ARS MATHEMATICA CONTEMPORANEA

ISSN 1855-3966 (printed edn.), ISSN 1855-3974 (electronic edn.)
ARS MATHEMATICA CONTEMPORANEA 14 (2018) 375–385
https://doi.org/10.26493/1855-3974.911.3b4
(Also available at http://amc-journal.eu)

A note on the directed genus of $K_{n,n,n}$ and K_n

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Received 7 August 2015, accepted 28 May 2017, published online 24 October 2017

Abstract

It is proved that a complete graph K_n can have an orientation whose minimum directed genus is $\lceil \frac{1}{12}(n-3)(n-4) \rceil$ if and only if $n \equiv 3, 7 \pmod{12}$. This answers a question of Bonnington et al. by using a method different from current graphs. It is also proved that a complete symmetric tripartite graph $K_{n,n,n}$ has an orientation whose minimum directed genus is $\frac{1}{2}(n-1)(n-2)$.

Keywords: Digraph, complete tripartite graph, directed genus, surfaces.

Math. Subj. Class.: 05C10, 05B05, 05B07

1 Introduction

Throughout this paper, all graphs are assumed to be finite, connected and simple. In a directed graph D, the number of in-arcs at a vertex v is called the *in-degree* of v which is denoted by $d^-(v)$; the number of out-arcs at v is called the *out-degree* of v, denoted by $d^+(v)$. The *degree* of v, denoted by $d(v)$, is the sum of $d^-(v)$ and $d^+(v)$. A digraph D is *Eulerian* if it is connected and every vertex has equal in-degree and out-degree. The underlying graph G of a digraph D is a graph obtained from D by suppressing all directions of the arcs in D. The *orientable surface* of genus h, denoted by S_h, is the sphere with h handles added. A graph is said to be *2-cell embedded* in a surface S, if it is embedded in a surface S such that each component, called a *region,* of $S \setminus D$ is homeomorphic to an open disk. A *2-cell directed embedding* (or *2-cell embedding*) of a digraph D on an orientable surface S means that it is a 2-cell embedding of its underlying graph of D in S such that each region is bounded by a directed cycle. In this paper, all embeddings of graphs

*The author expresses the sincere thanks to the anonymous referees for their constructive suggestions that greatly improve the quality of this paper. This work was supported by the National Natural Science Foundation of China (Nos. 11371052, 11731002), the Fundamental Research Funds for the Central Universities (Nos. 2016JBM071, 2016JBZ012) and the 111 Project of China (B16002).

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and digraphs are assumed to be 2-cell embedded on oriented surfaces. Let the genus of a surface S be denoted by $\gamma(S)$. The *directed genus* (or simply say genus) of an embeddable digraph D, denoted by $\gamma(D)$, is the smallest of the numbers $\gamma(S)$ for orientable surfaces S in which D can be directed embedded. Let $|X|$ be the cardinality of a set X.

The study of embeddings of a graph began with Euler. By now, there are many results about the genus ([14, 22, 23, 25, 26, 28, 27, 29]), the maximum genus ([24, 30]), and the genus distribution of a graph ([12, 13, 19, 20]). However, a study of the embeddings of a digraph was started in 2002 by Bonnington et al. in [2]. Bonnington, Hartsfield and Širáň ([3]) gave some obstructions for directed embeddings of digraphs and proved Kuratowski-type theorem for embeddings of digraphs in the plane. This area has remained almost uninvestigated. As we know, genera of only a few kinds of digraphs are known. Hales and Hartsfield calculated the directed genus of the de Bruijn graph in [15]. Hao et al. ([16, 17, 18]) obtained the embedding distributions of some digraphs and maximum embedding properties of digraphs. Chen, Gross and Hu ([4]) derived a splitting theorem for digraph embedding distributions that is analogous to the splitting theorems of [11] and [5] for graph embedding distributions.

