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A compact presentation for the alternating central extension of the positive part of $U_q(\widehat{\mathfrak{sl}}_2)$

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Abstract

This paper concerns the positive part U_q^+ of the quantum group $U_q(\widehat{\mathfrak{sl}}_2)$. The algebra U_q^+ has a presentation involving two generators that satisfy the cubic q-Serre relations. We recently introduced an algebra $\mathcal{U}_q^+$ called the alternating central extension of U_q^+. We presented $\mathcal{U}_q^+$ by generators and relations. The presentation is attractive, but the multitude of generators and relations makes the presentation unwieldy. In this paper we obtain a presentation of $\mathcal{U}_q^+$ that involves a small subset of the original set of generators and a very manageable set of relations. We call this presentation the compact presentation of $\mathcal{U}_q^+$.

Keywords: q-Onsager algebra, q-Serre relations, q-shuffle algebra, tridiagonal pair.

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1 Introduction

The algebra $U_q(\widehat{\mathfrak{sl}}_2)$ is well known in representation theory [15] and statistical mechanics [20]. This algebra has a subalgebra U_q^+ called the positive part. The algebra U_q^+ has a presentation involving two generators (said to be standard) and two relations, called the q-Serre relations. The presentation is given in Definition 2.1 below.

Our interest in U_q^+ is motivated by some applications to linear algebra and combinatorics; these will be described shortly. Before going into detail, we have a comment about q. In the applications, either q is not a root of unity, or q is a root of unity with exponent large enough to not interfere with the rest of the application. To keep things simple, throughout the paper we will assume that q is not a root of unity.

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Our first application has to do with tridiagonal pairs [17]. A tridiagonal pair is roughly described as an ordered pair of diagonalizable linear maps on a nonzero finite-dimensional vector space, that each act on the eigenspaces of the other one in a block-tridiagonal fashion [17, Definition 1.1]. There is a type of tridiagonal pair said to be q-geometric [18, Definition 2.6]; for this type of tridiagonal pair the eigenvalues of each map form a q^2-geometric progression. A finite-dimensional irreducible U_q^+-module on which the standard generators are not nilpotent, is essentially the same thing as a tridiagonal pair of q-geometric type [18, Theorem 2.7]; these U_q^+-modules are described in [18, Section 1]. See [13, 24] for more background on tridiagonal pairs.

Our next application has to do with distance-regular graphs [1, 14, 32]. Consider a distance-regular graph Γ that has diameter $d \geq 3$ and classical parameters (d, b, α, β) [14, p. 193] with $b = q^2$ and $\alpha = q^2 - 1$. The condition on α implies that Γ is formally self-dual in the sense of [14, p. 49]. Let A denote the adjacency matrix of Γ, and let A^* denote the dual adjacency matrix with respect to any vertex of Γ [19, Section 7]. Then by [19, Lemma 9.4], there exist complex numbers r, s, r^*, s^* with r, r^* nonzero such that $rA + sI$, $r^*A^* + s^*I$ satisfy the q-Serre relations. As mentioned in [19, Example 8.4], the above parameter restriction is satisfied by the bilinear forms graph [14, p. 280], the alternating forms graph [14, p. 282], the Hermitean forms graph [14, p. 285], the quadratic forms graph [14, p. 290], the affine E_6 graph [14, p. 340], and the extended ternary Golay code graph [14, p. 359].

Our next application has to do with uniform posets [23, 27]. Let $\mathrm{GF}(b)$ denote a finite field with b elements, and let N, M denote positive integers. Let H denote a vector space over $\mathrm{GF}(b)$ that has dimension $N + M$. Let h denote a subspace of H with dimension M. Let P denote the set of subspaces of H that have zero intersection with h. For $x, y \in P$ define $x \leq y$ whenever $x \subseteq y$. The relation $\leq$ is a partial order on P, and the poset P is ranked with rank N. The poset P is called an attenuated space poset, and denoted by $A_b(N, M)$ [21], [27, Example 3.1]. By [27, Theorem 3.2] the poset $A_b(N, M)$ is uniform in the sense of [27, Definition 2.2]. It is shown in [21, Lemma 3.3] that for $A_b(N, M)$ the raising matrix R and the lowering matrix L satisfy the q-Serre relations, provided that $b = q^2$.

Our last application has to do with q-shuffle algebras. Let $\mathbb{F}$ denote a field, and let x, y denote noncommuting indeterminates. Let V denote the free associative $\mathbb{F}$-algebra with generators x, y. By a letter in V we mean x or y. For an integer $n \geq 0$, by a word of length n in V we mean a product of letters $v_1 v_2 \cdots v_n$. The words in V form a basis for the vector space V. In [25, 26] M. Rosso introduced an algebra structure on V, called the q-shuffle algebra. For letters u, v their q-shuffle product is $u \star v = uv + q^{\langle u,v \rangle} vu$, where $\langle u, v \rangle = 2$ (resp. $\langle u, v \rangle = -2$) if $u = v$ (resp. $u \neq v$). By [25, Theorem 13], in the q-shuffle algebra V the elements x, y satisfy the q-Serre relations. Consequently there exists an algebra homomorphism $\natural$ from U_q^+ into the q-shuffle algebra V, that sends the standard generators of U_q^+ to x, y. By [26, Theorem 15] the map $\natural$ is injective.

Next we recall the alternating elements in U_q^+ [30]. Let $v_1 v_2 \cdots v_n$ denote a word in V. This word is called alternating whenever $n \geq 1$ and $v_{i-1} \neq v_i$ for $2 \leq i \leq n$. Thus the alternating words have the form $\cdots xyxy \cdots$. The alternating words are displayed below:

$$\begin{array}{lllll} x, & xyx, & xyxyx, & xyxyxyx, & \ldots \\ y, & yxy, & yxyxy, & yxyxyxy, & \ldots \\ yx, & yxyx, & yxyxyx, & yxyxyxyx, & \ldots \end{array}$$

$$xy, \qquad xyxy, \qquad xyxyxy, \qquad xyxyxyxy, \qquad \ldots$$

By [30, Theorem 8.3] each alternating word is contained in the image of $\natural$. An element of U_q^+ is called alternating whenever it is the $\natural$-preimage of an alternating word. For example, the standard generators of U_q^+ are alternating because they are the $\natural$-preimages of the alternating words x, y. It is shown in [30, Lemma 5.12] that for each row in the above display, the corresponding alternating elements mutually commute. A naming scheme for alternating elements is introduced in [30, Definition 5.2].

Next we recall the alternating central extension of U_q^+ [29]. In [30] we displayed two types of relations among the alternating elements of U_q^+; the first type is [30, Propositions 5.7, 5.10, 5.11] and the second type is [30, Propositions 6.3, 8.1]. The relations in [30, Proposition 5.11] are redundant; they follow from the relations in [30, Propositions 5.7, 5.10] as pointed out in [2, Propositions 3.1, 3.2] and [5, Remark 2.5]; see also Corollary 6.3 below. The relations in [30, Proposition 6.3] are also redundant; they follow from the relations in [30, Propositions 5.7, 5.10] as shown in the proof of [30, Proposition 6.3]. By [30, Lemma 8.4] and the previous comments, the algebra U_q^+ is presented by its alternating elements and the relations in [30, Propositions 5.7, 5.10, 8.1]. For this presentation it is natural to ask what happens if the relations in [30, Proposition 8.1] are removed. To answer this question, in [29, Definition 3.1] we defined an algebra $\mathcal{U}_q^+$ by generators and relations in the following way. The generators, said to be alternating, are in bijection with the alternating elements of U_q^+. The relations are the ones in [30, Propositions 5.7, 5.10]. By construction there exists a surjective algebra homomorphism $\mathcal{U}_q^+ \to U_q^+$ that sends each alternating generator of $\mathcal{U}_q^+$ to the corresponding alternating element of U_q^+. In [29, Lemma 3.6, Theorem 5.17] we adjusted this homomorphism to get an algebra isomorphism $\mathcal{U}_q^+ \to U_q^+ \otimes \mathbb{F}[z_1, z_2, \ldots]$, where $\{z_n\}_{n=1}^\infty$ are mutually commuting indeterminates. By [29, Theorem 10.2] the alternating generators form a PBW basis for $\mathcal{U}_q^+$. The algebra $\mathcal{U}_q^+$ is called the alternating central extension of U_q^+.

We mentioned above that the algebra $\mathcal{U}_q^+$ is presented by its alternating generators and the relations in [30, Propositions 5.7, 5.10]. This presentation is attractive, but the multitude of generators and relations makes the presentation unwieldy. In this paper we obtain a presentation of $\mathcal{U}_q^+$ that involves a small subset of the original set of generators and a very manageable set of relations. This presentation is given in Definition 3.1 below; we call it the compact presentation of $\mathcal{U}_q^+$. At first glance, it is not clear that the algebra presented in Definition 3.1 is equal to $\mathcal{U}_q^+$. So we denote by $\mathcal{U}$ the algebra presented in Definition 3.1, and eventually prove that $\mathcal{U} = \mathcal{U}_q^+$. After this result is established, we describe some features of $\mathcal{U}_q^+$ that are illuminated by the presentation in Definition 3.1.

Our investigation of $\mathcal{U}_q^+$ is inspired by some recent developments in statistical mechanics, concerning the q-Onsager algebra O_q. In [9] Baseilhac and Koizumi introduce a current algebra $\mathcal{A}_q$ for O_q, in order to solve boundary integrable systems with hidden symmetries. In [12, Definition 3.1] Baseilhac and Shigechi give a presentation of $\mathcal{A}_q$ by generators and relations. This presentation and the discussion in [12, Section 4] suggest that $\mathcal{A}_q$ is related to O_q in roughly the same way that $\mathcal{U}_q^+$ is related to U_q^+. The relationship between $\mathcal{A}_q$ and O_q was conjectured in [7, Conjectures 1, 2] and [28, Conjectures 4.5, 4.6, 4.8], before being settled in [31, Theorems 9.14, 10.2, 10.3, 10.4]. The articles [3, 4, 6, 7, 8, 9, 10, 11, 12] contain background information on O_q and $\mathcal{A}_q$.

Earlier in this section, we indicated how U_q^+ has applications to tridiagonal pairs, distance-regular graphs, and uniform posets. Possibly $\mathcal{U}_q^+$ appears in these applications, and this possibility should be investigated in the future.

This paper is organized as follows. In Section 2 we review some facts about U_q^+. In Section 3, we introduce the algebra $\mathcal{U}$ and give an algebra homomorphism $U_q^+ \to \mathcal{U}$. In Section 4, we introduce the alternating generators for $\mathcal{U}$ and establish some formulas involving these generators. In Sections 5, 6 we use these formulas and generating functions to show that the alternating generators for $\mathcal{U}$ satisfy the relations in [30, Propositions 5.7, 5.10]. Using this result, we prove that $\mathcal{U} = \mathcal{U}_q^+$. Theorem 6.2 and Corollary 6.5 are the main results of the paper. In Section 7 we describe some features of $\mathcal{U}_q^+$ that are illuminated by the presentation in Definition 3.1. Appendix A contains a list of relations involving the generating functions from Section 5.

2 The algebra U_q^+

We now begin our formal argument. For the rest of the paper, the following notational conventions are in effect. Recall the natural numbers $\mathbb{N} = \{0, 1, 2, \ldots\}$. Let $\mathbb{F}$ denote a field. Every vector space and tensor product mentioned is over $\mathbb{F}$. Every algebra mentioned is associative, over $\mathbb{F}$, and has a multiplicative identity. Fix a nonzero $q \in \mathbb{F}$ that is not a root of unity. Recall the notation

$$[n]_q = \frac{q^n - q^{-n}}{q - q^{-1}} \qquad\qquad n \in \mathbb{N}.$$

For elements X, Y in any algebra, define their commutator and q-commutator by

$$[X, Y] = XY - YX, \qquad\qquad [X, Y]_q = qXY - q^{-1}YX.$$

Note that

$$[X, [X, [X, Y]_q]_{q^{-1}}] = X^3Y - [3]_q X^2YX + [3]_q XYX^2 - YX^3.$$

Definition 2.1 ([22, Corollary 3.2.6])**.** Define the algebra U_q^+ by generators W_0, W_1 and relations

$$[W_0, [W_0, [W_0, W_1]_q]_{q^{-1}}] = 0, \qquad\qquad [W_1, [W_1, [W_1, W_0]_q]_{q^{-1}}] = 0. \tag{2.1}$$

We call U_q^+ the *positive part of* $U_q(\widehat{\mathfrak{sl}}_2)$. The generators W_0, W_1 are called *standard*. The relations (2.1) are called the *q-Serre relations*.

We will use the following concept.

Definition 2.2 ([16, p. 299])**.** Let $\mathcal{A}$ denote an algebra. A *Poincaré-Birkhoff-Witt* (or *PBW*) basis for $\mathcal{A}$ consists of a subset $\Omega \subseteq \mathcal{A}$ and a linear order $<$ on Ω such that the following is a basis for the vector space $\mathcal{A}$:

$$a_1 a_2 \cdots a_n \qquad n \in \mathbb{N}, \qquad a_1, a_2, \ldots, a_n \in \Omega, \qquad a_1 \leq a_2 \leq \cdots \leq a_n.$$

We interpret the empty product as the multiplicative identity in $\mathcal{A}$.

In [16, p. 299] Damiani obtains a PBW basis for U_q^+ that involves some elements

$$\{E_{n\delta+\alpha_0}\}_{n=0}^{\infty}, \qquad \{E_{n\delta+\alpha_1}\}_{n=0}^{\infty}, \qquad \{E_{n\delta}\}_{n=1}^{\infty}. \tag{2.2}$$

These elements are defined recursively as follows:

$$E_{\alpha_0} = W_0, \qquad E_{\alpha_1} = W_1, \qquad E_\delta = q^{-2}W_1W_0 - W_0W_1 \tag{2.3}$$

and for $n \geq 1$,

$$E_{n\delta+\alpha_0} = \frac{[E_\delta, E_{(n-1)\delta+\alpha_0}]}{q+q^{-1}}, \qquad E_{n\delta+\alpha_1} = \frac{[E_{(n-1)\delta+\alpha_1}, E_\delta]}{q+q^{-1}}, \tag{2.4}$$

$$E_{n\delta} = q^{-2}E_{(n-1)\delta+\alpha_1}W_0 - W_0E_{(n-1)\delta+\alpha_1}. \tag{2.5}$$

Proposition 2.3 ([16, p. 308]). *A PBW basis for U_q^+ is obtained by the elements* (2.2) *in the linear order*

$$E_{\alpha_0} < E_{\delta+\alpha_0} < E_{2\delta+\alpha_0} < \cdots < E_\delta < E_{2\delta} < E_{3\delta} < \cdots < E_{2\delta+\alpha_1} < E_{\delta+\alpha_1} < E_{\alpha_1}.$$

The elements (2.2) satisfy many relations [16]. We mention a few for later use.

Lemma 2.4 ([16, p. 300]). *For $i, j \in \mathbb{N}$ with $i > j$ the following hold in U_q^+.*

(i) *Assume that $i - j = 2r + 1$ is odd. Then*

$$E_{i\delta+\alpha_0}E_{j\delta+\alpha_0} = q^{-2}E_{j\delta+\alpha_0}E_{i\delta+\alpha_0} - (q^2-q^{-2})\sum_{\ell=1}^{r} q^{-2\ell}E_{(j+\ell)\delta+\alpha_0}E_{(i-\ell)\delta+\alpha_0},$$

$$E_{j\delta+\alpha_1}E_{i\delta+\alpha_1} = q^{-2}E_{i\delta+\alpha_1}E_{j\delta+\alpha_1} - (q^2-q^{-2})\sum_{\ell=1}^{r} q^{-2\ell}E_{(i-\ell)\delta+\alpha_1}E_{(j+\ell)\delta+\alpha_1}.$$

(ii) *Assume that $i - j = 2r$ is even. Then*

$$E_{i\delta+\alpha_0}E_{j\delta+\alpha_0} = q^{-2}E_{j\delta+\alpha_0}E_{i\delta+\alpha_0} - q^{j-i+1}(q-q^{-1})E^2_{(r+j)\delta+\alpha_0} - (q^2-q^{-2})\sum_{\ell=1}^{r-1} q^{-2\ell}E_{(j+\ell)\delta+\alpha_0}E_{(i-\ell)\delta+\alpha_0},$$

$$E_{j\delta+\alpha_1}E_{i\delta+\alpha_1} = q^{-2}E_{i\delta+\alpha_1}E_{j\delta+\alpha_1} - q^{j-i+1}(q-q^{-1})E^2_{(r+j)\delta+\alpha_1} - (q^2-q^{-2})\sum_{\ell=1}^{r-1} q^{-2\ell}E_{(i-\ell)\delta+\alpha_1}E_{(j+\ell)\delta+\alpha_1}.$$

Lemma 2.5. *The following* (i) – (iii) *hold in U_q^+.*

(i) (See [16, p. 307].) *For positive $i, j \in \mathbb{N}$,*

$$E_{i\delta}E_{j\delta} = E_{j\delta}E_{i\delta}. \tag{2.6}$$

(ii) (See [16, p. 307].) *For $i, j \in \mathbb{N}$,*

$$[E_{i\delta+\alpha_0}, E_{j\delta+\alpha_1}]_q = -qE_{(i+j+1)\delta}. \tag{2.7}$$

(iii) *For* $i \in \mathbb{N}$,

$$\frac{[W_0, E_{i\delta+\alpha_0}]_q}{q-q^{-1}} = \sum_{\ell=0}^{i} E_{\ell\delta+\alpha_0} E_{(i-\ell)\delta+\alpha_0}, \tag{2.8}$$

$$\frac{[E_{i\delta+\alpha_1}, W_1]_q}{q-q^{-1}} = \sum_{\ell=0}^{i} E_{(i-\ell)\delta+\alpha_1} E_{\ell\delta+\alpha_1}. \tag{2.9}$$

Proof. (iii) To verify (2.8) and (2.9), use Lemma 2.4 to write each term in the PBW basis for U_q^+ from Proposition 2.3. We give the details for (2.8). Referring to (2.8), let Δ denote the right-hand side minus the left-hand side. We show that $\Delta = 0$. This is quickly verified for $i = 0$, so assume that $i \geq 1$. For i even (resp. i odd) write $i = 2r$ (resp. $i = 2r + 1$). Using Lemma 2.4 we obtain $\Delta = \sum_{\ell=0}^{r} \alpha_\ell E_{\ell\delta+\alpha_0} E_{(i-\ell)\delta+\alpha_0}$, where for i even,

$$\alpha_0 = 1 + q^{-2} - \frac{q}{q-q^{-1}} + \frac{q^{-3}}{q-q^{-1}},$$

$$\alpha_\ell = 1 + q^{-2} - (q^2 - q^{-2}) \sum_{k=1}^{\ell} q^{-2k} - (q+q^{-1}) q^{-2\ell-1} \qquad (1 \leq \ell \leq r-1),$$

$$\alpha_r = 1 - (q-q^{-1}) \sum_{k=1}^{r} q^{1-2k} - q^{-i}$$

and for i odd,

$$\alpha_0 = 1 + q^{-2} - \frac{q}{q-q^{-1}} + \frac{q^{-3}}{q-q^{-1}},$$

$$\alpha_\ell = 1 + q^{-2} - (q^2 - q^{-2}) \sum_{k=1}^{\ell} q^{-2k} - (q+q^{-1}) q^{-2\ell-1} \qquad (1 \leq \ell \leq r).$$

For either case $\alpha_\ell = 0$ for $0 \leq \ell \leq r$, so $\Delta = 0$. We have verified (2.8). For (2.9) the details are similar, and omitted. □

3 An extension of U_q^+

In this section we introduce the algebra $\mathcal{U}$. In Section 6 we will show that $\mathcal{U}$ coincides with the alternating central extension $\mathcal{U}_q^+$ of U_q^+.

Definition 3.1. Define the algebra $\mathcal{U}$ by generators W_0, W_1, $\{\tilde{G}_{k+1}\}_{k\in\mathbb{N}}$ and relations

(i) $[W_0, [W_0, [W_0, W_1]_q]_{q^{-1}}] = 0$,

(ii) $[W_1, [W_1, [W_1, W_0]_q]_{q^{-1}}] = 0$,

(iii) $[\tilde{G}_1, W_1] = q\frac{[[W_0, W_1]_q, W_1]}{q^2-q^{-2}}$,

(iv) $[W_0, \tilde{G}_1] = q\frac{[W_0, [W_0, W_1]_q]}{q^2-q^{-2}}$,

(v) for $k \geq 1$,

$$[\tilde{G}_{k+1}, W_1] = \frac{[[[\tilde{G}_k, W_0]_q, W_1]_q, W_1]}{(1-q^{-2})(q^2-q^{-2})},$$

(vi) for $k \geq 1$,

$$[W_0, \tilde{G}_{k+1}] = \frac{[W_0, [W_0, [W_1, \tilde{G}_k]_q]_q]}{(1-q^{-2})(q^2-q^{-2})},$$

(vii) for $k, \ell \in \mathbb{N}$,

$$[\tilde{G}_{k+1}, \tilde{G}_{\ell+1}] = 0.$$

For notational convenience define $\tilde{G}_0 = 1$.

Note 3.2. Referring to Definition 3.1, the relation (iii) (resp. (iv)) is obtained from (v) (resp. (vi)) by setting $k = 0$.

Lemma 3.3. *There exists a unique algebra homomorphism* $\flat : U_q^+ \to \mathcal{U}$ *that sends* $W_0 \mapsto W_0$ *and* $W_1 \mapsto W_1$.

Proof. Compare Definitions 2.1, 3.1. □

In Corollary 6.7 we will show that $\flat$ is injective. Let $\langle W_0, W_1 \rangle$ denote the subalgebra of $\mathcal{U}$ generated by W_0, W_1. Of course $\langle W_0, W_1 \rangle$ is the $\flat$-image of U_q^+. For the elements (2.2) of U_q^+, the same notation will be used for their $\flat$-images in $\langle W_0, W_1 \rangle$.

4 Augmenting the generating set for $\mathcal{U}$

Some of the relations in Definition 3.1 are nonlinear. Our next goal is to linearize the relations by adding more generators.

Definition 4.1. We define some elements in $\mathcal{U}$ as follows. For $k \in \mathbb{N}$,

$$W_{-k} = \frac{[\tilde{G}_k, W_0]_q}{q - q^{-1}}, \tag{4.1}$$

$$W_{k+1} = \frac{[W_1, \tilde{G}_k]_q}{q - q^{-1}}, \tag{4.2}$$

$$G_{k+1} = \tilde{G}_{k+1} + \frac{[W_1, W_{-k}]}{1 - q^{-2}}. \tag{4.3}$$

For notational convenience define $G_0 = 1$.

Lemma 4.2. *For* $k \in \mathbb{N}$ *the following hold in* $\mathcal{U}$:

$$\tilde{G}_k W_0 = q^{-2} W_0 \tilde{G}_k + (1 - q^{-2}) W_{-k},$$
$$\tilde{G}_k W_1 = q^2 W_1 \tilde{G}_k + (1 - q^2) W_{k+1}.$$

Proof. These are reformulations of (4.1) and (4.2). □

The following is a generating set for $\mathcal{U}$:

$$\{W_{-k}\}_{k\in\mathbb{N}}, \qquad \{W_{k+1}\}_{k\in\mathbb{N}}, \qquad \{G_{k+1}\}_{k\in\mathbb{N}}, \qquad \{\tilde{G}_{k+1}\}_{k\in\mathbb{N}}. \tag{4.4}$$

The elements of this set will be called *alternating*. We seek a presentation of $\mathcal{U}$, that has the above generating set and all relations linear. We will obtain this presentation in Theorem 6.2.

Next we obtain some formulas that will help us prove Theorem 6.2. We will show that for $n \in \mathbb{N}$,

$$W_{n+1} = \sum_{k=0}^{n} \frac{E_{k\delta+\alpha_1}\tilde{G}_{n-k}(-1)^k q^k}{(q-q^{-1})^{2k}}, \tag{4.5}$$

$$W_{-n} = \sum_{k=0}^{n} \frac{E_{k\delta+\alpha_0}\tilde{G}_{n-k}(-1)^k q^{3k}}{(q-q^{-1})^{2k}}. \tag{4.6}$$

We will prove (4.5), (4.6) by induction on n. Note that (4.5), (4.6) hold for $n = 0$, since $W_1 = E_{\alpha_1}$ and $W_0 = E_{\alpha_0}$. We will give the main induction argument after a few lemmas. For the rest of this section k and ℓ are understood to be in $\mathbb{N}$.

Lemma 4.3. *Pick $n \in \mathbb{N}$, and assume that* (4.5), (4.6) *hold for $n, n-1, \dots, 1, 0$. Then*

$$[W_0, W_{n+1}] = [W_{-n}, W_1]. \tag{4.7}$$

Proof. The commutator $[W_0, W_{n+1}]$ is equal to

$$\begin{aligned}
&W_0W_{n+1} - W_{n+1}W_0 \\
&= \sum_{k=0}^{n} \frac{W_0E_{k\delta+\alpha_1}\tilde{G}_{n-k}(-1)^k q^k}{(q-q^{-1})^{2k}} - \sum_{k=0}^{n} \frac{E_{k\delta+\alpha_1}\tilde{G}_{n-k}W_0(-1)^k q^k}{(q-q^{-1})^{2k}} \\
&= \sum_{k=0}^{n} \frac{W_0E_{k\delta+\alpha_1}\tilde{G}_{n-k}(-1)^k q^k}{(q-q^{-1})^{2k}} \\
&\qquad\qquad - \sum_{k=0}^{n} \frac{E_{k\delta+\alpha_1}\big(q^{-2}W_0\tilde{G}_{n-k} + (1-q^{-2})W_{k-n}\big)(-1)^k q^k}{(q-q^{-1})^{2k}} \\
&= \sum_{k=0}^{n} \frac{\big(W_0E_{k\delta+\alpha_1} - q^{-2}E_{k\delta+\alpha_1}W_0\big)\tilde{G}_{n-k}(-1)^k q^k}{(q-q^{-1})^{2k}} - \sum_{k=0}^{n} \frac{E_{k\delta+\alpha_1}W_{k-n}(-1)^k q^{k-1}}{(q-q^{-1})^{2k-1}} \\
&= -\sum_{k=0}^{n} \frac{E_{(k+1)\delta}\tilde{G}_{n-k}(-1)^k q^k}{(q-q^{-1})^{2k}} - \sum_{k=0}^{n} \frac{E_{k\delta+\alpha_1}W_{k-n}(-1)^k q^{k-1}}{(q-q^{-1})^{2k-1}} \\
&= -\sum_{k=0}^{n} \frac{E_{(k+1)\delta}\tilde{G}_{n-k}(-1)^k q^k}{(q-q^{-1})^{2k}} \\
&\qquad\qquad - \sum_{k=0}^{n} \frac{E_{k\delta+\alpha_1}(-1)^k q^{k-1}}{(q-q^{-1})^{2k-1}} \sum_{\ell=0}^{n-k} \frac{E_{\ell\delta+\alpha_0}\tilde{G}_{n-k-\ell}(-1)^\ell q^{3\ell}}{(q-q^{-1})^{2\ell}} \\
&= -\sum_{p=0}^{n} \frac{E_{(p+1)\delta}\tilde{G}_{n-p}(-1)^p q^p}{(q-q^{-1})^{2p}} - \sum_{p=0}^{n} \Bigg(\sum_{k+\ell=p} q^{2\ell} E_{k\delta+\alpha_1}E_{\ell\delta+\alpha_0}\Bigg) \frac{\tilde{G}_{n-p}(-1)^p q^{p-1}}{(q-q^{-1})^{2p-1}}.
\end{aligned}$$

The commutator $[W_{-n}, W_1]$ is equal to

$$
\begin{aligned}
&W_{-n}W_1 - W_1W_{-n} \\
&= \sum_{k=0}^{n} \frac{E_{k\delta+\alpha_0}\tilde{G}_{n-k}W_1(-1)^k q^{3k}}{(q-q^{-1})^{2k}} - \sum_{k=0}^{n} \frac{W_1E_{k\delta+\alpha_0}\tilde{G}_{n-k}(-1)^k q^{3k}}{(q-q^{-1})^{2k}} \\
&= \sum_{k=0}^{n} \frac{E_{k\delta+\alpha_0}\big(q^2W_1\tilde{G}_{n-k} + (1-q^2)W_{n-k+1}\big)(-1)^k q^{3k}}{(q-q^{-1})^{2k}} \\
&\qquad\qquad - \sum_{k=0}^{n} \frac{W_1E_{k\delta+\alpha_0}\tilde{G}_{n-k}(-1)^k q^{3k}}{(q-q^{-1})^{2k}} \\
&= \sum_{k=0}^{n} \frac{\big(q^2E_{k\delta+\alpha_0}W_1 - W_1E_{k\delta+\alpha_0}\big)\tilde{G}_{n-k}(-1)^k q^{3k}}{(q-q^{-1})^{2k}} \\
&\qquad\qquad - \sum_{k=0}^{n} \frac{E_{k\delta+\alpha_0}W_{n-k+1}(-1)^k q^{3k+1}}{(q-q^{-1})^{2k-1}} \\
&= -\sum_{k=0}^{n} \frac{E_{(k+1)\delta}\tilde{G}_{n-k}(-1)^k q^{3k+2}}{(q-q^{-1})^{2k}} - \sum_{k=0}^{n} \frac{E_{k\delta+\alpha_0}W_{n-k+1}(-1)^k q^{3k+1}}{(q-q^{-1})^{2k-1}} \\
&= -\sum_{k=0}^{n} \frac{E_{(k+1)\delta}\tilde{G}_{n-k}(-1)^k q^{3k+2}}{(q-q^{-1})^{2k}} \\
&\qquad\qquad - \sum_{k=0}^{n} \frac{E_{k\delta+\alpha_0}(-1)^k q^{3k+1}}{(q-q^{-1})^{2k-1}} \sum_{\ell=0}^{n-k} \frac{E_{\ell\delta+\alpha_1}\tilde{G}_{n-k-\ell}(-1)^\ell q^\ell}{(q-q^{-1})^{2\ell}} \\
&= -\sum_{p=0}^{n} \frac{E_{(p+1)\delta}\tilde{G}_{n-p}(-1)^p q^{3p+2}}{(q-q^{-1})^{2p}} \\
&\qquad\qquad - \sum_{p=0}^{n} \left(\sum_{k+\ell=p} q^{2k} E_{k\delta+\alpha_0}E_{\ell\delta+\alpha_1} \right) \frac{\tilde{G}_{n-p}(-1)^p q^{p+1}}{(q-q^{-1})^{2p-1}}.
\end{aligned}
$$

By these comments

$$
[W_{-n}, W_1] - [W_0, W_{n+1}] = \sum_{p=0}^{n} \frac{C_p\tilde{G}_{n-p}(-1)^p q^p}{(q-q^{-1})^{2p}},
$$

where for $0 \le p \le n$,

$$
\begin{aligned}
C_p &= E_{(p+1)\delta} + q^{-1}(q-q^{-1}) \sum_{k+\ell=p} q^{2\ell} E_{k\delta+\alpha_1}E_{\ell\delta+\alpha_0} \\
&\qquad - q^{2p+2}E_{(p+1)\delta} - q(q-q^{-1}) \sum_{k+\ell=p} q^{2k} E_{k\delta+\alpha_0}E_{\ell\delta+\alpha_1} \\
&= (1-q^{2p+2})E_{(p+1)\delta} - (1-q^2) \sum_{k+\ell=p} q^{2\ell}\big(q^{-2}E_{k\delta+\alpha_1}E_{\ell\delta+\alpha_0} - E_{\ell\delta+\alpha_0}E_{k\delta+\alpha_1}\big) \\
&= (1-q^{2p+2})E_{(p+1)\delta} - (1-q^2) \sum_{k+\ell=p} q^{2\ell} E_{(p+1)\delta}
\end{aligned}
$$

$$= \left(1 - q^{2p+2} - (1-q^2)\sum_{\ell=0}^{p} q^{2\ell}\right) E_{(p+1)\delta}$$
$$= 0.$$

The result follows. □

Lemma 4.4. *Pick* $n \in \mathbb{N}$, *and assume that* (4.5), (4.6) *hold for* $n, n-1, \ldots, 1, 0$. *Then*

$$[\tilde{G}_n, E_\delta] = 0. \tag{4.8}$$

Proof. Using Lemma 4.3,

$$\begin{aligned} 0 &= (q-q^{-1})\big([W_{-n}, W_1] - [W_0, W_{n+1}]\big) \\ &= [[\tilde{G}_n, W_0]_q, W_1] - [W_0, [W_1, \tilde{G}_n]_q] \\ &= [\tilde{G}_n, [W_0, W_1]_q] \\ &= -q[\tilde{G}_n, E_\delta]. \end{aligned}$$

□

Lemma 4.5. *Pick* $n \in \mathbb{N}$, *and assume that* (4.5), (4.6) *hold for* $n, n-1, \ldots, 1, 0$. *Then*

$$[W_{-n}, W_0] = 0. \tag{4.9}$$

Proof. The commutator $[W_{-n}, W_0]$ is equal to

$$\begin{aligned} & W_{-n}W_0 - W_0W_{-n} \\ &= \sum_{k=0}^{n} \frac{E_{k\delta+\alpha_0}\tilde{G}_{n-k}W_0(-1)^k q^{3k}}{(q-q^{-1})^{2k}} - \sum_{k=0}^{n} \frac{W_0E_{k\delta+\alpha_0}\tilde{G}_{n-k}(-1)^k q^{3k}}{(q-q^{-1})^{2k}} \\ &= \sum_{k=0}^{n} \frac{E_{k\delta+\alpha_0}\big(q^{-2}W_0\tilde{G}_{n-k} + (1-q^{-2})W_{k-n}\big)(-1)^k q^{3k}}{(q-q^{-1})^{2k}} \\ &\qquad - \sum_{k=0}^{n} \frac{W_0E_{k\delta+\alpha_0}\tilde{G}_{n-k}(-1)^k q^{3k}}{(q-q^{-1})^{2k}} \\ &= \sum_{k=0}^{n} \frac{[W_0, E_{k\delta+\alpha_0}]_q\tilde{G}_{n-k}(-1)^{k-1} q^{3k-1}}{(q-q^{-1})^{2k}} + \sum_{k=0}^{n} \frac{E_{k\delta+\alpha_0}W_{k-n}(-1)^k q^{3k-1}}{(q-q^{-1})^{2k-1}} \\ &= \sum_{k=0}^{n} \frac{[W_0, E_{k\delta+\alpha_0}]_q\tilde{G}_{n-k}(-1)^{k-1} q^{3k-1}}{(q-q^{-1})^{2k}} \\ &\qquad + \sum_{k=0}^{n} \frac{E_{k\delta+\alpha_0}(-1)^k q^{3k-1}}{(q-q^{-1})^{2k-1}} \sum_{\ell=0}^{n-k} \frac{E_{\ell\delta+\alpha_0}\tilde{G}_{n-k-\ell}(-1)^\ell q^{3\ell}}{(q-q^{-1})^{2\ell}} \\ &= \sum_{p=0}^{n} \frac{[W_0, E_{p\delta+\alpha_0}]_q\tilde{G}_{n-p}(-1)^{p-1} q^{3p-1}}{(q-q^{-1})^{2p}} \\ &\qquad + \sum_{p=0}^{n} \left(\sum_{k+\ell=p} E_{k\delta+\alpha_0}E_{\ell\delta+\alpha_0}\right) \frac{\tilde{G}_{n-p}(-1)^p q^{3p-1}}{(q-q^{-1})^{2p-1}}. \end{aligned}$$

By these comments

$$[W_{-n}, W_0] = \sum_{p=0}^{n} \frac{S_p \tilde{G}_{n-p}(-1)^{p-1} q^{3p-1}}{(q-q^{-1})^{2p-1}}$$

where

$$S_p = \frac{[W_0, E_{p\delta+\alpha_0}]_q}{q-q^{-1}} - \sum_{k+\ell=p} E_{k\delta+\alpha_0} E_{\ell\delta+\alpha_0} \qquad (0 \le p \le n).$$

By (2.8) we have $S_p = 0$ for $0 \le p \le n$. The result follows. □

Lemma 4.6. *Pick $n \in \mathbb{N}$, and assume that* (4.5), (4.6) *hold for $n, n-1, \ldots, 1, 0$. Then*

$$[W_{n+1}, W_1] = 0. \tag{4.10}$$

Proof. The commutator $[W_{n+1}, W_1]$ is equal to

$$\begin{aligned}
&W_{n+1}W_1 - W_1 W_{n+1} \\
&= \sum_{k=0}^{n} \frac{E_{k\delta+\alpha_1} \tilde{G}_{n-k} W_1 (-1)^k q^k}{(q-q^{-1})^{2k}} - \sum_{k=0}^{n} \frac{W_1 E_{k\delta+\alpha_1} \tilde{G}_{n-k} (-1)^k q^k}{(q-q^{-1})^{2k}} \\
&= \sum_{k=0}^{n} \frac{E_{k\delta+\alpha_1} \big(q^2 W_1 \tilde{G}_{n-k} + (1-q^2) W_{n-k+1}\big)(-1)^k q^k}{(q-q^{-1})^{2k}} \\
&\qquad - \sum_{k=0}^{n} \frac{W_1 E_{k\delta+\alpha_1} \tilde{G}_{n-k} (-1)^k q^k}{(q-q^{-1})^{2k}} \\
&= \sum_{k=0}^{n} \frac{[E_{k\delta+\alpha_1}, W_1]_q \tilde{G}_{n-k} (-1)^k q^{k+1}}{(q-q^{-1})^{2k}} - \sum_{k=0}^{n} \frac{E_{k\delta+\alpha_1} W_{n-k+1} (-1)^k q^{k+1}}{(q-q^{-1})^{2k-1}} \\
&= \sum_{k=0}^{n} \frac{[E_{k\delta+\alpha_1}, W_1]_q \tilde{G}_{n-k} (-1)^k q^{k+1}}{(q-q^{-1})^{2k}} \\
&\qquad - \sum_{k=0}^{n} \frac{E_{k\delta+\alpha_1} (-1)^k q^{k+1}}{(q-q^{-1})^{2k-1}} \sum_{\ell=0}^{n-k} \frac{E_{\ell\delta+\alpha_1} \tilde{G}_{n-k-\ell} (-1)^\ell q^\ell}{(q-q^{-1})^{2\ell}} \\
&= \sum_{p=0}^{n} \frac{[E_{p\delta+\alpha_1}, W_1]_q \tilde{G}_{n-p} (-1)^p q^{p+1}}{(q-q^{-1})^{2p}} \\
&\qquad - \sum_{p=0}^{n} \left(\sum_{k+\ell=p} E_{k\delta+\alpha_1} E_{\ell\delta+\alpha_1} \right) \frac{\tilde{G}_{n-p} (-1)^p q^{p+1}}{(q-q^{-1})^{2p-1}}.
\end{aligned}$$

By these comments

$$[W_{n+1}, W_1] = \sum_{p=0}^{n} \frac{T_p \tilde{G}_{n-p} (-1)^p q^{p+1}}{(q-q^{-1})^{2p-1}}$$

where

$$T_p = \frac{[E_{p\delta+\alpha_1}, W_1]_q}{q-q^{-1}} - \sum_{k+\ell=p} E_{k\delta+\alpha_1} E_{\ell\delta+\alpha_1} \qquad (0 \le p \le n).$$

By (2.9) we have $T_p = 0$ for $0 \le p \le n$. The result follows. □

Proposition 4.7. *The equations* (4.5), (4.6) *hold in* $\mathcal{U}$ *for* $n \in \mathbb{N}$.

Proof. The proof is by induction on n. We assume that (4.5), (4.6) hold for $n, n-1, \dots, 1, 0$, and show that (4.5), (4.6) hold for $n+1$. Concerning (4.5),

$$\begin{aligned}
W_{n+2} &= \frac{qW_1\tilde{G}_{n+1} - q^{-1}\tilde{G}_{n+1}W_1}{q-q^{-1}} && \text{by (4.2)}\\
&= W_1\tilde{G}_{n+1} - q^{-1}\frac{[\tilde{G}_{n+1}, W_1]}{q-q^{-1}}\\
&= W_1\tilde{G}_{n+1} - \frac{[[[\tilde{G}_n, W_0]_q, W_1]_q, W_1]}{(q-q^{-1})^2(q^2-q^{-2})} && \text{by Definition 3.1(v)}\\
&= W_1\tilde{G}_{n+1} - \frac{[[W_{-n}, W_1]_q, W_1]}{(q-q^{-1})(q^2-q^{-2})} && \text{by (4.1)}\\
&= W_1\tilde{G}_{n+1} - \frac{[[W_{-n}, W_1], W_1]_q}{(q-q^{-1})(q^2-q^{-2})}\\
&= W_1\tilde{G}_{n+1} - \frac{[[W_0, W_{n+1}], W_1]_q}{(q-q^{-1})(q^2-q^{-2})} && \text{by Lemma 4.3}\\
&= W_1\tilde{G}_{n+1} - \frac{[[W_0, W_1]_q, W_{n+1}]}{(q-q^{-1})(q^2-q^{-2})} && \text{by Lemma 4.6}\\
&= W_1\tilde{G}_{n+1} + \frac{q[E_\delta, W_{n+1}]}{(q-q^{-1})(q^2-q^{-2})} && \text{by (2.3)}\\
&= W_1\tilde{G}_{n+1} + q\sum_{k=0}^{n} \frac{[E_\delta, E_{k\delta+\alpha_1}\tilde{G}_{n-k}](-1)^k q^k}{(q-q^{-1})^{2k+1}(q^2-q^{-2})} && \text{by (4.5) and induction}\\
&= W_1\tilde{G}_{n+1} + q\sum_{k=0}^{n} \frac{[E_\delta, E_{k\delta+\alpha_1}]\tilde{G}_{n-k}(-1)^k q^k}{(q-q^{-1})^{2k+1}(q^2-q^{-2})} && \text{by Lemma 4.4}\\
&= W_1\tilde{G}_{n+1} + \sum_{k=0}^{n} \frac{E_{(k+1)\delta+\alpha_1}\tilde{G}_{n-k}(-1)^{k+1} q^{k+1}}{(q-q^{-1})^{2k+2}} && \text{by (2.4)}\\
&= E_{\alpha_1}\tilde{G}_{n+1} + \sum_{k=1}^{n+1} \frac{E_{k\delta+\alpha_1}\tilde{G}_{n+1-k}(-1)^k q^k}{(q-q^{-1})^{2k}}\\
&= \sum_{k=0}^{n+1} \frac{E_{k\delta+\alpha_1}\tilde{G}_{n+1-k}(-1)^k q^k}{(q-q^{-1})^{2k}}.
\end{aligned}$$

We have shown that (4.5) holds for $n+1$. Concerning (4.6),

$$W_{-n-1} = \frac{q\tilde{G}_{n+1}W_0 - q^{-1}W_0\tilde{G}_{n+1}}{q-q^{-1}} \qquad \text{by (4.1)}$$

$$
\begin{aligned}
&= W_0\tilde{G}_{n+1} - q\frac{[W_0, \tilde{G}_{n+1}]}{q-q^{-1}} \\
&= W_0\tilde{G}_{n+1} - q^2\frac{[W_0, [W_0, [W_1, \tilde{G}_n]_q]_q]}{(q-q^{-1})^2(q^2-q^{-2})} && \text{by Definition 3.1(vi)} \\
&= W_0\tilde{G}_{n+1} - q^2\frac{[W_0, [W_0, W_{n+1}]_q]}{(q-q^{-1})(q^2-q^{-2})} && \text{by (4.2)} \\
&= W_0\tilde{G}_{n+1} - q^2\frac{[W_0, [W_0, W_{n+1}]]_q}{(q-q^{-1})(q^2-q^{-2})} \\
&= W_0\tilde{G}_{n+1} - q^2\frac{[W_0, [W_{-n}, W_1]]_q}{(q-q^{-1})(q^2-q^{-2})} && \text{by Lemma 4.3} \\
&= W_0\tilde{G}_{n+1} - q^2\frac{[W_{-n}, [W_0, W_1]_q]}{(q-q^{-1})(q^2-q^{-2})} && \text{by Lemma 4.5} \\
&= W_0\tilde{G}_{n+1} + q^3\frac{[W_{-n}, E_\delta]}{(q-q^{-1})(q^2-q^{-2})} && \text{by (2.3)} \\
&= W_0\tilde{G}_{n+1} + q^3\sum_{k=0}^{n}\frac{[E_{k\delta+\alpha_0}\tilde{G}_{n-k}, E_\delta](-1)^k q^{3k}}{(q-q^{-1})^{2k+1}(q^2-q^{-2})} && \text{by (4.6) and induction} \\
&= W_0\tilde{G}_{n+1} + q^3\sum_{k=0}^{n}\frac{[E_{k\delta+\alpha_0}, E_\delta]\tilde{G}_{n-k}(-1)^k q^{3k}}{(q-q^{-1})^{2k+1}(q^2-q^{-2})} && \text{by Lemma 4.4} \\
&= W_0\tilde{G}_{n+1} + \sum_{k=0}^{n}\frac{E_{(k+1)\delta+\alpha_0}\tilde{G}_{n-k}(-1)^{k+1} q^{3k+3}}{(q-q^{-1})^{2k+2}} && \text{by (2.4)} \\
&= E_{\alpha_0}\tilde{G}_{n+1} + \sum_{k=1}^{n+1}\frac{E_{k\delta+\alpha_0}\tilde{G}_{n+1-k}(-1)^k q^{3k}}{(q-q^{-1})^{2k}} \\
&= \sum_{k=0}^{n+1}\frac{E_{k\delta+\alpha_0}\tilde{G}_{n+1-k}(-1)^k q^{3k}}{(q-q^{-1})^{2k}}.
\end{aligned}
$$

We have shown that (4.6) holds for $n+1$. □

Lemma 4.8. *The following relations hold in $\mathcal{U}$. For $n \in \mathbb{N}$,*

$$
[W_0, W_{n+1}] = [W_{-n}, W_1], \qquad [\tilde{G}_n, E_\delta] = 0,
$$
$$
[W_{-n}, W_0] = 0, \qquad [W_{n+1}, W_1] = 0.
$$

Proof. By Lemmas 4.3 – 4.6 and Proposition 4.7. □

Lemma 4.9. *The following relations hold in $\mathcal{U}$. For $k \in \mathbb{N}$,*

(i) $[G_{k+1}, W_1]_q = [W_1, \tilde{G}_{k+1}]_q$;

(ii) $[W_0, G_{k+1}]_q = [\tilde{G}_{k+1}, W_0]_q$.

Proof. (i) We have

$$
[G_{k+1}, W_1]_q - [W_1, \tilde{G}_{k+1}]_q = \left[\tilde{G}_{k+1} + \frac{[W_1, W_{-k}]}{1-q^{-2}}, W_1\right]_q - [W_1, \tilde{G}_{k+1}]_q
$$

$$
\begin{aligned}
&= (q+q^{-1})[\tilde{G}_{k+1}, W_1] - \frac{[[W_{-k}, W_1], W_1]_q}{1-q^{-2}} \\
&= (q+q^{-1})[\tilde{G}_{k+1}, W_1] - \frac{[[W_{-k}, W_1]_q, W_1]}{1-q^{-2}} \\
&= (q+q^{-1})[\tilde{G}_{k+1}, W_1] - \frac{[[[\tilde{G}_k, W_0]_q, W_1]_q, W_1]}{(1-q^{-2})(q-q^{-1})} \\
&= 0.
\end{aligned}
$$

(ii) We have

$$
\begin{aligned}
[W_0, G_{k+1}]_q - [\tilde{G}_{k+1}, W_0]_q &= \left[W_0, \tilde{G}_{k+1} + \frac{[W_{k+1}, W_0]}{1-q^{-2}}\right]_q - [\tilde{G}_{k+1}, W_0]_q \\
&= (q+q^{-1})[W_0, \tilde{G}_{k+1}] - \frac{[W_0, [W_0, W_{k+1}]]_q}{1-q^{-2}} \\
&= (q+q^{-1})[W_0, \tilde{G}_{k+1}] - \frac{[W_0, [W_0, W_{k+1}]_q]}{1-q^{-2}} \\
&= (q+q^{-1})[W_0, \tilde{G}_{k+1}] - \frac{[W_0, [W_0, [W_1, \tilde{G}_k]_q]_q]}{(1-q^{-2})(q-q^{-1})} \\
&= 0.
\end{aligned}
$$

□

5 Generating functions

The alternating generators of $\mathcal{U}$ are displayed in (4.4). In the previous section we described how these generators are related to W_0 and W_1. Our next goal is to describe how the alternating generators are related to each other. It is convenient to use generating functions.

Definition 5.1. We define some generating functions in an indeterminate t. Referring to (4.4),

$$
G(t) = \sum_{n\in\mathbb{N}} G_n t^n, \qquad \tilde{G}(t) = \sum_{n\in\mathbb{N}} \tilde{G}_n t^n,
$$
$$
W^-(t) = \sum_{n\in\mathbb{N}} W_{-n} t^n, \qquad W^+(t) = \sum_{n\in\mathbb{N}} W_{n+1} t^n.
$$

Lemma 5.2. *For the algebra $\mathcal{U}$,*

$$
\frac{[W_0, G(t)]_q}{q-q^{-1}} = W^-(t), \qquad \frac{[\tilde{G}(t), W_0]_q}{q-q^{-1}} = W^-(t),
$$
$$
[W_0, W^-(t)] = 0, \qquad \frac{[W_0, W^+(t)]}{1-q^{-2}} = t^{-1}(\tilde{G}(t) - G(t))
$$

and

$$
\frac{[G(t), W_1]_q}{q-q^{-1}} = W^+(t), \qquad \frac{[W_1, \tilde{G}(t)]_q}{q-q^{-1}} = W^+(t),
$$
$$
[W_1, W^+(t)] = 0, \qquad \frac{[W_1, W^-(t)]}{1-q^{-2}} = t^{-1}(G(t) - \tilde{G}(t)).
$$

Proof. Use Definition 4.1 and Lemmas 4.8, 4.9. □

For the rest of this section, let s denote an indeterminate that commutes with t.

Lemma 5.3. *For the algebra $\mathcal{U}$,*

$$
\begin{aligned}
&[W^-(s), W^-(t)] = 0, \qquad [W^+(s), W^+(t)] = 0,\\
&[W^-(s), W^+(t)] + [W^+(s), W^-(t)] = 0,\\
&s[W^-(s), G(t)] + t[G(s), W^-(t)] = 0,\\
&s[W^-(s), \tilde{G}(t)] + t[\tilde{G}(s), W^-(t)] = 0,\\
&s[W^+(s), G(t)] + t[G(s), W^+(t)] = 0,\\
&s[W^+(s), \tilde{G}(t)] + t[\tilde{G}(s), W^+(t)] = 0,\\
&[G(s), G(t)] = 0, \qquad [\tilde{G}(s), \tilde{G}(t)] = 0,\\
&[\tilde{G}(s), G(t)] + [G(s), \tilde{G}(t)] = 0
\end{aligned}
$$

and also

$$
\begin{aligned}
&[W^-(s), G(t)]_q = [W^-(t), G(s)]_q, \qquad [G(s), W^+(t)]_q = [G(t), W^+(s)]_q,\\
&[\tilde{G}(s), W^-(t)]_q = [\tilde{G}(t), W^-(s)]_q, \qquad [W^+(s), \tilde{G}(t)]_q = [W^+(t), \tilde{G}(s)]_q,\\
&t^{-1}[G(s), \tilde{G}(t)] - s^{-1}[G(t), \tilde{G}(s)] = q[W^-(t), W^+(s)]_q - q[W^-(s), W^+(t)]_q,\\
&t^{-1}[\tilde{G}(s), G(t)] - s^{-1}[\tilde{G}(t), G(s)] = q[W^+(t), W^-(s)]_q - q[W^+(s), W^-(t)]_q,\\
&[G(s), \tilde{G}(t)]_q - [G(t), \tilde{G}(s)]_q = qt[W^-(t), W^+(s)] - qs[W^-(s), W^+(t)],\\
&[\tilde{G}(s), G(t)]_q - [\tilde{G}(t), G(s)]_q = qt[W^+(t), W^-(s)] - qs[W^+(s), W^-(t)].
\end{aligned}
$$

Proof. We refer to the generating functions $A(s,t), B(s,t), \ldots, S(s,t)$ from Appendix A. The present lemma asserts that for the algebra $\mathcal{U}$ these generating functions are all zero. To verify this assertion, we refer to the canonical relations in Appendix A. We will use induction with respect to the linear order

$$
\begin{aligned}
&I(s,t), M(s,t), N(s,t), A(s,t), B(s,t), Q(s,t), D(s,t), E(s,t), F(s,t),\\
&G(s,t), R(s,t), S(s,t), H(s,t), K(s,t), L(s,t), P(s,t), C(s,t), J(s,t).
\end{aligned}
$$

For each element in this linear order besides $I(s,t)$, there exists a canonical relation that expresses the given element in terms of the previous elements in the linear order. So in $\mathcal{U}$ the given element is zero, provided that in $\mathcal{U}$ every previous element is zero. Note that in $\mathcal{U}$ we have $I(s,t) = 0$ by Definition 3.1(vii). By these comments and induction, in $\mathcal{U}$ every element in the linear order is zero. We have shown that in $\mathcal{U}$ each of $A(s,t)$, $B(s,t), \ldots, S(s,t)$ is zero. □

6 The main results

In this section we present our main results, which are Theorem 6.2 and Corollary 6.5. Recall the alternating generators (4.4) for $\mathcal{U}$.

Lemma 6.1. *The following relations hold in* $\mathcal{U}$. *For* $k, \ell \in \mathbb{N}$ *we have*

$$[W_0, W_{k+1}] = [W_{-k}, W_1] = (1 - q^{-2})(\tilde{G}_{k+1} - G_{k+1}), \tag{6.1}$$

$$[W_0, G_{k+1}]_q = [\tilde{G}_{k+1}, W_0]_q = (q - q^{-1})W_{-k-1}, \tag{6.2}$$

$$[G_{k+1}, W_1]_q = [W_1, \tilde{G}_{k+1}]_q = (q - q^{-1})W_{k+2}, \tag{6.3}$$

$$[W_{-k}, W_{-\ell}] = 0, \qquad [W_{k+1}, W_{\ell+1}] = 0, \tag{6.4}$$

$$[W_{-k}, W_{\ell+1}] + [W_{k+1}, W_{-\ell}] = 0, \tag{6.5}$$

$$[W_{-k}, G_{\ell+1}] + [G_{k+1}, W_{-\ell}] = 0, \tag{6.6}$$

$$[W_{-k}, \tilde{G}_{\ell+1}] + [\tilde{G}_{k+1}, W_{-\ell}] = 0, \tag{6.7}$$

$$[W_{k+1}, G_{\ell+1}] + [G_{k+1}, W_{\ell+1}] = 0, \tag{6.8}$$

$$[W_{k+1}, \tilde{G}_{\ell+1}] + [\tilde{G}_{k+1}, W_{\ell+1}] = 0, \tag{6.9}$$

$$[G_{k+1}, G_{\ell+1}] = 0, \qquad [\tilde{G}_{k+1}, \tilde{G}_{\ell+1}] = 0, \tag{6.10}$$

$$[\tilde{G}_{k+1}, G_{\ell+1}] + [G_{k+1}, \tilde{G}_{\ell+1}] = 0 \tag{6.11}$$

and also

$$[W_{-k}, G_\ell]_q = [W_{-\ell}, G_k]_q, \qquad [G_k, W_{\ell+1}]_q = [G_\ell, W_{k+1}]_q, \tag{6.12}$$

$$[\tilde{G}_k, W_{-\ell}]_q = [\tilde{G}_\ell, W_{-k}]_q, \qquad [W_{\ell+1}, \tilde{G}_k]_q = [W_{k+1}, \tilde{G}_\ell]_q, \tag{6.13}$$

$$[G_k, \tilde{G}_{\ell+1}] - [G_\ell, \tilde{G}_{k+1}] = q[W_{-\ell}, W_{k+1}]_q - q[W_{-k}, W_{\ell+1}]_q, \tag{6.14}$$

$$[\tilde{G}_k, G_{\ell+1}] - [\tilde{G}_\ell, G_{k+1}] = q[W_{\ell+1}, W_{-k}]_q - q[W_{k+1}, W_{-\ell}]_q, \tag{6.15}$$

$$[G_{k+1}, \tilde{G}_{\ell+1}]_q - [G_{\ell+1}, \tilde{G}_{k+1}]_q = q[W_{-\ell}, W_{k+2}] - q[W_{-k}, W_{\ell+2}], \tag{6.16}$$

$$[\tilde{G}_{k+1}, G_{\ell+1}]_q - [\tilde{G}_{\ell+1}, G_{k+1}]_q = q[W_{\ell+1}, W_{-k-1}] - q[W_{k+1}, W_{-\ell-1}]. \tag{6.17}$$

Proof. The relations (6.1) – (6.3) are from Definition 4.1 and Lemmas 4.8, 4.9. The relations (6.4) – (6.17) follow from Definition 5.1 and Lemma 5.3. □

Theorem 6.2. *The algebra* $\mathcal{U}$ *has a presentation by generators*

$$\{W_{-k}\}_{k\in\mathbb{N}}, \qquad \{W_{k+1}\}_{k\in\mathbb{N}}, \qquad \{G_{k+1}\}_{k\in\mathbb{N}}, \qquad \{\tilde{G}_{k+1}\}_{k\in\mathbb{N}}$$

and the relations in Lemma 6.1.

Proof. It suffices to show that the relations in Definition 3.1 are implied by the relations in Lemma 6.1. The relation (iii) in Definition 3.1 is obtained from the equation on the left in (6.3) at $k = 0$, by eliminating G_1 using $[W_0, W_1] = (1-q^{-2})(\tilde{G}_1 - G_1)$. The relation (iv) in Definition 3.1 is obtained from the equation on the left in (6.2) at $k = 0$, by eliminating G_1 using $[W_0, W_1] = (1 - q^{-2})(\tilde{G}_1 - G_1)$. For $k \geq 1$ the relation (v) in Definition 3.1 is obtained from the equation on the left in (6.3), by eliminating G_{k+1} using $[W_{-k}, W_1] = (1 - q^{-2})(\tilde{G}_{k+1} - G_{k+1})$ and evaluating the result using $[\tilde{G}_k, W_0]_q = (q - q^{-1})W_{-k}$. For $k \geq 1$ the relation (vi) in Definition 3.1 is obtained from the equation on the left in (6.2), by eliminating G_{k+1} using $[W_0, W_{k+1}] = (1 - q^{-2})(\tilde{G}_{k+1} - G_{k+1})$ and evaluating the result using $[W_1, \tilde{G}_k]_q = (q - q^{-1})W_{k+1}$. The relation (vii) in Definition 3.1 is from (6.10). The relation (i) in Definition 3.1 is obtained from $[W_0, W_{-1}] = 0$, by eliminating W_{-1} using $[\tilde{G}_1, W_0]_q = (q - q^{-1})W_{-1}$ and evaluating the result using Definition 3.1(iv). The relation (ii) in Definition 3.1 is obtained from $[W_1, W_2] = 0$, by eliminating W_2 using $[W_1, \tilde{G}_1]_q = (q - q^{-1})W_2$ and evaluating the result using Definition 3.1(iii). □

It is apparent from the proof of Theorem 6.2 that the relations in Lemma 6.1 are redundant in the following sense.

Corollary 6.3. The relations in Lemma 6.1 are implied by the relations listed in (i) – (iii) below:

(i) (6.1) – (6.3);

(ii) (6.4) with $k = 0$ and $\ell = 1$;

(iii) the relations on the right in (6.10).

Proof. By Lemma 6.1 the relations (6.1) – (6.17) are implied by the relations in Definitions 3.1, 4.1. The relations listed in (i) – (iii) are used in the proof of Theorem 6.2 to obtain the relations in Definition 3.1. The relations listed in (i) imply the relations in Definition 4.1. The result follows. □

The relations in Lemma 6.1 first appeared in [30, Propositions 5.7, 5.10, 5.11]. It was observed in [2, Propositions 3.1, 3.2] and [5, Remark 2.5] that the relations (6.1) – (6.11) imply the relations (6.12) – (6.17). This observation motivated the following definition.

Definition 6.4 ([29, Definition 3.1]). Define the algebra $\mathcal{U}_q^+$ by generators

$$\{W_{-k}\}_{k\in\mathbb{N}}, \qquad \{W_{k+1}\}_{k\in\mathbb{N}}, \qquad \{G_{k+1}\}_{k\in\mathbb{N}}, \qquad \{\tilde{G}_{k+1}\}_{k\in\mathbb{N}}$$

and the relations (6.1) – (6.11). The algebra $\mathcal{U}_q^+$ is called the *alternating central extension of* U_q^+.

Corollary 6.5. *We have* $\mathcal{U} = \mathcal{U}_q^+$.

Proof. By Theorem 6.2, Corollary 6.3, and Definition 6.4. □

Definition 6.6. By the *compact* presentation of $\mathcal{U}_q^+$ we mean the presentation given in Definition 3.1. By the *expanded* presentation of $\mathcal{U}_q^+$ we mean the presentation given in Theorem 6.2.

Corollary 6.7. *The map* $\flat$ *from* Lemma 3.3 *is injective.*

Proof. By Corollary 6.5 and [29, Proposition 6.4]. □

7 The subalgebra of $\mathcal{U}_q^+$ generated by $\{\tilde{G}_{k+1}\}_{k\in\mathbb{N}}$

Let $\tilde{G}$ denote the subalgebra of $\mathcal{U}_q^+$ generated by $\{\tilde{G}_{k+1}\}_{k\in\mathbb{N}}$. In this section we describe $\tilde{G}$ and its relationship to $\langle W_0, W_1\rangle$.

The following notation will be useful. Let $z_1, z_2, \ldots$ denote mutually commuting indeterminates. Let $\mathbb{F}[z_1, z_2, \ldots]$ denote the algebra consisting of the polynomials in $z_1, z_2, \ldots$ that have all coefficients in $\mathbb{F}$. For notational convenience define $z_0 = 1$.

Lemma 7.1 ([29, Lemma 3.5]). *There exists an algebra homomorphism* $\mathcal{U}_q^+ \to \mathbb{F}[z_1, z_2, \ldots]$ *that sends*

$$W_{-n} \mapsto 0, \qquad W_{n+1} \mapsto 0, \qquad G_n \mapsto z_n, \qquad \tilde{G}_n \mapsto z_n$$

for $n \in \mathbb{N}$.

Proof. By Theorem 6.2 and the nature of the relations in Lemma 6.1. □

Corollary 7.2 ([29, Theorem 10.2])**.** *The generators* $\{\tilde{G}_{k+1}\}_{k\in\mathbb{N}}$ *of* $\tilde{G}$ *are algebraically independent.*

Proof. By Lemma 7.1 and since $\{z_{k+1}\}_{k\in\mathbb{N}}$ are algebraically independent. □

The following result will help us describe how $\tilde{G}$ is related to $\langle W_0, W_1\rangle$.

Lemma 7.3. *For* $n \in \mathbb{N}$,

$$\tilde{G}_n W_1 = W_1 \tilde{G}_n + \sum_{k=1}^{n} \frac{E_{k\delta+\alpha_1}\tilde{G}_{n-k}(-1)^{k+1}q^{k+1}}{(q-q^{-1})^{2k-1}}, \tag{7.1}$$

$$\tilde{G}_n W_0 = W_0 \tilde{G}_n + \sum_{k=1}^{n} \frac{E_{k\delta+\alpha_0}\tilde{G}_{n-k}(-1)^{k}q^{3k-1}}{(q-q^{-1})^{2k-1}}. \tag{7.2}$$

Proof. To obtain (7.1), eliminate W_{n+1} from (4.5) using (4.2), and solve the resulting equation for $\tilde{G}_n W_1$. To obtain (7.2), eliminate W_{-n} from (4.6) using (4.1), and solve the resulting equation for $\tilde{G}_n W_0$. □

Shortly we will describe how $\tilde{G}$ is related to $\langle W_0, W_1\rangle$. This description involves the center $\mathcal{Z}$ of $\mathcal{U}_q^+$. To prepare for this description, we have some comments about $\mathcal{Z}$. In [29, Sections 5, 6] we introduced some algebraically independent elements $Z_1, Z_2, \ldots$ that generate the algebra $\mathcal{Z}$. For notational convenience define $Z_0 = 1$. Using $\{Z_n\}_{n\in\mathbb{N}}$ we obtain a basis for $\mathcal{Z}$ that is described as follows. For $n \in \mathbb{N}$, a *partition of* n is a sequence $\lambda = \{\lambda_i\}_{i=1}^{\infty}$ of natural numbers such that $\lambda_i \geq \lambda_{i+1}$ for $i \geq 1$ and $n = \sum_{i=1}^{\infty}\lambda_i$. The set Λ_n consists of the partitions of n. Define $\Lambda = \cup_{n\in\mathbb{N}}\Lambda_n$. For $\lambda \in \Lambda$ define $Z_\lambda = \prod_{i=1}^{\infty} Z_{\lambda_i}$. The elements $\{Z_\lambda\}_{\lambda\in\Lambda}$ form a basis for the vector space $\mathcal{Z}$. Next we describe a grading for $\mathcal{Z}$. For $n \in \mathbb{N}$ let $\mathcal{Z}_n$ denote the subspace of $\mathcal{Z}$ with basis $\{Z_\lambda\}_{\lambda\in\Lambda_n}$. For example $\mathcal{Z}_0 = \mathbb{F}1$. The sum $\mathcal{Z} = \sum_{n\in\mathbb{N}}\mathcal{Z}_n$ is direct. Moreover $\mathcal{Z}_r\mathcal{Z}_s \subseteq \mathcal{Z}_{r+s}$ for $r, s \in \mathbb{N}$. By these comments the subspaces $\{\mathcal{Z}_n\}_{n\in\mathbb{N}}$ form a grading of $\mathcal{Z}$. Note that $Z_n \in \mathcal{Z}_n$ for $n \in \mathbb{N}$. Next we describe how $\mathcal{Z}$ is related to $\langle W_0, W_1\rangle$.

