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Independent coalition in graphs: existence and characterization

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Abstract

An independent coalition in a graph G consists of two disjoint sets of vertices V_1 and V_2 neither of which is an independent dominating set but whose union $V_1 \cup V_2$ is an independent dominating set. An independent coalition partition, abbreviated, ic-partition, in a graph G is a vertex partition $\pi = \{V_1, V_2, \dots, V_k\}$ such that each set V_i of π either is a singleton dominating set, or is not an independent dominating set but forms an independent coalition with another set $V_j \in \pi$. The maximum number of classes of an ic-partition of G is the independent coalition number of G, denoted by $IC(G)$. In this paper, we study the concept of ic-partition. In particular, we discuss the possibility of the existence of ic-partitions in graphs and introduce a family of graphs for which no ic-partition exists. We also determine the independent coalition number of some classes of graphs and investigate graphs G of order n with $IC(G) \in \{1, 2, 3, 4, n\}$ and the trees T of order n with $IC(T) = n - 1$.

Keywords: Independent coalition, independent coalition partition, independent dominating set, idomatic partition.

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1 Introduction

Let $G = (V, E)$ denote a simple graph of order n with vertex set $V = V(G)$ and edge set $E = E(G)$. The *open neighborhood* of a vertex $v \in V$ is the set $N(v) = \{u | \{u, v\} \in E\}$, and its *closed neighborhood* is the set $N[v] = N(v) \cup \{v\}$. Each vertex of $N(v)$ is called a *neighbor* of v, and the cardinality of $N(v)$ is called the *degree* of v, denoted by $\deg(v)$ or

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$\deg_G(v)$. A vertex v of degree 1 is called a *pendant vertex* or *leaf*, and its neighbor is called a *support vertex*. A vertex of degree $n-1$ is called a *full vertex* while a vertex of degree 0 is called an *isolated vertex*. The *minimum* and *maximum degree* of G are denoted by $\delta(G)$ and $\Delta(G)$, respectively. For a set S of vertices of G, the subgraph induced by S is denoted by $G[S]$. For two sets X and Y of vertices, let $[X, Y]$ denote the set of edges between X and Y. If every vertex of X is adjacent to every vertex of Y, we say that $[X, Y]$ is *full*, while if there are no edges between them, we say that $[X, Y]$ is *empty*. A subset $V_i \subseteq V$ is called a *singleton set* if $|V_i| = 1$, and is called a *non-singleton set* if $|V_i| \geq 2$. The *join* $G + H$ of two disjoint graphs G and H is the graph obtained from the union of G and H by adding every possible edge between the vertices of G and the vertices of H. We denote the family of paths, cycles, complete graphs and stars of order n by P_n, C_n, K_n and $K_{1,n-1}$, respectively, and the complete k-partite graph with partite sets of order $n_1, n_2, \ldots, n_k$, by $K_{n_1,\ldots,n_k}$. A double star with respectively p and q leaves connected to each support vertex is denoted by $S_{p,q}$. The complete graph K_3 is called a *triangle*, and a graph is *triangle-free* if it has no K_3 as an induced subgraph. The *girth* of a graph with a cycle is the length of its shortest cycle. For a graph G, the girth of G is denoted by $g(G)$. For a graph G of order n, let $\overline{G}$ denote the complement of G with $V(\overline{G}) = V(G)$ and $E(\overline{G}) = E(K_n) - E(G)$ [13].

A set $S \subseteq V$ in a graph $G = (V, E)$ is called a *dominating set* if every vertex $v \in V$ is either an element of S or is adjacent to an element of S. A set $S \subseteq V$ is called an *independent set* if its vertices are pairwise nonadjacent. The *vertex independence number*, denoted by $\alpha(G)$, is the maximum cardinality of an independent set of G. An *independent dominating set* in a graph G is a set which is both independent and dominating.

A partition of the vertices of G into dominating sets (independent dominating sets) is called a domatic partition (idomatic partition). The maximum number of classes of a domatic partition (idomatic partition) of G is called the domatic number (idomatic number) of G, denoted by $d(G)$ ($id(G)$). The concepts of domination and domatic partition and their variations have been studied widely in the literature. See, for example, [1, 2, 3, 4, 5, 6, 12].