Let $\gamma(G)$ denote the genus of a graph G. There are many results on computing genera of undirected graphs. For example, in [25], the genera of the complete graph K_n and the complete tripartite graph $K_{mn,n,n}$ were given as follows: $\gamma(K_n) = \lceil \frac{1}{12}(n-3)(n-4) \rceil$ and $\gamma(K_{mn,n,n}) = \frac{1}{2}(mn-2)(n-1)$. In [28], $\gamma(K_{n,n,n-2}) = \frac{1}{2}(n-2)^2$ for even $n \geq 2$ and $\gamma(K_{2n,2n,n}) = \frac{1}{2}(3n-2)(n-1)$ for $n \geq 1$ were derived. In [26], $\gamma(K_{n,n,n}) = \frac{1}{2}(n-2)(n-1)$ was obtained.

Up to now, the genera of only a few kinds of digraphs are known. For examples, the directed genus of the de Bruijn graph was derived in [15]. In [2], Bonnington et al. determined the genera of the cartesian product $C_n \times C_n$ of two directed cycles, the spoke digraph on $n = 2k+1$ vertices and the directed antiprism DA_k, which are $(n^2-3n+2)/2$, $k-1$ and 0, respectively. Let $\vec{K}_n$ and $\vec{K}_{n,n,n}$ be directed graphs gotten from the complete graph K_n and the complete tripartite $K_{n,n,n}$, respectively, by giving an orientation to each edge. In this paper, we aim to answer the following problem by using a method different from current graphs.

Problem 1.1 ([2])**.** Which kinds of $\vec{K}_n$ have $\gamma(\vec{G}) = \lceil \frac{1}{12}(n-3)(n-4) \rceil$, the genus of K_n.

A natural question analogue to Problem 1.1 is the following.

Problem 1.2. Which kinds of $\vec{K}_{n,n,n}$ with n vertices in each parts have directed genus $\frac{1}{2}(n-1)(n-2)$, the genus of $K_{n,n,n}$.

In this paper, we solve the Problems 1.1 and 1.2. Problem 1.2 is solved by giving the equivalent conditions for the minimum directed genus embedding of a directed graph $\vec{K}_{n,n,n}$ and a pair of biembeddable Latin squares with order n in an orientable surface. Furthermore, we prove that there is a one to one correspondence between the set of directed embeddings of a digraph D and the set of face-2-colorable embeddings of the underlying graph of D both on orientable surfaces. The result that there exists an orientation on edges of K_n such that the obtained tournament $\vec{K}_n$ has the directed genus $\lceil \frac{1}{12}(n-3)(n-4) \rceil$, when $n \equiv 3, 7 \pmod{12}$ is gotten which answer the Problem 1.1.

2 Alternating rotations, face-2-colorable embeddings, and Latin squares

An *alternating rotation* at a vertex v of D is a cyclic permutation of the arcs incident at v, such that in-arcs alternate with out-arcs. A list of alternating rotations, one for each vertex, is called an *alternating embedding scheme* (also called *alternating rotation system*) for the digraph D. There exists a one to one correspondence between the set of all embeddings (resp. directed embeddings) of a graph G (resp. a digraph D) on orientable surfaces and the set of the embedding schemes (resp. alternating embedding schemes) of G (resp. D). A *color class* is a set of faces with the same color. A *face-2-colorable embedding* of a graph G is an embedding which admits a 2-coloring of regions such that no two distinct regions of the same color shares a common edge. Two colors always mean black and white. Regions in an embedding of a graph are also called *faces*, while regions in a directed embedding of a digraph are partitioned into *faces* which use the arcs in the forward direction and *antifaces* which use arcs traversed against the given orientation.

An embedding is *triangular* if all regions are bounded by 3-cycles. Two face-2-colorable embeddings of K_n are said to be *isomorphic* if there exists a permutation on the n vertices (of the complete graph) such that it maps edges and faces of one embedding to edges and faces of the other one, respectively, see [2]. Equivalently, two face-2-colorable embeddings of K_n are isomorphic if and only if there exists a permutation on the n vertices such that it either preserves the color of the triangles or reverses the color. Let D_1 and D_2 be two digraphs. If D_1 is derived from D_2 by reversing all arcs of D_2, then we say these two digraphs have the *opposite orientation*.

