 \end{bmatrix} x^{l_1}\right)\cdots\left(\sum_{l_k=1}^{i_k}\begin{bmatrix} i_k \\ l_k \end{bmatrix} x^{l_k}\right)\right)\Bigg|_{x=N}\,. \end{aligned} \tag{6.9}$$

If $j = n$ in (6.9), we get the coefficient of the constant in λ as

$$d_{n,0} = n! \sum_{k=1}^{n} (-N!)^k \sum_{\substack{i_1+\cdots+i_k=n \\ i_1,\dots,i_k \ge 1}} \frac{n!}{n!(N+i_1)! \cdots (N+i_k)!} ,$$

which is equal to $B_{N,n}$ by (2.4). If $j = 0$ in (6.9), we get the leading coefficient in λ as

$$\begin{aligned} d_{n,n} &= n! \sum_{k=1}^{n} (-N!)^k (-1)^n \sum_{\substack{i_1+\cdots+i_k=n \\ i_1,\dots,i_k \ge 1}} \frac{(N)^{(i_1)}}{(N+i_1)!} \cdots \frac{(N)^{(i_k)}}{(N+i_k)!} \\ &= n! \sum_{k=1}^{n} (-1)^{n-k} \sum_{\substack{i_1+\cdots+i_k=n \\ i_1,\dots,i_k \ge 1}} \frac{N^k}{(N+i_1)\cdots(N+i_k)} , \end{aligned}$$

which is equal to $c_{N,n}$ in [1, 23].

However, it seems difficult to express other terms of $\beta_{N,n}(\lambda)$ in any explicit form, except the leading coefficient and the constant.

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Arc-transitive graphs of valency twice a prime admit a semiregular automorphism*

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Abstract

We prove that every finite arc-transitive graph of valency twice a prime admits a nontrivial semiregular automorphism, that is, a non-identity automorphism whose cycles all have the same length. This is a special case of the Polycirculant Conjecture, which states that all finite vertex-transitive digraphs admit such automorphisms.

Keywords: Arc-transitive graphs, polycirculant conjecture, semiregular automorphism.

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1 Introduction

All graphs in this paper are finite. In 1981, Marušič asked if every vertex-transitive digraph admits a nontrivial semiregular automorphism [13], that is, an automorphism whose cycles all have the same length. This question has attracted considerable interest and a generalisation of the affirmative answer is now referred to as the Polycirculant Conjecture [4]. See [1] for a recent survey on this problem.

One line of investigation of this question has been according to the valency of the graph or digraph. Every vertex-transitive graph of valency at most four admits such an automorphism [7, 14], and so does every vertex-transitive digraph of out-valency at most three [9].

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Every arc-transitive graph of prime valency has a nontrivial semiregular automorphism [10] and so does every arc-transitive graph of valency 8 [17]. Partial results were also obtained for arc-transitive graphs of valency a product of two primes [18]. We continue this theme by proving the following theorem.

Theorem 1.1. *Arc-transitive graphs of valency twice a prime admit a nontrivial semiregular automorphism.*

2 Preliminaries

If G is a group of automorphisms of a graph Γ and v is a vertex of Γ, we denote by G_v the stabiliser in G of v, by $\Gamma(v)$ the neighbourhood of v, and by $G_v^{\Gamma(v)}$ the permutation group induced by G_v on $\Gamma(v)$. We will need the following well-known results.