The term *coalition* was introduced by Haynes et al, [7] and has been studied further in [8, 9, 10, 11].

Definition 1.1 ([7]). A coalition in a graph G consists of two disjoint sets of vertices $V_1, V_2 \subset V$, neither of which is a dominating set but whose union $V_1 \cup V_2$ is a dominating set. We say that the sets V_1 and V_2 form a coalition, and are *coalition partners*.

Definition 1.2 ([7]). A *coalition partition*, henceforth called a *c-partition*, in a graph G is a *vertex partition* $\pi = \{V_1, V_2, \ldots, V_k\}$ such that every set V_i of π is either a *singleton dominating set*, or is not a dominating set but forms a coalition with another set V_j in π. The *coalition number* $C(G)$ equals the maximum order k of a c-partition of G, and a c-partition of G having order $C(G)$ is called a *$C(G)$-partition*.

Herein we will focus on coalitions involving independent dominating sets in graphs. In other words, we will study the concepts of independent coalition and independent coalition partition which have been introduced in [7] as an area for future research. We begin with the following definitions.

Definition 1.3. An *independent coalition* in a graph G consists of two disjoint sets of independent vertices V_1 and V_2, neither of which is an independent dominating set but whose union $V_1 \cup V_2$ is an independent dominating set. We say the sets V_1 and V_2 form an independent coalition, and are *independent coalition partners* (or *ic-partners*).

Definition 1.4. An *independent coalition partition*, abbreviated *ic-partition*, in a graph G is a vertex partition $\pi = \{V_1, V_2, \ldots, V_k\}$ such that every set V_i of π is either a singleton dominating set, or is not an independent dominating set but forms an independent coalition with another set $V_j \in \pi$. The *independent coalition number* $IC(G)$ equals the maximum number of classes of an ic-partition of G, and an ic-partition of G having order $IC(G)$ is called an *$IC(G)$-partition*.

Definition 1.5 ([8]). Let G be a graph of order n with vertex set $V = \{v_1, v_2, \ldots, v_n\}$. The *singleton partition*, denoted π_1, of G is the partition of V into n singleton sets, that is, $\pi_1 = \{\{v_1\}, \{v_2\}, \ldots, \{v_n\}\}$.

This paper is organized as follows. Section 1 is devoted to terminology and definitions. We discuss the possibility of the existence of ic-partitions in graphs and derive some bounds on independent coalition number in Section 2. In Section 3, we determine the independent coalition number of some classes of graphs. The graphs G with $IC(G) \in \{1, 2, 3, 4\}$ are investigated in Section 4. In Section 5, we characterize triangle-free graphs G with $IC(G) = n$ and trees T with $IC(T) = n - 1$. Finally, we end the paper with some research problems.

2 Independent coalition partition: existence and bound

This section is divided into two subsections. In the first subsection, we show that not all graphs admit an ic-partition, and in the second subsection, we present some bounds on $IC(G)$ whenever the graph G admits an ic-partition.

2.1 Existence

In the following definition, we construct graphs with arbitrarily large order for which no ic-partition exists.

Definition 2.1. Let $\mathcal{B}$ be the set of all graphs obtained from the complete graph K_n, $(n \geq 4)$ with the vertices v_i, $(1 \leq i \leq n)$, and two additional vertices v_{n+1}, v_{n+2} such that v_{n+1} and v_{n+2} are adjacent to v_n, and v_{n+1} is adjacent to v_{n-1}. Figure 1 illustrates such a graph for $n = 4$.

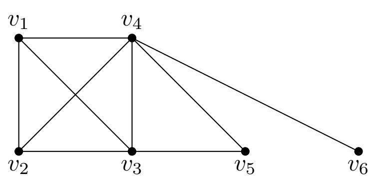

Figure 1: The graph G in $\mathcal{B}$ for $n = 4$.