A *transversal design* $TD(3,n)$ is an ordered triple $(V,\mathcal{G},\mathcal{B})$, where V is a $3n$-element set (the *points*), $\mathcal{G}$ is a partition of V into three disjoint sets (the *groups*) each of which has cardinality n, and $\mathcal{B}$ is a set of three-element subsets of V (the *triples*), such that every unordered pair of elements from V is either contained in precisely one triple or one group, but not both.

Example 2.1. An example of a $TD(3,n)$ of $n=3$. Let

$V=\{1,2,3,\dots,9\}$,
$\mathcal{G}=\{\{4,5,6\},\{7,8,9\},\{1,2,3\}\}$, and
$\mathcal{B}=\{(4,7,3),(4,8,1),(4,9,2),(5,7,1),(5,8,2),(5,9,3),(6,7,2),(6,8,3),(6,9,1)\}$.

Then $(V,\mathcal{G},\mathcal{B})$ is a transversal design $TD(3,3)$.

A *Latin square LS(n) of order* n is an $n\times n$ array filled with n different entries, each occurring exactly once in each row and exactly once in each column.

Example 2.2. A Latin square LS(n) of order n for $n=3$. Let

$$M=\begin{bmatrix}3&1&2\\1&2&3\\2&3&1\end{bmatrix}.$$

Then M is a Latin square LS(3).

There are relations among the face-2-colorable triangular embeddings of $K_{n,n,n}$ on an orientable surface, the transversal design $TD(3,n)$ and the Latin squares as follows.

For a given face 2-colourable triangular embeddings of $K_{n,n,n}$ on an orientable surface, it is proved in [10] that there exists a transversal design which is determined under one of the clockwise and counter-clockwise in each colour class. On the other hand, for a given transversal design $TD(3,n) = (V, \mathcal{G}, \mathcal{B})$, there is a Latin square determined by $TD(3,n)$ by assigning the three groups in $\mathcal{G}$ as labels for the row, columns and entries of the Latin square.

Two color classes $\mathcal{A}$ and $\mathcal{B}$ of a face-2-colorable triangular embedding of $K_{n,n,n}$ on an orientable surface give two Latin squares, corresponding to $\mathcal{A}$ and $\mathcal{B}$ respectively, which is considered as a *biembedding* of these two Latin squares with order n. Two Latin squares A and B are *biembeddable*, denoted by $A \bowtie B$, on an orientable surface S if there is a face-2-colorable (black and white) triangular embedding of $K_{n,n,n}$ in the orientable surface S such that the white face set is $\mathcal{A}$ and the black face set is $\mathcal{B}$. For more details, the readers are referred to [6, 7, 8, 9] and [21].

Example 2.3. Let V_1, V_2 and V_3 be a partition of $V(K_{3,3,3})$, where $V_1 = \{4,5,6\}, V_2 = \{7,8,9\}$ and $V_3 = \{1,2,3\}$. For a given embedding ρ of $K_{3,3,3}$ on an orientable surface, let ρ_v be the rotation at a vertex v. Let

$$\begin{array}{lll} \rho_1 = (7,5,9,6,8,4); & \rho_2 = (7,6,9,4,8,5); & \rho_3 = (7,4,9,5,8,6); \\ \rho_4 = (7,3,9,2,8,1); & \rho_5 = (7,1,9,3,8,2); & \rho_6 = (8,3,7,2,9,1); \\ \rho_7 = (1,5,2,6,3,4); & \rho_8 = (2,5,3,6,1,4); & \rho_9 = (2,4,3,5,1,6). \end{array}$$

Then $\rho = \{\rho_i : i \in \{1,\ldots,9\}\}$ is a face 2-colourable triangular embedding of $K_{3,3,3}$ on an orientable surface. In fact, a set of faces with the white color is

$$\mathcal{A}_1 = \{(5,7,1),(6,9,1),(4,8,1),(6,7,2),(4,9,2),(5,8,2),(4,7,3),(5,9,3),(6,8,3)\};$$

while a set of faces with the black color is

$$\mathcal{A}_2 = \{(9,5,1),(8,6,1),(7,4,1),(9,6,2),(8,4,2),(7,5,2),(9,4,3),(8,5,3),(7,6,3)\}.$$