Lemma 2.1. *Let Γ be a connected graph and $G \leqslant \mathrm{Aut}(\Gamma)$. If a prime p divides $|G_v|$ for some $v \in \mathrm{V}(\Gamma)$, then there exists $u \in \mathrm{V}(\Gamma)$ such that p divides $|G_u^{\Gamma(u)}|$.*

Proof. Since p divides $|G_v|$, there exists an element g of order p in G_v. As $g \neq 1$, there are vertices not fixed by g. Among these vertices, let w be one at minimal distance from v. Let P be a path of minimal length from v to w and let u be the vertex preceding w on P. By the definition of w, we have that u is fixed by g, so $g \in G_u$. On the other hand, g does not fix the neighbour w of u, so $g^{\Gamma(u)} \neq 1$ hence $|g^{\Gamma(u)}| = p$ and the result follows. ☐

Lemma 2.2. *Let G be a permutation group and let K be a normal subgroup of G such that G/K acts faithfully on the set of K-orbits. If G/K has a semiregular element Kg of order r coprime to $|K|$, then G has a semiregular element of order r.*

Proof. See for example [17, Lemma 2.3]. ☐

Lemma 2.3. *A transitive group of degree a power of a prime p contains a semiregular element of order p.*

Proof. In a transitive group of degree a power of a prime p, every Sylow p-subgroup is transitive. A non-trivial element of the center of this subgroup must be semiregular. ☐

Recall that a permutation group is *quasiprimitive* if every non-trivial normal subgroup is transitive, and *biquasiprimitive* if it is transitive but not quasiprimitive and every non-trivial normal subgroup has at most two orbits.

3 Arc-transitive graphs of prime valency

In the most difficult part of our proof, the arc-transitive graph of valency twice a prime will have a normal quotient with prime valency. We will thus need a lot of information about arc-transitive graphs of prime valency, which we collect in this section. We start with the following result, which is [3, Theorem 5]:

Theorem 3.1. *Let Γ be a connected G-arc-transitive graph of prime valency p such that the action of G on $\mathrm{V}(\Gamma)$ is either quasiprimitive or biquasiprimitive. Then one of the following holds:*

(1) G contains a semiregular element of odd prime order;

(2) $|\mathrm{V}(\Gamma)|$ *is a power of* 2*;*

(3) $\Gamma = \mathrm{K}_{12}$*,* $G = \mathrm{M}_{11}$ *and* $p = 11$*;*

(4) $|\mathrm{V}(\Gamma)| = (p^2-1)/2s$ *and* $G = \mathrm{PSL}_2(p)$ *or* $\mathrm{PGL}_2(p)$*, where* p *is a Mersenne prime and* s *is a proper divisor of* $(p-1)/2$ *but also a multiple of the product of the distinct prime divisors of* $(p-1)/2$*;*

(5) $|\mathrm{V}(\Gamma)| = (p^2-1)/s$ *and* $G = \mathrm{PGL}_2(p)$*, where* p *and* s *are as in part (4), and* Γ *is the canonical double cover of the graph given in (4).*

(Recall that the *canonical double cover* of a graph Γ is $\Gamma \times \mathrm{K}_2$.) We note that in cases (4) and (5), we must have $p \geqslant 127$, since this is the smallest Mersenne prime p such that $(p-1)/2$ is not squarefree. This fact will be used at the end of Section 4.

Corollary 3.2. *Let* Γ *be a connected* G*-arc-transitive graph of prime valency. Then one of the following holds:*

(1) G *contains a semiregular element of odd prime order;*

(2) $|\mathrm{V}(\Gamma)|$ *is a power of* 2*;*

(3) G *contains a normal* 2*-subgroup* P *such that* $(\Gamma/P, G/P)$ *is one of the graph-group pairs in (3–5) of Theorem 3.1.*

Proof. Suppose that $|\mathrm{V}(\Gamma)|$ is not a power of 2. If G is quasiprimitive or biquasiprimitive on $\mathrm{V}(\Gamma)$, then the result follows immediately from Theorem 3.1 (with $P = 1$ in case (3)). We thus assume that this is not the case and let P be a normal subgroup of G that is maximal subject to having at least three orbits on $\mathrm{V}(\Gamma)$. In particular, P is the kernel of the action of G on the set of P-orbits. Hence G/P acts faithfully, and quasiprimitively or biquasiprimitively on $\mathrm{V}(\Gamma/P)$. Since Γ has prime valency, is connected and G-arc-transitive, [12, Theorem 9] implies that P is semiregular. We may thus assume that P is a 2-group. (Otherwise P contains a semiregular element of odd prime order.) If G/P contains a semiregular element of odd prime order, then Lemma 2.2 implies that so does G. We may assume that this is not the case. Similarly, we may assume that $|\mathrm{V}(\Gamma/P)|$ is not a power of 2. (Otherwise, $|\mathrm{V}(\Gamma)|$ is a power of 2.) It follows that Γ/P and G/P are as is (3–5) of Theorem 3.1. $\square$