Proposition 2.2. *Let G be a graph. If $G \in \mathcal{B}$, then G has no ic-partition.*

Proof. Suppose, to the contrary, that G has an ic-partition π. The vertices $v_1, v_2, \ldots, v_{n-1}$ are pairwise adjacent, so they must be in different classes. Further, v_n is a full vertex, so it must be in a singleton class. Since v_{n-1} is adjacent to all vertices except v_{n+2}, and

$\{v_{n-1}, v_{n+2}\}$ dominates G, it follows that $\{v_{n-1}\} \in \pi$. Further, since $\{v_{n-1}\}$ can only form an independent coalition with $\{v_{n+2}\}$, it follows that $\{v_{n+2}\} \in \pi$. If $\{v_{n+1}\} \in \pi$, then π is a singleton partition. In this case, $\{v_{n+1}\}$ has no ic-partner, a contradiction. Hence, $\{v_{n+1}\} \notin \pi$. It follows that π consists of a non-singleton set $\{v_{n+1}, v_i\}$ such that $v_i \in \{v_1, v_2, \ldots, v_{n-2}\}$, and n singleton sets. Assume, without loss of generality, that $\{v_{n+1}, v_1\} \in \pi$. Now for each $2 \le i \le n-2$, the set $\{v_i\}$ has no ic-partner, a contradiction. □

2.2 Bounds

Definition 1.4 implies that an ic-partition of a graph G is also a c-partition. Further, we note that an ic-partition of G is a proper coloring as well. Hence, we have the following two sharp bounds on $IC(G)$. To see the sharpness of them, consider the complete graph K_n.

Observation 2.3. Let G be a graph. If G has an ic-partition, then $IC(G) \le C(G)$. Furthermore, this bound is sharp.

Observation 2.4. Let G be a graph. If G has an ic-partition, then $IC(G) \ge \chi(G)$. Furthermore, this bound is sharp.

Given a connected graph G and an ic-partition π of it, the following theorem shows that each set in π admits at most $\Delta(G)$ ic-partners.

Theorem 2.5. *Let G be a connected graph with maximum degree $\Delta(G)$, and let π be an ic-partition of G. If $X \in \pi$, then X is in at most $\Delta(G)$ independent coalitions. Furthermore, this bound is sharp.*

Proof. Let π be an ic-partition of G, and let X be a set in π. If X is a dominating set, then it has no ic-partner. Hence, we may assume that X does not dominate G. Let x be a vertex that is not dominated by X. Now every ic-partner of X must dominate x, that is, it must contain a vertex in $N[x]$. Hence, there are at most $|N[x]| \le \Delta(G) + 1$ sets in π that can form an independent coalition with X. Now we show that X cannot form an independent coalition with $\Delta(G) + 1$ sets. Suppose, to the contrary, that X has $\Delta(G) + 1$ ic-partners (name $V_1, V_2, \ldots, V_{\Delta+1}$). Consequently, $[X, V_i]$ is empty for each $1 \le i \le \Delta(G) + 1$. Let $U = \bigcup_{i=1}^{\Delta(G)+1} V_i$, and $G' = G[U]$. Consider an arbitrary vertex $v \in U$ (say $v \in V_i$) and an arbitrary set V_j such that $1 \le j \le \Delta(G) + 1$ and $j \ne i$. Since $X \cup V_j$ dominates G and $[X, V_i]$ is empty, it follows that v has a neighbor in V_j. Choosing V_j arbitrarily, we conclude that $\deg_{G'}(v) \ge \Delta(G)$, and so $\deg_{G'}(v) = \Delta(G)$. Hence, for each $v \in U$, we have $\deg_{G'}(v) = \Delta(G)$. Now since G is connected, there is a path $P = (v_0, v_1, \ldots, v_k)$ connecting U to X such that $v_0 \in U$ and $v_k \in X$. Note that $[U, X]$ is empty, and so $V(P) \setminus (U \cup X) \ne \emptyset$. Let i be the smallest index for which $v_i \notin U \cup X$. It follows that $v_{i-1} \in U$, and so $\deg_{G'}(v_{i-1}) = \Delta(G)$. Thus, we have $\deg_G(v_{i-1}) \ge \deg_{G'}(v_{i-1}) + 1 = \Delta(G) + 1$, a contradiction.