There exists a transversal design TD$(3,3)$, say $(V, \mathcal{G}, \mathcal{B}_1)$, which is determined under the clockwise in white colour class $\mathcal{A}_1$. That is,

$$\begin{aligned} V &= \{1,2,3,\ldots,9\}, \\ \mathcal{G} &= \{\{4,5,6\},\{7,8,9\},\{1,2,3\}\}, \text{ and} \\ \mathcal{B}_1 &= \{(5,7,1),(6,9,1),(4,8,1),(6,7,2),(4,9,2),(5,8,2),(4,7,3),(5,9,3),(6,8,3)\}. \end{aligned}$$

There exists another transversal design TD$(3,3)$, say $(V, \mathcal{G}, \mathcal{B}_2)$, which is determined under the counter-clockwise in black colour class $\mathcal{A}_2$. That is,

$$\begin{aligned} V &= \{1,2,3,\ldots,9\}, \\ \mathcal{G} &= \{\{4,5,6\},\{7,8,9\},\{1,2,3\}\}, \text{ and} \\ \mathcal{B}_2 &= \{(5,9,1),(6,8,1),(4,7,1),(6,9,2),(4,8,2),(5,7,2),(4,9,3),(5,8,3),(6,7,3)\}. \end{aligned}$$

Example 2.4. Let $(V, \mathcal{G}, \mathcal{B}_1)$ be a transversal design given in Example 2.3. Assume that $\{4,5,6\}$ labels for the row, $\{7,8,9\}$ labels for columns and $\{1,2,3\}$ labels for entries of the Latin square. Thus

$$\mathcal{B}_1 = \{(5,7,1),(6,9,1),(4,8,1),(6,7,2),(4,9,2),(5,8,2),(4,7,3),(5,9,3),(6,8,3)\}$$

determines the matrix A_1 as

$$\begin{array}{c} \\ 4 \\ 5 \\ 6 \end{array}\begin{array}{c} \begin{array}{ccc} 7 & 8 & 9 \end{array} \\ \begin{pmatrix} 3 & 1 & 2 \\ 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} \end{array}. \tag{2.1}$$

Thus there is a Latin square A_1 determined by $(V, \mathcal{G}, \mathcal{B}_1)$, where

$$A_1 = \begin{bmatrix} 3 & 1 & 2 \\ 1 & 2 & 3 \\ 2 & 3 & 1 \end{bmatrix}.$$

Similarly, for a transversal designs $(V, \mathcal{G}, \mathcal{B}_2)$ given in Example 2.3, there is a Latin square A_2 determined by $(V, \mathcal{G}, \mathcal{B}_2)$, where

$$A_2 = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{bmatrix}.$$

In fact, using $V_1 = \{4,5,6\}$ as labels for the row, $V_2 = \{7,8,9\}$ as labels for the columns, and $V_3 = \{1,2,3\}$ as labels for entries of the Latin square, thus

$$\mathcal{B}_2 = \{(5,9,1),(6,8,1),(4,7,1),(6,9,2),(4,8,2),(5,7,2),(4,9,3),(5,8,3),(6,7,3)\}$$

determines the matrix A_2 as

$$\begin{array}{c} \\ 4 \\ 5 \\ 6 \end{array}\begin{array}{c} \begin{array}{ccc} 7 & 8 & 9 \end{array} \\ \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{pmatrix} \end{array}. \tag{2.2}$$

As a result, a face-2-colorable triangular embedding ρ of $K_{3,3,3}$ on an orientable surface gives two Latin squares A_1 and A_2, corresponding to two color classes $\mathcal{A}_1$ and $\mathcal{A}_2$ respectively. And $A_1 \bowtie A_2$ is a biembedding of these two Latin squares with order 3.

Because an embedding of an embeddable digraph is an embedding of the underlying graph, the following version of Euler's polyhedral formula holds.

Lemma 2.5. *Let $D = (V, A)$ be an embedding digraph, then for any alternating embedding scheme ρ of D, we have*

$$|V| - |A| + |R| = 2 - 2g,$$

where $|R|$ is the number of regions in the embeding scheme ρ and g is the genus of the embedding surface.

