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On the A_α-spectral radius of connected graphs*

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Abstract

For a simple graph G, the generalized adjacency matrix $A_\alpha(G)$ is defined as $A_\alpha(G) = \alpha D(G) + (1-\alpha)A(G), \alpha \in [0,1]$, where $A(G)$ is the adjacency matrix and $D(G)$ is the diagonal matrix of the vertex degrees. It is clear that $A_0(G) = A(G)$ and $2A_{\frac{1}{2}}(G) = Q(G)$ implying that the matrix $A_\alpha(G)$ is a generalization of the adjacency matrix and the signless Laplacian matrix. In this paper, we obtain some new upper and lower bounds for the generalized adjacency spectral radius $\lambda(A_\alpha(G))$, in terms of vertex degrees, average vertex 2-degrees, the order, the size, etc. The extremal graphs attaining these bounds are characterized. We will show that our bounds are better than some of the already known bounds for some classes of graphs. We derive a general upper bound for $\lambda(A_\alpha(G))$, in terms of vertex degrees and positive real numbers b_i. As application, we obtain some new upper bounds for $\lambda(A_\alpha(G))$. Further, we obtain some relations between clique number $\omega(G)$, independence number $\gamma(G)$ and the generalized adjacency eigenvalues of a graph G.

Keywords: Adjacency matrix, signless Laplacian matrix, generalized adjacency matrix, spectral radius, degree sequence, clique number, independence number.

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1 Introduction

Let $G = (V(G), E(G))$ be a simple connected graph with vertex set $V(G) = \{v_1, v_2, \dots, v_n\}$ and edge set $E(G)$. The *order* of G is the number $n = |V(G)|$ and its *size* is the number $m = |E(G)|$. The set of vertices adjacent to $v \in V(G)$, denoted by $N(v)$, refers to the *neighborhood* of v. The *degree* of v, denoted by $d_G(v)$ (we simply write d_v if it is clear from the context) means the cardinality of $N(v)$. A graph is called *regular* if all vertices have the same degree. The graph $\overline{G}$ is the *complement* of the graph G. Moreover, the complete graph K_n, the complete bipartite graph $K_{s,t}$, the path P_n, the cycle C_n and the star S_n are defined in the conventional way. The *distance* between two vertices $u, v \in V(G)$, denoted by d_{uv}, is defined as the length of a shortest path between u and v in G. The *diameter* of G is the maximum distance between any two vertices of G. Let m_i be the average degree of the adjacent vertices of vertex v_i in G. If v_i is an isolated vertex in G, then we assume that $m_i = 0$. Hence we can write

$$m_i = \begin{cases} 0 & d_i = 0. \\ \frac{1}{d_i} \sum_{j:j\sim i} d_j & \text{otherwise.} \end{cases}$$

Let p_i be the average degree of the vertices non-adjacent to vertex v_i in G. If v_i is adjacent to all the remaining vertices, then we assume that $p_i = 0$. Then we can write

$$p_i = \begin{cases} 0 & d_i = n-1. \\ \frac{\sum_{j:j\nsim i,\, j\neq i} d_j}{n-d_i-1} & \text{otherwise.} \end{cases}$$

Let $D(G)$ be the diagonal matrix of vertex degrees and $A(G)$ be the adjacency matrix of G. The signless Laplacian matrix of G is $Q(G) = D(G) + A(G)$. Its eigenvalues can be arranged as: $q_1(G) \geq q_2(G) \geq \cdots \geq q_n(G)$. In [20], Nikiforov proposed the following matrix:

$$A_\alpha(G) = \alpha D(G) + (1-\alpha)A(G), \quad 0 \leq \alpha \leq 1,$$

calling it the *generalized adjacency matrix* of G. Obviously, $A_0(G) = A(G)$, $2A_{\frac{1}{2}}(G) = Q(G)$, $A_1(G) = D(G)$ and $A_\alpha(G) - A_\beta(G) = (\alpha - \beta)L(G)$, where $L(G)$ is the well-studied Laplacian matrix of G, defined as $L(G) = D(G) - A(G)$. Therefore, the family $A_\alpha(G)$ can extend both $A(G)$ and $Q(G)$. The matrix $A_\alpha(G)$ is a real symmetric matrix, therefore we can arrange its eigenvalues as $\lambda_1(A_\alpha(G)) \geq \lambda_2(A_\alpha(G)) \geq \cdots \geq \lambda_n(A_\alpha(G))$, where $\lambda_1(A_\alpha(G))$ is called the *generalized adjacency spectral radius* of G. Afterwards, we will denote $\lambda_1(A_\alpha(G))$ by $\lambda(A_\alpha(G))$. If G is a connected graph and $\alpha \neq 1$, then the matrix $A_\alpha(G)$ is non-negative and irreducible. Therefore by the Perron-Frobenius theorem, $\lambda(A_\alpha(G))$ is the simple eigenvalue and there is a unique positive unit eigenvector $\mathbf{x}$ corresponding to $\lambda(A_\alpha(G))$, which is called the *generalized adjacency Perron vector* of G.

A column vector $\mathbf{x} = (x_1, x_2, \dots, x_n)^T \in \mathbb{R}^n$ can be considered as a function defined on $V(G)$ which maps vertex v_i to x_i, i.e., $\mathbf{x}(v_i) = x_i$ for $i = 1, 2, \dots, n$. Then,

$$\langle \mathbf{x}, A_\alpha \mathbf{x} \rangle = \mathbf{x}^T A_\alpha(G)\mathbf{x} = \alpha \sum_{i=1}^{n} d_i x_i^2 + 2(1-\alpha) \sum_{i\sim j} x_i x_j,$$

and λ is an eigenvalue of $A_\alpha(G)$ corresponding to the eigenvector $\mathbf{x}$ if and only if $\mathbf{x} \neq \mathbf{0}$ and

$$\lambda x_i = \alpha d_i x_i + (1-\alpha) \sum_{j \sim i} x_j, \; i = 1, 2, \ldots, n.$$

These equations are called the (λ, x)-*eigenequations* of G. For a normalized column vector $\mathbf{x} \in \mathbb{R}^n$, by the Rayleigh's principle, we have

$$\lambda(A_\alpha(G)) \geq \mathbf{x}^T A_\alpha(G)\mathbf{x}$$

with equality if and only if $\mathbf{x}$ is the generalized adjacency Perron vector of G.

The research on the (adjacency, signless Laplacian) spectrum is an intriguing topic during past two decades [4, 10, 22]. At the same time, the adjacency or signless Laplacian spectral radius have attracted many interests among the mathematical literature including linear algebra and graph theory. An interesting problem in the spectral graph theory is to obtain bounds for the (adjacency, signless Laplacian) spectral radius connecting it with different parameters associated with the graph. Another interesting problem which is worth to mention is to characterize the extremal graphs for the (adjacency, signless Laplacian) spectral radius among all graphs of order n or among a special class of graphs of order n. The spectral radius $\lambda(G)$ of the adjacency matrix $A(G)$, called the spectral radius (or adjacency spectral radius) of the graph G and the spectral radius $q_1(G)$ of the signless Laplacian matrix $Q(G)$, called signless Laplacian spectral radius of the graph G, are both well studied and their spectral theories are well developed. Various papers can be found in the literature regarding the establishment of bounds for $\lambda(G)$ and $q_1(G)$ connecting them with different parameters associated with the structure of the graph G. Since the matrix $A_\alpha(G)$ is a generalization of the matrices $A(G)$ and $Q(G)$, therefore it will be interesting to see whether the results which already hold for the spectral radius of the matrices $A(G)$ and/or $Q(G)$ can be extended to the spectral radius of the $A_\alpha(G)$. This is one of the motivation to study the spectral radius of the matrix $A_\alpha(G)$.

Let $A(G)$ be the adjacency matrix of the graph G and let B be a real diagonal matrix of order n. In 2002, Bapat et al. [1] defined the matrix $L^{'} = B - A(G)$ and called it the perturbed Laplacian matrix of the graph G. The aim of introducing this matrix was to generalize the results that hold for the adjacency matrix and the Laplacian matrix $L(G)$ of the graph to some general class of matrices. For $\alpha \neq 1$, it is easy to see that

$$A_\alpha(G) = \alpha D(G) + (1-\alpha)A(G) = (\alpha - 1)\Big(\frac{\alpha}{\alpha - 1} D(G) - A(G)\Big).$$

Clearly $\frac{\alpha}{\alpha-1} D(G)$ is a diagonal matrix with real entries, giving that the matrix $A_\alpha(G)$ is a scaler multiple of a perturbed Laplacian matrix. This is another motivation to study the spectral properties of the matrix $A_\alpha(G)$.

Although the generalized adjacency matrix $A_\alpha(G)$ of a graph G was introduced in 2017, but a large number of papers can be found in the literature regarding the spectral properties of this matrix. Like other graph matrices, most of these papers are regarding the generalized spectral radius $\lambda(A_\alpha(G))$. In fact, various upper and lower bounds connecting $\lambda(A_\alpha(G))$ with different graph parameters and the graphs attaining these bounds can be found in the literature. For some recent works regarding the spectral properties of $A_\alpha(G)$, we refer to [8, 9, 11, 13, 14, 15, 16, 17, 21, 23, 24].

The rest of this paper is organized as follows. In Section 2, we obtain some new upper and lower bounds for $\lambda(A_\alpha(G))$, in terms of vertex degrees, average vertex 2-degrees, the order, the size, etc. The extremal graphs attaining these bounds are characterized. We will show that our bounds are better than some of the already known bounds for some classes of graphs. In Section 3, we derive a general upper bound for $\lambda(A_\alpha(G))$, in terms of vertex degrees and positive real numbers b_i. As application, we obtain some new upper bounds for $\lambda(A_\alpha(G))$. In Section 4, we obtain some relations between clique number $\omega(G)$, independence number $\gamma(G)$ and the generalized adjacency eigenvalues. We conclude this paper by a remark in Section 5.

2 Bounds on generalized adjacency spectral radius

The average 2-degree of a vertex $v_i \in V(G)$ is denoted by $m_i = m(v_i)$ and is defined as $m_i = \sum_{k:k\sim i} \frac{d_k}{d_i}$, where d_k is the degree of the vertex v_k.