Lemma 7.4 ([29, Proposition 6.5])**.** *The multiplication map*

$$\begin{aligned}\langle W_0, W_1\rangle \otimes \mathcal{Z} &\to \mathcal{U}_q^+\\ w\otimes z &\mapsto wz\end{aligned}$$

is an algebra isomorphism.

For $n \in \mathbb{N}$ let $\mathcal{U}_n$ denote the image of $\langle W_0, W_1\rangle \otimes \mathcal{Z}_n$ under the multiplication map. By construction the sum $\mathcal{U}_q^+ = \sum_{n\in\mathbb{N}}\mathcal{U}_n$ is direct.

In the next two lemmas we describe how $\tilde{G}$ is related to $\mathcal{Z}$.

Lemma 7.5 ([29, Lemmas 3.6, 5.9])**.** *For* $n \in \mathbb{N}$,

$$\tilde{G}_n \in \sum_{k=0}^{n}\langle W_0, W_1\rangle Z_k, \qquad\qquad \tilde{G}_n - Z_n \in \sum_{k=0}^{n-1}\langle W_0, W_1\rangle Z_k.$$

For $\lambda \in \Lambda$ define $\tilde{G}_\lambda = \prod_{i=1}^{\infty} \tilde{G}_{\lambda_i}$. By Corollary 7.2 the elements $\{\tilde{G}_\lambda\}_{\lambda\in\Lambda}$ form a basis for the vector space $\tilde{G}$.

Lemma 7.6. *For $n \in \mathbb{N}$ and $\lambda \in \Lambda_n$,*

$$\tilde{G}_\lambda \in \sum_{k=0}^{n} \mathcal{U}_k, \qquad\qquad \tilde{G}_\lambda - Z_\lambda \in \sum_{k=0}^{n-1} \mathcal{U}_k.$$

Proof. By Lemma 7.5 and our comments above Lemma 7.4 about the grading of $\mathcal{Z}$. □

Next we describe how $\tilde{G}$ is related to $\langle W_0, W_1\rangle$.

Proposition 7.7. *The multiplication map*

$$\begin{aligned} \langle W_0, W_1\rangle \otimes \tilde{G} &\to \mathcal{U}_q^+ \\ w \otimes g &\mapsto wg \end{aligned}$$

is an isomorphism of vector spaces.

Proof. The multiplication map is $\mathbb{F}$-linear. The multiplication map is surjective by Lemma 7.3 and since $\mathcal{U}_q^+$ is generated by W_0, W_1, $\tilde{G}$. We now show that the multiplicaton map is injective. Consider a vector $v \in \langle W_0, W_1\rangle \otimes \tilde{G}$ that is sent to zero by the multiplication map. We show that $v = 0$. Write $v = \sum_{\lambda\in\Lambda} a_\lambda \otimes \tilde{G}_\lambda$, where $a_\lambda \in \langle W_0, W_1\rangle$ for $\lambda \in \Lambda$ and $a_\lambda = 0$ for all but finitely many $\lambda \in \Lambda$. To show that $v = 0$, we must show that $a_\lambda = 0$ for all $\lambda \in \Lambda$. Suppose that there exists $\lambda \in \Lambda$ such that $a_\lambda \neq 0$. Let C denote the set of natural numbers m such that Λ_m contains a partition λ with $a_\lambda \neq 0$. The set C is nonempty and finite. Let n denote the maximal element of C. By construction $\sum_{\lambda\in\Lambda_n} a_\lambda \otimes Z_\lambda$ is nonzero. By Lemma 7.4,

$$\sum_{\lambda\in\Lambda_n} a_\lambda Z_\lambda \neq 0. \tag{7.3}$$

By construction

$$0 = \sum_{\lambda\in\Lambda} a_\lambda \tilde{G}_\lambda = \sum_{k=0}^{n} \sum_{\lambda\in\Lambda_k} a_\lambda \tilde{G}_\lambda = \sum_{\lambda\in\Lambda_n} a_\lambda \tilde{G}_\lambda + \sum_{k=0}^{n-1} \sum_{\lambda\in\Lambda_k} a_\lambda \tilde{G}_\lambda. \tag{7.4}$$

Using (7.4),

$$\sum_{\lambda\in\Lambda_n} a_\lambda Z_\lambda = \sum_{\lambda\in\Lambda_n} a_\lambda (Z_\lambda - \tilde{G}_\lambda) - \sum_{k=0}^{n-1} \sum_{\lambda\in\Lambda_k} a_\lambda \tilde{G}_\lambda. \tag{7.5}$$

The left-hand side of (7.5) is contained in $\mathcal{U}_n$. By Lemma 7.6 the right-hand side of (7.5) is contained in $\sum_{k=0}^{n-1} \mathcal{U}_k$. The subspaces $\mathcal{U}_n$ and $\sum_{k=0}^{n-1} \mathcal{U}_k$ have zero intersection because the sum $\sum_{k=0}^{n} \mathcal{U}_k$ is direct. This contradicts (7.3), so $a_\lambda = 0$ for $\lambda \in \Lambda$. Consequently $v = 0$, as desired. We have shown that the multiplication map is injective. By the above comments the multiplication map is an isomorphism of vector spaces. □

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Appendix A

Recall the algebra $\mathcal{U}$ from Definition 3.1. In this appendix we list some relations that hold in $\mathcal{U}$. We will define an algebra $\mathcal{U}^\vee$ that is a homomorphic preimage of $\mathcal{U}$. All the results in this appendix are about $\mathcal{U}^\vee$.

Define the algebra $\mathcal{U}^\vee$ by generators

$$\{W_{-k}\}_{k\in\mathbb{N}}, \qquad \{W_{k+1}\}_{k\in\mathbb{N}}, \qquad \{G_{k+1}\}_{k\in\mathbb{N}}, \qquad \{\tilde{G}_{k+1}\}_{k\in\mathbb{N}}$$

and the following relations. For $k \in \mathbb{N}$,

$$[W_0, W_{k+1}] = [W_{-k}, W_1] = (1-q^{-2})(\tilde{G}_{k+1} - G_{k+1}), \tag{A.1}$$

$$[W_0, G_{k+1}]_q = [\tilde{G}_{k+1}, W_0]_q = (q-q^{-1})W_{-k-1}, \tag{A.2}$$

$$[G_{k+1}, W_1]_q = [W_1, \tilde{G}_{k+1}]_q = (q-q^{-1})W_{k+2}, \tag{A.3}$$

$$[W_0, W_{-k}] = 0, \qquad [W_1, W_{k+1}] = 0. \tag{A.4}$$

For notational convenience, define $G_0 = 1$ and $\tilde{G}_0 = 1$.

For $\mathcal{U}^\vee$ we define the generating functions $W^-(t)$, $W^+(t)$, $G(t)$, $\tilde{G}(t)$ as in Definition 5.1. In terms of these generating functions, the relations (A.1) – (A.4) become the relations in Lemma 5.2. Let s denote an indeterminate that commutes with t. Define

$$\begin{aligned}
A(s,t) &= [W^-(s), W^-(t)],\\
B(s,t) &= [W^+(s), W^+(t)],\\
C(s,t) &= [W^-(s), W^+(t)] + [W^+(s), W^-(t)],\\
D(s,t) &= s[W^-(s), G(t)] + t[G(s), W^-(t)],\\
E(s,t) &= s[W^-(s), \tilde{G}(t)] + t[\tilde{G}(s), W^-(t)],\\
F(s,t) &= s[W^+(s), G(t)] + t[G(s), W^+(t)],\\
G(s,t) &= s[W^+(s), \tilde{G}(t)] + t[\tilde{G}(s), W^+(t)],\\
H(s,t) &= [G(s), G(t)],\\
I(s,t) &= [\tilde{G}(s), \tilde{G}(t)],\\
J(s,t) &= [\tilde{G}(s), G(t)] + [G(s), \tilde{G}(t)]
\end{aligned}$$

and also

$$\begin{aligned}
K(s,t) &= [W^-(s), G(t)]_q - [W^-(t), G(s)]_q,\\
L(s,t) &= [G(s), W^+(t)]_q - [G(t), W^+(s)]_q,\\
M(s,t) &= [\tilde{G}(s), W^-(t)]_q - [\tilde{G}(t), W^-(s)]_q,\\
N(s,t) &= [W^+(s), \tilde{G}(t)]_q - [W^+(t), \tilde{G}(s)]_q,\\
P(s,t) &= t^{-1}[G(s), \tilde{G}(t)] - s^{-1}[G(t), \tilde{G}(s)] - q[W^-(t), W^+(s)]_q + q[W^-(s), W^+(t)]_q,\\
Q(s,t) &= t^{-1}[\tilde{G}(s), G(t)] - s^{-1}[\tilde{G}(t), G(s)] - q[W^+(t), W^-(s)]_q + q[W^+(s), W^-(t)]_q,\\
R(s,t) &= [G(s), \tilde{G}(t)]_q - [G(t), \tilde{G}(s)]_q - qt[W^-(t), W^+(s)] + qs[W^-(s), W^+(t)],\\
S(s,t) &= [\tilde{G}(s), G(t)]_q - [\tilde{G}(t), G(s)]_q - qt[W^+(t), W^-(s)] + qs[W^+(s), W^-(t)].
\end{aligned}$$

By linear algebra,

$$C(s,t) = \frac{(q+q^{-1})(P(s,t)+Q(s,t)) - (s^{-1}+t^{-1})(R(s,t)+S(s,t))}{(q^2-s^{-1}t)(q^2-st^{-1})q^{-1}}, \tag{A.5}$$

$$J(s,t) = \frac{(q+q^{-1})(R(s,t)+S(s,t)) - (s+t)(P(s,t)+Q(s,t))}{(q^2-s^{-1}t)(q^2-st^{-1})q^{-2}}. \tag{A.6}$$

Using Lemma 5.2 we routinely obtain

$$[W_0, A(s,t)] = 0, \qquad \frac{[W_0, B(s,t)]}{1-q^{-2}} = \frac{G(s,t)-F(s,t)}{st},$$

$$\frac{[W_0, C(s,t)]}{1-q^{-2}} = \frac{E(s,t)-D(s,t)}{st}, \qquad \frac{[W_0, D(s,t)]_q}{q-q^{-1}} = (s+t)A(s,t),$$

$$\frac{[E(s,t), W_0]_q}{q-q^{-1}} = (s+t)A(s,t), \qquad \frac{[W_0, F(s,t)]_q}{1-q^{-2}} = S(s,t) - (q+q^{-1})H(s,t),$$

$$\frac{[G(s,t), W_0]_q}{1-q^{-2}} = S(s,t) - (q+q^{-1})I(s,t), \qquad \frac{[W_0, H(s,t)]_{q^2}}{q-q^{-1}} = K(s,t),$$

$$\frac{[I(s,t), W_0]_{q^2}}{q-q^{-1}} = M(s,t), \qquad \frac{[W_0, J(s,t)]}{q-q^{-1}} = M(s,t) - K(s,t)$$

and

$$\frac{[W_0, K(s,t)]_q}{q^2-q^{-2}} = A(s,t), \qquad \frac{[W_0, L(s,t)]_q}{q-q^{-1}} = P(s,t) - (s^{-1}+t^{-1})H(s,t),$$

$$\frac{[M(s,t), W_0]_q}{q^2-q^{-2}} = A(s,t), \qquad \frac{[N(s,t), W_0]_q}{q-q^{-1}} = Q(s,t) - (s^{-1}+t^{-1})I(s,t),$$

$$\frac{[P(s,t), W_0]}{q-q^{-1}} = (s^{-1}+t^{-1})K(s,t) - (q+q^{-1})s^{-1}t^{-1}E(s,t),$$

$$\frac{[W_0, Q(s,t)]}{q-q^{-1}} = (s^{-1}+t^{-1})M(s,t) - (q+q^{-1})s^{-1}t^{-1}D(s,t),$$

$$\frac{[W_0, R(s,t)]}{q-q^{-1}} = (s^{-1}+t^{-1})(E(s,t)-D(s,t)),$$

$$\frac{[W_0, S(s,t)]}{q^2-q^{-2}} = M(s,t) - K(s,t)$$

and

$$\frac{[W_1, A(s,t)]}{1-q^{-2}} = \frac{D(s,t)-E(s,t)}{st}, \qquad [W_1, B(s,t)] = 0,$$

$$\frac{[W_1, C(s,t)]}{1-q^{-2}} = \frac{F(s,t)-G(s,t)}{st}, \qquad \frac{[D(s,t), W_1]_q}{1-q^{-2}} = R(s,t) - (q+q^{-1})H(s,t),$$

$$\frac{[W_1, E(s,t)]_q}{1-q^{-2}} = R(s,t) - (q+q^{-1})I(s,t), \qquad \frac{[F(s,t), W_1]_q}{q-q^{-1}} = (s+t)B(s,t),$$

$$\frac{[W_1, G(s,t)]_q}{q-q^{-1}} = (s+t)B(s,t), \qquad \frac{[H(s,t), W_1]_{q^2}}{q-q^{-1}} = L(s,t),$$

$$\frac{[W_1, I(s,t)]_{q^2}}{q-q^{-1}} = N(s,t), \qquad \frac{[W_1, J(s,t)]}{q-q^{-1}} = L(s,t) - N(s,t)$$

and

$$\frac{[K(s,t),W_1]_q}{q-q^{-1}} = P(s,t) - (s^{-1}+t^{-1})H(s,t), \qquad \frac{[L(s,t),W_1]_q}{q^2-q^{-2}} = B(s,t),$$
$$\frac{[W_1,M(s,t)]_q}{q-q^{-1}} = Q(s,t) - (s^{-1}+t^{-1})I(s,t), \qquad \frac{[W_1,N(s,t)]_q}{q^2-q^{-2}} = B(s,t),$$
$$\frac{[W_1,P(s,t)]}{q-q^{-1}} = (s^{-1}+t^{-1})L(s,t) - (q+q^{-1})s^{-1}t^{-1}G(s,t),$$
$$\frac{[Q(s,t),W_1]}{q-q^{-1}} = (s^{-1}+t^{-1})N(s,t) - (q+q^{-1})s^{-1}t^{-1}F(s,t),$$
$$\frac{[W_1,R(s,t)]}{q^2-q^{-2}} = L(s,t) - N(s,t), \qquad \frac{[W_1,S(s,t)]}{q-q^{-1}} = (s^{-1}+t^{-1})(F(s,t)-G(s,t)).$$

We just listed 38 relations, including (A.5), (A.6). These 38 relations are called *canonical*.

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The adjacency dimension of graphs*

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Abstract

It is known that the problem of computing the adjacency dimension of a graph is NP-hard. This suggests finding the adjacency dimension for special classes of graphs or obtaining good bounds on this invariant. In this work we obtain general bounds on the adjacency dimension of a graph G in terms of known parameters of G. We discuss the tightness of these bounds and, for some particular classes of graphs, we obtain closed formulae. In particular, we show the close relationships that exist between the adjacency dimension and other parameters, like the domination number, the location-domination number, the 2-domination number, the independent 2-domination number, the vertex cover number, the independence number and the super domination number.

Keywords: Adjacency dimension, metric dimension, location-domination number, independence number, super domination number.

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1 Introduction

The metric dimension of a general metric space was introduced in 1953 by Blumenthal [2, p. 95]. A *metric generator* for a metric space (X, d) is a set $S \subseteq X$ of points in the space with the property that every point of X is uniquely determined by the distances from the elements of S, *i.e.*, $S \subseteq X$ is a metric generator for X if for any pair of distinct points $x, x' \in X$ there exists $s \in S$ such that $d(x, s) \neq d(x', s)$. A metric generator of minimum cardinality in X is called a *metric basis*, and its cardinality, which is denoted by $\dim(X)$, is called the *metric dimension* of X. The notion of metric dimension of a graph was introduced by Slater in [23], where the metric generators were called *locating sets*. Harary and Melter [11] independently introduced the same concept, where metric generators were called *resolving sets*. Given a simple and connected graph $G = (V, E)$, we consider the function $d\colon V \times V \to \mathbb{N} \cup \{0\}$, where $d(x, y)$ is the length of a shortest path in G between u and v and $\mathbb{N}$ is the set of positive integers. Since (V, d) is a metric space, a metric generator for a graph $G = (V, E)$ is simply a metric generator for the metric space (V, d) and we will use the notation $\dim(G)$ instead of $\dim(V)$ for the metric dimension of G.

Several variations of metric generators have been introduced and studied, namely, resolving dominating sets [3], locating-dominating sets [24, 25], independent resolving sets [5], local metric sets [18], strong resolving sets [22], adjacency generators [15, 16], k-adjacency generators [6], k-metric generators [1, 7, 8], simultaneous metric generators [21] etc. In this article, we are interested in the study of adjacency generators.

The notion of adjacency generator was introduced by Jannesari and Omoomi in [16] as a tool to study the metric dimension of lexicographic product graphs. This concept has been studied further by Fernau and Rodríguez-Velázquez in [9, 10] where they showed that the (local) metric dimension of the corona product of a graph of order n and some non-trivial graph H equals n times the (local) adjacency dimension of H. As a consequence of this strong relation they showed that the problem of computing the adjacency dimension is NP-hard. This suggests finding the adjacency dimension for special classes of graphs or obtaining good bounds on this invariant. In this work we obtain general bounds on the adjacency dimension of a graph G in terms of known parameters of G, while for some particular cases we obtain closed formulae.

In order to introduce the concept of adjacency generator for a graph $G = (V, E)$, we define the distance function $d_2\colon V \times V \to \mathbb{N} \cup \{0\}$, where

$$d_2(x, y) = \min\{d(x, y), 2\}.$$

An *adjacency generator* for a graph $G = (V, E)$ is a metric generator for the metric space (V, d_2). Hence, the *adjacency dimension* of $G = (V, E)$, denoted by $\operatorname{adim}(G)$, equals the metric dimension of (V, d_2).

Notice that $S \subseteq V$ is an adjacency generator for $G = (V, E)$ if for every pair of vertices $x, y \in V \setminus S$ there exists $s \in S$ which is adjacent to exactly one of these two vertices x and

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y. Therefore, S is an adjacency generator for G if and only if S is an adjacency generator for its complement $\overline{G}$. Consequently,

$$\mathrm{adim}(G) = \mathrm{adim}(\overline{G}). \tag{1.1}$$

From the definition of adjacency and metric bases, we deduce that S is an adjacency basis of a graph G of diameter at most two if and only if S is a metric basis of G. In these cases, $\mathrm{adim}(G) = \dim(G)$. The reader is referred to [6, 10, 15, 16, 20] for known results on the adjacency dimension.

The paper is organized as follows. Section 2 is devoted to study the variation of the adjacency dimension of a graph by removing a set of edges. In particular, we wonder how far can decrease the adjacency dimension by removing edges from a complete graph and we obtain a lower bound on the adjacency dimension of any graph in terms of the order. In Section 3 we show the close relationships that exist between the adjacency dimension and other parameters, like the domination number, the location-domination number, the 2-domination number, the independent 2-domination number, the vertex cover number, the independence number and the super domination number.

We will use the notation K_n, $K_{r,n-r}$, C_n, P_n and N_n for complete graphs, complete bipartite graphs, cycle graphs, path graphs and empty graphs of order n, respectively. We use the notation $u \sim v$ if u and v are adjacent vertices and $G \cong H$ if G and H are isomorphic graphs. For a vertex v of a graph G, $N(v)$ will denote the set of neighbours or *open neighborhood* of v in G, *i.e.*, $N(v) = \{u \in V(G) : u \sim v\}$. The *closed neighborhood*, denoted by $N[v]$, equals $N(v) \cup \{v\}$. We also define $\deg(v) = |N(v)|$ as the degree of $v \in V(G)$, as well as, $\delta = \min_{v \in V(G)}\{\deg(v)\}$ and $\Delta = \max_{v \in V(G)}\{\deg(v)\}$. For the remainder of the paper, definitions will be introduced whenever a concept is needed.

2 The effect of removing edges and bounds in terms of the order

The following theorem is an important tool to derive some of our results.

Theorem 2.1 ([16]). *Let G be a graph of order n. Then the following statements hold.*

(i) $\mathrm{adim}(G) = 1$ *if and only if* $n \in \{1, 2, 3\}$, $G \ncong K_3$ *and* $G \ncong N_3$.

(ii) $\mathrm{adim}(G) = n - 1$ *if and only if* $G \cong K_n$ *or* $G \cong N_n$.

(iii) *If* $n \geq 3$ *and* $t \in \{1, \ldots, n-1\}$, *then* $\mathrm{adim}(K_{t,n-t}) = n - 2$.

(iv) *If* $n \geq 4$, *then* $\mathrm{adim}(P_n) = \mathrm{adim}(C_n) = \left\lfloor \frac{2n+2}{5} \right\rfloor$.

In this section we show the effect, on the adjacency dimension, of an operation which removes a set of edges from a graph. Given a non-empty graph $G = (V, E)$ and an edge $e \in E$ we denote by $G - e = (V, E \setminus \{e\})$ the subgraph obtained by removing the edge e from G. In general, given a set of edges $E_k = \{e_1, \ldots, e_k\} \subseteq E$ we denote by $G - E_k = (V, E \setminus E_k)$ the subgraph obtained by removing the k edges in E_k from G. By analogy we define the supergraphs $G + e = (V, E \cup \{e\})$ and $G + E_k = (V, E \cup E_k)$, where $\{e\}$ and E_k are sets of edges of the complement of G.

Theorem 2.2. *Let $G = (V, E)$ be a non-empty graph. For any set $E_k = \{e_1, \ldots, e_k\} \subseteq E$,*

$$\mathrm{adim}(G) - k \leq \mathrm{adim}(G - E_k) \leq \mathrm{adim}(G) + k.$$

Proof. Since $(G - E_{k-1}) - e_k = G - E_k$, it is enough to prove that, for any $e \in E$,

$$\operatorname{adim}(G) - 1 \leq \operatorname{adim}(G - e) \leq \operatorname{adim}(G) + 1.$$

Let S be an adjacency basis of $G - e$, where $e = xy$. Since $S \cup \{y\}$ is an adjacency generator for G, we have that $\operatorname{adim}(G) \leq |S \cup \{y\}| \leq |S| + 1 = \operatorname{adim}(G - e) + 1$. Hence, $\operatorname{adim}(G) - 1 \leq \operatorname{adim}(G - e)$.

Finally, let us observe that $\operatorname{adim}(G - e) = \operatorname{adim}(\overline{G - e}) = \operatorname{adim}(\overline{G} + e) \leq \operatorname{adim}((\overline{G} + e) - e) + 1 = \operatorname{adim}(\overline{G}) + 1 = \operatorname{adim}(G) + 1$. Therefore, the result follows. □

Since $\operatorname{adim}(G - E_k) = \operatorname{adim}(\overline{G - E_k}) = \operatorname{adim}(\overline{G} + E_k)$, we conclude that $\operatorname{adim}(G - E_k) = \operatorname{adim}(G) - k$ if and only if the graph $H = \overline{G} + E_k$ satisfies $\operatorname{adim}(H - E_k) = \operatorname{adim}(H) + k$. Therefore, in order to show that the inequalities above are tight, we only need to consider one of them. For instance, $\operatorname{adim}(K_n - e) = n - 2 = \operatorname{adim}(K_n) - 1$. With the aim of showing a more general example, let us consider s stars $H_i \cong K_{1,r}$, $r \geq 4$, such that v_i is the center and $u_{i_1}, \ldots, u_{i_r}$ are the leaves of H_i, for $i \in \{1, \ldots, s\}$. Let $e_i = u_{i_1}u_{i_2}$, $G_i = H_i + e_i$ and $M = \{v_iv_{i+1} : 1 \leq i < s\}$, and define $G_{r,s} = (V, E)$, where $V = \bigcup_{i=1}^{s} V(G_i)$ and $E = (\bigcup_{i=1}^{s} E(G_i)) \cup M$. It is readily seen that $\operatorname{adim}(G_{r,s}) = s(r-1) - 1$, while for any $k \leq s$ and $E_k = \{e_1, \ldots, e_k\}$, $\operatorname{adim}(G_{r,s} - E_k) = s(r-1) - 1 + k = \operatorname{adim}(G_{r,s}) + k$.

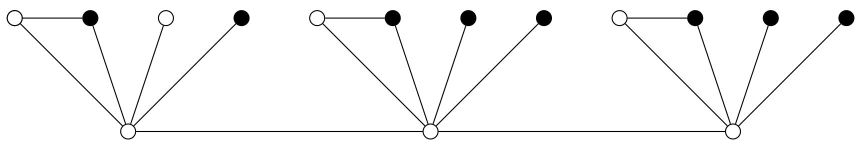

Figure 1: The set of black-colored vertices is an adjacency basis of $G_{4,3}$.

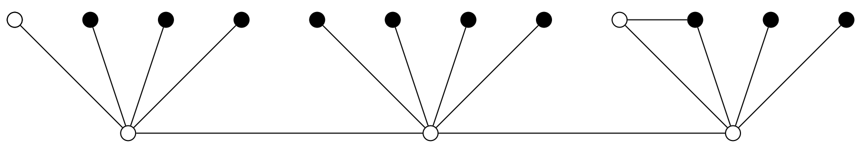

Figure 2: The set of black-colored vertices is an adjacency basis of $G_{4,3} - E_2$.

Figure 1 shows the graph $G_{4,3}$, while Figure 2 shows the graph $G_{4,3} - E_2$. In this case, $\operatorname{adim}(G_{4,3} - E_2) = 10 = \operatorname{adim}(G_{4,3}) + 2$.

All graphs of order n are obtained by successive elimination of edges from a complete graph (or by successive addition of edges to an empty graph). We know from Theorem 2.1 that for any graph G of order n, $\operatorname{adim}(G) \leq n - 1$ and the equality holds if and only if $G \cong K_n$ or $G \cong N_n$. Hence, by Theorem 2.2 we conclude that $\operatorname{adim}(K_n - e) = n - 2$ for every $e \in E(K_n)$. Now we wonder how far can decrease the adjacency dimension by removing edges from K_n, *i.e.*, which is the lower bound for the adjacency dimension in terms of the order of the graph. This problem is addressed in Propositions 2.3 and 2.4. Before stating it we need to introduce the following notation.

Given a positive integer s, let $\mathcal{G}'$ be the family of all graphs of order s and $\mathcal{G}''$ the family of all graphs of order 2^s. We can assume that the graphs in $\mathcal{G}'$ are defined on

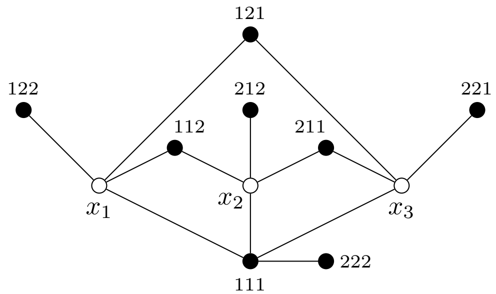

Figure 3: A graph $G \in \mathcal{G}_3$ constructed from $G' \cong N_3 \in \mathcal{G}'$ and $G'' \cong (K_2 \cup N_6) \in \mathcal{G}''$.

$S = \{x_1, \ldots, x_s\}$ and the graphs in $\mathcal{G}''$ are defined on the set $\{1,2\}^s$ of binary words of length s. Let $\mathcal{G}_s$ be the family of graphs constructed from $\mathcal{G}'$ and $\mathcal{G}''$ as follows. We say that $G \in \mathcal{G}_s$ if and only if there exist $G' \in \mathcal{G}'$ and $G'' \in \mathcal{G}''$ such that $V(G) = V(G') \cup V(G'')$ and $E(G) = E(G') \cup E(G'') \cup E^*$, where E^* is the set of edges connecting vertices of G' with vertices of G'' in such a way that x_i is adjacent to $y \in \{1,2\}^s$ if and only if the i-th letter of y is 1. Notice that S is an adjacency generator for every $G \in \mathcal{G}_s$. Figure 3 shows a graph $G \in \mathcal{G}_3$ constructed from $G' \cong N_3 \in \mathcal{G}'$ and $G'' \cong (K_2 \cup N_6) \in \mathcal{G}''$.

The following inequality appeared recently in [15], but we characterize here all graphs satisfying the equality.

Proposition 2.3. *For any graph G of order n,*

$$2^{\mathrm{adim}(G)} + \mathrm{adim}(G) \geq n. \tag{2.1}$$

Furthermore, a graph G of order $n = 2^s + s$ satisfies $\mathrm{adim}(G) = s$ if and only if $G \in \mathcal{G}_s$.

Proof. As we mentioned before, the inequality was proved in [15]. By definition of $\mathcal{G}_s$, if $G \in \mathcal{G}_s$, then $\{x_1, \ldots, x_s\}$ is an adjacency generator. Now, if $\mathrm{adim}(G) = r < s$, then Equation (2.1) leads to $n = s + 2^s > r + 2^r \geq n$, which is a contradiction. Therefore, $G \in \mathcal{G}_s$ leads to $\mathrm{adim}(G) = s$.

Conversely, suppose that G has order $n = 2^s + s$ and $\mathrm{adim}(G) = s$. In this case, for any adjacency basis $S = \{x_1, \ldots, x_s\}$ of G, the function $\psi\colon V(G)\backslash S \longrightarrow \{1,2\}^s$ defined by

$$\psi(x) = (d_2(x, x_1), \ldots, d_2(x, x_s)),$$

is bijective, as it is injective and $|V(G)\backslash S| = 2^s$. Hence, taking $G' \in \mathcal{G}'$ as the subgraph of G induced by S, $G'' \in \mathcal{G}''$ as the subgraph of G induced by $V(G) \setminus S$ and E^* as the set of edges connecting vertices in S with vertices in $V(G)\backslash S$, we can conclude that $G \in \mathcal{G}_s$. □

Proposition 2.4. *For any graph G of order n,* $\mathrm{adim}(G) \geq \left\lceil \frac{\ln(\frac{2n}{3})}{\ln(2)} \right\rceil$.

Proof. If G is a graph with order n and $\mathrm{adim}(G) = k$, since

$$n \leq 2^k + k \leq 2^k + 2^{k-1} = 2^k\left(1 + \frac{1}{2}\right) = 2^k\left(\frac{3}{2}\right),$$

we conclude that $k \geq \frac{\ln(\frac{2n}{3})}{\ln(2)}$. □

The bound above is tight. It is achieved, for instance, for the family $\mathcal{G}_s$ of graphs constructed prior to Proposition 2.3. These graphs have order $n = s + 2^s$ and metric dimension s. To check the tightness of the bound we only need to observe that $\frac{2(s+2^s)}{3} > 2^{s-1}$, for every positive integer s. Examples of graphs of small order for which the bound is achieved are the path P_3, the cycles C_r $(4 \leq r \leq 6)$, and the cube $Q_3 = K_2 \square K_2 \square K_2$, as $\operatorname{adim}(P_3) = 1$, $\operatorname{adim}(C_4) = \operatorname{adim}(C_5) = \operatorname{adim}(C_6) = 2$ and $\operatorname{adim}(Q_3) = 3$.

By Theorem 2.1, for any non-complete and non-empty graph of order n, $\operatorname{adim}(G) \leq n - 2$. The characterization for graphs G such that $\operatorname{adim}(G) = n - 2$ appeared recently in [15].

Theorem 2.5. *Let G be a connected graph of order $n \geq 5$. Then $\operatorname{adim}(G) = n - 2$ if and only if one of the following conditions holds.*

(i) $G \cong K_{t,n-t}$, *for some* $t \in \{1, \dots, n-1\}$.

(ii) $G \cong K_{n-t} + N_t$, *for some* $t \in \{2, \dots, n-2\}$.

(iii) $G \cong (K_1 \cup K_t) + K_{n-t-1}$, *for some* $t \in \{2, \dots, n-2\}$.

We conclude this section with a characterization of all graphs G satisfying that $\operatorname{adim}(G) = 2$, which also appeared in [15].

Theorem 2.6. *Let G be a connected graph of order n. Then $\operatorname{adim}(G) = 2$ if and only if one of the following conditions holds for G (or $\overline{G}$).*

(a) $G \cong K_3$.

(b) $n = 4$ *and* $G \ncong K_4$.

(c) $n = 5$ *and* $G \ncong K_5$, $G \ncong K_{t,5-t}$ *for* $t \in \{1, \dots, 4\}$, $G \ncong K_{5-t} + N_t$ *and* $G \ncong (K_1 \cup K_t) + K_{4-t}$ *for* $t \in \{2, 3\}$.

(d) $n = 6$ *and* $G \in \mathcal{G}_2$.

3 Relationship between the adjacency dimension and other parameters

A set $D \subseteq V(G)$ is a *dominating set* of G if $N(x) \cap D \neq \emptyset$ for every vertex $x \in V(G) \setminus D$. The *domination number*, $\gamma(G)$, is the minimum cardinality among all dominating sets of G. A dominating set of cardinality $\gamma(G)$ is called a $\gamma(G)$-set. The reader is referred to the books [12, 13] on the domination theory.

The following result is immediate from Equation (1.1) and the fact that at most one vertex of G is not dominated by the vertices in an adjacency generator of G.

Remark 3.1 ([10])**.** For any graph G,

$$\operatorname{adim}(G) \geq \max\{\gamma(G), \gamma(\overline{G})\} - 1.$$

The bound above is tight. For instance, it is attained by the corona graphs $K_r \odot K_1$, $r \geq 3$, as $\operatorname{adim}(K_r \odot K_1) = r - 1$ and $\gamma(K_r \odot K_1) = r$. Another example is any graph $G \in \mathcal{G}$ with $\gamma(G) = s + 1$. A particular case is shown in Figure 3.

A *locating-dominating* set is a dominating set D that locates/distinguishes all the vertices in the sense that every vertex not in D is uniquely determined by its neighbourhood in D, i.e., $N(u) \cap D \neq N(v) \cap D$ for every pair of vertices $u, v \in V(G) \setminus D$. The *location-domination number* of G, denoted $\lambda(G)$, is the minimum cardinality among all locating-dominating sets in G. A locating-dominating set of cardinality $\lambda(G)$ is called a $\lambda(G)$-set. The concept of a locating-dominating set was introduced and first studied by Slater [24, 25] and studied, for instance, in [4, 14, 19] and elsewhere.

Since every locating-dominating set is an adjacency generator and any adjacency basis S dominates at least all but one vertex in $V(G) \setminus S$, we deduce the following remark.

Remark 3.2. For any graph G,

$$\operatorname{adim}(G) \leq \lambda(G) \leq \operatorname{adim}(G) + 1.$$

Furthermore, $\lambda(G) = \operatorname{adim}(G) + 1$ if and only if no adjacency basis of G is a dominating set.

In general, for non-connected graphs we can state the following remark.

Remark 3.3. Let $\{G_1, \ldots, G_k\}$ be the set of components of a graph G. If there exists at least one component where no adjacency basis is a dominating set, then

$$\operatorname{adim}(G) = -1 + \sum_{i=1}^{k} \lambda(G_i).$$

Otherwise,

$$\operatorname{adim}(G) = \sum_{i=1}^{k} \lambda(G_i) = \sum_{i=1}^{k} \operatorname{adim}(G_i).$$

Furthermore, if there are exactly $t \geq 1$ components where no adjacency basis is a dominating set, then

$$\operatorname{adim}(G) = t - 1 + \sum_{i=1}^{k} \operatorname{adim}(G_i).$$

According to the two remarks above, tight bounds on $\operatorname{adim}(G)$ impose good bounds on $\lambda(G)$. In any case, the problem of obtaining the location-domination number of a graph G from the adjacency dimension of G, forces us to know whether G has dominating basis or not. Therefore, we can state the following open problem.

Problem 3.4. Characterize the graphs where no adjacency basis is a dominating set.

In order to show some families of graphs where every adjacency basis is a dominating set, we proceed to state the following lemma obtained previously in [20].

Lemma 3.5 ([20])**.** *Let G be a connected graph. If has diameter $\mathcal{D} \geq 6$, or $G \cong C_n$ with $n \geq 7$, or G is a graph of girth* $\mathsf{g} \geq 5$ *and minimum degree $\delta \geq 3$, then for every adjacency generator B for G and every $v \in V(G)$, $B \nsubseteq N(v)$.*

Theorem 3.6. *Let G be a connected graph. If G has diameter $\mathcal{D} \geq 6$, or $G \cong C_n$ with $n \geq 7$, or G is a graph of girth* $\mathsf{g} \geq 5$ *and minimum degree $\delta \geq 3$, then*

$$\operatorname{adim}(G) = \lambda(\overline{G}).$$

Proof. Let G be a graph satisfying the hypothesis and let S be an adjacency basis of G. By Lemma 3.5 we deduce that S is a dominating set of $\overline{G}$ and, since S is an adjacency basis of $\overline{G}$, we can conclude that S is a locating-dominating set of $\overline{G}$. Therefore, $\operatorname{adim}(G) = \lambda(\overline{G})$, as required. □

Theorem 3.7. *Let G be a graph of order n and maximum degree Δ. If $\Delta \ln(2) < \ln\left(\frac{2n}{3}\right)$, then $\operatorname{adim}(G) = \lambda(\overline{G})$.*

Proof. Let S be an adjacency basis of G. If $\Delta \ln(2) < \ln\left(\frac{2n}{3}\right)$, then Proposition 2.4 leads to $\deg(u) \leq \Delta < \frac{\ln\left(\frac{2n}{3}\right)}{\ln(2)} \leq |S|$ for every $u \in V(G) \setminus S$, concluding that S is a locating-dominating set of $\overline{G}$. Therefore, $\operatorname{adim}(G) = \lambda(\overline{G})$. □

The following result is a direct consequence of the theorem above.

Corollary 3.8. *Let G be a graph of order n and minimum degree δ. If $\delta > n - \left\lceil \frac{\ln\left(\frac{2n}{3}\right)}{\ln(2)} \right\rceil - 1$, then $\operatorname{adim}(G) = \lambda(G)$.*

Theorem 3.9. *Given a graph G of order n, the following assertions hold.*

(i) *If G has at most one isolated vertex, then $\operatorname{adim}(G) \leq n - \gamma(G)$.*

(ii) *If G has at most one vertex of degree $n-1$, then $\operatorname{adim}(G) \leq n - \gamma(\overline{G})$.*

(iii) *If G has no isolated vertices, then $\lambda(G) \leq n - \gamma(G)$.*

Proof. In this proof, the number of edges of a graph H will be denoted by $m(H)$. Let G be a graph having at most one isolated vertex and let S be a $\gamma(G)$-set such that for any $\gamma(G)$-set S' it is satisfied $m(\langle S \rangle) \geq m(\langle S' \rangle)$. We shall show that $V(G) \setminus S$ is an adjacency generator. Suppose to the contrary that $V(G) \setminus S$ is not an adjacency generator. In such a case, there exist $x, y \in S$ such that for every $z \in V(G) \setminus S$, either $x, y \in N(z)$ or $x, y \notin N(z)$. As a result, neither x nor y has any private neighbour (with respect to S) in $V(G) \setminus S$. We can assume that x is not an isolated vertex. Now, if $N(x) \cap S \neq \emptyset$, then $S \setminus \{x\}$ is a dominating set, which is a contradiction. If $N(x) \cap S = \emptyset$, then taking any $z \in N(x)$ we have that $S' = (S \setminus \{x\}) \cup \{z\}$ is a $\gamma(G)$-set such that $m(\langle S' \rangle) > m(\langle S \rangle)$, which is a contradiction. Therefore, $V(G) \setminus S$ is an adjacency generator, and so (i) and (ii) follow.

Furthermore, if G has no isolated vertices, then the complement of every $\gamma(G)$-set is a dominating set, which implies that $V(G) \setminus S$ is a locating-dominating set. Therefore, (iii) follows. □

The bounds above are tight. For instance, bounds (i) and (iii) are achieved by $G \cong K_n$, $G \cong P_4$ and $K_{p,q}$ $(2 \leq p \leq q)$. Bound (i) is also achieved by $G \cong K_1 \cup K_r$ $(r \geq 2)$, as $\operatorname{adim}(K_1 \cup K_r) = r - 1$ and $\gamma(K_1 \cup K_r) = 2$, and (iii) is also achieved by any corona graph $G \cong H \odot K_1$, as in this case $\lambda(G) = |V(H)| = \gamma(G) = \frac{n}{2}$. Obviously, bound (i) is achieved by a graph G if and only if bound (ii) is achieved by $\overline{G}$.

We now emphasize two well-known bounds on the domination number.

Theorem 3.10 ([26]). *For any graph G of order n and maximum degree $\Delta \geq 1$,*

$$\gamma(G) \geq \left\lceil \frac{n}{\Delta + 1} \right\rceil.$$

A graph invariant closely related to the domination number is the 2-packing number. A set $S \subseteq V(G)$ is a 2-*packing* if for each pair of vertices $u, v \in S$, $N[u] \cap N[v] = \emptyset$. The 2-*packing number* $\rho(G)$ is the cardinality of a maximum 2-packing.

Theorem 3.11 ([13])**.** *For any graph G,*

$$\gamma(G) \geq \rho(G).$$

The following result is a direct consequence of combining Remark 3.1 and Theorems 3.9, 3.10 and 3.11.

Theorem 3.12. *Let G be a non-empty graph of order n, maximum degree Δ and minimum degree δ. The following assertions hold.*

(a) $\mathrm{adim}(G) \geq \max\left\{\left\lceil \frac{\delta}{n-\delta} \right\rceil, \left\lceil \frac{n-\Delta-1}{\Delta+1} \right\rceil\right\}.$

(b) $\mathrm{adim}(G) \geq \max\{\rho(G), \rho(\overline{G})\} - 1.$

(c) *If* $\delta \geq 1$*, then* $\lambda(G) \leq n - \max\left\{\rho(G), \left\lceil \frac{n}{\Delta+1} \right\rceil\right\}.$

(d) *If G has at most one isolated vertex, then* $\mathrm{adim}(G) \leq n - \max\left\{\rho(G), \left\lceil \frac{n}{\Delta+1} \right\rceil\right\}.$

(e) *If G has at most one vertex of degree $n-1$, then*

$$\mathrm{adim}(G) \leq n - \max\left\{\rho(\overline{G}), \left\lceil \frac{n}{n-\delta} \right\rceil\right\}.$$

Bound (a) is achieved by complete graphs, while bounds (b) and (c) are achieved by the corona graphs $K_r \odot K_1$, $r \geq 3$, as in this case $\mathrm{adim}(K_r \odot K_1) = r-1$ and $\rho(K_r \odot K_1) = r = \lambda(K_r \odot K_1)$. Bounds (c) and (d) are achieved by $G = K_n$. Obviously, bound (e) is achieved by a graph G is and only if bound (d) is achieved by $\overline{G}$.

A set $S \subseteq V(G)$ is a k-*dominating set* if $|N(v) \cap S| \geq k$ for every $v \in V(G) \setminus S$. The minimum cardinality among all k-dominating sets is called the k-*domination number* of G and it is denoted by $\gamma_k(G)$. A set $S \subseteq V(G)$ is an *independent k-dominating set* if it is both an independent set and a k-dominating set. The minimum cardinality among all independent k-dominating sets is called the *independent k-domination number* of G and it is denoted by $i_k(G)$.

Theorem 3.13. *If G is a non-trivial graph which does not have cycles of length four, then* $\lambda(G) \leq \gamma_2(G)$.

Proof. Let S be a 2-dominating set. If S is not an adjacency basis, then there exist $u, v \in V \setminus S$ such that $N(u) \cap S = N(v) \cap S$. Since $|N(v) \cap S| \geq 2$, there exists a cycle with four vertices, which is a contradiction. □

The inequality above is tight. For instance, for the graph shown in Figure 4 we have that $\mathrm{adim}(G) = \lambda(G) = \gamma_2(G) = 4$.

Theorem 3.14. *Let G be a graph which does not have cycles of length four, and let S be a $\gamma_2(G)$-set. If there exists $s \in S$ such that $N[s] \cap S \neq N(x) \cap S$ for every $x \in N(s) \setminus S$, then* $\mathrm{adim}(G) \leq \gamma_2(G) - 1$.

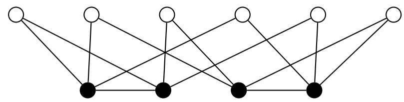

Figure 4: The set of black-colored vertices is an adjacency basis and a 2-dominating set. Hence, $\operatorname{adim}(G) = \lambda(G) = \gamma_2(G) = 4$.

Proof. Let $s \in S$ such that $N[s] \cap S \neq N(x) \cap S$ for every $x \in N(s) \setminus S$. Let us see that $S' = S \setminus \{s\}$ is an adjacency generator. Suppose to the contrary, that S' is not an adjacency generator. In such a case, there exist $u, v \in V(G) \setminus S'$ such that $N(u) \cap S' = N(v) \cap S'$. We differentiate three cases for these two vertices.
Case 1: $u = s$. In this case $v \notin N(s)$ and so $|N(v) \cap S'| \geq 2$. Hence, there exists a cycle with four vertices, which is a contradiction.
Case 2: $u \notin N[s]$. Since $|N(u) \cap S'| \geq 2$, there exists a cycle with four vertices, which is a contradiction.
Case 3: $u, v \in N(s)$. Since $|N(u) \cap S| \geq 2$, there exists a cycle with four vertices, which is a contradiction.
Therefore, the result follows. □

In the next result we are assuming that any acyclic graph has girth $\mathrm{g} = +\infty$.

Corollary 3.15. *Let G be a graph of minimum degree $\delta \geq 1$. Then the following assertions hold.*

(i) *If G has girth* $\mathrm{g} \geq 5$*, then* $\operatorname{adim}(G) \leq \gamma_2(G) - 1$.

(ii) *If G has an independent 2-dominating set and does not have cycles of length four, then* $\operatorname{adim}(G) \leq i_2(G) - 1$.

The bounds above are tight. For instance, for $3 \leq k \leq 7$ we have that $i_2(C_{2k}) = \gamma_2(C_{2k}) = k$ and $\operatorname{adim}(C_{2k}) = k - 1$.

Recall that a set S of vertices of G is a *vertex cover* of G if every edge of G is incident with at least one vertex of S. The *vertex cover number* of G, denoted by $\beta(G)$, is the smallest cardinality of a vertex cover of G. We refer to a $\beta(G)$-set in a graph G as a vertex cover of cardinality $\beta(G)$. The largest cardinality of a set of vertices of G, no two of which are adjacent, is called the *independence number* of G and it is denoted by $\alpha(G)$.

The following well-known result, due to Gallai, states the relationship between the independence number and the vertex cover number of a graph.

Theorem 3.16. (Gallai's theorem) *For any graph G of order n,*

$$\alpha(G) + \beta(G) = n.$$

A *leaf* is a vertex of degree one and a *strong support vertex* is a vertex which is adjacent to more than one leaf.

Theorem 3.17. *Let G be a graph of order n without isolated vertices. If G does not have neither cycles of four vertices nor strong support vertices, then*

$$\lambda(G) \leq \beta(G) = n - \alpha(G).$$

Proof. Let S be a $\beta(G)$-set. Since $V(G) \setminus S$ is an independent set and G does not have isolated vertices, S is a dominating set. Suppose to the contrary that S is not an adjacency generator. In such a case, there exist $u, v \in V(G) \setminus S$ such that $N(u) \cap S = N(v) \cap S$. If $|N(v) \cap S| \geq 2$, then there exists a cycle with four vertices, which is a contradiction. Now, if $|N(v) \cap S| = \{w\}$, then w is a strong support vertex, which is a contradiction again. Therefore, the results follows. □

To see that the above inequality is tight, we can consider the graph shown in Figure 4. In this case, the set of black-colored vertices is a $\beta(G)$-set and $\mathrm{adim}(G) = \lambda(G) = \beta(G) = n - \alpha(G) = 4$.

A set $S \subseteq V(G)$ is called a *super dominating set* of G if for every vertex $u \in V(G) \setminus S$, there exists $u' \in S$ such that $N(u') \setminus S = \{u\}$. The *super domination number* of G, denoted by $\gamma_{\mathrm{sp}}(G)$, is the minimum cardinality among all super dominating sets in G. A super dominating set of cardinality $\gamma_{\mathrm{sp}}(G)$ is called a $\gamma_{\mathrm{sp}}(G)$-set. The study of super domination in graphs was introduced in [17].

Theorem 3.18. *For any graph G,*

$$\lambda(G) \leq \gamma_{\mathrm{sp}}(G).$$

Furthermore, if G has minimum degree $\delta \geq 3$ and does not have cycles of length four, then

$$\lambda(G) \leq \gamma_{\mathrm{sp}}(G) - 1.$$

Proof. Let S be a $\gamma_{\mathrm{sp}}(G)$-set, $C = V(G) \setminus S$ and the function $f\colon C \longrightarrow S$ where $f(u)$ is one of the vertices in S satisfying that $N(f(u)) \setminus S = \{u\}$. Since, $f(u)$ distinguishes $u \in C$ from any $v \in C \setminus \{u\}$, we conclude that S is a locating-dominating set of G. Hence, $\lambda(G) \leq |S| = \gamma_{\mathrm{sp}}(G)$.

Assume that G has minimum degree $\delta \geq 3$ and does not have cycles of length four. Let $A = f(C)$ be the image of f and $B = S \setminus A$. We differentiate the following two cases.
Case 1: There exists $u \in C$ such that $N(u) \cap B \neq \emptyset$. We claim that $S' = S \setminus \{f(u)\}$ is a locating-dominating set. Since $N(f(u)) \cap C = \{u\}$ and $\deg(f(u)) \geq 3$, we have that $|N(f(u)) \cap S'| \geq 2$. Hence, S' is a dominating set. Now, every $v \in C \setminus \{u\}$

is distinguished from u by $f(v) \in S'$. Finally, if $f(u)$ and $v \in C$ are not distinguished by some vertex in S', then v, $f(u)$ and two vertices belonging to $N(f(u)) \cap S'$ form a cycle of length four, which is a contradiction. Therefore, S' is a locating-dominating set, and so $\lambda(G) \leq |S'| = \gamma_{\mathrm{sp}}(G) - 1$.
Case 2: $N(u) \cap B = \emptyset$ for every $u \in C$. Notice that $|N(f(u)) \cap S| \geq 2$ for every $u \in C$. Let $u, v \in C$ be two adjacent vertices. We claim that $S' = (S \setminus \{f(u), f(v)\}) \cup \{v\}$ is a locating-dominating set. Obviously, S' is a dominating set. Now, u is distinguished from any $u' \in C \setminus \{u, v\}$ by $f(u') \in S'$, and v distinguishes $f(u)$ from $f(v)$. Notice also that if $x \in C \setminus \{u, v\}$, then $|N(x) \cap (S' \setminus \{v\})| = 1$ and, since $u \sim v$, we have that $f(u) \not\sim f(v)$, which implies that $|N(y) \cap (S' \setminus \{v\})| \geq 2$ for every $y \in \{f(u), f(v)\}$. Thus, if $x \in C \setminus \{v\}$ and $y \in \{f(u), f(v)\}$, then $N(x) \cap S' \neq N(y) \cap S'$. In summary, S' is a locating-dominating set and, as a result, $\lambda(G) \leq |S'| = \gamma_{\mathrm{sp}}(G) - 1$. □

To show that the inequality $\lambda(G) \leq \gamma_{\mathrm{sp}}(G)$ is tight we consider the following cases: $\lambda(K_n) = \gamma_{\mathrm{sp}}(K_n) = n - 1$, $\lambda(K_{1,n-1}) = \gamma_{\mathrm{sp}}(K_{1,n-1}) = n - 1$, $\lambda(K_{r,n-r}) = \gamma_{\mathrm{sp}}(K_{r,n-r}) = n - 2$ for $2 \leq r \leq n - 2$ and $\lambda(H \odot N_t) = \gamma_{\mathrm{sp}}(H \odot N_t) = |V(H)|t$. For the Petersen graph, shown in Figure 5, we have that $\lambda(G) = \gamma_{\mathrm{sp}}(G) - 1 = 3$.

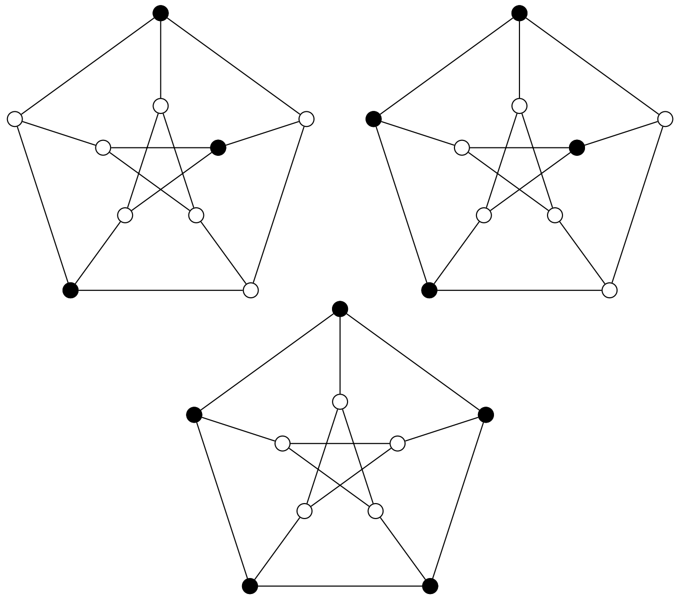

Figure 5: This figure shows three copies of the Petersen graph. The set of black-coloured vertices, on the left forms an adjacency basis, on the center forms a $\lambda(G)$-set, while on the right forms a $\gamma_{\rm sp}(G)$-set.

Lemma 3.19. *Let G be a graph with two adjacent vertices $x, y \in V(G)$ such that* $\deg(x) = 1$ *and* $\deg(y) = 2$*. If $G' = G - \{x, y\}$, then* $\operatorname{adim}(G) \le \operatorname{adim}(G') + 1$ *and* $\gamma_{\rm sp}(G) = \gamma_{\rm sp}(G') + 1$.

Proof. If S is an adjacency basis of G', then $S \cup \{x\}$ is an adjacency generator of G, which implies that $\operatorname{adim}(G) \le \operatorname{adim}(G') + 1$.

Assume that D' is a $\gamma_{\rm sp}(G')$-set and $u \in V(G')$ is adjacent to y in G. If $u \in D'$, then $D' \cup \{y\}$ is a super dominating set of G, while if $u \notin D'$, then $D' \cup \{x\}$ is a super dominating set of G. Therefore, $\gamma_{\rm sp}(G) \le \gamma_{\rm sp}(G') + 1$. Now, let D be a $\gamma_{\rm sp}(G)$-set and $v \in N(y) \setminus \{x\}$. If $x, y \in D$, then $v \notin D$ and $(D \cup \{v\}) \setminus \{x, y\}$ is a super dominating set of G', which implies that $\gamma_{\rm sp}(G') \le \gamma_{\rm sp}(G) - 1$. Now, if $|D \cap \{x, y\}| = 1$, $D \setminus \{x, y\}$ is a super dominating set of G' and so $\gamma_{\rm sp}(G') \le \gamma_{\rm sp}(G) - 1$. □

We know that adim$(P_4) = \lambda(P_4) = \gamma_{\rm sp}(P_4) = 2$, adim$(K_n) = \lambda(K_n) = \gamma_{\rm sp}(K_n) = n - 1$, adim$(K_{p,q}) = \lambda(K_{p,q}) = \gamma_{\rm sp}(K_{p,q}) = p + q - 2$ ($2 \le p \le q$). We proceed to show that for the remaining graphs, $\operatorname{adim}(G) \le \gamma_{\rm sp}(G) - 1$.

Theorem 3.20. *For any connected graph $G \notin \{P_4, K_n, K_{p,q}\}$, with $2 \le p \le q$,*

$$\operatorname{adim}(G) \le \gamma_{\rm sp}(G) - 1.$$

Proof of Theorem 3.20. Let G be a connected graph such that $G \notin \{P_4, K_n, K_{p,q}\}$ for $2 \le p \le q$. If $G' = G - \{x, y\}$, where $\deg(x) = 1$ and $\deg(y) = 2$, then we have the following:

- If $G' \cong P_4$, then $\operatorname{adim}(G) = 2 < 3 = \gamma_{\rm sp}(G)$.
- If $G' \cong K_1$, then $\operatorname{adim}(G) = 1 < 2 = \gamma_{\rm sp}(G)$.
- If $G' \cong K_{n-2}$ ($n \geq 5$), then $\operatorname{adim}(G) = n-3 < n-2 = \gamma_{\rm sp}(G)$.
- If $G' \cong K_{p,q}$ ($2 \leq p \leq q$), then $\operatorname{adim}(G) = p+q-2 < p+q-1 = \gamma_{\rm sp}(G)$.

Hence, by Lemma 3.19 we only need to consider the case where G does not have vertices of degree one which are adjacent to vertices of degree two.

Let D be a $\gamma_{\rm sp}(G)$-set, $C = V(G) \setminus D$ and $f\colon C \longrightarrow D$ a function such that, for every $u \in C$, $f(u)$ is one of the vertices in S satisfying that $N(f(u)) \setminus S = \{u\}$. Let $A = f(C)$ be the image of f and $B = S \setminus A$. Notice that $D_c = C \cup B$ is also a $\gamma_{\rm sp}(G)$-set, so any condition given on A could be also considered on C.

Suppose to the contrary that $\operatorname{adim}(G) \geq \gamma_{\rm sp}(G)$. With the assumptions above in mind, we proceed to prove the following eight claims.

Claim 1. For any vertex $x \in C$, $|N(x) \cap C| \leq 1$ and $|N(f(x)) \cap A| \leq 1$.

Proof of Claim 1. If there exist $y, z \in C$ such that $f(y), f(z) \in N(f(x)) \cap A$, then x and $f(x)$ are distinguished by $f(y)$; x and any $u \in C \setminus \{x\}$ are distinguished by $f(u)$; while $f(x)$ and any $u \in C \setminus \{x\}$ are distinguished by $f(y)$ or by $f(z)$. Hence, $D \setminus \{f(x)\}$ is an adjacency generator, which is a contradiction.

If $|N(x) \cap C| \leq 1$, then we proceed by analogy to the proof above using D_c instead of D. □

Claim 2. For any vertex $x \in C$, $\deg(x) \geq 2$ and $\deg(f(x)) \geq 2$.

Proof of Claim 2. Suppose that there exists $x \in C$ such that $\deg(x) = 1$. If $N(f(x)) \cap B = \emptyset$, then (by the connectivity of G) Claim 1 leads to $\deg(f(x)) = 2$, which is a contradiction with our assumptions. Now, if there exists $v \in N(f(x)) \cap B$, then $f(x)$ and x are distinguished by v; for any $y \in C \setminus \{x\}$, $f(y)$ and $f(x)$ are distinguished by y; while $f(y)$ and x are distinguished by y. Thus, $D_c \setminus \{x\}$ is an adjacency generator, which is a contradiction.

If $\deg(f(x)) = 1$, then we proceed by analogy to the proof above using D instead of D_c. □

Claim 3. Let $x \in C$. If $N(x) \cap C = \emptyset$ or $N(f(x)) \cap A = \emptyset$, then $N(x) \cap B = N(f(x)) \cap B$.

Proof of Claim 3. If $N(f(x)) \cap A = \emptyset$, then for any $z \in C \setminus \{x\}$, $f(x)$ and z are distinguished by $f(z)$. Since $D \setminus \{f(x)\}$ is not an adjacency generator, $N(f(x)) \cap B = N(x) \cap B$. A similar argument works for the case $N(x) \cap C = \emptyset$. □

Claim 4. Let $x, y \in C$. If $N(f(x)) \cap A = \{f(y)\}$, then $N(f(x)) \cap B = N(y) \cap B$ and $N(f(y)) \cap B = N(x) \cap B$.

Proof of Claim 4. Since $D \setminus \{f(x)\}$ is not an adjacency generator, if $N(f(x)) \cap A = \{f(y)\}$, then $N(f(x)) \cap B = N(y) \cap B$. Furthermore, by Claim 1, $N(f(x)) \cap A = \{f(y)\}$ leads to $N(f(y)) \cap A = \{f(x)\}$, and since $D \setminus \{f(y)\}$ is not an adjacency generator, we have that $N(f(y)) \cap B = N(x) \cap B$. □

Claim 5. If $v \in B$, then $|N(v) \cap A| = 1$ and $|N(v) \cap C| = 1$.

Proof of Claim 5. If $v \in B$ and $N(v) \cap A = \emptyset$, then v and any $x \in C$ are distinguished by $f(x)$. Now, if $v \in B$ and and there exist $y, z \in C$ such that $f(y), f(z) \in N(v) \cap A$, then v and any $x \in C$ are distinguished by $f(y)$ or by $f(z)$. In both cases, $D \setminus \{v\}$ is an adjacency generator, which is a contradiction. Therefore, $|N(v) \cap A| = 1$. By analogy we deduce that $|N(v) \cap C| = 1$. □

Claim 6. If $v_1, v_2 \in B$ are adjacent vertices, $N(v_1) \cap A = \{f(x)\}$ and $N(v_1) \cap C = \{y\}$, then $N(v_2) \cap A = \{f(y)\}$ and $N(v_2) \cap C = \{x\}$.

Proof of Claim 6. Assume that $v_1, v_2 \in B$ are adjacent vertices, $N(v_1) \cap A = \{f(x)\}$ and $N(v_1) \cap C = \{y\}$. Since $D \setminus \{v_1\}$ is not an adjacency generator and $f(x)$ distinguishes v_1 and z for every $z \in C \setminus \{x\}$, we have that $x \in N(v_2)$. Thus, by Claim 5, $N(v_2) \cap C = \{x\}$. Furthermore, since $D \setminus \{v_2\}$ is not an adjacency generator and v_1 distinguishes v_2 and z for every $z \in C \setminus \{y\}$, we have that $f(y) \in N(v_2)$. Hence, by Claim 5 we conclude that $N(v_2) \cap A = \{f(y)\}$. □

Claim 7. If there exists $x \in C$ such that $N(x) \cap C = \emptyset$ and $N(f(x)) \cap A = \emptyset$, then $|C| = 1$ and G is a complete graph.

Proof of Claim 7. Assume that there exists a vertex $x \in C$ such that $N(x) \cap C = \emptyset$ and $N(f(x)) \cap A = \emptyset$. By Claim 3, $N(x) \cap B = N(f(x)) \cap B$. Let $B_x = N(x) \cap B$, which is nonempty, as G is connected and $G \not\cong K_2$. If there exist two nonadjacent vertices $v_r, v_s \in B_x$, then $D \setminus \{v_s\}$ is an adjacency generator because x and v_s are distinguished by v_r, and any $u \in C \setminus \{x\}$ is distinguished from v_s by $f(u)$. Therefore, $X = \{x, f(x)\} \cup B_x$ induces a complete graph. Now, by the connectivity of G, if $V(G) \neq X$, then there exist two adjacent vertices $b, b' \in B$ such that $b \in B_x$ and $b' \in B \setminus B_x$. In such a case, applying Claim 6 for $x = y$, we conclude that $b' \in B_x$, which is a contradiction. Therefore, $V(G) = X$, $|C| = 1$ and G is a complete graph. □

Claim 8. If there exist $x, y \in C$ such that $N(f(x)) \cap A = \{f(y)\}$, then $G \cong K_{p,q}$, where $2 \leq p \leq q$.

Proof of Claim 8. We differentiate two cases.

Case 1: $N(x) \cap C = \{y\}$. Since the subgraph induced by $U = \{x, f(x), y, f(y)\}$ is isomorphic to $K_{2,2}$, $V(G) \setminus U \neq \emptyset$. By Claims 1, 4 and 5, every vertex in $V(G) \setminus U$ which is adjacent to some vertex in U has to belong to $B_1 = B \cap N(f(x)) \cap N(y)$ or to $B_2 = B \cap N(x) \cap N(f(y))$. Notice that $B_1 \cap B_2 = \emptyset$. Let $X_1 = \{x, f(y)\} \cup B_1$ and $X_2 = \{f(x), y\} \cup B_2$. Let us see that G is a complete bipartite graph. Firstly, if there exist two adjacent vertices $u \in V(G) \setminus (X_1 \cup X_2)$ and $v \in B_1 \cup B_2$, by the definition of B_1 and B_2, we know that $u \notin A \cup C$. Hence, if v belongs, for instance, to B_1, by Claim 6, $u \in B_2$, which is a contradiction. Consequently, $V(G) = X_1 \cup X_2$. Secondly, if there exist two adjacent vertices $u, v \in B$, by Claim 6, either $u \in B_1$ and $v \in B_2$ or $u \in B_2$ and $v \in B_1$. Finally, if there exist two nonadjacent vertices $u \in B_1$ and $v \in B_2$, since u and x are distinguished by v, while u and any $z \in C \setminus \{x\}$ are distinguished by $f(z)$, we have that $D \setminus \{u\}$ is an adjacency generator, which is a contradiction. Therefore, $G = (X_1 \cup X_2, E)$ is a complete bipartite graph with $|X_1| \geq 2$ and $|X_2| \geq 2$.