Lemma 2.6 ([7]). *There is a unique regular triangular embedding of a complete tripartite graph $K_{n,n,n}$ on an orientable surface for $n \geq 2$.*

Lemma 2.7 ([6]). *For a triangular embedding of $K_{n,n,n}$, it is orientable if and only if it is face-2-colorable embedding.*

The readers are referred to [1] for any undefined notations.

3 The directed genus of $\vec{K}_{n,n,n}$

For an embedding σ of a given digraph $\vec{K}_{n,n,n}$, the alternating embedding scheme is denoted by ρ_σ, the alternating rotation at a vertex $v \in V(D)$ is denoted by $\rho_\sigma(v)$ (or simply ρ_v).

Recall that $K_{n,n,n}$ is a complete tripartite graph. A complete tripartite digraph, denoted by $\vec{K}_{n,n,n}$, obtained from $K_{n,n,n}$ by giving an orientation for each edge in $K_{n,n,n}$. In the following, we find an orientation $\vec{K}_{n,n,n}$ of $K_{n,n,n}$ such that $\vec{K}_{n,n,n}$ has the directed genus $\frac{1}{2}(n-1)(n-2)$, the same as the genus of $K_{n,n,n}$.

Theorem 3.1. *The following two conditions on an orientation $\vec{K}_{n,n,n}$ of the complete tripartite graph $K_{n,n,n}$ are equivalent:*

(1) $\vec{K}_{n,n,n}$ has a directed embedding on the orientable surface of genus $\frac{1}{2}(n-1)(n-2)$, for which we call the sets of faces and antifaces $\mathcal{A}$ and $\mathcal{B}$, respectively.

(2) The sets $\mathcal{A}$ and $\mathcal{B}$ of white faces and black faces for a face-2-colorable triangular embedding of $K_{n,n,n}$ correspond to a pair of biembeddable Latin squares A and B of order n.

Proof. We first show that (1) implies (2).

Assume $\vec{K}_{n,n,n}$ has a directed embedding on an orientable surface of genus $\frac{1}{2}(n-1)(n-2)$ such that the sets of faces and antifaces $\mathcal{A}$ and $\mathcal{B}$, respectively. Let $\phi\colon \vec{K}_{n,n,n} \to S$ be this directed embedding of $\vec{K}_{n,n,n}$ and ρ_ϕ be the alternating embedding scheme of ϕ. Note that $\vec{K}_{n,n,n}$ has $3n$ vertices, $3n^2$ arcs and the embedding genus $\frac{1}{2}(n-1)(n-2)$. By Euler's formula of Lemma 2.5, the number of regions in ρ_ϕ is $2n^2$. This implies that each region is bounded by a directed 3-cycle because there are no i-cycles for $i = 1, 2$.

Let the embedding scheme ρ of $K_{n,n,n}$ be the same as ρ_ϕ without considering the directions of arcs, then $\mathcal{A} \cup \mathcal{B}$ is the facial set of the embedding ρ of $K_{n,n,n}$. We color faces in $\mathcal{A}$ with white and antifaces in $\mathcal{B}$ with black. By the definition of a directed embedding, each arc appears once in exactly one facial boundary and exactly one antifacial boundary. That is, no two distinct faces in $\mathcal{A}$ (resp. $\mathcal{B}$) are incident to the same edge. So ρ of $K_{n,n,n}$ is a face-2-colorable triangle embedding with two color classes $\mathcal{A}$ and $\mathcal{B}$ with $|\mathcal{A}| = |\mathcal{B}| = n^2$. Note that two color classes $\mathcal{A}$ and $\mathcal{B}$ of a face-2-colorable triangular embedding of $K_{n,n,n}$ on an orientable surface give two Latin squares, say A and B, corresponding to $\mathcal{A}$ and $\mathcal{B}$ respectively, which is a biembedding of these two Latin squares A and B. The result (2) is obtained.

Secondly, we show that (2) implies (1).

