

# Unifying adjacency, Laplacian, and signless Laplacian theories\*

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## Abstract

Let  $G$  be a simple graph with associated diagonal matrix of vertex degrees  $D(G)$ , adjacency matrix  $A(G)$ , Laplacian matrix  $L(G)$  and signless Laplacian matrix  $Q(G)$ . Recently, Nikiforov proposed the family of matrices  $A_\alpha(G)$  defined for any real  $\alpha \in [0, 1]$  as  $A_\alpha(G) := \alpha D(G) + (1 - \alpha) A(G)$ , and also mentioned that the matrices  $A_\alpha(G)$  can underpin a unified theory of  $A(G)$  and  $Q(G)$ . Inspired from the above definition, we introduce the  $B_\alpha$ -matrix of  $G$ ,  $B_\alpha(G) := \alpha A(G) + (1 - \alpha) L(G)$  for  $\alpha \in [0, 1]$ . Note that  $L(G) = B_0(G)$ ,  $D(G) = 2B_{\frac{1}{2}}(G)$ ,  $Q(G) = 3B_{\frac{2}{3}}(G)$ ,  $A(G) = B_1(G)$ . In this article, we study several spectral properties of  $B_\alpha$ -matrices to unify the theories of adjacency, Laplacian, and signless Laplacian matrices of graphs. In particular, we prove that each eigenvalue of  $B_\alpha(G)$  is continuous on  $\alpha$ . Using this, we characterize positive semidefinite  $B_\alpha$ -matrices in terms of  $\alpha$ . As a consequence, we provide an upper bound of the independence number of  $G$ . Besides, we establish some bounds for the largest and the smallest eigenvalues of  $B_\alpha(G)$ . As a result, we obtain a bound for the chromatic number

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of  $G$  and deduce several known results. In addition, we present a Sachs-type result for the characteristic polynomial of a  $B_\alpha$ -matrix.

*Keywords:* Adjacency matrix, Laplacian matrix, signless Laplacian matrix, convex combination,  $B_\alpha$ -matrix,  $A_\alpha$ -matrix, chromatic number, independence number.

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# Poenotenje teorij sosednosti, Laplaciana in nepredznačenega Laplaciana\*

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## Povzetek

Naj bo  $G$  enostaven graf s pridruženo diagonalno matriko točkovnih stopenj  $D(G)$ , matriko sosednosti  $A(G)$ , Laplaceovo matriko  $L(G)$  in nepredznačeno Laplaceovo matriko  $Q(G)$ . Nedavno je Nikiforov predlagal raziskovanje družine matrik  $A_\alpha(G)$ , definiranih za vsako realno število  $\alpha \in [0, 1]$  kot  $A_\alpha(G) := \alpha D(G) + (1 - \alpha) A(G)$ , omenil pa je tudi, da so matrike  $A_\alpha(G)$  lahko podlaga za poenoteno teorijo matrik  $A(G)$  in  $Q(G)$ . Po zgledu zgornje definicije uvedemo  $B_\alpha$ -matriko grafa  $G$ ,  $B_\alpha(G) := \alpha A(G) + (1 - \alpha) L(G)$  za  $\alpha \in [0, 1]$ . Upoštevajte, da  $L(G) = B_0(G)$ ,  $D(G) = 2B_{\frac{1}{2}}(G)$ ,  $Q(G) = 3B_{\frac{2}{3}}(G)$ ,  $A(G) = B_1(G)$ . V tem članku raziskujemo več spektralnih lastnosti  $B_\alpha$ -matrik s ciljem poenotenja teorij sosednosti, Laplaceovih in nepredznačenih Laplaceovih matrik grafov. Dokažemo, da je vsaka lastna vrednost  $B_\alpha(G)$  zvezna na  $\alpha$ . Z uporabo tega karakteriziramo pozitivne semidefinitne  $B_\alpha$ -matrike v smislu njihovega parametra  $\alpha$ . Kot posledico predstavimo zgornjo mejo za neodvisnostno število grafa  $G$ . Določimo tudi nekaj mej za največje in najmanjše lastne vrednosti  $B_\alpha(G)$ . Tako dobimo mejo za kromatsko število grafa  $G$  in izpeljemo več znanih rezultatov. Poleg tega, predstavimo rezultat Sachsovega tipa za karakteristični polinom  $B_\alpha$ -matrike.

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*Ključne besede: Matrika sosednosti, Laplaceova matrika, nepredznačena Laplaceova matrika, konveksna kombinacija,  $B_\alpha$ -matrika,  $A_\alpha$ -matrika, kromatsko število, neodvisnostno število.*

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