We will now prove some more results about the graphs that appear in (3–5) of Theorem 3.1. Let us first recall the notion of *coset graphs*. Let G be a group with a subgroup H and let $g \in G$ such that $g^2 \in H$ but $g \notin \mathrm{N}_G(H)$. The graph $\mathrm{Cos}(G, H, HgH)$ has vertices the right cosets of H in G, with two cosets Hx and Hy adjacent if and only if $xy^{-1} \in HgH$. Observe that the action of G on the set of vertices by right multiplication induces an arc-transitive group of automorphisms such that H is the stabiliser of a vertex. Moreover, every arc-transitive graph can be constructed in this way [16, Theorem 2].

Lemma 3.3. *The graphs in (3) and (4) of Theorem 3.1 have a* 3*-cycle.*

Proof. Clearly K_{12} has a 3-cycle so suppose that Γ is one of the graphs given in (4). Let G be as in Theorem 3.1 and let $v \in \mathrm{V}(\Gamma)$. Then G is one of $\mathrm{PSL}_2(p)$ or $\mathrm{PGL}_2(p)$ and acts arc-transitively on Γ. In both cases, let $X = \mathrm{PSL}_2(p)$, so $G = X$ or $|G : X| = 2$. By [3, Lemma 5.3], we have that $G_v \cong \mathrm{C}_p \rtimes \mathrm{C}_s$ if $G = \mathrm{PSL}_2(p)$, and $\mathrm{C}_p \rtimes \mathrm{C}_{2s}$ if

$G = \mathrm{PGL}_2(p)$. By [5, pp. 285–286], $\mathrm{PGL}_2(p)$ has a unique conjugacy class of subgroups of order p and the normaliser of such a subgroup is isomorphic to $\mathrm{C}_p \rtimes \mathrm{C}_{p-1}$, which is the stabiliser in $\mathrm{PGL}_2(p)$ of a 1-dimensional subspace of the natural 2-dimensional vector space. The intersection of this subgroup with X is isomorphic to $\mathrm{C}_p \rtimes \mathrm{C}_{(p-1)/2}$, which has odd order. It follows that if $|G : X| = 2$, then G_v is not contained in X. Thus, in both cases, X is transitive on $\mathrm{V}(\Gamma)$. Since X is normal in G, $X_v \neq 1$ and Γ has prime valency, it follows that X is arc-transitive on Γ and so $\Gamma = \mathrm{Cos}(X, H, HgH)$ where $H \cong \mathrm{C}_p \rtimes \mathrm{C}_s$ and $g \in X \backslash \mathrm{N}_X(H)$ such that HgH is a union of p distinct right cosets of H.