The following gives an upper bound for the generalized adjacency spectral radius $\lambda(A_\alpha(G))$ of a graph in terms of the vertex degrees, the average vertex 2-degrees and the parameter α.

Theorem 2.1. *Let G be a graph of order n having vertex degrees d_i, vertex average 2-degrees m_i, $1 \le i \le n$, and let $\alpha \in [0,1]$. Then*

$$\lambda(A_\alpha(G)) \le \max_{1\le i\le n} \left\{ \alpha d_i + (1-\alpha)\sqrt{d_i m_i} \right\}.$$

Moreover, the equality holds if G is a k-regular graph.

Proof. Let $\mathbf{x} = (x_1, \dots, x_n)$ be the generalized adjacency Perron vector of G and let $\|\mathbf{x}\| = 1$. For any $v_i \in V(G)$, we have $\lambda(A_\alpha(G))x_i = \alpha d_i x_i + (1-\alpha)\sum_{j:j\sim i} x_j$. Hence

$$\lambda^2(A_\alpha(G))x_i^2 = \alpha^2 d_i^2 x_i^2 + 2\alpha(1-\alpha)d_i x_i \sum_{j:j\sim i} x_j + (1-\alpha)^2 \left(\sum_{j:j\sim i} x_j \right)^2. \tag{2.1}$$

By Cauchy-Schwarz inequality, we obtain

$$\left(\sum_{j:j\sim i} x_j \right)^2 \le d_i \sum_{j:j\sim i} x_j^2. \tag{2.2}$$

Therefore from (2.1) and (2.2), we get

$$\lambda^2(A_\alpha(G))x_i^2 \le \alpha^2 d_i^2 x_i^2 + 2\alpha\, d_i x_i \Big[\lambda(A_\alpha(G))x_i - \alpha d_i x_i \Big] + (1-\alpha)^2 d_i \sum_{j:j\sim i} x_j^2.$$

Thus, taking sum over all $v_i \in V(G)$, we get

$$\begin{aligned} &\sum_{v_i\in V(G)} \lambda^2(A_\alpha(G))x_i^2 \\ &\le \sum_{v_i\in V(G)} \Big[2\alpha d_i \lambda(A_\alpha(G)) - \alpha^2 d_i^2 \Big] x_i^2 + (1-\alpha)^2 \sum_{v_i\in V(G)} d_i \sum_{j:j\sim i} x_j^2 \\ &= \sum_{v_i\in V(G)} \Big[2\alpha d_i \lambda(A_\alpha(G)) - \alpha^2 d_i^2 \Big] x_i^2 + (1-\alpha)^2 \sum_{v_i\in V(G)} d_i m_i x_i^2 \\ &= \sum_{v_i\in V(G)} \Big[2\alpha d_i \lambda(A_\alpha(G)) - \alpha^2 d_i^2 + (1-\alpha)^2 d_i m_i \Big] x_i^2 \end{aligned}$$

as
$$\sum_{v_i\in V(G)} d_i \sum_{j:j\sim i} x_j^2 = \sum_{v_i\in V(G)} x_i^2 \sum_{j:j\sim i} d_j = \sum_{v_i\in V(G)} d_i m_i x_i^2.$$
From the above result, we obtain
$$\sum_{v_i\in V(G)} \Big(\lambda^2(A_\alpha(G)) - 2\alpha d_i \lambda(A_\alpha(G)) + \alpha^2 d_i^2 - (1-\alpha)^2 d_i m_i\Big) x_i^2 \le 0.$$
This is only true if there exist a vertex, say $v_j \in V(G)$, such that
$$\lambda^2(A_\alpha(G)) - 2\alpha d_j \lambda(A_\alpha(G)) + \alpha^2 d_j^2 - (1-\alpha)^2 d_j m_j \le 0,$$
hence, we get
$$\lambda(A_\alpha(G)) \le \alpha d_j + (1-\alpha)\sqrt{d_j m_j} \le \max_{1\le i\le n} \left\{\alpha d_i + (1-\alpha)\sqrt{d_i m_i}\right\}.$$
Now, suppose that G is a k-regular graph. So, for $i = 1,\dots,n$, we have $d_i = m_i = k$, then $\alpha d_i + (1-\alpha)\sqrt{d_i m_i} = k$ and $\lambda(A_\alpha(G)) = k$. This shows that equality occurs for a regular graph. □

For $\alpha = 0$, the upper bound given by Theorem 2.1 reduces to the upper bound in the following corollary.

Corollary 2.2 ([5]). *Let G be a graph of order n having vertex degrees d_i, vertex average 2-degrees m_i, $1 \le i \le n$. Then*
$$\lambda(A(G)) \le \max_{1\le i\le n} \sqrt{d_i\, m_i}.$$

The following upper bound for the generalized adjacency spectral radius $\lambda(A_\alpha(G))$, in terms of vertex degrees and average vertex 2-degrees was obtained in [20]:

Theorem 2.3. *If G is a graph with no isolated vertices, then*
$$\lambda(A_\alpha(G)) \le \max_{v_j\in V} \Big\{\alpha d_j + (1-\alpha) m_j\Big\}.$$
If $\alpha \in \left(\frac{1}{2}, 1\right)$ and G is connected, equality holds if and only if G is regular.

Remark 2.4. For non-regular graphs the upper bound given by Theorem 2.1 and the upper bound given by Theorem 2.3 are incomparable for different values of α. For example, consider the graph $G = K_4 - e$. For this graph we have $d_1 = 2, d_2 = 3, d_3 = 2, d_4 = 3, m_1 = 3, m_2 = \frac{7}{3}, m_3 = 3$ and $m_4 = \frac{7}{3}$. By Theorem 2.3, we have
$$\lambda(A_\alpha(G)) \le \max\Big\{3-\alpha, \frac{7}{3} + \frac{2}{3}\alpha\Big\}.$$
It is easy to see that
$$\max\Big\{3-\alpha, \frac{7}{3} + \frac{2}{3}\alpha\Big\} = \begin{cases} \frac{7}{3} + \frac{2}{3}\alpha & \text{for } \alpha > 0.4, \\ 3-\alpha & \text{for } \alpha \le 0.4. \end{cases}$$

Also, by Theorem 2.1, we have

$$\lambda(A_\alpha(G)) \leq \max\left\{\sqrt{6}+(2-\sqrt{6})\alpha, \sqrt{7}+(3-\sqrt{7})\alpha\right\} = \sqrt{7}+(3-\sqrt{7})\alpha.$$

For $\alpha \leq 0.4$, we have $3-\alpha > \sqrt{7}+(3-\sqrt{7})\alpha$ giving that $\alpha < \frac{5-\sqrt{7}}{9} \approx 0.2615$. This gives that for $0 \leq \alpha < \frac{5-\sqrt{7}}{9}$, the upper bound given by Theorem 2.1 is better than the upper bound given by Theorem 2.3; while as for $\frac{5-\sqrt{7}}{9} \leq \alpha \leq 0.4$, the upper bound given by Theorem 2.3 is better than the upper bound given by Theorem 2.1 for the graph $K_4 - e$.

For the graph $G = K_{1,3}$, we have $d_1 = 3, d_2 = 1, d_3 = 1, d_4 = 1, m_1 = 1, m_2 = 3, m_3 = 3$ and $m_4 = 3$. By Theorem 2.3, we have

$$\lambda(A_\alpha(G)) \leq \max\left\{1+2\alpha, 3-2\alpha\right\}.$$

It is easy to see that

$$\max\left\{1+2\alpha, 3-2\alpha\right\} = \begin{cases} 3-2\alpha & \text{for } \alpha < 0.5, \\ 1+2\alpha & \text{for } \alpha \geq 0.5. \end{cases}$$

Also, by Theorem 2.1, we have

$$\lambda(A_\alpha(G)) \leq \max\left\{\sqrt{3}+(3-\sqrt{3})\alpha, \sqrt{3}+(1-\sqrt{3})\alpha\right\} = \sqrt{3}+(3-\sqrt{3})\alpha.$$

For $\alpha < 0.5$, we have $3-2\alpha > \sqrt{3}+(3-\sqrt{3})\alpha$ giving that $\alpha < \frac{3-\sqrt{3}}{5-\sqrt{3}} \approx 0.38799$. This gives that for $0 \leq \alpha < \frac{3-\sqrt{3}}{5-\sqrt{3}}$, the upper bound given by Theorem 2.1 is better than the upper bound given by Theorem 2.3; while as for $\frac{3-\sqrt{3}}{5-\sqrt{3}} \leq \alpha < 0.5$, the upper bound given by Theorem 2.3 is better than the upper bound given by Theorem 2.1 for the graph $K_{1,3}$.

The following gives another upper bound for the generalized adjacency spectral radius $\lambda(A_\alpha(G))$ of a graph G in terms of the vertex degrees, the average vertex 2-degrees and the unknown parameter β.

Theorem 2.5. *Let G be a connected graph of order n having vertex degrees d_i, average vertex 2-degrees m_i, $1 \leq i \leq n$, and let $\alpha \in [0,1)$. Then*

$$\lambda(A_\alpha(G)) \leq \max_{1\leq i\leq n}\left\{\frac{-\beta+\sqrt{\beta^2+4d_i(\alpha d_i+(1-\alpha)m_i+\beta)}}{2}\right\}, \tag{2.3}$$

where $\beta \geq 0$ is an unknown parameter. Equality occurs if and only if G is a regular graph.