To prove the sharpness, let G be the graph that is obtained from the complete graph K_n with vertices v_i, $(1 \le i \le n)$, and a path $P_2 = (a, b)$, where b is adjacent to v_1. Let $A = \{a\}$, $B = \{b\}$ and $V_i = \{v_i\}$, for $1 \le i \le n$. One can observe that $\Delta(G) = n$ and that the singleton partition $\pi_1 = \{V_1, V_2, \ldots, V_n, A, B\}$ is an ic-partition of G such that A forms an independent coalition with V_i, for each $1 \le i \le n$. This completes the proof. □

Note that the bound presented in Theorem 2.5 does not hold for disconnected graphs. As a counterexample, consider the graph $G = K_2 \cup K_2$ and the singleton partition π_1 of it. On can verify that π_1 is an ic-partition of G such that each set in π_1 has two ic-partners, while $\Delta(G) = 1$.

The next bound relates independent coalition number of a graph to its idomatic number. As we will see in the proof of Theorem 2.6, any graph admitting an idomatic partition has an ic-partition. However, the converse is not necessarily true. For example, the singleton partition of the cycle C_5 is an ic-partition of it, while C_5 has no idomatic partition. Or the cycle C_7 has the ic-partition $\pi = \{\{v_1, v_5\}, \{v_2, v_4\}, \{v_3\}, \{v_6\}, \{v_7\}\}$, while it has no idomatic partition.

Theorem 2.6. *Let G be a connected graph, and let $r \geq 0$ be the number of full vertices of G. If G admits an idomatic partition, then $IC(G) \geq 2id(G) - r$.*

Proof. Let $F = \{v_1, v_2, \ldots, v_r\}$ be the set of full vertices of G, and let $\pi = \{V_1, V_2, \ldots, V_{id(G)}\}$ be an idomatic partition of G of order $id(G)$. Note that each full vertex must be in a singleton set of π. Without loss of generality, assume that $v_i \in V_i$, for each $1 \leq i \leq r$. It follows that for each $r + 1 \leq i \leq k$, we have $|V_i| \geq 2$. Now for each $r + 1 \leq i \leq k$, we partition V_i into two nonempty subsets $V_{i,1}$ and $V_{i,2}$. Note that no proper subset of V_i is a dominating set. Thus, neither $V_{i,1}$ nor $V_{i,2}$ is an independent dominating set, and so $V_{i,1}$ and $V_{i,2}$ are ic-partners. It follows that the partition $\pi' = \{V_1, V_2, \ldots, V_r, V_{r+1,1}, V_{r+1,2}, V_{r+2,1}, V_{r+2,2}, \ldots, V_{id(G),1}, V_{id(G),2}\}$ is an ic-partition of G of order $2id(G) - r$. Hence, $IC(G) \geq 2id(G) - r$. □

3 Independent coalition number for some classes of graphs

Let us begin this section with some routine results.

Observation 3.1. For $n \geq 1$, we have $IC(K_n) = n$.

Observation 3.2. For $n \geq 3$, we have $IC(K_{1,n-1}) = 3$.

Observation 3.3. For $p, q \geq 1$, we have $IC(S_{p,q}) = 4$.

For complete multipartite graphs, the following result is obtained.

Proposition 3.4. *Let $G = K_{n_1,n_2,\ldots,n_k}$ be a complete k-partite graph with $m \geq 0$ full vertices (m partite sets of cardinality 1). Then $IC(G) = 2k - m$.*