Case 2: For any $x, y \in C$ such that $N(f(x)) \cap A = \{f(y)\}$, the subgraph induced by $\{x, f(x), y, f(y)\}$ is not isomorphic to $K_{2,2}$. By Claim 2, for every $x \in C$, $\deg(x) \geq 2$

and $\deg(f(x)) \geq 2$. Since $\operatorname{adim}(C_n) = \left\lfloor \frac{2n+2}{5} \right\rfloor$ for $n \geq 4$ and $\gamma_{\rm sp}(C_n) = \left\lceil \frac{n}{2} \right\rceil$ for $n \geq 3$, we have that $\operatorname{adim}(C_n) \leq \gamma_{\rm sp}(C_n) - 1$ for any $n \geq 5$. Hence, $G \not\cong C_n$ and so $B \neq \emptyset$. By Claim 7, for every $x \in C$ either $N(x) \cap C \neq \emptyset$ or $N(f(x)) \cap A \neq \emptyset$. Suppose that there exist $x, y \in C$ such that $y \notin N(x)$ and $N(f(x)) \cap A = \{f(y)\}$. If there exists $b \in B \cap N(x) \cap N(f(y))$, then $D' = (D \cup \{y\}) \setminus \{b\}$, is also a $\gamma_{\rm sp}(G)$-set. In such a case, we define a new function $f' \colon (C \cup \{b\}) \setminus \{y\} \longrightarrow D'$ where $f'(b) = f(y)$ and $f'(w) = f(w)$ for every $w \in C \setminus \{y\}$. Since the subgraph induced by $\{x, f'(x), b, f'(b)\}$ is isomorphic to $K_{2,2}$, we can conclude the proof using again Case 1. Analogously, suppose that there exist $x, y \in C$ such that $f(y) \notin N(f(x))$ and $N(x) \cap C = \{y\}$. If there exists $b \in B \cap N(x) \cap N(f(y))$, then we can take the $\gamma_{\rm sp}(G)$-set $D' = (D_c \cup \{f(x)\}) \setminus \{b\}$ and $f' \colon (A \cup \{b\}) \setminus \{f(x)\} \longrightarrow D'$ where $f'(b) = x$ and $f'(f(z)) = z$ for every $z \in C \setminus \{x\}$ to obtain a subgraph isomorphic to $K_{2,2}$. Applying again Case 1 we get the result. □

End of *Proof of* Theorem 3.20. □

The bound above is tight. For instance, for any graph H, $\operatorname{adim}(H \odot N_t) = \gamma_{\rm sp}(H \odot N_t) - 1 = |V(H)|t - 1$ and for the Petersen graph shown in Figure 5 we have $\operatorname{adim}(G) = \gamma_{\rm sp}(G) - 1 = 3$.

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On the chromatic index of generalized truncations

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Abstract

We examine the chromatic index of generalized truncations of graphs and multigraphs. The insertion graphs considered are complete graphs, cycles, regular graphs and forests.

Keywords: generalized truncation, chromatic index

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1 Introduction

A broad definition of generalized truncations of graphs was introduced in [1]. We give this definition now for completeness, but first a brief word about terminology is in order. The term *multigraph* is used if multiple edges are allowed. Thus, a graph does not have multiple edges. We use $V(X)$ to denote the set of vertices of a multigraph X and $E(X)$ to denote the set of edges. The *order* of X is $|V(X)|$ and the *size* of X is $|E(X)|$. Finally, the *valency* of a vertex u, denoted $\mathrm{val}(u)$, is the number of edges incident with u. The term k-*valent* multigraph is used for regular multigraphs of valency k. Throughout the paper, $\Delta(X)$ denotes the maximum valency of the multigraph X and Δ often is used when the graph X involved is apparent.

Given a multigraph X, a *generalized truncation* of X is obtained as follows via a two-step operation. The first step is the *excision step*. Let M denote an auxiliary matching (no two edges have a vertex in common) of size $|E(X)|$. Let $F\colon E(X) \to M$ be a bijective function and for $[u, v] \in E(X)$, label the ends of the edge $F([u, v])$ with u and v. Let

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M_F denote the vertex-labelled matching thus obtained. So M_F represents the edges of X completely disassembled.

The second step is the *assemblage step*. For each $v \in V(X)$, the set of vertices of M_F labelled with v is called the *cluster at* v and is denoted $\mathrm{cl}(v)$. Insert an arbitrary graph on $\mathrm{cl}(v)$. The inserted graph on $\mathrm{cl}(v)$ is called the *constituent graph at* v and is denoted $\mathrm{con}(v)$. The resulting graph

$$M_F \cup_{v \in V(X)} \mathrm{con}(v)$$

is a *generalized truncation* of X. We usually think of the labels on the vertices of M_F as being removed following the assemblage stage, but there are many times when the labels are useful in the exposition. We use $\mathrm{TR}(X)$ to denote a generalized truncation of the multigraph X.

Truncations arise via action involving the edges incident with a vertex. Consequently, isolated vertices are useless and we make the important convention that the multigraphs from which we are forming generalized truncations do not have isolated vertices. This will not be mentioned in the subsequent material, but is required for the validity of a few statements.

Recall that a *proper edge coloring* of a multigraph X is a coloring of the edges so that adjacent edges do not have the same color. The *chromatic index* of X, denoted $\chi'(X)$, is the fewest number of colors for which a proper edge coloring exists. We now state Vizing's well-known theorem [4] which plays a fundamental role in studying chromatic index.

Theorem 1.1. *If X is a graph, then its chromatic index is either $\Delta(X)$ or $\Delta(X) + 1$.*

The preceding theorem leads to a classification of graphs as follows. A graph is *class* I if its chromatic index is equal to its maximum valency Δ and is *class* II otherwise. There is a generalization for multigraphs and to ease subsequent exposition we shall say a multigraph is class I if its chromatic index equals its maximum valency.

The following result was proved in [1]. A generalized truncation $\mathrm{TR}(X)$ is called *complete* when every constituent is a complete subgraph.

Theorem 1.2. *If X is a class* I *graph, then its complete truncation also is class* I. *If X is a class* II *graph and its maximum valency is even, then its complete truncation is class* I.

The purpose of this paper is to extend the preceding result and explore edge colorings of generalized truncations in more detail.

2 Some useful results

In the following material we shall be considering the relationship between multigraphs and their generalized truncations that are class I. Doing so requires the use of some well-known edge coloring results which we now give for completeness.

Let σ be a permutation of the vertex set of the complete graph K_n of order n. If Y is a subgraph of K_n, then $\sigma(Y)$ denotes the subgraph of K_n whose edge set is $\{[\sigma(u), \sigma(v)] : [u, v] \in E(Y)\}$. Let n be even and ρ be the permutation defined by $\rho = (u_0)(u_1\ u_2\ u_3\ \cdots\ u_{n-1})$. So ρ fixes one vertex and cyclically rotates the remaining vertices. Let Y be the perfect matching of K_n consisting of the edges

$$[u_0, u_1], [u_2, u_{n-1}], [u_3, u_{n-2}], \ldots, [u_{n/2}, u_{(n+2)/2}].$$

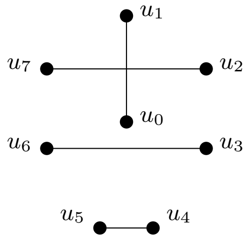

Figure 1: Canonical edge coloring.

We then obtain a proper edge coloring of K_n by letting the color classes be $Y, \rho(Y), \rho^2(Y), \ldots, \rho^{n-2}(Y)$. We call this proper edge coloring the *canonical edge coloring* of K_n. Figure 1 shows Y for K_8. We obtain a proper edge coloring of K_n with n colors, n odd, by starting with the canonical edge coloring of K_{n+1} and then removing the central vertex u_0 and the edges incident with it. Note that each vertex is missing an edge of one color and the set of missing colors has cardinality n. This fact is true for any proper edge coloring of K_n, n odd.

Because we use the preceding canonical edge coloring for both even and odd orders, we describe it by referring to the edge of length 1 as the *anchor*. When n is even, the anchor is the edge $[u_{n/2}, u_{1+n/2}]$. When n is odd, the anchor is the edge $[u_{(n+1)/2}, u_{(n+3)/2}]$. To obtain proper edge colorings of complete graphs, we cyclically rotate the canonical edge coloring which, of course, cyclically rotates the anchor. So we may determine a color class by specifying the anchor.

Lemma 2.1. *Every proper edge coloring of K_n with n colors, n odd, has the property that there is one color missing on the edges incident with a given vertex, and the set of missing colors at the vertices has cardinality n.*

Proof. There are at most $(n-1)/2$ edges of a fixed color in a proper edge coloring of K_n because n is odd. Thus, there is no proper edge coloring using just $n-1$ colors because $(n-1)^2/2 < \binom{n}{2}$. There is a proper edge coloring using n colors by Vizing's Theorem. Hence, each color class contains precisely $(n-1)/2$ edges from which the conclusion follows. □

We use a result about list chromatic index so we discuss it briefly here and present the result. Given a graph X and for each edge $[u, v]$ a list $L_{[u,v]}$ of colors we may use for the edge $[u, v]$, a *proper list edge coloring* is a proper edge coloring of X so that the color on each edge $[u, v]$ belongs to $L_{[u,v]}$. The *list chromatic index* of X is the smallest N such that if every list $L_{[u,v]}$ has cardinality N, then X admits a proper list edge coloring. We denote this value by $\chi'_L(X)$. The following important result was proved by Häggkvist and Jansen in [2].

Theorem 2.2. *The complete graph K_n satisfies $\chi'_L(K_n) \leq n$.*

3 A general result

We are interested in determining which generalized truncations of a given multigraph X are class I. Theorem 3.1 below provides a general answer and in subsequent sections we use the theorem to describe class I generalized truncations of specific types. Some definitions and notation are required before stating the theorem.

Given a generalized truncation $\mathrm{TR}(X)$, for convenience we call the subgraph induced by the edges of $\mathrm{con}(v)$ together with the edges of M_F incident with the vertices of $\mathrm{con}(v)$ the *sun centered at* $\mathrm{con}(v)$. Given that $\mathrm{con}(v)$ is regular of valency $d-1$, when is the sun centered at $\mathrm{con}(v)$ class I? If it is class I, then there is a proper edge coloring using d colors. We use a vector of length d to describe the number of edges of M_F of the various colors in the sun centered at $\mathrm{con}(v)$.

Given a vertex $v \in X$ of valency r, we say that a vector $(x_1, x_2, \ldots, x_d)$ is *admissible* if $\sum_i x_i = r$, the edges of M_F are colored according to the vector, that is, x_i edges have color i, and there is a $(d-1)$-regular graph on $\mathrm{con}(v)$ such that the sun centered at $\mathrm{con}(v)$ is class I. We say the vector is *totally inadmissible* if there is no $(d-1)$-regular graph on $\mathrm{con}(v)$ which makes the sun class I.

Theorem 3.1. *If $(x_1, x_2, \ldots, x_d)$ satisfies $x_1+x_2+\cdots+x_d = r \geq d$, then $(x_1, x_2, \ldots, x_d)$ is admissible if and only if all of $x_1, x_2, \ldots, x_d$ and r have the same parity and when the parity is odd, d also is odd. Otherwise, the vector is totally inadmissible.*

Proof. Suppose that $\mathrm{val}(v) = r$ in a multigraph X and that there is a class I sun centered at $\mathrm{con}(v)$ which is regular of valency d. Let $(x_1, x_2, \ldots, x_d)$ be the vector for the colors of M_F. For an arbitrary color $c(i)$, if the number of edges of color $c(i)$ in $\mathrm{con}(v)$ is k, then the number of edges of color $c(i)$ in M_F must be $r - 2k$. It then follows that $x_1, x_2, \ldots, x_d$ and r all have the same parity. Moreover, if r is odd, then the graph $\mathrm{con}(v)$ is regular of valency $d-1$ and order r which implies that $d-1$ is even. This completes the necessity.

Now suppose that all coordinates of $(x_1, x_2, \ldots, x_d)$ are odd, r is odd and $x_1 + x_2 + \cdots + x_d = r$. In this case we also have that d is odd. Label the r vertices of the constituent $u_1, u_2, u_3, \ldots, u_r$ so that the first x_1 vertices are incident with the x_1 edges of M_F of color $c(1)$, the next x_2 vertices are incident with the x_2 edges of M_F of color $c(2)$ and continue in this way. Also carry out subscript arithmetic modulo r on the residues $1, 2, 3, \ldots, r$.

Consider the x_i successive vertices incident with the x_i edges of M_F of color $c(i)$. Because x_i is odd, there is a central edge of M_F of color $c(i)$ and let it be incident with u_a. Letting $\alpha = (x_i - 1)/2$, the vertices incident with edges of M_F of color $c(i)$ are $u_{a-\alpha}, u_{a-\alpha+1}, \ldots, u_a, u_{a+1}, \ldots, u_{a+\alpha}$.

Consider the canonical color class whose anchor is $[u_{a+(r-1)/2}, u_{a+(r+1)/2}]$. Color the edges of this class with color $c(i)$ starting with the anchor and stopping with the edge $[u_{a-\alpha-1}, u_{a+\alpha+1}]$. This yields edges colored with $c(i)$ so that each vertex of the constituent is incident with one edge of color $c(i)$. After doing this for each of the d colors, the resulting sun centered at the constituent is class I and the constituent is regular of valency $d-1$.

This leaves us with the case that all the coordinates of $(x_1, x_2, \ldots, x_d)$ are even so that their sum r also is even. There is no restriction on the parity of d in this case. Also note that $x_i = 0$ is possible for various values of i. Without loss of generality, let $x_1, x_2, \ldots, x_a$ be the non-zero entries of the vector.

We label the vertices of the constituent a little differently using the canonical edge coloring of Figure 1 as a template. The central vertex is labelled u_0 and the others are labelled $u_1, u_2, \ldots, u_{r-1}$.

Let the x_1 edges of M_F of color $c(1)$ be incident with $u_0, u_1, \ldots, u_{x_1-1}$. Let the remaining edges of M_F be incident with vertices of the constituent as in the preceding case, that is, edges of the same color are incident with successively labelled vertices.

The edges of color $c(1)$ have a different form than the other colored edges making the completion of the coloring a little different for them. Vertices $u_1, u_2, \ldots, u_{x_1-1}$ are incident with edges of M_F of color $c(1)$. This corresponds to the color class with anchor $[u_{(x_1+r-4)/2}, u_{(x_1+r-2)/2}]$. So color the edges of that class with color $c(1)$ starting with the anchor until finishing with $[u_{r-1}, u_{x_1}]$. Every vertex of the constituent now is incident with an edge of color $c(1)$.

For all the other colors $c(j)$, $1 < j \leq a$, there are an even number of successive vertices, say $u_j, u_{j+1}, \ldots, u_{j+t}$, incident with edges of M_F of color $c(j)$. Then the edge $[u_{j+(t-1)/2}, u_{j+(t+1)/2}]$ is the anchor of a color class. So complete the coloring of the edges of this color class starting with $[u_{j-1}, u_{j+t+1}]$ moving away from the anchor. For $x_i = 0$, simply include an entire color class from the canonical coloring scheme using an unused anchor.

Doing the preceding yields a class I coloring of the sun centered at the constituent so that the constituent graph is regular of valency $d - 1$ and completes the proof. □

We now obtain three corollaries from Theorem 3.1, but first require a couple of definitions. An edge coloring of a multigraph X is said to be *parity-balanced* if for each vertex v of X, the parity of the number of edges of each color incident with v is the same as the parity of $\mathrm{val}(v)$. A *regular truncation* is a generalized truncation for which every vertex has the same valency. A generalized truncation is said to be *semiregular* if each sun centered at a constituent $\mathrm{con}(v)$ is class I and the subgraph $\mathrm{con}(v)$ is regular. The distinction between a semiregular truncation and a regular truncation is that valencies of regularity for a semiregular truncation may differ over the constituents. Note that the source multigraph need not be regular in order to have a regular truncation.

Corollary 3.2. *Let every entry of the feasible vector $(x_1, x_2, \ldots, x_d)$ be even and $x_1 + x_2 + \cdots + x_d = r$. If d' is the number of non-zero entries of the vector, then there are class* I *suns centered at the appropriate constituent such that the constituents are regular of every valency from d' through $r - 1$.*

Proof. This result follows from the proof of Theorem 3.1 rather than the statement. Because the number of colors on edges of M_F is d', the scheme used in the proof of Theorem 3.1 introduces no new colors so that the valency of regularity is d'. Because r is even, we may add canonical coloring classes one at a time until reaching K_r. This completes the proof. □

Corollary 3.3. *A multigraph X has a class* I *semiregular truncation if and only if it has a parity-balanced edge coloring.*

Proof. If X has a parity-balanced edge coloring, it follows immediately from Theorem 3.1 that it has a semiregular truncation. On the other hand, if X has a semiregular truncation, then performing the standard contraction and retention of the colors on the edges of M_F produces a parity-balanced edge coloring of X. □

Corollary 3.4. *A multigraph X has a class* I *regular truncation of valency d if and only if it has a parity-balanced coloring and one of the following conditions holds:*

(i) *When d is odd, every vertex of odd valency has precisely d colors on its incident edges, and every vertex of even valency has valency at least $d + 1$ and at most d colors on its incident edges ; or*

(ii) *When d is even, every vertex of X has even valency at least d.*

Proof. Let X be a mutigraph and suppose it has a generalized truncation Y which is regular of valency d. Consider the case that d is even. This means that a constituent $\mathrm{con}(v)$ is regular of valency $d - 1$ and the latter is odd. Thus, the constituent has even order. This implies that every vertex of X has even valency. Clearly every valency is at least d.

When d is odd, each constituent must itself be regular of valency $d - 1$. Because $d - 1$ is even, there is no restriction on the order N of the constituent other than it must be at least d. If N is odd, then for each of the d distinct colors, there must be at least one edge of that color belonging to the edges of M_F incident with vertices of the constituent. So the corresponding vertex of X is incident with edges of precisely d different colors. When N is even there is no restriction on the number of colors on edges of M_F incident with vertices of the constituent other than it is at most d.

For the other direction, when d is even, every constituent has even order and by Corollary 3.2 it is easy to obtain regular valency of $d - 1$ on the constituent. When d is odd, the result follows from Theorem 3.1. □

4 Complete truncations

Let X be a multigraph with maximum valency Δ. If Δ is odd and X has an edge coloring with Δ colors such that the edges incident with every vertex of valency Δ have distinct colors, then we say that X is *edge-feasible*. The next result characterizes graphs whose complete truncations are class I. Of course, a complete truncation is a semi-regular truncation.

Theorem 4.1. *The complete truncation of a multigraph X is class* I *if and only if either the maximum valency Δ of X is even, or Δ is odd and X is edge-feasible.*

Proof. This theorem is an improvement on Theorem 1.2 as the latter does not include multigraphs as part of the hypotheses. Because a complete truncation is a semiregular truncation, Corollary 3.3 implies that the coloring of the edges in M_F must correspond to a parity-balanced coloring of X. When Δ is even, coloring all the edges of M_F with a single color corresponds to a parity-balanced coloring of X. Any constituent of order Δ can be properly edge-colored with $\Delta - 1$ colors because Δ is even. Any constituent of order less than Δ can be properly edge-colored with at most $\Delta - 1$ colors. Hence, the complete truncation of X is class I.

For the remainder of the proof we assume that Δ is odd. First suppose that the complete truncation $\mathrm{TR}(X)$ is class I. If $u \in V(X)$ has valency Δ, then $\mathrm{con}(u) = K_\Delta$ the complete graph of order Δ. We conclude that the edges of M_F incident with $\mathrm{con}(u)$ all have different colors by Lemma 2.1. So if we contract each constituent to a single vertex, delete all the loops formed and keep the colors of the edges of M_F, we obtain an edge coloring of X. Clearly, all the edges incident to any vertex of valency Δ in X have distinct colors. Thus, X is edge-feasible.

Now let X be edge-feasible and choose an edge coloring of X which is edge-feasible. Let $\mathrm{TR}(X)$ be the complete truncation of X. Color the edges of M_F with the same color they had in the edge coloring of X. These are the edges between the constituents all of which are complete subgraphs. At this point we have used Δ colors. It suffices to show that we can color the edges of the constituents without introducing any new colors so that the resulting edge coloring of $\mathrm{TR}(X)$ is proper.

Any constituent $\mathrm{con}(u)$ of order Δ corresponds to a vertex u of X of valency Δ. This implies that the edges of M_F incident with the vertices of $\mathrm{con}(u)$ all have distinct colors because the coloring of X was edge-feasible. From Lemma 2.1 it is clear that we may color the edges of $\mathrm{con}(u)$ with Δ colors so that the missing color at each vertex is the color of the edge of M_F at the vertex. This coloring of the edges does not violate the definition of a proper edge coloring.

Now consider any constituent $\mathrm{con}(u)$ of order $r \leq \Delta - 2$. Because each edge of $\mathrm{con}(u)$ has one edge of M_F at each of its end vertices, the number of possible colors for the edge that do not violate the proper edge coloring condition is $\Delta - 2$. So each edge has a list of $\Delta - 2$ possible colors, and Theorem 2.2 implies that we may color the edges of $\mathrm{con}(u)$ without violating the proper edge coloring condition because $r \leq \Delta - 2$.

This leaves us with the case that $\mathrm{con}(u)$ has order $\Delta - 1$ and this is the most complicated case. The first observation we make is that the coloring pattern of the edges of M_F incident with the vertices of $\mathrm{con}(u)$ can vary all the way from having $\Delta - 1$ distinct colors to every color being the same. So we introduce a sequence describing the color pattern. Let $(s_1, s_2, \ldots, s_t)$ satisfy $s_1 \leq s_2 \leq \cdots \leq s_t$ and $s_1 + s_2 + \cdots + s_t = \Delta - 1$. The sequence means there are t distinct colors on the edges of M_F incident with vertices of $\mathrm{con}(u)$ and s_i such edges have the same color $c(i)$ for $i = 1, 2, \ldots, t$.

We need to show that no matter what the color sequence is we may color the edges of $\mathrm{con}(u)$ so that the sun centered at $\mathrm{con}(u)$ is properly edge colored with Δ colors. As a first step, label the vertices of $\mathrm{con}(u)$ with $v_0, v_1, \ldots, v_a$, where $a = \Delta - 2$. Let the vertices $v_1, v_2, \ldots, v_{s_1}$ be incident with the s_1 edges of M_F of color $c(1)$. Let the next s_2 vertices be incident with the s_2 edges of M_F of color $c(2)$. Continue labelling the vertices in the obvious manner and let the edge of M_F incident with v_0 have color $c(t)$.

Carry out an initial coloring of the edges of $\mathrm{con}(u)$ using the canonical edge coloring described in Section 2 with v_0 acting as the fixed vertex and $\rho = (v_0)(v_1 \ \ v_2 \ \ \cdots \ \ v_a)$. The strategy now is to choose the colors for the color classes in such a way that we may re-color some edges to obtain a proper edge coloring for the sun centered at $\mathrm{con}(u)$.

The first observation we make is that only $\Delta - 2$ colors are required for a canonical edge coloring of $\mathrm{con}(u)$. So we do not use the colors $c(t-1)$ or $c(t)$ for the canonical edge coloring of $\mathrm{con}(u)$. Hence, if $t = 1$ or $t = 2$, we already have an edge coloring of $\mathrm{con}(u)$ that does not violate the conditions for a proper edge coloring. Thus, we assume that $t \geq 3$.

The anchor for a color class plays an active role as follows. Suppose we require two edges of a color class so that the two edges use four successive vertices under the cyclic labelling of $\{v_1, v_2, \ldots, v_a\}$. If the anchor is $[v_i, v_{i+1}]$, then adding the edge $[v_{i-1}, v_{i+2}]$ from the same color class easily does the job. It is now easy to see how we may obtain k edges from the same color class so that they cover $2k$ successive vertices. With this in mind, we now describe an iterative process for determining the colors of certain color classes.

If s_1 is odd, then we color the color class for which $[v_{(s_1+1)/2}, v_{(s_1+3)/2}]$ is the anchor with $c(1)$. If s_1 is even, then we color the color class for which $[v_{s_1/2}, v_{(s_1+2)/2}]$ is the anchor with $c(1)$. There are two points to observe about the preceding choices. When s_1 is odd, the edge from v_1 to v_{s_1+1} has color $c(1)$ and note that the edge of M_F incident with v_{s_1+1} has color $c(2)$. When s_1 is even, the edge from v_1 to v_{s_1} has color $c(1)$ and vertex v_{s_1} is the last vertex for which the edge of M_F incident with it has color $c(1)$.

We continue in the manner suggested by the preceding paragraph, but discuss it further to make it clearer. When s_i is odd and the first vertex incident with an edge of M_F of color $c(i)$ is v_d, then the anchor is the edge $[v_{d+(s_i-1)/2}, v_{d+(s_i+1)/2}]$ and we color this color class with $c(i)$. Note that the edge $[v_d, v_{d+s_i}]$ is in this color class, but the edge of M_F incident with v_{d+s_i} has color $c(i+1)$.

When s_i is even and the first vertex incident with an edge of M_F of color $c(i)$ is v_d, then the anchor is $[v_{d+(s_i-2)/2}, v_{d+s_i/2}]$ and we color this color class with $c(i)$. In this case, the edge from v_d to v_{d+s_i-1} is colored $c(i)$ and v_{d+s_i-1} is the last vertex incident with an edge of M_F of color $c(i)$.

The preceding procedure is carried out for the colors $c(1)$ through $c(t-2)$ at which point it stops because colors $c(t-1)$ and $c(t)$ are not used for the canonical edge coloring of $\mathrm{con}(u)$. The edges of $\mathrm{con}(u)$ that need to be re-colored are those that are adjacent with edges of M_F having the same color. We have seen that when s_i is odd, there is an edge in $\mathrm{con}(u)$ of color $c(i)$ with one end vertex incident with an edge of M_F of color $c(i)$ and the other end vertex incident with an edge of M_F of color $c(i+1)$. So at the end vertex incident with an edge of M_F of color $c(i+1)$, the procedure gives another edge of color $c(i+1)$ which also needs to be re-colored.

Thus, the edges that need to be recolored are isolated and possibly paths. Luckily we have two colors to use for the re-coloring and we arbitrarily re-color isolated edges with either $c(t-1)$ or $c(t)$, and alternately re-color the edges of the paths making certain that if one of the paths terminates with a vertex whose incident edge from M_F has color $c(t-1)$, we color the edge of the path terminating there with $c(t)$. This removes all potential color conflicts for $\mathrm{con}(u)$ and completes the proof of the theorem. □

Corollary 4.2. *Let X be a class I graph and $\mathrm{TR}(X)$ a generalized truncation. If $\Delta(X) = \Delta(\mathrm{TR}(X))$, then $\mathrm{TR}(X)$ is class I.*

Proof. Let $\mathrm{TR}(X)$ be a generalized truncation of X satisfying $\Delta(\mathrm{TR}(X)) = \Delta(X)$. Then a proper edge coloring of $\mathrm{TR}(X)$ requires at least Δ colors. We know the complete truncation Y of X is class I by Theorem 4.1, that is, it has a proper edge coloring using Δ colors. Remove any edges of Y not belonging to $\mathrm{TR}(X)$ and we are left with a proper edge coloring of $\mathrm{TR}(X)$ using Δ colors. So $\mathrm{TR}(X)$ is class I. □

It is well known that both the Petersen graph and its complete truncation are class II graphs. The extension to a regular multigraph is an immediate corollary of Theorem 4.1.

Corollary 4.3. *A regular multigraph of odd valency is class* I *if and only if its complete truncation is class* I.

5 Cyclic truncations

Complete truncations have been used by various authors, and there is another generalized truncation that has been employed frequently. Namely, the generalized truncation obtained by letting each constituent graph be a cycle. We shall call these *cyclic truncations*. Probably the best known example of this is the cube-connected cycles graph first introduced in [3].

Of course, the ancient Greeks studied truncations of Platonic and Archimedian solids, and these resulted in cyclic truncations.

When moving from a multigraph X to a generalized truncation Y, we frequently wish Y to inherit properties of X. This can be a problem for cyclic truncations. For example, poorly chosen cyclic truncations can play havoc with automorphisms.

We are interested in determining conditions for which a cyclic truncation is class I. Of course, a cyclic truncation is a regular truncation so that we may use Theorem 3.1. We use the same vector describing the numbers of colors used on the edges of M_F belonging to a sun centered at a constituent. In this case the vector has length 3 as we are looking at cyclic truncations all of which are 3-valent. There is a new term not used before, namely, a vector (x_1, x_2, x_3) is *universal* if the sun centered at the corresponding constituent is class I for every cycle on the vertices of the constituent.

Corollary 5.1. *If (x_1, x_2, x_3) satisfies $x_1 + x_2 + x_3 = d \geq 3$, then (x_1, x_2, x_3) is admissible if and only x_1, x_2, x_3 and d all have the same parity. Otherwise, the vector is totally inadmissible. Moreover, if just one of x_1, x_2 and x_3 is non-zero, then (x_1, x_2, x_3) is universal when d is even.*

Proof. The portion of the statement prior to the sentence about universality is simply Theorem 3.1 restricted to $d = 3$. When the vector has the form $(x_1, 0, 0)$ and $x_1 = d$ is even, this corresponds to all the edges of M_F incident with the vertices of the constituent having the same color. Clearly, no matter which even length cycle we form on the vertices of the constituent, the resulting sun is class I. □

Corollary 5.1 gives us an easy way to construct a class I cyclic truncation of a multigraph X based on parity conditions for the colors on the edges of M_F. Thus, we need to concentrate on coloring the edges in the source multigraph.

Corollary 5.2. *If every vertex of the multigraph X has even valency, then every cyclic truncation of X is class I.*

Proof. This follows from Corollary 5.1 by coloring all edges of M_F with a single color. □

Corollary 5.3. *If X is a regular multigraph of odd valency $d > 1$ and is class* I*, then there are class* I *cyclic truncations of X.*

Proof. Let X be a class I regular multigraph of odd valency $d \geq 3$. Choose a proper edge coloring of X using d colors. In forming a cyclic truncation Y of X, color the edges of M_F according to the colors $c(1), c(2), \ldots, c(d)$ the edges have in the proper edge coloring of X. Now change the color of any edge of colors $c(4), c(5), \ldots, c(d)$ to $c(3)$. The vector for each constituent becomes $(1, 1, d-2)$ all of which are odd. We now may find a cycle for each constituent such that the cyclic truncation is class I by Corollary 5.1. □

We now give a sufficient condition for a multigraph to have a class I cyclic truncation. A definition is required. Let X be an arbitrary multigraph and V_0, V_1, V_2 and V_3 be a partition of $V(X)$, where V_i contains the vertices whose valencies are congruent to i modulo 4. Given a submultigraph Y of X, let $X \setminus Y$ be the submultigraph obtained by removing the edges of Y from X. Moreover, let V_0', V_1', V_2' and V_3' denote the vertices of $V(X)$ whose valencies in $X \setminus Y$ are congruent to i modulo 4. A submultigraph Y of X is called an *enabling submultigraph* if it satisfies:

- if $v \in V_0$, then $v \in V_0'$;
- if $v \in V_1$, then $v \in V_2'$;
- if $v \in V_2$, then $v \in V_0'$; and
- if $v \in V_3$, then $v \in V_2'$.

The preceding conditions are not as complicated as they may look at first. The first thing to notice is that the components of $X \setminus Y$ are Eulerian. Second, for many graphs the conditions are simple. For example, if X is 3-regular, then a perfect matching is an enabling subgraph.

Corollary 5.4. *Let X be a multigraph with an enabling submultigraph Y. If every component of $X \setminus Y$ has even size, then there is a class I cyclic truncation of X.*

Proof. Because each component of $X \setminus Y$ has even size, we may alternately color the edges of an Euler tour of the component with two colors so that we finish with same number of colors on the edges incident with every vertex. Doing this for each component yields a 2-coloring so that each vertex of $X \setminus Y$ has the same number of colors incident with it.

If we now color all the edges of Y with a third color, it is easy to verify that the conditions of Corollary 5.1 are met for X. We check one possibility for illustrative purposes. If $v \in V_3$, it belongs to V_2' in $X \setminus Y$. Thus, the number of edges of Y incident with v is congruent to 1 modulo 4, that is, it is incident to an odd number of edges of the third color in X. Because it is in V_2', it is incident with an odd number of edges of each of the first two colors. Thus, the vector for v has three odd components. □

Proposition 5.5. *If a trivalent graph X has a cut edge, then X is class* II.

The proof of the preceding proposition is immediate and leads to examples such as the following. Let X be the graph of order 10 obtained by taking two vertex-disjoint copies of K_5 and joining the two copies with a single edge from a vertex of one copy to a vertex of the other copy. Lemma 2.1 implies that X is class I. If we take any cyclic truncation $\mathrm{TR}(X)$ of X, then $\mathrm{TR}(X)$ is class II by Proposition 5.5 because the edge of M_F connecting the two copies still is a cut edge in $\mathrm{TR}(X)$.

6 Arboreal truncations

We now examine another generalized truncation. An *arboreal truncation* is a generalized truncation for which every constituent graph is a forest. The following result is useful for arboreal truncations.

Theorem 6.1. *Let $\mathrm{TR}(X)$ be a generalized truncation of a multigraph X. If the maximum valency of $\mathrm{TR}(X)$ is Δ and every constituent of $\mathrm{TR}(X)$ having a vertex of valency Δ is class* I*, then $\mathrm{TR}(X)$ is class* I.

Proof. Let $\mathrm{con}(v)$ have a vertex of valency Δ in $\mathrm{TR}(X)$. Then the maximum valency in the subgraph $\mathrm{con}(v)$ is $\Delta - 1$. Its edges may be properly edge-colored with $\Delta - 1$ colors because it is class I. Any constituent not having a vertex of valency Δ may have its edges properly colored with at most $\Delta - 1$ colors because its maximum valency is $\Delta - 2$. Then the edges of M_F may be colored with a new color yielding a proper edge coloring of $\mathrm{TR}(X)$ with Δ colors. The conclusion now follows. □

Corollary 6.2. *Every arboreal truncation of a multigraph X is class I.*

Proof. This follows immediately from Theorem 6.1 because forests are class I. □

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Convex drawings of the complete graph: topology meets geometry*

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Abstract

In a geometric drawing of K_n, trivially each 3-cycle bounds a convex region: if two vertices are in that region, then so is the (geometric) edge between them. We define a topological drawing D of K_n in the sphere to be *convex* if each 3-cycle bounds a closed region R (either of the two sides of the 3-cycle) such that any two vertices in R have the (topological) edge between them contained in R.

While convex drawings generalize geometric drawings, they specialize topological ones. Therefore it might be surprising if all *optimal* (that is, crossing-minimal) topological drawings of K_n were convex. However, we take a first step to showing that they are convex: we show that if D has a non-convex K_5 all of whose extensions to a K_7 have no other non-convex K_5, then D is not optimal (without reference to the conjecture for the crossing number of K_n). This is the first example of non-trivial local considerations providing sufficient conditions for suboptimality. At our request, Aichholzer has computationally verified that, up to $n = 12$, every optimal drawing of K_n is convex.

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Convexity naturally lends itself to refinements, including *hereditarily convex* (h-convex) and *face convex* (f-convex). The hierarchy rectilinear $\subseteq$ f-convex $\subseteq$ h-convex $\subseteq$ convex $\subseteq$ topological provides links between geometric and topological drawings. It is known that f-convex is equivalent to pseudolinear (generalizing rectilinear) and h-convex is equivalent to pseudospherical (generalizing spherical geodesic). We characterize h-convexity by three forbidden (topological) subdrawings.

This hierarchy provides a framework to consider generalizations of other geometric questions for point sets in the plane. We provide two examples of such questions, namely numbers of empty triangles and existence of convex k-gons.

Keywords: Simple drawings, complete graphs, convex drawings.

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1 Introduction

Hill's long-standing conjecture [14, 10] asserts that the crossing number $\mathrm{cr}(K_n)$ of K_n is equal to

$$H(n) := \frac{1}{4}\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor\left\lfloor\frac{n-2}{2}\right\rfloor\left\lfloor\frac{n-3}{2}\right\rfloor.$$

To date, Hill's conjecture has only been verified up to $n \leq 12$. Moreover, current proofs for $n = 11, 12$ rely on extensive computer searches, therefore providing limited explanation for the elegance of the expression in Hill's conjecture. Guy's [13] original proof for $n = 9, 10$ also relied on an extensive case analysis, with most details left to the reader, similar to a computer proof.

The main point of this work is the introduction of the class of convex drawings of K_n. It turns out that, of the (up to spherical homeomorphisms) five drawings of K_5 in the sphere, the drawings $\widetilde{\mathbb{K}}_5^3$ and $\widetilde{\mathbb{K}}_5^5$ in Figure 1 are not convex in our sense. Furthermore, an elementary but principal result of this work is to characterize (rather than define) a spherical drawing of K_n as convex if and only if neither $\widetilde{\mathbb{K}}_5^3$ nor $\widetilde{\mathbb{K}}_5^5$ occurs as a subdrawing.

Our study of these drawings was motivated by a couple of specific events. One was the computer-free proof by two of the authors that the crossing number of K_9 is 36 [21]. As part of that proof, the two drawings $\widetilde{\mathbb{K}}_5^3$ and $\widetilde{\mathbb{K}}_5^5$ were both shown not to occur in any *optimal* (that is, fewest crossings) drawing of K_7. Another was the question, "Is there, for some n, an optimal drawing (or even one with the conjectured fewest crossings) of K_n that contains $\widetilde{\mathbb{K}}_5^5$?"

One of us asked Tilo Wiedera to check by computer if any optimal drawing of K_9 contains a $\widetilde{\mathbb{K}}_5^5$. Not only was the answer negative as expected, but Wiedera also found that the smallest number of crossings in a drawing of K_9 that contains $\widetilde{\mathbb{K}}_5^5$ has 40 crossings – a surprising 4 more than optimal! At the Crossing Number Workshop in Osnäbruck (May 2017), the authors then asked Aichholzer for the smallest n for which an optimal drawing of K_n could contain it. After checking for both $\widetilde{\mathbb{K}}_5^3$ and $\widetilde{\mathbb{K}}_5^5$ among all optimal drawings for K_n with $n \leq 12$, he announced at the workshop his findings, implying that if such n exists, it must be at least 13. Theorem 5.1 below gives further evidence that the answer to the question is "no".

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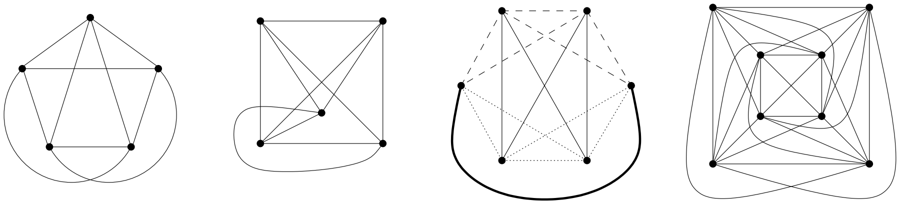

Figure 1: Drawings of interest: $\widetilde{\mathbb{K}}_5^3$, $\widetilde{\mathbb{K}}_5^5$, $\mathbb{K}_6^{11}$, and TC_8.

Thus, we were quite naturally led to the class of simple (i.e., no edge crosses itself and no two closed edges intersect twice; often referred to as "good") drawings of K_n in which neither $\widetilde{\mathbb{K}}_5^3$ nor $\widetilde{\mathbb{K}}_5^5$ occurs; we first thought of these as "locally rectilinear", as, of the (up to spherical homeomorphisms) five drawings of K_5 in the sphere, these are the two that are not isomorphic (via a homeomorphism from the sphere with an appropriate point deleted to the plane) to rectilinear drawings of K_5. We became especially interested in them when we realized they have a topological characterization (Theorem 2.6, below), which we have since taken to be the definition of convex drawings in Definition 1.1 below.

If D is a drawing of a graph G, and H is a subgraph of G (or even a set of vertices and edges of G), then we let $D[H]$ denote the drawing of H induced by D. In a simple drawing D of a graph G, for a 3-cycle T of G, $D[T]$ is a simple closed curve.

Definition 1.1. Let D be a simple drawing of K_n in the sphere.

1. If T is a 3-cycle in K_n, then a closed disc Δ bounded by $D[T]$ is *a convex side of* T if, for any distinct vertices x and y of K_n such that $D[x]$ and $D[y]$ are both contained in Δ, then $D[xy]$ is also contained in Δ.

2. The drawing D is *convex* if every 3-cycle of K_n has a convex side.

Evidently every rectilinear or spherical geodesic drawing is convex. Therefore, the "tin can" drawing TC_8 of K_8 shown in Figure 1 is convex (see Section 2); it is not homeomorphic to any rectilinear drawing. Indeed, K_8 is known to have rectilinear crossing number of 19, while TC_8 is optimal with the minimum of 18 crossings.

Wagner [25] showed that establishing the Hill Conjecture for spherical geodesic drawings is equivalent to the special case of the Spherical Generalized Upper Bound Conjecture for arrangements of n hemispheres in an $(n-4)$-dimensional sphere. In particular, proving the convex or h-convex crossing number of K_n is $H(n)$ would establish this case of the SGUBC. Flag algebras are used by Balogh et al. [7] to show that $\lim_{n\to\infty} \mathrm{cr}(K_n)/H(n) > 0.985$. Restricted to convex drawings, the same technique gets a lower bound of 0.996. The 0.996 is also the lower bound for spherical geodesic drawings (these are all necessarily convex).

There are two natural refinements of Definition 1.1; the first is satisfied by spherical geodesic drawings, while the second holds for rectilinear and pseudolinear drawings.

Definition 1.2. Let D be a convex drawing of K_n.

1. Then D is *hereditarily convex* (abbreviated to h-convex) if, for every 3-cycle T, there is a choice Δ_T of a convex side, such that, if T_1 and T_2 are 3-cycles with $D[T_2] \subseteq \Delta_{T_1}$, then $\Delta_{T_2} \subseteq \Delta_{T_1}$.

2. Then D is *face convex* (abbreviated to f-convex) if there is a face Γ of D such that, for every 3-cycle T of K_n, the side of $D[T]$ disjoint from Γ is convex.

The drawing $\mathbb{K}_6^{11}$ of K_6 shown in Figure 1 is convex but not h-convex. The (optimal) drawing TC_8 of K_8 is h-convex but not f-convex.

It is an easy exercise to prove that an f-convex drawing is also h-convex. Moreover, every rectilinear (or, more generally, pseudolinear) drawing of K_n is f-convex, with (in both cases) Γ being the unbounded face. In fact, Aichholzer et al. [4] and, independently, the current authors [5], have shown that f-convex is equivalent to pseudolinear. Generalizing spherical geodesic drawings, Arroyo et al. [6] have introduced a natural notion of "pseudospherical drawings" of K_n in the sphere; surprisingly, they are exactly the h-convex drawings.

It would be very interesting to obtain an analogous "geometric-style" generalization that characterizes convexity. At this time, we have no suggestion for what this might be.

Thus, our definitions regarding drawings of complete graphs correspond precisely to geometric descriptions of point-sets. These geometric connections open up new possibilities for studying geometric questions to see how the results differ for convex drawings. For complete graphs, we now have a geometrically meaningful hierarchy of drawings. It is, from most to least restrictive: rectilinear $\subseteq$ f-convex (= pseudolinear) $\subseteq$ h-convex (= pseudospherical) $\subseteq$ convex $\subseteq$ topological.

One question of long-standing interest is: given n points in general position in the plane, how many of the 3-tuples (that is, triangles) have none of the other points inside the triangle (empty triangle)? Currently, we know that there can be as few as about $1.6n^2+\mathrm{o}(n^2)$ empty triangles [9] and every set of n points has at least $n^2+O(n)$ empty triangles ($n^2+o(n^2)$ first proved in [8]). In [5], we proved the $n^2+o(n^2)$ bound also holds for f-convex drawings. At the other extreme, Harborth [15] presented an example of a topological drawing of K_n having only $2n-4$ empty triangles, while Aichholzer et al. [3] show that every topological drawing of K_n has at least n empty triangles. We have shown in [5] that every convex drawing of K_n has at least $\frac{1}{3}n^2+O(n)$ empty triangles. For h-convex, it is shown in [6], using the f-convex result and other facts about h-convex drawings, that there are at least $\frac{3}{4}n^2+o(n^2)$ empty triangles. We would be interested in progress related to the coefficients $\frac{1}{3}$ and $\frac{3}{4}$.

Another question of interest is: given n points in general position in the plane, what is the largest k so that k of the n points are the corners of a k-gon in convex position? In Theorem 3.4, we generalize to convex drawings the Erdős-Szekeres theorem [12] that, for every k, there is an n such that every set of n points in the plane in general position has a set of k points that are the corners of a convex k-gon. Finding the least such n is of current interest. Suk [24] has shown that $2^{k+\mathrm{o}(k)}$ points suffices in the geometric case. For $k=5$, 9 points is best possible in the rectilinear case (see Bonnice [11] for a short proof).

For a general drawing D of K_n, we can ask whether there is a subdrawing $D[K_k]$ such that one face is bounded by a k-cycle: this is a *natural drawing of* K_k. (In [21], these drawings are quite appropriately labelled "convex". We think convex is very descriptive of the drawings considered in this work and expect there to be no confusion with the two quite different uses of the term "convex".) Bonnice's proof adapts easily to the pseudolinear case (that is, the f-convex case). Aichholzer (personal communication) has verified by computer that 11 points is best possible for $k=5$ for the convex drawings considered in this article. A trivial consequence of our Theorem 4.5 characterizing h-convex drawings is that any convex, but not h-convex, drawing has a natural K_5; therefore, 11 is also best possible for h-convex drawings. For general drawings of K_n, there need not be a natural K_5. In Harborth's example [15] (originally from [16]), every K_5 is isomorphic to $\widetilde{\mathbb{K}}_5^5$.

We remark that Pach et al. [23] show that, for any fixed positive integer r, there is a large enough integer $N = N(r)$ such that, for $n \geq N$, every simple drawing of K_n contains either a natural K_r or Harborth's generalization of $\widetilde{\mathbb{K}}_5^5$ on r vertices. In Harborth's generalization, every K_5 is isomorphic to $\widetilde{\mathbb{K}}_5^5$. These are the "twisted" K_rs in [23].

Section 2 introduces many fundamental properties of a convex drawing D of K_n, including showing convexity of D is equivalent to not containing either of the two non-rectilinear drawings $\widetilde{\mathbb{K}}_5^3$ and $\widetilde{\mathbb{K}}_5^5$ of K_5 (the first two drawings in Figure 1). Another equivalence is that every 3-cycle T has a side such that every vertex v on that side is such that $D[T + v]$ is a non-crossing K_4.

Section 3 proves that a convex drawing D of K_n has a particularly nice structure: there is a natural K_r such that $D[K_r]$ has a face Γ bounded by an r-cycle C; if $D[v]$ is in Γ and $D[w]$ is in the closure of Γ, then $D[vw]$ is in $\Gamma \cup \{D[w]\}$; and if $D[v]$ and $D[w]$ are in the closure of the complement of Γ, then $D[vw]$ is in the complement of Γ. This structure theorem may provide a strategy for showing that a convex drawing of K_n has at least $H(n)$ crossings.

Section 4 treats h-convex drawings. The main result here is that a convex drawing D is h-convex if and only if there is no K_6 such that $D[K_6]$ is $\mathbb{K}_6^{11}$ in Figure 1. We do not know a comparable result distinguishing f-convex drawings from h-convex. The tin can drawing TC_8 of K_8 in Figure 1 is one such (as are the larger tin can drawings). However it is not clear to us whether TC_8 is the only minimal one or, indeed, if there are only finitely many minimal distinguishing examples. The final result of the section is that testing a set of convex sides for h-convexity is also a "Four Point Property", which is to say that it can be verified by checking all sets of four points.

Finally, in Section 5, we prove the principal result Theorem 5.1, showing that some non-convex drawings of K_n are not optimal.

Table 1: The convexity hierarchy.

level	characterization	distinguish
general	edges share ≤ 1 point	
convex	general, no $\widetilde{\mathbb{K}}_5^3$, $\widetilde{\mathbb{K}}_5^5$	$\widetilde{\mathbb{K}}_5^3$
h-convex	convex, no $\mathbb{K}_6^{11}$	$\mathbb{K}_6^{11}$
f-convex	h-convex + ??	TC_8
rectilinear	unlikely	Pappus

This work will provide characterizations of the different kinds of convexity and distinguishing between them by examples and theorems. A summary is given in Table 1.

Matoušek [19] gives a nice exposition of the theorem of Mnëv [22] that testing the stretchability of an arrangement of pseudolines in the plane is $\exists\mathbb{R}$-complete. A straightforward application of Levi's Enlargement Lemma [18] (see also [5]) turns such an arrangement with n pseudolines into a pseudolinear drawing of K_{2n} that is not stretchable. Hence, unless P=NP=∃ℝ, there are infinitely many non-stretchable f-convex drawings of K_n.

2 Convex drawings

In this section we introduce the basics of convexity. We already mentioned in the introduction that the two drawings $\widetilde{\mathbb{K}}_5^3$ and $\widetilde{\mathbb{K}}_5^5$ of K_5 in Figure 1 are not convex. In fact, their absence characterizes convexity. We first prove some intermediate results that make this completely clear. Our first observation is immediate from the definition of convex side and is surprisingly useful.

Observation 2.1. *If J is such that $D[J]$ is a crossing K_4, and T is a 3-cycle in J, then the side of $D[T]$ containing the fourth vertex in J is not convex.* □

We had some difficulty deciding on the right definition of convexity. At the level of individual 3-cycles, the definition given in the introduction makes more sense. At the level of a drawing being convex, there is a simpler one, as shown in the next lemma and, more particularly, its Corollary 2.4: we only need to test single points in the closed disc Δ and how they connect to the three corners.

Definition 2.2. Let D be a drawing of K_n, let T be a 3-cycle in K_n, and let Δ be a closed disc bounded by $D[T]$. Then Δ has the *Four Point Property* if, for every vertex v of K_n not in T such that $D[v] \in \Delta$, $D[T+v]$ is a non-crossing K_4.

Lemma 2.3. *Let D be a drawing of K_5 such that the side Δ of a 3-cycle T has the Four Point Property. Suppose u and v are vertices of K_5 such that $D[u], D[v] \in \Delta$. If $D[uv]$ is not contained in Δ, then there is a vertex b of T such that neither side of the 3-cycle induced by u, v, and b satisfies the Four Point Property; in particular, neither side is convex.*

Proof. Since Δ_T has the Four Point Property, neither u nor v is in T. Because $D[u]$ and $D[v]$ are both on the same side of $D[T]$, $D[uv]$ crosses $D[T]$ an even number of times. However, $D[uv]$ crosses each of the three sides of $D[T]$ at most once, so $D[uv]$ crosses $D[T]$ at most three times. Thus, $D[uv]$ crosses $D[T]$ either 0 or 2 times.

As $D[uv]$ is not contained in Δ_T, $D[uv]$ crosses $D[T]$ a positive number of times. We conclude they cross exactly twice. Label the vertices of T as a, b, and c so that $D[uv]$ crosses both $D[ab]$ and $D[ac]$.

Since $D[T+u]$ is a non-crossing K_4, the three edges of $T+u$ incident with u partition Δ_T into three faces, each incident with a different two of a, b, and c. Because $D[uv]$ crosses $D[ab]$ and $D[ac]$, but not any of the three edges of $T+u$ incident with u, v must be in one of the faces of $D[T+u]$ incident with a. We choose the labelling so that v is in the face of $D[T+u]$ incident with both a and c.

The Four Point Property implies $D[vb]$ is contained in Δ_T. It must cross either $D[ua]$ or $D[uc]$. To show that it crosses $D[uc]$, we assume by way of contradiction that it crosses $D[ua]$. Let $\times$ be the point where $D[ab]$ crosses $D[uv]$. Then $D[vb]$ must exit the region incident with a, u, and $\times$, but it cannot cross either $D[ab]$ or $D[uv]$, and it cannot cross $D[au]$ a second time. This contradiction shows $D[vb]$ crosses $D[uc]$.

Therefore $D[T+\{u,v\}]$ is isomorphic to $\widetilde{\mathbb{K}}_5^3$, with u and v being the upper left and upper right vertices, respectively, and a the peak in Figure 1. Letting b and c be the lower left and lower right vertices, respectively, the 3-cycles uvb and uvc have no convex side. □

Obviously, if Δ is a convex side of $D[T]$, then Δ has the Four Point Property. The following converse is an immediate consequence of Lemma 2.3.

Corollary 2.4. *Let D be a drawing of K_n and, for each 3-cycle T in K_n, let Δ_T be a closed disc bounded by $D[T]$. Suppose, for each T, Δ_T has the Four Point Property. Then each Δ_T is convex; in particular, D is convex.* □

Our next two corollaries yield additional characterizations of convexity.

Corollary 2.5. *Let D be a drawing of K_n.*

(a) *Then D is not convex if and only if there exists a 3-cycle T of K_n and vertices u, w of K_n, one in the interior of each side of $D[T]$, such that both $D[T+u]$ and $D[T+w]$ are crossing K_4's.*

(b) *If D is convex and T is a 3-cycle in K_n, then a side Δ of $D[T]$ is convex if and only if it satisfies the Four Point Property.*

Proof. For Item (a), Observation 2.1 shows that if D is convex, then no such 3-cycle can exist. Conversely, Corollary 2.4 implies that some 3-cycle T of K_n does not have a side that satisfies the Four Point Property. This implies that, for each side Δ of $D[T]$, there is a vertex v_Δ such that $D[T + v_\Delta]$ is a crossing K_4, as required.

The proof of Item (b) is direct from the definition of convex side and Lemma 2.3. □

We came to the concept of convexity by considering drawings of K_n without the two drawings $\widetilde{\mathbb{K}}_5^3$ and $\widetilde{\mathbb{K}}_5^5$ (see Figure 1) of K_5 for reasons that have been subsumed by some of the developments described in this article. Since the remaining drawings of K_5 are rectilinear, we think of such drawings of K_n as locally rectilinear. Our next result is the surprising equivalence with convexity and this led us to consider convexity and its strengthenings to h- and f-convex.

Theorem 2.6. *A drawing D of K_n is convex if and only if, for every subgraph J of K_n isomorphic to K_5, $D[J]$ is not isomorphic to either $\widetilde{\mathbb{K}}_5^3$ or $\widetilde{\mathbb{K}}_5^5$.*

Proof. In the drawing of $\widetilde{\mathbb{K}}_5^3$ in Figure 1, we see that a 3-cycle consisting of one of the edges that is not a straight segment together with the longer horizontal edge has no convex side. In the drawing of $\widetilde{\mathbb{K}}_5^5$, there are two 3-cycles that have the "interior vertex" in their interiors. Neither of these 3-cycles is convex. Thus, these two drawings of K_5 cannot occur in a convex drawing of K_n.

Conversely, in a rectilinear drawing of K_5, the bounded side of each 3-cycle has the Four Point Property. Thus, Corollary 2.4 shows a rectilinear drawing of K_5 is convex. On the other hand, Corollary 2.5 shows every non-convex drawing of K_n contains a non-convex drawing of K_5. Such a drawing is either $\widetilde{\mathbb{K}}_5^3$ or $\widetilde{\mathbb{K}}_5^5$. □

Theorem 2.6 has the following simple corollary. (To help with phrasing, we refer to an *isomorph J of K_r* to mean J is a specific one of the complete subgraphs of K_n having r vertices.)

Corollary 2.7. *Let D be a drawing of K_n. Suppose, for every isomorph J of K_5, there is, for some $m \leq 12$, an isomorph L of K_m containing J such that $D[L]$ is an optimal drawing of K_m. Then D is convex.*

Proof. Aichholzer [2] has verified that $D[L]$ is convex, so $D[J]$ is a convex K_5. □

As illustration of the utility of Corollary 2.7, Ábrego et al. [1] exhibit, for odd n, a drawing $N_{n,n,1}$ of K_{2n+1} having $H(2n+1)$ crossings, with every edge involved in at least one crossing. We argue here that it is convex; furthermore, we believe that analogous arguments apply to any of the known examples of drawings of K_n having $H(n)$ crossings.

We start with the tin can drawing TC_{2n} of K_{2n} (TC_8 is illustrated in Figure 1; in this figure, the labelling described below is counterclockwise). This has concentric regular n-gons, the outside one labelled $u_0, \dots, u_{n-1}$ and the inside one labelled $v_0, \dots, v_{n-1}$. For each i, u_i and v_i are roughly "antipodal", so u_i is closest to $v_{i+\lfloor n/2 \rfloor}$. We use the non-self-crossing perfect matching consisting of the edges $u_i v_{i+\lfloor n/2 \rfloor}$ to work from (the indices are always read modulo n). We join u_i to $v_{i+\lfloor n/2 \rfloor+1}, v_{i+\lfloor n/2 \rfloor+2}, \dots, v_{i-1}$ in one direction from $u_i v_{i+\lfloor n/2 \rfloor}$ around the cylinder, and to $v_{i+\lfloor n/2 \rfloor-1}, v_{i+\lfloor n/2 \rfloor-2}, \dots, v_{i+1}, v_i$ in the other direction. The two sides are as equal as possible. Thus, the rotation at u_i is

$$u_{i+1}, u_{i+2}, \dots, u_{i-1}, v_i, v_{i+1}, \dots v_{i-1} .$$

We mentioned in the introduction that this is homeomorphic to a spherical drawing and therefore convex. However, an earlier referee requested more detail for proofs of convexity, so we provide the argument for TC_{2n} and adapt it to prove the convexity of $N_{n,n,1}$.

To argue that TC_{2n} is convex, let V be any set of five vertices in TC_{2n}. These vertices are covered by at most five edges of the matching M consisting of the antipodal edges $u_i v_i$. Add one or two more edges from M as needed to get up to five edges, and use the induced drawing on these 10 vertices. This is TC_{10} because the induced rotations of the edges at the 10 vertices satisfy the rotation described in the preceding paragraph. This is an optimal drawing of K_{10}, so Corollary 2.7 shows TC_{2n} is convex.

To get the Ábrego et al. drawing $N_{n,n,1}$, add a vertex w to TC_{2n} near the centre of the concentric n-gons. This is joined to the $2n$ vertices by straight line segments, creating the drawing TC^+_{2n}. The drawing $N_{n,n,1}$ is completed by redrawing the edges $u_i v_i$ of TC^+_{2n} as illustrated in Figure 2; these are the green edges in their Figure 4. Their labelling and ours coincide except that our indices on u_i and v_i are one less than theirs.

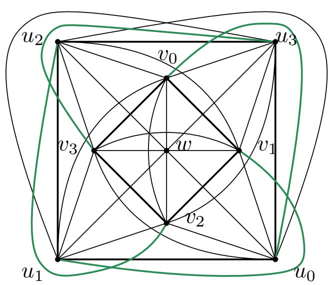

Figure 2: $N_{4,4,1}$.

Many of the K_5's in $N_{n,n,1}$ are exactly the same in TC^+_{2n}; all of these are convex. This is even true for a K_5 that contains just one green edge, say $u_0 v_0$ and no vertex from

$u_1, u_2, \ldots, u_{\lfloor n/2 \rfloor}$: the drawings of this K_5 in $N_{n,n,1}$ and TC^+_{2n} are isomorphic. This is helpful in dealing with the case there is only one green edge: we may assume such a K_5 also has at least one of the u_j listed in the preceding paragraph.

In case of a single green edge and the green edge is not involved in a crossing, we make use of the following simple observation.

Observation 2.8. *Let D be a convex drawing of K_5 and let D' be another (simple) drawing of K_5 obtained from D by rerouting a single edge e. If e is uncrossed in D', then D' is convex.*

Proof. If e is uncrossed in D, then D and D' are homeomorphic drawings and the result is trivial. Otherwise, the edge e is crossed in D, so $\mathrm{cr}(D') < \mathrm{cr}(D)$. These numbers are both odd (Kleitman [17]) and at most 5. Since the spherical drawing of K_5 with one crossing is convex, the only non-trivial case is D is the natural drawing of K_5.

In this case, e is one of the crossed edges; these are all the same. Only the face of $G - e$ bounded by the 5-cycle is incident with both ends of e. Thus, D' is a homeomorph of the rectilinear drawing whose outer face is bounded by a 4-cycle. □

Next, consider a K_5 having a unique green edge crossed in the K_5. With 4 vertices determined, if the fifth vertex is w or v_j, then the K_5 is optimal and hence convex. Otherwise, there are four "u vertices" and it is easy to see that it is one of the rectilinear K_5's.

Lastly, there may be two green edges. Except for one exceptional case when n is even, these edges must cross. Moreover, they give us four of the vertices of the K_5 and it is not difficult to check the different possible locations for the remaining vertex.

The following definition and corollary to Lemma 2.3 will be used in the next section for the structural description of a convex drawing.

Definition 2.9. Let D be a drawing of K_n, let u be a vertex of K_n, and let J be a complete subgraph of $K_n - u$. If $D[J]$ is natural and $D[u]$ is in the face of $D[J]$ bounded by a $|V(J)|$-cycle, then u is *planarly joined to* J if no edge from $D[u]$ to $D[J]$ crosses any edge of J.

Corollary 2.10. *Let D be a convex drawing of K_5 with vertices u, v such that $D - \{u, v\}$ is the 3-cycle T. If $D[u]$ and $D[v]$ are in the same face of $D[T]$ and u and v are both planarly joined to T, then $D[uv]$ is in the same face of $D[T]$ as $D[u]$ and $D[v]$.* □

A further perusal of the five drawings of K_5 shows that the following further refinement of forbidden substructures is possible. This configuration was mentioned at Crossing Number Workshop 2015 (Rio de Janeiro) in the context of being one forbidden configuration for a drawing of an arbitrary graph to be pseudolinear. The proof is quite straightforward.

Lemma 2.11. *Let D be a drawing of K_n. Then D is convex if and only if, for every path P of length 4, $D[P]$ is not isomorphic to $\widetilde{\mathbb{P}}_4$.* □

We will use the following observation in Section 5. Its proof, left to the reader, is a good exercise in using the fact that no two closed edges can have two points in common.

Observation 2.12. *Let D be a drawing of K_5 in which some 3-cycle is crossed three times by a single edge. Then D is $\widetilde{\mathbb{K}}^5_5$ (as in* Figure 1*).* □

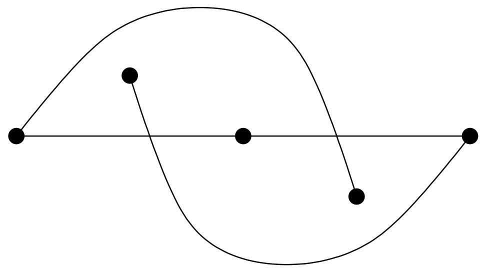

Figure 3: The drawing $\widetilde{\mathbb{P}}_4$.

3 Convexity and natural drawings of K_n

We recall from Section 1 that a *natural drawing* of K_n is a drawing in which an n-cycle bounds a face Γ. It is easy to see that, in any natural drawing of K_n, every 3-cycle T has a side Δ_T that is disjoint from Γ and there is no vertex of K_n in the interior of Δ_T. Thus, Γ and the Δ_T show that a natural drawing of K_n is f-convex.

In this section, we show that if D is a convex drawing of K_n with the maximum number $\binom{n}{4}$ of crossings, then D is a natural drawing of K_n. This leads us to a structure theorem for convex drawings of K_n whose central piece is, for some $r \geq 4$, a natural drawing of K_r. It also leads to the Erdős-Szekeres Theorem for convex drawings: for every $r \geq 5$, if n is sufficiently large, then every convex drawing of K_n contains a natural K_r.

Lemma 3.1. *Let D be a drawing of K_n. Then D is a convex drawing of K_n with $\binom{n}{4}$ crossings if and only if D is a natural drawing of K_n.*

Proof. One direction is trivial; for the other, let D be a convex drawing of K_n with $\binom{n}{4}$ crossings; all K_4's are crossing in D. We proceed by induction on n, the cases $n \leq 5$ being trivial. Let v_0 be any vertex of K_n and let H be the Hamilton cycle of $K_n - v_0$ such that $D[H]$ bounds a face F of $D[K_n - v_0]$.

Claim 3.2. If T is a 3-cycle in K_n, then one side of $D[T]$ has no vertex drawn in its interior.

Proof. Otherwise, the convex side Δ of $D[T]$ has a vertex v with $D[v]$ in the interior of Δ. This yields the contradiction that $D[T + v]$ is a non-crossing K_4. □

In particular, Claim 3.2 applied to all the 3-cycles in $K_n - v_0$ shows v_0 is in F.

Claim 3.3. Suppose x, y, z are distinct vertices of $K_n - v_0$ such that $yz \in E(H)$ and the edge $D[v_0x]$ crosses the edge $D[yz]$. Then, for any vertex w of $K_n - \{v_0, y, z\}$:

(a) $D[v_0w]$ crosses $D[yz]$; and

(b) yz is the only edge of H crossed by v_0w.

Proof. For (a), if $D[v_0w]$ does not cross $D[yz]$, then the 3-cycle $D[v_0wx]$ has $D[y]$ and $D[z]$ on different sides, contradicting Claim 3.2.

For (b), in traversing v_0w, let ab be the first edge of $K_n - v_0$ such that $D[v_0w]$ crosses $D[ab]$. Then ab is in H and v_0w does not cross either aw or bw. That is, the portion of $D[v_0w]$ from the crossing with $D[ab]$ to $D[w]$ is contained inside the 3-cycle $D[abw]$. □

Since every K_4 is crossing, v_0 is not planarly joined to $D - v_0$. Therefore, there is an edge v_0x that crosses an edge yz of H. Claim 3.3(a) shows that, for any vertex w of $D - \{v_0, y, z\}$, v_0w also crosses yz.

If v_0y crosses some edge $y'z'$ of H, then Claim 3.3(a) shows that, for any vertex w of $D - \{v_0, y', z'\}$ also crosses $y'z'$. Note that $y \notin \{y', z'\}$. Since $n \geq 6$, there is a w_0 different from all of v_0, y, z, y', z', so v_0w_0 crosses both yz and $y'z'$, contradicting Claim 3.3(b).