Proof. Let $\mathbf{x} = (x_1, \ldots, x_n)$ be the generalized adjacency Perron vector of G and let

$$x_i = \max_{1\leq j\leq n} x_j.$$

Since

$$\begin{aligned}\lambda^2(A_\alpha(G))\mathbf{x} &= \left(A_\alpha(G)\right)^2\mathbf{x} = (\alpha D+(1-\alpha)A)^2\mathbf{x} \\ &= \alpha^2D^2\mathbf{x}+\alpha(1-\alpha)DA\mathbf{x}+\alpha(1-\alpha)AD\mathbf{x}+(1-\alpha)^2A^2\mathbf{x},\end{aligned}$$

we have

$$\lambda^2(A_\alpha(G))x_i = \alpha^2 d_i^2 x_i + \alpha(1-\alpha)d_i \sum_{j:j\sim i} x_j + \alpha(1-\alpha)\sum_{j:j\sim i} d_j x_j + (1-\alpha)^2 \sum_{j:j\sim i}\sum_{k:k\sim j} x_k.$$

Now, we consider a simple quadratic function of $\lambda(A_\alpha(G))$:

$$\big(\lambda^2(A_\alpha(G)) + \beta\lambda(A_\alpha(G))\big)\,\mathbf{x} = \Big(\alpha^2 D^2\mathbf{x} + \alpha(1-\alpha)DA\mathbf{x} + \alpha(1-\alpha)AD\mathbf{x} + (1-\alpha)^2 A^2\mathbf{x}\Big) + \beta(\alpha D\mathbf{x} + (1-\alpha)A\mathbf{x}).$$

Considering the i-th equation, we have

$$\big(\lambda^2(A_\alpha(G)) + \beta\lambda(A_\alpha(G))\big)\,x_i = \alpha^2 d_i^2 x_i + \alpha(1-\alpha)d_i \sum_{j:j\sim i} x_j + \alpha(1-\alpha)\sum_{j:j\sim i} d_j x_j + (1-\alpha)^2 \sum_{j:j\sim i}\sum_{k:k\sim j} x_k + \beta\Big(\alpha d_i x_i + (1-\alpha)\sum_{j:j\sim i} x_j\Big).$$

One can easily see that

$$\alpha(1-\alpha)d_i \sum_{j:j\sim i} x_j \le \alpha(1-\alpha)d_i^2 x_i, \quad \alpha(1-\alpha)\sum_{j:j\sim i} d_j x_j \le \alpha(1-\alpha)d_i m_i x_i,$$
$$(1-\alpha)^2 \sum_{j:j\sim i}\sum_{k:k\sim j} x_k \le (1-\alpha)^2 d_i m_i x_i, \quad (1-\alpha)\sum_{j:j\sim i} x_j \le (1-\alpha)d_i x_i.$$

Hence, we obtain

$$\big(\lambda^2(A_\alpha(G)) + \beta\lambda(A_\alpha(G))\big)\,x_i \le d_i(\alpha d_i + (1-\alpha)m_i)x_i + \beta d_i x_i,$$

$$\text{that is,}\quad \lambda^2(A_\alpha(G)) + \beta\lambda(A_\alpha(G)) - d_i(\alpha d_i + (1-\alpha)m_i + \beta) \le 0,$$

$$\text{that is,}\quad \lambda(A_\alpha(G)) \le \frac{-\beta + \sqrt{\beta^2 + 4d_i(\alpha d_i + (1-\alpha)m_i + \beta)}}{2}.$$

From this the inequality (2.3) follows.

Suppose that equality occurs in (2.3). Then all the inequalities in the above argument occur as equalities. Thus we obtain

$$\alpha(1-\alpha)d_i \sum_{j:j\sim i} x_j = \alpha(1-\alpha)d_i^2 x_i, \quad \alpha(1-\alpha)\sum_{j:j\sim i} d_j x_j = \alpha(1-\alpha)d_i m_i x_i,$$
$$(1-\alpha)^2 \sum_{j:j\sim i}\sum_{k:k\sim j} x_k = (1-\alpha)^2 d_i m_i x_i, \quad (1-\alpha)\sum_{j:j\sim i} x_j = (1-\alpha)d_i x_i.$$

Therefore we must have $x_j = x_i$ for any $j : j \sim i$ and $x_k = x_i$ for any $k : k \sim j, j \sim i$. Let $U = \{v_\ell : x_\ell = x_i\}$. Now we have to prove that $U = V(G)$. Assume to the contrary that $U \neq V(G)$. Then there exists a vertex r in U such that $N(r) \subseteq U$ and $t \in V(G)\backslash U$ with $t \sim s$, where $s \in N(r)$. Then $x_t < x_i$. One can easily see that

$$\begin{aligned}\lambda(A_\alpha(G)) &< \frac{-\beta + \sqrt{\beta^2 + 4d_r(\alpha d_r + (1-\alpha)m_r + \beta)}}{2} \\ &\leq \max_{1\leq i\leq n} \left\{ \frac{-\beta + \sqrt{\beta^2 + 4d_i(\alpha d_i + (1-\alpha)m_i + \beta)}}{2} \right\},\end{aligned}$$

a contradiction as the equality holds in (2.3). Therefore $U = V(G)$. Then $x_1 = x_2 = \cdots = x_n$ and $\lambda(A_\alpha(G)) = d_i$, $i = 1, 2, \ldots, n$. Hence G is a regular graph.

Conversely, let G be a r-regular graph. Then

$$\lambda(A_\alpha(G)) = r = \max_{1\leq i\leq n} \left\{ \frac{-\beta + \sqrt{\beta^2 + 4d_i(\alpha d_i + (1-\alpha)m_i + \beta)}}{2} \right\}.$$

This completes the proof of the theorem. □

The following upper bound for the generalized adjacency spectral radius $\lambda(A_\alpha(G))$, in terms of vertex degrees and average vertex 2-degrees was obtained in [20]:

$$\lambda(A_\alpha(G)) \leq \max_{1\leq i\leq n} \left\{ \sqrt{\alpha d_i^2 + (1-\alpha)w_i} \right\}, \tag{2.4}$$

where $w_i = d_i m_i$ for $i = 1, \ldots, n$. Also, equality holds if and only if $\alpha d_i^2 + (1-\alpha)w_i$ is same for all i.

Remark 2.6. For a connected graph G of order n, the upper bound given by Theorem 2.5 reduces to the upper bound given by (2.4) for $\beta = 0$. For $\beta \neq 0$, the upper bound given by Theorem 2.5 is incomparable with the upper bound given by (2.4). For example, consider the graph $G = K_{1,3}$. For this graph, the upper bound (2.4) gives

$$\lambda(A_\alpha(G)) \leq \max\left\{\sqrt{3+6\alpha}, \sqrt{3-3\alpha}\right\} = \sqrt{3+6\alpha}.$$

While as the upper bound given by Theorem 2.5 gives

$$\begin{aligned}\lambda(A_\alpha(G)) &\leq \max\left\{ \frac{-\beta + \sqrt{\beta^2 + 12\beta + 12\alpha + 12}}{2}, \frac{-\beta + \sqrt{\beta^2 + 4\beta - 8\alpha + 12}}{2} \right\} \\ &= \frac{-\beta + \sqrt{\beta^2 + 12\beta + 12\alpha + 12}}{2}.\end{aligned}$$

Taking $\beta = 1$, we have $\frac{-\beta + \sqrt{\beta^2 + 12\beta + 12\alpha + 12}}{2} = \frac{-1 + \sqrt{25 + 12\alpha}}{2} < \sqrt{3+6\alpha}$ giving that $3\alpha^2 - 8\alpha + 2 < 0$. This last inequality holds provided that $\alpha > \frac{4-\sqrt{10}}{3} \approx 0.279240$. This shows that for $\beta = 1$, the upper bound given by Theorem 2.5 is better than the upper bound given by (2.4) for $\alpha > \frac{4-\sqrt{10}}{3}$. Taking $\beta = 0.5$, it can be seen that the upper bound given by Theorem 2.5 is better than the upper bound given by (2.4) for $\alpha > 0.177$ and for $\beta = 0.1$, it can be seen that the upper bound given by Theorem 2.5 is better than the upper bound given by (2.4) provided that $\alpha > 0.008$.

Since for $\beta = 0$, the upper bounds given by Theorem 2.5 and inequality (2.4) are same and for the graph $K_{1,3}$, it follows from the above discussion that for small value of β the upper bound given by Theorem 2.5 behaves well for all α, incomparable to the upper bound given by (2.4). This gives that the choice of parameter β in the upper bound given by Theorem 2.5 can be helpful to obtain a better upper bound.

Let $x_i = \min\{x_j, j = 1, \dots, n\}$ be the minimum among the entries of the generalized distance Perron vector $\mathbf{x} = (x_1, \dots, x_n)$ of the graph G. Proceeding similar to Theorem 2.5, we obtain the following lower bound for $\lambda(A_\alpha(G))$, in terms of the vertex degrees, the average vertex 2-degrees and the unknown parameter β.

Theorem 2.7. *Let G be a connected graph of order n having vertex degrees d_i, average vertex 2-degrees m_i, $1 \le i \le n$, and let $\alpha \in [0, 1)$. Then*

$$\lambda(A_\alpha(G)) \ge \min_{1\le i\le n} \left\{ \frac{-\beta + \sqrt{\beta^2 + 4d_i(\alpha d_i + (1-\alpha)m_i + \beta)}}{2} \right\},$$

where $\beta \ge 0$ is an unknown parameter. Equality occurs if and only if G is a regular graph.

The following lower bound for the generalized adjacency spectral radius $\lambda(A_\alpha(G))$, in terms of vertex degrees and average vertex 2-degrees was obtained in [20]:

$$\lambda(A_\alpha(G)) \ge \min_{1\le i\le n} \left\{ \sqrt{\alpha d_i^2 + (1-\alpha)w_i} \right\}, \tag{2.5}$$

where $w_i = d_i m_i$ for $i = 1, \dots, n$. Equality occurs if and only if $\alpha d_i^2 + (1-\alpha)w_i$ is same for all i.

Remark 2.8. For a connected graph G of order n, the lower bound given by Theorem 2.7 reduces to the lower bound given by (2.5), for $\beta = 0$. For $\beta \ne 0$, the lower bound given by Theorem 2.7 is incomparable with the lower bound given by (2.5). For example, consider the graph $G = K_{1,3}$. For this graph, the lower bound (2.5) gives

$$\lambda(A_\alpha(G)) \ge \min\left\{\sqrt{3+6\alpha}, \sqrt{3-3\alpha}\right\} = \sqrt{3-3\alpha}.$$

While as the lower bound given by Theorem 2.7 gives

$$\begin{aligned}\lambda(A_\alpha(G)) &\ge \min\left\{ \frac{-\beta+\sqrt{\beta^2+12\beta+12\alpha+12}}{2}, \frac{-\beta+\sqrt{\beta^2+4\beta-8\alpha+12}}{2} \right\}\\ &= \frac{-\beta+\sqrt{\beta^2+4\beta-8\alpha+12}}{2}.\end{aligned}$$

Taking $\beta = 1$, we have $\frac{-\beta+\sqrt{\beta^2+4\beta-8\alpha+12}}{2} = \frac{-1+\sqrt{17-8\alpha}}{2} > \sqrt{3-3\alpha}$ giving that $4\alpha^2 + 20\alpha - 8 > 0$. This last inequality holds provided that $\alpha > \frac{\sqrt{33}-5}{2} \approx 0.372281$. This shows that for $\beta = 1$, the lower bound given by Theorem 2.7 is better than the lower bound given by (2.5) for $\alpha > \frac{\sqrt{33}-5}{2}$. Taking $\beta = 0.1$, it can be seen that the lower bound given by Theorem 2.7 is better than the lower bound given by (2.5) for

$\alpha > 0.09$ and for $\beta = 0.01$, it can be seen that the lower bound given by Theorem 2.7 is better than the lower bound given by (2.5) provided that $\alpha > 0.023$.