Since H has a characteristic subgroup Y of order p, it follows that $\mathrm{N}_X(H)$ normalises Y and so $\mathrm{N}_X(H) \leqslant \mathrm{N}_X(Y) \cong \mathrm{C}_p \rtimes \mathrm{C}_{(p-1)/2}$. Since $\mathrm{N}_X(Y)/Y$ is cyclic it follows that $\mathrm{N}_X(H) = \mathrm{N}_X(Y)$. Also note that $\mathrm{N}_X(H)$ is the stabiliser in X of a 1-dimensional subspace and so the action of X on the set of right cosets of $\mathrm{N}_X(H)$ is 2-transitive and the stabiliser of any two 1-dimensional subspaces is isomorphic to $\mathrm{C}_{(p-1)/2}$. Let $x \in X$. The stabiliser in X of the coset Hx is H^x and so the orbit of Hx under H has length $|H : H \cap H^x|$. In particular, H fixes the coset Hx if and only if $x \in \mathrm{N}_X(H)$, and so H fixes $|\mathrm{N}_X(H) : H| = (p(p-1)/2)/ps = (p-1)/2s$ points of $\mathrm{V}(\Gamma)$. Moreover, since the stabiliser of two 1-dimensional subspaces is isomorphic to $\mathrm{C}_{(p-1)/2}$ it follows that if $x \notin \mathrm{N}_X(H)$ then $H \cap H^x \cong \mathrm{C}_{(p-1)/2}$ and so $|H : H \cap H^x| = p$. Thus the points of $\mathrm{V}(\Gamma)$ that are not fixed by H are permuted by H in orbits of size p and so for any $g \notin \mathrm{N}_X(H)$ we have that HgH is a union of p distinct right cosets of H.

For each $x \in \mathrm{N}_X(H)$ define the bijection λ_x of $\mathrm{V}(\Gamma)$ by $Hy \mapsto x^{-1}Hy = Hx^{-1}y$. Since X acts on $\mathrm{V}(\Gamma)$ by right multiplication, we see that λ_x commutes with each element of X. Moreover, λ_x is nontrivial if and only if $x \notin H$. Let $C = \{\lambda_x \mid x \in \mathrm{N}_G(H)\} \leqslant \mathrm{Sym}(\mathrm{V}(\Gamma))$. Since X acts transitively on $\mathrm{V}(\Gamma)$ and C centralises X, it follows from [6, Theorem 4.2A] that C acts semiregularly on $\mathrm{V}(\Gamma)$. Now $|C| = |\mathrm{N}_X(H) : H| = (p-1)/2s$ and $X \times C \leqslant \mathrm{Sym}(\mathrm{V}(\Gamma))$. Since $C \trianglelefteq X \times C$, the set of orbits of C forms a system of imprimitivity for $X \times C$ and hence for X. Moreover, since C is semiregular, comparing orders yields that C has $|\mathrm{V}(\Gamma)|/|C| = p + 1$ orbits. One of these orbits is the set of fixed points of H and H transitively permutes the remaining p orbits of C. In particular, it follows that C transitively permutes the nontrivial orbits of H and so the isomorphism type of Γ does not depend on the choice of the double coset HgH.

Let Z be the subgroup of scalar matrices in $\mathrm{SL}_2(p)$ and let $\hat{H}$ be the subgroup of the stabiliser in $\mathrm{SL}_2(p)$ of the 1-dimensional subspace $\langle(1, 0)\rangle$ such that $H = \hat{H}/Z$. Note that $\hat{h} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \in \hat{H}$ and let $\hat{g} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. In particular, letting $h = \hat{h}Z$ and $g = \hat{g}Z$ we have that $g \notin \mathrm{N}_X(H)$ and so we may assume that $\Gamma = \mathrm{Cos}(X, H, HgH)$. Now Hg and Hgh are both adjacent to H and one can check that $g(gh)^{-1} = hgh \in HgH$ and so $\{H, Hg, Hgh\}$ is a 3-cycle in Γ. □

Definition 3.4. Let Γ be a graph and let S_0 be a subset of $\mathrm{V}(\Gamma)$. Let $S = S_0$. If a vertex u outside S has at least two neighbours in S, add u to S. Repeat this procedure until no more vertices outside S have this property. If at the end of the procedure, we have $S = \mathrm{V}(\Gamma)$, then we say that Γ is *dense with respect to* S_0.

It is an easy exercise to check that denseness is well-defined.