Replacing the edge yz of H with the path y, v_0, z yields a Hamilton cycle in D that proves D is a natural drawing. □

The convex version of the Erdős–Szekeres Theorem is an immediate consequence of the main result of Pach et al. [23]. Their bounds are better than those coming from our Ramsey argument. Lemma 3.1 and Ramsey's Theorem also easily prove the convex version of the Erdős–Szekeres Theorem. We suppose $r \geq 5$ is an integer and choose n large enough so that some subset of $V(K_n)$ of size r is such that every K_4 in the K_r is crossing. (For $r \geq 5$, they cannot all be non-crossing.) If the drawing D of K_n is convex, Lemma 3.1 implies $D[K_r]$ is natural. We state the theorem here for reference.

Theorem 3.4. *Let $r \geq 5$ be an integer. Then there is an integer $N = N(r)$ such that, if $n \geq N$ and D is a convex drawing of K_n, then there is a subgraph J of K_n isomorphic to K_r such that $D[J]$ is a natural K_r.* □

The remainder of this section is devoted to a structure theorem for convex drawings. Let D be a convex drawing of K_n and, for some $r \geq 4$, let J be a K_r in K_n such that $D[J]$ is natural. We set C_J to be the facial r-cycle in $D[J]$. We refer to the face of $D[J]$ bounded by C_J as the *outside* of J and the other side of C_J as the *inside* of J.

The proof uses the following elementary observations that are somewhat interesting and otherwise useful in their own right.

Lemma 3.5. *Let D be a convex drawing of K_n and, for some $r \geq 4$, let J be a K_r such that $D[J]$ is natural, with facial r-cycle C_J.*

(a) *If u is inside J, then, for each $v \in V(J)$, $D[uv]$ is inside J.*

(b) *If u and v are both inside J, then $D[uv]$ is inside J.*

(c) *If u and v are both outside J and planarly joined to J, then $D[uv]$ is contained in the outside of J.*

(d) *Let u be outside of J and suppose there is a vertex v of J such that $D[uv]$ crosses C_J. Then $D[uv]$ crosses C_J exactly once.*

(e) *Suppose u is outside of J but, for vertices v and w of J, $D[uv]$ and $D[uw]$ both cross C_J. Let e and f be the edges of C_J crossed by $D[uv]$ and $D[uw]$. Then v and w are in the same component of $C_J - \{e, f\}$.*

(f) *Suppose u is outside of J, v is a vertex of J, and $D[uv]$ crosses C_J on the edge ab. Then $D[ua]$ and $D[ub]$ are contained in the outside of J.*

Proof. We start with (a). If we consider the edges of J incident with v, they partition the inside of J into discs bounded by 3-cycles. As $|V(J)| \geq 4$, the disc containing u is the convex side of its bounding 3-cycle. Thus, $D[uv]$ is inside this disc and so is inside J.

For (b), we present an argument suggested by Kasper Szabo Lyngsie that simplifies our original. There is an edge xy in C_J such that v is in the side Δ of $D[uxy]$ that has no vertices of $J - \{x, y\}$. If there is an edge of J incident with either x or y that crosses the 3-cycle uxy, then v is in the crossing side of a natural K_4 containing u, x, and y. In this case, Δ is the convex side of uxy, so $D[uv]$ is inside Δ.

In the other case, let x' and y' be the neighbours of x and y, respectively, in $C_J - xy$. Then Δ is contained in the convex side Δ' of the 3-cycle $x'xy$, and again $D[uv]$ is contained in Δ' and consequently inside J. (We remark that, in fact, $D[uv]$ is contained inside Δ, but vx or vy might cross uxy, so Δ need not be the convex side of uxy.)

Moving on to (c), let x, y, z be any three vertices of J and let L be the K_5 induced by u, v, x, y, z. Then $D[L]$ is a convex drawing of K_5. Let T be the 3-cycle (x, y, z). The assumption that u and v are planarly joined to T in D shows that the side Δ_T of T that contains u and v satisfies the Four Point Property in $D[L]$.

Corollary 2.10 implies that $D[uv]$ is contained in Δ_T. This is true for every three vertices of J, so $D[uv]$ is contained in the intersection of all the Δ_T's; this is precisely the closure of the face of $D[J]$ containing $D[u]$ and $D[v]$, as required.

In the proof of (d), we suppose the first crossing of uv is with the edge xy of C_J. The 3-cycle xyv is inside J and, by the definition of drawing, uv cannot cross xyv a second time.

Turning to (e), we suppose that v and w are in different components of $C_J - \{e, f\}$ and that uv crosses e, while uw crosses f. Let x be the end of e in the component of $C_J - \{e, f\}$ containing v and let y be the end of f in the component of $C_J - \{e, f\}$ containing w. By the definition of drawing, $x \neq v$ and $y \neq w$.

The edge xw crosses uv and the edge yv crosses uw. Moreover, x and y are on different sides of the 3-cycle uvw, so uvw has no convex side, a contradiction.

For (f), it suffices by symmetry to show $D[ua]$ is outside J. In the alternative, ua crosses C_J. Since it cannot cross uv by goodness, it must cross the av-subpath of $C_J - ab$. But now ua and uv violate (e). □

We now turn to the basic ingredient in the structure theorem. Let D be a convex drawing of K_n and, for some $r \geq 4$, let J be a K_r such that $D[J]$ is natural. The *J-induced drawing* $\bar{J}$ consists of the subdrawing induced by $D[J]$ and all vertices inside of J. The following is the main point in the proof of the structure theorem.

Lemma 3.6. *Let D be a convex drawing of K_n and, for some $r \geq 4$, let J be a K_r such that $D[J]$ is natural. If there is a vertex u outside J and a vertex v of J such that $D[uv]$ crosses C_J, then there is, for some $s \geq 4$, a K_s-subgraph J' including u such that $D[J']$ is natural and $\bar{J} \subset \bar{J}'$.*

Proof. Let ab be the edge of C_J crossed by uv. Lemma 3.5(f) implies that $D[ua]$ and $D[ub]$ are contained in the outside of J. It follows that, in the av-subpath of $C_J - ab$, there is a vertex w_a nearest v such that $D[uw_a]$ is contained in the outside of J. Likewise, there is a nearest such vertex w_b in the bv-subpath.

For any internal vertex x in the $w_a w_b$-subpath P of $C_J - ab$, the edge ux must cross C_J; Lemma 3.5(d) and (f) imply ux must cross $w_a w_b$. It follows that ux does not cross P. Thus, the cycle consisting of u, together with P, makes the facial cycle for a natural K_s ($s = 1 + |V(P)| \geq 4$) and all the points of $\bar{J}$ are on or inside this K_s. □

Our structure theorem is an immediate consequence of Lemmas 3.5(c) and 3.6.

Theorem 3.7 (Structure Theorem)**.** *Let $n \geq 5$ and let D be a convex drawing of K_n. Then, for some $r \geq 4$, there is a K_r-subgraph J such that $D[J]$ is natural, every vertex outside of J is planarly joined to J, and any two vertices outside J are joined outside J.* □

As a consequence of the Structure Theorem, we have the following straightforward, though not trivial, observation. As it is not of immediate interest, we give only a proof sketch.

Theorem 3.8. *Let $n \geq 5$ and let D be a drawing of K_n. Suppose that, for every subgraph J of K_n that is isomorphic to a K_4 and $D[J]$ has a crossing, there are no vertices of K_n inside $D[J]$. Then $D[K_n]$ is convex and either:*

1. *a natural K_n; or*
2. *a natural K_{n-1} with one vertex outside that is planarly joined to the K_{n-1}; or*
3. *the unique drawing of K_6 with three crossings.*

Proof sketch. The hypothesis on the crossing K_4's implies the drawing is convex: in both $\widetilde{\mathbb{K}}_5^3$ and $\widetilde{\mathbb{K}}_5^5$, there is a crossing K_4 with a vertex inside the K_4.

Apply the Structure Theorem 3.7 to D to get a subgraph J of K_n such that $D[J]$ is a natural K_r, with $r \geq 4$ and every other vertex of K_n is either inside $D[J]$ or is outside J and planarly joined to J.

Any vertex inside $D[J]$ is in a face that is incident with a crossing of some crossing K_4 involving four vertices in J. Since this is forbidden, there is no vertex inside $D[J]$.

If there are three vertices of K_n outside $D[J]$, then there is a crossing K_4 with a vertex inside.

If there are two vertices u, v of K_n outside $D[J]$ and some edge from u to J crosses two edges from v to J, then there is a crossing K_4 with a vertex inside. In particular, if $r \geq 5$, then there is at most one vertex outside J.

The remaining case is $r = 4$ and no uJ-edge crosses two vJ-edges and no vJ-edge crosses two uJ-edges. This is the unique drawing of K_6 with three crossings. □

In general, if, in a convex drawing of K_n, we bound by a non-negative integer p the number of vertices allowed inside any natural K_4, there is a theorem in the spirit of Theorem 3.8. There are more special cases with n small, but if n is large enough (on the order of $3p$), the structure is: a natural K_r, with r at least roughly $p/3$, and at most one of the remaining points is outside the natural K_r.

4 h-convex drawings

In this section, we investigate h-convex drawings. Our main result is a characterization of h-convex drawings; this immediately yields a polynomial time algorithm for determining if a drawing is h-convex.

Consider the drawing $\mathbb{K}_6^{11}$. It is convex, but not h-convex. To see that it is convex, it suffices to check the six K_5's and observe that none of them is either $\widetilde{\mathbb{K}}_5^3$ or $\widetilde{\mathbb{K}}_5^5$. To see that it is not h-convex, consider the dashed K_4 (including the thick edge) highlighted in Figure 1. For this K_4, either of the 3-cycles T containing the thick edge has its bounded (in the figure) side convex, while its unbounded side is not convex. A similar statement holds for the a 3-cycle in the dotted K_4 that contains the thick edge (bounded and unbounded sides reversing roles). These 3-cycles show that D is not h-convex.

Definition 4.1. Let D be a drawing of K_n and let J and J' be distinct K_4's in D such that both $D[J]$ and $D[J']$ are crossing K_4's. For 3-cycles T and T' in J and J', respectively, let Δ_T and $\Delta_{T'}$ be the sides of T and T', respectively, not containing the fourth vertex of J and J', respectively. Then J and J' are *inverted K_4's in D* if there are 3-cycles T in J and T' in J' such that $D[T] \subseteq \Delta_{T'}$ but $\Delta_T \not\subseteq \Delta_{T'}$.

The following observation is immediate from the definition.

Observation 4.2. *Let J and J' be inverted K_4's in a drawing D of K_n and let T and T' be 3-cycles in J and J', respectively. Let Δ_T and $\Delta_{T'}$ be the sides of T and T', respectively, not containing the fourth vertex of J and J', respectively. If $D[T] \subseteq \Delta_{T'}$ but $\Delta_T \not\subseteq \Delta_{T'}$, then $D[T'] \subseteq \Delta_T$ but $\Delta_{T'} \not\subseteq \Delta_T$.* □

We are ready for our first characterization of h-convex drawings.

Lemma 4.3. *Let D be a convex drawing of K_n. Then D is h-convex if and only if there are no inverted K_4's.*

Proof. It is clear that if D is h-convex, then there are no inverted K_4's.

For the converse, we shall inductively obtain a list $\mathcal{C}$ of convex sides, one for each 3-cycle of K_n. Along the way, the list $\mathcal{C}$ will have convex sides for some, but not all, of the 3-cycles of K_n. Such a partial list is *hereditary* if, for any 3-cycles T and T' having convex sides Δ_T and $\Delta_{T'}$, respectively, in $\mathcal{C}$, if $D[T] \subseteq \Delta_{T'}$, then $\Delta_T \subseteq \Delta_{T'}$.

Our initial list $\mathcal{C}_0$ consists of the convex sides for every 3-cycle that is in a crossing K_4. The assumption that there are no inverted K_4's immediately implies $\mathcal{C}_0$ is hereditary.

Let $T_1, \dots, T_r$ be the 3-cycles in K_n such that, for $i = 1, 2, \dots, r$, T_i is not in any crossing K_4. For $j \geq 1$, suppose that $\mathcal{C}_{j-1}$ is a hereditary list of convex sides that includes $\mathcal{C}_0$ and a convex side for each of $T_1, \dots, T_{j-1}$.

If there is a convex side $\Delta_T \in \mathcal{C}_{j-1}$ such that $D[T_j] \subseteq \Delta_T$, then we choose Δ_{T_j} so that $\Delta_{T_j} \subseteq \Delta_T$. Otherwise, we choose Δ_{T_j} arbitrarily from the two sides of $D[T_j]$. Set $\mathcal{C}_j = \mathcal{C}_{j-1} \cup \{\Delta_{T_j}\}$.

We show that $\mathcal{C}_j$ is hereditary. If not, then, since $\mathcal{C}_{j-1}$ is hereditary, there is a 3-cycle T with a convex side $\Delta_T \in \mathcal{C}_{j-1}$ such that either $D[T] \subseteq \Delta_{T_j}$ and $\Delta_T \not\subseteq \Delta_{T_j}$ or $D[T_j] \subseteq \Delta_T$ and $\Delta_{T_j} \not\subseteq \Delta_T$. The second case implies that $D[T] \subseteq \Delta_{T_j}$ and $\Delta_T \not\subseteq \Delta_{T_j}$, which is the first case.

Thus, in both cases, we have that $D[T] \subseteq \Delta_{T_j}$ and $\Delta_T \not\subseteq \Delta_{T_j}$. By the choice of Δ_{T_j}, there is a second already considered triangle T' such that $D[T']$ is contained in the other side $\overline{\Delta}_{T_j}$ of $D[T_j]$ but $\Delta_{T'} \not\subseteq \overline{\Delta}_{T_j}$.

Evidently, $D[T] \subseteq \Delta_{T'}$ and $\Delta_T \not\subseteq \Delta_{T'}$, yielding the contradiction that $\mathcal{C}_{j-1}$ is not hereditary. □

Lemma 4.3 yields an $O(n^8)$ algorithm for determining whether a drawing is h-convex. Also, a similar argument proves the following analogous fact for f-convexity. This is essentially the characterization of pseudolinearity due to Aichholzer et al. [4].

Theorem 4.4 ([4]). *Let D be a drawing of K_n. Then D is f-convex if and only if there is a face Γ such that, for every isomorph J of K_4 for which $D[J]$ is a crossing K_4, Γ is contained in the face of $D[J]$ bounded by the 4-cycle.* □

There is a colourful way to understand this theorem. For each isomorph J of K_4 for which $D[J]$ is a crossing K_4, let C_J be the 4-cycle in J that bounds a face of $D[J]$. Paint the side of $D[C_J]$ that contains the crossing of $D[J]$. If the whole sphere is painted, then D is not f-convex. Otherwise, with respect to any face F of $D[K_n]$ that is not painted, F witnesses that D is f-convex.

Our next result gives a surprising characterization of h-convex drawings of K_n by a single additional forbidden configuration.

Theorem 4.5. *Let D be a convex drawing of K_n. Then D is h-convex if and only if, for each isomorph J of K_6 in K_n, $D[J]$ is not isomorphic to $\mathbb{K}_6^{11}$.*

Proof. Since h-convexity is evidently inherited by induced subgraphs, no h-convex drawing of K_n can contain $\mathbb{K}_6^{11}$. Conversely, suppose D is not h-convex; we show D contains $\mathbb{K}_6^{11}$.

By Lemma 4.3, there exist isomorphs J_1 and J_2 of K_4 that are inverted in D. For $i = 1, 2$, let T_i be a 3-cycle in J_i with convex side Δ_{T_i} such that $D[T_1] \subseteq \Delta_{T_2}$ and $\Delta_{T_1} \not\subseteq \Delta_{T_2}$.

Let w be the vertex of J_1 not in T_1; $D[w]$ is separated from $D[T_2]$ by $D[T_1]$. Let x be the vertex of T_1 such that $D[wx]$ crosses $D[T_1]$. Complete $D[wx]$ to a simple closed curve γ by adding a segment on the non-convex side of $D[T_1]$ joining $D[w]$ and $D[x]$. Clearly γ separates the two vertices of $T_1 - x$. Moreover, $D[T_1]$ and, therefore, $D[w]$ as well, are all contained in Δ_2. Convexity implies $D[J_1] \subseteq \Delta_2$. Thus, γ also separates one of the vertices of $T_1 - x$ from $D[T_2]$; let z be the one separated from T_2 by γ and let y be the other.

Since $D[T_1] \subseteq \Delta_{T_2}$, $D[T_2 + z]$ is a non-crossing K_4. If any of the edges from z to T_2 crosses T_1, then we have proof that the side Δ_{T_1} of $D[T_1]$ is not convex, a contradiction. Therefore, $D[T_1]$ is contained in a face Γ of $D[T_2 + z]$ that is incident with z. It follows that w is also in Γ.

Let a be the vertex of T_2 not incident with Γ. The edge wx has both its ends in Γ. Since γ separates z from T_2, γ must cross za and, therefore, is not contained in Γ. It follows that Γ is not the convex side of the 3-cycle T_3 that bounds Γ.

Evidently, $D[T_3] \subseteq \Delta_{T_2}$ and $\Delta_{T_3} \not\subseteq \Delta_{T_2}$. Corollary 2.5(b) implies that there is a vertex v_3 such that $v_3 \in \Gamma$ and $D[T_3 + v_3]$ is a crossing K_4. Because T_2 is in the isomorph J_2 of K_4, there is a vertex v_2 in J_2 that is not in T_2. Since $D[J_2]$ is a crossing K_4, $D[v_2] \notin \Delta_{T_2}$.

We now consider the isomorph of K_6 consisting of $(T_2 \cup T_3) + \{v_2, v_3\}$. Because $D[(T_2 \cup T_3) + v_2]$ is contained in Δ_{T_3}, no edge from v_2 to a vertex in $T_2 \cup T_3$ can cross $D[T_3]$. In particular, (recall that a is the vertex of T_2 not in T_3) $D[v_2 a]$ does not cross $D[T_2 \cup T_3]$. Let b be the vertex of T_2 such that $D[v_2 b]$ crosses $D[T_2]$ and let c be the third vertex of T_2.

Symmetrically, z is the vertex of T_3 that is not in T_2, so $D[v_3 z]$ does not cross $D[T_2 \cup T_3]$. As both $D[z]$ and $D[v_2]$ are in Δ_{T_3}, the edge zv_2 cannot cross T_3. Since $D[v_2 b]$ crosses $D[ac]$ but not $D[T_3]$, it must also cross $D[az]$. It follows that $D[v_2 z]$ crosses only $D[ac]$.

Let b' be the one of b and c such that $D[v_3 b']$ crosses $D[T_3]$ and let c' be the other. There are two cases to consider: $b = b'$ and $c = c'$; or $b = c'$ and $c = b'$. Note that, in each case, convexity and the definitions of b and b' determine the routings of all the edges except $v_2 v_3$ and $v_3 a$.

Let T_4 be the 3-cycle induced by b, v_2 and c. Since $D[ac]$ crosses $D[v_2 b]$, the convex side of $D[T_4]$ is the side that contains v_3. Thus, $D[v_2 v_3]$ must be contained in this side of

$D[T_4]$. In the case $b = b'$, $D[v_3 b]$ crosses $D[cz]$, $D[zv_2]$, and $D[az]$. Thus, the only routing for $D[v_2 v_3]$ is across $D[ac]$ and $D[zc]$. In the case $b = c'$, the only routing for $D[v_2 v_3]$ is across $D[ac]$, $D[az]$, and $D[zb]$. In both cases there is only one routing available for $D[v_3 a]$.

To see in each case that these drawings are both $\mathbb{K}_6^{11}$, focus on the face-bounding 4-cycles induced by b, a', v_3, c and b, a, v_2, c. □

The previous results were about a drawing being h-convex. The following result characterizes when a collection of sides, one for each 3-cycle of K_n, is a set of h-convex sides. Its proof is similar to the proof of Theorem 4.5.

Lemma 4.6. *Let D be a drawing of K_n and, for each 3-cycle T of K_n, let Δ_T be one of the closed discs bounded by $D[T]$. Let $\mathcal{C}$ be the set of all these Δ_T. Then $\mathcal{C}$ is a set of h-convex sides if and only if both of the following hold:*

(1) *each Δ_T has the Four Point Property; and*

(2) *for each non-crossing K_4, at least three of the four (closed) faces of the non-crossing K_4 are in $\mathcal{C}$.*

Proof. Let $\mathcal{C}$ be a set of h-convex sides. They are also convex sides, so obviously satisfy (1). If J is a non-crossing K_4 and T is a 3-cycle in J, then the side Δ_T of T that is in $\mathcal{C}$ is either empty or contains J; in the latter case all the empty sides of the other 3-cycles in J are in $\mathcal{C}$ by heredity. Thus, at most one of the four 3-cycles in J has non-empty side in $\mathcal{C}$, which is (2).

Conversely, let $\mathcal{C}$ satisfy (1) and (2). Then $\mathcal{C}$ is a set of convex sides by Corollary 2.4. Now let T, T' be 3-cycles such that $D[T'] \subseteq \Delta_T$.

Suppose there is a $t \in T \setminus T'$ such that $T' + t$ is a crossing K_4. Then the convex side $\Delta_{T'}$ of T' is the side that does not contain t. Because $D[T'] \subseteq \Delta(T)$, this implies $\Delta_{T'} \subseteq \Delta(T)$, as required. Thus, we may assume T' is not crossed in D.

Let v be any vertex of $T \setminus T'$. Then $D[T + v]$ is a non-crossing K_4. By (2), each of the three faces of $D[T + v]$ incident with v is the side of the bounding 3-cycle that is in $\mathcal{C}$.

We proceed by induction on the number of vertices of T' that are not in T; this number being at least 1. Because T' is not crossed in D, the remaining vertices of T' are either in or incident with the same face F of $T + v$. Let T'' be the 3-cycle bounding F.

The base of the induction is that $T' - v \subseteq T$. In this case, $T' = T''$ and $\Delta_{T'}$ is F; thus, $\Delta_{T'} \subseteq \Delta_T$, as required.

For the induction step, suppose $|T' \setminus T| \geq 2$. We have already seen that the 3-cycle T'' bounding F has $\Delta_{T''} = F$. For this induction step, there is a vertex w of T' in the interior of F, so the side of $\Delta_{T''}$ in $\mathcal{C}$ is not a face of $T'' + w$. Now $T' \subseteq \Delta_{T''}$ and $|T' \setminus T''| < |T' \setminus T|$. By induction, $\Delta_{T'} \subseteq \Delta_{T''}$. Since $\Delta_{T''} \subseteq \Delta_T$, we have shown $\mathcal{C}$ is the set of h-convex sides, as required. □

Although Lemma 4.6 shows that h-convexity is determined by considering all sets of four points, it is not evident that there is an $\mathrm{O}(n^4)$ algorithm to test whether a drawing is h-convex. Theorem 4.5 makes it clear that there is an $\mathrm{O}(n^6)$ algorithm to determine if a drawing of K_n is h-convex.

It is $\mathrm{O}(n^4)$ to check that a drawing is convex. To see this, we use the Four Point Property (Corollary 2.4): for each 3-cycle T and each vertex v, we determine which side of T v is on and whether $T + v$ is a planar K_4.

We conclude this section with an observation related to the Structure Theorem 3.7.

Lemma 4.7. *Let D be an h-convex drawing of K_n consisting of a natural K_r (with $r \geq 4$) and all other points inside the natural K_r. Then D is f-convex.*

Proof. Any triangle of the K_r has its convex side determined; it is the side that avoids the face F bounded by the cycle C_r. Let T be any other triangle. If an edge of K_r crosses T, then the convex side of T must also avoid F. If this does not happen, then T is inside a triangle of K_r and the result follows from the hereditary property. □

5 Suboptimal drawings of K_n having either $\widetilde{\mathbb{K}}_5^3$ or $\widetilde{\mathbb{K}}_5^5$

In this section, we prove that a broad class of "locally determined" drawings of K_n are suboptimal. This is the first theorem of its type. The theorem requires the presence of either $\widetilde{\mathbb{K}}_5^3$ or $\widetilde{\mathbb{K}}_5^5$ in the drawing, but, for at least one such K_5, the occurrence is restricted. This might be a first step towards showing that all optimal drawings of K_n are convex.

This line of research was stimulated by Tilo Wiedera's computation (personal communication) showing that any drawing of K_9 that contains a $\widetilde{\mathbb{K}}_5^5$ has at least 40 crossings. This is in line with Aichholzer's later computations (see the remark following the statement of Theorem 5.1 below).

We also rethink the approach in [21] that $\operatorname{cr}(K_9) = 36$. This was done before convexity became known to us. Using the fact that $\operatorname{cr}(K_7) = 9$, it is easy to see that $\operatorname{cr}(K_9) \geq 34$. At the end of this section, we show easily by hand that there is no non-convex drawing D of K_9 such that $\operatorname{cr}(D) = 34$. Thus, to prove that $\operatorname{cr}(K_9) = 36$, it suffices to consider convex drawings of K_9.

A principal result of this work is the following, which shows that if D is a drawing of K_n such that some non-convex K_5 intersects every other non-convex K_5 in at most two vertices, then D is not optimal.

Theorem 5.1. *Let D be a drawing of K_n such that there is an isomorph J of K_5 with $D[J]$ either $\widetilde{\mathbb{K}}_5^3$ or $\widetilde{\mathbb{K}}_5^5$. Suppose, for every isomorph H of K_7 in K_n containing J, $D[J]$ is the only non-convex K_5 in $D[H]$.*

1. *If J is $\widetilde{\mathbb{K}}_5^3$, then there is a drawing D' of K_n such that* $\operatorname{cr}(D') \leq \operatorname{cr}(D) - 2$.
2. *If J is $\widetilde{\mathbb{K}}_5^5$, there is a drawing D' of K_n such that* $\operatorname{cr}(D') \leq \operatorname{cr}(D) - 4$. *If, in addition, n is even, then* $\operatorname{cr}(D') \leq \operatorname{cr}(D) - 5$.

We remark that the lower bounds 2, 4, and 5 for $\operatorname{cr}(D) - \operatorname{cr}(D')$ exhibited in Theorem 5.1 are precisely the smallest differences found by Aichholzer (private communication) between any drawing, for $n \leq 12$, of K_n that has either a $\widetilde{\mathbb{K}}_5^3$ or a $\widetilde{\mathbb{K}}_5^5$ and an optimal drawing of K_n.

Proof of Theorem 5.1. We use the labelling of J as shown in Figure 4. We first deal with the case $J = \widetilde{\mathbb{K}}_5^3$.

(I) $J = \widetilde{\mathbb{K}}_5^3$.

Claim 5.2. *There is no vertex of $D[K_n]$ in the side of any of the 3-cycles $D[stw]$, $D[suv]$, and $D[tuv]$ that has no vertex of $D[J]$.*

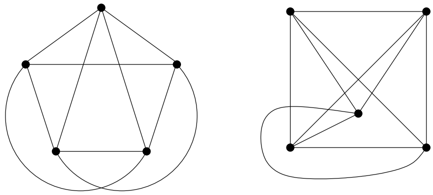

Figure 4: Labelled $\widetilde{\mathbb{K}}_5^3$ and $\widetilde{\mathbb{K}}_5^5$ for the proof of Theorem 5.1.

Proof of Claim 5.2. We start with $D[stw]$. Similar arguments apply to $D[suv]$. Finally, symmetry shows that $D[tuv]$ also does not have a vertex on the side empty in $D[J]$.

Suppose to the contrary that there is a vertex x of K_n such that $D[x]$ is in the side of $D[stw]$ that is empty in $D[J]$. By hypothesis, the K_5 consisting of $J - w$ plus x is convex in D. Since $D[x]$ is incident with a face of $D[J - w]$ that is incident with the crossing of $D[J - w]$, Observation 2.1 and convexity imply $D[xu]$ does not cross the 4-cycle $D[stuv]$.

Likewise, the K_5 consisting of $J - v$ together with x is convex in D. Again, $D[x]$ is in a face of $D[J - v]$ incident with a crossing, so $D[xu]$ does not cross the 4-cycle $D[wtus]$. However, $D[x]$ and $D[u]$ are in different faces of $D[stuv] \cup D[wtus]$, so $D[xu]$ must cross at least one of the two 4-cycles.

The same deletions show that any vertex in the empty side of $D[suv]$ cannot connect to t. □

There are two remaining regions of interest. Let $\times$ be the crossing of su with tv. Let R_1 be the region bounded by $D[wt{\times}sw]$ that does not contain $D[u]$ and $D[v]$; R_2 is the region bounded by $D[stuv]$ that does not contain $D[w]$.

The following observations follow immediately from the convexity of the corresponding K_5 and the knowledge of the crossings.

(Obs. 1) If $x \in R_1$, then $J - w + x$ is convex, so the routings of x to s, t, u, v are determined.

(Obs. 2) If $y \in R_2$, then $J - u + y$ and $J - v + y$ are convex, so the routings of y to J are determined.□

These observations yield all of (1) – (5) in the following assertions except for (3).

Claim 5.3. *If $D[x] \in R_1$ and $D[y] \in R_2$, then:*

(1) *$D[xu]$ crosses $D[J]$ only on $D[tv]$ and $D[xv]$ crosses $D[J]$ only on $D[su]$;*

(2) *$D[xs]$ and $D[xt]$ do not cross $D[J]$;*

(3) *$D[xw]$ either does not cross $D[J]$, or crosses $D[st]$ and at least one of $D[su]$ and $D[tv]$;*

(4) *$D[ys]$, $D[yt]$, $D[yu]$, and $D[yv]$ cross $D[J]$ at most in either $D[uw]$ or $D[vw]$ (or both);*

(5) *$D[yw]$ crosses only $D[st]$.*

Moreover, if zz' is an edge of G with neither z nor z' in J and T is one of the 3-cycles stw, suv, and tuv, then either $D[zz']$ does not cross $D[T]$ or it crosses the one of $D[st]$, $D[su]$, and $D[tv]$ that is in $D[T]$.

Proof of Claim 5.3. For xw, the following argument is due to Matthew Sullivan, simplifying our original. Consider the isomorph L of $K_{2,4}$ with x and w on one side and s, t, u, v on the other side. Then $D[xw]$ does not cross (the planar drawing) $D[L]$ and so is contained in one of the four faces of $D[L]$. The face of $D[L]$ bounded by $swtx$ is disjoint from $D[J]$. In each of the other three faces, $D[xw]$ must cross $D[st]$. In two of these three faces, it also crosses exactly one of $D[su]$ and $D[tv]$. In the third, it crosses both $D[su]$ and $D[tv]$.

For the moreover, we consider the remaining three types of edges z_1z_2: $D[z_1]$ and $D[z_2]$ can both be in R_1; both in R_2; or one in each. In all three cases for z_1, z_2 and all three cases for the three-cycle T, $D[z_1]$ and $D[z_2]$ are on the same side of $D[T]$. In the event that $D[z_1]$ and $D[z_2]$ are both planarly joined to $D[T]$, Corollary 2.10 applies to show $D[z_1z_2]$ does not cross the 3-cycle.

In the remaining cases, we assume that $D[z_1]$ is not planarly joined to $D[T]$. If $T = stw$, then Claim 5.3(3) and (5) imply the only possible crossing with $D[T]$ is $D[z_1w]$ crossing $D[st]$. As $D[z_1z_2]$ has either 0 or 2 crossings with $D[stw]$, but does not cross $D[z_1w]$, the two crossings of $D[z_1z_2]$ and $D[T]$ cannot be on $D[ws]$ and $D[wt]$. For $T = suv$ and $T = tuv$, the edges z_1u and z_1v, respectively, produce analogous results. □

We are now prepared for the final part of the proof. For $i = 1, 2$, let r_i be the number of vertices of $D[K_n]$ that are in (the interior of) R_i.

Let D' be the drawing of K_n obtained from D by making the following two changes:

(C1) reroute st alongside the path $D[swt]$, so as to not cross $D[wu]$ and $D[wv]$; and

(C2) reroute su alongside the path $D[svu]$ so as to cross $D[vw]$.

We first consider the changes in crossings arising from rerouting st. There are at least $2+r_2$ crossing pairs of edges in D that do not cross in D': two from $D[st]$ crossing $D[wv]$ and $D[wu]$, plus all the crossings of $D[st]$ from those edges incident with $D[w]$ that cross $D[st]$. There are at least r_2 of these latter crossings, as every vertex z such that $D[z]$ is in R_2 has $D[zw]$ crossing $D[st]$.

On the other hand, there is a set of at most r_1 crossing pairs in D' that do not cross in D. These arise from the the edges joining a vertex drawn in R_1 to $D[w]$; these might not intersect $D[J]$. Those that do intersect $D[J]$ cross $D[st]$ and, therefore, yield further savings.

We show that every other edge z_1z_2 has no more crossings in D' than it has in D.

Case 1: *z_1, say, is in J.*

In this case, we use Claim 5.3. Items (1), (2), (4), and (5) show that no such edge has more crossings in D' than in D, except possibly xw.

If $D[xw]$ does not cross $D[J]$, then $D'[xw]$ also does not cross $D'[J - st]$, as required. If $D[xw]$ crosses $D[J]$, then Claim 5.3(3) implies that $D[xw]$ crosses $D[st]$. Thus, $D[xw]$ crosses both $D[su]$ and one of $D[sv]$ and $D[tu]$; in this case, the same is true of $D'[xw]$ in D', as required.

Case 2: *neither z_1 nor z_2 is in J.*

If $D[z_1z_2]$ crosses the 3-cycle $D[stw]$, then the moreover part of Claim 5.3 shows it crosses $D[st]$. Therefore, it crosses exactly one of $D[sw]$ and $D[wt]$, showing that $D'[z_1z_2]$ crosses $D'[st]$ and the same one of $D'[sw]$ and $D'[wt]$. That is, z_1z_2 crosses the same two edges in both drawings, and we are done.

The net result is that the rerouting of st contributes at least $2 + (r_2 - r_1)$ to $\mathrm{cr}(D) - \mathrm{cr}(D')$.

Now we turn our attention to the changes in crossings from rerouting su. The crossing of $D[su]$ with $D[tv]$ is replaced by a crossing of $D'[su]$ with $D'[vw]$. In addition, r_2 edges incident with v do not cross $D[su]$, but cross $D'[su]$, while r_1 edges incident with v cross $D[su]$, but do not cross $D'[su]$.

The only additional remark special to this rerouting is the observation that, for z in R_1, Claim 5.3 implies that if $D[zw]$ crosses the 3-cycle $D[suv]$, then zw crosses su. This shows that $D'[zw]$ also crosses $D'[suv]$ twice, so there are no other "new" crossings.

Therefore, the su rerouting contributes at least $r_1 - r_2$ to the difference $\mathrm{cr}(D) - \mathrm{cr}(D')$. Combining this with the contribution from st, we get that $\mathrm{cr}(D) - \mathrm{cr}(D') \geq (2 + r_2 - r_1) + (r_1 - r_2) = 2$. That is, $\mathrm{cr}(D') \leq \mathrm{cr}(D) - 2$, as required.

(II) $J = \widetilde{\mathbb{K}}_5^5$.

In this case, there is a homeomorphism Θ of the sphere to itself that is an involution that restricts to J as, using the labelling in Figure 4: $s \leftrightarrow w$; $t \leftrightarrow v$; and u is fixed. This will be helpful at several points in the following discussion. The outline of the argument is the same as for $\widetilde{\mathbb{K}}_5^3$, but there are some interesting differences.

Let R_1 be the face of $D[J]$ incident with all three points in $D[\{s, t, u\}]$ (the unbounded face in the diagram) and let R_2 be the face of $D[J]$ incident with all three points in $D[\{u, v, w\}]$ (note that $R_2 = \Theta(R_1)$).

Claim 5.4. *If z is a vertex of K_n not in J, then $D[z] \in R_1 \cup R_2$.*

Proof of Claim 5.4. Suppose x is a vertex of $K_n - V(J)$ such that $D[x]$ is not in $R_1 \cup R_2$. Suppose first that $D[x]$ is in the region bounded by the 4-cycle $D[wtsv]$.

The convexity of $D[(J - s) + x]$ and of $D[(J - w) + x]$ imply that $D[xu]$ does not cross the 4-cycles $D[twvu]$ and $D[stuv]$, respectively. However, $D[x]$ is not in a face of $D[twvu] \cup D[stuv]$ incident with $D[u]$, a contradiction.

The remaining possibility is that $D[x]$ is in the face F that is both distinct from R_1 and either incident with $D[ut]$ or, symmetrically, incident with uv; we assume the former. The convexity of $D[(J-t)+x]$ and $D[(J-v)+x]$ show that $D[xw]$ does not cross the 4-cycles $D[swvu]$ and $D[swut]$, respectively. However, $D[x]$ is not in a face of $D[swvu] \cup D[swut]$ incident with $D[w]$, a contradiction. □

We next move to the routings of the edges from a vertex $D[x]$ in $R_1 \cup R_2$ to $D[J]$.

Claim 5.5. *If $D[x] \in R_1$, then:*

(1) *$D[xu]$ and $D[xs]$ do not cross $D[J]$;*

(2) *$D[xv]$ crosses $D[J]$ only on $D[uw]$, and $D[xw]$ crosses $D[J]$ only on $D[sv]$ and $D[tv]$; and*

(3) *$D[xt]$ either does not cross $D[J]$ or it crosses $D[J]$ precisely on $D[sv]$, $D[sw]$, and $D[su]$.*

Furthermore, if $D[x], D[x'] \in R_1$, then $D[xx'] \subseteq R_1$.

Proof of Claim 5.5. The convexity of $D[(J-t)+x]$ and $D[(J-u)+x]$ shows $D[xs]$ does not cross the 4-cycles $D[suvw]$ and $D[stwv]$, respectively. The convexity of $J-s+x$ shows xu does not cross $J-s$. Also xu does not cross su by simplicity, so xu does not cross any edge incident with s.

The convexity of $J-t+x$ shows xv, respectively xw, crosses $J-t$ only uw, respectively tv. Also xv, respectively xw, does not cross vt, respectively wt, by simplicity, so xv, respectively xw, does not cross any edge incident with t.

The convexity of $D[(J-s)+x]$ determines the routing of $D[xt]$, except with respect to $D[s]$, leaving the two options described.

For the furthermore conclusion, $D[x]$ and $D[x']$ are planarly joined to the 3-cycle $D[svu]$. Corollary 2.10 shows that $D[xx']$ is disjoint from $D[svu]$. In the same way, $D[xx']$ is disjoint from $D[swu]$, and $D[tvu]$. Thus, $D[xx']$ can only cross $D[J]$ on $D[st]$. However, letting $\times$ denote the crossing of $D[su]$ with $D[tv]$, $D[xx']$ must cross the 3-cycle $D[st\times]$ an even number of times and it can only cross it on $D[st]$, which is impossible. □

The homeomorphism Θ implies a completely symmetric statement when $x \in R_2$. We provide it here for ease of reference.

Claim 5.6. *If $D[x] \in R_2$, then, in $D[J+x]$:*

(1) *$D[xu]$ and $D[xw]$ do not cross $D[J]$;*

(2) *$D[xt]$ crosses $D[J]$ only on $D[us]$, and $D[xs]$ crosses $D[J]$ only on $D[tw]$ and $D[tv]$; and*

(3) *$D[xv]$ either does not cross $D[J]$ or it crosses $D[J]$ precisely on $D[wt]$, $D[ws]$, and $D[wu]$.*

Furthermore, if $D[x], D[x'] \in R_2$, then $D[xx'] \subseteq R_2$.

Using the homeomorphism Θ, we may choose the labelling of J so that the number r_1 of vertices of $D[K_n]$ drawn in R_1 is at most the number r_2 drawn in R_2.

Our next claim was somewhat surprising to us in the strength of its conclusion.

Claim 5.7. *If there is a vertex x of $K_n - V(J)$ such that $D[x] \in R_1$ and $D[xt]$ crosses $D[sv]$, $D[sw]$, and $D[su]$, then there is a drawing D' of K_n such that $\mathrm{cr}(D') \leq \mathrm{cr}(D) - 4$ and, if n is even, $\mathrm{cr}(D') \leq \mathrm{cr}(D) - 5$.*

Proof of Claim 5.7. Choose such an x so that $D[xt]$ crosses $D[sv]$, $D[sw]$, and $D[su]$ and such that, among all such x, the crossing of $D[xt]$ with $D[sv]$ is as close to $D[s]$ on $D[sv]$ as possible. Let Δ be the closed disc bounded by the 3-cycle $D[sxt]$ that does not contain the vertices $D[\{v,u,w\}]$.

If there is a vertex y of K_n such that $D[y]$ is in the interior of Δ, then $D[y]$ is in the face of $D[J+x]$ contained in Δ and incident with $D[sx]$. However, the convexity in D of $(J-\{u,w\})+\{x,y\}$ implies $D[yt]$ crosses $D[sv]$ closer to s in $D[sv]$ than $D[xt]$ does, contradicting the choice of x. Therefore, no vertex of $D[K_n]$ is in Δ.

The drawing D' is obtained from D by rerouting xt to go alongside the path $D[xst]$, on the side not in Δ. (That is, $D[xt]$ is pushed to the other side of $D[s]$.)

The hardest part of the analysis of the crossings of D' compared to D is determining what happens to an edge of $D[K_n]$ that crosses $D[st]$. No edge of $D[J]$ crosses $D[st]$. Claims 5.5 and 5.6 imply that: no edge from a vertex in $R_1 \cup R_2$ to a vertex in $D[J]$ crosses $D[st]$; and no edge with both incident vertices in the same one of R_1 or R_2 crosses $D[st]$. Thus, the only possible crossing of $D[st]$ is by an edge $D[yz]$, with $D[y] \in R_1$ and $D[z] \in R_2$.

Because of the routing of $D[sz]$, $D[yz]$ cannot also cross $D[xs]$. Therefore, $D[yz]$ also crosses $D[xt]$. It follows that such an edge has the same number of crossings of xt in both D and D'. Therefore, any edge that crosses $D[xs]$ crosses $D[xt]$ and so has the same number of crossings with $D[xt]$ and $D'[xt]$.

The only changes then are in the number of crossings of $D[xt]$ with edges incident with $D[s]$ and the number of crossings with $D[J]$. There are 3 fewer of the latter. From R_1 to $D[s]$, there are at most $r_1 - 1$ crossings of $D'[xt]$. From R_2 to $D[s]$, we have lost r_2 crossings of $D[xt]$. Thus, D' has at least $(r_2 - (r_1 - 1)) + 3 = (r_2 - r_1) + 4$ fewer crossings than D. This proves the first conclusion.

Since $n = 5 + r_1 + r_2$, if n is even, then $r_1 \neq r_2$ and, therefore, $r_2 - r_1 \geq 1$. In this case D' has at least 5 fewer crossings, as claimed. □

It follows from Claim 5.7 that we may assume that, for $D[x] \in R_1$, $D[xt]$ is disjoint from $D[J]$. Combining this with the other information from Claim 5.5, we may assume the following property.

$\mathbf{R_1}$ Assumption: If $D[x] \in R_1$, then $D[x]$ is planarly joined to $D[J - w]$.

Let D' be obtained from D by rerouting $D[tv]$ on the other side of the path $D[tsv]$. There are two claims that complete the proof of Theorem 5.1. The first, similar to Claim 5.7, shows that there are at least 2 fewer crossings in D' (3 if n is even). The second shows that D' satisfies the hypotheses of Theorem 5.1. Therefore, there is a third drawing D'' with at least two fewer crossings than D', as required.

Claim 5.8. $\operatorname{cr}(D') \leq \operatorname{cr}(D) - ((r_2 - r_1) - 2)$.

Proof of Claim 5.8. The proof is very similar to that of Claim 5.7. The main point is to see that no edge e can have $D[e]$ cross both $D[ts]$ and $D[sv]$. Let x be incident with e. If $D[x] \in R_2$, then the routing of $D[xs]$ is known; since $D[xs]$ does not cross $D[e]$, the crossings of $D[e]$ with $D[ts]$ and $D[sv]$ are joined by an arc of $D[e]$ outside $D[stuv]$. But then $D[e]$ has both ends in $J \cup R_2$. This contradicts Claim 5.6, so $D[x] \notin R_2$.

Likewise, Claim 5.5 shows $D[x] \notin R_1$ and clearly e is not in J. Therefore, there is no such e.

It is now easy to see that there are $(r_2 - r_1) + 2$ fewer crossings of $D'[tv]$ with edges incident with s than there are of $D[tv]$. All other crossings of $D'[tv]$ pair off with crossings of $D[tv]$. □

Finally, we show that the drawing D' satisfies the hypotheses of Theorem 5.1. It is routine to verify that $D'[J]$ is $\widetilde{\mathbb{K}}_5^3$. Now let N be a K_5 in K_n such that $N \cap J$ has 3 or 4 vertices.

If any of s, t, v is not in N, then $D'[N]$ is homeomorphic to $D[N]$ and so is convex. Thus, we may assume s, t, v are all in N.

Case 1: *$N \cap J$ has four vertices.*

In this case, there is a vertex x not in J such that N is either $(J-w)+x$ or $(J-u)+x$. If $D[x]$ is in R_1, then the routings are determined and we can see by inspection that $D'[N]$ is, respectively, the K_5 with 1 crossing or the convex K_5 with 3 crossings.

If $D[x] \in R_2$, then Claim 5.6 shows that $D'[(J-w)+x]$ is the K_5 with one crossing. However, Claim 5.6(3) shows $D'[(J-u)+x]$ has two possibilities for xv, depending on which way around w it goes. If it crosses both wt and ws, then the drawing is the natural one. The alternative is to reroute it to not cross $D'[N-x]$. In this case, Observation 2.8 shows $D'[N]$ is convex.

Case 2: *$N \cap J$ has 3 vertices.*

In this case $N = (J - \{u, w\}) + \{x, y\}$. Since $D[x], D[y] \in R_1 \cup R_2$, they are both on the same side of $D[stv]$. The routings from either to $D[J]$ are determined by Claims 5.5 and 5.6 and the assumption following the proof of Claim 5.7. Only when $D[x]$ and $D[y]$ are in different ones of R_1 and R_2 is it possible that $D[xy]$ crosses $D[stv]$.

We consider the three possibilities for $D[x]$ and $D[y]$.

Subcase 2.1: *$D[x] \in R_1$ and $D[y] \in R_2$.*

All routings in $D'[N]$ are determined except for $D[xy]$. The 4-cycle $D[xvyt]$ is uncrossed in $D[N-xy]$. As D is a drawing, $D[xy]$ does not cross $D[xvyt]$. Therefore, either $D[xvyt]$ or $D[xvy]$ is a face of $D'[N]$, showing $D'[N]$ is convex.

Subcase 2.2: *$D[x]$ and $D[y]$ are both in R_2.*

Since $D[x]$ and $D[y]$ are both planarly joined to $D'[stv]$ and $D[xy]$ does not cross $D'[stv]$, $D'[stv]$ bounds a face of $D'[N]$. Thus, $D'[N]$ is convex.

Subcase 2.3: *$D[x], D[y]$ are both in R_1.*

Suppose $D'[N]$ is not convex. Then Corollary 2.5(a) implies there is a 3-cycle T in N such that the two vertices z, z' of N not in T are in different faces of $D'[T]$ and both $D'[T+z]$ and $D'[T+z']$ are crossing K_4's.

Since both x and y are in the same face of $D'[stv]$, $T \neq stv$. If $a \in \{s, t, v\}$, then the routings of the edges from x and y to stv show that the two vertices in $\{s, t, v\} \setminus \{a\}$ are on the same side of $D'[xya]$, so $xya \neq T$. The only remaining possibility is that T has x, say, and two of s, t, v.

Claim 5.9. *The 3-cycle $D'[tvx]$ has no convex side.*

Proof of Claim 5.9. In the alternative, T is either stx or svx. These two situations are very similar, so we treat only stx, leaving the completely analogous argument for svx to the reader. Our strategy is to show that assuming that stx has no convex side in D' implies that tvx has no convex side in D' either.

The vertices v and y are on different sides of $D'[stx]$ and $D[vt]$ crosses $D[sx]$, showing that the side of $D'[stx]$ containing $D[v]$ is not convex. The edge $D[sx]$ also shows that the side of $D'[tvx]$ containing $D[s]$ is not convex.

Likewise, there is an edge e incident with y to one of s, t, and x such that $D[e]$ crosses $D'[stx]$. Notice that $D[xy]$ does not cross $D[xs]$ and $D[xt]$ by definition of drawing and $D[xy]$ does not cross $D[st]$ by the R_1 Assumption. Therefore, $D[xy]$ does not cross $D[st]$ and we conclude that $D[xy]$ does not cross $D[stx]$.

Next suppose that that $D[yt]$ crosses $D[xs]$. The R_1 Assumption shows that $D[yt]$ does not cross $D[stv]$ and so $D[yt]$ crosses $D[vx]$. Therefore, this side of $D'[tvx]$ is also not convex. Combined with the second paragraph of this proof, $D'[tvx]$ is not convex.

In the final case, $D[ys]$ crosses $D[xt]$. As we traverse $D[ys]$ from $D[y]$, there is the crossing with $D[xt]$. A point of $D[ys]$ just beyond this crossing is on the other side of $D[tvx]$ from both y and s.

The edge $D[yv]$ is contained on the same side of the 3-cycle $D[sty]$ as $D[v]$. Therefore, $D[yv]$ must also cross $D[xt]$, showing that the $D[y]$-side of $D[tvx]$ is also not convex, as required. □

Notice that $D[y]$ is in one side of $D'[tvx]$ and $D[s]$ is on the other. Since $s \notin \{t, v, x, y\}$, $D[\{t, v, x, y\}]$ and $D'[\{t, v, x, y\}]$ are homeomorphic. Thus, the side of $D[tvx]$ that contains $D[y]$ is not convex in D.

On the other hand, we know that, in D, $D[w]$ is on the other side of $D[tvx]$ from $D[y]$. However, Claim 5.5(2) shows that $D[wx]$ crosses $D[tv]$. This shows that the side of $D[tvx]$ containing $D[w]$ is not convex. Combined with the preceding paragraph, the K_5 induced by t, v, w, x, y is not convex in D, contradicting the hypothesis of the theorem. This completes the proof of Subcase 2.3 and the theorem. □

It would be significant progress to prove some analogue of Theorem 5.1 with a weaker hypothesis on extensions J. Indeed, one might expect that no hypothesis beyond the existence of J is required, as is easily verified for $n = 7$ (Lemmas 7.5 and 7.7 [21] prove this for K_7 as a simple consequence of the theory developed).

Suppose a drawing D of K_8 has a non-convex K_5. This K_5 is in three different K_7's, each having at least 11 crossings. Lemma 5.10 below shows D has at least 20 crossings, in agreement with Aichholzers's computations.

Lemma 5.10. *Let n be an integer, $n \geq 4$, and let D be a drawing of K_n. Then* $(n-4)\,\mathrm{cr}(D) = \sum_{v \in V(K_n)} \mathrm{cr}(D - v)$. □

A similar argument shows that a non-convex drawing of K_9 cannot have 34 crossings. Let J be any non-convex K_5 in a drawing D of K_9 having 34 crossings. Then J is contained in four K_8's in the K_9. The preceding paragraph shows each of these K_8's has at least 20 crossings. Lemma 5.10 and the assumption that D has only 34 crossings shows that the five remaining K_8's are optimal and hence convex. Thus, J is the only non-convex K_5 in D and so the hypothesis of Theorem 5.1 trivially holds.

The hypothesis of Theorem 5.1 is stronger than we would like and stronger than needed for the preceding argument for K_8. It is not so strong, however, as to force a single non-convex K_5 in a drawing. For example, we have a drawing of K_8—a modified TC_8—having two different non-convex K_5's and satisfying the hypotheses of Theorem 5.1.

6 Questions and conjectures

We conclude with a few questions and conjectures.

1. In Section 1 we presented a table with the convexity hierarchy. One obvious omission is a forbidden drawing characterization of when an h-convex drawing is f-convex. We pointed out that TC_8 is one example of h-convex that is not f-convex. Rerouting some of the edges between the central and outer crossing K_4's produces a few more examples.

Question 6.1. Is there a characterization by forbidden subdrawings of those h-convex drawings of K_n that are f-convex?

2. The *deficiency* $\delta(D)$ of a drawing D of K_n is the number $\mathrm{cr}(D)-H(n)$. The drawing D has the *natural deficiency property* if, for every vertex v of K_n, $\delta(D-v) \leq 2\delta(D)$. If the Hill Conjecture is true for $n = 2k+1$, then every drawing of K_{2k} has the natural deficiency property. We prove this in the Appendix.

   Conjecture 6.2. *For every $k \geq 2$, every simple drawing of K_{2k} has the natural deficiency property.*

   This seems to be an interesting weakening of the Hill Conjecture; it came up tangentially in the proof that $\mathrm{cr}(K_{13}) > 217$ [20].

3. Pach, Solymosi, and Tóth [23] proved that, for each positive integer r, there is an $N(r) = O(2^{r^8})$ such that, for every $n \geq N(r)$, every drawing D of K_n contains either the natural K_r or the Harborth K_r [15]. If D is convex, then it must be the natural K_r.

   Question 6.3. Can the $O(2^{r^8})$ bound be improved for convex drawings?

4. The big question is: Is every optimal drawing of K_n convex? While we believe this is quite conceivable, Ramsey theory suggests other possibilities. Hedging our bets, we have the following conjecture.

   Conjecture 6.4. *Exactly one of the following holds:*

   (a) *for all $n \geq 5$, no optimal drawing of K_n contains $\widetilde{\mathbb{K}}_5^5$; and*

   (b) *for any $p \geq 1$ and any drawing D of K_p, there is some $n \geq p$ and an optimal drawing of K_n (or at least one with at most $H(n)$ crossings) that contains $D[K_p]$.*

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Appendix

Here we prove the claim made in the discussion about the natural deficiency property, related to Conjecture 6.2: if the Hill Conjecture is true for $n = 2k + 1$, then every drawing of K_{2n} has the natural deficiency property.

Let D be a simple drawing of K_{2n}, let v be any vertex of K_{2n}, and let $r(v)$ denote the total number of crossings in D involving edges incident with v. Duplicating v to get the drawing $D + v$ of K_{2n+1}, we get $\operatorname{cr}(D + v) = \operatorname{cr}(D) + W(2n - 1) + r(v)$, where $W(m) = \lfloor \frac{m}{2} \rfloor \lfloor \frac{m-1}{2} \rfloor$ is the smallest number of crossings in a drawing of $K_{2,m}$ such that the vertices on the 2-side have the same clockwise rotation of the m remaining vertices.

By assumption, $\operatorname{cr}(D+v) \geq H(2n+1)$, so $H(2n+1) \leq \operatorname{cr}(D)+W(2n-1)+r(v)$. On the other hand, arithmetic shows that $H(2n + 1) = H(2n) + W(2n - 1) + (H(2n) - H(2n - 1))$. Therefore,

$$\begin{aligned} 2H(2n) + W(2n-1) - H(2n-1) &= H(2n+1) \\ &\leq \operatorname{cr}(D) + W(2n-1) + r(v)\,. \end{aligned}$$

Cancelling the common $W(2n - 1)$ and rearranging yields

$$r(v) + H(2n-1) \geq 2H(2n) - \operatorname{cr}(D)\,. \tag{A.1}$$

Inequality (A.1) implies

$$\begin{aligned} \delta(D - v) &= \operatorname{cr}(D - v) - H(2n-1) \\ &= \operatorname{cr}(D) - r(v) - H(2n-1) \\ &\leq 2\operatorname{cr}(D) - 2H(2n) \\ &= 2\delta(D)\,, \end{aligned}$$

as claimed.

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On the essential annihilating-ideal graph of commutative rings*

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Abstract

Let R be a commutative ring with unity, $A(R)$ be the set of annihilating-ideals of R and $A^*(R) = A(R) \setminus \{0\}$. In this paper, we introduced and studied the *essential annihilating-ideal graph* of R, denoted by $\mathcal{EG}(R)$, with vertex set $A^*(R)$ and two distinct vertices I_1 and I_2 are adjacent if and only if $Ann(I_1I_2)$ is an essential ideal of R. We prove that $\mathcal{EG}(R)$ is a connected graph with diameter at most three and girth at most four if $\mathcal{EG}(R)$ contains a cycle. Furthermore, the rings R are characterized for which $\mathcal{EG}(R)$ is a star or a complete graph. Finally, we classify all the Artinian rings R for which $\mathcal{EG}(R)$ is isomorphic to some well-known graphs.

Keywords: Annihilating-ideal graph, zero-divisor graph, complete graph, planar graph, genus of a graph.

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1 Introduction

Throughout this paper all rings are commutative rings (not a field) with unit element such that $1 \neq 0$. For a commutative ring R, we use $\mathbb{I}(R)$ to denote the set of ideals of R and $\mathbb{I}^*(R) = \mathbb{I}(R) \setminus \{0\}$. An ideal I of R is said to be *non-trivial* if it is nonzero and proper both. An ideal I of R is said to be *annihilator ideal* if there is a nonzero ideal J of R such that $IJ = 0$. For $X \subseteq R$, we define $annihilator$ of X as $Ann(X) = \{r \in R : rX = 0\}$. We use $A(R)$ to denote the set of annihilator ideas of R and $A^*(R) = A(R) \setminus \{0\}$. We denote the set of zero-divisors, the set of nilpotent elements, the set of maximal ideals, the set of minimal prime ideals, and the set of Jacobson radical of a ring R by $Z(R)$, $Nil(R)$,

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$Max(R)$, $Min(R)$ and $J(R)$, respectively. A nonzero ideal I of R is called *essential*, denoted by $I \leq_e R$, if I has a nonzero intersection with every nonzero ideal of R. Also, if I is not an essential ideal of R then, it is denoted by $I \not\leq_e R$. A ring R is said to be reduced, if it has no nonzero nilpotent element. For a nonzero nilpotent element x of R, we use η to denote the index of nilpotency of x. If S is any subset of R, then S^* denote the set $S \setminus \{0\}$. For any undefined notation or terminology in ring theory, we refer the reader to see [9].

Let G be a graph with vertex set $V(G)$. The $distance$ between two vertices u and v of G denoted by $d(u, v)$, is the smallest path from u to v. If there is no such path, then $d(u, v) = \infty$. The $diameter$ of G is defined as $diam(G) = \sup\{d(u, v) : u, v \in V(G)\}$. A $cycle$ is a closed path in G. The $girth$ of G denoted by $gr(G)$ is the length of a shortest cycle in G ($gr(G) = \infty$ if G contains no cycle). A graph is said to be *complete* if all its vertices are adjacent to each other. A complete graph with n vertices is denoted by K_n. If G is a graph such that the vertices of G can be partitioned into two nonempty disjoint sets U_1 and U_2 such that vertices u and v are adjacent if and only if $u \in U_1$ and $v \in U_2$, then G is called a *complete bipartite graph*. A complete bipartite graph with disjoint vertex sets of size m and n, respectively, is denoted by $K_{m,n}$. We write $K_{n,\infty}$ (respectively, $K_{\infty,\infty}$) if one (respectively, both) of the disjoint vertex sets is infinite. A complete bipartite graph of the form $K_{1,n}$ is called a *star graph*. A graph G is said to be *planar* if it can be drawn in the plane so that its edges intersect only at their ends. A subdivision of a graph is a graph obtained from it by replacing edges with pairwise internally-disjoint paths. A remarkably simple characterization of planar graphs was given by Kuratowski in 1930. Kuratowski's Theorem says that a graph G is planar if and only if it contains no subdivision of K_5 or $K_{3,3}$. The *genus* of a graph G, denoted by $\gamma(G)$, is the minimum integer k such that the graph can be drawn without crossing itself on a sphere with k handles (i.e. an oriented surface of genus k). Thus, a planar graph has genus 0, because it can be drawn on a sphere without self-crossing. For more details on graph theory, we refer to reader to see [21, 22].

The concept of *zero-divisor graph* of a commutative ring R, denoted by $\Gamma(R)$, was introduced by I. Beck [10]. The vertex set of $\Gamma(R)$ is $Z^*(R) = Z(R) \setminus \{0\}$ (set of nonzero zero-divisors of R) and two distinct vertices x and y are adjacent if and only if $xy = 0$, for details see [5, 8, 7]. In [14], Dolžan and Oblak also obtained several interesting results related with zero-divisor graph of rings and semirings. The zero-divisor graph of a noncommutative ring has been introduced and studied by Redmond [18], whereas the same concept for semigroup by Demeyer et al. [13].

In [11], Behboodi et al. generalized the zero-divisor graph to ideals by defining *the annihilating-ideal graph* $AG(R)$, with vertex set is $A^*(R)$ and two distinct vertices I_1 and I_2 are adjacent if and only if $I_1I_2 = 0$. For more details on annihilating-ideal graph, we refer the reader to see [1, 2, 3, 4, 6, 12, 16].

In [17], M. Nikmehr et al. introduced *the essential graph* $EG(R)$ with vertex set $Z^*(R) = Z(R)\setminus\{0\}$ and two distinct vertices x and y are adjacent if and only if $ann_R(xy)$ is an essential ideal of R.

Motivated by [17], we define *the essential annihilating-ideal graph* of R denoted by $\mathcal{EG}(R)$ with vertex set $A^*(R)$ and two distinct vertices I_1 and I_2 adjacent if and only if $Ann(I_1I_2)$ is an essential ideal of R. In this paper we first prove that $AG(R)$ is a subgraph of $\mathcal{EG}(R)$ and then studied some basic properties of $\mathcal{EG}(R)$ such as connectedness, diameter, girth and shows that $\mathcal{EG}(R)$ is a connected graph with $diam(\mathcal{EG}(R)) \leq 3$ and $gr(\mathcal{EG}(R)) \leq 4$, if $\mathcal{EG}(R)$ contains a cycle. In the third section, we determine some conditions on R under which $\mathcal{EG}(R)$ is a star graph or a complete graph. In the last, we identify all the Artinian rings R for which $\mathcal{EG}(R)$ is isomorphic to some well-known graphs.

2 Basic properties of essential annihilating-ideal graph

We begin this section with the following lemma given by [17].

Lemma 2.1 ([17, Lemma 2.1])**.** *Let R be a commutative ring and I be an ideal of R. Then*

(1) *$I + Ann(I)$ is an essential ideal of R.*

(2) *If $I^2 = (0)$, then $Ann(I)$ is an essential ideal of R.*

(3) *If R contains no proper essential ideals, then $J(R) = (0)$.*

The following lemma is analogue of [17, Lemma 2.2].

Lemma 2.2. *Let R be a commutative ring. Then*

(1) *If I_1 and I_2 are adjacent in $AG(R)$, then I_1 and I_2 are also adjacent in $\mathcal{EG}(R)$.*

(2) *If $I^2 = 0$ for some $I \in A^*(R)$, then I is adjacent to every other vertex in $\mathcal{EG}(R)$.*

Proof. (1) Suppose I_1 and I_2 are adjacent in $AG(R)$, then $I_1 I_2 = 0$ and so $Ann(I_1 I_2) = R$, is an essential ideal of R. Thus I_1 and I_2 are also adjacent in $\mathcal{EG}(R)$.
(2) Suppose that $I^2 = 0$ for some $I \in A^*(R)$. Then by Lemma 2.1(2), $Ann(I)$ is an essential ideal of R. Since $Ann(I) \subseteq Ann(IJ)$ for every $J \in A^*(R)$, therefore $Ann(IJ)$ is also an essential ideal of R. Thus I is adjacent to every other vertex of $\mathcal{EG}(R)$. □

Let R be a commutative ring. By [11, Theorem 2.1], the annihilating ideal graph $AG(R)$ is a connected graph with $diam(AG(R)) \leq 3$. Moreover, if $AG(R)$ contains a cycle, then $gr(AG(R)) \leq 4$.

In view of part (1) of Lemma 2.2, we have the following result.

Theorem 2.3. *Let R be a commutative ring. Then $\mathcal{EG}(R)$ is connected with $diam(\mathcal{EG}(R)) \leq 3$. Moreover, if $\mathcal{EG}(R)$ contain a cycle, then $gr(\mathcal{EG}(R)) \leq 4$.*

In Lemma 2.2(1), we proved that $AG(R)$ is a spanning subgraph of $\mathcal{EG}(R)$ but this containment may be proper. The following examples shows that $AG(R)$ and $\mathcal{EG}(R)$ are not identical.

Example 2.4.

1. If $R = \mathbb{Z}_{16}$, then $AG(R)$ is P_3 and $\mathcal{EG}(R)$ is K_3.

2. If $R = \mathbb{Z}_{p^5}$, where p is a prime number. Then $AG(R)$ is the following graph and $\mathcal{EG}(R)$ is K_4.

Theorem 2.5. *Let R be a commutative reduced ring. Then $\mathcal{EG}(R) = AG(R)$.*

Proof. Clearly, $AG(R) \subseteq \mathcal{EG}(R)$. We just have to prove that $\mathcal{EG}(R)$ is a subgraph of $AG(R)$. Suppose on contrary that $I_1 \sim I_2$ is an edge of $\mathcal{EG}(R)$ such that $I_1 I_2 \neq 0$. Since R is a reduced ring, then $I_1 I_2 \cap Ann(I_1 I_2) = 0$, which implies that $Ann(I_1 I_2)$ is not an essential ideal of R, a contradiction. Thus $I_1 I_2 = 0$ and $\mathcal{EG}(R) = AG(R)$. □

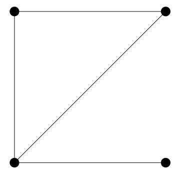

Figure 1: The graph $AG(\mathbb{Z}_{p^5})$.