Again, it follows from the above discussion that for small value of β the lower bound given by Theorem 2.7 behaves well for all α, incomparison to the lower bound given by (2.5) for the graph $K_{1,3}$. This gives that the choice of parameter β in the lower bound given by Theorem 2.7 can be helpful to obtain a better lower bound.

We note that if, in particular we take the parameter β in Theorem 2.5/Theorem 2.7 equal to the vertex covering number, the edge covering number, the clique number, the independence number, the domination number, the generalized adjacency rank, minimum degree, maximum degree, etc., then Theorems 2.5/ Theorem 2.7 gives upper bound/lower bound for $\lambda(A_\alpha(G))$, in terms of the vertex covering number, the edge covering number, the clique number, the independence number, the domination number, the generalized adjacency rank, minimum degree, maximum degree, etc.

Let S_n be the class of graphs of order n with maximum degree $n-1$. Clearly, $K_{1,n-1}$, $K_n \in S_n$. The following result gives an upper bound for $\max_{v_j \in V}\{d_j + m_j\}$ in terms of order n and size m.

Lemma 2.9 ([3]). *Let G be a graph of order n with m edges. Then*

$$\max_{1\le j\le n}\left\{d_j + m_j\right\} \le \frac{2m}{n-1} + n - 2, \tag{2.6}$$

with equality if and only if $G \in S_n$ or $G \cong K_{n-1} \cup K_1$.

We now generalize the above result.

Theorem 2.10. *Let G be a graph of order n with m edges and real numbers β, θ with $\beta \ge \theta > 0$. Then*

$$\max_{1\le j\le n}\left\{\beta d_j + \theta\, m_j\right\} \le \frac{2m\,\theta}{n-1} + \beta\,(n-1) - \theta, \tag{2.7}$$

with equality if and only if $G \in S_n$ or $G \cong K_{n-1} \cup K_1$ $(\beta = \theta)$.

Proof. If $\beta = \theta > 0$, then by Lemma 2.9, we get the required result in (2.7). Moreover, the equality holds if and only if $G \in S_n$ or $G \cong K_{n-1} \cup K_1$ $(\beta = \theta)$. Otherwise, $\beta > \theta > 0$. Let v_i be the vertex in G such that

$$\max_{1\le j\le n}\left\{\beta d_j + \theta\, m_j\right\} = \beta d_i + \theta\, m_i.$$

We have $2m = d_i + d_i m_i + (n - d_i - 1)\,p_i$, where p_i is the average of the degrees of the vertices non-adjacent to vertex v_i in G. We consider the following two cases:

Case 1: $d_i = n-1$. One can easily see that

$$\max_{1\le j\le n}\left\{\beta d_j + \theta\, m_j\right\} = \beta d_i + \theta\, m_i = \frac{2m\,\theta}{n-1} + \beta\,(n-1) - \theta.$$

In this case $G \in S_n$.

Case 2: $d_i \le n-2$. Now, to arrive at (2.7), we need to show that

$$\beta d_i + \theta\, m_i \le \frac{d_i + d_i m_i + (n - d_i - 1)p_i}{n-1}\,\theta + \beta\,(n-1) - \theta,$$

that is,

$$(n-d_i-1)\left((n-1)\,\beta+(p_i-1-m_i)\,\theta\right)\geq 0,$$

that is,

$$(n-1)\,\beta+(p_i-1-m_i)\,\theta\geq 0,$$

that is,

$$(n-1)\,\beta-(\Delta-\delta+1)\,\theta\geq 0, \tag{2.8}$$

as $m_i\leq\Delta$ and $p_i\geq\delta$. We consider the following two subcases:

Subcase 2.1: G is disconnected. Then $\Delta\leq n-2$. From (2.8), we obtain $(n-1)\,(\beta-\theta)>0$, which is true always as $\beta>\theta>0$. This shows that the inequality (2.8) strictly holds in this case.

Subcase 2.2: G is connected. In this case $\Delta-\delta\leq n-2$, again it follows from (2.8) that $(n-1)\,(\beta-\theta)>0$, which is true always as $\beta>\theta>0$. This shows that the inequality (2.8) strictly holds in this case as well. □

As an immediate consequence of Theorem 2.10, we get the following corollary.

Corollary 2.11. *Let G be a graph of order n with m edges and real number $\alpha\geq\frac{1}{2}$. Then*

$$\max_{1\leq j\leq n}\left\{\alpha d_j+(1-\alpha)\,m_j\right\}\leq\frac{2m\,(1-\alpha)}{n-1}+\alpha\,n-1, \tag{2.9}$$

with equality if and only if $G\in S_n$ or $G\cong K_{n-1}\cup K_1$ $(\alpha=1/2)$.

Combining Theorem 2.3 with Corollary 2.11, we get the following result, which gives an upper bound for the generalized adjacency spectral radius $\lambda(A_\alpha(G))$, in terms of the order n, the size m and the parameter α.

Theorem 2.12. *Let G be a graph of order n with m edges, with no isolated vertices and let $\alpha\in\left[\frac{1}{2},1\right]$. Then*

$$\lambda(A_\alpha(G))\leq\frac{2m\,(1-\alpha)}{n-1}+\alpha\,n-1.$$

If $\alpha\in\left(\frac{1}{2},1\right)$ and G is connected, equality holds if and only if $G=K_n$.

Let Γ be the class of graphs $G=(V,E)$ such that the maximum degree vertex (of degree Δ) are adjacent to the vertices of degree Δ and non-adjacent to the vertices of degree δ. If m is the number of edges in G $(\in\Gamma)$, then

$$2m=\Delta\,(\Delta+1)+(n-\Delta-1)\,\delta.$$

The following result gives an upper bound for d_i+m_i in terms of the order n, the size m, the maximum degree Δ and the minimum degree δ.

Lemma 2.13 ([3]). *Let G be a graph of order n with m edges having maximum degree Δ and minimum degree δ. Then*

$$d_i+m_i\leq\frac{2m}{n-1}+\Delta-\delta+\frac{\Delta}{n-1}\Big[n-2-(\Delta-\delta)\Big],$$

with equality if and only if $G\in S_n$ or $G\in\Gamma$.

The following result gives an upper bound for $\max_{v_j\in V}\{\beta\, d_j + \theta\, m_j\}$ in terms of the order n, the size m, the maximum degree Δ, the minimum degree δ and the parameters β, θ.

Theorem 2.14. *Let G be a graph of order n with m edges and real numbers β, θ with $\beta \geq \theta > 0$. Then*

$$\max_{v_j\in V}\{\beta d_j + \theta m_j\} \leq \frac{2m\theta}{n-1} + \theta\,(\Delta-\delta) + \frac{\Delta}{n-1}\Big[\beta\,(n-1) - \theta(\Delta-\delta+1)\Big] \tag{2.10}$$

with equality if and only if $G \in S_n$ or $G \in \Gamma$.

Proof. Let v_i be a vertex in G such that

$$\max_{v_j\in V}\{\beta\, d_j + \theta\, m_j\} = \beta\, d_i + \theta\, m_i.$$

First we assume that $\beta = \theta$. Then by Lemma 2.13, we obtain

$$\begin{aligned}\max_{v_j\in V}\{\beta d_j + \theta m_j\} &= \beta\,(d_i + m_i) \leq \beta\left[\frac{2m}{n-1} + \Delta - \delta + \frac{\Delta}{n-1}\Big(n-2-(\Delta-\delta)\Big)\right]\\ &= \frac{2m\theta}{n-1} + \theta\,(\Delta-\delta) + \frac{\Delta}{n-1}\Big[\beta\,(n-1) - \theta(\Delta-\delta+1)\Big],\end{aligned}$$

as $\beta > 0$. Moreover, the equality holds in (2.10) if and only if $G \in S_n$ or $G \in \Gamma$.
Next, we assume that $\beta > \theta$. We consider the following two cases:
Case 1: $d_i = n-1$. In this case

$$\beta\, d_i + \theta\, m_i = \beta\,(n-1) + \theta\,\frac{2m-(n-1)}{n-1} = \frac{2m\,\theta}{n-1} + \beta\,(n-1) - \theta,$$

and so it is clear that the equality holds in (2.10) as $\Delta = n-1$.