Proof. Let $\pi = \{V_1, V_2, \ldots, V_k\}$ be the partition of G into its partite sets. Assume, without loss of generality, that the sets V_i, for $1 \leq i \leq m$, are those containing full vertices. Now for each $m + 1 \leq i \leq k$, we partition V_i into two sets $V_{i,1}$ and $V_{i,2}$. Observe that $V_{i,1}$ and $V_{i,2}$ are ic-partners, and so the partition $\pi' = \{\{V_1\}, \{V_2\}, \ldots, \{V_m\}, \{V_{m+1,1}, V_{m+1,2}\}, \{V_{m+2,1}, V_{m+2,2}\}, \ldots, \{V_{k,1}, V_{k,2}\}\}$ is an ic-partition of G of order $2k - m$. Thus, $IC(G) \geq 2k - m$. Now let π'' be an ic-partition of G. We note that π'' has the following properties:

- For any set $S \in \pi''$, all vertices in S are in the same partite set of G.
- For any set $V_i \in \pi$, the vertices in V_i are in at most two sets of π''.

Hence, we have $IC(G) \leq 2k - m$, and so $IC(G) = 2k - m$. □

Next we determine the independent coalition number of all paths and cycles.

Lemma 3.5 ([7]). *For any path P_n, $C(P_n) \le 6$.*

Theorem 3.6. *For the path P_n,*

$$IC(P_n) = \begin{cases} n & \text{if } n \le 4; \\ 4 & \text{if } n = 5; \\ 5 & \text{if } n = 6, 7, 8, 9; \\ 6 & \text{if } n \ge 10. \end{cases}$$

Proof. It is clear that for $1 \le n \le 4$, we have $IC(P_n) = n$. Now let $n = 5$. Consider the path P_5 with $V(P_5) = \{v_1, v_2, v_3, v_4, v_5\}$. It is easily seen that $IC(P_5) \ne 5$. Thus, $IC(P_5) \le 4$. The partition $\{\{v_1, v_3\}, \{v_2\}, \{v_4\}, \{v_5\}\}$ is an ic-partition of P_5, so $IC(P_5) = 4$. Now assume $n = 6$. Consider the path P_6 with $V(P_6) = \{v_1, v_2, v_3, v_4, v_5, v_6\}$. It is clear that $IC(P_6) \ne 6$. The partition $\{\{v_1, v_6\}, \{v_2\}, \{v_3\}, \{v_4\}, \{v_5\}\}$ is an ic-partition of P_6, so $IC(P_6) = 5$. Next assume $n = 7$. Consider the path P_7 with $V(P_7) = \{v_1, v_2, v_3, v_4, v_5, v_6, v_7\}$. By Lemma 3.5 and Observation 2.3, we have $IC(P_n) \le 6$. Now we show that $IC(P_7) \ne 6$. Suppose that $IC(P_7) = 6$. Let π be an $IC(P_7)$-partition. We note that π consists of a set (name A) of cardinality 2 and five singleton sets. Since $\gamma_i(P_7) = 3$, each singleton set must be an ic-partner of A. On the other hand, Theorem 2.5 implies that A has at most two ic-partners, a contradiction. The partition $\{\{v_1, v_6\}, \{v_2, v_7\}, \{v_3\}, \{v_4\}, \{v_5\}\}$ is an ic-partition of P_7. Therefore, $IC(P_7) = 5$. Next we assume $n = 8$. Consider the path P_8 with $V(P_8) = \{v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8\}$. By Lemma 3.5 and Observation 2.3, we have $IC(P_n) \le 6$. Now we show that $IC(P_8) \ne 6$. Suppose that $IC(P_8) = 6$. Let π be an $IC(P_8)$-partition. We consider two cases.

Case 1. π consists of a set (name A) of cardinality 3 and five singleton sets. Since $\gamma_i(P_8) = 3$, each singleton set must be an ic-partner of A. On the other hand, Theorem 2.5 implies that A has at most two ic-partners, a contradiction.

Case 2. π consists of two sets of cardinality 2 and four singleton sets. Since $\gamma_i(P_8) = 3$, each singleton set must be an ic-partner of a set of cardinality 2. Therefore, using Theorem 2.5, we deduce that for any two ic-partners C and D, it holds that $|C \cup D| = 3$. On the other hand, v_3 and v_6 are not present in any independent dominating set of cardinality 3, a contradiction.