Theorem 2.6 ([12, Theorem 1.9(3)]). *Let R be a commutative ring with finitely many minimal primes. Then $diam(AG(R)) = 2$ if and only if either R is reduced with exactly two minimal primes and at least three nonzero annihilating-ideals, or R is not reduced, $Z(R)$ is an ideal whose square is not (0) and for each pair of annihilating-ideals I_1 and I_2, $I_1 + I_2$ is an annihilating-ideal.*

Theorem 2.7. *Let R be a commutative ring with $|Min(R)| < \infty$. Then*

(1) *If R is reduced ring, then $diam(\mathcal{EG}(R)) = 2$ if and only if $|Min(R)| = 2$ and R has at least three nonzero annihilating-ideals. Moreover, in this case $gr(\mathcal{EG}(R)) \in \{4, \infty\}$.*

(2) *If R is non-reduced, then $diam(\mathcal{EG}(R)) \leq 2$. Moreover, in this case $gr(\mathcal{EG}(R)) \in \{3, \infty\}$.*

Proof. (1) First part is clear from Theorems 2.5 and 2.6. Now, let $Min(R) = \{P_1, P_2\}$, then $\mathcal{EG}(R)$ is a complete bipartite graph with partitions $V_1 = \{I \in V(\mathcal{EG}) : I \subseteq P_1\}$ and $V_2 = \{I \in V(\mathcal{EG}) : I \subseteq P_2\}$ by [12, Theorem 1.2]. Hence $gr(\mathcal{EG}(R)) \in \{4, \infty\}$.

(2) Since R is a non-reduced ring, then there is $I_1 \in A^*(R)$ such that $I_1^2 = 0$. Thus by Lemma 2.2(2), I_1 is adjacent to every other vertex of $\mathcal{EG}(R)$. Hence $diam(\mathcal{EG}(R)) \leq 2$. Also, if there are $I, J \in V(\mathcal{EG}(R)) \setminus \{I_1\}$ such that $I \sim J$ is an edge of $\mathcal{EG}(R)$, then $I_1 \sim I \sim J \sim I_1$ is a triangle in $\mathcal{EG}(R)$. Thus, $gr(\mathcal{EG}(R)) = 3$, otherwise $gr(\mathcal{EG}(R)) = \infty$. □

3 Completeness of essential annihilating-ideal graph

In this section, we characterize commutative rings R for which $\mathcal{EG}(R)$ is a star graph or a complete graph. We begin with the following lemma.

Lemma 3.1. *Let R be a commutative nonreduced ring. Then*

(1) *For every nilpotent ideal I_1 of R, I_1 is adjacent to every other vertex of $\mathcal{EG}(R)$.*

(2) *The subgraph induced by the nilpotent ideals of R is a complete subgraph of $\mathcal{EG}(R)$.*

Proof. (1) Suppose that I_1 be any nilpotent ideal of R. Let $I_2 \in A^*(R)$. We show that $Ann(I_1 I_2) \leq_e R$. Since $Ann(I_1) \subseteq Ann(I_1 I_2)$, then it is enough to show that $Ann(I_1) \leq_e R$. Suppose on contrary that $Ann(I_1) \not\leq_e R$, then there exists $I_3 \in \mathbb{I}^*(R)$ such that $Ann(I_1) \cap I_3 = 0$, which implies that $rI_1 \neq 0$ for every $r \in I_3^*$. Since $0 \neq rI_1 \subseteq I_3$, then $I_1 \cdot rI_1 = rI_1^2 \neq 0$. Continuing this process, we get $rI_1^n \neq 0$, for every positive integer n, which is a contradiction. This complete the proof.
(2) It is clear from (1). □

Lemma 3.2. *Let $(R, \mathfrak{m})$ be a commutative Artinian local ring. Then $\mathcal{EG}(R)$ is a complete graph.*

Proof. Follows from Lemma 3.1. □

Lemma 3.3. *Let R be a commutative decomposable ring. Then $\mathcal{EG}(R)$ is a star graph if and only if $R = F \times D$, where F is a field and D is an integral domain.*

Proof. $(\Rightarrow)$ Suppose that $\mathcal{EG}(R)$ is a star graph and let $R = R_1 \times R_2$, where R_1 and R_2 are commutative rings. If R_1 and R_2 both are not fields and $I_1 \in \mathbb{I}^*(R_1)$, $I_2 \in \mathbb{I}^*(R_2)$, then $(R_1 \times (0)) \sim ((0) \times R_2) \sim (I_1 \times (0)) \sim ((0) \times I_2) \sim (R_1 \times (0))$ is a cycle of length 4 in $\mathcal{EG}(R)$, a contradiction. Thus, without loss of generality we can assume that R_1 is a field. We claim that R_2 is an integral domain. Suppose on contrary that R_2 is not an integral domain, then there exists $I_3, I_4 \in \mathbb{I}^*(R_1)$ such that $I_3 I_4 = 0$. If $I_3 \neq I_4$, then $(R_1 \times (0)) \sim ((0) \times I_3) \sim ((0) \times I_4) \sim (R_1 \times (0))$ is a triangle in $\mathcal{EG}(R)$, a contradiction. Also, if $I_3 = I_4$, then by Lemma 3.1, $(R_1 \times (0)) \sim ((0) \times I_3) \sim ((0) \times R_2) \sim (R_1 \times (0))$ is a triangle in $\mathcal{EG}(R)$, again a contradiction. This complete the proof.
$(\Leftarrow)$ is clear. □

Theorem 3.4. *Let R be an Artinian commutative ring with atleast two non-trivial ideals. Then $\mathcal{EG}(R)$ is a star graph if and only if $\mathcal{EG}(R) \cong K_2$.*

Proof. $(\Rightarrow)$ Suppose $\mathcal{EG}(R)$ is a star graph. If R is a local ring, then from Lemma 3.2, $\mathcal{EG}(R)$ is a complete graph. Since $\mathcal{EG}(R)$ is a star graph, therefore $\mathcal{EG}(R) \cong K_2$. If R is non-local ring, then it is decomposable. Thus by Lemma 3.3, $R = F \times D$, where F is a field and D is an integral domain. Since R is Artinian ring, then D is Artinian and hence is a field. Thus $\mathcal{EG}(R) \cong K_2$.
$(\Leftarrow)$ is evident. □

Theorem 3.5. *Let R be a commutative ring with at least two non-trivial ideals. Then $\mathcal{EG}(R)$ is a star graph if and only if one of the following holds:*

(1) *R has exactly two non-trivial ideals.*

(2) *$R = F \times D$, where F is a field and D is an integral domain which is not a field.*

(3) *R has a minimal ideal I_1 such that I_1 is not an essential ideal of R, $I_1^2 = 0$ and for any nonzero annihilating ideal I_2 of R, $Ann(I_2) = I_1$.*

Proof. $(\Rightarrow)$ Suppose $\mathcal{EG}(R)$ is a star graph. If $|A^*(R)| < \infty$, then from [11, Theorem 1.1], R is an Artinian ring. Thus, by Theorem 3.4, $\mathcal{EG}(R) \cong K_2$ and hence (1) hold.
Now, let $|A^*(R)| = \infty$ and I_1 is adjacent to every other vertex of $\mathcal{EG}(R)$. We show that I_1 is minimal ideal of R. Suppose on contrary that there exists $I_2 \in \mathbb{I}^*(R)$ such that $I_2 \subset I_1$. Let $I_3 \in A^*(R) \setminus \{I_1, I_2\}$, then $Ann(I_1 I_3) \leq_e R$. Since $I_2 I_3 \subseteq I_1 I_3$, then $Ann(I_2 I_3)$ is also essential ideal of R. This implies that I_2 is also adjacent to every other vertex of $\mathcal{EG}(R)$, a contradiction. Now, following two cases occur:
Case I: $I_1^2 \neq 0$. Then $I_1^2 = I_1$, thus by Brauer's Lemma [15, p. 172, Lemma 10.22], R is decomposable. Since $|A^*(R)| = \infty$ and $\mathcal{EG}(R)$ is a star graph. Then from Lemma 3.3, $R = F \times D$, where F is a field and D is an integral domain which is not a field. Hence (2) hold.
Case II: $I_1^2 = 0$. Let $I_2 \in A^*(R) \setminus \{I_1\}$. Then $I_2 \neq Ann(I_2)$, otherwise $I_2^2 = 0$

implies that I_2 is also adjacent to every other vertex of $\mathcal{EG}(R)$, a contradiction. Now, since $I_2 \sim Ann(I_2)$, then $Ann(I_2) = I_1$. If I_1 is an essential ideal of R, then $Ann(I_2)$ is also an essential ideal of R. This shows that I_2 is also adjacent with every other vertex of $\mathcal{EG}(R)$, which is a contradiction to our assumption that $\mathcal{EG}(R)$ is a star graph because we are assuming that I_1 is adjacent with every other vertex of $\mathcal{EG}(R)$ and $I_1 \neq I_2$. Hence I_1 is not an essential ideal of R.

($\Leftarrow$) If R has exactly two non-trivial ideals, then R is Artinian ring with $|A^*(R)| = 2$. Since $\mathcal{EG}(R)$ is connected, therefore $\mathcal{EG}(R) \cong K_2$. If $R = F \times D$, where F is a field and D is an integral domain which is not a field, then from Lemma 3.3, $\mathcal{EG}(R)$ is a star graph. Now, suppose that R has a minimal ideal I_1 such that I_1 is not an essential ideal of R, $I_1^2 = 0$ and for any nonzero annihilating ideal I_2 of R, $Ann(I_2) = I_1$. Let $I_2, I_3 \in A^*(R) \setminus \{I_1\}$ such that $I_2 \sim I_3$ in $\mathcal{EG}(R)$. This implies that $Ann(I_2I_3) \leq_e R$ and $Ann(I_2) = I_1 = Ann(I_3)$. Since $Ann(I_2) = Ann(I_3)$ is not an essential of R, there exists a nonzero ideal I_4 of R such that $Ann(I_2) \cap I_4 = Ann(I_3) \cap I_4 = 0$. This shows that $rI_2 \neq 0$ and $rI_3 \neq 0$ for every $r \in I_4^*$. On the other hand, since $Ann(I_2I_3) \leq_e R$, then $Ann(I_2I_3) \cap I_4 \neq 0$. That is there exists $s \in I_4^*$ such that $sI_2I_3 = 0$. Now, observe that $sI_2 \subseteq I_4^*$ satisfies $sI_2 \subseteq Ann(I_3)$, which implies that $Ann(I_3) \cap I_4 \neq 0$, a contradiction. This complete the proof. □

Theorem 3.6. *Let R be a commutative Artinian ring. Then $\mathcal{EG}(R)$ is a complete graph if and only if one of the following holds:*

(1) *$R = F_1 \times F_2$, where F_1 and F_2 are fields.*

(2) *R is a local ring.*

Proof. ($\Rightarrow$) Suppose that $\mathcal{EG}(R)$ is a complete graph. Since R is Artinian, then $R \cong R_1 \times R_2 \times \cdots \times R_n$, where R_i is Artinian local ring for each $1 \leq i \leq n$. The following cases occur:

Case I: $n \geq 3$. Then $R_1 \times (0) \times \cdots \times (0)$ and $R_1 \times (0) \times R_3 \times \cdots \times (0)$ are nonzero annihilating ideals of R such that $(R_1 \times (0) \times \cdots \times (0)) \nsim (R_1 \times (0) \times R_3 \times \cdots \times (0))$ in $\mathcal{EG}(R)$, a contradiction.

Case II: $n = 2$. We show that R_1 and R_2 are fields. Suppose on contrary that R_1 is not a field with non-trivial maximal ideal $\mathfrak{m}$. Then $Ann(((0) \times R_2) \cdot (\mathfrak{m} \times R_2)) = Ann((0) \times R_2) = R_1 \times (0)$, which is not an essential ideal of R. Thus $((0) \times R_2) \nsim (\mathfrak{m} \times R_2)$ in $\mathcal{EG}(R)$, a contradiction. Hence (2) holds.

Case III: $n = 1$. Then R is Artinian local ring and (1) holds.

($\Leftarrow$) If R is local, then from Lemma 3.2, $\mathcal{EG}(R)$ is a complete graph. If $R = F_1 \times F_2$, where F_1 and F_2 are fields, then $\mathcal{EG}(R) \cong K_2$. □

Theorem 3.7. *Let R be a commutative ring with at least one minimal ideal. Then $\mathcal{EG}(R) \cong K_{m,n}$, where $m, n \geq 2$ if and only if $R = D \times S$, where D and S are integral domains which are not fields.*

Proof. ($\Rightarrow$) Suppose that $\mathcal{EG}(R) \cong K_{m,n}$, where $m, n \geq 2$. Let I_1 be minimal ideal of R. If $I_1^2 = 0$, then from Lemma 2.2, I_1 is adjacent to every other vertex, a contradiction. Thus $I_1^2 \neq 0$. Since I_1 is minimal, therefore $I_1^2 = I_1$. Therefore, Brauer's Lemma [15, p. 172, Lemma 10.22], $R = R_1 \times R_2$, where R_1 and R_2 are commutative rings. Now, our objective is to show that R_1 and R_2 are integral domains. Suppose on contrary that

R_1 is not an integral domain with nonzero annihilating ideal I_2. As above, $I_2^2 \neq 0$ which implies that $I_2 \notin Ann(I_2)$. Thus $(I_2 \times (0)) \sim ((0) \times R_2) \sim (Ann(I_2) \times (0)) \sim (I_2 \times (0))$ is a triangle in $\mathcal{EG}(R)$, a contradiction. Hence R_1 is an integral domain. Similarly, one can prove that R_2 is an integral domain. Since $m, n \geq 2$, therefore R_1 and R_2 are not fields.
($\Leftarrow$) Suppose that $R = D \times S$, where D and S are integral domains which are not fields. Let $U = \{I_1 \times (0) : I_1 \in \mathbb{I}^*(D)\}$ and $V = \{(0) \times I_2 : I_2 \in \mathbb{I}^*(S)\}$. Then $A^*(R) = U \cup V$ such that no two vertices of U or V are adjacent in $\mathcal{EG}(R)$. Also, every vertex of U is adjacent to every vertex of V in $\mathcal{EG}(R)$. Thus, $\mathcal{EG}(R) \cong K_{m,n}$. Since D and S are not fields, therefore $m, n \geq 2$. □

Lemma 3.8. *Let R be a commutative ring. Then*

(1) *Let $I_1, I_2, I_3 \in A^*(R)$ such that $Ann(I_1) = Ann(I_2)$. Then $I_1 \sim I_3$ is an edge of $\mathcal{EG}(R)$ if and only if $I_2 \sim I_3$ is an edge of $\mathcal{EG}(R)$.*

(2) *Let $I \in A^*(R)$. Then $Ann(I) \leq_e R$ if and only if $Ann(I^n) \leq_e R$ for every $n \geq 2$. In particular, if $Ann(I^3) \leq_e R$, then $Ann(I^n) \leq_e R$ for every $n \geq 1$.*

Proof. (1) ($\Rightarrow$) Suppose that $I_1 \sim I_3$ is an edge of $\mathcal{EG}(R)$, then $Ann(I_1I_3) \leq_e R$. We have to show that $Ann(I_2I_3) \leq_e R$. Suppose on contrary that $Ann(I_2I_3)$ is not an essential ideal of R, then there exits $I_4 \in \mathbb{I}^*(R)$ such that $Ann(I_2I_3) \cap I_4 = 0$. This implies that $rI_2I_3 \neq 0$ for all $r \in I_4^*$. On the other hand, since $Ann(I_1I_3)$ is an essential ideal of R, then $Ann(I_1I_3) \cap I_4 \neq 0$. That is there exists some $s \in I_4^*$ such that $sI_1I_3 = 0$. Now, observe that $sI_3 \subseteq I_4^*$ satisfies $sI_3 \subseteq Ann(I_1) = Ann(I_2)$, which implies that $sI_2I_3 = 0$, a contradiction.
($\Leftarrow$) Using similar argument as above we get the required result.
(2) ($\Rightarrow$) is clear.
($\Leftarrow$) Suppose on contrary that $Ann(I)$ is not an essential ideal of R, then there exists nonzero ideal I_1 of R such that $Ann(I) \cap I_1 = 0$. This implies that $rI \neq 0$ for all $r \in I_1^*$. On the other hand, since $Ann(I^2) \leq_e R$, then $Ann(I^2) \cap I_1 \neq 0$. That is there exists some $s \in I_1^*$ such that $sI^2 = 0$. Now, observe that $r = sI \subseteq I_1^*$ such that $rI = 0$, a contradiction.
For the particular case, we need to show that $Ann(I^2) \leq_e R$. Suppose on contrary that there is some $I_1 \in \mathbb{I}^*(R)$ such that $Ann(I^2) \cap I_1 = 0$, which implies that $rI^2 \neq 0$ for all $r \in I_1^*$. On the other hand, since $Ann(I^3) \leq_e R$, then $Ann(I^3) \cap I_1 \neq 0$. That is there exists some $s \in I_1^*$ such that $sI^3 = 0$. Now, observe that $r = sI^2 \subseteq I_1^*$ such that $rI = 0$, which implies that $Ann(I) \cap I_1 \neq 0$. Since $Ann(I)$ is a subset of $Ann(I^2)$, then $Ann(I^2) \cap I_1 \neq 0$, a contradiction. □

Theorem 3.9. *Let R be a commutative non-reduced ring. Then $\mathcal{EG}(R)$ is a complete graph if and only if $Ann(I) \leq_e R$ for every $I \in A^*(R)$.*

Proof. ($\Rightarrow$) Suppose that $\mathcal{EG}(R)$ is a complete graph. We claim that R is indecomposable ring. Suppose on contrary that $R = R_1 \times R_2$, where R_1 and R_2 are commutative rings. Since R is non-reduced ring, without loss of generality, we can assume that R_1 is non-reduced ring with nonzero nilpotent element x. Let $I_1 = xR_1$. Then $Ann((I_1 \times R_2) \cdot ((0) \times R_2)) = Ann((0) \times R_2) = R_1 \times (0)$, is not an essential ideal of R, a contradiction to the completeness of $\mathcal{EG}(R)$. Let $I \in A^*(R)$ be arbitrary. If I is nilpotent ideal, then from Lemma 3.1(1), $Ann(I) \leq_e R$. Suppose I is not nilpotent ideal.

Since R is indecomposable, then $I^2 \neq I$, which implies that $Ann(I^3) \leq_e R$. Hence by Lemma 3.8(2), $Ann(I) \leq_e R$.
($\Leftarrow$) is evident. □

4 Essential annihilating-ideal graph as some special type of graphs

In this section, we characterize all the Artinian rings R for which $\mathcal{EG}(R)$ is a tree, a unicycle graph, a split graph, a outerplanar graph, a planar graph and a toroidal graph.

Theorem 4.1. *Let R be a commutative Artinian ring (not a field). Then $\mathcal{EG}(R)$ is a tree if and only if either $R \cong F_1 \times F_2$, where F_1 and F_2 are fields or R is a local ring with at most two non-trivial ideals.*

Proof. Suppose that $\mathcal{EG}(R)$ is a tree. Since R is an Artinain ring, then $R \cong R_1 \times R_2 \times \cdots \times R_n$, where each R_i is an Artinian local ring. If $n \geq 3$. Consider $I_1 = R_1 \times (0) \times \cdots \times (0)$, $I_2 = (0) \times R_2 \times (0) \times \cdots \times (0)$ and $I_3 = (0) \times (0) \times R_3 \times (0) \times \cdots \times (0)$. Then $I_1 \sim I_2 \sim I_3 \sim I_1$ is a cycle of in $\mathcal{EG}(\mathcal{R})$, a contradiction.
Suppose $n = 2$, then we show that R_1 and R_2 both are fields. Suppose on contrary that R_1 is not a field with nonzero maximal ideal $\mathfrak{m}$. Consider $J_1 = (0) \times R_2$, $J_2 = \mathfrak{m} \times (0)$, $J_3 = \mathfrak{m} \times R_2$ and $J_4 = R_1 \times (0)$. Then $J_1 \sim J_2 \sim J_3 \sim J_4 \sim J_1$ is a cycle in $\mathcal{EG}(R)$, a contradiction.
If $n = 1$, then R is Artinian local ring. Thus by Lemma 3.2, $\mathcal{EG}(R)$ is a complete graph. Since $\mathcal{EG}(R)$ is a tree, therefore R has at most two non-trivial ideal.
Converse is clear. □

Theorem 4.2. *Let R be a commutative Artinian ring (not a field). Then $\mathcal{EG}(R)$ is unicycle if and only if either $R \cong F_1 \times F_2 \times F_3$, where F_i is a field for each $1 \leq i \leq 3$ or R is an Artinain local ring with exactly three non-trivial ideals.*

Proof. Suppose that $\mathcal{EG}(R)$ is unicycle. Since R is Artinian ring, then $R \cong R_1 \times R_2 \times \cdots \times R_n$, where R_i is Artinian local ring for each $1 \leq i \leq n$. Let $n \geq 4$. Consider $I_1 = R_1 \times (0) \times \cdots \times (0)$, $I_2 = (0) \times R_2 \times (0) \times \cdots \times (0)$, $I_3 = (0) \times (0) \times R_3 \times (0) \times \cdots \times (0)$ and $J_1 = (0) \times (0) \times R_3 \times (0) \times \cdots \times (0)$, $J_2 = R_1 \times R_2 \times (0) \times \cdots \times (0)$, $J_3 = (0) \times (0) \times (0) \times R_4 \times (0) \times \cdots \times (0)$. Then $I_1 \sim I_2 \sim I_3 \sim I_1$ as well as $J_1 \sim J_2 \sim J_3 \sim J_1$ are two different cycles in $\mathcal{EG}(R)$, a contradiction. Hence $n \leq 3$.
First, let $n = 3$ and suppose on contrary that R_2 is not a field with nonzero maximal ideal $\mathfrak{m}$. Consider $I_1 = R_1 \times (0) \times (0)$, $I_2 = (0) \times R_2 \times (0)$, $I_3 = (0) \times (0) \times R_3$ and $J_1 = R_1 \times (0) \times (0)$, $J_2 = (0) \times \mathfrak{m} \times (0)$, $J_3 = (0) \times (0) \times R_3$. Then $I_1 \sim I_2 \sim I_3 \sim I_1$ and $J_1 \sim J_2 \sim J_3 \sim J_1$ are two different cycles in $\mathcal{EG}(R)$, a contradiction. Hence R_i is a field for each $1 \leq i \leq 3$.
Now, let $n = 2$. If R_1 and R_2 both are fields then $\mathcal{EG}(R) \cong K_2$, a contradiction. Thus one of R_i say R_2 is not a field with nonzero maximal ideal $\mathfrak{m}$. Then $(R_1 \times (0)) \sim ((0) \times \mathfrak{m}) \sim ((0) \times R_2) \sim (R_1 \times (0))$ as well as $(R_1 \times \mathfrak{m} \sim ((0) \times \mathfrak{m}) \sim ((0) \times R_2) \sim (R_1 \times \mathfrak{m})$ are two different cycles in $\mathcal{EG}(R)$, again a contradiction.
If $n = 1$, then R is an Artinian local ring. Thus, by Lemma 3.2, $\mathcal{EG}(R)$ is a complete graph. Since $\mathcal{EG}(R)$ is unicycle, R have exactly three non-trivial ideals. □

Theorem 4.3 ([21]). *Let G be a connected graph. Then G is a split graph if and only if G contains no induced subgraph isomorphic to $2K_2$, C_4, C_5.*

Theorem 4.4. *Let R be a commutative Artinian non-local ring. Then $\mathcal{EG}(R)$ is split graph if and only if either $R \cong F_1 \times F_2 \times F_3$ or $R \cong F_1 \times F_2$, where F_i is a field for each $1 \leq i \leq 3$.*

Proof. Suppose that $\mathcal{EG}(R)$ is a split graph. Since R is Artinian non-local ring, then $R \cong R_1 \times R_2 \times \cdots \times R_n$, where each R_i is an Artinian local ring and $n \geq 2$. If $n \geq 4$, then $I_1 = R_1 \times R_2 \times (0) \times \cdots \times (0) \sim J_1 = (0) \times (0) \times R_3 \times R_4 \times (0) \times \cdots \times (0)$ and $I_2 = R_1 \times (0) \times R_3 \times (0) \times \cdots \times (0) \sim J_2 = (0) \times R_2 \times (0) \times R_4 \times (0) \times \cdots \times (0)$ induces $2K_2$ in $\mathcal{EG}(R)$, a contradiction. Hence $n = 2$ or 3. We have following cases:
Case I: If $n = 3$, then we show that each R_i ia a field. Suppose on contrary that R_1 is not a field with nonzero maximal ideal $\mathfrak{m}$. Then $(R_1 \times (0) \times (0)) \sim ((0) \times R_2 \times R_3) \sim (\mathfrak{m} \times (0) \times (0)) \sim ((0) \times R_2 \times (0)) \sim (R_1 \times (0) \times (0))$ is C_4 in $\mathcal{EG}(R)$, a contradiction. Hence R_i is a field for each $1 \leq i \leq 3$.
Case II: Let $n = 2$ and suppose that R_2 is not a field with nonzero maximal ideal $\mathfrak{m}'$. Then $(R_1 \times (0)) \sim ((0) \times R_2) \sim (R_1 \times \mathfrak{m}') \sim ((0) \times \mathfrak{m}') \sim (R_1 \times (0))$ is C_4 in $\mathcal{EG}(R)$, a contradiction. Hence R_1 and R_2 both are fields.
Converse is clear. □

Theorem 4.5 ([22])**.** *A graph G is outerplanar if and only if it does not contain a subdivision of K_4 or $K_{2,3}$.*

Theorem 4.6. *Let R be a commutative Artinian ring. Then $\mathcal{EG}(R)$ is outerplanar if and only if one of the following holds:*

(1) *$R = F_1 \times F_2 \times F_3$, where F_i is a field for each $1 \leq i \leq 3$.*

(2) *$R = F_1 \times F_2$, where F_1 and F_2 are fields.*

(3) *$R = F \times R_1$, where F is a field and $(R_1, \mathfrak{m})$ is a local ring with $\mathfrak{m}$ is the only non-trivial ideal of R_1.*

(4) *R is a local ring with at most three non-trivial ideals.*

Proof. Suppose that $\mathcal{EG}(R)$ is outerplanar. Since R is Artinian ring, then $R \cong R_1 \times R_2 \times \cdots \times R_n$, where each R_i is Artinian local ring. If $n \geq 4$, then the set $\{I_1 = R_1 \times (0) \times \cdots \times (0), I_2 = (0) \times R_2 \times (0) \times \cdots \times (0), I_3 = (0) \times (0) \times R_3 \times (0) \times \cdots \times (0), I_4 = (0) \times (0) \times (0) \times R_4 \times (0) \times \cdots \times (0)\}$ induces K_4 in $\mathcal{EG}(R)$, a contradiction. Hence $n \leq 3$. The following cases occur:
Case I: $n = 3$. We claim that R_i is a field for each $1 \leq i \leq 3$. Suppose on contrary that R_2 is not a field with nonzero maximal ideal $\mathfrak{m}$. Then the set $\{R_1 \times (0) \times (0), R_1 \times \mathfrak{m} \times (0), (0) \times \mathfrak{m} \times (0), (0) \times (0) \times R_3, (0) \times R_2 \times R_3\}$ induces a copy of $K_{2,3}$ with partition sets $A = \{(0) \times (0) \times R_3, (0) \times R_2 \times R_3\}$ and $B = \{R_1 \times (0) \times (0), R_1 \times \mathfrak{m} \times (0), (0) \times \mathfrak{m} \times (0)\}$, a contradiction. Therefore R_i is a field for each $1 \leq i \leq 3$.
Case II: $n = 2$ and let R_i is not a field with nonzero maximal ideal $\mathfrak{m}_i$ for each $i = 1, 2$. Then the set $\{R_1 \times (0), (0) \times R_2, \mathfrak{m}_1 \times (0), (0) \times \mathfrak{m}_2\}$ induces a copy of K_4 in $\mathcal{EG}(R)$, a contradiction. Hence one of R_i (say R_1) must be a field. Let I be a non-trivial ideal of R_2 other than maximal ideal $\mathfrak{m}_2$. Then the set $\{R_1 \times (0), R_1 \times \mathfrak{m}_2, (0) \times R_2, (0) \times \mathfrak{m}_2, (0) \times I\}$ induces a copy of $K_{2,3}$ with partition sets $A = \{R_1 \times (0), R_1 \times \mathfrak{m}_2\}$ and $B = \{(0) \times R_2, (0) \times \mathfrak{m}_2, (0) \times I\}$ in $\mathcal{EG}(R)$, a contradiction. Hence R_2 is a field or

has unique non-trivial ideal.

Case III: $n = 1$, then R is an Artinian local ring. Thus by Lemma 3.2, $\mathcal{EG}(R)$ is a complete graph. Since $\mathcal{EG}(R)$ is outerplanar, R have at most three non-trivial ideals.

Converse follows from Lemma 3.2, Theorem 4.5, Figures 2 and 3. □

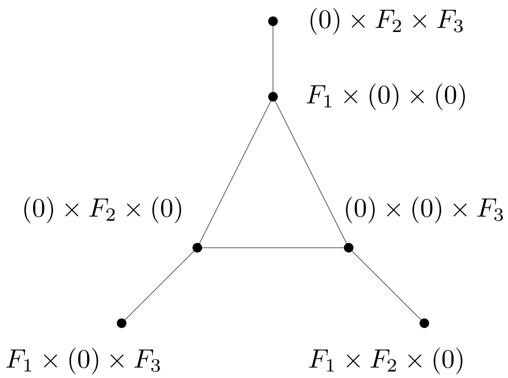

Figure 2: The graph $\mathcal{EG}(F_1 \times F_2 \times F_3)$.

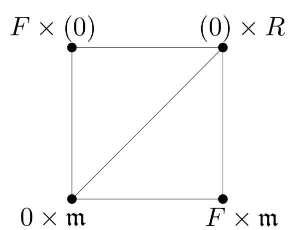

Figure 3: The graph $\mathcal{EG}(F \times R_1)$, where $\mathfrak{m}$ is the only non-trivial ideal of R_1.

Lemma 4.7 ([20, Proposition 2.7])**.** *If $(R, \mathfrak{m})$ is an Artinian local ring and there is an ideal I of R such that $I \neq \mathfrak{m}^i$ for every i, then R has at least three distinct non-trivial ideals J, K and L such that $J, K, L \neq \mathfrak{m}^i$ for each i.*

Theorem 4.8 (Kuratowski's Theorem)**.** *A graph G is planar if and only if it contains no subdivision of K_5 or $K_{3,3}$.*

Lemma 4.9. *Let $(R, \mathfrak{m})$ be a commutative Artinian local ring. Then $\mathcal{EG}(R)$ is planar if and only if R have at most four non-trivial ideals.*

Proof. It is clear from Lemma 3.2 and Theorem 4.8. □

Theorem 4.10. *Let R be a commutative Artinian ring. Then $\mathcal{EG}(R)$ is planar graph if and only if one of the following hold:*

(1) $R = F_1 \times F_2 \times F_3$*, where F_i is a field for each $1 \leq i \leq 3$.*

(2) *R has at most four non-trivial ideals.*

Proof. Suppose that $\mathcal{EG}(R)$ is a planar graph. If $|A^*(R)| \leq 4$, then (2) holds. Thus, we assume that $|A^*(R)| \geq 5$. Since R is Artinian ring, then $R \cong R_1 \times R_2 \times \cdots \times R_n$, where each R_i is Artinian local ring. If $n \geq 4$, then the set $\{R_1 \times (0) \times \cdots \times (0), R_1 \times R_2 \times (0) \times \cdots \times (0), (0) \times R_2 \times (0) \times \cdots \times (0)\} \cup \{(0) \times (0) \times R_3 \times R_4 \times (0) \times \cdots \times (0), (0) \times (0) \times R_3 \times (0) \times \cdots \times (0), (0) \times (0) \times (0) \times R_4 \times (0) \times \cdots \times (0)\}$ induces a copy of $K_{3,3}$ in $\mathcal{EG}(R)$, a contradiction. Hence $n \leq 3$. The following cases occur:
Case I: $n = 3$. We claim that R_i is a field for each $1 \leq i \leq 3$. Suppose on contrary that one of R_i say R_2 is not a field with nonzero maximal ideal $\mathfrak{m}$. Then the set $\{R_1 \times (0) \times (0), R_1 \times \mathfrak{m} \times (0), (0) \times \mathfrak{m} \times (0), (0) \times \mathfrak{m} \times R_3, (0) \times (0) \times R_3, (0) \times R_2 \times R_3\}$ induces a copy of $K_{3,3}$ with partition sets $A = \{R_1 \times (0) \times (0), R_1 \times \mathfrak{m} \times (0), (0) \times \mathfrak{m} \times (0)\}$ and $B = \{(0) \times (0) \times R_3, (0) \times \mathfrak{m} \times R_3, (0) \times R_2 \times R_3\}$ in $\mathcal{EG}(R)$, a contradiction. Hence, (1) satisfied.
Case II: $n = 2$. Since $|A^*(R)| \geq 5$, then one of R_i is not a field for some $i = 1, 2$. Suppose that R_1 is not a field with nonzero maximal ideal $\mathfrak{m}_1$. If R_2 is a field, then $|A^*(R)| \geq 5$ shows that R_1 have at least two non-trivial ideals. Let I be a non-trivial ideal of R_1 other than the maximal ideal. Then the set $\{R_1 \times (0), \mathfrak{m}_1 \times (0), I \times (0)\} \cup \{(0) \times R_2, \mathfrak{m}_1 \times R_2, I \times R_2\}$ induces a copy of $K_{3,3}$ in $\mathcal{EG}(R)$, a contradiction.

Now, if R_2 is not a field with nonzero maximal ideal $\mathfrak{m}_2$, then the set $\{R_1 \times (0), (0) \times \mathfrak{m}_2, R_1 \times \mathfrak{m}_2\} \cup \{(0) \times R_2, \mathfrak{m}_1 \times (0), \mathfrak{m}_1 \times R_2\}$ induces a copy of $K_{3,3}$ in $\mathcal{EG}(R)$, again a contradiction.
Case III: $n = 1$. Then R is an Artinian local ring. Thus, by Lemma 3.2, $\mathcal{EG}(R)$ is a complete graph. Since $|A^*(R)| \geq 5$, then $\mathcal{EG}(R)$ contains a copy of K_5, which is a contradiction.

Conversely, If R is an Artinian ring with at most four non-trivial ideals, then by Theorem 4.8, $\mathcal{EG}(R)$ is planar. Also, if $R = F_1 \times F_2 \times F_3$, where F_i is a field for each $1 \leq i \leq 3$, then from Figure 2, $\mathcal{EG}(R)$ is planar. □

Lemma 4.11 ([22])**.** $\gamma(K_n) = \lceil \frac{1}{12}(n-3)(n-4) \rceil$, *where* $\lceil x \rceil$ *is the least integer that is greater than or equal to* x*. In particular,* $\gamma(K_n) = 1$ *if* $n = 5, 6, 7$.

Lemma 4.12 ([22])**.** $\gamma(K_{m,n}) = \lceil \frac{1}{4}(m-2)(n-2) \rceil$, *where* $\lceil x \rceil$ *is the least integer that is greater than or equal to* x*. In particular,* $\gamma(K_{4,4}) = \gamma(K_{3,n}) = 1$ *if* $n = 3, 4, 5, 6$.

Theorem 4.13. *Let* $(R, \mathfrak{m})$ *be a commutative Artinian local ring. Then* $\gamma(\mathcal{EG}(R)) = 1$ *if and only if* R *have at least five and at most seven non-trivial ideals.*

Proof. Since $(R, \mathfrak{m})$ is an Artinian local ring, then from Lemma 3.2, $\mathcal{EG}(R)$ is a complete graph. Thus, by Lemma 4.11, $5 \leq r \leq 7$, where r is the number of non-trivial ideals of R. □

Theorem 4.14. *Let* R *be a commutative Artinian ring such that* $R = F_1 \times F_2 \times \cdots \times F_n$, *where* $n \geq 4$ *and* F_i *is a field for each* $1 \leq i \leq n$*. Then* $\gamma(\mathcal{EG}(R)) = 1$ *if and only if* $n = 4$.

Proof. Since R is a reduced ring, $\mathcal{EG}(R) = AG(R)$ by Theorem 2.5. Hence the result follows from [19, Theorem 2].

□

Theorem 4.15. *Let R be a commutative Artinian ring such that $R = R_1 \times R_2 \times \cdots \times R_n$, where $n \geq 2$ and each $(R_i, \mathfrak{m}_i)$ is an Artinian local ring with $m_i \neq 0$. Let η_i be the nilpotency of $\mathfrak{m}_i$. Then $\gamma(\mathcal{EG}(R)) = 1$ if and only if $n = 2$ and $\mathfrak{m}_1$ and $\mathfrak{m}_2$ are the only non-trivial ideals of R_1 and R_2 respectively.*

Proof. Suppose that $\gamma(\mathcal{EG}(R)) = 1$. If $n \geq 3$, then the set $\{\mathfrak{m}_1^{\eta_1-1} \times (0) \times \cdots \times (0), (0) \times \mathfrak{m}_2^{\eta_2-1} \times (0) \times \cdots \times (0), \mathfrak{m}_1^{\eta_1-1} \times \mathfrak{m}_2^{\eta_2-1} \times (0) \times \cdots \times (0)\} \cup \{(0) \times (0) \times R_3 \times (0) \times \cdots \times (0), (0) \times (0) \times \mathfrak{m}_3 \times (0) \times \cdots \times (0), \mathfrak{m}_1 \times \mathfrak{m}_2 \times \mathfrak{m}_3 \times (0) \times \cdots \times (0), \mathfrak{m}_1 \times (0) \times \mathfrak{m}_3 \times (0) \times \cdots \times (0), (0) \times \mathfrak{m}_2 \times \mathfrak{m}_3 \times (0) \times \cdots \times (0), \mathfrak{m}_1 \times (0) \times R_3 \times (0) \times \cdots \times (0), (0) \times \mathfrak{m}_2 \times R_3 \times (0) \times \cdots \times (0)\}$ induces a copy of $K_{3,7}$ in $\mathcal{EG}(R)$. Thus, from Lemma 4.12, $\gamma(\mathcal{EG}(R)) > 1$, a contradiction. Hence $n = 2$.
Suppose I is non-trivial ideal of R_1 such that $I \neq \mathfrak{m}_1$. Then the set $\{R_1 \times (0), \mathfrak{m}_1 \times (0), R_1 \times \mathfrak{m}_2, I \times (0), I \times \mathfrak{m}_2\} \cup \{(0) \times R_2, (0) \times \mathfrak{m}_2, \mathfrak{m}_1 \times R_2, \mathfrak{m}_1 \times \mathfrak{m}_2\}$ induces a copy of $K_{4,5}$ in $\mathcal{EG}(R)$. By Lemma 4.12, $\gamma(\mathcal{EG}(R)) > 1$, a contradiction. Hence R_1 has unique non-trivial ideal $\mathfrak{m}_1$. Similarly, we can show that R_2 has unique non-trivial ideal $\mathfrak{m}_2$.
Conversely, let $R = R_1 \times R_2$, where $\mathfrak{m}_1$ and $\mathfrak{m}_2$ are the only non-trivial ideals of R_1 and R_2 respectively, then $|A^*(R)| = 7$. It is easy to see that the set $\{R_1 \times (0), \mathfrak{m}_1 \times (0), R_1 \times \mathfrak{m}_2\} \cup \{(0) \times R_2, (0) \times \mathfrak{m}_2, \mathfrak{m}_1 \times R_2\}$ induces a copy of $K_{3,3}$, which implies that $K_{3,3} \leq \mathcal{EG}(R) \leq K_7$. Hence, by Lemma 4.11 and 4.12, $\gamma(\mathcal{EG}(R)) = 1$. □

Theorem 4.16 ([19, Theorem 4])**.** *Let $R = R_1 \times R_2 \times F$ be a commutative ring, where each $(R_i, \mathfrak{m}_i)$ is a local ring with $\mathfrak{m}_i \neq 0$ and F is a field. Let η_i be the nilpotency of $\mathfrak{m}_i$. Then $\gamma(AG(R)) > 1$.*

Theorem 4.17 ([19, Theorem 5])**.** *Let $R = R_1 \times F_1 \times F_2 \times \cdots \times F_m$ be a commutative ring, where each $(R_1, \mathfrak{m}_1)$ is a local ring with $\mathfrak{m}_1 \neq 0$ and each F_j is a field. Let η_1 be the nilpotency of $\mathfrak{m}_1$ and $m \geq 3$. Then $\gamma(AG(R)) > 1$.*

Theorem 4.18. *Let R be a commutative Artinian ring such that $R = R_1 \times R_2 \times \cdots \times R_n \times F_1 \times F_2 \times \cdots \times F_m$, where each $(R_i, \mathfrak{m}_i)$ is an Artinian local ring with $\mathfrak{m}_i \neq 0$ and each F_j is a field. Let η_i be the nilpotency of $\mathfrak{m}_i$ and $n \geq 2$ or $m \geq 3$. Then $\gamma(\mathcal{EG}(R)) > 1$.*

Proof. Follows from Theorems 4.16 and 4.17. □

Theorem 4.19. *Let R be a commutative Artinian ring such that $R = R_1 \times F_1 \times F_2$, where $(R_1, \mathfrak{m})$ is an Artinian local ring and F_1 and F_2 are fields. Let η be the nilpotency of $\mathfrak{m}$. Then $\gamma(\mathcal{EG}(R)) = 1$ if and only if $\eta = 2$ and $\mathfrak{m}$ is the only non-trivial ideal of R_1.*

Proof. Suppose that $\eta = 2$ and $\mathfrak{m}$ is the only non-trivial ideal of R_1. Then from Figure 5, we get $\gamma(\mathcal{EG}(R)) = 1$, where $a = \mathfrak{m} \times (0) \times (0)$, $b = R_1 \times (0) \times (0)$, $c = \mathfrak{m} \times F_1 \times F_2$, $d = (0) \times F_1 \times F_2$, $e = \mathfrak{m} \times (0) \times F_2$, $f = (0) \times F_1 \times (0)$, $g = R_1 \times F_1 \times (0)$, $h = R_1 \times (0) \times F_2$, $i = (0) \times (0) \times F_2$, $j = \mathfrak{m} \times F_1 \times (0)$.
Conversely, assume that $\gamma(\mathcal{EG}(R)) = 1$. Let J be a non-trivial ideal of R_1 such that $J \neq \mathfrak{m}$. Then the set $\{\mathfrak{m} \times (0) \times (0), \mathfrak{m} \times F_1 \times (0), J \times F_1 \times (0), (0) \times F_1 \times (0)\} \cup \{J \times (0) \times (0), \mathfrak{m} \times (0) \times F_2, J \times (0) \times F_2, (0) \times (0) \times F_2, R_1 \times (0) \times (0)\}$ induces a copy of $K_{4,5}$ in $\mathcal{EG}(R)$, which is a contradiction. Hence $\mathfrak{m}$ is the only non-trivial ideal of R_1. □

Theorem 4.20. *Let R be a commutative Artinian ring such that $R = R_1 \times F$, where $(R_1, \mathfrak{m})$ is an Artinian local ring and F is a field. Let η be the nilpotency of $\mathfrak{m}$. Then $\gamma(\mathcal{EG}(R)) = 1$ if and only if one of the following holds:*

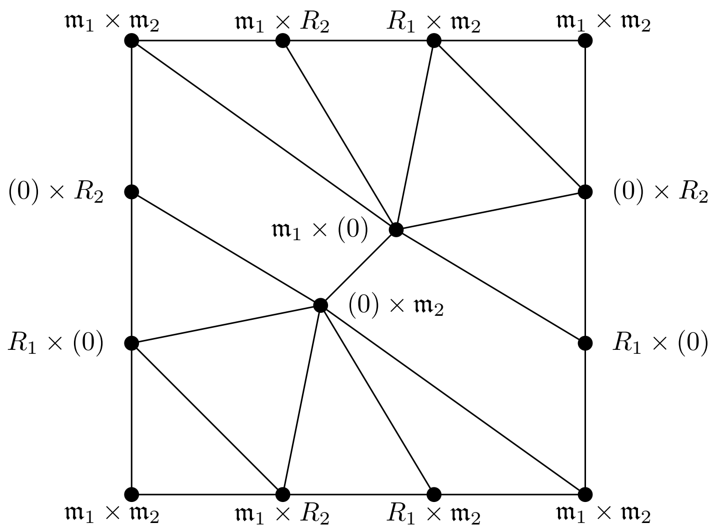

Figure 4: Toroidal embedding of $\mathcal{EG}(R_1 \times R_2)$, where $\mathfrak{m}_i$ is the only non-trivial ideal of R_i for $i = 1, 2$.

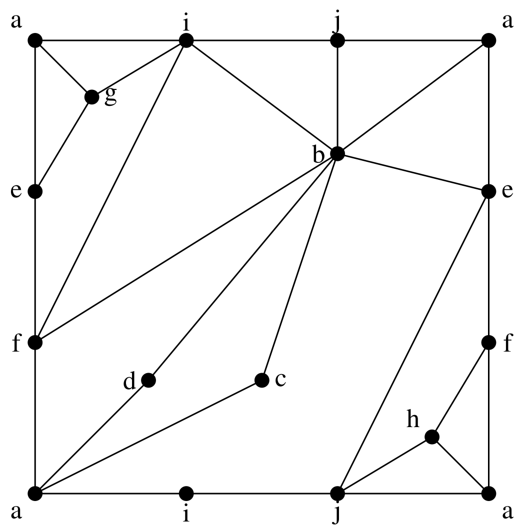

Figure 5: Toroidal embedding of $\mathcal{EG}(R_1 \times F_1 \times F_2)$, where $\mathfrak{m}$ is the only non-trivial ideal of R_1.

(1) $\eta = 3$ *and* $\mathfrak{m}$ *and* $\mathfrak{m}^2$ *are the only non-trivial ideals of* R_1.

(2) $\eta = 4$ *and* $\mathfrak{m}$, $\mathfrak{m}^2$ *and* $\mathfrak{m}^3$ *are the only non-trivial ideals of* R_1.

Proof. Suppose that $\gamma(\mathcal{EG}(R)) = 1$. If $\eta \geq 5$, then the set $\{\mathfrak{m}^{\eta-1} \times (0), \mathfrak{m}^{\eta-2} \times (0), \mathfrak{m}^{\eta-3} \times (0)\} \cup \{R_1 \times (0), \mathfrak{m} \times (0), (0) \times F, \mathfrak{m}^{\eta-1} \times F, \mathfrak{m}^{\eta-2} \times F, \mathfrak{m}^{\eta-3} \times F, \mathfrak{m} \times F\}$

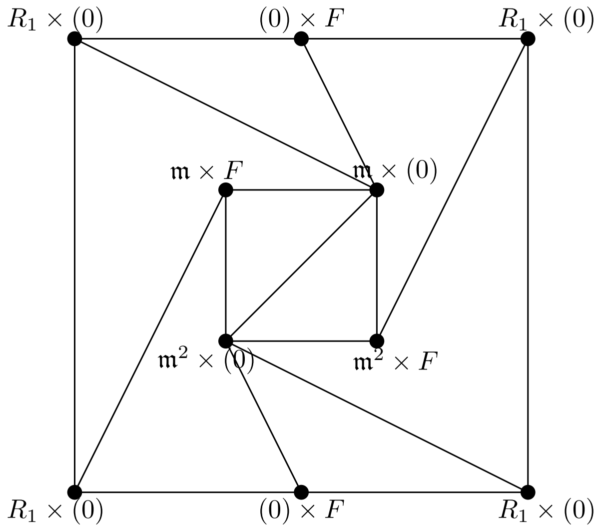

Figure 6: Toroidal embedding of $\mathcal{EG}(R_1 \times F)$, where $\mathfrak{m}$ and $\mathfrak{m}^2$ are only non-trivial ideals of R_1.

induces a copy of $K_{3,7}$. Thus, by Lemma 4.12, $\gamma(\mathcal{EG}(R)) > 1$, a contradiction. Hence $\eta \leq 4$. We have following cases:

Case I: $\eta = 2$. Let J be a non-trivial ideal of R_1 such that $J \neq \mathfrak{m}$. Then by Lemma 4.7, R_1 has at least three non-trivial ideals I_1, I_2 and I_3 such that $I_1, I_2, I_3 \neq \mathfrak{m}$. We can see that the set $\{R_1 \times (0), J \times (0), I_1 \times (0), I_2 \times (0)\} \cup \{(0) \times F, J \times F, I_1 \times F, I_2 \times F, \mathfrak{m} \times F\}$ induces a copy of $K_{3,7}$ in $\mathcal{EG}(R)$, a contradiction. Hence $\mathfrak{m}$ is the only non-trivial ideal of R_1. It follows from Theorem 4.10 that $\mathcal{EG}(R)$ is a planar graph, a contradiction.

Case II: $\eta = 3$. Let I be a non-trivial ideal of R_1 such that $I \neq \mathfrak{m}, \mathfrak{m}^2$. Then by Lemma 4.7, R_1 has at least three non-trivial ideals I_1, I_2 and I_3 such that $I_1, I_2, I_3 \neq \mathfrak{m}, \mathfrak{m}^2$. It is easy to see that the set $\{R_1 \times (0), \mathfrak{m} \times (0), \mathfrak{m}^2 \times (0)\} \cup \{I \times (0), I_1 \times (0), I_2 \times (0), 0 \times F, \mathfrak{m} \times F, \mathfrak{m}^2 \times F, I \times F\}$ induces a copy of $K_{3,7}$ in $\mathcal{EG}(R)$, a contradiction. Hence $\mathfrak{m}$ and $\mathfrak{m}^2$ are the only non-trivial ideals of R_1.

Case III: $\eta = 4$. Let I be a non-trivial ideal of R_1 such that $I \neq \mathfrak{m}^i$ for each $i = 1, 2, 3$. Then by Lemma 4.7, R_1 has at least three non-trivial ideals I_1, I_2 and I_3 such that $I_1, I_2, I_3 \neq \mathfrak{m}^i$ for each $i = 1, 2, 3$. It is easy to see that the set $\{\mathfrak{m} \times (0), \mathfrak{m}^2 \times (0), \mathfrak{m}^3 \times (0)\} \cup \{R_1 \times (0), I \times (0), I_1 \times (0), I_2 \times (0), \mathfrak{m} \times F, \mathfrak{m}^2 \times (0), \mathfrak{m}^3 \times (0)\}$ induces a copy of $K_{3,7}$ in $\mathcal{EG}(R)$, a contradiction. Hence $\mathfrak{m}$, $\mathfrak{m}^2$ and $\mathfrak{m}^3$ are the only non-trivial ideals of R_1.

Conversely, if $\mathfrak{m}$ and $\mathfrak{m}^2$ are the only non-trivial ideals of R_1, then $|A^*(R)| = 6$ and the set $\{R_1 \times (0), \mathfrak{m} \times (0), \mathfrak{m}^2 \times (0)\} \cup \{(0) \times F, \mathfrak{m} \times F, \mathfrak{m}^2 \times F\}$ induces a copy of $K_{3,3}$ in $\mathcal{EG}(R)$. Thus, $K_{3,3} \leq \mathcal{EG}(R) \leq K_6$, which implies that $\gamma(\mathcal{EG}(R)) = 1$.

Now, if $\mathfrak{m}$, $\mathfrak{m}^2$ and $\mathfrak{m}^3$ are the only non-trivial ideals of R_1. Then from Figure 7, $\gamma(\mathcal{EG}(R)) = 1$. □

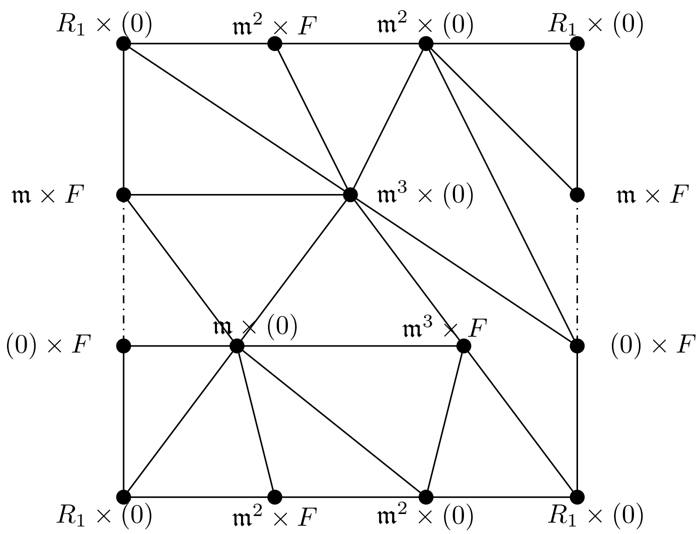

Figure 7: Toroidal embedding of $\mathcal{EG}(R_1 \times F)$, where $\mathfrak{m}$, $\mathfrak{m}^2$ and $\mathfrak{m}^3$ are non-trivial ideals of R_1.

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Cell reducing and the dimension of the C^1 bivariate spline space

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Abstract

In this paper, the problem of determining the dimension of the space $S_n^1(\triangle), n \geq 3$ of bivariate C^1 splines of degree $\leq n$ over a triangulation $\triangle$ is considered. The piecewise polynomials are represented as blossoms, and the smoothness conditions are written as a system of linear equations. The rank of the system matrix is analysed by repeatedly reducing small subtriangulations (cells) at the boundary of a triangulation. It is shown that the dimension of the bivariate spline space $S_n^1(\triangle), n \geq 3$ is equal to Schumaker's lower bound for a large class of triangulations.

Keywords: Dimension, spline space, triangulation, cell.

Math. Subj. Class. (2020): 65D05, 65D07, 65D17, 15A03

1 Introduction

In the last 40 years the problem of determining the dimension of the bivariate spline space has received a considerable attention. For a given triangulation $\triangle$ of a polygonal region $\Omega \subset \mathbb{R}^2$ with N triangles Ω_i, *the bivariate spline space* of degree n and smoothness r is defined as

$$S_n^r(\triangle) := \{f \in C^r(\Omega); \quad f|_{\Omega_i} \in \Pi_n(\mathbb{R}^2), \quad i = 1, 2, \ldots, N\},$$

where $\Pi_n(\mathbb{R}^2)$ denotes the space of bivariate polynomials of total degree $\leq n$. In contrast to the univariate case, the bivariate spline space has a much more complex structure and even such basic problems as determining its dimension or construction of its basis are surprisingly hard to tackle. Even more surprising is the fact that the "simplest" spaces of splines of the lowest degrees are the most complex. For example, for the most interesting

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case - the space of cubic C^1 splines $S_3^1(\triangle)$, quite frequently used in practical applications, the dimension is still unknown in general, even though a great deal of research has been done on the topic.

But it is essential that the dimension is known in advance in some important applications, in particular for Lagrange interpolation by bivariate splines.

In general, the problem has been solved for a spline space of degree n and smoothness r over a regular triangulation $\triangle$, $S_n^r(\triangle)$, where the degree n is large in comparison to the smoothness r ($n \geq 3r+2$ ([6]), $n = 4, r = 1$ ([1])). Recall that a triangulation is *regular*, if two adjacent triangles Ω_i, Ω_j can have only one vertex or the whole edge in common.

The dimension of the spline space $S_3^1(\triangle)$ is known for particular classes of triangulations only [4, 7, 5], etc. It has been conjectured that the dimension is equal to Schumaker's lower bound ([12, 13])

$$\dim S_3^1(\triangle) \geq 3V_B(\triangle) + 2V_I(\triangle) + \sigma(\triangle) + 1, \tag{1.1}$$

where $V_B(\triangle)$ denotes the number of boundary vertices, $V_I(\triangle)$ the number of internal vertices, and $\sigma(\triangle) = \sum_{i=1}^{V_I(\triangle)} \sigma_i$,

$$\sigma_i = \begin{cases} 1, & \text{if vertex is singular,} \\ 0, & \text{otherwise.} \end{cases}$$

A vertex is *singular* if it is obtained as an intersection of exactly two lines.

Suppose that a triangulation $\triangle$ consists of a set of triangles that all have one common vertex v. Suppose every triangle in $\triangle$ has at least one neighbour with which it shares a common edge. Then we call $\triangle$ *a cell*. If v is an interior vertex, then $\triangle$ is *an interior cell*, otherwise it is *a boundary cell* (see [11]). *Cell degree* is the degree (valency) of the vertex v.

The main obstacle in the study of the dimension problem is the fact that the dimension depends not only on the topology of the triangulation $\triangle$ but also on its geometry. It has been conjectured (see [14]) that the dimension is equal to Schumaker's lower bound for $n \geq 2r+1$ and that the dimension jump occurs only for singular vertices.

Various approaches have been applied to tackle the dimension problem (tools from linear algebra, algebraic topology, graph theory, symbolic computation and computer aided design), but the problem is very hard. It has been compared even with the well-known Four Color Map Problem.

For the space of cubic C^1 splines, the dimension has been determined for special triangulations only (triangulations of type 1 and 2 ([12, 13]), nested polygon triangulations ([4]), reducible triangulations ([7]), etc.). Dimension is known also for generic triangulations, i.e., such triangulations, that if the dimension of the spline space exceeds the lower bound, a small perturbation of vertices of the triangulation causes the dimension to match the lower bound (see [2, 15], and more elementary proof in [11]). In all the cases the dimension equals the lower bound (1.1).

In this paper, the blossoming approach is used (see [3, 7]). The idea is to study the smoothness conditions between polynomial patches, written as their blossoms ([10]). This is a dual approach to the well known classical approach (see [11], e.g.) and brings a new insight to the dimension problem. An overview of cell reduction at the boundary of the triangulation is given. Thus sufficient conditions for an inductive approach for determining whether the dimension of $S_n^1(\triangle), n \geq 3$, is equal to Schumaker's lower bound for a large class of triangulations $\triangle$ are obtained. It is shown that interior cells of degrees $k = 4, 5, \ldots, 8$ can be tackled, but the reduction can be applied only in the case $k = 4$ and for special cases for $k = 5$. For $k = 6, 7, 8$, a negative result is proven. Furthermore, interior cells with more than 2 free boundary edges are studied. Since it is possible to reduce most of the cases by methods for boundary cells, we focus the study to the cases with collinearities. It is proven that a cell of degree 4 with 1 common edge and a cell of degree 5 with 2 common edges with the rest of the triangulation can be reduced.

An algorithm that extends the results of [7] is presented, and it is proven that the results can be generalized to $S_n^1(\triangle), n > 3$.

The structure of the paper is as follows. In Section 2, an overview of the blossoming approach to the dimension problem is given. In Section 3, the cell reduction is studied and the main results of the paper are presented. In Section 4, an efficient algorithm for determining whether a given triangulation belongs to the observed class of triangulations is presented. Section 5 extends the results from the cubic ($S_3^1(\triangle)$) to the general case $n \geq 3$ ($S_n^1(\triangle)$). The paper is concluded by a proof of one of the main results.

2 Blossoming approach to the dimension problem

First, let us recall the blossoming approach. It is well known that there exists a bijective correspondence between a bivariate polynomial and its blossom: for every polynomial $p \in \Pi_n(\mathbb{R}^m)$ there exists a unique symmetric n-affine polynomial

$$B_n(p)\Big(x^{(1)}, x^{(2)}, \ldots, x^{(n)}\Big), \quad x^{(i)} \in \mathbb{R}^m,$$

with a diagonal property

$$B_n(p)\Big(\underbrace{x, x, \ldots, x}_{n}\Big) = p(x), \quad x \in \mathbb{R}^m.$$

The polynomial $B_n(p)$ is called the *blossom* of the polynomial p.

For example, the blossom of the polynomial

$$p_2(x, y) = a_0 + a_1 x + a_2 y + a_3 x^2 + a_4 xy + a_5 y^2$$

is

$$B_2(p_2)\Big((u_1, v_1), (u_2, v_2)\Big) = \\ a_0 + a_1 \frac{u_1 + u_2}{2} + a_2 \frac{v_1 + v_2}{2} + a_3 u_1 u_2 + a_4 \frac{u_1 v_2 + u_2 v_1}{2} + a_5 v_1 v_2.$$

The blossom generalizes a given polynomial. One can study the smoothness conditions between polynomial patches over adjacent triangles, expressed in the blossoming form. In order to determine the dimension of the spline space, smoothness conditions over all inner edges of the triangulation need to be studied.

By studying the dual representation of a triangulated graph [3], it can be seen that some smoothness conditions are independent and can be omitted. The study of the rest results into the dimension equation

$$\dim S_n^r(\triangle) = N\binom{n+2}{2} - \binom{r+2}{2}(E_I - V_I) - \operatorname{rank} M_n. \tag{2.1}$$

Here N denotes the number of triangles, E_I the number of inner edges, and V_I the number of inner vertices of the triangulation. The matrix M_n describes the smoothness relations that need to be studied. It has a particular structure:

$$M_n := (M_{km})_{k=1;m=1}^{n-r;n-r} := (M_{r,km})_{k=1;m=1}^{n-r;n-r} \tag{2.2}$$

is a block upper triangular matrix with blocks M_{km} of size $(r+1)E_I \times (m+r+1)(N-1)$. The matrix M_{km} is also a block matrix with E_I block rows and $N-1$ block columns: ℓ-th block row corresponds to smoothness conditions across the edge $e_\ell = (i,j)$ between triangles Ω_i and Ω_j, and it has at most two nonzero blocks $Q_{\ell i} := Q_{km,\ell i}$, $Q_{\ell j} := Q_{km,\ell j}$, with $Q_{km,\ell i} + Q_{km,\ell j} = 0$. Blocks $Q_{km,\ell i}$ are circulant matrices of size $(r+1) \times (m+r+1)$. Theirs first row is defined by

$$\binom{n-r}{k} \sum_{|\gamma|=k} \binom{k}{\gamma} \binom{n-r-k}{\beta-\gamma} v_\ell^\gamma u_\ell^{\beta-\gamma} = f \cdot \sum_{|\gamma|=k} \binom{k}{\gamma} \binom{m-k}{\beta-\gamma} v_\ell^\gamma u_\ell^{\beta-\gamma},$$
$$\beta = (b_1, m-b_1),\ b_1 = m, m-1, \ldots, 0,$$

with

$$f := f_{n,r,k,m} = \binom{n-r-k}{m-k}.$$

The rest r elements of the row are zeros. Here u_ℓ denotes an arbitrary point on the edge e_ℓ with the normalized directional vector v_ℓ. The standard multiindex notation is used. For more details, see [3]. In the following section, the matrix M_n for the case $n = 3$ will be described in detail.

Thus the main problem is how to determine the rank of a large symbolic matrix M_n that depends on the geometry and the topology of a triangulation. Such a problem is very hard to tackle in general. A natural idea is to reduce the problem to a smaller one, if particular assumptions are satisfied.

3 Cell reduction

The idea of the blossoming approach to the dimension problem is to inductively reduce the problem from the given triangulation $\triangle$ to its subtriangulation $\triangle \backslash \triangle_1$ for a proper choice of the subtriangulation $\triangle_1$ (see Figure 1).

Let $\mathcal{B}$ denote the intersection of $\triangle_1$ and $\triangle \backslash \triangle_1$. We call the subtriangulation $\triangle_1$ *proper*, if it is simply connected, has a vertex T_0 on the outer face of $\triangle$ and contains all the triangles in $\triangle$ with the vertex T_0, no triangle in $\triangle_1$ has two edges on $\mathcal{B}$, there are no consecutive pairs of collinear edges at the vertices of degree 3 in $\triangle_1$ on $\mathcal{B}$, and every singular vertex in $\triangle_1$ lies in the interior of $\triangle_1$.