Case 2: $d_i \leq n-2$. Then there is at least one vertex non-adjacent to v_i in G. Let G' be the graph obtained from the graph G by adding edges between v_i and the vertices non-adjacent to v_i in G. Let d_i' and m_i' be the degree of the vertex v_i and the average degree of the vertices adjacent to the vertex v_i in G', respectively. Then $d_i' = n-1$ and hence $G' \in S_n$. Now,

$$\begin{aligned}\beta d_i' + \theta\, m_i' &= \beta(n-1) + \theta\left(\frac{2m+2(n-d_i-1)-(n-1)}{n-1}\right)\\ &= \beta\,(n-1) - \theta + \frac{2\theta\,(m+n-d_i-1)}{n-1}.\end{aligned} \tag{2.11}$$

Let p_i be the average degree of the vertices non-adjacent to vertex v_i in the graph G. Hence

$$\begin{aligned}&\beta\, d_i' + \theta\, m_i' - (\beta\, d_i + \theta\, m_i)\\ &= \beta\,(d_i' - d_i) + \theta\,(m_i' - m_i)\\ &= \beta\,(n-d_i-1) + \theta\left(\frac{2m+(n-d_i-1)-(n-1)}{n-1} - m_i\right)\\ &= \beta(n-d_i-1) + \theta\left(\frac{d_i m_i + (n-d_i-1)(p_i+1)}{n-1} - m_i\right).\end{aligned}$$

Since $\beta \geq \theta > 0$ and $\Delta - \delta \leq n-2$, we have $\beta\,(n-1) \geq \theta\,(\Delta - \delta + 1)$. Moreover, we have $m_i \leq \Delta$ and $p_i \geq \delta$ for any vertex $v_i \in V(G)$. Using these results, we obtain

$$\begin{aligned}
&\beta d_i + \theta\, m_i \\
&= \beta\, d_i - \theta + \frac{2\theta\,(m+n-d_i-1)}{n-1} + \theta\,\left(m_i - \frac{d_i m_i + (n-d_i-1)(p_i+1)}{n-1}\right) \\
&= \frac{2m\,\theta}{n-1} + \frac{d_i}{n-1}\,\Big(\beta\,(n-1) - \theta\Big) + \theta\Big(1 - \frac{d_i}{n-1}\Big)\,(m_i - p_i) \\
&\leq \frac{2m\,\theta}{n-1} + \frac{d_i}{n-1}\,\Big(\beta\,(n-1) - \theta\Big) + \theta\Big(1 - \frac{d_i}{n-1}\Big)\,(\Delta - \delta) \qquad (2.12) \\
&= \frac{2m\,\theta}{n-1} + \theta\,(\Delta - \delta) + \frac{d_i}{n-1}\,\Big(\beta\,(n-1) - \theta - \theta\,(\Delta - \delta)\Big) \\
&\leq \frac{2m\,\theta}{n-1} + \theta\,(\Delta - \delta) + \frac{\Delta}{n-1}\,\Big(\beta\,(n-1) - \theta\,(\Delta - \delta + 1)\Big) \qquad (2.13)
\end{aligned}$$

as $d_i \leq \Delta$. The first part of the proof is done.

Now, suppose that equality in (2.10) holds with $\beta > \theta$. Then all the above inequalities must be equalities. If $d_i = n-1$, then $G \in S_n$. Otherwise, $d_i \leq n-2$. From the equality in (2.12), we have $m_i = \Delta$ and $p_i = \delta$. Since $\beta > \theta$, we have $\beta\,(n-1) > \theta\,(\Delta - \delta + 1)$. From the equality in (2.13), we have $d_i = \Delta$. Therefore all the vertices those are adjacent to the vertex v_i are of degree Δ and those are non-adjacent to the vertex v_i are of degree δ. Hence $G \in \Gamma$.

Conversely, let $G \in S_n$. Then $\Delta = n-1$ and hence

$$\begin{aligned}
\max_{v_j \in V}\{\beta d_j + \theta m_j\} &= \frac{2m\theta}{n-1} + \beta\,(n-1) - \theta \\
&= \frac{2m\theta}{n-1} + \theta\,(\Delta - \delta) + \frac{\Delta}{n-1}\,\Big[\beta\,(n-1) - \theta(\Delta - \delta + 1)\Big].
\end{aligned}$$

Let $G \in \Gamma$. Then $2m = \Delta\,(\Delta + 1) + (n - \Delta - 1)\,\delta$ and hence

$$\max_{v_j \in V}\{\beta d_j + \theta m_j\} = (\beta + \theta)\,\Delta = \frac{2m\theta}{n-1} + \theta\,(\Delta - \delta) + \frac{\Delta}{n-1}\,\Big[\beta\,(n-1) - \theta(\Delta - \delta + 1)\Big].$$

This completes the proof. $\square$

Corollary 2.15. *Let G be a graph of order n with m edges and let $\alpha \geq \frac{1}{2}$. Let Δ and δ are respectively, the maximum degree and the minimum degree of G. Then*

$$\max_{1 \leq j \leq n}\Big\{\alpha\, d_j + (1-\alpha)\, m_j\Big\} \leq \frac{2m\,(1-\alpha)}{n-1} + \frac{\alpha n - 1}{n-1}\Delta + (1-\alpha)\left(1 - \frac{\Delta}{n-1}\right)(\Delta - \delta) \qquad (2.14)$$

with equality if and only if $G \in S_n$ or $G \in \Gamma$.

Combining Theorem 2.3 with Corollary 2.15, we get the following result, which gives an upper bound for the generalized adjacency spectral radius $\lambda(A_\alpha(G))$, in terms of the order n, the size m, the maximum degree Δ, the minimum degree δ and the parameter α.

Theorem 2.16. *Let G be a graph of order n, with m edges and let $\alpha \geq \frac{1}{2}$. Let Δ and δ are respectively, the maximum degree and the minimum degree of G. Then*

$$\lambda(A_\alpha(G)) \leq \frac{2m\,(1-\alpha)}{n-1} + \frac{\alpha n - 1}{n-1}\Delta + (1-\alpha)\left(1 - \frac{\Delta}{n-1}\right)(\Delta - \delta).$$

If $\alpha \in \left(\frac{1}{2}, 1\right)$ and G is connected, equality holds if and only if $G \cong K_n$.

The following result gives a Nordhaus–Gaddum type upper bound for the generalized adjacency spectral radius $\lambda(A_\alpha(G))$, in terms of the order n, the size m, the minimum degree δ, the maximum degree Δ and the parameter α.

Theorem 2.17. *Let G be a graph of order n, with m edges and let $\alpha \geq \frac{1}{2}$. Let Δ and δ are respectively, the maximum degree and the minimum degree of G. Then*

$$\lambda(A_\alpha(G)) + \lambda(A_\alpha(\bar{G})) \leq n - 1 + \frac{(1-\alpha)(\Delta-\delta)}{n-1}\left(n + \delta - \Delta - 1 + \frac{\alpha n - 1}{1-\alpha}\right) \tag{2.15}$$

If $\alpha \in \left(\frac{1}{2}, 1\right)$ and G is connected, equality holds if and only if $G = K_n$.

Proof. Following Theorem 2.16, we have

$$\begin{aligned}
\lambda(A_\alpha(G)) + \lambda(A_\alpha(\bar{G})) &\leq (1-\alpha)\frac{2m + 2\bar{m}}{n-1} + \frac{\alpha n - 1}{n-1}(\Delta + \bar{\Delta}) \\
&\quad + (1-\alpha)\left(1 - \frac{\Delta}{n-1}\right)(\Delta - \delta) + (1-\alpha)\left(1 - \frac{\bar{\Delta}}{n-1}\right)(\bar{\Delta} - \bar{\delta}) \\
&= (1-\alpha)n + \frac{\alpha n - 1}{n-1}(\Delta - \delta + n - 1) \\
&\quad + (1-\alpha)(\Delta - \delta)\left(1 - \frac{\Delta}{n-1} + \frac{\delta}{n-1}\right) \\
&= n - 1 + \frac{(1-\alpha)(\Delta-\delta)}{n-1}\left(n + \delta - \Delta - 1 + \frac{\alpha n - 1}{1-\alpha}\right),
\end{aligned}$$

since $m + \bar{m} = \frac{n(n-1)}{2}$, $\bar{\Delta} = n - 1 - \delta$ and $\bar{\delta} = n - 1 - \Delta$.

Now, we consider the equality case in (2.15). If G is regular, then both sides of (2.15) are equal to $n-1$. Now, assume that equality occurs in (2.15) for G. Then the equalities must hold in (2.15) for both G and $\bar{G}$. Hence $G \cong K_n$. □

3 A general upper bound for the generalized adjacency spectral radius

In this section, we obtain a general upper bound for the generalized adjacency spectral radius in terms of vertex degrees and arbitrary positive real numbers b_i. If we replace b_i by some graph parameters, then we can derive some upper bounds for $\lambda(A_\alpha(G))$, in terms of vertex degrees. For this we need the following result:

Lemma 3.1 ([7]). *Let $D = (d_{i,j})$ be an $n \times n$ irreducible non-negative matrix with spectral radius σ and let $R_i(D) = \sum\limits_{j=1}^{n} d_{i,j}$ be the i-th row sum of D. Then*

$$\min\{R_i(D) \,:\, 1 \leq i \leq n\} \leq \sigma \leq \max\{R_i(D) \,:\, 1 \leq i \leq n\}\,. \tag{3.1}$$

Moreover, if the row sums of D are not all equal, then the both inequalities in (3.1) are strict.

The following result gives an upper bound for $\lambda(A_\alpha(G))$, in terms of vertex degrees and the arbitrary positive real numbers b_i.

Theorem 3.2. *Let G be a connected graph of order n and $0 < \alpha < 1$. Let $d_1 \geq d_2 \geq \cdots \geq d_n$ be the vertex degrees of G. Then*

$$\lambda(A_\alpha(G)) \leq \max_{1\leq i\leq n} \left\{ \frac{\alpha d_i + \sqrt{\alpha^2 d_i^2 + \frac{4}{b_i} \sum\limits_{j:j\sim i} b_j(1-\alpha)(\alpha d_j + (1-\alpha)b'_j)}}{2} \right\}, \tag{3.2}$$

where $b_i \in \mathbb{R}^+$ and $b'_i = \frac{1}{b_i} \sum\limits_{j:j\sim i} b_j$. Moreover, the equality holds if and only if $\alpha d_1 + (1-\alpha)b'_1 = \alpha d_2 + (1-\alpha)b'_2 = \cdots = \alpha d_n + (1-\alpha)b'_n$.