Suppose (2) holds. Note that there exists a face-2-colorable triangular embedding, say ϕ, of $K_{n,n,n}$ on an orientable surface with two facial color classes $\mathcal{A}$ and $\mathcal{B}$ which corresponds a pair of biembeddable Latin squares A and B of order n, respectively. Assume the embedding scheme of the embedding ϕ is ρ_ϕ and the rotation at vertex v in $K_{n,n,n}$ is denoted by $\rho_\phi(v)$. Let $V(K_{n,n,n}) = V_1 \cup V_2 \cup V_3$, where $\{V_1, V_2, V_3\}$ is a partition of $V(K_{n,n,n})$. Suppose $V_1 = \{a_1, a_2, \ldots, a_n\}$, $V_2 = \{b_1, b_2, \ldots, b_n\}$ and $V_3 = \{c_1, c_2, \ldots, c_n\}$.

Note that $\mathcal{A}$ and $\mathcal{B}$ determine transversal designs $(V, \mathcal{G}, \mathcal{A})$ and $(V, \mathcal{G}, \mathcal{B})$ respectively, where $V = V(K_{n,n,n})$, $\mathcal{G} = \{V_1, V_2, V_3\}$ and the faces in each color class form the triples in $\mathcal{A}$ and $\mathcal{B}$ of the transversal designs.

For every edge $uv \in E(K_{n,n,n})$, without loss of generality, let $u = a_i \in V_1$, $v = b_j \in V_2$. By the definition of a transversal design, there is only one triple in $\mathcal{A}$ containing a_i, b_j, say $\{a_i, b_j, c_x\}$ for some $c_x \in V_3$. Thus, vertices b_j and c_x are neighbors of a_i. Without loss of generality, let c_x be the closest successor of b_j in the rotation $\rho_\phi(a_i)$ along the counter-clockwise and the color of the region corresponding to the triple $\{a_i, b_j, c_x\}$ be white. On the other hand, there is exactly one triple in $\mathcal{B}$ containing a_i, b_j, say $\{a_i, b_j, c_y\}$ with $c_y \in V_3$, so b_j is the closest successor of c_y in the rotation $\rho_\phi(a_i)$ along the counter-clockwise and the color of the region corresponding to the triple $\{a_i, b_j, c_y\}$ is black which is illustrated in the left one of Figure 1.

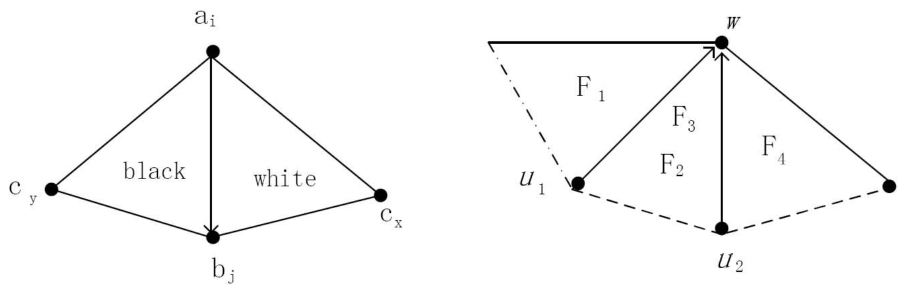

Figure 1: The rotations at vertices a_i and w respectively.

Give the orientation of the edge $uv = a_ib_j$ from $u = a_i$ to $v = b_j$, i.e., the color of the left region of the arc $\overrightarrow{uv}$ is white and the color of the right region is black. By the random choice of uv, all edges in $K_{n,n,n}$ are oriented and the obtained digraph is $\vec{K}_{n,n,n}$.