Let $v_\ell = (\alpha_\ell, \beta_\ell)$ denote a normalized directional vector of the edge e_ℓ between triangles Ω_i and Ω_j. Further, let

$$v_i \times v_j := \alpha_i \beta_j - \alpha_j \beta_i$$

be the planar vector product. The matrix $M := M_n$ could be written as

$$M = M(\triangle) = \begin{bmatrix} M(\triangle_1) & 0 \\ M(\triangle_1, \triangle) & M(\triangle, \triangle_1) \\ 0 & M(\triangle \backslash \triangle_1) \end{bmatrix},$$

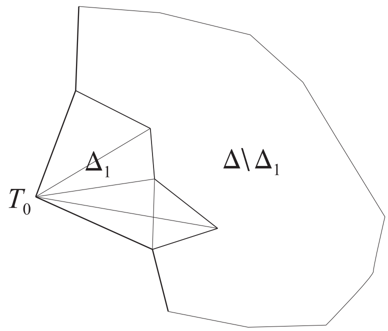

Figure 1: Reduction of a triangulation.

where the matrices $M(\triangle_1, \triangle), M(\triangle, \triangle_1)$ represent the common part of the smoothness conditions between $\triangle_1$ and $\triangle\backslash\triangle_1$, $M(\triangle_1)$ represents the conditions inside $\triangle_1$, and $M(\triangle\backslash\triangle_1)$ the conditions inside $\triangle\backslash\triangle_1$. Let

$$\widetilde{M}(\triangle_1, \triangle) := \begin{bmatrix} M(\triangle_1) \\ M(\triangle_1, \triangle) \end{bmatrix} \in \mathbb{R}^{r\times c}.$$

Theorem 3.1 ([7, Theorem 1]). *Suppose that $\triangle_1$ is a proper subtriangulation of $\triangle$ that satisfies $V_B(\triangle_1) \leq 2V_I(\triangle_1) + 6$. If*

$$\operatorname{rank} \widetilde{M}(\triangle_1, \triangle) = r - \sigma(\triangle_1) \tag{3.1}$$

and $\dim S_3^1(\triangle\backslash\triangle_1)$ *is equal to the lower bound, then the dimension* $\dim S_3^1(\triangle)$ *is equal to the lower bound* (1.1) *too.*

The matrices $\widetilde{M}(\triangle_1, \triangle)$ that have to be studied, have a block structure, based on the incidence matrix of the underlying graph of the triangulation. They consist of blocks

$$Q_{11,\ell i} = -Q_{11,\ell j} = \begin{bmatrix} \alpha_\ell & \beta_\ell & 0 \\ 0 & \alpha_\ell & \beta_\ell \end{bmatrix},$$

$$Q_{22,\ell i} = -Q_{22,\ell j} = \begin{bmatrix} \alpha_\ell^2 & 2\alpha_\ell\beta_\ell & \beta_\ell^2 & 0 \\ 0 & \alpha_\ell^2 & 2\alpha_\ell\beta_\ell & \beta_\ell^2 \end{bmatrix},$$

and blocks $Q_{12,\ell j}$ that depend not only on the directions but also on the vertices of the triangulation and some arbitrary additional points. More precisely, by using [3, Lemma 3.1] it is possible to simplify some of the blocks in M_{12} without changing the rank of M by choosing the points $t_\ell := (c_\ell, d_\ell)$ as the inner vertices of the triangulation, and by choosing some arbitrary additional points $z_k := (x_k, y_k) \in \mathbb{R}^2$, $k = 1, 2, \ldots, N$ for the faces Ω_k. The block $Q_{12,\ell i}$ is of the form

$$\begin{bmatrix} \alpha_\ell(c_\ell - x_i) & \alpha_\ell(d_\ell - y_i) + \beta_\ell(c_\ell - x_i) & \beta_\ell(d_\ell - y_i) & 0 \\ 0 & \alpha_\ell(c_\ell - x_i) & \alpha_\ell(d_\ell - y_i) + \beta_\ell(c_\ell - x_i) & \beta_\ell(d_\ell - y_i) \end{bmatrix}, \tag{3.2}$$

and the block $Q_{12,\ell j}$ reads

$$-\begin{bmatrix} \alpha_\ell(c_\ell - x_j) & \alpha_\ell(d_\ell - y_j) + \beta_\ell(c_\ell - x_j) & \beta_\ell(d_\ell - y_j) & 0 \\ 0 & \alpha_\ell(c_\ell - x_j) & \alpha_\ell(d_\ell - y_j) + \beta_\ell(c_\ell - x_j) & \beta_\ell(d_\ell - y_j) \end{bmatrix}, \tag{3.3}$$

so the choice $z_k = t_\ell$, $k \in \{i, j\}$ reduces (3.2) or (3.3) to zero block. Of course, not all blocks can be simplified in this way.

The following theorem gives conditions on boundary cells $\triangle_1$ where the reduction can be applied.

Theorem 3.2 ([7, Theorem 2]). *Let $\triangle_1$ be a proper subtriangulation of $\triangle$ with a vertex T_0 of degree $s+1$ on the outer face of $\triangle$. If*

(1) $s \le 4,\ |\triangle_1| = s$, *or*

(2) $s = 2, 3,\ |\triangle_1| = s + 2$, *and $\triangle_1$ includes an interior cell, and $\dim S_3^1(\triangle \backslash \triangle_1)$ is equal to the lower bound, then the dimension $\dim S_3^1(\triangle)$ is equal to the lower bound* (1.1) *too.*

Therefore, as a natural extension, it is interesting to study interior cell reduction at the boundary of the triangulation $\triangle$. Usually, the main obstacle are collinear edges. Those are the only special cases for $n \ge 3r + 2$ and for known results for $n = 2r + 1$. For $n = 2r$, there exist other geometric configurations, that result in a change of the dimension (Morgan-Scott triangulation [3], e.g.).

By studying interior cells, the collinear edges can be included in the subtriangulation $\triangle_1$. This will enable us to study the dimension problem on a wider class of triangulations.

The conditions of Theorem 3.1 allow us to study cells of degree k, $k = 4, 5, \ldots, 8$. In light of the previous discussion, the study will be limited to particularly interesting cells where there is a collinearity of edges between the inner vertex and the vertices, adjacent to the boundary vertex.

Theorem 3.3. *Let $\triangle_1$ be an interior cell of degree k, $k = 5, 6, 7, 8$, and let $\triangle_1$ be a proper subtriangulation of the triangulation $\triangle$ with a vertex T_0 of degree 3 at the outer face of the triangulation $\triangle$ and an inner vertex T_1. Let the edges adjacent to T_1 be denoted in clockwise direction as $e_1 = T_0T_1, e_2, \ldots, e_k$ (see* Figures 3, 5, 6, 7*). Let there be the collinearity $e_2 || e_k$.*

(1) *If $k = 5$ and there is a collinearity of edges $e_1 || e_3$ or $e_1 || e_4$* (Figure 3)*, and $\dim S_3^1(\triangle \backslash \triangle_1)$ is equal to Schumaker's lower bound, then also the dimension $\dim S_3^1(\triangle)$ is equal to the lower bound* (1.1).

(2) *If $k = 6, 7, 8$, then* Theorem 3.1 *can not be applied* (Figures 5, 6, 7).

Proof. Proof of the theorem is given as the last section of the paper. □

By Theorem 3.1 we have to study ranks of certain matrices $\widetilde{M}(\triangle_1, \triangle)$ that belong to cells considered. Ranks will be obtained by studying appropriate minors. Since the matrices are large and symbolic, symbolic computer algebra tools and [9] will be used for the computation of determinants and the simplification of huge symbolic expressions. Here, properties of triangulation's topology and geometry have to be used very carefully, since otherwise computations are futile because of the huge time and memory requirements and enormous symbolic expressions. Note that the determinants obtained can be easily verified by evaluating the polynomials in enough number of points, exact numerical determinant calculation, and the well-known results on multivariate Lagrange polynomial interpolation (see [8, Lemma 1]).

Remark 3.4. Interior cells of degree 4 and boundary cells of degree ≤ 5 are covered by Theorem 3.2.

Remark 3.5. The study of interior cells in the general position is much more difficult.

1. For $k = 5, 6$, the appropriate minors can be computed, but the obtained polynomial expressions are difficult to simplify enough to be able to prove that the polynomial is nonzero for all geometrically admissible edge directions and vertex positions.

2. For $k = 7$ and $k = 8$ also in the general case the appropriate matrices are not of the full rank and the considered cell reducing approach can not be applied. The proof of Theorem 3.3 holds in general, since the collinearity of the edges is not used.

Remark 3.6. In general, it is interesting to consider interior cells of degree k at the boundary of triangulation, where there are $\ell = 1, 2, \ldots, k - 2$ edges in the common border $\mathcal{B}$. It can be easily seen from (3.1), that the matrix dimensions limit the consideration to $\ell \leq \min\{k - 2, \lfloor 3/4k \rfloor\}$. It is important to notice that most of the cases can be reduced by methods for boundary cells (Theorem 3.3). As expected, the most interesting cases with collinearities are left for the study.

Theorem 3.7. *For interior cells of small degrees $k = 4, 5, 6$ there is only one problematic configuration, namely, an interior cell with $\ell = k - 3$ edges in $\mathcal{B}$ and two collinearities in the interior. For $k = 4, 5$ such configurations can be reduced.*

Proof. For $k = 4$, by [7] the matrix obtained is not of full rank because of the interior singular vertex. Consider Figure 2. Let $T_1 = (c_1, d_1)$ be the interior point of the cell, and $T_2 = (c_2, d_2)$ be an additional point on $\mathcal{B}$, the join of e_5 and e_6. Further, let $w_6 := (\beta_6, -\alpha_6)$. The minor, obtained by omitting rows, corresponding to the boundary edge e_5 and additionally row 1 and columns $1, 2, 3, 4, 13, 14, 15, 16, 26$, is

$$\alpha_3\alpha_6\beta_4^2\beta_6^2(v_3 \times v_4)^9(v_6 \times v_3)^2(\beta_6(-c_1 + c_2) + \alpha_6(d_1 - d_2)),$$

which is nonzero, since $(\beta_6(-c_1 + c_2) + \alpha_6(d_1 - d_2)) = \langle w_6, T_1 - T_2\rangle$ and w_6 is perpendicular to $v_6 = (\alpha_6, \beta_6)$.

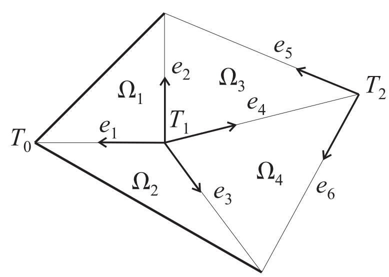

Figure 2: An interior cell of degree 4 at the boundary of the triangulation.

For $k = 5$ we similarly first omit the block rows and columns belonging to e_8 (see Figure 3), apply the same simplification as in the proof of [7, Theorem 2], and then omit

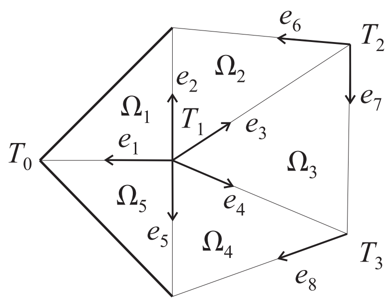

Figure 3: A cell of degree 5 at the boundary of the triangulation.

the columns $15, 16, 19, 20$ and further the columns $3, 14, 15, 16, 18, 19, 24$ in the obtained matrix. The resulting minor is

$$6g^2\alpha_1\alpha_2^3\alpha_7^2\beta_1^2\beta_2^2(v_2 \times v_1)^2(v_6 \times v_3)^5(v_7 \times v_3)^3(v_7 \times v_6)(v_1 \times v_8)(v_2 \times v_8). \quad (3.4)$$

Here g denotes the length of $T_2 - T_1 = gv_3$. Since by assumption the cell is proper, $v_7 \times v_6 \neq 0$, thus the expression (3.4) is nonzero. □

Remark 3.8. For interior cells of higher degrees $k \geq 6$ the matrices obtained are unfortunately too large to study with current computational facilities.

Numerical computations suggest that in the case $k = 5$ the interior cell reduction can be applied for all but one configuration. Unfortunately this seems hard to prove, and we will state it as the following conjecture with a discussion on its possible proof.

Conjecture. Let $k = 5$ and let the assumptions of Theorem 3.3 be fulfilled. Then the cell reduction can be applied for almost every position of edges of the interior cell. There exists a unique configuration of edges where the reduction can not be used.

A computation of all possible $\binom{35}{32} = 6545$ minors of size 32 and a careful simplification of nonzero expressions reveals that all contain the same linear expression in α_1 and β_1. This results in a unique condition on the positions of the edges of the cell, where the matrix considered, $\widetilde{M}(\triangle_1, \triangle)$, is not of full rank. This configuration is geometrically admissible. Thus the reduction of the interior cell can not be applied in this special case. Unfortunately, the symbolic polynomial expressions are huge and it is not feasible to write them down. Thus it is difficult to find a proper minor and prove that it is nonzero for all possible geometrically admissible edge directions, with the exception of the considered one.

4 An algorithm for the reduction of the triangulation

By using Theorem 3.2, Theorem 3.3 and Theorem 3.7, we can construct an algorithm that determines if the triangulation $\triangle$ belongs to the class of triangulations where the dimension $\dim S_3^1(\triangle)$ can be obtained by sequential reductions of the triangulation $\triangle$. This algorithm improves the algorithm in [7].

We are given a triangulation $\triangle$. First, it is rotated to a general position, such that no edge lies on the coordinate axes. Then the reduction step can be used on every boundary vertex that satisfies one of the conditions of Theorem 3.2, Theorem 3.3, item 1 or Theorem 3.7. In the algorithm, let B denote the intersection of a current boundary or interior cell $\triangle_1$ and $\triangle\backslash\triangle_1$. Let $|B|$ denote the number of edges in B.

An algorithm for the reduction of the triangulation

```
// T - a given triangulation
// deg(v) - degree of the vertex v
list<vertex> L = {list of all vertices of degree <=5 on the
outer face of the triangulation T};
while (!L.empty()) {
    if (T= interior cell or T=triangle) break(success);
    //check for possible reductions using L
    v=L.pop();
    check for collinearities between neighbours of v in B;
    //boundary cells
    if (no collinearities && all neighbours of v are in B){
        T.delete_vertex(v);
        if any neighbour(v) has new degree <=5, add it to L;
        }
    //interior cells
    else if ((collinearity at the inner vertex z, deg(z)=4,
    deg(v)<5), |B|=1 or 2 ||
    (collinearity at the inner vertex z, deg(v)=3, deg(z)=5,
    edge vz is collinear with another edge from z,
    |B|=2 or 3)){
        T.delete_vertex(v);
        T.delete_vertex(z);
        add vertices in B with new degree <=5 to L;
        }
    else {
        L.append(v);
        }
    }
if (success) {
    print("Dimension equals Schumaker's lower bound");
    }
else {
    print("Algorithm can not be used");
    }
}
```

If all the vertices in the list `L` were checked, and no reduction could be applied, the list `L` remains the same, and the algorithm stops. In such a case (because of too large degrees of the boundary vertices or because of the collinearities) this method can not be applied for the study of the dimension. If the while loop successfully terminates, the dimension $\dim S_3^1(\triangle)$ is equal to Schumaker's lower bound (1.1). The answer of the algorithm is quite clearly independent of the enumeration of the vertices, i.e., on a particular sequence of reductions.

As an example, consider the triangulation in Figure 4. It is derived from a nested polygonal configuration (see [4] and an example in [7]). Because of a slight modification (deletion of some vertices and edges, and perturbation of vertices), the tools in [4] can not be applied. Similarly, the algorithm from [7] can not tackle it because of collinearities along reduction boundaries at all boundary vertices. But the algorithm, presented in this paper, enables the reduction of the triangulation. Thus the dimension of the C^1 cubic spline space on the triangulation in Figure 4 is equal to Schumaker's lower bound.

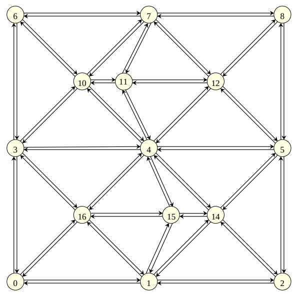

Figure 4: A triangulation $\triangle$, where the algorithm determines $\dim S_3^1(\triangle)$.

5 Generalization to $n \geq 3$

In [3], it has been shown how particular triangulations can be tackled under the assumption, that no inner edges that share a common vertex have the same slope. We will show that this assumption can be omitted, and apply this to generalize our results to $n \geq 3$. For the sake of completeness, we will include the relevant results from [3].

First, let us recall Schumaker's lower bound in general,

$$\dim S_n^r(\triangle) \geq LB_n^r(\triangle) := \binom{n+2}{2} + \binom{n-r+1}{2} E_I - \left(\binom{n+2}{2} - \binom{r+2}{2} \right) V_I + \sum_{i=1}^{V_I} \sigma_i, \tag{5.1}$$

where

$$\sigma_i = \sum_{j=1}^{n-r} (r + j + 1 - j\, e_i)_+, \quad i = 1, 2, \ldots, V_I, \tag{5.2}$$

and E_I denotes the number of interior edges, V_I the number of inner vertices, and e_i the number of edges with distinct slopes in an inner vertex v_i.

Let $\sigma_i(n)$ denote the number σ_i, defined in (5.2), that belong to the vertex i and the considered spline space $S_n^r(\triangle)$.

Lemma 5.1. *Let* $n > 2r$. *Then* $\sigma_i(n) = \sigma_i(2r)$.

Proof. The number $\sigma_i(n)$ can be written as

$$\sigma_i(n) = \sum_{j=1}^{n-r}(r+j+1-j\,e_i)_+ = \sigma_i(2r) + \sum_{j=r+1}^{n-r}(r+j+1-j\,e_i)_+,$$

where e_i is the number of edges with different edge slopes at the inner vertex i. Note that in the second term $j \geq r+1$. Now let us consider the expression

$$r+j+1-j\,e_i = r+1-j(e_i-1) \leq r+1-(r+1)(e_i-1) = (r+1)(2-e_i).$$

Since for the inner vertices of the triangulation in general position $e_i \geq 3$ and for the singular vertices $e_i = 2$,

$$(r+1)(2-e_i) \leq 0.$$

Thus σ_i does not change if the polynomial degree increases. □

Now we can show that [3, Theorem 3.2] and [3, Theorem 3.3] hold also for triangulations that contain collinearities.

Recall the definition of the matrix M_n in (2.2).

Theorem 5.2 ([3, Theorem 3.2]). *Let $\triangle$ be a regular triangulation and $n \geq n_0 \geq 2r$. Then*

$$\operatorname{rank} M_n \geq \operatorname{rank} M_{n_0} + (r+1)E_I(n-n_0). \tag{5.3}$$

Proof. The matrix M_n can be written as

$$M_n = \begin{bmatrix} M_{n-1} & X \\ 0 & M_{n-k,n-k} \end{bmatrix}.$$

Clearly, $\operatorname{rank} M_n \geq \operatorname{rank} M_{n-1} + \operatorname{rank} M_{n-k,n-k}$. We need to prove that

$$\operatorname{rank} M_{n-k,n-k} = (r+1)E_I,$$

i.e., the rank of the matrix $M_{n-k,n-k}$ is equal to the number of rows. The number of columns of $M_{n-k,n-k}$ is equal to

$$(n+1)(N-1) \geq 2(r+1)(N-1) \geq \frac{4}{3}(r+1)E_I > (r+1)E_I,$$

since $3\,N \geq 2\,E_I + 3$, thus it suffices to prove, that the rows are linearly independent.

Suppose that the rows are not independent. Then there exists a vector $x \in \mathbb{R}^{(r+1)E_I}$, $x \neq 0$, such that $x^T M_{n-k,n-k} = 0$. Since the rows that correspond to boundary triangles with only one inner edge are clearly independent, the corresponding components of the vector x are zero. Thus we can assume, that all boundary triangles have only one outer edge. Let Ω_i be such a boundary triangle in $\triangle$. Then $e_{\ell_1} = (j_1, i)$, $e_{\ell_2} = (i, j_2)$ for some j_1, j_2, and the block matrices $Q_{\ell_1 i}, Q_{\ell_2 i}$ are the only nonzero blocks in the block column i. Let $x|_{\ell_j}$ denote the ℓ_j-th block row in x. Then

$$[x|_{\ell_1}, x|_{\ell_2}]^T \begin{bmatrix} Q_{\ell_1 i} \\ Q_{\ell_2 i} \end{bmatrix} = 0.$$

It can easily be seen that the matrix

$$\begin{bmatrix} Q_{\ell_1 i} \\ Q_{\ell_2 i} \end{bmatrix} \in \mathbb{R}^{2(r+1)\times(n+1)} \tag{5.4}$$

where $n+1 \geq 2r+1+1 = 2(r+1)$, is of full rank. Indeed, the rank of the matrix (5.4) stays unchanged, if $n-2r-1$ zero columns are added. If we append another $n-2r-1$ rows, such that they cyclically continue $Q_{\ell_1 i}$, and $n-2r-1$ rows by cyclical continuation of $Q_{\ell_2 i}$, the rank of the matrix (5.4) increases at most by $2(n-2r-1)$. The resulting matrix is of dimension $2(n-r)\times 2(n-r)$. Its determinant is equal to the resultant of polynomials

$$p_{\ell_1}(x) := (v_{1,\ell_1}x + v_{2,\ell_1})^{n-r}, \;\; p_{\ell_2}(x) := (v_{1,\ell_2}x + v_{2,\ell_2})^{n-r},$$

where $v_{\ell_i} := (v_{1,\ell_i}, v_{2,\ell_i})$ is the direction of e_{ℓ_i}. Since the directions v_{ℓ_1} and v_{ℓ_2} are different, the polynomials p_{ℓ_1} and p_{ℓ_2} can not have common zeros. Thus theirs resultant $(v_{1,\ell_1}v_{2,\ell_2} - v_{2,\ell_1}v_{1,\ell_2})^{(n-r)^2}$ is nonzero, hence the matrix (5.4) is of full rank. Thus $x|_{\ell_1} = x|_{\ell_2} = 0$. So one can study $\triangle\backslash\{\Omega_i\}$ only, and use the previous result on the reduced triangulation. The procedure can be repeated until only one triangle is left. This concludes the proof. □

Theorem 5.3 ([3, Theorem 3.3])**.** *Let $\triangle$ be a triangulation, and $n \geq n_0 \geq 2r$. The function*

$$\theta(n, n_0, r) := \dim S_n^r(\triangle) - N\left(\binom{n+2}{2} - \binom{n_0+2}{2}\right) + (r+1)E_I(n-n_0)$$

is nonincreasing function of n, and

$$LB_{n_0}^r(\triangle) \leq \theta(n, n_0, r) \leq \dim S_{n_0}^r(\triangle). \tag{5.5}$$

Proof. It is enough to consider $n > n_0$. The first claim follows from (2.1) and Theorem 5.2. Recall Schumaker's lower bound (5.1). By Lemma 5.1, σ_i stays unchanged for any $n > n_0$. Now it is straightforward to apply (2.1) and (5.3) to obtain (5.5). □

This approach has been used in [3] in order to determine the dimension of $S_n^2(\triangle_{MS})$, where $\triangle_{MS}$ denotes the Morgan-Scott triangulation. The key observation is the fact, that if in (5.5) the right inequality reduces to equality, so does the left. Of course, the main problem is how to determine rank M_n.

From Theorem 5.3 it follows that the results of Theorem 3.2 and Theorem 3.3 hold not only for the cubic case, but also for spline spaces of higher degrees.

Remark 5.4. The algorithm, given in Section 4, determines whether the dimension $\dim S_n^1(\triangle)$, $n \geq 3$, equals Schumaker's lower bound for a large class of triangulations $\triangle$. If the answer is in the affirmative, $\dim S_3^1(\triangle)$ agrees with Schumaker's lower bound, and we can apply Theorem 5.3. Of course, for $n \geq 4$, the dimension $S_n^1(\triangle)$ is known for any triangulation $\triangle$ (see [1] and [6]). In this case, the blossoming approach does not yield anything new. It is promising for use on the spline spaces with higher degrees of smoothness $r \geq 1$, as observed for the special case $S_{2r}^r(\triangle)$ in [3].

6 Proof of Theorem 3.3

Proof. Let us shorten the notation by $Q_{ij}^{k\ell} := Q_{k\ell,ij}$. It turns out that instead of considering all the cells at once, it is more reasonable to choose different points for each case $k = 5, 6, \ldots, 8$ in order to simplify non-diagonal blocks as much as possible. Thus each case will be studied separately.

First, let us consider the case $k = 5$. The points on the edges are chosen as $e_1 : T_1,\ e_2 : T_1,\ e_3 : T_1,\ e_4 : T_1,\ e_5 : T_1,\ e_6 : T_2,\ e_7 : T_2,\ e_8 : T_3$, and the points for the faces as $\Omega_1 : T_1,\ \Omega_2 : T_1,\ \Omega_3 : T_1,\ \Omega_4 : T_1,\ \Omega_5 : T_1$ (Figure 3). This implies that in the matrix M_{12} only three nonzero blocks remain: $Q_{62}^{12}, Q_{73}^{12}, Q_{84}^{12}$. The matrix $\widetilde{M}(\triangle_1, \triangle)$ is

$$
\begin{bmatrix}
Q_{11}^{11} & 0 & 0 & 0 & Q_{15}^{11} & 0 & 0 & 0 & 0 & 0 \\
Q_{21}^{11} & Q_{22}^{11} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & Q_{32}^{11} & Q_{33}^{11} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & Q_{43}^{11} & Q_{44}^{11} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & Q_{54}^{11} & Q_{55}^{11} & 0 & 0 & 0 & 0 & 0 \\
0 & Q_{62}^{11} & 0 & 0 & 0 & 0 & Q_{62}^{12} & 0 & 0 & 0 \\
0 & 0 & Q_{73}^{11} & 0 & 0 & 0 & 0 & Q_{73}^{12} & 0 & 0 \\
0 & 0 & 0 & Q_{84}^{11} & 0 & 0 & 0 & 0 & Q_{84}^{12} & 0 \\
0 & 0 & 0 & 0 & 0 & Q_{11}^{22} & 0 & 0 & 0 & Q_{15}^{22} \\
0 & 0 & 0 & 0 & 0 & Q_{21}^{22} & Q_{22}^{22} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & Q_{32}^{22} & Q_{33}^{22} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & Q_{43}^{22} & Q_{44}^{22} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & Q_{54}^{22} & Q_{55}^{22} \\
0 & 0 & 0 & 0 & 0 & 0 & Q_{62}^{22} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & Q_{73}^{22} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & Q_{84}^{22} & 0
\end{bmatrix}_{32\times 35}.
$$

Since

$$
\det \begin{bmatrix} Q_{11}^{22} \\ Q_{21}^{22} \end{bmatrix}_{4\times 4} = (v_2 \times v_1)^4 \neq 0,
$$

the rows $17, 18, 19, 20$, and columns $16, 17, 18, 19$, can be omitted without changing the rank of the matrix $\widetilde{M}(\triangle_1, \triangle)$. If the columns $3, 30$ and 31 in the new matrix are omitted, in the case of the collinearity $e_1 || e_3$ we obtain the minor

$$
\|e_4\|^2 \alpha_2^6 (v_2 \times v_3)(v_2 \times v_4)(v_4 \times v_3)^2 (v_3 \times v_6)^5 \cdot \\
(v_7 \times v_4)^2 (v_6 \times v_7)(v_8 \times v_4)^3 (v_8 \times v_7)^3 \neq 0,
$$

and in the case of the collinearity $e_1 || e_4$ the minor

$$
\|e_4\|^2 \alpha_2^6 (v_2 \times v_3)(v_2 \times v_4)(v_4 \times v_3)^2 (v_3 \times v_6)^3 \cdot \\
(v_7 \times v_4)^2 (v_6 \times v_7)^3 (v_4 \times v_8)^5 (v_7 \times v_8) \neq 0.
$$

Therefore the matrix $\widetilde{M}(\triangle_1, \triangle)$ is of full rank in the considered special cases. The conditions of Theorem 3.1 are fulfilled.

In the case $k = 6$ we pick the points on the edges as $e_1 : T_1,\ e_2 : T_1,\ e_3 : T_1,\ e_4 : T_1,\ e_5 : T_1,\ e_6 : T_1,\ e_7 : T_2,\ e_8 : T_2,\ e_9 : T_3,\ e_{10} : T_3$, and the points for the faces $\Omega_1 : T_1,\ \Omega_2 : T_1,\ \Omega_3 : T_1,\ \Omega_4 : T_1,\ \Omega_5 : T_1,\ \Omega_6 : T_1$ (Figure 5). This choice implies that in the matrix M_{12} only 4 nonzero blocks remain: $Q_{72}^{12}, Q_{83}^{12}, Q_{94}^{12}, Q_{10,5}^{12}$. The matrix $\widetilde{M}(\triangle_1, \triangle)$ is

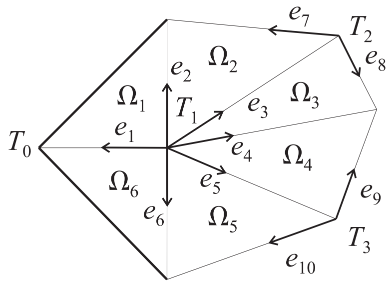

Figure 5: A cell of degree 6 at the boundary of the triangulation.

$$\begin{bmatrix}
Q^{11}_{11} & 0 & 0 & 0 & 0 & Q^{11}_{16} & 0 & 0 & 0 & 0 & 0 & 0 \\
Q^{11}_{21} & Q^{11}_{22} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & Q^{11}_{32} & Q^{11}_{33} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & Q^{11}_{43} & Q^{11}_{44} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & Q^{11}_{54} & Q^{11}_{55} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & Q^{11}_{65} & Q^{11}_{66} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & Q^{11}_{72} & 0 & 0 & 0 & 0 & 0 & Q^{12}_{72} & 0 & 0 & 0 & 0 \\
0 & 0 & Q^{11}_{83} & 0 & 0 & 0 & 0 & 0 & Q^{12}_{83} & 0 & 0 & 0 \\
0 & 0 & 0 & Q^{11}_{94} & 0 & 0 & 0 & 0 & 0 & Q^{12}_{94} & 0 & 0 \\
0 & 0 & 0 & 0 & Q^{11}_{10,5} & 0 & 0 & 0 & 0 & 0 & Q^{12}_{10,5} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & Q^{22}_{11} & 0 & 0 & 0 & 0 & Q^{22}_{16} \\
0 & 0 & 0 & 0 & 0 & 0 & Q^{22}_{21} & Q^{22}_{22} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & Q^{22}_{32} & Q^{22}_{33} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & Q^{22}_{43} & Q^{22}_{44} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & Q^{22}_{54} & Q^{22}_{55} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & Q^{22}_{65} & Q^{22}_{66} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & Q^{22}_{72} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & Q^{22}_{83} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & Q^{22}_{94} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & Q^{22}_{10,5} & 0
\end{bmatrix}_{40\times 42}.$$

Since the submatrix

$$\begin{bmatrix} Q^{22}_{11} \\ Q^{22}_{21} \end{bmatrix},$$

that belong to the block column for Ω_1 in M_{22}, is nonsingular, the block rows for e_1, e_2 and the block column for Ω_1 in M_{12} and M_{22} can be omitted. A matrix of dimension 36×38 remains. It is enough to compute 6 minors, where 2 of the columns that belong to the last block column of the matrix are omitted (the rest of the minors are 0 because of the linearly dependent columns in the last block column). A quick computation shows that all minors are 0. Therefore the matrix $\widetilde{M}(\triangle_1, \triangle)$ is not of full rank.

In the case $k = 7$ we choose the points on the edges as $e_1 : T_1,\ e_2 : T_1,\ e_3 : T_1,\ e_4 : T_1,\ e_5 : T_1,\ e_6 : T_1,\ e_7 : T_1,\ e_8 : T_2,\ e_9 : T_2,\ e_{10} : T_3,\ e_{11} : T_3,\ e_{12} : T_4$, and the points for the faces $\Omega_1 : T_1,\ \Omega_2 : T_1,\ \Omega_3 : T_1,\ \Omega_4 : T_1,\ \Omega_5 : T_1,\ \Omega_6 : T_1,\ \Omega_7 : T_1$ (Figure 6). Therefore, in the matrix M_{12} only 5 nonzero blocks re-

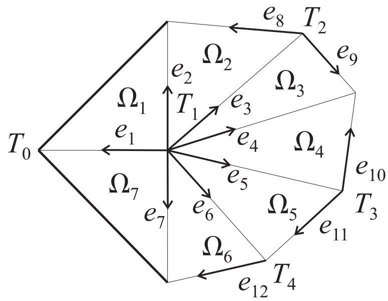

Figure 6: A cell of degree 7 at the boundary of the triangulation.

main: $Q^{12}_{82}, Q^{12}_{93}, Q^{12}_{10,4}, Q^{12}_{11,5}, Q^{12}_{12,6}$. The matrix $\widetilde{M}(\triangle_1, \triangle)$ is

$$\begin{bmatrix}
Q^{11}_{11} & 0 & 0 & 0 & 0 & 0 & Q^{11}_{17} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
Q^{11}_{21} & Q^{11}_{22} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & Q^{11}_{32} & Q^{11}_{33} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & Q^{11}_{43} & Q^{11}_{44} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & Q^{11}_{54} & Q^{11}_{55} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & Q^{11}_{65} & Q^{11}_{66} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & Q^{11}_{76} & Q^{11}_{77} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & Q^{11}_{82} & 0 & 0 & 0 & 0 & 0 & 0 & Q^{12}_{82} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & Q^{11}_{93} & 0 & 0 & 0 & 0 & 0 & 0 & Q^{12}_{93} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & Q^{11}_{10,4} & 0 & 0 & 0 & 0 & 0 & 0 & Q^{12}_{10,4} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & Q^{11}_{11,5} & 0 & 0 & 0 & 0 & 0 & 0 & Q^{12}_{11,5} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & Q^{11}_{12,6} & 0 & 0 & 0 & 0 & 0 & 0 & Q^{12}_{12,6} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & Q^{22}_{11} & 0 & 0 & 0 & 0 & 0 & Q^{22}_{17} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & Q^{22}_{21} & Q^{22}_{22} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & Q^{22}_{32} & Q^{22}_{33} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & Q^{22}_{43} & Q^{22}_{44} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & Q^{22}_{54} & Q^{22}_{55} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & Q^{22}_{65} & Q^{22}_{66} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & Q^{22}_{76} & Q^{22}_{77} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & Q^{22}_{82} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & Q^{22}_{93} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & Q^{22}_{10,4} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & Q^{22}_{11,5} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & Q^{22}_{12,6} & 0
\end{bmatrix}_{48\times 49}.$$

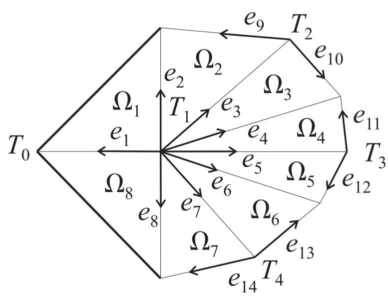

Figure 7: A cell of degree 8 at the boundary of the triangulation.

In the case $k = 8$ we choose the points on the edges as $e_1 : T_1,\ e_2 : T_1,\ e_3 : T_1,$

$e_4 : T_1,\ e_5 : T_1,\ e_6 : T_1,\ e_7 : T_1,\ e_8 : T_1,\ e_9 : T_2,\ e_{10} : T_2,\ e_{11} : T_3,\ e_{12} : T_3,$ $e_{13} : T_4,\ e_{14} : T_4$, and the points for the faces as $\Omega_1 : T_1,\ \Omega_2 : T_1,\ \Omega_3 : T_1,$ $\Omega_4 : T_1,\ \Omega_5 : T_1,\ \Omega_6 : T_1,\ \Omega_7 : T_1,\ \Omega_8 : T_1$ (Figure 7). Therefore, in the matrix M_{12} only 6 nonzero blocks remain: $Q^{12}_{92}, Q^{12}_{10,3}, Q^{12}_{11,4}, Q^{12}_{12,5}, Q^{12}_{13,6}, Q^{12}_{14,7}$. The matrix $\widetilde{M}(\triangle_1, \triangle)$ is

$$\begin{bmatrix}
Q^{11}_{11} & 0 & 0 & 0 & 0 & 0 & 0 & Q^{11}_{18} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
Q^{11}_{21} & Q^{11}_{22} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & Q^{11}_{32} & Q^{11}_{33} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & Q^{11}_{43} & Q^{11}_{44} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & Q^{11}_{54} & Q^{11}_{55} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & Q^{11}_{65} & Q^{11}_{66} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & Q^{11}_{76} & Q^{11}_{77} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & Q^{11}_{87} & Q^{11}_{88} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & Q^{11}_{92} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & Q^{12}_{92} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & Q^{11}_{10,3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & Q^{12}_{10,3} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & Q^{11}_{11,4} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & Q^{12}_{11,4} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & Q^{11}_{12,5} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & Q^{12}_{12,5} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & Q^{11}_{13,6} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & Q^{12}_{13,6} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & Q^{11}_{14,7} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & Q^{12}_{14,7} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & Q^{22}_{11} & 0 & 0 & 0 & 0 & 0 & 0 & Q^{22}_{18} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & Q^{22}_{21} & Q^{22}_{22} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & Q^{22}_{32} & Q^{22}_{33} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & Q^{22}_{43} & Q^{22}_{44} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & Q^{22}_{54} & Q^{22}_{55} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & Q^{22}_{65} & Q^{22}_{66} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & Q^{22}_{76} & Q^{22}_{77} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & Q^{22}_{87} & Q^{22}_{88} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & Q^{22}_{92} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & Q^{22}_{10,3} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & Q^{22}_{11,4} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & Q^{22}_{12,5} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & Q^{22}_{13,6} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & Q^{22}_{14,7} & 0
\end{bmatrix}_{56\times 56}.$$

The cases $k = 7$ and $k = 8$ can be considered simultaneously. Since the submatrix

$$\begin{bmatrix} Q^{22}_{11} \\ Q^{22}_{21} \end{bmatrix},$$

that belong to the block column for Ω_1 in M_{22}, is nonsingular, Gaussian eliminations on the rows transform it to the identity matrix. Then the eliminations, applied on the columns, can be used to set all the rest of the elements in the block rows for e_1 and e_2 in M_{22} to zero. This simplifies the matrix, the considered block rows and the block column for Ω_1 can be omitted. In the last block column only the nonzero block Q^{22}_{77} (Q^{22}_{88}) of dimension 2×4 remains. In order to choose minors to prove full rank of the matrix, we have to omit two of the columns in the last block column, otherwise the columns are linearly dependent. But this is not possible because of the matrix dimensions. For $k = 7$ one and for $k = 8$ none of the critical columns can be removed. Therefore the matrices $\widetilde{M}(\triangle_1, \triangle)$ for $k = 7, 8$ are not of full rank. □

7 Conclusion

The problem of determining the dimension of the cubic C^1 bivariate spline space over triangulations may seem easy. But for over 40 years it remains unsolved. Various mathematical tools were applied, from numerical mathematics, algebra and graph theory. A possible way in practice is by introducing triangle subdivision. Unfortunately this modifies the triangulation and significantly increases the number of triangles, and thus influences further numerical computations. In this paper an approach by using cell reduction at the boundary of a triangulation is presented. If the triangulation is reduced to a single triangle, this proves the dimension result. Otherwise it could be combined with some other method, that would yield the result for the remaining subtriangulation. Larger interior cells remain to be analysed, since the applied technique has a very large memory and processor power requirements.

An another interesting research topic is the study of the space $S_2^1(\triangle)$ and its generalization $S_{2r}^r(\triangle), r \geq 1$, where very little is known in general. In a more general setting, not much is known on very complex spline spaces on higher dimensional simplical complexes, where tools from commutative algebra show a lot of promise.

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Avoidance in bowtie systems

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Abstract

There are ten configurations of two bowties that can arise in a bowtie system. The avoidance spectrum for three of these was determined in a previous paper (Aequat. Math. **85** (2013), 347–358). In this paper the avoidance spectrum for a further five configurations is determined.

Keywords: Bowtie system, configuration, avoidance, Steiner triple system.

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1 Introduction

Let $X = (V, E)$ be the graph with vertex set $V = \{x, a, b, c, d\}$ and edge set $E = \{xa, xb, xc, xd, ab, cd\}$. Such a graph is called a *bowtie* and will be represented throughout this paper by the notation $a, b - x - c, d$. The vertex x is called the *centre* of the bowtie and the other vertices are called *endpoints*. A decomposition of the complete graph K_n into subgraphs isomorphic to X is called a *bowtie system* of order n and denoted by $\mathrm{BTS}(n)$. An elementary counting argument shows that a necessary condition for the existence of a $\mathrm{BTS}(n)$ is $n \equiv 1$ or $9 \pmod{12}$. In a $\mathrm{BTS}(n)$, if every vertex of the complete graph K_n occurs the same number of times as the centre of a bowtie, then the bowtie

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system is said to be *balanced*, otherwise the system is said to be *unbalanced*. A necessary condition for the existence of a balanced $\mathrm{BTS}(n)$ is $n \equiv 1 \pmod{12}$.

It is easy to see that, given a $\mathrm{BTS}(n)$, by regarding each of the two triangles of every bowtie as separate entities, we have a Steiner triple system $\mathrm{STS}(n)$. We call this the *associated* Steiner triple system of the bowtie system. Conversely, if $n \equiv 1$ or $9 \pmod{12}$, it is also true that the triangles of every $\mathrm{STS}(n)$ can be amalgamated to form bowties. This is a consequence of the fact that the block intersection graph of every Steiner triple system is Hamiltonian, see for example [2, Section 13.6]. If $n \equiv 1 \pmod{12}$, there exists a cyclic $\mathrm{STS}(n)$, see also [2, Section 7.2], and this system will have an even number of full orbits. It is then immediate that we can amalgamate triangles from pairs of orbits to form a balanced $\mathrm{BTS}(n)$. Hence the necessary conditions for both $\mathrm{BTS}(n)$ and balanced $\mathrm{BTS}(n)$ given above are also sufficient.

A *configuration* in a bowtie system (resp. Steiner triple system) is a small collection of bowties (resp. triangles) which may occur in the system. The study of configurations in $\mathrm{STS}(n)$ is now well established and the whole of Chapter 13 of [2] is devoted to various results about them and in particular includes formulae for the number of occurrences of all possible configurations of four or fewer triangles. Those for configurations of one, two or three triangles are functions of n. Such configurations are called *constant* because the number of occurrences is independent of the structure of the $\mathrm{STS}(n)$. Other configurations are *variable*. There are 16 non-isomorphic configurations of four triangles of which 5 are constant and 11 are variable. An important concept is that of avoidance; given any particular configuration in a bowtie system (resp. Steiner triple system), to determine the spectrum of n for which there exists a $\mathrm{BTS}(n)$ (resp. $\mathrm{STS}(n)$) which does not contain that configuration. Avoidance sets for all configurations of four or fewer triangles in Steiner triple systems are known. Most, particularly those for constant configurations, are easy to determine but that for the so-called *Pasch configuration* (four triangles isomorphic to $\{a, b, c\}$, $\{a, y, z\}$, $\{x, b, z\}$, $\{x, y, c\}$) was more challenging. It is $n \equiv 1$ or $3 \pmod{6}$, $n \neq 7, 13$ and a complete solution appears in the two papers [7] and [6].

In this paper we will be concerned with the avoidance sets of configurations of two bowties in a BTS(n). There are ten such configurations which were determined in [3] and are illustrated in Figure 1. In this figure each triangle of a bowtie is represented by a path on three vertices and, in each case, one bowtie is represented by solid lines and the second by dashed lines. The intersection of two solid lines or two dashed lines is the centre of the bowtie and the other four points are the endpoints. The ten configurations are each labelled $\hat{C}_i$ for some value of i, $1 \leq i \leq 16$, to reflect the fact that the bowtie configuration with that label gives the configuration C_i in the standard listing of configurations of four triangles in Steiner triple systems as given in [5] or [2, Section 13.1]. Indeed it was by examining all 16 possible configurations of four triangles in a Steiner triple system and identifying which could be obtained from two bowties that the ten possible configurations of two bowties were obtained.

There are four equations which connect the number of occurrences of the various configurations of two bowties and these were proved in [3]. Denoting the number of occurrences of the configuration $\hat{C}_i$ by c_i, the equations are the following.

$$4c_7 + c_8 + c_{11} + c_{15} = n(n-1)(n-5)/24. \tag{1.1}$$

$$c_{11} + c_{12} + 2c_{14} + 3c_{15} + 4c_{16} = n(n-1)/3. \tag{1.2}$$

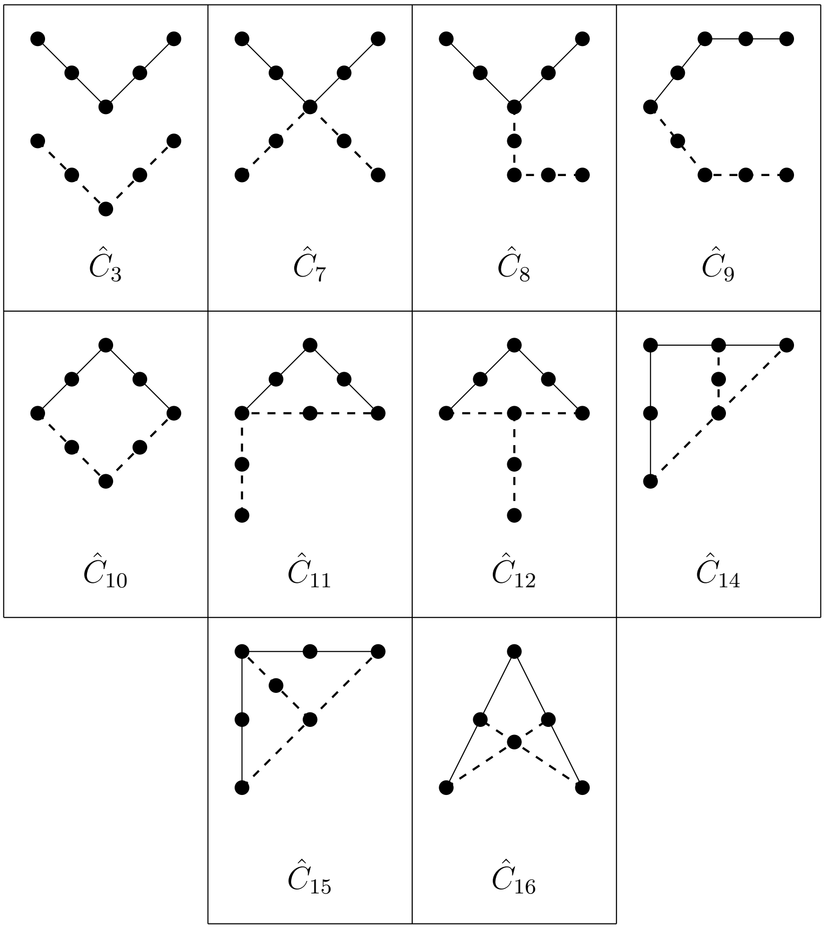

Figure 1: Configurations of two bowties.

$$c_8 + c_9 + 2c_{10} + c_{11} + c_{12} + c_{14} = n(n-1)(n-7)/12. \tag{1.3}$$

$$4c_3 + c_8 + 2c_9 + c_{12} = n(n-1)(n-7)(n-9)/72. \tag{1.4}$$

If the bowtie system is balanced, there is a further equation.

$$c_7 = n(n-1)(n-13)/288. \tag{1.5}$$

All configurations are variable except that $\hat{C}_7$ is constant in balanced bowtie systems.

Avoidance sets for the three most compact configurations, $\hat{C}_{14}, \hat{C}_{15}$ and $\hat{C}_{16}$ have already been determined in [3]. The following theorem was proved.

Theorem 1.1. *For each* $n \equiv 1 \pmod{12}$ *there exists both a balanced and an unbalanced* $\mathrm{BTS}(n)$ *simultaneously avoiding* $\hat{C}_{14}, \hat{C}_{15}$ *and* $\hat{C}_{16}$. *For each* $n \equiv 9 \pmod{12}$, $n \neq 9$ *there exists a (necessarily unbalanced)* $\mathrm{BTS}(n)$ *simultaneously avoiding* $\hat{C}_{14}, \hat{C}_{15}$ *and* $\hat{C}_{16}$.

Thus not only can each of these three configurations be avoided for all values of n for which both balanced $\mathrm{BTS}(n)$ and unbalanced $\mathrm{BTS}(n)$ exist except for $n = 9$, they can all be avoided simultaneously. There are precisely 12 non-isomorphic $\mathrm{BTS}(9)$s which were enumerated in [4]. All avoid $\hat{C}_{16}$, none avoid $\hat{C}_{15}$ and just one avoids $\hat{C}_{14}$. The details are in [3].

In this paper, we consider five further configurations. In particular we show that $\mathrm{BTS}(n)$ avoiding three of the least compact configurations $\hat{C}_3$, $\hat{C}_7$ and $\hat{C}_8$ do not exist if $n > 13$. Our main results are that for each of the configurations $\hat{C}_{11}$ and $\hat{C}_{12}$, and for all admissible values of n, there exists a $\mathrm{BTS}(n)$ avoiding that configuration, with the single exception of $\hat{C}_{11}$ when $n = 13$. The situation for the two configurations $\hat{C}_9$ and $\hat{C}_{10}$ remains unresolved.

2 Avoiding $\hat{C}_3$, $\hat{C}_7$ and $\hat{C}_8$

We begin with $\hat{C}_7$. The number of bowties in a $\mathrm{BTS}(n)$ is $n(n-1)/12$. Hence if $n > 13$, there will be two bowties with a common centre. So the only possible systems which may avoid $\hat{C}_7$ are balanced $\mathrm{BTS}(13)$s, and indeed all such systems do avoid $\hat{C}_7$, and $\mathrm{BTS}(9)$s. Checking the data of the 12 non-isomorphic $\mathrm{BTS}(9)$s from [3] shows that six of these do avoid $\hat{C}_7$ and the other six do not. We state this formally as a theorem.

Theorem 2.1. *The only bowtie systems to avoid* $\hat{C}_7$ *are six of the twelve non-isomorphic* $\mathrm{BTS}(9)$*s and all balanced* $\mathrm{BTS}(13)$*s.*

Next we consider $\hat{C}_8$ and begin with some observations. First, if $a, b - x - c, d$ is a bowtie in a $\mathrm{BTS}(n)$ which has no $\hat{C}_8$ configurations, then there are at most two bowties whose centre is a. This is because any such bowtie must intersect the bowtie $a, b - x - c, d$ in a further point which can only be c or d. Similarly, there are at most two bowties whose centre is b, c or d.

Secondly, in any $\mathrm{BTS}(n)$, a point x can be the centre of at most $(n-1)/4$ bowties. Thus if the $\mathrm{BTS}(n)$ has no $\hat{C}_8$ configurations and x is the centre of less then $(n-1)/4$ bowties, then it is an endpoint of at least one other bowtie and so, by the above, there are at most two bowties whose centre is x. As a consequence, in a $\mathrm{BTS}(n)$ which has no $\hat{C}_8$ configurations, each point x is the centre of 0, 1, 2 or $(n-1)/4$ bowties. Furthermore, if a point is the centre of $(n-1)/4$ bowties, then all remaining points are the centre of at most two bowties. We can now prove the following theorem.

Theorem 2.2. *A* $\mathrm{BTS}(n)$ *avoiding* $\hat{C}_8$ *can only exist if* $n \leq 13$.

Proof. Subtracting equation 1.2 from equation 1.1 and re-arranging terms gives

$$c_8 = n(n-1)(n-13)/24 - 4c_7 + c_{12} + 2c_{14} + 2c_{15} + 4c_{16}.$$

Hence $c_8 \geq n(n-1)(n-13)/24 - 4c_7$.

Now let a_x be the number of bowties in a $\mathrm{BTS}(n)$ whose centre is x. Then $c_7 = \sum_{x \in V} \binom{a_x}{2}$ where V denotes the set of n points in the design. Suppose that $n > 13$ and that the $\mathrm{BTS}(n)$ has no $\hat{C}_8$ configurations. Let m be the maximum number of bowties centred on any point in the $\mathrm{BTS}(n)$. Then from the argument above either $m = 2$ or $m = (n-1)/4$ and all but one point is the centre of at most two bowties.
In either case

$$c_7 = \sum_{x \in V} \binom{a_x}{2} \leq \binom{(n-1)/4}{2} + (n-1) = (n-1)(n+27)/32.$$

Hence

$$c_8 \geq n(n-1)(n-13)/24 - (n-1)(n+27)/8 = (n-1)(n^2 - 16n - 81)/24.$$

The right hand side of this expression is strictly positive for $n \geq 21$, and the result follows. □

In order to complete the avoidance spectrum for the configuration $\hat{C}_8$, we have the following result.

Theorem 2.3. *All* $\mathrm{BTS}(9)$*s avoid* $\hat{C}_8$ *but no balanced* $\mathrm{BTS}(13)$ *avoids* $\hat{C}_8$. *There exist unbalanced* $\mathrm{BTS}(13)$*s which avoid* $\hat{C}_8$.

Proof. Checking the data of the 12 non-isomorphic $\mathrm{BTS}(9)$s from [3] shows that all avoid $\hat{C}_8$. The fact that no balanced $\mathrm{BTS}(13)$ avoids $\hat{C}_8$ follows from an exhaustive computer search of all $1,411,422$ non-isomorphic systems identified in [4]. Two unbalanced $\mathrm{BTS}(13)$s on the point set $\{0, 1, 2, \ldots, 12\}$ which avoid $\hat{C}_8$ are given below. In the first case the associated $\mathrm{STS}(13)$ is cyclic and in the second case it is non-cyclic.

(1) $0,4-1-2,5$; $0,7-2-3,6$; $2,9-4-3,7$;
$0,6-8-1,3$; $4,5-8-9,12$; $1,7-9-5,6$;
$0,9-10-6,7$; $2,8-10-3,5$; $0,5-11-1,10$;
$2,12-11-4,6$; $3,9-11-7,8$; $0,3-12-4,10$;
$1,6-12-5,7$.

(2) $1,4-0-2,7$; $6,8-0-9,10$; $0,12-3-1,8$;
$2,6-3-4,7$; $2,9-4-5,8$; $1,2-5-3,10$;
$1,7-9-5,6$; $2,8-10-6,7$; $0,5-11-1,6$;
$2,12-11-4,10$; $3,9-11-7,8$; $1,10-12-5,7$;
$4,6-12-8,9$. □

Finally in this section we consider $\hat{C}_3$. We have a parallel result to Theorem 2.2 for the configuration $\hat{C}_8$.

Theorem 2.4. *A* $\mathrm{BTS}(n)$ *avoiding* $\hat{C}_3$ *can only exist if* $n \leq 13$.

Proof. Assume that $n > 13$, so that from Theorem 2.2, $c_8 > 0$. From equation 1.3, $c_9 < n(n-1)(n-7)/12$ and $c_8 + c_9 + c_{12} \leq n(n-1)(n-7)/12$. So by addition $c_8 + 2c_9 + c_{12} < n(n-1)(n-7)/6$. From equation 1.4,

$$4c_3 = n(n-1)(n-7)(n-9)/72 - (c_8 + 2c_9 + c_{12}).$$

Therefore $4c_3 > n(n-1)(n-7)(n-21)/72$. Throughout this proof all inequalities are strict and since $n > 13$, i.e. $n \geq 21$, we have that $c_3 > 0$. □

Again, to complete the avoidance spectrum for the configuration $\hat{C}_3$, we have the following result.

Theorem 2.5. *The avoidance spectrum of the configuration* $\hat{C}_3$ *is the set* $\{9, 13\}$.

Proof. The configuration $\hat{C}_3$ has ten points so all $\mathrm{BTS}(9)$s avoid $\hat{C}_3$. A balanced $\mathrm{BTS}(13)$ on the set Z_{13} which avoid $\hat{C}_3$ is the set of bowties $(i+1), (i+4) - i - (i+2), (i+7)$, $0 \leq i \leq 12$, with arithmetic modulo 13. An unbalanced $\mathrm{BTS}(13)$ can be obtained by replacing the bowties $1, 4 - 0 - 2, 7$ and $7, 10 - 6 - 8, 0$ with the bowties $1, 4 - 0 - 6, 8$ and $6, 10 - 7 - 0, 2$. □

3 Avoiding $\hat{C}_{11}$ and $\hat{C}_{12}$

The method we use to construct bowtie systems which avoid the configurations $\hat{C}_{11}$ and $\hat{C}_{12}$ is similar to how we proved Theorem 1.1 on avoiding $\hat{C}_{14}, \hat{C}_{15}$ and $\hat{C}_{16}$ and uses standard techniques involving group divisible designs. It is however more intricate. We note that all GDDs used in this paper exist (see [1, Section IV 4.1]). An essential component of the construction is the following $\mathrm{BTS}(9)$ which is System (a)(I) in [4] and avoids both $\hat{C}_{11}$ and $\hat{C}_{12}$.

$$\begin{array}{lll} 1,2-0-3,6; & 4,8-0-5,7; & 3,5-4-1,7; \\ 6,7-8-2,5; & 5,6-1-3,8; & 3,7-2-4,6. \end{array}$$

We begin by proving the following result.

Theorem 3.1. *For each* $n \equiv 1, 9 \pmod{24}$*, there exists a BTS*(n) *avoiding* $\hat{C}_{12}$.

Proof. Take a 3-GDD of type 4^t, where $t = 3s$ or $3s+1$ and $s \geq 1$. Denote the points of the i^{th} group, $1 \leq i \leq t$, by $(i,1), (i,2), (i,3)$ and $(i,4)$. Inflate each point to two points, i.e. a point (i,j) becomes two points (i,j) and (i,j'). Add a single new point ∞. On each inflated group of 8 points augmented with the ∞ point place a copy of the $\mathrm{BTS}(9)$ above, identifying the points as follows.

$$\begin{array}{lllll} \infty = 0, & (i,1) = 1, & (i,1') = 3, & (i,2) = 2, & (i,2') = 6, \\ & (i,3) = 4, & (i,3') = 5, & (i,4) = 8, & (i,4') = 7. \end{array}$$

On each of the original blocks of the GDD, say $\{(i_1,j_1), (i_2,j_2), (i_3,j_3)\}$, where $i_1 \neq i_2 \neq i_3 \neq i_1$, place the two bowties $(i_2,j_2), (i_3,j_3) - (i_1,j_1) - (i_2,j_2'), (i_3,j_3')$ and $(i_2,j_2), (i_3,j_3') - (i_1,j_1') - (i_2,j_2'), (i_3,j_3)$. The bowties in the resulting $\mathrm{BTS}(8t+1)$ can be thought of as being of two types; (i) those resulting from a $\mathrm{BTS}(9)$ which we will call

BTS *bowties* and (ii) those resulting from the blocks of the GDD which we will call GDD *bowties*. We need to consider pairs of bowties which arise from all possibilities. There are five cases to consider.

(1) Two GDD bowties which come from the same block of the GDD. By the construction these form a configuration $\hat{C}_{16}$.

(2) Two GDD bowties which come from different blocks of the GDD. There are four possible scenarios.

  (a) If the two bowties are disjoint then they form a configuration $\hat{C}_3$.

  (b) If the centres of the two bowties are the same, then they have no further points in common and we have a configuration $\hat{C}_7$.

  (c) If the centre of one of the bowties is an endpoint of the other bowtie, then again they have no further points in common and we have a configuration $\hat{C}_8$.

  (d) If the two bowties have an endpoint in common, then they also have a further endpoint in common and they form a configuration $\hat{C}_{10}$.

(3) Two BTS bowties which come from the same $\mathrm{BTS}(9)$. The configuration they form is completely determined by the structure of the $\mathrm{BTS}(9)$ and so avoids $\hat{C}_{12}$ (and $\hat{C}_{11}$).

(4) Two BTS bowties which come from different $\mathrm{BTS}(9)$s. If the two bowties are disjoint then they form a configuration $\hat{C}_3$. Otherwise they can only intersect in the point ∞ which will be the centre of both bowties and we have a configuration $\hat{C}_7$.

(5) A BTS bowtie and a GDD bowtie. If the two bowties are disjoint then they form a configuration $\hat{C}_3$. If they have just one point in common then they also avoid $\hat{C}_{12}$. Otherwise they have two points in common and these points will both be endpoints of the GDD bowtie. Further, the two points will be (i, j) and (i, j') for some i, j such that $1 \le i \le t$ and $1 \le j \le 4$. If either of these points is the centre of the BTS bowtie, then the other point is an endpoint and we have a configuration $\hat{C}_{11}$. Otherwise both points are endpoints of the BTS bowtie and, because of the way in which the points of the $\mathrm{BTS}(9)$ were assigned to the points ∞, (i, j) and (i, j'), they are in different triangles. Hence we have a configuration $\hat{C}_{10}$. □

We now prove a parallel result for the configuration $\hat{C}_{11}$.

Theorem 3.2. *For each* $n \equiv 1, 9 \pmod{24}$*, there exists a* $\mathrm{BTS}(n)$ *avoiding* $\hat{C}_{11}$.

Proof. This follows the same steps as the previous theorem. However the way in which each inflated group of 8 points augmented with the ∞ point is identified with the points of the $\mathrm{BTS}(9)$ is different. In this case it is as follows.

$$\begin{aligned} \infty = 0, \quad (i,1) = 1, \quad (i,1') = 2, \quad (i,2) = 3, \quad (i,2') = 6, \\ (i,3) = 4, \quad (i,3') = 8, \quad (i,4) = 5, \quad (i,4') = 7. \end{aligned}$$

The construction of the GDD bowties is the same. Also, in the analysis of pairs of bowties, the first four cases are the same. So we only need to consider case (5) of a BTS bowtie and a GDD bowtie. Again, if the two bowties are disjoint then they form a configuration $\hat{C}_3$.

If they have just one point in common then they also avoid $\hat{C}_{11}$. Otherwise they have two points in common and they are (i, j) and (i, j') as before. Because of the way in which the points of the $\mathrm{BTS}(9)$ were assigned to the points ∞, (i, j) and (i, j'), no BTS bowtie has its centre at a point (i, j) (resp. (i, j')) and an endpoint at the point (i, j') (resp. (i, j)). So both points are endpoints of the BTS bowtie. If they are in the same triangle then we have a configuration $\hat{C}_{12}$. If they are in different triangles then we have a configuration $\hat{C}_{10}$. □

We next consider the cases $n \equiv 13, 21 \pmod{24}$. In order to deal with bowtie systems in these residue classes avoiding $\hat{C}_{12}$, the following further $\mathrm{BTS}(13)$ is used.

$$\begin{array}{lll}
1,4-0-9,10; & 2,7-0-6,8; & 3,12-0-5,11; \\
1,5-2-3,6; & 1,8-3-5,10; & 2,10-8-4,5; \\
7,9-1-10,11; & 1,12-6-7,10; & 2,4-9-5,6; \\
4,7-3-9,11; & 2,12-11-4,6; & 4,10-12-8,9; \\
5,12-7-8,11. & &
\end{array}$$

This system avoids the configuration $\hat{C}_{12}$ and has the property that one point, namely 0, is at the centre of three bowties and never appears as an endpoint. We can now prove the following result.

Theorem 3.3. *For each* $n \equiv 13, 21 \pmod{24}$*, there exists a* $\mathrm{BTS}(n)$ *avoiding* $\hat{C}_{12}$.

Proof. Take a 3-GDD of type $4^t 6^1$, where $t = 3s$ or $3s + 1$ and $s \geq 1$. Proceed as in Theorem 3.1 where in addition the points of the long group are denoted by $(t + 1, j)$, $1 \leq j \leq 6$. On this inflated group of 12 points augmented with the ∞ point place a copy of the $\mathrm{BTS}(13)$ above, identifying the points as follows.

$$\begin{array}{llll}
\infty = 0, & & & \\
(t+1,1) = 1, & (t+1,1') = 10, & (t+1,2) = 4, & (t+1,2') = 9, \\
(t+1,3) = 2, & (t+1,3') = 6, & (t+1,4) = 7, & (t+1,4') = 8, \\
(t+1,5) = 3, & (t+1,5') = 5, & (t+1,6) = 12, & (t+1,6') = 11.
\end{array}$$

The proof now follows that of Theorem 3.1 . This proves the result for all stated values of n except $n = 21$. A solution for this value is the following.

$$\begin{array}{lll}
15,9-3-11,17; & 17,9-5-11,15; & 18,10-3-14,19; \\
19,10-5-14,18; & 16,12-3-13,20; & 20,12-5-13,16; \\
18,9-4-11,19; & 19,9-8-11,18; & 16,10-4-14,20; \\
20,10-8-14,16; & 15,12-4-13,17; & 17,12-8-13,15; \\
16,9-6-11,20; & 20,9-7-11,16; & 15,10-6-14,17; \\
17,10-7-14,15; & 18,12-6-13,19; & 19,12-7-13,18; \\
0,7-3-1,5; & 2,3-6-5,7; & 2,7-8-3,4; \\
6,8-1-4,7; & 0,6-4-2,5; & 2,9-12-11,13; \\
2,13-14-9,10; & 12,14-1-10,13; & 0,14-11-1,9; \\
0,12-10-2,11; & 0,19-15-1,17; & 2,15-18-17,19; \\
2,19-20-15,16; & 18,20-1-16,19; & 0,18-16-2,17; \\
1,2-0-9,13; & 8,5-0-20,17. &
\end{array}$$

□

Turning our attention to avoiding $\hat{C}_{11}$, we have shown by an exhaustive computer search that there is no $\mathrm{BTS}(13)$ that avoids this configuration. So for the residue classes 13 and 21 $\pmod{24}$ we use the modified constructions given in Theorems 3.4 and 3.5. For balanced $\mathrm{BTS}(13)$s the minimum number of $\hat{C}_{11}$ configurations is 10 for both associated cyclic and non-cyclic $\mathrm{STS}(13)$s. For unbalanced systems with the associated cyclic $\mathrm{STS}(13)$, we find that the minimum is 5, but for unbalanced systems with the associated non-cyclic $\mathrm{STS}(13)$, we find that the minimum is 4 and an example is given below.

$$\begin{array}{lll} 0,12-3-1,8; & 2,6-3-4,7; & 3,5-10-6,7; \\ 2,4-9-3,11; & 2,7-0-5,11; & 0,10-9-8,12; \\ 0,8-6-4,12; & 1,7-9-5,6; & 0,4-1-10,12; \\ 2,5-1-6,11; & 2,11-12-5,7; & 2,8-10-4,11; \\ 4,5-8-7,11. & & \end{array}$$

Theorem 3.4. *For each* $n \equiv 21 \pmod{24}$*, there exists a BTS*(n) *avoiding* $\hat{C}_{11}$.

Proof. Take a 3-GDD of type 3^t, where $t = 4s+3$ and $s \geq 0$. Denote the points of the i^{th} group, $1 \leq i \leq t$, by $(i,1), (i,2)$ and $(i,3)$. As before inflate each point to two points, i.e. a point (i,j) becomes two points (i,j) and (i,j'). Add three new points ∞_0, ∞_1 and ∞_2. On each inflated group of 6 points augmented with the three ∞ points first place a copy of the $\mathrm{BTS}(9)$ at the beginning of this Section, identifying the points as follows.

$$\begin{array}{llll} \infty_0 = 0, & \infty_1 = 1, & \infty_2 = 2, & \\ (i,1) = 3, & (i,1') = 6, & (i,2) = 4, & (i,2') = 8, \\ (i,3) = 5, & (i,3') = 7. & & \end{array}$$

The triangle $\{\infty_0, \infty_1, \infty_2\}$ now occurs $4s+3$ times. Remove the bowties

$$\infty_1, \infty_2 - \infty_0 - (i,1), (i,1')$$

for all i such that $2 \leq i \leq 4s+3$ and replace them by the bowties

$$(2i,1), (2i,1') - \infty_0 - (2i+1,1), (2i+1,1'), \quad 1 \leq i \leq 2s+1.$$

We call these $\mathrm{BTS}^\star$ *bowties*. The construction of the GDD bowties is as in the previous three theorems.