Proof. Let $B = \mathrm{diag}(b_1, b_2, \ldots, b_n)$, where $b_i \in \mathbb{R}^+$ are positive real number. Since the matrices $A_\alpha(G)$ and $B^{-1}A_\alpha(G)B$ are similar and similar matrices have same spectrum, it follows that if $\lambda(A_\alpha(G))$ is the largest eigenvalue of $A_\alpha(G)$, then it is also the largest eigenvalue of $B^{-1}A_\alpha(G)B$. Let $\mathbf{x} = (x_1, x_2, \ldots, x_n)^T$ be an eigenvector corresponding to the eigenvalue $\lambda(A_\alpha(G))$ of $B^{-1}A_\alpha(G)B$. We assume that one eigencomponent x_i is equal to 1 and the other eigencomponents are less than or equal to 1. The (i,j)-th entry of $B^{-1}A_\alpha(G)B$ is

$$\begin{cases} \alpha\, d_i & \text{if} \quad i = j, \\ (1-\alpha)\, \dfrac{b_j}{b_i} & \text{if} \quad j \sim i, \\ 0 & \text{otherwise.} \end{cases}$$

We have

$$B^{-1}A_\alpha(G)B\mathbf{x} = \lambda(A_\alpha(G))\mathbf{x}. \tag{3.3}$$

From the i-th equation of (3.3), we have

$$\lambda(A_\alpha(G))x_i = \alpha d_i x_i + (1-\alpha) \sum_{j:j\sim i} \frac{b_j}{b_i} x_j,$$

$$\text{i.e.,} \;\; \lambda(A_\alpha(G)) = \alpha d_i + (1-\alpha) \sum_{j:j\sim i} \frac{b_j}{b_i} x_j. \tag{3.4}$$

Again from the j-th equation of (3.3),

$$\lambda(A_\alpha(G))x_j = \alpha d_j x_j + (1-\alpha) \sum_{k:k\sim j} \frac{b_k}{b_j} x_k.$$

Multiplying both sides of (3.4) by $\lambda(A_\alpha(G))$ and substituting this value $\lambda(A_\alpha(G))x_j$, we get

$$\begin{aligned}
\lambda^2(A_\alpha(G)) &= \alpha d_i\lambda(A_\alpha(G)) + (1-\alpha)\sum_{j:j\sim i}\frac{b_j}{b_i}\left[\alpha d_j x_j + (1-\alpha)\sum_{k:k\sim j}\frac{b_k}{b_j}x_k\right]\\
&= \alpha d_i\lambda(A_\alpha(G)) + \alpha(1-\alpha)\sum_{j:j\sim i}\frac{b_j d_j}{b_i}x_j + (1-\alpha)^2\sum_{j:j\sim i}\sum_{k:k\sim j}\frac{b_k}{b_i}x_k\\
&\le \alpha d_i\lambda(A_\alpha(G)) + \alpha(1-\alpha)\sum_{j:j\sim i}\frac{b_j d_j}{b_i} + (1-\alpha)^2\sum_{j:j\sim i}\frac{b_j b'_j}{b_i} \qquad (3.5)\\
&= \alpha d_i\lambda(A_\alpha(G)) + \sum_{j:j\sim i}\frac{b_j(1-\alpha)(\alpha d_j + (1-\alpha)b'_j)}{b_i},
\end{aligned}$$

as $b_i\, b'_i = \sum_{j:j\sim i} b_j$. Hence we get the upper bound.

Suppose that the equality holds in (3.2). Then all inequalities in the above argument must be equalities. Since $0<\alpha<1$, from equality in (3.5), we get $x_j = 1$ for all j such that $j\sim i$, and $x_k = 1$ for all k such that $k\sim j$ and $j\sim i$. From the above, one can easily prove that $x_i = 1$ for all $i\in V(G)$, that is, $\alpha d_1 + (1-\alpha)b'_1 = \alpha d_2 + (1-\alpha)b'_2 = \cdots = \alpha d_n + (1-\alpha)b'_n$.

Conversely, let G be a connected graph such that $\alpha d_1+(1-\alpha)b'_1 = \alpha d_2+(1-\alpha)b'_2 = \cdots = \alpha d_n + (1-\alpha)b'_n$ $(b_i\in\mathbb{R}^+)$. Since $\lambda(A_\alpha(G)) = \lambda(B^{-1}A_\alpha(G)B)$, then by Lemma 3.1, we obtain

$$\begin{aligned}
\lambda(A_\alpha(G)) &= \alpha\, d_\ell + (1-\alpha)\, b'_\ell\\
&= \max_{1\le i\le n}\left\{\frac{\alpha d_i + \sqrt{\alpha^2 d_i^2 + \frac{4}{b_i}\sum_{j:j\sim i} b_j(1-\alpha)(\alpha d_j + (1-\alpha)b'_j)}}{2}\right\}
\end{aligned}$$

for $1\le\ell\le n$. □

Taking $b_i = d_i$ in (3.2), and noting that $b'_i = \frac{1}{b_i}\sum_{j:j\sim i} b_j = \frac{1}{d_i}\sum_{j:j\sim i} d_j = m_i$, we obtain the following upper bound for $\lambda(A_\alpha(G))$, in terms of vertex degrees and average vertex 2-degrees.

Corollary 3.3. *Let G be a connected graph of order n having vertex degrees d_i, average vertex 2-degrees m_i $(1\le i\le n)$ and $0<\alpha<1$. Then*

$$\lambda(A_\alpha(G)) \le \max_{1\le i\le n}\left\{\frac{\alpha d_i + \sqrt{\alpha^2 d_i^2 + \frac{4(1-\alpha)}{d_i}\sum_{j:j\sim i} d_j[\alpha d_j + (1-\alpha)m_j]}}{2}\right\}.$$

Equality holds if and only if $\alpha d_1+(1-\alpha)m_1 = \alpha d_2+(1-\alpha)m_2 = \cdots = \alpha d_n+(1-\alpha)m_n$.

Taking $b_i = \sqrt{d_i}$ in (3.2), and noting that $b'_i = \frac{1}{b_i}\sum_{j:j\sim i} b_j = \frac{1}{\sqrt{d_i}}\sum_{j:j\sim i}\sqrt{d_j} = m'_i$ (say), we obtain the following upper bound for $\lambda(A_\alpha(G))$, in terms of vertex degrees and m'_i.

Corollary 3.4. *Let G be a connected graph of order n having vertex degrees d_i and let $m_i^{'} = \frac{1}{\sqrt{d_i}} \sum_{j:j\sim i} \sqrt{d_j}$, $1 \le i \le n$ and $0 < \alpha < 1$. Then*

$$\lambda(A_\alpha(G)) \le \max_{1\le i\le n} \left\{ \frac{\alpha d_i + \sqrt{\alpha^2 d_i^2 + \frac{4(1-\alpha)}{\sqrt{d_i}} \sum_{j:j\sim i} \sqrt{d_j}[\alpha d_j + (1-\alpha)m_j^{'}]}}{2} \right\}.$$

Equality holds if and only if $\alpha d_1 + (1-\alpha)m_1^{'} = \alpha d_2 + (1-\alpha)m_2^{'} = \cdots = \alpha d_n + (1-\alpha)m_n^{'}$.

Taking $b_i = 1$ in (3.2), and noting that $b_i' = \frac{1}{b_i} \sum_{j:j\sim i} b_j = \sum_{j:j\sim i} 1 = d_i$, we obtain the following upper bound for $\lambda(A_\alpha(G))$, in terms of vertex degrees and average vertex 2-degrees. We note that this upper bound was recently obtained in [15].

Corollary 3.5 ([15]). *Let G be a connected graph of order n having vertex degrees d_i, average vertex 2-degrees m_i $(1 \le i \le n)$ and $0 < \alpha < 1$. Then*

$$\lambda(A_\alpha(G)) \le \max_{1\le i\le n} \left\{ \frac{\alpha d_i + \sqrt{\alpha^2 d_i^2 + 4(1-\alpha) d_i m_i}}{2} \right\}.$$

Equality holds if and only if $d_1 = d_2 = \cdots = d_n$.

Taking $b_i = m_i$ in (3.2), and noting that $b_i' = \frac{1}{m_i} \sum_{j:j\sim i} m_j = \bar{m}_i$, we obtain the following upper bound for $\lambda(A_\alpha(G))$, in terms of vertex degrees and the quantity $\bar{m}_i$.

Corollary 3.6. *Let G be a connected graph of order n having vertex degrees d_i, average vertex 2-degrees m_i $(1 \le i \le n)$ and $0 < \alpha < 1$. Then*

$$\lambda(A_\alpha(G)) \le \max_{1\le i\le n} \left\{ \frac{\alpha d_i + \sqrt{\alpha^2 d_i^2 + \frac{4(1-\alpha)}{m_i} \sum_{j:j\sim i} m_j[\alpha d_j + (1-\alpha)\bar{m}_j]}}{2} \right\},$$

where $\bar{m}_i = \frac{1}{m_i} \sum_{j:j\sim i} m_j$. Equality holds if and only if $\alpha d_1 + (1-\alpha)\bar{m}_1 = \alpha d_2 + (1-\alpha)\bar{m}_2 = \cdots = \alpha d_n + (1-\alpha)\bar{m}_n$.

Taking $b_i = d_i + m_i$, $b_i = d_i + \sqrt{m_i}$, $b_i = \sqrt{d_i} + m_i$, $b_i = \sqrt{d_i} + \sqrt{m_i}$, $b_i = \frac{1}{\sqrt{d_i}}$, $b_i = \frac{1}{\sqrt{m_i}}$, $b_i = \frac{1}{d_i^2}$, $b_i = d_i^2$, etc, and proceeding similarly as above we can obtain some other new upper bounds for $\lambda(A_\alpha(G))$.

4 Relation between $\omega(G)$, $\gamma(G)$ and the generalized adjacency eigenvalues

For a graph G, define $\omega(G)$ and $\gamma(G)$, the *clique number* and the *independence number* of G to be the numbers of vertices of the largest clique and the largest independent set in G, respectively. In this section, we give bounds for clique number and independence number of (regular) graph G involving generalized adjacency eigenvalues.

The following lemma, due to Motzkin and Straus [19], links the spectrum of graphs to its structure.

Lemma 4.1 ([19]). *Let $F = \{x = (x_1, x_2, \ldots, x_n)^T \,|\, x_i \ge 0,\ \sum_{i=1}^{n} x_i = 1\}$. Then*

$$1 - \frac{1}{\omega(G)} = \max_{x\in F} \langle x, Ax \rangle.$$

The following result gives a lower bound for $\omega(G)$, in terms of the size m, the generalized adjacency spectral radius $\lambda(A_\alpha(G))$, the maximum degree Δ and the parameter α.