Corollary 3.5. *Let Γ be a graph in (3) or (4) of Theorem 3.1 and let $S_0 = \{u, v\}$ be an edge of Γ. Then Γ is dense with respect to S_0.*

Proof. Since Γ is arc-transitive of prime valency p, the local graph at v (that is, the subgraph induced on $\Gamma(v)$) is a vertex-transitive graph of order p and thus vertex-primitive. By Lemma 3.3, Γ has a 3-cycle so the local graph has at least one edge and thus must be connected. It follows that, running the process described in Definition 3.4 starting at $S = S_0$, eventually S will contain all neighbours of v. Repeating this argument and using connectedness of Γ yields the desired conclusion. □

The following is immediate from Definition 3.4.

Lemma 3.6. *Let* Γ *be a graph and* S_0 *be a set of vertices such that* Γ *is dense with respect to* S_0*. Then the canonical double cover of* Γ*, with vertex-set* $\mathrm{V}(\Gamma) \times \{0,1\}$*, is dense with respect to* $S_0 \times \{0,1\}$.

Proof. Let S_i be the sequence of subsets of $\mathrm{V}(\Gamma)$ obtained when running the procedure from Definition 3.4 starting with S_0 and ending with S_n for some n. Since Γ is dense with respect to S_0, we have $S_n = \mathrm{V}(\Gamma)$. For $i \in \{1, \dots, n\}$, let $v_i = S_i \setminus S_{i-1}$. (In other words, v_1 is the first vertex added to S_0 to get S_1, then v_2 is added to S_1 to get S_2, etc.)

Now, let $\Gamma' = \Gamma \times \mathrm{K}_2$ be the canonical double cover of Γ and let $S_0' = S_0 \times \{0,1\} \subseteq \mathrm{V}(\Gamma')$. We now run the procedure from Definition 3.4 starting at S_0'. At the first step, we note that, since v_1 was added to S_0, it must have at least two neighbours in S_0, say u_1 and w_1. It follows that both $(v_1, 0)$ and $(v_1, 1)$ also have at least two neighbours in S_0' (for example, $(u_1, 1)$ and $(w_1, 1)$, and $(u_1, 0)$ and $(w_1, 0)$, respectively). We thus add $(v_1, 0)$ and $(v_1, 1)$ to S_0' to get $S_1' = S_0' \cup \{(v_1, 0), (v_1, 1)\}$. Note that $S_1' = S_1 \times \{0,1\}$. We then repeat this procedure, preserving the condition $S_i' = S_i \times \{0,1\}$ at each iteration. At the end of this process, we have $S_n' = S_n \times \{0,1\} = \mathrm{V}(\Gamma) \times \{0,1\} = \mathrm{V}(\Gamma')$ and so Γ' is dense with respect to $S_0 \times \{0,1\}$. □

4 Proof of Theorem 1.1

Let p be a prime, let Γ be an arc-transitive graph of valency $2p$ and let $G = \mathrm{Aut}(\Gamma)$. We may assume that Γ is connected. If G is quasiprimitive or bi-quasiprimitive, then G contains a nontrivial semiregular element, by [8, Theorem 1.2] and [10, Theorem 1.4]. We may thus assume that G has a minimal normal subgroup N such that N has at least three orbits. In particular, Γ/N has valency at least 2.

If N is nonabelian, then G has a nontrivial semiregular element by [18, Theorem 1.1]. We may therefore assume that N is abelian and, in particular, N is an elementary abelian q-group for some prime q.

We may also assume that N is not semiregular that is, $N_v \neq 1$ for some vertex v. It follows from Lemma 2.1 that $1 \neq N_v^{\Gamma(v)} \trianglelefteq G_v^{\Gamma(v)}$. As Γ is G-arc-transitive, we have that $G_v^{\Gamma(v)}$ is transitive and so the orbits of $N_v^{\Gamma(v)}$ all have the same size, either 2 or p. Since N is a q-group, this size is equal to q. Writing d for the valency of Γ/N, we have that either $(d, q) = (2, p)$ or $(d, q) = (p, 2)$.