We need to prove that a bowtie system constructed in this way avoids configuration $\hat{C}_{11}$. The proof for the five cases involving just BTS bowties and GDD bowties is as in Theorem 3.2. So any putative configuration $\hat{C}_{11}$ must contain a $\mathrm{BTS}^\star$ bowtie. We show that this is impossible. A configuration $\hat{C}_{11}$ consists of two bowties isomorphic to $c, y-x-b, z$ and $a, z-y-d, e$. The centre of every $\mathrm{BTS}^\star$ bowtie is ∞_0; however this point never occurs as the endpoint of any bowtie. So $y \neq \infty_0$. Now suppose that $x = \infty_0$ and that $c, y-x-b, z$ is a $\mathrm{BTS}^\star$ bowtie. Then without loss of generality $y = (2i,1)$ and $z = (2i+1,1)$ for some i such that $1 \leq i \leq 2s+1$, say $i = q$. Therefore the bowtie $a, z-y-d, e$ is a GDD bowtie and either d or $e = (2q+1,1') = b$ which means that we do not have a configuration $\hat{C}_{11}$. □

We note that, by using a 3-GDD of type 3^t where $t = 4s+1$, $s \geq 1$, the above theorem can also be used to provide an alternative proof of the existence of a $\mathrm{BTS}(n)$ avoiding $\hat{C}_{11}$ for the residue class $9 \pmod{24}$.

A $\mathrm{BTS}(21)$ avoiding $\hat{C}_{11}$ from the above theorem is given below. This will be needed in the proof of the final theorem. It has the crucial property that one point, again namely 0, is at the centre of five bowties and never appears as an endpoint.

$$\begin{array}{lll}
1,2-0-3,6; & 4,8-0-5,7; & 10,14-0-11,13; \\
16,20-0-17,19; & 9,12-0-15,18; & \\
3,5-4-1,7; & 9,11-10-1,13; & 15,17-16-1,19; \\
6,7-8-2,5; & 12,13-14-2,11; & 18,19-20-2,17; \\
5,6-1-3,8; & 11,12-1-9,14; & 17,18-1-15,20; \\
3,7-2-4,6; & 9,13-2-10,12; & 15,19-2-16,18; \\
9,15-3-12,18; & 10,17-3-14,19; & 11,16-3-13,20; \\
9,18-6-12,15; & 10,19-6-14,17; & 11,20-6-13,16; \\
10,16-4-14,20; & 11,15-4-13,18; & 9,17-4-12,19; \\
10,20-8-14,16; & 11,18-8-13,15; & 9,19-8-12,17; \\
11,17-5-13,19; & 9,16-5-12,20; & 10,15-5-14,18; \\
11,19-7-13,17; & 9,20-7-12,16; & 10,18-7-14,15.
\end{array}$$

Theorem 3.5. *For each* $n \equiv 13 \pmod{24}$*, except for* $n = 13$*, there exists a* $\mathrm{BTS}(n)$ *avoiding* $\hat{C}_{11}$.

Proof. Take a 3-GDD of type $4^t 10^1$, where $t = 3s + 2$, $s \geq 1$. Proceed as in Theorem 3.2 where the points of the long group are denoted by $(t+1, j)$, $1 \leq j \leq 10$. On this inflated group of 20 points augmented with the ∞ point place a copy of the $\mathrm{BTS}(21)$ above, identifying the points as follows.

$$\begin{array}{llll}
\infty = 0, & & & \\
(t+1,1) = 1, & (t+1,1') = 2, & (t+1,2) = 3, & (t+1,2') = 6, \\
(t+1,3) = 4, & (t+1,3') = 8, & (t+1,4) = 5, & (t+1,4') = 7, \\
(t+1,5) = 10, & (t+1,5') = 14, & (t+1,6) = 11, & (t+1,6') = 13, \\
(t+1,7) = 16, & (t+1,7') = 20, & (t+1,8) = 17, & (t+1,8') = 19, \\
(t+1,9) = 9, & (t+1,9') = 12, & (t+1,10) = 15, & (t+1,10') = 18.
\end{array}$$

The proof now follows that of Theorem 3.2. This proves the result for all stated values of n except $n = 37$. A solution for this value is given in Table 1 below. □

Finally, we again note that, by using a 3-GDD of type $4^t 10^1$ where $t = 3s$, $s \geq 1$, the above theorem can also be used to provide an alternative proof of the existence of a $\mathrm{BTS}(n)$ avoiding $\hat{C}_{11}$ for the residue class $21 \pmod{24}$.

Table 1: A BTS(37) avoiding $\hat{C}_{11}$.

16, 34 − 0 − 17, 35;	18, 36 − 0 − 1, 19;	8, 26 − 0 − 9, 27;
10, 28 − 0 − 11, 29;	12, 30 − 0 − 13, 31;	14, 32 − 0 − 15, 33;
2, 20 − 0 − 7, 25;	3, 21 − 0 − 4, 22;	5, 23 − 0 − 6, 24;
7, 18 − 6 − 36, 1;	25, 36 − 24 − 18, 19;	24, 1 − 7 − 19, 36;
6, 19 − 25 − 1, 18;	9, 8 − 1 − 26, 27;	27, 8 − 19 − 26, 9;
7, 5 − 2 − 23, 25;	25, 5 − 20 − 23, 7;	14, 13 − 3 − 31, 32;
32, 13 − 21 − 31, 14;	15, 11 − 4 − 29, 33;	33, 11 − 22 − 29, 15;
12, 10 − 6 − 28, 30;	30, 10 − 24 − 28, 12;	11, 10 − 1 − 28, 29;
29, 10 − 19 − 28, 11;	15, 13 − 2 − 31, 33;	33, 13 − 20 − 31, 15;
6, 5 − 3 − 23, 24;	24, 5 − 21 − 23, 6;	12, 8 − 4 − 26, 30;
30, 8 − 22 − 26, 12;	14, 9 − 7 − 27, 32;	32, 9 − 25 − 27, 14;
13, 12 − 1 − 30, 31;	31, 12 − 19 − 30, 13;	11, 9 − 2 − 27, 29;
29, 9 − 20 − 27, 11;	7, 4 − 3 − 22, 25;	25, 4 − 21 − 22, 7;
15, 10 − 5 − 28, 33;	33, 10 − 23 − 28, 15;	14, 8 − 6 − 26, 32;
32, 8 − 24 − 26, 14;	15, 14 − 1 − 32, 33;	33, 14 − 19 − 32, 15;
6, 4 − 2 − 22, 24;	24, 4 − 20 − 22, 6;	11, 8 − 3 − 26, 29;
29, 8 − 21 − 26, 11;	12, 9 − 5 − 27, 30;	30, 9 − 23 − 27, 12;
13, 10 − 7 − 28, 31;	31, 10 − 25 − 28, 13;	18, 17 − 16 − 35, 36;
36, 17 − 34 − 35, 18;	2, 3 − 1 − 21, 16;	20, 21 − 19 − 3, 34;
19, 16 − 2 − 34, 21;	1, 34 − 20 − 16, 3;	10, 14 − 4 − 32, 16;
28, 32 − 22 − 14, 34;	22, 16 − 10 − 34, 32;	4, 34 − 28 − 16, 14;
8, 13 − 5 − 31, 16;	26, 31 − 23 − 13, 34;	23, 16 − 8 − 34, 31;
5, 34 − 26 − 16, 13;	9, 15 − 6 − 33, 16;	27, 33 − 24 − 15, 34;
24, 16 − 9 − 34, 33;	6, 34 − 27 − 16, 15;	11, 12 − 7 − 30, 16;
29, 30 − 25 − 12, 34;	25, 16 − 11 − 34, 30;	7, 34 − 29 − 16, 12;
4, 5 − 1 − 23, 17;	22, 23 − 19 − 5, 35;	19, 17 − 4 − 35, 23;
1, 35 − 22 − 17, 5;	12, 14 − 2 − 32, 17;	30, 32 − 20 − 14, 35;
20, 17 − 12 − 35, 32;	2, 35 − 30 − 17, 14;	9, 10 − 3 − 28, 17;
27, 28 − 21 − 10, 35;	21, 17 − 9 − 35, 28;	3, 35 − 27 − 17, 10;
11, 13 − 6 − 31, 17;	29, 31 − 24 − 13, 35;	24, 17 − 11 − 35, 31;
6, 35 − 29 − 17, 13;	8, 15 − 7 − 33, 17;	26, 33 − 25 − 15, 35;
25, 17 − 8 − 35, 33;	7, 35 − 26 − 17, 15;	8, 10 − 2 − 28, 18;
26, 28 − 20 − 10, 36;	20, 18 − 8 − 36, 28;	2, 36 − 26 − 18, 10;
12, 15 − 3 − 33, 18;	30, 33 − 21 − 15, 36;	21, 18 − 12 − 36, 33;
3, 36 − 30 − 18, 15;	9, 13 − 4 − 31, 18;	27, 31 − 22 − 13, 36;
22, 18 − 9 − 36, 31;	4, 36 − 27 − 18, 13;	11, 14 − 5 − 32, 18;
29, 32 − 23 − 14, 36;	23, 18 − 11 − 36, 32;	5, 36 − 29 − 18, 14.

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Two families of pseudo metacirculants*

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Abstract

A split weak metacirculant which is not metacirculant is simply called a *pseudo metacirculant*. In this paper, two infinite families of pseudo metacirculants are constructed.

Keywords: Metacirculant, metacyclic, split weak metacirculant.

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1 Introduction

Metacirculant graphs were introduced by Alspach and Parsons [1]. In 2008 Marušič and Šparl [11] gave an equivalent definition of metacirculant graphs as follows. Let $m \geq 1$ and $n \geq 2$ be integers. A graph Γ of order mn is called [11] an (m, n)-*metacirculant graph* (in short (m, n)-*metacirculant*) if it has an automorphism σ of order n such that $\langle \sigma \rangle$ is semiregular on the vertex set of Γ, and an automorphism τ normalizing $\langle \sigma \rangle$ and cyclically permuting the m orbits of $\langle \sigma \rangle$ such that τ has a cycle of size m in its cycle decomposition. A graph is called a *metacirculant* if it is an (m, n)-metacirculant for some m and n.

It follows from the definition above that a metacirculant Γ has a vertex-transitive automorphism group $\langle \sigma, \tau \rangle$ which is metacyclic. If we, instead, require that the graph has a vertex-transitive metacyclic group of automorphisms, then we obtained the so-called *weak metacirculants*, which were introduced by Marušič and Šparl [11] in 2008. In [10] Li *et al.* initiated the study of relationship between metacirculants and weak metacirculants, and they divided the weak metacirculants into the following two subclasses: A weak metacirculant which has a vertex-transitive split metacyclic automorphism group is called *split weak metacirculant*. Otherwise, a weak metacirculant Γ is called a *non-split weak metacirculant* if its full automorphism group does not contain any split metacyclic subgroup which is vertex-transitive. In [10, Lemma 2.2] it was proved that every metacirculant is a split weak

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metacirculant, but it was unknown whether the converse of this statement is true. In [14] Sanming Zhou and the second author asked the following question:

Question 1.1 ([14, Question A])**.** Is it true that any split weak metacirculant is a metacirculant?

If the graph under consideration is of prime-power order, it was shown by the authors [5, 14] that the answer to the above question is positive. However, in [6] we show that there do exist infinitely many split weak metacirculants which are not metacirculants. For convenience, we shall say that a split metacirculant is a *pseudo metacirculant* if it is not metacirculant. To the best of our knowledge, up to now the only known pseudo metacirculants are the graphs constructed in [6]. So it might be interesting to find some other families of pseudo metacirculants. Furthermore, it seems that the existence of pseudo metacirculants is closely related to the orders of graphs. Motivated by this, it is natural to consider the following problem.

Problem 1.2. Characterize those integers n for which there is a pseudo metacirculant of order n.

There are some partial answers to Problem 1.2. By [5, 14], there do not exist a pseudo metacirculant with a prime-power order. The construction of pseudo metacirculants in [6] shows that for two any primes p, q with $q \mid p-1$, there exists a pseudo metacirculant of order $p^m q$ for each $m \geq 3$. In this paper, two new infinite families of pseudo metacirculants are constructed. Our construction implies that for any primes p, q, if either $p^{\lfloor \frac{m}{2} \rfloor + 1} \mid q-1$ or $p = 2$ and $4 \mid q-1$, then there exists a pseudo metacirculant of order $p^m q$ with $m \geq 3$.

Our research also shows that the three families of pseudo metacirculants constructed in [6] and this paper are crucial for solving Problem 1.2, and we shall use them to give a complete solution of Problem 1.2 in our subsequent paper [4].

2 Preliminaries

2.1 Definitions and notation

For a positive integer n, we denote by C_n the cyclic group of order n, by $\mathbb{Z}_n$ the ring of integers modulo n, by $\mathbb{Z}_n^*$ the multiplicative group of $\mathbb{Z}_n$ consisting of numbers coprime to n, and by D_{2n} the dihedral group of order $2n$. For two groups M and N, $N : M$ denotes a semidirect product of N by M. Given a group G, denote by 1, $\mathrm{Aut}(G)$, and $Z(G)$ the identity element, full automorphism group and center of G, respectively. Denote by $o(x)$ the order of an element x of G. For a subgroup H of G, denote by $C_G(H)$ the centralizer of H in G. A group G is called *metacyclic* if it contains a normal cyclic subgroup N such that G/N is cyclic. In other words, a metacyclic group G is an extension of a cyclic group $N \cong C_n$ by a cyclic group $G/N \cong C_m$, written as $G \cong C_n.C_m$. If this extension is split, namely $G \cong C_n : C_m$, then G is called a *split metacyclic group*.

Let G be a permutation group on a set Ω and $\alpha \in \Omega$. Denote by G_α the stabilizer of α in G, that is, the subgroup of G fixing the point α. We say that G is *semiregular* on Ω if $G_\alpha = 1$ for every $\alpha \in \Omega$ and *regular* if G is transitive and semiregular. For any subset Δ of Ω, use G_Δ and $G_{(\Delta)}$ to denote the subgroups of G fixing Δ setwise and pointwise, respectively. A *block of imprimitivity* of G on Ω is a subset Δ of Ω with $1 < |\Delta| < |\Omega|$ such that for any $g \in G$, either $\Delta^g = \Delta$ or $\Delta^g \cap \Delta = \emptyset$. In this case the *blocks* $\Delta^g,\ g \in G$ form a *G-invariant partition* of Ω.

All graphs in this paper are finite, simple and undirected. For a graph Γ, we denote its vertex set and edge set by $V(\Gamma)$ and $E(\Gamma)$, respectively. Given two adjacent vertices u, v of Γ, denote by $\{u, v\}$ the edge between u and v. Denote by $\Gamma(v)$ the neighbourhood of v, and by $\Gamma[B]$ the subgraph of Γ induced by a subset B of $V(\Gamma)$. An *s-cycle* in Γ, denoted by C_s, is an $(s+1)$-tuple of pairwise distinct vertices $(v_0, v_1, \ldots, v_s)$ such that $\{v_{i-1}, v_i\} \in E(\Gamma)$ for $1 \leq i \leq s$ and $\{v_s, v_0\} \in E(\Gamma)$. Denote by $\mathbf{K}_n$ the complete graph of order n, and by $\mathbf{K}_{n,n}$ the complete bipartite graph with biparts of cardinality n. The full automorphism group of Γ is denoted by $\mathrm{Aut}(\Gamma)$.

2.2 Quotient graph

Let Γ be a connected vertex-transitive graph, and let $G \leq \mathrm{Aut}(\Gamma)$ be vertex-transitive on Γ. A partition $\mathcal{B}$ of $V(\Gamma)$ is said to be *G-invariant* if for any $B \in \mathcal{B}$ and $g \in G$ we have $B^g \in \mathcal{B}$. For a G-invariant partition $\mathcal{B}$ of $V(\Gamma)$, the *quotient graph* $\Gamma_{\mathcal{B}}$ is defined as the graph with vertex set $\mathcal{B}$ such that, for any two different vertices $B, C \in \mathcal{B}$, B is adjacent to C if and only if there exist $u \in B$ and $v \in C$ which are adjacent in Γ. Let N be a normal subgroup of G. Then the set $\mathcal{B}$ of orbits of N on $V(\Gamma)$ is a G-invariant partition of $V(\Gamma)$. In this case, the symbol $\Gamma_{\mathcal{B}}$ will be replaced by Γ_N, and the original graph Γ is said to be a *cover* of Γ_N if Γ and Γ_N have the same valency.

2.3 Cayley graph

Given a finite group G and an inverse closed subset $S \subseteq G \setminus \{1\}$, the *Cayley graph* $\mathrm{Cay}(G, S)$ on G with respect to S is a graph with vertex set G and edge set $\{\{g, sg\} \mid g \in G, s \in S\}$. For any $g \in G$, $R(g)$ is the permutation of G defined by $R(g)\colon x \mapsto xg$ for $x \in G$. Set $R(G) := \{R(g) \mid g \in G\}$. It is well-known that $R(G)$ is a subgroup of $\mathrm{Aut}(\mathrm{Cay}(G, S))$. A Cayley graph $\mathrm{Cay}(G, S)$ is said to be *normal* if $R(G)$ is normal in $\mathrm{Aut}(\mathrm{Cay}(G, S))$. This concept was introduced by Xu in [13], and for more results about normal Cayley graphs, we refer the reader to [7].

The following proposition determines the normalizer of $R(G)$ in the full automorphism group of $\mathrm{Cay}(G, S)$.

Proposition 2.1 ([8, Lemma 2.1])**.** *Let $\Gamma = \mathrm{Cay}(G, S)$ be a Cayley graph on G with respect to S. Then $N_{\mathrm{Aut}(\Gamma)}(R(G)) = R(G) : \mathrm{Aut}(G, S)$, where $\mathrm{Aut}(G, S)$ is the group of automorphisms of G fixing the set S setwise.*

2.4 Coset graph

Let G be a group and for a subgroup H of G, let $\Omega = [G : H] = \{Hx \mid x \in G\}$, the set of right cosets of H in G. For $g \in G$, define $R_H(g)\colon Hx \mapsto Hxg$, $x \in G$, and set $R_H(G) = \{R_H(g) \mid g \in G\}$. The map $g \mapsto R_H(g)$, $g \in G$, is a homomorphism from G to S_Ω and it is called the *coset action* of G relative to H. The kernel of the coset action is $H_G = \cap_{g \in G} H^g$, the largest normal subgroup of G contained in H, and $G/H_G \cong R_H(G)$. It is well-known that any transitive action of G on Ω is equivalent to the coset action of G relative the subgroup G_α for any given $\alpha \in \Omega$. If $H_G = 1$, we say that H is *core-free* in G.

Let D be a union of several double-cosets of the form HgH with $g \notin H$ such that $D = D^{-1}$. The *coset graph* $\Gamma = \mathrm{Cos}(G, H, D)$ of G with respect to H and D is defined as the graph with vertex set $V(\Gamma) = [G : H]$, and edge set $E(\Gamma) = \{\{Hg, Hdg\} \mid g \in G, d \in D\}$. It is easy to see that Γ is well defined and has valency $|D|/|H|$, and Γ is connected if and only if D generates G. Further, $R_H(G) \leq \mathrm{Aut}(\Gamma)$, and hence Γ is vertex-transitive.

Let $\mathrm{Aut}(G, H, D) = \{\alpha \in \mathrm{Aut}(G) \mid H^\alpha = H, D^\alpha = D\}$. For any $\alpha \in \mathrm{Aut}(G, H, D)$, define $\alpha_H \colon Hx \mapsto Hx^\alpha$, $x \in G$, and consider the action of $\mathrm{Aut}(G, H, D)$ on $[G : H]$ induced by $\alpha \mapsto \alpha_H$. It follows that $\mathrm{Aut}(G, H, D)/L \cong \mathrm{Aut}(G, H, D)_H$, where $\mathrm{Aut}(G, H, D)_H = \{\alpha_H \mid \alpha \in \mathrm{Aut}(G, H, D)\}$ and L is the kernel of the action. Furthermore, it is easy to see that $\mathrm{Aut}(G, H, D)_H \leq \mathrm{Aut}(\Gamma)$ and $\mathrm{Aut}(G, H, D)_H$ fixes the vertex H. For $h \in H$, let $\sigma(h)$ be the inner automorphism of G induced by h, that is, $\sigma(h) \colon g \mapsto h^{-1}gh$, $g \in G$. One may show that $\sigma(H) = \{\sigma(h) \mid h \in H\}$ is a subgroup of $\mathrm{Aut}(G, H, D)$ and hence $R_H(H) = \{R_H(h) \mid h \in H\}$ is a subgroup of $\mathrm{Aut}(G, H, D)_H$.

The following proposition determines the normalizer of $R_H(G)$ in the full automorphism group of $\mathrm{Cos}(G, H, D)$.

Proposition 2.2 ([12, Lemma 2.10]). *Let G be a finite group, H a core-free subgroup of G and D a union of several double-cosets HgH such that $H \not\subseteq D$. Let $\Gamma = \mathrm{Cos}(G, H, D)$ and $A = \mathrm{Aut}(\Gamma)$. Then $R_H(G) \cong G$, $\mathrm{Aut}(G, H, D)_H \cong \mathrm{Aut}(G, H, D)$, $R_H(H) \cong \sigma(H)$, and $N_A(R_H(G)) = R_H(G)\,\mathrm{Aut}(G, H, D)_H$ with $R_H(G) \cap \mathrm{Aut}(G, H, D)_H = R_H(H)$.*

Below, we prove a technical lemma.

Lemma 2.3. *Let G be a finite group and let H be a core-free subgroup of G. Let S be a set of non-identity elements of G such that S is self-inverse, and let T be any self-inverse subset of S. Let $D = HSH$ and let $C = HTH$. Set*

$$\Gamma = \mathrm{Cos}(G, H, D) \text{ and } \Sigma = \mathrm{Cos}(G, H, C).$$

If the vertex-stabilizer $\mathrm{Aut}(\Gamma)_H$ fixes $\Sigma(H) = \{Hd \mid d \in C\}$ setwise, then $\mathrm{Aut}(\Gamma) \leq \mathrm{Aut}(\Sigma)$.

Proof. Let $A = \mathrm{Aut}(\Gamma)$. Suppose that A_H fixes $\Sigma(H) = \{Hd \mid d \in C\}$ setwise. Then for any $g \in G$, we have $A_{Hg} = (A_H)^{R_H(g)}$, and so A_{Hg} fixes the following set setwise:

$$\{Hd \mid d \in C\}^{R_H(g)} = \{Hdg \mid d \in C\} = \Sigma(Hg).$$

Take $x \in A$ and take any edge $e = \{Hg, Hdg\}$ of Σ. To show $A \leq \mathrm{Aut}(\Sigma)$, it suffices to show that $e^x \in E(\Sigma)$. Since G acts transitively on $V(\Gamma)$ by right multiplication, there exists $g' \in G$ such that $(Hg)^x = Hgg'$, and then $(Hg)^{xR_H((g')^{-1})} = Hg$. It follows that $(Hdg)^{xR_H((g')^{-1})} \in \Sigma(Hg)$ and so $(Hdg)^x \in (\Sigma(Hg))^{R_H((g'))} = \Sigma(Hgg') = \Sigma((Hg)^x)$. Hence, we have $e^x \in E(\Sigma)$, and consequently, $A \leq \mathrm{Aut}(\Sigma)$. □

2.5 Circulants

A *circulant* of order n is a Cayley graph over a cyclic group of order n. The following proposition gives a classification of arc-transitive circulants. Before stating this result, we introduce several concepts.

If a graph Γ has $n > 1$ connected components, each of which is isomorphic to a graph Σ, then we shall write $\Gamma = n\Sigma$. The *lexicographic* (or *wreath*) *product* of graphs Γ_1 and Γ_2 is a graph $\Gamma_1 \circ \Gamma_2$ with vertex set $V(\Gamma_1) \times V(\Gamma_2)$ such that $\{(x_1, x_2), (y_1, y_2)\} \in E(\Gamma_1 \circ \Gamma_2)$ if and only if either $x_1 = y_1$ and $\{x_2, y_2\} \in E(\Gamma_2)$, or $\{x_1, y_1\} \in E(\Gamma_1)$. If Γ_2 is of order m with $V(\Gamma_2) = \{y_1, y_2, \ldots, y_m\}$, then we have a natural embedding of $m\Gamma_1$ in

$\Gamma_1 \circ \Gamma_2$, where, for $1 \le i \le m$, the ith copy of Γ_1 is the subgraph induced by the subset of vertices of $\Gamma_1 \circ \Gamma_2$. The *deleted lexicographic product* of graphs Γ_1 and Γ_2, denoted by $\Gamma_1 \circ \Gamma_2 - m\Gamma_1$, is the graph obtained by deleting from $\Gamma_1 \circ \Gamma_2$ the edges of $m\Gamma_1$.

Proposition 2.4 ([9, Theorem 1]). *Let Γ be a connected arc-transitive circulant of order n. Then one of the following holds:*

(a) $\Gamma = \mathbf{K}_n$;

(b) Γ *is a normal circulant;*

(c) $\Gamma = \Sigma \circ \bar{\mathbf{K}}_d$, *where $n = md$ and Σ is a connected arc-transitive circulant of order m;*

(d) $\Gamma = \Sigma \circ \bar{\mathbf{K}}_d - d\Sigma$, *where $n = md$, $d > 3$, $\mathrm{gcd(d,m)} = 1$ and Σ is a connected arc-transitive circulant of order m.*

3 Pseudo metacirculants–Family A

Consruction A. *Let p be a prime such that $4 \mid p-1$ and let $n \ge 2$ be an integer. If $p > 5$, then let $r \in \mathbb{Z}_p^*$ be such that $r^2 \equiv -1 \pmod p$, and if $p = 5$, then let $r = 2$. Let*

$$G_{2,n,p,r} = \langle a, b, c \mid a^{2^n} = b^p = c^4 = 1, ab = ba, c^{-1}ac = a^{-1}, c^{-1}bc = b^r \rangle.$$

Let

$$\Gamma_{2,n,p,r} = \mathrm{Cos}(G_{2,n,p,r}, H, H\{(ab)^{\pm1}, (abc)^{\pm1}, (bc)^{\pm1}, c^{\pm1}\}H),$$

where $H = \langle a^{2^{n-1}} c^2 \rangle$.

We first prove a lemma.

Lemma 3.1. *Let $G = G_{2,n,p,r}$, $H = \langle a^{2^{n-1}} c^2 \rangle$, and let $D = H\{(ab)^{\pm1}, (abc)^{\pm1}, (bc)^{\pm1}, c^{\pm1}\}H$. If $n = 2$ and $p > 5$, then $\mathrm{Aut}(G, H, D)$ contains exactly one involution.*

Proof. Since $H \cong C_2$, by Proposition 2.2, $\mathrm{Aut}(G, H, D)$ has an involution. Take an involution $\alpha \in \mathrm{Aut}(G, H, D)$. Since $\langle b \rangle$ is a normal Sylow p-subgroup of G, $\langle b \rangle$ is characteristic in G, implying that $b^\alpha \in \langle b \rangle$. Since α has order 2, one has $b^\alpha = b$ or b^{-1}. By a direct computation, we see that $C_G(b) = \langle a \rangle \times \langle b \rangle$. Then

$$a^\alpha \in C_G(b^\alpha) = C_G(b) = \langle a \rangle \times \langle b \rangle.$$

It follows that $a^\alpha \in \langle a \rangle$. Since α has order 2 and $n = 2$, one has $a^\alpha = a$ or a^{-1}, and hence $(a^2)^\alpha = a^2$.

Assume that $c^\alpha = a^i b^j c^k$ for some $i \in \mathbb{Z}_{2^n}$, $j \in \mathbb{Z}_p$ and $k \in \mathbb{Z}_4^*$. Considering the image of the equality $c^{-1}bc = b^r$ under α, we obtain that $(a^i b^j c^k)^{-1}(b^{\pm1})a^i b^j c^k = b^{\pm r}$, and hence $b^{r^k} = b^r$. It follow that $k = 1$ in $\mathbb{Z}_4$, and so $c^\alpha = a^i b^j c$. Moreover, $(c^2)^\alpha = b^{j(1-r)}c^2$. From $H^\alpha = H$ we obtain that $(a^2c^2)^\alpha = a^2c^2$. As $(a^2)^\alpha = a^2$, one has $(c^2)^\alpha = c^2$. Thus, $j = 0$ in $\mathbb{Z}_p$, and so $c^\alpha = a^i c$.

Now by a direct computation, we have

$$ac, a^{-1}c, ba^2c, b^{-1}a^2c, a^{-1}bc, a^{-1}b^{-1}c \notin D. \tag{3.1}$$

Remember that $D^\alpha = D$. Since $c \in D$, $c^\alpha = a^i c$ implies that $a^i c \in D$. So the only possibility is either $c^\alpha = c$ or $c^\alpha = a^2 c$. If the latter happens, then $(bc)^\alpha = ba^2c$ or $b^{-1}a^2c$, and since $bc \in D$, either ba^2c or $b^{-1}a^2c$ belongs to D, contrary to Equation (3.1). Thus, $c^\alpha = c$.

Recall that $a^\alpha = a^{-1}$ or a. For the former, we have $(abc)^\alpha = a^{-1}bc$ or $a^{-1}b^{-1}c$, and since $abc \in D$, either $a^{-1}bc$ or $a^{-1}b^{-1}c$ belongs to D. This is again impossible by Equation (3.1). Thus, $a^\alpha = a$, and hence we have that

$$\alpha\colon a \mapsto a, b \mapsto b^{-1}, c \mapsto c.$$

This implies that $\mathrm{Aut}(G, H, D)$ has exactly one involution. □

Below we shall determine the full automorphism group of $\Gamma_{2,n,p,r}$.

Lemma 3.2. *Let* $\Gamma = \Gamma_{2,n,p,r}$ *and let* $G = G_{2,n,p,r}$. *Then* $\mathrm{Aut}(\Gamma) = R_H(G) \cong G$.

Proof. Let $A = \mathrm{Aut}(\Gamma)$. It is easy to see that H is a non-normal subgroup of G, and so H is core-free in G. It follows that G acts faithfully and transitively on $V(\Gamma)$ by right multiplication, and so we may view G as a transitive subgroup of A. If $n = 2$ and $p = 5$, then by Magma [3], we obtain that $\mathrm{Aut}(\Gamma) = G$. In what follows, we shall always assume that either $p > 5$ or $n > 2$.

Noting that $ab = ba$, we have $\langle ab\rangle \cong C_{2^n p}$. Clearly, $\langle ab\rangle \trianglelefteq G$, so $\langle ab\rangle$ is semiregular on $V(\Gamma)$. Since $|V(\Gamma)| = |G : H| = 2^{n+1}p$, $\langle ab\rangle$ has two orbits on $V(\Gamma)$ which are listed as follows:

$$V_0 = \{Ha^ib^j \mid i \in \mathbb{Z}_{2^n}, j \in \mathbb{Z}_p\} \text{ and } V_1 = \{Ha^ib^jc \mid i \in \mathbb{Z}_{2^n}, j \in \mathbb{Z}_p\}.$$

The kernel of G acting on $\{V_0, V_1\}$ is $\langle ab\rangle : \langle c^2\rangle$. We can also easily obtain the following two observations:

$$\begin{aligned} &\forall i \in \mathbb{Z}_{2^n}, j \in \mathbb{Z}_p, && Ha^ib^jc^2 = Hc^2a^i(c^2b^jc^2) = Ha^{2^{n-1}+i}b^{-j},\\ &\forall i_1, i_2 \in \mathbb{Z}_{2^n}, j_1, j_2 \in \mathbb{Z}_p, k \in \mathbb{Z}_2 && Ha^{i_1}b^{j_1}c^k = Ha^{i_2}b^{j_2}c^k \Leftrightarrow \begin{cases} i_1 \equiv i_2 \pmod{2^n},\\ j_1 \equiv j_2 \pmod{p}. \end{cases} \end{aligned} \tag{3.2}$$

Set $\Delta_1 = \{Hd \mid d \in H\{(ab)^{\pm 1}\}H\}$ and $\Delta_2 = \{Hd \mid d \in H\{(abc)^{\pm 1}, (bc)^{\pm 1}, c^{\pm 1}\}H\}$. Then $\Gamma(H) = \Delta_1 \cup \Delta_2$. Furthermore, an easy computation shows that

$$\begin{aligned} \Delta_1 &= \{Hab, H(ab)^{-1}, Hab^{-1}, Ha^{-1}b\},\\ \Delta_2 &= \{Habc, Ha^{1+2^{n-1}}b^rc, Hab^{-1}c, Ha^{1+2^{n-1}}b^{-r}c,\\ &\qquad Hbc, Ha^{2^{n-1}}b^rc, Hb^{-1}c, Ha^{2^{n-1}}b^{-r}c, Hc, Ha^{2^{n-1}}c\}. \end{aligned}$$

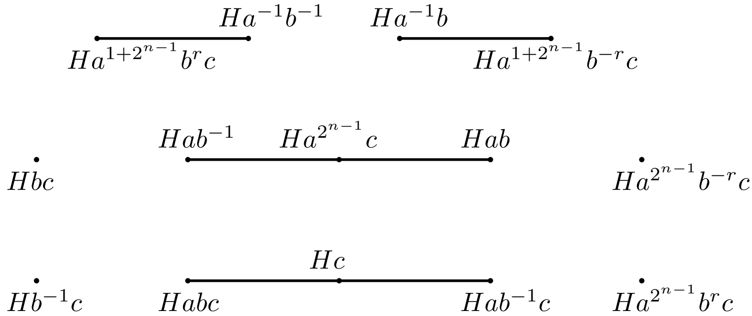

Figure 1: The subgraph induced by $\Gamma(H)$ when $p > 5$ and $n > 2$.

So for any $i \in \mathbb{Z}_{2^n}, j \in \mathbb{Z}_p$, we have

$$\begin{aligned}\Gamma(Ha^ib^j) = \{&Ha^{i+1}b^{j+1}, Ha^{i-1}b^{j-1}, Ha^{i+1}b^{j-1}, Ha^{i-1}b^{j+1}, Ha^{1-i}b^{1-jr}c,\\ &Ha^{2^{n-1}+1-i}b^{r-jr}c, Ha^{1-i}b^{-1-jr}c, Ha^{1+2^{n-1}-i}b^{-r-jr}c,\\ &Ha^{-i}b^{1-jr}c, Ha^{2^{n-1}-i}b^{r-jr}c, Ha^{-i}b^{-1-jr}c, Ha^{2^{n-1}-i}b^{-r-jr}c,\\ &Ha^{-i}b^{-jr}c, Ha^{2^{n-1}-i}b^{-jr}c\},\\ \Gamma(Ha^ib^jc) = \{&Ha^{i+1}b^{j+1}c, Ha^{i-1}b^{j-1}c, Ha^{i+1}b^{j-1}c, Ha^{i-1}b^{j+1}c, Ha^{2^{n-1}+1-i}b^{jr-1},\\ &Ha^{1-i}b^{jr-r}, Ha^{2^{n-1}+1-i}b^{1+jr}, Ha^{1-i}b^{r+jr}, Ha^{2^{n-1}-i}b^{jr-1},\\ &Ha^{-i}b^{jr-r}, Ha^{2^{n-1}-i}b^{1+jr}, Ha^{-i}b^{r+jr}, Ha^{2^{n-1}-i}b^{jr}, Ha^{-i}b^{jr}\}.\end{aligned}$$

We shall finish the proof by the following four steps.

Step 1: Let $\Sigma = \mathrm{Cos}(G, H, H\{(bc)^{\pm 1}, c^{\pm 1}\}H)$ and let $M = \langle bc, c, H\rangle$. Then $A \leq \mathrm{Aut}(\Sigma)$. In particular, the orbit $H^M = \{Hg \mid g \in M\}$ of M on $V(\Gamma)$ containing H is a block of imprimitivity of A on $V(\Gamma)$.

By direct computations, we may depict the subgraph induced by $\Gamma(H)$ as in Figures 1 – 3. From these three figures one may see that the vertex-stabilizer A_H fixes the following set setwise:

$$\begin{aligned}\Sigma(H) &= \{Hd \mid d \in H\{(bc)^{\pm 1}, c^{\pm 1}\}H\}\\ &= \{Hbc, Ha^{2^{n-1}}b^rc, Hb^{-1}c, Ha^{2^{n-1}}b^{-r}c, Hc, Ha^{2^{n-1}}c\}.\end{aligned}$$

From Lemma 2.3 it follows that $A \leq \mathrm{Aut}(\Sigma)$.

Since $M = \langle bc, c, H\rangle = \langle a^{2^{n-1}}\rangle \times \langle b, c\rangle \cong C_2 \times (C_p : C_4)$, the coset graph

$$\Delta = \mathrm{Cos}(M, H, H\{(bc)^{\pm 1}, c^{\pm 1}\}H\})$$

is just a component of Σ, and since A is transitive on $V(\Sigma) = V(\Gamma)$, the orbit of M containing H is a block of imprimitivity of A on $V(\Gamma)$.

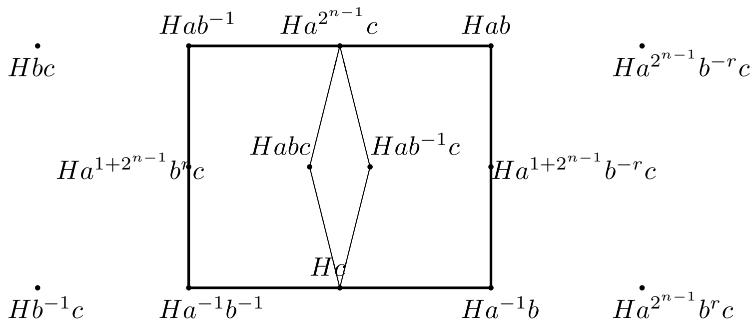

Figure 2: The subgraph induced by $\Gamma(H)$ when $p > 5$ and $n = 2$.

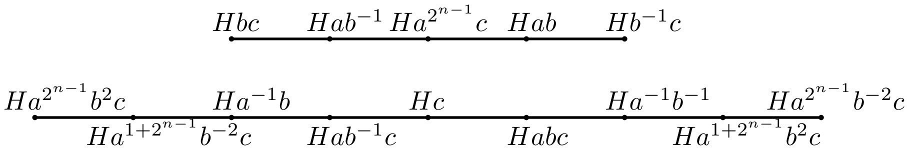

Figure 3: The subgraph induced by $\Gamma(H)$ when $p = 5$, $r = 2$ and $n > 2$.

Step 2: Set $N = \langle a^{2^{n-1}} \rangle \times \langle b \rangle$. Then each orbit of N is a block of imprimitivity of A on $V(\Gamma)$.

Let O be the orbit of M on $V(\Gamma)$ containing H. Then $O = \{Hg \mid g \in M\} = V(\Delta)$. By Step 1, O is a block of imprimitivity of A on $V(\Gamma)$. Let $S = \{Og \mid g \in G\}$. Then S is an imprimitivity block system of A on $V(\Gamma)$. Let K be the kernel of A acting on S. Note that $N \trianglelefteq G$. Since $N \leq M$, N fixes every block Og in S. It follows that $N \leq K$. Note that the subgraph $\Sigma[Og]$ of Σ is just a component which is isomorphic to Δ. It is easy to see that N acts on each Og semiregularly with two orbits $\{Hng \mid n \in N\}$ and $\{Htcg \mid t \in N\}$. As

$$\Sigma(Hg) = \{Hbcg, Ha^{2^{n-1}}b^rcg, Hb^{-1}cg, Ha^{2^{n-1}}b^{-r}cg, Hcg, Ha^{2^{n-1}}cg\},$$

the component $\Sigma[Og]$ is a bipartite graph with the two orbits of N on Og as its two parts. This implies that every orbit of N on $V(\Gamma)$ is a block of imprimitivity of A on $V(\Gamma)$.

Step 3: Take two adjacent orbits B_0, B_1 of N on $V(\Gamma)$ such that $B_0 \subseteq V_0$ and $B_1 \subseteq V_1$. Then we have $A_{(B_0)} = A_{(B_1)}$. In particular, A_H fixes $\Delta_1 = \{Hab, H(ab)^{-1}, Hab^{-1}, Ha^{-1}b\} = \{Hd \mid d \in H\{(ab)^{\pm 1}\}H\}$ setwise. Since G acts transitively on $V(\Gamma)$, we may let $B_0 = B = \{Hn \mid n \in N\}$. Since $N \trianglelefteq G$, one has $B = NH \leq G$. Recall that

$$\begin{aligned}\Gamma(H) = \{&Hab, H(ab)^{-1}, Hab^{-1}, Ha^{-1}b,\\ &Habc, Ha^{1+2^{n-1}}b^rc, Hab^{-1}c, Ha^{1+2^{n-1}}b^{-r}c,\\ &Hbc, Ha^{2^{n-1}}b^rc, Hb^{-1}c, Ha^{2^{n-1}}b^{-r}c, Hc, Ha^{2^{n-1}}c\}.\end{aligned}$$

Each orbit of N on $V(\Gamma)$ is also an independent subset of Γ. Consider the orbits Ba, Ba^{-1}, Bc and Bac of N. Since $B = NH$, one has

$$\begin{aligned}
&Hab, Hab^{-1} \in Ba,\\
&Ha^{-1}b, Ha^{-1}b^{-1} \in Ba^{-1},\\
&Habc, Ha^{1+2^{n-1}}b^r c, Hab^{-1}c, Ha^{1+2^{n-1}}b^{-r}c \in Bac,\\
&Hbc, Ha^{2^{n-1}}b^r c, Hb^{-1}c, Ha^{2^{n-1}}b^{-r}c, Hc, Ha^{2^{n-1}}c \in Bc.
\end{aligned}$$

So Ba, Ba^{-1}, Bc and Bac are all orbits of N adjacent to B. Furthermore, it is easy to check that if $n > 2$, then Ba, Ba^{-1}, Bc and Bac are four pair-wise different orbits of N, and if $n = 2$, then $Ba = Ba^{-1}, Bc$ and Bac are three pair-wise different orbits of N. Clearly, $Bc, Bac \subseteq V_1$. So $B_1 = Bc$ or Bac. Note that $\Gamma[B, Bc]$ has valency 6 and $\Gamma[B \cup Bac]$ has valency 4.

If $n > 2$, then both $\Gamma[B \cup Ba]$ and $\Gamma[B \cup Ba^{-1}]$ have valency 2. This implies that A_B also fixes Bc and Bac. If $n = 2$, then $\Gamma[B \cup Ba]$ has valency 4, and from Figure 2 one may see that A_H fixes $\{Hab, Ha^{-1}b^{-1}, Hab^{-1}, Ha^{-1}b\}$ setwise and so A_H fixes Ba. Again, we have A_B also fixes each of Bc and Bac.

Thus, we always have that A_B also fixes each of Bc and Bac. Clearly, the subgraph $\Gamma[B \cup B_1]$ is bipartite, where B_1 is either Bc or Bac. Let K be the subgroup of $\mathrm{Aut}(\Gamma[B \cup B_1])$ fixing B setwise. Then B and B_1 are two orbits of K. Let $K_{(B)}$ be the kernel of K acting on B. If $K_{(B)}$ does not fix every vertex of B_1, then each orbit of $K_{(B)}$ on B_1 has length $2p$, p or 2. Take two vertices u, v of B_1 such that u, v are in the same orbit of $K_{(B)}$. Then u, v will share the common neighborhood in $\Gamma[B \cup B_1]$. Without loss of generality, we may assume that H is one of their common neighbors.

If $B_1 = Bac$, then $u, v \in \{Habc, Ha^{1+2^{n-1}}b^r c, Hab^{-1}c, Ha^{1+2^{n-1}}b^{-r}c\}$. Note that

$$\begin{aligned}
\Gamma(Habc) \cap B &= \{Ha^{2^{n-1}}b^{r-1}, H, Ha^{2^{n-1}}b^{1+r}, Hb^{2r}\},\\
\Gamma(Ha^{1+2^{n-1}}b^r c) \cap B &= \{Hb^{-2}, Ha^{2^{n-1}}b^{-1-r}, H, Ha^{2^{n-1}}b^{r-1}\},\\
\Gamma(Hab^{-1}c) \cap B &= \{Ha^{2^{n-1}}b^{-r-1}, Hb^{-2r}, Ha^{2^{n-1}}b^{1-r}, H\},\\
\Gamma(Ha^{1+2^{n-1}}b^{-r}c) \cap B &= \{H, Ha^{2^{n-1}}b^{1-r}, Hb^2, Ha^{2^{n-1}}b^{r+1}\}.
\end{aligned}$$

It is easy to see no two of the above four sets are the same, and so no two vertices in $\{Habc, Ha^{1+2^{n-1}}b^r c, Hab^{-1}c, Ha^{1+2^{n-1}}b^{-r}c\}$ share the common neighborhood in $\Gamma[B, Bac]$, a contradiction.

If $B_1 = Bc$, then $u, v \in \{Hbc, Ha^{2^{n-1}}b^r c, Hb^{-1}c, Ha^{2^{n-1}}b^{-r}c, Hc, Ha^{2^{n-1}}c\}$. Note that

$$\begin{aligned}
\Gamma(Hbc) \cap B &= \{Ha^{2^{n-1}}b^{r-1}, H, Ha^{2^{n-1}}b^{1+r}, Hb^{2r}, Ha^{2^{n-1}}b^r, Hb^r\},\\
\Gamma(Ha^{2^{n-1}}b^r c) \cap B &= \{Hb^{-2}, Ha^{-2^{n-1}}b^{-1-r}, H, Ha^{-2^{n-1}}b^{r-1}, Hb^{-1}, Ha^{-2^{n-1}}b^{-1}\},\\
\Gamma(Hb^{-1}c) \cap B &= \{Ha^{2^{n-1}}b^{-r-1}, Hb^{-2r}, Ha^{2^{n-1}}b^{1-r}, H, Ha^{2^{n-1}}b^{-r}, Hb^{-r}\},\\
\Gamma(Ha^{2^{n-1}}b^{-r}c) \cap B &= \{H, Ha^{-2^{n-1}}b^{1-r}, Hb^2, Ha^{-2^{n-1}}b^{r+1}, Hb, Ha^{-2^{n-1}}b\},\\
\Gamma(Hc) \cap B &= \{Ha^{2^{n-1}}b^{-1}, Hb^{-r}, Ha^{2^{n-1}}b, Hb^r, Ha^{2^{n-1}}, H\},\\
\Gamma(Ha^{2^{n-1}}c) \cap B &= \{Hb^{-1}, Ha^{2^{n-1}}b^{-r}, Hb, Ha^{-2^{n-1}}b^r, H, Ha^{-2^{n-1}}\}.
\end{aligned}$$

It is easily checked that the above six sets are pair-wise different, and no two vertices in $\{Hbc, Ha^{2^{n-1}}b^rc, Hb^{-1}c, Ha^{2^{n-1}}b^{-r}c, Hc, Ha^{2^{n-1}}c\}$ share the common neighborhood in $\Gamma[B, Bc]$, a contradiction. Thus, $K_{(B)}$ also fixes every vertex of B_1. Consequently, we have $A_{(B_0)} = A_{(B)} = A_{(B_1)}$ with $B_1 = Bc$ or Bac.

Step 4: $A = G$.

By Step 3, A_H fixes Δ_1 setwise and so fixes Δ_2 setwise. By Lemma 2.3, we have $A \leq \mathrm{Aut}(\Lambda)$, where $\Lambda = \mathrm{Cos}(G, H, D_2)$ with $D_2 = H\{(abc)^{\pm 1}, (bc)^{\pm 1}, c^{\pm 1}\}H$. It is easy to see that Λ is a connected bipartite graph with V_0 and V_1 as its two parts. Let K be the kernel of A acting on $\{V_0, V_1\}$. Again, by Step 3, we obtain that K acts faithfully on V_0. It is easy to see that $\Gamma[V_0] \cong \Theta = \mathrm{Cay}(\langle ab\rangle, \{ab, (ab)^{-1}, ab^{-1}, a^{-1}b\})$. By [2, Corollary 1.3], Θ is a normal Cayley graph on $\langle ab\rangle$. Then $\mathrm{Aut}(\langle ab\rangle, \{ab, (ab)^{-1}, ab^{-1}, a^{-1}b\}) \cong C_2 \times C_2$ is regular on $\{ab, (ab)^{-1}, ab^{-1}, a^{-1}b\}$. So, $|K| \leq 4|V_0|$. It follows that $|A| \leq 4|V(\Gamma)|$ and hence $|A : G| \leq 2$. Consequently, we have $G \trianglelefteq A$. By Proposition 2.2, we have $A_H = \mathrm{Aut}(G, H, D) \leq \mathrm{Aut}(\langle ab\rangle, \{ab, (ab)^{-1}, ab^{-1}, a^{-1}b\})$. If $n > 2$, then from Figures 1 and 3, we can see that A_H is intransitive on the set of four neighbors of H contained in V_0. It follows that $A_H \cong C_2$ and hence $A = G$. If $n = 2$ and $p > 5$, then by Lemma 3.1, we must have $\mathrm{Aut}(G, H, D) \cong C_2$, implying that $A = G$. □

Corollary 3.3. *The graph* $\Gamma_{2,n,p,r}$ *is non-Cayley.*

Proof. Let $\Gamma = \Gamma_{2,n,p,r}$, and let $A = \mathrm{Aut}(\Gamma)$. Suppose on the contrary that Γ is a Cayley graph. Then A has a regular subgroup, say T, and then $A = T : A_H$. Since A is metacyclic, every Sylow 2-subgroup of T must be cyclic. It follows that every Sylow 2-subgroup of A has a cyclic maximal subgroup. However, this is impossible because from the Construction B we know that every Sylow 2-subgroup of A is isomorphic to $C_{2^n} : C_4$, a contradiction. □

Theorem 3.4. *The graph* $\Gamma_{2,n,p,r}$ *is a pseudo metacirculant.*

Proof. Let $\Gamma = \Gamma_{2,n,p,r}$, and let $G = G_{2,n,p,r}$. Note that G acts faithfully and transitively on $V(\Gamma)$ by right multiplication. Since $G = \langle ab\rangle : \langle c\rangle \cong C_{2^n \cdot p} : C_4$, Γ is a split weak metacirculant.

Suppose that Γ is also a metacirculant. Then by the definition of metacirculant, Γ has two automorphisms σ, τ satisfying the following conditions:

(1) $\langle\sigma\rangle$ is semiregular and has m orbits on $V(\Gamma)$,

(2) τ normalizes $\langle\sigma\rangle$ and cyclically permutes the m orbits of $\langle\sigma\rangle$,

(3) τ has a cycle of size m in its cycle decomposition.

By Lemma 3.2, we have $\mathrm{Aut}(\Gamma) = G$. By Corollary 3.3, Γ is a non-Cayley graph, and then we have $G = \langle\sigma, \tau\rangle$. Thus, $\tau^m \neq 1$ and hence $\langle\tau^m\rangle \cong C_2$. Since G is transitive on $V(\Gamma)$, we may assume that $\langle\tau^m\rangle = G_H = \langle a^{2^{n-1}}c^2\rangle$. Then there would exist an element x of G of order 4 such that $x^2 = \tau^m = a^{2^{n-1}}c^2$. By a direct computation, we have $C_G(a^{2^{n-1}}c^2) = \langle a, c\rangle$. Then $x \in \langle a, c\rangle$ and so $x = a^ic^j$ for some integers i, j. However, $x^2 = c^{2j}$ due to $c^{-1}ac = a^{-1}$. A contradiction occurs. Thus, Γ is not a metacirculant. □

4 Pseudo metacirculants—Family B

Consruction B. *Let $m > n > 1$ be positive integers and let p, q be primes such that $p^m \mid q-1$. Let $r \in \mathbb{Z}_q^*$ be of order p^m, and let*

$$G_{p,q,m,n,r} = \langle a, b, c \mid a^{p^n} = b^q = c^{p^m} = 1, ab = ba, ac = ca, c^{-1}bc = b^r \rangle.$$

Let

$$\Gamma_{p,q,m,n,r} = \mathrm{Cos}(G_{p,q,m,n,r}, H, H\{(ab)^{\pm 1}, c^{\pm 1}\}H),$$

where $H = \langle c^{p^{m-1}} a^{p^{n-1}} \rangle$.

We shall first give some basic properties of G.

Lemma 4.1. *Let $G = G_{p,q,m,n,r}$, and let $D = H\{(ab)^{\pm 1}, c^{\pm 1}\}H$. Then the following hold.*

(1) *$|H| = p$, H is non-normal in G and $C_G(H) = \langle a, c \rangle$.*

(2) *$C_G(b) = \langle a, b \rangle$.*

(3) *For any $g \in G$, if $\langle g \rangle \trianglelefteq G$, then $g \in \langle a, b \rangle$.*

(4) *$D = (\bigcup_{k \in \mathbb{Z}_p} H(ab^{r^{kp^{m-1}}})) \cup (\bigcup_{k \in \mathbb{Z}_p} H(ab^{r^{kp^{m-1}}})^{-1}) \cup Hc \cup Hc^{-1}$.*

(5) *$\mathrm{Aut}(G, H, D) \cong C_p$.*

Proof. Note that $G = \langle a \rangle \times (\langle b \rangle : \langle c \rangle) \cong \mathbb{Z}_{p^n} \times (\mathbb{Z}_q : \mathbb{Z}_{p^m})$. Let $P = \langle a, c \rangle$. Clearly, $P = \langle a \rangle \times \langle c \rangle \cong C_{p^n} \times C_{p^m}$, so $H = \langle c^{p^{m-1}} a^{p^{n-1}} \rangle$ has order p. If $H \trianglelefteq G$, then b centralizes $c^{p^{m-1}} a^{p^{n-1}}$ and then centralizes $c^{p^{m-1}}$ since $a \in Z(G)$. This is impossible because $c^{-p^{m-1}} b c^{p^{m-1}} = b^{r^{p^{m-1}}} \neq b$. Thus, H is non-normal in G. It follows that $\langle a, c \rangle \leq C_G(H) < G$. Observing that $|G : \langle a, c \rangle| = q$, we have $C_G(H) = \langle a, c \rangle$. Therefore, item (1) holds.

For (2) , it is easy to see that $\langle a, b \rangle \leq C_G(b)$ and $G = \langle a, b \rangle : \langle c \rangle$. Recall that $c^{-1}bc = b^r$ with r an element of $\mathbb{Z}_q^*$ of order p^m. This implies that $C_G(b) = \langle a, b \rangle$.

For (3) , recall that $G/\langle a \rangle \cong \langle b \rangle : \langle c \rangle \cong \mathbb{Z}_q : \mathbb{Z}_{p^m}$, and $\langle b \rangle$ is self-centralized in $\langle b \rangle : \langle c \rangle$. This implies that $\langle b \rangle$ is the unique non-trivial normal cyclic subgroup of $\langle b \rangle : \langle c \rangle$. For any $g \in G$, if $\langle g \rangle \trianglelefteq G$, then $\langle g \rangle \langle a \rangle / \langle a \rangle$ is normal in $G/\langle a \rangle$, and then $\langle g \rangle \langle a \rangle / \langle a \rangle \leq \langle b \rangle \langle a \rangle / \langle a \rangle$. So $g \in \langle a, b \rangle$, as desired.

For (4) , we have $D = H\{(ab)^{\pm 1}, c^{\pm 1}\}H = H\{ab, (ab)^{-1}\}H \cup H\{c, c^{-1}\}H$. Since c centralizes H, one has $H\{c, c^{-1}\}H = Hc \cup Hc^{-1}$. Clearly,

$$H\{ab, (ab)^{-1}\}H = HabH \cup H(ab)^{-1}H.$$

Then $HabH = \bigcup_{k \in \mathbb{Z}_p} Hab(c^{p^{m-1}} a^{p^{n-1}})^k = \bigcup_{k \in \mathbb{Z}_p} Habc^{kp^{m-1}} a^{kp^{n-1}}$. Since $c^{-1}bc = b^r$, one has $bc^{kp^{m-1}} = c^{kp^{m-1}} b^{r^{kp^{m-1}}}$. As $a \in Z(G)$, it follows that

$$HabH = \bigcup_{k \in \mathbb{Z}_p} Habc^{kp^{m-1}} a^{kp^{n-1}} = \bigcup_{k \in \mathbb{Z}_p} Hab^{r^{kp^{m-1}}}.$$

Similarly, $H(ab)^{-1}H = \bigcup_{k\in\mathbb{Z}_p} H(ab^{r^{kp^{m-1}}})^{-1}$. (4) is proved.

Finally, we shall prove (5) . Take $\alpha \in \mathrm{Aut}(G, H, D)$. Then $H^\alpha = H$ and $D^\alpha = D$. Observe that $\langle b\rangle$ is a normal Sylow q-subgroup of G and $\langle a\rangle$ is just the center of G. It follows that $b^\alpha \in \langle b\rangle$ and $a^\alpha \in \langle a\rangle$. Since $H^\alpha = H$, one has $c^\alpha \in C_G(H) = \langle a, c\rangle$. Assume that

$$a^\alpha = a^i, b^\alpha = b^j, c^\alpha = a^s c^t, \text{ with } i \in \mathbb{Z}^*_{p^n}, j \in \mathbb{Z}^*_q, s \in \mathbb{Z}_{p^n}, t \in \mathbb{Z}^*_{p^m}.$$

Considering the image of the equality $c^{-1}bc = b^r$ under α, we obtain that

$$(a^s c^t)^{-1} b^j (a^s c^t) = b^{jr},$$

and hence $b^{jr^t} = b^{jr}$. It follows that $r^t \equiv r \pmod q$. Since r is an element of $\mathbb{Z}^*_{p^m}$ of order p^m and $t \in \mathbb{Z}^*_{p^m}$, the equality $r^t \equiv r \pmod q$ implies that $t = 1$ in $\mathbb{Z}_{p^m}$, and so $c^\alpha = a^s c$. Consequently, $a^s c = c^\alpha \in D^\alpha = D$. By a direct computation, we have $a^\ell c \notin D$ for any $\ell \neq 0$ (in $\mathbb{Z}_{p^n}$). So we must have $c^\alpha = c$.

Since $ab \in D$, one has $a^i b^j = (ab)^\alpha \in D$. Since $j \in \mathbb{Z}^*_q$, one has $a^i b^j \notin Hc \cup Hc^{-1}$ because $(Hc \cup Hc^{-1}) \subseteq \langle a, c\rangle$. Then by (4) we have

$$a^i b^j \in (\bigcup_{k\in\mathbb{Z}_p} H(ab^{r^{kp^{m-1}}})) \cup (\bigcup_{k\in\mathbb{Z}_p} H(ab^{r^{kp^{m-1}}})^{-1}).$$

Note that $\langle a, b\rangle \cap H = 1$. It follows that $(ab)^\alpha = a^i b^j \in \{(ab^{r^{kp^{m-1}}})^{\pm 1} \mid k \in \mathbb{Z}_p\}$, and so $a^i b^j = ab^{r^{kp^{m-1}}}$ or $(ab^{r^{kp^{m-1}}})^{-1}$ for some $k \in \mathbb{Z}_p$. Since $\langle a, b\rangle = \langle a\rangle \times \langle b\rangle \cong \mathbb{Z}_{p^n} \times \mathbb{Z}_q$, one has

$$(a, b)^\alpha = (a^i, b^j) = (a, b^{r^{kp^{m-1}}}) \text{ or } (a^{-1}, b^{-r^{kp^{m-1}}}).$$

Since $H^\alpha = H$, one has $c^{p^{m-1}}(a^i)^{p^{n-1}} = (c^{p^{m-1}} a^{p^{n-1}})^\alpha \in H$, and hence $a^\alpha = a$. It follows that $(a, b)^\alpha = (a^i, b^j) = (a, b^{r^{kp^{m-1}}})$. Hence $|\mathrm{Aut}(G, H, D)| \leq p$. Note that $H \leq \mathrm{Aut}(G, H, D)$. Then $\mathrm{Aut}(G, H, D) \cong C_p$. □

Next we shall determine the full automorphism group of $\Gamma_{p,q,n,m,r}$.

Lemma 4.2. *Let* $\Gamma = \Gamma_{p,q,m,n,r}$ *and let* $G = G_{p,q,m,n,r}$. *Then* $\mathrm{Aut}(\Gamma) = R_H(G) \cong G$. *Moreover,* Γ *is a non-Cayley graph.*

Proof. We shall first prove three claims.

Claim 1. *Let* $\Lambda = \mathrm{Cos}(G, H, H\{ab, (ab)^{-1}\}H)$. *Then* $\mathrm{Aut}(\Gamma) \leq \mathrm{Aut}(\Lambda)$.

Proof of Claim 1. By Lemma 4.1(4), the neighborhood of H in Γ is

$$\Gamma(H) = \{H(ab^{r^{kp^{m-1}}})^{\pm 1}, Hc^{\pm 1} \mid k \in \mathbb{Z}_p\},$$

and the neighborhood of H is Λ is

$$\Lambda(H) = \{H(ab^{r^{kp^{m-1}}})^{\pm 1} \mid k \in \mathbb{Z}_p\}.$$

By direct computations, we see that for any $k \in \mathbb{Z}_p$,

$$\{H, Ha^2b^{1+r^{kp^{m-1}}} \mid k \in \mathbb{Z}_p\} \subseteq \Gamma(Hab^{r^{kp^{m-1}}}) \cap \Gamma(Hab),$$
$$\{H, H(a^2b^{1+r^{kp^{m-1}}})^{-1} \mid k \in \mathbb{Z}_p\} \subseteq \Gamma(H(ab^{r^{kp^{m-1}}})^{-1}) \cap \Gamma(H(ab)^{-1}),$$

and moreover, for any $Hx \in \{H(ab^{r^{kp^{m-1}}})^{\pm 1} \mid k \in \mathbb{Z}_p\}$ and $Hc^\ell \in \{Hc, Hc^{-1}\}$, we have

$$\Gamma(Hx) \cap \Gamma(Hc^\ell) = \{H\}.$$

It then follows that the vertex-stabilizer $\mathrm{Aut}(\Gamma)_H$ fixes the following set setwise:

$$\Lambda(H) = \{H(ab^{r^{kp^{m-1}}})^{\pm 1} \mid k \in \mathbb{Z}_p\}.$$

By Lemma 2.3, we have $\mathrm{Aut}(\Gamma) \leq \mathrm{Aut}(\Lambda)$. □

Claim 2. *Let $V_i = \{Ha^jb^kc^i \mid j \in \mathbb{Z}_{p^n}, k \in \mathbb{Z}_q\}$ with $i \in \mathbb{Z}_{p^{m-1}}$. Then the following hold.*

(1) *V_i is a block of imprimitivity of $\mathrm{Aut}(\Gamma)$ on $V(\Gamma)$.*

(2) *The edges of Γ between V_i and V_{i+1} form a perfect matching, where the subscripts are modulo p^{m-1}.*

(3) *Let $\mathcal{B} = \{V_0, V_1, \ldots, V_{p^{m-1}-1}\}$. Then the quotient graph $\Gamma_{\mathcal{B}}$ is isomorphic to $\mathrm{C}_{p^{m-1}}$.*

Proof of Claim 2. By Claim 1, we have $\mathrm{Aut}(\Gamma) \leq \mathrm{Aut}(\Lambda)$. Recall that

$$\Lambda = \mathrm{Cos}(G, H, H\{ab, (ab)^{-1}\}H).$$

Then Λ is disconnected, and the coset graph

$$\Delta = \mathrm{Cos}(\langle ab, H\rangle, H, H\{ab, (ab)^{-1}\}H)$$

is a component of Λ. Consequently, $V_0 = V(\Delta) = \{Hg \mid g \in \langle ab\rangle\}$ is a block of imprimitivity of $\mathrm{Aut}(\Gamma)$ on $V(\Lambda) = V(\Gamma)$. Clearly, each V_i is an orbit of $\langle ab\rangle$ on $V(\Gamma)$. Since $\langle ab\rangle \trianglelefteq G$ and G is transitive on $V(\Gamma)$, V_i is a block of imprimitivity of $\mathrm{Aut}(\Gamma)$ on $V(\Gamma)$.

For (2), observing that $V_1 \cap \Gamma(H) = \{Hc\}$, the edges between V_0 and V_1 form a perfect matching. As c cyclically permutates the orbits $V_0, V_1, \ldots, V_{p^{m-1}-1}$ of $\langle ab\rangle$, for every $i \in \mathbb{Z}_{p^{m-1}}$, the subgraph of Γ induces by the edges between V_i and V_{i+1} form a perfect matching, where the subscripts are modulo p^{m-1}.