Theorem 4.2. *Let G be a graph of order n, with m edges and maximum degree Δ. Then*

$$\omega(G) \geq \frac{2(1-\alpha)^2 m}{2(1-\alpha)^2 m - (\lambda(A_\alpha(G)) - \alpha\Delta)^2}.$$

Proof. Let $\mathbf{x} = (x_1, x_2, \ldots, x_n)^T$ be the normalized eigenvector corresponding to $\lambda(A_\alpha(G))$. Then

$$\begin{aligned}\lambda(A_\alpha(G)) &= \alpha \sum_{i=1}^{n} d_i x_i^2 + 2(1-\alpha) \sum_{j:j\sim i} x_i x_j \\ &\leq \alpha\Delta \sum_{i=1}^{n} x_i^2 + 2(1-\alpha) \sum_{j:j\sim i} x_i x_j \\ &= \alpha\Delta + 2(1-\alpha) \sum_{j:j\sim i} x_i x_j.\end{aligned}$$

Since $\lambda(A_\alpha(G)) \geq \alpha(\Delta+1)$, for $\alpha \in [0, \frac{1}{2}]$, (see [20]), by Cauchy-Schwarz inequality, we obtain

$$(\lambda(A_\alpha(G)) - \alpha\Delta)^2 \leq \left(2(1-\alpha) \sum_{j:j\sim i} x_i x_j\right)^2 \leq 2(1-\alpha)^2 m \left(2 \sum_{j:j\sim i} x_i^2 x_j^2\right).$$

Note that $(x_1^2, x_2^2, \ldots, x_n^2)^T \geq 0$ and $x_1^2 + x_2^2 + \cdots + x_n^2 = 1$. Hence, by Lemma 4.1, we have

$$2 \sum_{j:j\sim i} x_i^2 x_j^2 \leq 1 - \frac{1}{\omega(G)},$$

then

$$\frac{(\lambda(A_\alpha(G)) - \alpha\Delta)^2}{2(1-\alpha)^2 m} \leq 1 - \frac{1}{\omega(G)},$$

that is,

$$\omega(G) \geq \frac{2(1-\alpha)^2 m}{2(1-\alpha)^2 m - (\lambda(A_\alpha(G)) - \alpha\Delta)^2}.$$

This completes the proof. □

Note that Theorem 4.2 extends the Theorem 4.1 proved in [12] for the signless Laplacian spectral radius to generalized adjacency spectral radius.

The following result gives a lower bound for $\omega(G)$, when G is a regular graph, in terms of the order n, the second smallest generalized adjacency eigenvalue $\lambda_{n-1} = \lambda_{n-1}(A_\alpha(G))$ and the parameter α

Theorem 4.3. *Let G be a r-regular graph of order $n \geq 3$. Then*

$$\omega(G) \geq \frac{(1-\alpha)n^2}{(1-\alpha)\left(n^2 - nr\right) + S^2(\alpha r - \lambda_{n-1})},$$

where $S = \min_{y_i \neq 0} \frac{1}{|y_i|}$ and $u_{n-1} = (y_1, y_2, \ldots, y_n)^T$ is the normalized eigenvector corresponding to λ_{n-1}, the second smallest eigenvalue of $A_\alpha(G)$.

Proof. Since G is a r-regular graph, we have $\lambda(A_\alpha(G)) = r$ and the normalized eigenvector corresponding to $\lambda(A_\alpha(G))$ is $u_1 = \frac{e}{\sqrt{n}}$, where $e = (1, 1, \ldots, 1)^T$. Let $\Theta = \frac{S}{n}$ and $\mathbf{x} = \frac{e}{n} + \Theta u_{n-1}$. Then $\Theta y_i \geq -\frac{1}{n}$ $(i = 1, 2, \ldots, n)$. Since $\sum_{i=1}^n \lambda_i(G) = 2\alpha m = \alpha n r$ and $n \geq 3$, we have $\lambda(A_\alpha(G)) \neq \lambda_{n-1}(G)$ and $\langle e, u_{n-1}\rangle = 0$. So, $\mathbf{x} \in \left\{(x_1, x_2, \ldots, x_n)^T; x_i \geq 0,\ \sum_{i=1}^n x_i = 1\right\}$. By Lemma 4.1, we have

$$\begin{aligned}
\langle \mathbf{x}, A_\alpha \mathbf{x}\rangle &= \alpha\langle \mathbf{x}, D\mathbf{x}\rangle + (1-\alpha)\langle \mathbf{x}, A\mathbf{x}\rangle \\
&\leq r\alpha\langle \mathbf{x}, \mathbf{x}\rangle + (1-\alpha)\left(1 - \frac{1}{\omega(G)}\right) \\
&= \alpha r\left(\frac{1}{n} + \Theta^2\right) + (1-\alpha)\left(1 - \frac{1}{\omega(G)}\right).
\end{aligned}$$

On the other hand

$$\begin{aligned}
\langle \mathbf{x}, A_\alpha \mathbf{x}\rangle &= \left\langle \frac{e}{n} + \Theta u_{n-1}, A_\alpha\left(\frac{e}{n} + \Theta u_{n-1}\right)\right\rangle \\
&= \left\langle \frac{e}{n}, A_\alpha \frac{e}{n}\right\rangle + \left\langle \frac{e}{n}, A_\alpha \Theta u_{n-1}\right\rangle + \left\langle \Theta u_{n-1}, A_\alpha \frac{e}{n}\right\rangle + \langle \Theta u_{n-1}, A_\alpha \Theta u_{n-1}\rangle \\
&= \frac{nd}{n^2} + \Theta^2 \lambda_{n-1}.
\end{aligned}$$

Then

$$\frac{d}{n} + \Theta^2\lambda_{n-1} \leq \alpha r\left(\frac{1}{n} + \Theta^2\right) + (1-\alpha)\left(1 - \frac{1}{\omega(G)}\right),$$

that is,

$$\omega(G) \geq \frac{1-\alpha}{(1-\alpha)\left(1 - \frac{r}{n}\right) + \Theta^2(\alpha r - \lambda_{n-1})}.$$

Since $\Theta = \frac{S}{n}$ and $S = \min_{y_i \neq 0} \frac{1}{|y_i|}$, we have

$$\omega(G) \geq \frac{(1-\alpha)n^2}{(1-\alpha)\left(n^2 - nr\right) + S^2(\alpha r - \lambda_{n-1})}.$$

This completes the proof. □

Note that Theorem 4.3 extends the Theorem 4.4 proved in [12] for the signless Laplacian spectral radius to generalized adjacency spectral radius.

Consider two sequences of real numbers $\xi_1 \geq \xi_2 \geq \cdots \geq \xi_n$ and $\eta_1 \geq \eta_2 \geq \cdots \geq \eta_t$ with $t < n$. The second sequence is said to *interlace* the first one whenever

$$\xi_i \geq \eta_i \geq \xi_{n-t+i},$$

for $i = 1, 2, \ldots, t$. The interlacing is called *tight* if there exists an integer $k \in [0, t]$ such that $\xi_i = \eta_i$ for $1 \leq i \leq k$ and $\xi_{n-t+i} = \eta_i$ for $k+1 \leq i \leq t$. Suppose rows and columns of the matrix M are partitioned according to a partitioning of $\{1, 2, \ldots, n\}$. The partition is called *regular* if each block of M has constant row (and column) sum. The following lemma can be found in [6].

Lemma 4.4 ([6])**.** *Let B be the matrix whose entries are the average row sums of the blocks of a symmetric partitioned matrix of M. Then*

(i) *the eigenvalues of B interlace the eigenvalues of M,*

(ii) *if the interlacing is tight, then the partition is regular.*

Next result gives a lower bound for $\gamma(G)$, in terms of the order n, the sum of first two largest generalized adjacency eigenvalues, the maximum degree Δ, the minimum degree δ and the parameter α.

Theorem 4.5. *Let G be a simple graph of order n with at least one edge, with minimum degree δ and maximum degree Δ. Let $\lambda_1(G)$ and $\lambda_2(G)$ are respectively the first and the second largest eigenvalue of $A_\alpha(G)$. If $\lambda_1(G) + \lambda_2(G) - (1+\alpha)\delta \leq 0$, then*

$$\gamma(G) \geq \frac{\lambda_1(G) + \lambda_2(G) - (1+\alpha)\delta}{\delta} \times \frac{n\Delta}{\lambda_1(G) + \lambda_2(G) - 2\Delta}. \tag{4.1}$$

Proof. Let G be a simple graph with order n and a partition $V(G) = V_1 \cup V_2$. Let G_i $(i = 1, 2)$ be the subgraph of G induced by V_i with $n_i < n$ vertices and average degree r_i $(n_1 + n_2 = n)$. Let $t_i = \frac{\sum_{v \in V_i} d(v)}{n_i}$ for $i = 1, 2$. Note that

$$A_\alpha(G) = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} = \begin{pmatrix} \alpha D_{11} + (1-\alpha)A(G_1) & (1-\alpha)A_{12} \\ (1-\alpha)A_{21} & \alpha D_{22} + (1-\alpha)A(G_2) \end{pmatrix},$$

where $D_{11} = \operatorname{diag}(d(v_1), \ldots, d(v_{n_1}))$, $D_{22} = \operatorname{diag}(d(v_{n_1+1}), \ldots, d(v_n))$ and $A_{21} = A_{12}^T$. Put $M = \left(\frac{m_{ij}}{n_i}\right)$, where m_{ij} is the sum of the entries in $A_{ij}(G)$. Hence

$$M = \begin{pmatrix} \alpha t_1 + (1-\alpha)r_1 & (1-\alpha)(t_1 - r_1) \\ (1-\alpha)(t_2 - r_2) & \alpha t_2 + (1-\alpha)r_2 \end{pmatrix}$$

and

$$\begin{aligned} |\phi I - M| &= \phi^2 - (\alpha t_1 + (1-\alpha)r_1 + \alpha t_2 + (1-\alpha)r_2)\phi \\ &\quad - (1-\alpha)^2(t_1 - r_1)(t_2 - r_2) + (\alpha t_1 + (1-\alpha)r_1)(\alpha t_2 + (1-\alpha)r_2). \end{aligned}$$

Then by Lemma 4.4, we have $\phi_1(M) \leq \lambda_1(G)$ and $\phi_2(M) \leq \lambda_2(G)$, hence

$$\phi_1(M) + \phi_2(M) = \alpha t_1 + (1-\alpha)r_1 + \alpha t_2 + (1-\alpha)r_2 \leq \lambda_1(G) + \lambda_2(G).$$

Note that $2(n_2 t_2 - n_1 t_1) = n_2(t_2 + r_2) - n_1(t_1 + r_1)$, and hence $n_2 t_2 - n_1 t_1 = n_2 r_2 - n_1 r_1$.