In the following, we only need to show that this orientation makes the in-arcs and out-arcs alternating at $\rho_\phi(v)$ for any $v \in V(K_{n,n,n})$. By the contrary, suppose there exists a vertex, say $w \in V$, such that in-arcs and out-arcs at w are not alternative. Without loss of generality, suppose two arcs, say $\overrightarrow{u_1w}, \overrightarrow{u_2w}$, are two neighbor in-arcs of w in $\rho_\phi(w)$ and $\rho_\phi(w) = (\ldots, u_1, u_2, \ldots)$ along counter-clockwise. Let the left face and right face of $\overrightarrow{u_1w}$ going from u_1 to w be F_1 and F_2 respectively and the left face and right face of $\overrightarrow{u_2w}$ going along the direction from u_2 to w be F_3 and F_4 respectively. Then $F_2 = F_3$. By the principle of the orientation, F_2 is colored black because of the direction of arc $\overrightarrow{u_1w}$ and F_3 is colored white because of the direction of arc $\overrightarrow{u_2w}$, which is shown in the right graph of Figure 1. It contradicts with face-2-colorable because $F_2 = F_3$. As a result, this orientation makes in-arcs and out-arcs alternating at every vertex $w \in V$ along the rotation $\rho_\phi(w)$.

As a result, $\vec{K}_{n,n,n}$, obtained from $K_{n,n,n}$ by this orientation, has an alternating embedding scheme determined by ϕ such that the sets of faces and antifaces of this directed embedding of $\vec{K}_{n,n,n}$ are $\mathcal{A}$ and $\mathcal{B}$, respectively.

Since each region of this directed embedding of $\vec{K}_{n,n,n}$ is a 3-cycle, the number of regions is $2n^2$. By $|V| = 3n$, the cardinality of arcs in $\vec{K}_{n,n,n}$ being $3n^2$ and Lemma 2.5, it follows $3n - 3n^2 + 2n^2 = 2 - 2g$, where g is the genus of this directed embedding. So $g = \frac{1}{2}(n-1)(n-2)$. Since neither loop nor 2-cycle is in $\vec{K}_{n,n,n}$, the minimum directed genus of $\vec{K}_{n,n,n}$ is $\frac{1}{2}(n-1)(n-2)$. Thus $\vec{K}_{n,n,n}$ has a directed embedding in the orientable surface of genus $\frac{1}{2}(n-1)(n-2)$, for which we call the sets of faces and antifaces $\mathcal{A}$ and

$\mathcal{B}$, respectively. □

Theorem 3.2. *Let $K_{n,n,n}$ be the complete tripartite graph. Then there exists an orientation of $K_{n,n,n}$ such that the obtained digraph $\vec{K}_{n,n,n}$ has the directed genus $\frac{1}{2}(n-1)(n-2)$, the same as the genus of $K_{n,n,n}$.*

Proof. Let $\vec{K}_{n,n,n}$ be the digraph obtained from $K_{n,n,n}$ by giving the orientation to each edge in $K_{n,n,n}$ and g be the directed genus of $\vec{K}_{n,n,n}$.

(1) If $n = 1$, then $K_{n,n,n} = K_{1,1,1}$ is a triangle. Let $\vec{K}_{1,1,1}$ be the digraph obtained by giving an orientation of $K_{1,1,1}$ such that it is a directed 3-cycle. Hence $g = 0$.

(2) If $n \geq 2$, by Lemma 2.6, there is a unique regular triangular embedding of a complete tripartite graph $K_{n,n,n}$ on an orientable surface. By Lemma 2.7, this regular triangular embedding of a complete tripartite graph $K_{n,n,n}$ must be a face-2-colorable embedding and two set of color faces are denoted by $\mathcal{A}$ and $\mathcal{B}$ respectively. By Theorem 3.1 , there is an orientation for $K_{n,n,n}$ such that the resulting digraph $\vec{K}_{n,n,n}$ has a directed embedding in the orientable surface of genus $\frac{1}{2}(n-1)(n-2)$, the set of faces is $\mathcal{A}$ and the set of antifaces is $\mathcal{B}$. Thus the result holds. □

4 The number of different orientations of K_n

Theorem 3.1 for a directed triangular embedding of the directed complete tripartite graph can be generalized to Lemma 4.1 for directed embedding of a general digraph.

Lemma 4.1. *The following two conditions on an orientation $\vec{G}$ of a graph G are equivalent.*

(1) $\vec{G}$ has a directed embedding on an orientable surface of genus g.

(2) G has a face-2-colorable embedding on an orientable surface of genus g.

Proof. We first show that (1) implies (2).