For (3), noting that Γ has valency $2p+2$ while Δ has valency $2p$, the quotient graph $\Gamma_{\mathcal{B}}$ must be a cycle of length p^{m-1}. □

Claim 3. *Let $\Delta = \mathrm{Cos}(\langle ab, H\rangle, H, H\{ab, (ab)^{-1}H\})$. Then $|\mathrm{Aut}(\Delta)| = 2p^{n+1}q$.*

Proof of Claim 3. It is easy to see that Δ is isomorphic to the following Cayley graph

$$\Theta = \mathrm{Cay}(\langle ab\rangle, \{ab^{r^{kp^{m-1}}}, (ab^{r^{kp^{m-1}}})^{-1} \mid k \in \mathbb{Z}_p\}).$$

Let $M = \langle a, b\rangle (\cong \mathbb{Z}_{p^n q})$ and $S = \{ab^{r^{kp^{m-1}}}, (ab^{r^{kp^{m-1}}})^{-1} \mid k \in \mathbb{Z}_p\}$. The maps $\alpha\colon a \mapsto a, b \mapsto b^{r^{p^{m-1}}}$ and $:a \mapsto a^{-1}, b \mapsto b^{-1}$ induce two automorphisms of M which fix S setwise. So, $\langle \alpha, \beta\rangle \leq \mathrm{Aut}(M, S)$. It is easy to see that $\langle \alpha, \beta\rangle$ acts transitively on S. Hence Θ is a connected arc-transitive Cayley graph on M. Suppose that Θ is not normal. It is obvious that $\Theta \not\cong \mathbf{K}_{p^n q}$. By Proposition 2.4, there exists a connected arc-transitive circulant Σ of order t such that one of the following happens:

(i) $\Theta \cong \Sigma \circ \bar{\mathbf{K}}_d$ with $p^n q = td$,

(ii) $\Theta \cong \Sigma \circ \bar{\mathbf{K}}_d - d\Sigma$, where $p^n q = td$, $d > 3$ and $\gcd(\mathrm{d}, \mathrm{t}) = 1$.

Suppose that (i) happens. Let k be the valency of Σ. Then $\Sigma[\bar{\mathbf{K}}_d]$ has valency kd. Noting that Θ has valency $2p$, $\Theta \cong \Sigma \circ \bar{\mathbf{K}}_d$ implies that $kd = 2p$. As $p^n q = td$, one has $d = p$, and hence $k = 2$. It follows that Σ is a cycle of length t. Also, $\Theta \cong \Sigma \circ \bar{\mathbf{K}}_p$ implies that there exist t independent subsets, say $D_0, D_1, D_2, \ldots, D_{t-1}$ of $V(\Theta)$ of cardinality p such that the subgraph induced by $D_i \cup D_{i+1}$ is isomorphic to $\mathbf{K}_{p,p}$, where the subscripts are modulo t. Furthermore, these t subsets are also blocks of imprimitivity of $\mathrm{Aut}(\Theta)$ on $V(\Theta)$. Assume that D_0 contains the identity of M. Since M acts on $V(\Theta)$ by right multiplication, D_0 will be a subgroup of M of order p, and then $D_0 = \langle a^{p^{n-1}}\rangle$ and then each D_i is a coset of D_0 in M. Recall that the only two blocks adjacent to D_0 are D_1 and D_2, and that any two adjacent blocks induce a subgraph isomorphic to $\mathbf{K}_{p,p}$. Then $D_1 \cup D_{t-1} = S$. We may assume that $ab \in D_1$. Then $D_1 = D_0 ab$ and $D_{t-1} = D_0(ab)^{-1}$. This is contrary to the fact that $S = \{ab^{r^{kp^{m-1}}}, (ab^{r^{kp^{m-1}}})^{-1} \mid k \in \mathbb{Z}_p\}$.

Suppose now that (ii) holds. Observing that the valency of $\Sigma \circ \bar{\mathbf{K}}_d - d\Sigma$ is a multiple of $d-1$, one has $d - 1 \mid 2p$, and hence $d - 1 = p$ due to $d > 3$. As $p^n q = td$ and $d = p + 1$, one has $p + 1 \mid p^n q$, implying $p + 1 = q$. This is contrary to the fact that p, q are odd.

Thus, Θ is a normal Cayley graph on M. By Proposition 2.1, we have $\mathrm{Aut}(\Theta) = R(M) : \mathrm{Aut}(M, S)$. As M is cyclic, $\mathrm{Aut}(M, S)$ is abelian, and so $\mathrm{Aut}(M, S)$ acts regularly on S. It follows that $|\mathrm{Aut}(M, S)| = 2p$, and so $|\mathrm{Aut}(\Theta)| = 2p^{n+1}q$, as claimed. □

Proof of Lemma 4.2*, continued:* Now we are ready to finish the proof. Let $A = \mathrm{Aut}(\Gamma)$. By Claim 2(1), $\mathcal{B} = \{V_0, V_1, \ldots, V_{p^{m-1}-1}\}$ is a system of blocks of A. Let K be the kernel of A acting on $\mathcal{B}$. By Claim 2(3), we have $A/K \leq \mathrm{Aut}(\Gamma_{\mathcal{B}}) \cong D_{2p^{m-1}}$. By Claim 2(2), one may see that K acts faithfully on V_0. So K can be viewed as a subgroup of the full automorphism group of the subgraph Δ of Γ induced by V_0. By Claim 3, we have $|\mathrm{Aut}(\Delta)| = 2p^{n+1}q$, and so $|K| \leq 2p^{n+1}q$.

From Lemma 4.1(1) we know that H is non-normal in G, and so G acts faithfully on $V(\Gamma)$ by right multiplication. Therefore, we may identify G with $R_H(G)$, and then G is a vertex-transitive subgroup of A. Then $GK/K \cong C_{p^{m-1}}$ which is normal in $A/K \leq D_{2p^{m-1}}$. In particular, $GK \trianglelefteq A$. Furthermore, $|GK/K| = p^{m-1}$ implies that $|GK| = p^{m-1}|K| \leq 2p^{m+n}q$. So $|GK : G| \leq 2$, and hence G is the unique Hall $\{p, q\}$-subgroup of GK. Thus, G is characteristic in GK, and so normal in A since $GK \trianglelefteq A$. By Proposition 2.2, the stabilizer of H in A is $A_H = \mathrm{Aut}(G, H, D)$. By Lemma 4.1(5), we have $\mathrm{Aut}(G, H, D) \cong C_p$. This implies that $G = A$.

Finally, suppose that Γ is a Cayley graph. Then A has a regular subgroup, say T, and then $A = T : A_H$. Since $A = G$ is metacyclic, every Sylow p-subgroup of T must be cyclic because $A_H \cong C_p$. It follows that every Sylow p-subgroup of A has a cyclic maximal subgroup. However, this is impossible because from the Construction B we know that every Sylow p-subgroup of A is isomorphic to $C_{p^n} : C_{p^m}$ with $m > n > 1$. $\square$

Theorem 4.3. *The graph $\Gamma_{p,q,m,n,r}$ is a pseudo metacirculant.*

Proof. Let $\Gamma = \Gamma_{p,q,n,m,r}$, and let $G = G_{p,q,n,m,r}$. Note that G acts faithfully and transitively on $V(\Gamma)$ by right multiplication. Since $G = \langle ab\rangle : \langle c\rangle \cong C_{p^n q} : C_{p^m}$, Γ is a split weak metacirculant.

Suppose that Γ is also a metacirculant. Then by the definition of metacirculant, Γ has two automorphisms σ, τ satisfying the following conditions:

(1) $\langle\sigma\rangle$ is semiregular and has t orbits on $V(\Gamma)$,

(2) τ normalizes $\langle\sigma\rangle$ and cyclically permutes the t orbits of $\langle\sigma\rangle$,

(3) τ has a cycle of size t in its cycle decomposition.

By Lemma 4.2, we have $A = G$ and Γ is a non-Cayley graph. Since $|G| = |V(\Gamma)|p$, we must have $G = \langle\sigma, \tau\rangle$. Thus, $\tau^t \neq 1$ and hence $\langle\tau^t\rangle \cong C_p$. Since G is transitive on $V(\Gamma)$, we may assume that $\langle\tau^t\rangle = G_H = H = \langle a^{p^{n-1}} c^{p^{m-1}}\rangle$. By Lemma 4.1(3), we have $\sigma \in \langle a, b\rangle$, and so $o(\sigma) \mid p^n q$. Consequently, $p^m \mid o(\tau)$. Let $x \in \langle\tau\rangle$ be of order p^m such that $x^{p^{m-1}} = \tau^t = a^{p^{n-1}} c^{p^{m-1}}$. Then $x \in C_G(H)$, and by Lemma 4.1(1), $C_G(H) = \langle a, c\rangle$. So $x = a^i c^j$ for some integers i, j. However, $x^{p^{m-1}} = a^{ip^{m-1}} c^{jp^{m-1}} = c^{jp^{m-1}}$ due to $m > n$. A contradiction occurs. Thus, Γ is not a metacirculant. $\square$

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Partial-dual Euler-genus distributions for bouquets with small Euler genus*

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Abstract

For an arbitrary ribbon graph G, the partial-dual Euler-genus polynomial of G is a generating function that enumerates partial duals of G by Euler genus. When G is an orientable ribbon graph, the partial-dual orientable genus polynomial of G is a generating function that enumerates partial duals of G by orientable genus. Gross, Mansour, and Tucker inaugurated these partial-dual Euler-genus and orientable genus distribution problems in 2020. A bouquet is a one-vertex ribbon graph. Given a ribbon graph G, its partial-dual Euler-genus polynomial is the same as that of some bouquet; this motivates our focus on bouquets. We obtain the partial-dual Euler-genus polynomials for all the bouquets with Euler genus at most two.

Keywords: Ribbon graph, partial dual, Euler-genus polynomial, orientable genus polynomial.

Math. Subj. Class. (2020): 05C10, 05C30, 05C31, 57M15

1 Introduction

Ribbon graphs are well-known to be equivalent to 2-cell embeddings of graphs. Let G be a ribbon graph, $V(G)$ and $E(G)$ denote its vertex-disk set and edge-ribbon (or simply ribbon) set, respectively. In 2009, Chmutov [1] defined the partial dual of G when he was studying signed Bollobás–Riordan polynomials. For any $A \subseteq E(G)$, the partial dual ribbon graph with respect to A is denoted by G^A. As a generalization of the geometric duality, partial

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duality has developed into a topic of independent interest for its applications in graph theory and topology. The reader is referred to [1, 3] for more background about ribbon graphs and partial duals.

Given a ribbon graph G, there are $2^{|E(G)|}$ partial duals of G in total, the problem of the enumeration of its partial duals G^A according to their Euler-genus $\varepsilon(G^A)$ or orientable genus $\gamma(G^A)$ was inaugurated by Gross, Mansour, and Tucker [5] in 2020. They defined the partial-dual Euler-genus (and orientable genus) polynomial of a ribbon graph G, which was motivated by the Euler-genus (and orientable genus) polynomial [4] of a graph.

Definition 1.1 ([5])**.** The *partial-dual Euler-genus polynomial* of a ribbon graph G is the generating function

$$^{\partial}\varepsilon_G(z) = \sum_{A \subseteq E(G)} z^{\varepsilon(G^A)}$$

that enumerates partial duals by Euler genus. The *partial-dual orientable genus polynomial* of a ribbon graph G is the generating function

$$^{\partial}\gamma_G(z) = \sum_{A \subseteq E(G)} z^{\gamma(G^A)}$$

that enumerates partial duals by orientable genus.

In [5], the authors discussed some properties of each polynomial, gave a recursion for subdivision of an edge, and obtained the partial-dual Euler-genus (or orientable genus) polynomials for some infinite families of ribbon graphs. In [8], Yan and Jin found some bouquets having a non-constant partial-dual orientable genus polynomial with only one non-zero coefficient, which disproved one of conjectures provided in [5], they also obtained the partial-dual Euler-genus polynomials for all bouquets with the number of loops at most 3 and the partial-dual orientable genus polynomials for all bouquets with the number of loops at most 4. They also found two infinite families of counterexamples to another conjecture provided in [5] about the interpolating property of partial-dual Euler-genus polynomials of non-orientable ribbon graphs in [10]. Then Chmutov and Vignes-Tourneret [2] and Yan and Jin [9] proved that the family of counterexamples given in [8] are the only counterexamples for that conjecture, independently.

A *bouquet* is a one-vertex ribbon graph, we denote the bouquet with n loops by B_n. The bouquets with orientable genus 0 and 1 are called *plane bouquets* and *toroidal bouquets*, respectively; the bouquets with nonorientable genus 1 and 2 are called *projective planar bouquets* and *Klein bottle bouquets*, respectively. In this paper we focus on the partial-dual Euler-genus polynomial for these bouquets. The following two lemmas motivate our interest in bouquets.

Lemma 1.2 ([8])**.** *If G is a ribbon graph and $A \subseteq E(G)$, then $^{\partial}\varepsilon_G(z) = {^{\partial}\varepsilon_{G^A}}(z)$. When G is orientable, $^{\partial}\gamma_G(z) = {^{\partial}\gamma_{G^A}}(z)$.*

Lemma 1.3 ([3, 5])**.** *If A is a spanning tree for G, then the partial dual G^A has only one vertex.*

From Lemmas 1.2 and 1.3, given a ribbon graph G, its partial-dual Euler-genus polynomial and partial-dual orientable genus polynomial when G is orientable will be the same as that of some bouquet. From this point of view, for the partial-dual Euler-genus (and orientable genus) distribution problems, we only need to focus on bouquets.

Given a bouquet B_n, we label each loop by distinct letter. By reading these letters we meet when travelling around the boundary of the vertex-disk of B_n, we obtain a cyclic order of $2n$ letters, call it the *rotation* of B_n. A loop in B_n is called a *twisted loop* if its neighbourhood is homeomorphic to a Mӧbius band, and *untwisted loop* if its neighbourhood is homeomorphic to an annulus. To distinguish between untwisted loops and twisted loops, we give the same signs $+$ to the two same letters which correspond to a common untwisted loop, and give two different signs (one $+$, the other $-$) to the two same letters which correspond to a common twisted loop. For simplicity, we usually omit the sign $+$. We call the cyclic order of $2n$ signed letters the *signed rotation* of B_n. It is easy to check that there is a 1-to-1 correspondence between the signed rotations and bouquets.

Two loops a and b in a bouquet are *interlaced* if the rotation of this bouquet is of form $a\cdots b\cdots a\cdots b\cdots$, otherwise *parallel*. A loop e in a bouquet is *trivial* if e is untwisted and not interlaced with any other loops; otherwise *nontrivial*. Notice that, all the twisted loops are nontrivial in this paper, which is different with that in [8] (where a loop in a bouquet is trivial if it is not interlaced with any other loops, so both trivial untwisted loops and trivial twisted loops may exist). An *essential bouquet* is a bouquet whose loops are all nontrivial. A ribbon graph H is a *ribbon subgraph* of G if H can be obtained by deleting vertices and ribbons of G. Given a bouquet B, by deleting all trivial loops, we obtain an essential bouquet B' which is a ribbon subgraph of B, and we call B' the *maximum essential subgraph* of B.

The *join* of two disjoint ribbon graphs G_1 and G_2, denoted by $G_1 \vee G_2$, is a ribbon graph which can be obtained by the following way. Firstly we choose an arc p_1 on the boundary of a vertex-disk of G_1 that lies between two consecutive ribbon ends, and choose another such arc p_2 on the boundary of a vertex-disk of G_2. Next by identifying the arcs p_1 and p_2, we merge the two vertex-disks from G_1 and G_2 into one new vertex-disk, the ribbon graph obtained is the graph $G_1 \vee G_2$. For the partial-dual Euler-genus polynomial of the graph $G_1 \vee G_2$, the following result had been obtained by Gross, Mansour, and Tucker.

Lemma 1.4 ([5]). *If G_1 and G_2 are disjoint ribbon graphs, then*

$$^{\partial}\varepsilon_{G_1\vee G_2}(z) = {^{\partial}\varepsilon_{G_1}}(z)\cdot {^{\partial}\varepsilon_{G_2}}(z).$$

For a bouquet, it can be obtained by the join of its maximum essential subgraph with each trivial loops successively. From Lemma 1.4, the following result can be deduced.

Lemma 1.5 ([8]). *Let G be a bouquet and G' be the maximum essential subgraph of G. If $|E(G)| - |E(G')| = i$, then*

$$^{\partial}\varepsilon_{G}(z) = 2^i\cdot {^{\partial}\varepsilon_{G'}}(z).$$

From Lemma 1.5, we can reduce the problem of partial-dual Euler-genus polynomial of a bouquet to that of its maximum essential subgraph. We also note that the Euler genus of a bouquet is equal to that of its maximum essential subgraph, due to the additivity of Euler genus over the join operation.

If $A \subseteq E(G)$, the *induced ribbon subgraph* $G|_A$ is a ribbon subgraph of G whose ribbon set is A and whose the vertex-disk set consists of all ends of ribbons in A. We use $A^c := E(G)\setminus A$ to denote the complement of $A \subseteq E(G)$. By the following lemma, the Euler genus of a partial dual of a bouquet can be computed from the Euler genus of its two ribbon subgraphs.

Lemma 1.6 ([5]). *If G is a bouquet and $A \subseteq E(G)$, then*

$$\varepsilon(G^A) = \varepsilon(G|_A) + \varepsilon(G|_{A^C}).$$

We note that $\varepsilon(G) = 2\gamma(G)$ for orientable ribbon graphs, the following two lemmas can be deduced easily.

Lemma 1.7 ([5]). *If G is an orientable bouquet and $A \subseteq E(G)$, then*

$$\gamma(G^A) = \gamma(G|_A) + \gamma(G|_{A^C}).$$

Lemma 1.8 ([5]). *If G is an orientable ribbon graph, then ${}^{\partial}\gamma_G(z) = {}^{\partial}\varepsilon_G(z^2)$.*

In this paper we deduce the forms of the signed rotations of maximum essential subgraphs for all projective planar bouquets, Klein bottle bouquets, plane bouquets, and toroidal bouquets, respectively, and obtain the partial-dual Euler-genus polynomials for these bouquets.

2 Partial-dual Euler-genus polynomials of projective planar bouquets and Klein bottle bouquets

Recall that a ribbon graph is equivalent to a 2-cell embedding of a graph. From the Proposition 4.1.5 in [6], we get the following fact easily.

Lemma 2.1 ([6]). *If H is a subgraph of a ribbon graph G, then $\varepsilon(H) \leq \varepsilon(G)$.*

Lemma 2.2. *The bouquet B_n is a plane bouquet if and only if all of its loops are trivial.*

Proof. This lemma follows from the definition of trivial loops in B_n. □

Lemma 2.3. *For a nonorientable bouquet B_n,*

(i) *if there exists a pair of parallel twisted loops, then $\varepsilon(B_n) \geq 2$;*

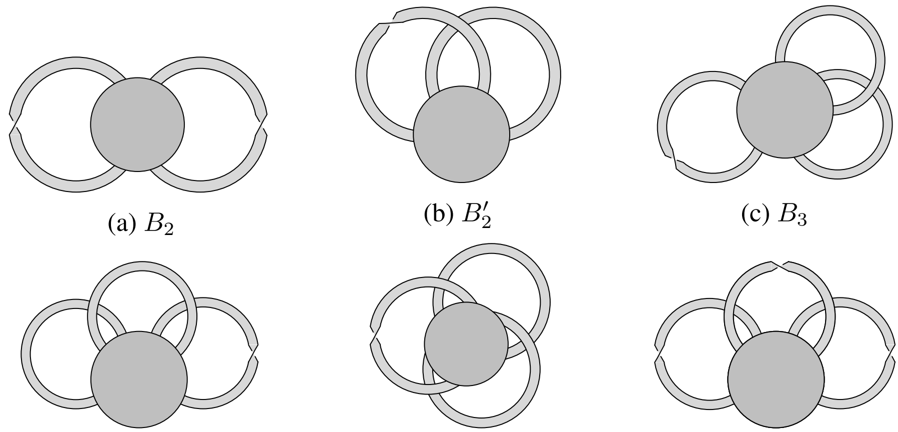
(ii) *if there exists a pair of interlaced loops, in which one is twisted and the other one is untwisted, then $\varepsilon(B_n) \geq 2$;*

(iii) *if there exists a pair of interlaced untwisted loops, then $\varepsilon(B_n) \geq 3$.*

Proof. For (i) and (ii), the bouquet B_n has ribbon subgraphs B_2 and B_2', as shown in Figure 1(a) and (b), respectively. It is easy to check that $\varepsilon(B_2) = \varepsilon(B_2') = 2$, then $\varepsilon(B_n) \geq 2$ by Lemma 2.1.

For (iii), there exists a ribbon subgraph with three ribbons e_1, e_2 and e_3 in which e_1 and e_2 are interlaced untwisted loops and e_3 is a twisted loop (for any nonorientable bouquet, a twisted loop must exist). The loop e_3 can be interlaced with the other i $(0 \leq i \leq 2)$ loops in this ribbon subgraph, by a case analysis, there are three nonequivalent such ribbon subgraphs B_3, B_3' and B_3'', as shown in Figure 1(c)-(e). One can check that $\varepsilon(B_3) = \varepsilon(B_3') = \varepsilon(B_3'') = 3$, then $\varepsilon(B_n) \geq 3$ follows from Lemma 2.1. □

The following is easy to prove by induction and simple facial walk counting. We present it as a technical lemma for the convenience of our later proof. It in fact reveals some important differences between twisted and untwisted loops in a bouquet.



(a) B_2 (b) B_2' (c) B_3

(d) B_3' (e) B_3'' (f) B_3'''

Figure 1: Some bouquets.

Lemma 2.4. *If G is a bouquet and $A \subseteq E(G)$, then*

(i) *If A (A^C, respectively) consists of k untwisted and pairwise parallel loops, then $\varepsilon(G|_A) = 0$ ($\varepsilon(G|_{A^C}) = 0$, respectively);*

(ii) *If A (A^C, respectively) consists of k twisted and pairwise parallel loops, then $\varepsilon(G|_A) = k$ ($\varepsilon(G|_{A^C}) = k$, respectively);*

(iii) *If A (A^C, respectively) consists of k twisted and pairwise interlaced loops, then $\varepsilon(G|_A) = 1$ ($\varepsilon(G|_{A^C}) = 1$, respectively).*

We first consider the partial-dual Euler-genus polynomials for projective planar bouquets.

Lemma 2.5. *If B_n is a projective planar bouquet and B_k is the maximum essential subgraph of B_n, then the signed rotation of B_k has the form*

$$a_1 \dots a_k(-a_1)\dots(-a_k), (1 \le k \le n). \tag{1}$$

Proof. Firstly a bouquet with signed rotation (1) has Euler genus 1, from Lemma 2.4(iii). From items (ii) and (iii) in Lemma 2.3, all the untwisted loops in B_n are trivial loops, so all the loops in B_k are twisted loops. From item (i) in Lemma 2.3, any pair of twisted loops in B_k cannot be paralleled to each other, so its signed rotation must have the form (1). The lemma follows. □

Lemma 2.6. *If G is a bouquet with the signed rotation $a_1 \dots a_t(-a_1)\dots(-a_t)$, then*

$$^{\partial}\varepsilon_G(z) = 2z + (2^t - 2)z^2.$$

Proof. Let $X \subseteq E(G)$ and $x = |X|$. When $x = 0$ or $x = t$, $\varepsilon(G^X) = \varepsilon(G) = 1$. When $0 < x < t$, the signed rotations of both $G|_X$ and $G|_{X^C}$ have the form (1), then

$\varepsilon(G|_X) = \varepsilon(G|_{X^C}) = 1$. By Lemma 1.6, $\varepsilon(G^X) = 2$. Since there are $\binom{t}{x}$ ways to choose the x edges,

$$^{\partial}\varepsilon_G(z) = 2z + \sum_{x=1}^{t-1} \binom{t}{x} z^2 = 2z + (2^t - 2)z^2. \qquad \square$$

Theorem 2.7. *If B_n is a projective planar bouquet with k $(1 \leq k \leq n)$ twisted loops, then*

$$^{\partial}\varepsilon_{B_n}(z) = 2^{n-k}(2z + (2^k - 2)z^2).$$

Proof. From Lemma 2.5, the k twisted loops induce the maximum essential subgraph of B_n whose signed rotation has form (1). Combining it with Lemmas 1.5 and 2.6, the theorem follows. $\square$

We now consider the partial-dual Euler-genus polynomials for Klein bottle bouquets. Firstly, we deduce the form of the signed rotation of maximum essential subgraphs for Klein bottle bouquets. We use the model of disk sum of two projective planes for the Klein bottle. The fundamental group of Klein bottle is generated by two topologically disjoint orientation reversing loops a and b (called *twisted* in this paper). There are three non-separating loops a, b and d up to homeomorphism, where $d = ab$ is orientation preserving (called *untwisted* in this paper).

Lemma 2.8. *If B_n is a Klein bottle bouquet and B_k is the maximum essential subgraph of B_n, then the signed rotation of B_k has the form*

$$a_1 \dots a_i d_1 \dots d_q(-a_1) \dots (-a_i) b_1 \dots b_j d_q \dots d_1(-b_1) \dots (-b_j) \qquad (2)$$

in which $k = i + j + q$; and

(i) $i, j, q \geq 1$, *or*

(ii) $i, j \geq 1$ *and* $q = 0$, *or*

(iii) $i, q \geq 1$ *and* $j = 0$, *or*

(iv) $j, q \geq 1$ *and* $i = 0$.

Proof. It is routine check that the bouquet whose signed rotation with form (2) has Euler genus two. Notice that Case (iii) and Case (iv) are symmetric and we only need to consider Case (iii). We have the following observations:

(α): From Lemma 2.4(ii), there are no three twisted loops that are paralleled to each other, otherwise the Euler genus of B_k is at least three. And there is no subgraph B_3''' as shown in Figure 1(f), because $\varepsilon(B_3''') = 3$. Let B_k' be the ribbon subgraph induced by all the twisted loops in B_k, the signed rotation of B_k' is of form

$$a_1 \dots a_i(-a_1) \dots (-a_i) b_1 \dots b_j(-b_1) \dots (-b_j),$$

in which $i + j \geq 1$.

(β): From Lemma 2.3(iii), there exist no pairs of interlaced untwisted loops in a Klein bottle bouquet. Each untwisted loop must be interlaced with some twisted loops, otherwise the untwisted loop is trivial.

(γ): Each nontrivial and untwisted loop must be interlaced with all twisted loops, otherwise there exits a ribbon subgraph $\hat{B}$ whose signed rotation is $dad(-a)b(-b)$, and $\varepsilon(\hat{B}) = 3$.

Combining observations (α), (β), and (γ), the lemma follows. $\square$

According to the four cases listed in Lemma 2.8, the signed rotation of B_k (the maximum essential subgraph of a Klein bottle bouquet) has one of the following three forms,

$$a_1 \dots a_i(-a_1)\dots(-a_i)b_1 \dots b_j(-b_1)\dots(-b_j), \tag{3}$$

$$a_1 \dots a_i d_1 \dots d_q(-a_1)\dots(-a_i)d_q \dots d_1, \tag{4}$$

$$a_1 \dots a_i d_1 \dots d_q(-a_1)\dots(-a_i)b_1 \dots b_j d_q \dots d_1(-b_1)\dots(-b_j) \tag{5}$$

in which $k = i + j$, $i, j \geq 1$ in form (3); $k = i + q$, $i, q \geq 1$ in form (4); and $k = i + j + q$, $i, j, q \geq 1$ in form (5).

Next we compute the partial-dual Euler-genus polynomials for bouquets whose signed rotation has form (3), (4) and (5) respectively. Let $A = \{a_1, \dots, a_t\}, B = \{b_1, \dots, b_m\}$, $D = \{d_1, \dots, d_s\}$, $X \subseteq E(G)$ and $x = |X|$.

Lemma 2.9. *If* G *is a bouquet with the signed rotation*

$$a_1 \dots a_t(-a_1)\dots(-a_t)b_1 \dots b_m(-b_1)\dots(-b_m), (t, m \geq 1),$$

then

$$^{\partial}\varepsilon_G(z) = 4z^2 + 2(2^t + 2^m - 4)z^3 + (2^t - 2)(2^m - 2)z^4.$$

Proof. Let B_t and B_m be projective planar bouquets whose signed rotation have forms $a_1 \dots a_t(-a_1)\dots(-a_t)$ and $b_1 \dots b_m(-b_1)\dots(-b_m)$, respectively. Then we have $G = B_t \vee B_m$. From Lemmas 1.4 and 2.6, the result follows. □

Lemma 2.10. *If* G *is a bouquet with the signed rotation*

$$a_1 \dots a_t d_1 \dots d_s(-a_1)\dots(-a_t)d_s \dots d_1, (t, s \geq 1),$$

then

$$^{\partial}\varepsilon_G(z) = 2z + 2(2^s - 1)z^2 + 2(2^t - 2)z^3 + (2^t - 2)(2^s - 2)z^4.$$

Proof. When $x = 0$ or $t + s$, $\varepsilon(G^X) = \varepsilon(G) = 2$. When $0 < x < t + s$, we have the following three cases.

Case 1: Assume $X \subseteq A$. If $X = A$, then X consists of t twisted and pairwise interlaced loops and X^C consists of s untwisted and pairwise parallel loops. From Lemma 2.4, we have $\varepsilon(G|_X) = 1$ and $\varepsilon(G|_{X^C}) = 0$. Then from Lemma 1.6, we have $\varepsilon(G^X) = 1$.

If $X \subset A$, then the signed rotations of the induced ribbon subgraphs $G|_X$ and $G|_{X^C}$ have form (1) and (4) respectively, so we have $\varepsilon(G|_X) = 1$, $\varepsilon(G|_{X^C}) = 2$ and $\varepsilon(G^X) = 3$. There are $\sum_{x=1}^{t-1} \binom{t}{x} = 2^t - 2$ ways to choose the x edges. Hence in this case, we have one partial dual with Euler genus one, and $2^t - 2$ partial duals with Euler genus three.

Case 2: Assume $X \subseteq D$. The argument is similar to that for Case 1, by using Lemmas 2.4, 2.8 and 1.6, we get $\varepsilon(G|_X)$, $\varepsilon(G|_{X^C})$, and $\varepsilon(G^X)$. Then we calculate the number of ways to choose the x edges, as shown in Table 1.

Table 1: Euler genus distribution in Case 2.

X	$\varepsilon(G\|_X)$	$\varepsilon(G\|_{X^C})$	$\varepsilon(G^X)$	number
$X = D$	0	1	1	1
$X \subset D$	0	2	2	$2^s - 2$

Table 2: Euler genus distribution in Case 3.

X	$\varepsilon(G\|_X)$	$\varepsilon(G\|_{X^C})$	$\varepsilon(G^X)$	number
$A \subset X$ and $D \not\subset X$	2	0	2	$2^s - 2$
$D \subset X$ and $A \not\subset X$	2	1	3	$2^t - 2$
$A \not\subset X$ and $D \not\subset X$	2	2	4	$(2^t - 2)(2^s - 2)$

Case 3: Assume $X \cap A \neq \emptyset$ and $X \cap D \neq \emptyset$. We discuss three subcases as shown in Table 2, in which we use Lemmas 2.4, 2.5 and 2.8 to compute $\varepsilon(G|_X)$ and $\varepsilon(G|_{X^C})$.

Summarizing the above, we have ${}^{\partial}\varepsilon_G(z) = 2z + 2(2^s - 1)z^2 + 2(2^t - 2)z^3 + (2^t - 2)(2^s - 2)z^4$. □

Lemma 2.11. *If G is a bouquet with the signed rotation*

$$a_1 \dots a_t d_1 \dots d_s(-a_1) \dots (-a_t) b_1 \dots b_m d_s \dots d_1 (-b_1) \dots (-b_m), (t, m, s \geq 1),$$

then

$${}^{\partial}\varepsilon_G(z) = 2^{s+1}z^2 + (2^{t+1} + 2^{m+1} - 4)z^3 + (2^{t+m+s} - 2^{t+1} - 2^{m+1} - 2^{s+1} + 4)z^4.$$

Proof. We consider the Euler genus of G^X with the following three cases.
Case 1: Ribbons in X come from one of the three ribbon sets A, B and D. We discuss two subcases as shown in Table 3. And notice that, subcase $X \subseteq D$ includes $X = \emptyset$.

Table 3: Euler genus distribution in Case 1.

X	$\varepsilon(G\|_X)$	$\varepsilon(G\|_{X^C})$	$\varepsilon(G^X)$	number
$X \subseteq D$	0	2	2	2^s
$X \neq \emptyset$; and $X \subseteq A$ or $X \subseteq B$	1	2	3	$2^t + 2^m - 2$

Case 2: Ribbons in X come from two of the three ribbon sets A, B and D. We discuss three subcases: $X \subseteq A \cup B$, $X \subseteq A \cup D$ and $X \subseteq B \cup D$ respectively, as shown in Table 4.

Table 4: Euler genus distribution in Case 2.

X	$\varepsilon(G\|_X)$	$\varepsilon(G\|_{X^C})$	$\varepsilon(G^X)$	number
$X = A \cup B$	2	0	2	1
$X \subset A \cup B$	2	2	4	$(2^t - 1)(2^m - 1) - 1$
$X = A \cup D$ or $X = B \cup D$	2	1	3	2
$X \subset A \cup D$ or $X \subset B \cup D$	2	2	4	$(2^t - 1)(2^s - 1) + (2^m - 1)(2^s - 1) - 2$

Case 3: Ribbons in X come from all of three ribbon sets A, B and D. We discuss three subcases as shown in Table 5. Notice that, subcase $X^C \subset D$ includes $X^C = \emptyset$. And when ribbons in X^C come from at least two of A, B and D, the signed rotations of both $G|_X$

and $G|_{X^C}$ have form (2), thus $\varepsilon(G^X) = \varepsilon(G|_X) + \varepsilon(G|_{X^C}) = 2 + 2 = 4$. There are

$$\begin{aligned}
&\sum_{x=1}^{t-1}\binom{t}{x}\sum_{x=1}^{m-1}\binom{m}{x}\sum_{x=1}^{s-1}\binom{s}{x} + \sum_{x=1}^{m-1}\binom{m}{x}\sum_{x=1}^{s-1}\binom{s}{x} \\
&\qquad + \sum_{x=1}^{t-1}\binom{t}{x}\sum_{x=1}^{s-1}\binom{s}{x} + \sum_{x=1}^{t-1}\binom{t}{x}\sum_{x=1}^{m-1}\binom{m}{x} \\
&= (2^t-2)(2^m-2)(2^s-2) + (2^m-2)(2^s-2) \\
&\qquad + (2^t-2)(2^s-2) + (2^t-2)(2^m-2) \\
&= 2^{t+m+s} - 2^{t+m} - 2^{t+s} - 2^{m+s} + 4
\end{aligned}$$

ways to choose X.

Table 5: Euler genus distribution in Case 3.

X	$\varepsilon(G\vert_X)$	$\varepsilon(G\vert_{X^C})$	$\varepsilon(G^X)$	number
$X^C \subset D$	2	0	2	$2^s - 1$
$X^C \neq \emptyset$; and $X^C \subseteq A$ or $X^C \subseteq B$	2	1	3	$2^t + 2^m - 4$
Ribbons in X^C come from at least two of A, B and D	2	2	4	$2^{t+m+s} - 2^{t+m} - 2^{t+s} - 2^{m+s} + 4$

Summarizing the above, we have

$$\begin{aligned}
{}^{\partial}\varepsilon_G(z) &= (2^s + 1 + 2^s - 1)z^2 + (2^t + 2^m - 2 + 2 + 2^t + 2^m - 4)z^3 \\
&\quad + ((2^t-1)(2^m-1) + (2^t-1)(2^s-1) + (2^m-1)(2^s-1) - 3 \\
&\quad + 2^{t+m+s} - 2^{t+m} - 2^{t+s} - 2^{m+s} + 4)z^4 \\
&= 2^{s+1}z^2 + (2^{t+1} + 2^{m+1} - 4)z^3 \\
&\quad + (2^{t+m+s} - 2^{t+1} - 2^{m+1} - 2^{s+1} + 4)z^4.
\end{aligned}$$

□

From Lemmas 1.5, 2.9, 2.10 and 2.11 the following three theorems can be deduced.

Theorem 2.12. *If B_n is a Klein bottle bouquet and the signed rotation of its maximum essential subgraph is of form $a_1 \dots a_i(-a_1)\dots(-a_i)b_1 \dots b_j(-b_1)\dots(-b_j)$, $(i, j \geq 1)$, then*

$${}^{\partial}\varepsilon_{B_n}(z) = 2^{n-i-j}(4z^2 + 2(2^i + 2^j - 4)z^3 + (2^i - 2)(2^j - 2)z^4).$$

Theorem 2.13. *If B_n is a Klein bottle bouquet and the signed rotation of its maximum essential subgraph is of form $a_1 \dots a_i d_1 \dots d_q(-a_1)\dots(-a_i)d_q \dots d_1$, $(i, q \geq 1)$, then*

$${}^{\partial}\varepsilon_{B_n}(z) = 2^{n-i-q}(2z + 2(2^q - 1)z^2 + 2(2^i - 2)z^3 + (2^i - 2)(2^q - 2)z^4).$$

Theorem 2.14. *If B_n is a Klein bottle bouquet and the signed rotation of its maximum essential subgraph is of form*

$$a_1 \dots a_i d_1 \dots d_q(-a_1)\dots(-a_i)b_1 \dots b_j d_q \dots d_1(-b_1)\dots(-b_j), (i, j, q \geq 1),$$

then

$${}^{\partial}\varepsilon_{B_n}(z) = 2^{n-i-j-q}(2^{q+1}z^2 + (2^{i+1} + 2^{j+1} - 4)z^3 + (2^{i+j+q} - 2^{i+1} - 2^{j+1} - 2^{q+1} + 4)z^4).$$

3 Partial-dual orientable genus polynomials of plane bouquets and toroidal bouquets

In this section, we will deduce the partial-dual orientable genus polynomials of plane bouquets and toroidal bouquets, then their partial-dual Euler-genus polynomials follow.

Theorem 3.1. *If B_n is a plane bouquet, then* $^{\partial}\gamma_{B_n}(z) = {}^{\partial}\varepsilon_{B_n}(z) = 2^n$.

Proof. From Lemma 2.2, the maximum essential subgraph of any plane bouquet is a bouquet with no loop. Combining it with Lemmas 1.5 and 1.8, the theorem follows. □

For toroidal bouquets, in order to obtain the signed rotation of their maximum essential subgraphs, we will use the structure of the maximal nonhomotopic loop system of the torus. Assume x is a point on surface S. *A nonhomotopic loop system*, denoted by $\mathfrak{L} = \{l_i : i = 1, \ldots, t\}$, is a collection of nonocontractible loops with base point x such that l_i and l_j only intersect at x and are not homotopic to each other for $1 \leq i < j \leq t$. A nonhomotopic loop system $\mathfrak{L}$ is *maximal* if adding any noncontractible loop l with x as the base point to $\mathfrak{L}$ then l will either be homotopic to some loop l_i in $\mathfrak{L}$ or intersect l_i at some point other than x. Let $\rho(S) = \max\{|\mathfrak{L}|\}$, where $\mathfrak{L}$ is a maximal nonhomotopic loop system of S, $|\mathfrak{L}|$ is the number of loops in $\mathfrak{L}$, and the maximality is taken over all such system of S. For the torus, the following result can be found in both [6] (in Chapter 4) and [7].

Lemma 3.2 ([6, 7])**.** *If S_1 is the torus, then $\rho(S_1) = 3$.*

Lemma 3.3. *If B_n is a toroidal bouquet, then all the nontrivial loops can be partitioned into at least two homotopy classes and at most three homotopy classes. Furthermore, any pair of nonhomotopic nontrivial loops interlace mutually.*

Proof. For B_n, viewing it as an embedding of one-vertex graph on torus, all the nontrivial loops are nonocontractible and all the trivial loops are contractible. By Lemma 3.2 all the nontrivial loops of B_n can be partitioned into at most three homotopy classes. The fundamental group of torus is $Z \oplus Z$, generated by two elements which are not homotopic and intersect each other transversely, so all the nontrivial loops of B_n can be partitioned into at least two homotopy classes. On the torus, two nontrivial loops are homotopic if and only if they are homotopically disjoint. Therefore any pair of nonhomotopic and nontrivial loops interlace mutually. □

Lemma 3.4. *If B_n is a toroidal bouquet, then the signed rotation of its maximum essential subgraph has the form*

$$a_1 \ldots a_i b_1 \ldots b_j d_1 \ldots d_q a_i \ldots a_1 b_j \ldots b_1 d_q \ldots d_1, \tag{6}$$

in which $i, j \geq 1, q \geq 0$ and $i + j + q \leq n$.

Proof. From Lemma 3.3, loops in the maximum essential subgraph are partitioned into two or three homotopy classes, the loops in the same homotopy class are paralleled with each other, and loops from different homotopy classes are interlaced with each other, the theorem follows. □

In the following, we will discuss the partial dual distributions of toroidal bouquets in two cases depending on whether $q = 0$ or not in Lemma 3.4.

Lemma 3.5. *If G is a bouquet with the signed rotation*

$$a_1 \dots a_t b_1 \dots b_m a_t \dots a_1 b_m \dots b_1, (t, m \geq 1), \tag{7}$$

then

$$^{\partial}\gamma_G(z) = 2 + 2(2^t + 2^m - 3)z + (2^t - 2)(2^m - 2)z^2.$$

Proof. When $x = 0$ or $t + m$, $\gamma(G^X) = \gamma(G) = 1$. When $0 < x < t + m$, we have the following three cases as shown in Table 6, in which we get $\gamma(G|_X)$, $\gamma(G|_{X^C})$ by Lemmas 2.4 and 3.4, and get $\gamma(G^X)$ by Lemma 1.7.

Table 6: Orientable genus distribution when $0 < x < t + m$.

X	$\gamma(G\|_X)$	$\gamma(G\|_{X^C})$	$\gamma(G^X)$	number
$X = A$ or $X = B$	0	0	0	2
$X \subset A$, or $X \subset B$, or $X^C \subset A$, or $X^C \subset B$	0	1	1	$2\sum_{x=1}^{t-1}\binom{t}{x} + 2\sum_{x=1}^{m-1}\binom{m}{x} = 2(2^t + 2^m - 4)$
$X \cap A \neq \emptyset$, $X \cap B \neq \emptyset$, $X^C \cap A \neq \emptyset$, and $X^C \cap B \neq \emptyset$	1	1	2	$\sum_{x=1}^{t-1}\binom{t}{x}\sum_{x=1}^{m-1}\binom{m}{x} = (2^t - 2)(2^m - 2)$

Summarizing the above, we have $^{\partial}\gamma_G(z) = 2 + 2(2^t + 2^m - 3)z + (2^t - 2)(2^m - 2)z^2$. □

Lemma 3.6. *If G is a bouquet with the signed rotation*

$$a_1 \dots a_t b_1 \dots b_m d_1 \dots d_s a_t \dots a_1 b_m \dots b_1 d_s \dots d_1, (t, m, s \geq 1), \tag{8}$$

then

$$^{\partial}\gamma_G(z) = 2(2^t + 2^m + 2^s - 2)z + (2^{t+m+s} - 2^{t+1} - 2^{m+1} - 2^{s+1} + 4)z^2.$$

Proof. We will consider the orientable genus of G^X into the following three cases.
Case 1: Ribbons in X come from one of the three ribbon sets A, B and D.
Case 2: Ribbons in X come from two of the three ribbon sets A, B and D.
In this case, we could only consider that ribbons in X come from both A and B, by symmetry. Then we have the following two subcases, i.e.,

Subcase 2a: $X = A \cup B$ and **Subcase 2b**: $X \neq A \cup B$.

Case 3: Ribbons in X come from all of the three ribbon sets A, B and D.
In this case, we have the following two subcase, i.e.,

Subcase 3a: $X^C \subset A$, or B, or D and **Subcase 3b**: Otherwise.

In each case, we discuss $\gamma(G|_X)$, $\gamma(G|_{X^C})$ and $\gamma(G^X)$, and compute the number of ways to choose X, as shown in Table 7.

Notice that, in Case 1, because the case $X = \emptyset$ has been counted three times in the three summations, we subtract 2 from the counting. So we have $2^t + 2^m + 2^s - 2$ partial duals with orientable genus one in this case. In Case 2, by symmetry, we have three partial duals with orientable genus one and

$$\begin{aligned}&(2^t - 1)(2^m - 1) - 1 + (2^t - 1)(2^s - 1) - 1 + (2^m - 1)(2^s - 1) - 1\\ &= 2^{t+m} + 2^{m+s} + 2^{t+s} - 2^{t+1} - 2^{m+1} - 2^{s+1}\end{aligned}$$

Table 7: Some orientable genus distribution of G^X.

X	$\gamma(G\|_X)$	$\gamma(G\|_{X^C})$	$\gamma(G^X)$	number
Case 1	0	1	1	$\sum_{x=0}^{t}\binom{t}{x}+\sum_{x=0}^{m}\binom{m}{x}+\sum_{x=0}^{s}\binom{s}{x}-2 = 2^t+2^m+2^s-2$
Subcase 2a	1	0	1	1
Subcase 2b	1	1	2	$\sum_{x=1}^{t}\binom{t}{x}\sum_{x=1}^{m}\binom{m}{x}-1 = (2^t-1)(2^m-1)-1$
Subcase 3a	1	0	1	$\sum_{x=1}^{t-1}\binom{t}{x}+\sum_{x=1}^{m-1}\binom{m}{x}+\sum_{x=1}^{s-1}\binom{s}{x}+1 = 2^t+2^m+2^s-5$
Subcase 3b	1	1	2	$2^{t+m+s}-2^{t+m}-2^{t+s}-2^{m+s}+4$

partial duals with orientable genus two. In Subcase 3a, notice that, we haven't include $X^C=\emptyset$ in any of the three summations, so we add one to the counting. In Subcase 3b, there are

$$\begin{aligned}
&\sum_{x=1}^{t-1}\binom{t}{x}\sum_{x=1}^{m-1}\binom{m}{x}\sum_{x=1}^{s-1}\binom{s}{x}+\sum_{x=1}^{m-1}\binom{m}{x}\sum_{x=1}^{s-1}\binom{s}{x}\\
&\qquad\qquad+\sum_{x=1}^{t-1}\binom{t}{x}\sum_{x=1}^{s-1}\binom{s}{x}+\sum_{x=1}^{t-1}\binom{t}{x}\sum_{x=1}^{m-1}\binom{m}{x}\\
&=(2^t-2)(2^m-2)(2^s-2)+(2^m-2)(2^s-2)\\
&\qquad\qquad+(2^t-2)(2^s-2)+(2^t-2)(2^m-2)\\
&=2^{t+m+s}-2^{t+m}-2^{t+s}-2^{m+s}+4
\end{aligned}$$

ways to choose the x edges. Hence, in Case 3, we have $2^t+2^m+2^s-5$ partial duals with orientable genus one and $2^{t+m+s}-2^{t+m}-2^{t+s}-2^{m+s}+4$ partial duals with orientable genus two.

Summarizing the above, we have

$$\begin{aligned}
{}^{\partial}\gamma_G(z) &= (2^t+2^m+2^s-2+3+2^t+2^m+2^s-5)z\\
&\quad+(2^{t+m}+2^{m+s}+2^{t+s}-2^{t+1}-2^{m+1}\\
&\quad-2^{s+1}+2^{t+m+s}-2^{t+m}-2^{t+s}-2^{m+s}+4)z^2\\
&=2(2^t+2^m+2^s-2)z+(2^{t+m+s}-2^{t+1}-2^{m+1}-2^{s+1}+4)z^2. \qquad \square
\end{aligned}$$

From Lemmas 1.5, 1.8, 3.5 and 3.6, the following two theorems can be deduced.

Theorem 3.7. *If B_n is a toroidal bouquet and the signed rotation of its maximum essential subgraph has the form $a_1\dots a_ib_1\dots b_ja_i\dots a_1b_j\dots b_1$ in which $i,j\geq 1$ and $i+j\leq n$, then*

$${}^{\partial}\gamma_{B_n}(z)=2^{n-i-j}(2+2(2^i+2^j-3)z+(2^i-2)(2^j-2)z^2),$$

and

$${}^{\partial}\varepsilon_{B_n}(z)=2^{n-i-j}(2+2(2^i+2^j-3)z^2+(2^i-2)(2^j-2)z^4).$$

Theorem 3.8. *If B_n is a toroidal bouquet and the signed rotation of its maximum essential subgraph has the form $a_1 \dots a_i b_1 \dots b_j d_1 \dots d_q a_i \dots a_1 b_j \dots b_1 d_q \dots d_1$, in which $i, j, q \geq 1$ and $i + j + q \leq n$, then*

$$^{\partial}\gamma_{B_n}(z) = 2^{n-i-j-q}(2(2^i + 2^j + 2^q - 2)z + (2^{i+j+q} - 2^{i+1} - 2^{j+1} - 2^{q+1} + 4)z^2),$$

and

$$^{\partial}\varepsilon_{B_n}(z) = 2^{n-i-j-q}(2(2^i + 2^j + 2^q - 2)z^2 + (2^{i+j+q} - 2^{i+1} - 2^{j+1} - 2^{q+1} + 4)z^4).$$

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The (non-)existence of perfect codes in Lucas cubes*

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Abstract

The *Fibonacci cube* of dimension n, denoted as Γ_n, is the subgraph of the n-cube Q_n induced by vertices with no consecutive 1's. Ashrafi and his co-authors proved the non-existence of perfect codes in Γ_n for $n \geq 4$. As an open problem the authors suggest to consider the existence of perfect codes in generalizations of Fibonacci cubes. The most direct generalization is the family $\Gamma_n(1^s)$ of subgraphs induced by strings without 1^s as a substring where $s \geq 2$ is a given integer. In a precedent work we proved the existence of a perfect code in $\Gamma_n(1^s)$ for $n = 2^p - 1$ and $s \geq 3.2^{p-2}$ for any integer $p \geq 2$. The Lucas cube Λ_n is obtained from Γ_n by removing vertices that start and end with 1. Very often the same problems are studied on Fibonacci cubes and Lucas cube. In this note we prove the non-existence of perfect codes in Λ_n for $n \geq 4$ and prove the existence of perfect codes in some generalized Lucas cube $\Lambda_n(1^s)$.

Keywords: Error correcting codes, perfect code, Fibonacci cube.

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1 Introduction and notations

An interconnection topology can be represented by a graph $G = (V, E)$, where V denotes the processors and E the communication links. The hypercube Q_n is a popular interconnection network because of its structural properties.

The Fibonacci cube was introduced in [8] as a new interconnection network. This graph is an isometric subgraph of the hypercube which is inspired in the Fibonacci numbers. It has attractive recurrent structures such as its decomposition into two subgraphs which are also Fibonacci cubes by themselves. Structural properties of these graphs were more extensively studied afterwards. See [12] for a survey.

*Footnote on the title.

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Lucas cubes, introduced in [17], have attracted the attention as well due to the fact that these cubes are closely related to the Fibonacci cubes. They have also been widely studied [3, 4, 6, 11, 13, 14, 18, 20, 23].

We will next define some concepts needed in this paper. Let G be a connected graph. The *open neighbourhood* of a vertex u is $N_G(u)$ the set of vertices adjacent to u. The *closed neighbourhood* of u is $N_G[u] = N_G(u) \cup \{u\}$. The *distance* between two vertices denoted $d_G(x, y)$ is the length of a shortest path between x and y. We have thus $N_G[u] = \{v \in V(G); d_G(u, v) \leq 1\}$. We will use the notations $d(x, y)$ and $N[u]$ when the graph is unambiguous.

A *dominating set* D of G is a set of vertices such that every vertex of G belongs to the closed neighbourhood of at least one vertex of D. In [2], Biggs initiated the study of perfect codes in graphs as a generalization of classical 1-error perfect correcting codes. A *code* C in G is a set of vertices C such that for every pair of distinct vertices c, c' of C we have $N_G[c] \cap N_G[c'] = \emptyset$ or equivalently such that $d_G(c, c') \geq 3$.

A *perfect code* of a graph G is both a dominating set and a code. It is thus a set of vertices C such that every vertex of G belongs to the closed neighbourhood of exactly one vertex of C. A perfect code is also known as an efficient dominating set. The existence or non-existence of perfect codes have been considered for many graphs. See the introduction of [1] for some references.

The vertex set of the *n-cube* Q_n is the set $\mathbb{B}_n$ of binary strings of length n, two vertices being adjacent if they differ in precisely one position. Classical 1-error correcting codes and perfect codes are codes and perfect codes in the graph Q_n. The *weight* of a binary string is the number of 1s. The concatenation of strings $\boldsymbol{x}$ and $\boldsymbol{y}$ is denoted $\boldsymbol{x}||\boldsymbol{y}$ or just $\boldsymbol{xy}$ when there is no ambiguity. A string $\boldsymbol{f}$ is a *substring* of a string $\boldsymbol{s}$ if there exist strings $\boldsymbol{x}$ and $\boldsymbol{y}$, may be empty, such that $\boldsymbol{s} = \boldsymbol{xfy}$.

A *Fibonacci string* of length n is a binary string $\boldsymbol{b} = b_1 \ldots b_n$ with $b_i \cdot b_{i+1} = 0$ for $1 \leq i < n$. In other words a Fibonacci string is a binary string without 11 as substring. The *Fibonacci cube* Γ_n ($n \geq 1$) is the subgraph of Q_n induced by the Fibonacci strings of length n. Adjacent vertices in Γ_n differ in one bit. Because of the empty string, $\Gamma_0 = K_1$.

A Fibonacci string of length n is a *Lucas string* if $b_1 \cdot b_n \neq 1$. That is, a Lucas string has no two consecutive 1's including the first and the last elements of the string. The *Lucas cube* Λ_n is the subgraph of Q_n induced by the Lucas strings of length n. We have $\Lambda_0 = \Lambda_1 = K_1$.

Let $\mathcal{F}_n$ and $\mathcal{L}_n$ be the set of strings of Fibonacci strings and Lucas strings of length n.

By $\Gamma_{n,k}$ and $\Lambda_{n,k}$ we denote the vertices of of weight k in respectively Γ_n and Λ_n.

Since

$$\mathcal{L}_n = \{0\boldsymbol{s}; \boldsymbol{s} \in \mathcal{F}_{n-1}\} \cup \{10\boldsymbol{s}0; \boldsymbol{s} \in \mathcal{F}_{n-3}\}$$

and

$$|\Gamma_{n,k}| = \binom{n-k+1}{k}$$

it is immediate to derive the following classical result.

Proposition 1.1. *Let $n \geq 1$. The number of vertices of weight $k \leq n$ in Λ_n is*

$$|\Lambda_{n,k}| = \binom{n-k}{k} + \binom{n-k-1}{k-1}.$$

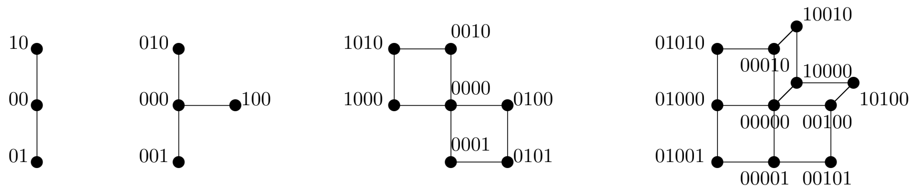

Figure 1: $\Gamma_2 = \Lambda_2$, Λ_3, Λ_4 and Λ_5.

It will be convenient to consider the binary strings of length n as vectors of $\mathbb{F}^n$ the vector space of dimension n over the field $\mathbb{F} = \mathbb{Z}_2$ thus to associate to a string $x_1x_2 \dots x_n$ the vector $\theta(x_1x_2 \dots x_n) = (x_1, x_2, \dots, x_n)$. The *Hamming distance* between two vectors $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{F}^n$, $d(\boldsymbol{x}, \boldsymbol{y})$ is the number of coordinates in which they differ. By the correspondence θ we can define the binary sum $\boldsymbol{x} + \boldsymbol{y}$ and the Hamming distance $d(\boldsymbol{x}, \boldsymbol{y})$ of strings in $\mathbb{B}_n$. Note that the Hamming distance is the usual graph distance in Q_n.

We will first recall some basic results about perfect codes in Q_n. Since Q_n is a regular graph of degree n the existence of a perfect code of cardinality $|C|$ implies $|C|(n+1) = 2^n$ thus a necessary condition of existence is that $n + 1$ is a power of 2 thus that $n = 2^p - 1$ for some integer p.

For any integer p Hamming [7] constructed, a linear subspace of $\mathbb{F}^{2^p-1}$ which is a perfect code. It is easy to prove that all linear perfect codes are Hamming codes. Notice that 1^n belongs to the Hamming code of length n.

In 1961 Vasilev [22], and later many authors, see [5, 21] for a survey, constructed perfect codes which are not linear codes.

In a recent work [1] Ashrafi and his co-authors proved the non-existence of perfect codes in Γ_n for $n \geq 4$. As an open problem the authors suggest to consider the existence of perfect codes in generalizations of Fibonacci cubes. The most complete generalization proposed in [9] is, for a given string $\boldsymbol{f}$, to consider $\Gamma_n(\boldsymbol{f})$ the subgraph of Q_n induced by strings that do not contain $\boldsymbol{f}$ as substring. Since Fibonacci cubes are $\Gamma_n(11)$ the most immediate generalization [15, 19] is to consider $\Gamma_n(1^s)$ for a given integer s. In [16] we proved the existence of a perfect code in $\Gamma_n(1^s)$ for $n = 2^p - 1$ and $s \geq 3.2^{p-2}$ for any integer $p \geq 2$.

In the next section we will prove the main result of this note.

Theorem 1.2. *The Lucas cube $\Lambda_n, n \geq 0$, admits a perfect code if and only if $n \leq 3$.*

2 Perfect codes in Lucas cube

It can be easily checked by hand that $\{0^n\}$ is a perfect code of Λ_n for $n \leq 3$ and that Λ_4 and Λ_5 do not contain a perfect code (Figure 1).

Assume thus $n \geq 6$.

Note first that from Proposition 1.1 we have

$$|\Lambda_{n,2}| = \frac{n(n-3)}{2} \text{ and } |\Lambda_{n,3}| = \frac{n(n-4)(n-5)}{6}.$$

Therefore $\Lambda_{n,2}$ and $\Lambda_{n,3}$ are none empty.

Let $\Lambda^1_{n,k}$ be the vertices of $\Lambda_{n,k}$ that start with 1. Since $\mathcal{L}_n = \{0\boldsymbol{s}; \boldsymbol{s} \in \mathcal{F}_{n-1}\} \cup \{10\boldsymbol{s}0; \boldsymbol{s} \in \mathcal{F}_{n-3}\}$ the number of vertices in $\Lambda^1_{n,k}$ is

$$|\Lambda^1_{n,k}| = |\Gamma_{n-3,k-1}| = \binom{n-1-k}{k-1}.$$

Lemma 2.1. *If $n \geq 6$ and C is a perfect code of Λ_n then $0^n \in C$.*

Proof. Suppose on the contrary that $0^n \notin C$. Since 0^n must be dominated there exists a vertex in $\Lambda_{n,1} \cap C$. This vertex is unique and because of the circular symmetry of Λ_n we can assume $10^{n-1} \in C$.

Since $0^n \notin C$ the other vertices of $\Lambda_{n,1}$ must be dominated by vertices in $\Lambda_{n,2}$. But a vertex in $\Lambda_{n,2}$ has precisely two neighbors in $\Lambda_{n,1}$ thus n must be odd and

$$|\Lambda_{n,2} \cap C| = \frac{n-1}{2}.$$

The unique vertex 10^{n-1} in $\Lambda_{n,1} \cap C$ has exactly $n-3$ neighbors in $\Lambda_{n,2}$. Let D be the vertices of $\Lambda_{n,2}$ not in C and not dominated by 10^{n-1}. Vertices in D must be dominated by vertices in $\Lambda_{n,3} \cap C$. Each vertex of $\Lambda_{n,3} \cap C$ has exactly exacty three neighbors in $\Lambda_{n,2}$. Thus 3 divides the number of vertices in D. This number is

$$|D| = |\Lambda_{n,2}| - (n-3) - \frac{n-1}{2} = \frac{n^2 - 6n + 7}{2}.$$

This is not possible since there exists no odd integer n such that 6 divides $n^2 + 1$. Indeed since n is odd, 6 does not divide n thus divides $(n+1)(n-1) = n^2 - 1$ or $(n+2)(n-2) = n^2 - 4$ or $(n+3)(n-3) = n^2 - 9$ thus cannot divide $n^2 + 1$. □

We are now going to prove Theorem 1.2.

Proof of Theorem 1.2. Let $n \geq 6$ and C be a perfect code. Since $0^n \in C$ all vertices of $\Lambda_{n,1}$ are dominated by 0^n and thus $\Lambda_{n,2} \cap C = \Lambda_{n,1} \cap C = \emptyset$. Consequently, each vertex of $\Lambda_{n,2}$ must be dominated by a vertex in $\Lambda_{n,3}$. Since each vertex in $\Lambda_{n,3}$ has precisely three neighbors in $\Lambda_{n,2}$ we obtain that

$$|\Lambda_{n,3} \cap C| = \frac{|\Lambda_{n,2}|}{3}.$$

This number must be an integer thus 3 divides $|\Lambda_{n,2}| = \frac{n(n-3)}{2}$ and therefore 3 divides $n(n-3)$. This is only possible if n is a multiple of 3.

Each vertex of $\Lambda^1_{n,2}$ must be dominated by a vertex in $\Lambda^1_{n,3}$. Furthermore a vertex in $\Lambda^1_{n,3}$ has precisely two neighbors in $\Lambda^1_{n,2}$. Therefore $|\Lambda^1_{n,2}| = n - 3$ must be even and thus $n = 6p + 3$ for some integer $p \geq 1$.

Let E be the set of vertices of $\Lambda_{n,3}$ not in C. Vertices in E must be dominated by a vertex in $\Lambda_{n,4}$. Furthermore each vertex in $\Lambda_{n,4}$ has precisely four neighbors in $\Lambda_{n,3}$.

Therefore 4 divides $|E|$ with

$$|E| = |\Lambda_{n,3}| - |\Lambda_{n,3} \cap C| = \frac{n(n-4)(n-5)}{6} - \frac{n(n-3)}{6} = \frac{n(n^2 - 10n + 23)}{6}.$$

Replacing n by $6p + 3$ we obtain that 4 divides the odd number $(2p+1)(18p^2 - 12p + 1)$. This contradiction proves Theorem 1.2. □

3 Perfect codes in generalized Lucas cube

The analogous of the generalisation of Fibonacci cube $\Gamma_n(1^s)$ for Lucas cube is the family $\Lambda_n(1^s)$ of subgraphs of Q_n induced by strings without 1^s as a substring in a circular manner where $s \geq 2$ is a given integer. More formally [10] for any binary strings $b_1b_2\ldots b_n$ and each $1 \leq i \leq n$, call $b_ib_{i+1}\ldots b_nb_1\ldots b_{i-1}$ the *i-th circulation* of $b_1b_2\ldots b_n$. The *generalized Lucas cube* $\Lambda_n(1^s)$ is the subgraph of Q_n induced by strings without a circulation containing 1^s as a substring.

In [16] the existence of a perfect code in $\Gamma_n(1^s)$ is proved for $n = 2^p - 1$ and $s \geq 3.2^{p-2}$ for any integer $p \geq 2$.

The strategy used in this construction is to build a perfect code C in Q_n such that no vertex of C contains 1^s as substring. The set C is also a perfect code in $\Gamma_n(1^s)$ since each vertex of $\Gamma_n(1^s)$ belongs to the unique closed neighbourhood in Q_n thus in $\Gamma_n(1^s)$ of a vertex in C. Because of the following proposition we cannot use the same idea for $\Lambda_n(1^s)$ and $s \leq n-1$.

Proposition 3.1. *Let n be an integer and $2 \leq s \leq n-1$. If C is a perfect code in Q_n then some word of C contains a circulation of 1^s as a substring.*

Proof. Let C be a such a perfect code in Q_n then $1^n \notin C$. Thus 1^n must be neighbour of a vertex c in C. Since $c = 1^i01^{n-1-i}$ for some integer i the $i+1$th-circulation of c is $1^{n-1}0$. □

We can supplement this proposition by the two following results.

Proposition 3.2. *Let $p \geq 2$ and $n = 2^p - 1$ then there exists a perfect code in $\Lambda_n(1^n)$ of order $|C| = \frac{2^n}{n+1}$.*

Proof. Let D be a Hamming code of length n and $C = \{d + (0^{n-1}1); d \in D\}$. Since $1^n \in D$, the set C is a perfect code of Q_n such that $1^n \notin C$. Since $\Lambda_n(1^n)$ is obtained from Q_n by the deletion of 1^n every vertex of $\Lambda_n(1^n)$ is in the closed neighbourhood of exactly one vertex of C. □

Proposition 3.3. *Let $p \geq 2$ and $n = 2^p - 1$ then there exists a perfect code in $\Lambda_n(1^{n-1})$ and in $\Lambda_n(1^{n-2})$ of order $|C| = \frac{2^n}{n+1} - 1$.*

Proof. Let D be a Hamming code of length n. Then D is a perfect code of Q_n such that $1^n \in D$. Since $\Lambda_n(1^{n-1})$ is obtained from Q_n by the deletion of the closed neighbourhood of 1^n every vertex of $\Lambda_n(1^{n-1})$ is in the closed neighbourhood of exactly one vertex of $C = D - \{1^n\}$. Furthermore since $1^n \in D$ there is no vertex of weight $n-2$ in D. Let u be a vertex of $\Lambda_n(1^{n-2})$ and $f(u)$ be the vertex in D such that $u \in N_{Q^n}[u]$. Since there is no vertex in D with weight $n-1$ or $n-2$ there is no circulation of $f(u)$ containing 1^{n-2} as a substring. Therefore $f(u)$ is a vertex of $\Lambda_n(1^{n-2})$ and $u \in N_{\Lambda_n(1^{n-2})}[f(u)]$. Since a code in Q_n is a code in each of its subgraph C is a perfect code of $\Lambda_n(1^{n-2})$. □

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