If $d = 2$ and $q = p$, then it follows from [15, Theorem 1] that Γ is isomorphic to the graph denoted by $\mathrm{C}(p, r, s)$ in [15]. By [15, Theorem 2.13], $\mathrm{Aut}(\mathrm{C}(p, r, s))$ contains the nontrivial semiregular automorphism ϱ defined in [15, Lemma 2.5].

We may thus assume that $d = p$ and $q = 2$. In this case, if u is adjacent to v, then u has exactly $2 = 2p/d$ neighbours in v^N. Let K be the kernel of the action of G on N-orbits. By the previous observation, the orbits of $K_v^{\Gamma(v)}$ have size 2 and thus it is a 2-group. It follows from Lemma 2.1 that K_v is a 2-group and thus so is $K = NK_v$.

Now, G/K is an arc-transitive group of automorphisms of Γ/N, so we may apply Corollary 3.2. If G/K has a semiregular element of odd prime order, then so does G, by Lemma 2.2. If $|\mathrm{V}(\Gamma/N)|$ is a power of 2, then so is $|\mathrm{V}(\Gamma)|$ and, in this case, G contains a semiregular involution by Lemma 2.3. We may thus assume that we are in case (3) of Corollary 3.2, that is, G/K contains a normal 2-subgroup P/K such that $(\Gamma/P, G/P)$ is one of the graph-group pairs in (3–5) of Theorem 3.1. Note that P is a 2-group. Let M be a minimal normal subgroup of G contained in the centre of P. Note that M is an elementary abelian 2-group. We may assume that M is not semiregular hence $M_v \neq 1$ and so by Lemma 2.1, $M_v^{\Gamma(v)} \neq 1$. Moreover, $|M| \neq 2$ as otherwise $M_v = M$ and we would deduce that M fixes each element of $\mathrm{V}(\Gamma)$, a contradiction. Since M is central in P, M_v fixes every vertex in v^P.

Note that the G-conjugates of M_v must cover M, otherwise M contains a nontrivial semiregular element. By the previous paragraph, the number of conjugates of M_v is bounded above by the number of P-orbits, that is $|\mathrm{V}(\Gamma/P)|$, so we have

$$|M| \leqslant |M_v||\mathrm{V}(\Gamma/P)|.$$

Since Γ is connected and G-arc-transitive, there are no edges within P-orbits. As $M_v^{\Gamma(v)} \neq 1$, there exists $g \in M_v$ such that w and w^g are distinct neighbours of v. Let u be the other neighbour of w in v^P. Since M_v fixes every element of v^P it follows that u is also a neighbour of w and w^g and so $\{v, w, u, w^g\}$ is a 4-cycle in Γ. Thus the graph induced between adjacent P-orbits is a union of C_4's.

If x is a vertex and y^P is a P-orbit adjacent to x, then there is a unique C_4 containing x between x^P and y^P, and thus a unique vertex z antipodal to x in this C_4. We say that z is the *buddy of x with respect to y^P*. The set of buddies of v is equal to $\Gamma_2(v) \cap v^P$, which is clearly fixed setwise by G_v. Moreover, each vertex has the same number of buddies. Furthermore, since G_v transitively permutes the set of p P-orbits adjacent to v^P, either v has a unique buddy or it has exactly p buddies.

If v has a unique buddy z, then $\Gamma(v) = \Gamma(z)$, and so swapping every vertex with its unique buddy is a nontrivial semiregular automorphism. Thus it remains to consider the case where v has p buddies. We first prove the following.