Let $G = (V, E)$ be a graph with n vertices, $\vec{G} = (V, A)$ be a digraph obtained from G by giving an orientation to each edge. So $|V| = n$ and $|E| = |A|$. By (1), $\vec{G}$ has a directed embedding on an orientable surface of genus g. Let ρ be the alternating embedding scheme and $\mathcal{F}_1$ and $\mathcal{F}_2$ be the set of faces and antifaces in $\vec{G}$, respectively. Note that a directed embedding of $\vec{G}$ is an embedding of G and $\mathcal{F}_1 \cup \mathcal{F}_2$ is the set of faces of this embedding of G. We color regions in $\mathcal{F}_1$ with white and rigions in $\mathcal{F}_2$ with black. From the definition of directed embedding, each arc in $\vec{G}$ is incident to exactly one face and exactly one antiface in the directed embedding ρ of $\vec{G}$, so there is no two distinct regions of the same color sharing a common edge in this embedding of G. It implies that this embedding of G is the face-2-colorable embedding on an orientable surface with genus g. So condition (2) holds.

Secondly, we show that (2) implies (1).

Suppose that (2) holds. Let ρ be the embedding scheme of a face-2-colorable embedding of a graph $G = (V, E)$ on an orientable surface S of genus g. And all regions of the embedding ρ can be colored by white and black. Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be the set of white and black regions, respectively. For each edge $e \in E(G)$, there are exactly two regions sharing the edge e, denoted by F_e^1 and F_e^2. By the definition of the face-2-colorable embedding, F_e^1 and F_e^2 have different colors. Without loss of generality, suppose that $F_e^1 \in \mathcal{F}_1$ and $F_e^2 \in \mathcal{F}_2$. We give the orientation of e such that the left is white region F_e^1 and the right is black region F_e^2 (this is known as *orientational principle*). Since each edge can be oriented, one can obtain a digraph, denoted by $\vec{G}$, from the graph G by this orientational principle. Let the alternating embedding scheme of $\vec{G}$ be the same as ρ. By the orientational principle and face-2-colorability, the in-arcs and out-arcs alternate at each vertex in ρ of $\vec{G}$. Thus this embedding scheme is an alternating embedding scheme of $\vec{G}$ as a directed embedding in the same surface S with genus g, so condition (1) holds. □

Theorem 4.2. *There is a one to one correspondence between the set of directed embeddings of a digraph D on orientable surfaces and the set of face-2-colorable embeddings of the underlying graph of D on orientable surfaces.*

Proof. Let D be a digraph and the underlying graph of D be obtained from D by ignoring the direction of arcs. Theorem 4.2 is obtained directly from Lemma 4.1. □

The following Theorem 4.3 give an answer to the problem in [2].

Theorem 4.3. *If $n \equiv 3, 7 \pmod{12}$, then there exists an orientation on edges of K_n such that the obtained tournament $\vec{K}_n$ has directed genus $\lceil \frac{1}{12}(n-3)(n-4) \rceil$.*

Proof. From Ringel and Youngs' results in [25] and [31], if $n \equiv 3, 7 \pmod{12}$, there exists a face-2-colorable triangular embedding of K_n on an orientable surface. By Lemma 4.1, there exists an orientation on edges of K_n such that the obtained digraph $\vec{K}_n$ has a directed triangular embedding on an orientable surface. By Euler's formula, digraph $\vec{K}_n$ has directed genus $\lceil \frac{1}{12}(n-3)(n-4) \rceil$. □

5 Concluding remarks

In this paper, we show that there is a one to one correspondence between the set of directed embeddings of a digraph D and the set of face-2-colorable embeddings of the underlying graph of D on orientable surfaces. Furthermore, we show that there exist orientations on $K_{n,n,n}$ and K_n such that the obtained graph $\vec{K}_{n,n,n}$ has the directed genus $\frac{1}{2}(n-1)(n-2)$ for $n \geq 1$ and $\vec{K}_n$ has directed genus $\lceil \frac{1}{12}(n-3)(n-4) \rceil$ for $n \equiv 3, 7 \pmod{12}$ which answers the problem about tournaments given in [2] by using a method different from current graphs which were discussed by the same author et al.

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