Let V_{G_1} be the largest independent set of G, then $r_1 = 0$ and $\gamma(G) = 0$, we have $r_2 = t_2 - \frac{n_1}{n_2} t_1$, and

$$\alpha t_1 + \alpha t_2 + (1-\alpha)\left(t_2 - \frac{n_1}{n_2} t_1\right) = \alpha t_1 + t_2 - (1-\alpha)\frac{n_1}{n_2} t_1 \leq \lambda_1(G) + \lambda_2(G).$$

By $n = n_1 + n_2$, we get

$$\frac{\lambda_1(G) + \lambda_2(G) - t_2 - \alpha t_1}{t_1} n \geq \frac{\lambda_1(G) + \lambda_2(G) - t_2 - t_1}{t_1} n_1.$$

Since G has at least one edge, $n_1 < n$. Also we have $\delta \leq t_1, t_2 \leq \Delta$, hence

$$\frac{\lambda_1(G) + \lambda_2(G) - (1+\alpha)\delta}{\delta} n \geq \frac{\lambda_1(G) + \lambda_2(G) - 2\Delta}{\Delta} n_1.$$

Thus

$$\gamma(G) = n_1 \geq \frac{\lambda_1(G) + \lambda_2(G) - (1+\alpha)\delta}{\delta} \times \frac{n\Delta}{\lambda_1(G) + \lambda_2(G) - 2\Delta}.$$

This completes the proof. □

Again, we note that Theorem 4.5 extends the Theorem 4.5 proved in [12] for the signless Laplacian spectral radius to generalized adjacency spectral radius.

Remark 4.6. Note that if $\lambda_1(G) + \lambda_2(G) - (1+\alpha)\delta > 0$, then $\frac{\lambda_1(G)+\lambda_2(G)-(1+\alpha)\delta}{\delta} \times \frac{n\Delta}{\lambda_1(G)+\lambda_2(G)-2\Delta} < 0$, and the inequality in (4.1) is trivial. Hence, we add the restriction $\lambda_1(G) + \lambda_2(G) - (1+\alpha)\delta \leq 0$, in Theorem 4.5. One can easily see that there exists graphs with the property that $\lambda_1(G) + \lambda_2(G) - (1+\alpha)\delta \leq 0$. For example, we have $spec_{A_\alpha}(K_n) = \{n-1, \alpha n - 1^{[n-1]}\}$. Hence, $\lambda_1(K_3) + \lambda_2(K_3) - (1+\alpha)\delta(K_3) = 2 + 3\alpha - 1 - 2(1+\alpha) = \alpha - 1 \leq 0$.

If G is an r-regular graph, then $\lambda_1(G) = r$ and $\Delta = \delta = r$. Hence, by Theorem 4.5, we get the following bound.

Corollary 4.7. *Let G be a simple r-regular graph of order n with at least one edge. Then*

$$\gamma(G) \geq \frac{n(\lambda_2(G) - \alpha r)}{\lambda_2(G) - r},$$

where $\lambda_2(G)$ is the second largest eigenvalue of $A_\alpha(G)$.

5 Some conclusions

As mentioned in the introduction, for $\alpha = 0$, the generalized adjacency matrix $A_\alpha(G)$ is same as the adjacency matrix $A(G)$ and for $\alpha = \frac{1}{2}$, twice the generalized adjacency matrix $A_\alpha(G)$ is same as the signless Laplacian matrix $Q(G)$. Therefore, if in particular, we put $\alpha = 0$ and $\alpha = \frac{1}{2}$, in all the results obtained in Sections 2, 3 and 4, we obtain the corresponding bounds for the adjacency spectral radius $\lambda(A(G))$ and the signless Laplacian spectral radius $\lambda(Q(G))$. We note most of these results we obtained in Section 2, 3 and 4 has been already discussed for the adjacency spectral radius $\lambda(A(G))$ or/and for the signless Laplacian spectral radius $\lambda(Q(G))$. Therefore, in this setting our results are the generalization of these known results.

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References

[1] R. B. Bapat, S. J. Kirkland and S. Pati, The perturbed Laplacian matrix of a graph, *Linear Multilinear Algebra* **49** (2001), 219–242, doi:10.1080/03081080108818697, https://doi.org/10.1080/03081080108818697.

[2] A. E. Brouwer and W. H. Haemers, *Spectra of Graphs*, Springer, New York, 2012, doi:10.1007/978-1-4614-1939-6, https://doi.org/10.1007/978-1-4614-1939-6.

[3] K. C. Das, Maximizing the sum of the squares of the degrees of a graph, *Discrete Math.* **285** (2004), 57–66, doi:10.1016/j.disc.2004.04.007, https://doi.org/10.1016/j.disc.2004.04.007.

[4] F. Duan, Q. Huang and X. Huang, On graphs with exactly two positive eigenvalues, *Ars Math. Contemp.* **17** (2019), 319–347, doi:10.26493/1855-3974.1516.58f, https://doi.org/10.26493/1855-3974.1516.58f.

[5] O. Favaron, M. Mahéo and J.-F. Saclé, Some eigenvalue properties in graphs (conjectures of Graffiti — ii), *Discrete Mathematics* **111** (1993), 197–220, doi:10.1016/0012-365x(93)90156-n, https://doi.org/10.1016/0012-365x(93)90156-n.

[6] W. H. Haemers, Interlacing eigenvalues and graphs, *Linear Algebra Appl.* **226-228** (1995), 593–616, doi:10.1016/0024-3795(95)00199-2, https://doi.org/10.1016/0024-3795(95)00199-2.

[7] R. A. Horn and C. R. Johnson, *Matrix Analysis*, Cambridge University Press, New York, 2013, doi:10.1017/cbo9780511810817, https://doi.org/10.1017/cbo9780511810817.

[8] X. Huang, H. Lin and J. Xue, The Nordhaus-Gaddum type inequalities of a_α-matrix, *Appl. Math. Comput.* **365** (2020), 8, doi:10.1016/j.amc.2019.124716, id/No 124716, https://doi.org/10.1016/j.amc.2019.124716.

[9] Y. Ji, J. Wang, M. Brunetti and N. Bian, a_α-spectral radius and measures of graph irregularity, *Linear Multilinear Algebra* (2022), doi:10.1080/03081087.2021.1910617, https://doi.org/10.1080/03081087.2021.1910617.

[10] S. Kubota, T. Taniguchi and K. Yoshino, On graphs with the smallest eigenvalue at least $-1-\sqrt{2}$, part iii, *Ars Math. Contemp.* **17** (2019), 555–579, doi:10.26493/1855-3974.1581.b47, https://doi.org/10.26493/1855-3974.1581.b47.

[11] D. Li, Y. Chen and J. Meng, The a_α-spectral radius of trees and unicyclic graphs with given degree sequence, *Appl. Math. Comput.* **363** (2019), 9, doi:10.1016/j.amc.2019.124622, id/No 124622, https://doi.org/10.1016/j.amc.2019.124622.

[12] S. Li and Y. Tian, Some results on the bounds of signless Laplacian eigenvalues, *Bull. Malays. Math. Sci. Soc. (2)* **38** (2015), 131–141, doi:10.1007/s40840-014-0008-x, https://doi.org/10.1007/s40840-014-0008-x.

[13] H. Lin, X. Huang and J. Xue, A note on the A_α-spectral radius of graphs, *Linear Algebra Appl.* **557** (2018), 430–437, doi:10.1016/j.laa.2018.08.008, https://doi.org/10.1016/j.laa.2018.08.008.

[14] H. Lin, X. Liu and J. Xue, Graphs determined by their A_α-spectra, *Discrete Math.* **342** (2019), 441–450, doi:10.1016/j.disc.2018.10.006, https://doi.org/10.1016/j.disc.2018.10.006.

[15] S. Liu, K. C. Das and J. Shu, On the eigenvalues of A_α-matrix of graphs, *Discrete Math.* **343** (2020), 11, doi:10.1016/j.disc.2020.111917, id/No 111917, https://doi.org/10.1016/j.disc.2020.111917.

[16] S. Liu, K. C. Das, S. Sun and J. Shu, On the least eigenvalue of A_α-matrix of graphs, *Linear Algebra Appl.* **586** (2020), 347–376, doi:10.1016/j.laa.2019.10.025, https://doi.org/10.1016/j.laa.2019.10.025.

[17] X. Liu and S. Liu, On the A_α-characteristic polynomial of a graph, *Linear Algebra Appl.* **546** (2018), 274–288, doi:10.1016/j.laa.2018.02.014, https://doi.org/10.1016/j.laa.2018.02.014.

[18] M. Lu, H. Liu and F. Tian, Laplacian spectral bounds for clique and independence numbers of graphs, *J. Comb. Theory, Ser. B* **97** (2007), 726–732, doi:10.1016/j.jctb.2006.12.003, https://doi.org/10.1016/j.jctb.2006.12.003.

[19] T. S. Motzkin and E. G. Straus, Maxima for graphs and a new proof of a theorem of Turán, *Can. J. Math.* **17** (1965), 533–540, doi:10.4153/cjm-1965-053-6, https://doi.org/10.4153/cjm-1965-053-6.

[20] V. Nikiforov, Merging the a-and q-spectral theories, *Appl. Anal. Discrete Math.* **11** (2017), 81–107, doi:10.2298/aadm1701081n, https://doi.org/10.2298/aadm1701081n.

[21] V. Nikiforov and O. Rojo, On the α-index of graphs with pendent paths, *Linear Algebra Appl.* **550** (2018), 87–104, doi:10.1016/j.laa.2018.03.036, https://doi.org/10.1016/j.laa.2018.03.036.

[22] M. R. Oboudi, Characterization of graphs with exactly two non-negative eigenvalues, *Ars Math. Contemp.* **12** (2017), 271–286, doi:10.26493/1855-3974.1077.5b6, https://doi.org/10.26493/1855-3974.1077.5b6.

[23] S. Wang, D. Wong and F. Tian, Bounds for the largest and the smallest A_α eigenvalues of a graph in terms of vertex degrees, *Linear Algebra Appl.* **590** (2020), 210–223, doi:10.1016/j.laa.2019.12.039, https://doi.org/10.1016/j.laa.2019.12.039.

[24] F. Xu, D. Wong and F. Tian, On the multiplicity of α as an eigenvalue of the a_α matrix of a graph in terms of the number of pendant vertices, *Linear Algebra Appl.* **594** (2020), 193–204, doi:10.1016/j.laa.2020.02.025, https://doi.org/10.1016/j.laa.2020.02.025.