Claim. *If X is a subgroup of M that fixes pointwise both a^P and b^P, and c^P is a P-orbit adjacent to both a^P and b^P, then X fixes c^P pointwise.*

Proof. Suppose that some $x \in X$ does not fix c. Now x fixes a^P pointwise, so c^x must be the buddy of c with respect to a^P. Similarly, c^x must be the buddy of c with respect to b^P. These are distinct, which is a contradiction. It follows that X fixes c and, since $X \leqslant M$, also c^P. □

Let $s \geqslant 1$, let $\alpha = (v_0, \ldots, v_s)$ be an s-arc of Γ and let $\alpha' = (v_0, \ldots, v_{s-1})$. Now $|{v_s}^{M_{v_{s-1}}}| = 2$, so $|M_{v_{s-1}} : M_{v_{s-1}v_s}| = 2$ and $|M_{\alpha'} : M_\alpha| \leqslant 2$. Applying induction yields that

$$|M_{v_0} : M_\alpha| \leqslant 2^s. \tag{4.1}$$

We first assume that Γ/P and G/P are as in (3) or (4) of Theorem 3.1. Let $\{u, v\}$ be an edge of Γ. By the previous paragraph, we have $|M_v : M_{uv}| \leqslant 2$. Recall that M_v fixes all vertices in v^P, so M_{uv} fixes all vertices in $v^P \cup u^P$. Combining the claim with Corollary 3.5

yields that $M_{uv} = 1$ and thus $|M_v| = 2$. It follows that $|M| \leqslant |M_v||\mathrm{V}(\Gamma/P)|$ so $|M| \leqslant 2|\mathrm{V}(\Gamma/P)|$. Since M is minimal normal in G, it is an irreducible G-module over $\mathrm{GF}(2)$, of dimension at least two. In fact, since M is central in P, it is also an irreducible (G/P)-module. Since G/P is nonabelian simple or has a nonabelian simple group as an index two subgroup, this implies that M is a faithful irreducible (G/P)-module over $\mathrm{GF}(2)$. If $G/P = \mathrm{M}_{11}$, then $|M| \geqslant 2^{10}$ [11, Theorem 8.1], contradicting $|M| \leqslant 2 \cdot 12 = 24$. If $G/P = \mathrm{PSL}_2(p)$ or $\mathrm{PGL}_2(p)$ then by [2, Section VIII], $|M| \geqslant 2^{(p-1)/2}$. Recall that $p \geqslant 127$ and so this contradicts $|M| \leqslant 2(p^2-1)/2s < p^2 - 1$.

Finally, we assume that Γ/P is in (5) of Theorem 3.1, that is, Γ/P is the canonical double cover of a graph Γ' which appears in (4) of Theorem 3.1. In particular, $\mathrm{V}(\Gamma/P) = \mathrm{V}(\Gamma') \times \{0,1\}$. By Lemma 3.3, Γ' has a 3-cycle, say (u, v, w). By Corollary 3.5 and Lemma 3.6, Γ/P is dense with respect to $\{u, v\} \times \{0, 1\}$. Now, let

$$\overline{\alpha} = ((u,0),(v,1),(w,0),(u,1),(v,0)).$$

Since $\overline{\alpha}$ contains $\{u, v\} \times \{0, 1\}$, Γ/P is dense with respect to $\overline{\alpha}$. Note that $\overline{\alpha}$ is a 4-arc of Γ/P. Let α be a 4-arc of Γ that projects to $\overline{\alpha}$. Since Γ/P is dense with respect to $\overline{\alpha}$, arguing as in the last paragraph yields $M_\alpha = 1$. On the other hand, if $v \in \mathrm{V}(\Gamma)$ is the the initial vertex of α, then by (4.1), we have $|M_v : M_\alpha| \leqslant 2^4$ and thus $|M_v| \leqslant 2^4$. Since $|M| \leqslant |M_v||\mathrm{V}(\Gamma/P)|$ it follows that $|M| \leqslant 2^4(p^2-1)/s$. As above, M is a faithful irreducible (G/P)-module over $\mathrm{GF}(2)$ of dimension at least two. Since $G/P = \mathrm{PGL}_2(p)$ we have from [2] that $|M| \geqslant 2^{(p-1)/2}$, which again contradicts $|M| \leqslant 2^4(p^2-1)/s < 2^4(p^2-1)$.

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