The partition $\{\{v_1, v_3, v_6\}, \{v_2, v_7\}, \{v_8\}, \{v_4\}, \{v_5\}\}$ is an ic-partition of P_8. Therefore, $IC(P_8) = 5$.

Now let $n = 9$. Consider the path P_9 with $V(P_9) = \{v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8, v_9\}$. By Lemma 3.5 and Observation 2.3, we have $IC(P_n) \le 6$. Now we show that $IC(P_9) \ne 6$. Suppose that $IC(P_9) = 6$. Let π be an $IC(P_9)$-partition. There exist three cases.

Case 1. π consists of a set (name A) of cardinality 4 and five singleton sets. Since $\gamma_i(P_9) = 3$, each singleton set must be an ic-partner of A. On the other hand, by Theorem 2.5, A has at most two ic-partners, a contradiction.

Case 2. π consists of a set (name A) of cardinality 3, a set (name B) of cardinality 2 and four singleton sets. Since $\gamma_i(P_9) = 3$, no two singleton sets in π are ic-partners. Furthermore, by Theorem 2.5, A has at most two ic-partners, so at least two singleton sets of π must be ic-partners of B, which is impossible, as P_9 has a unique independent dominating set of cardinality 3.

Case 3. π consists of three sets of cardinality 2, and three singleton sets. We note that each singleton set in π must be an ic-partner of a set of cardinality 2, which is impossible, as P_9 has a unique independent dominating set of cardinality 3.

The partition $\{\{v_1, v_3, v_5\}, \{v_2, v_4, v_9\}, \{v_6\}, \{v_7\}, \{v_8\}\}$ is an ic-partition of P_9. Therefore, $IC(P_9) = 5$.

Finally, let $n \geq 10$. Consider the path P_n with $V(P_n) = \{v_1, v_2, \ldots, v_n\}$. By Lemma 3.5 and Observation 2.3, we have $IC(P_n) \leq 6$. Now we consider the sets $V_1 = \{v_1, v_6\} \cup \{v_{2n-1} : n \geq 5\}$, $V_2 = \{v_2, v_5\} \cup \{v_{2n} : n \geq 5\}$, $V_3 = \{v_3\}$, $V_4 = \{v_4\}$, $V_5 = \{v_7\}$, $V_6 = \{v_8\}$. Then $\pi = \{V_1, V_2, V_3, V_4, V_5, V_6\}$ is an ic-partition of P_n, where V_3 and V_4 are ic-partners of V_1, and V_5 and V_6 are ic-partners of V_2. So the proof is complete. □

Lemma 3.7 ([7]). *For any cycle C_n, $C(C_n) \leq 6$.*

Lemma 3.7 and Observation 2.3 imply the following result.

Lemma 3.8. *For any cycle C_n, $IC(C_n) \leq 6$.*

Lemma 3.9. *For any cycle C_n with $n \geq 8$ and $n \equiv 0 \pmod 2$, it holds that $IC(C_n) = 6$.*

Proof. Let $V(C_n) = \{v_1, v_2, \ldots, v_n\}$. Consider the sets $V_1 = \{v_1, v_6\} \cup \{v_{2n-1} : n \geq 5\}$, $V_2 = \{v_2, v_5\} \cup \{v_{2n} : n \geq 5\}$, $V_3 = \{v_3\}$, $V_4 = \{v_4\}$, $V_5 = \{v_7\}$, $V_6 = \{v_8\}$. Then $\pi = \{V_1, V_2, V_3, V_4, V_5, V_6\}$ is an ic-partition of C_n, for $n \geq 8$, where V_3 and V_4 are ic-partners of V_1, and V_5 and V_6 are ic-partners of V_2. Hence, by Lemma 3.8 and Observation 2.3, we have $IC(C_n) = 6$. □

Lemma 3.10. *For any cycle C_n with $n \geq 8$ and $n \equiv 0 \pmod 3$, it holds that $IC(C_n) = 6$.*

Proof. Let $V(C_n) = \{v_1, v_2, \ldots, v_{3k}\}$. Consider the sets $V_1 = \{v_{3i+1}\}$, $V_2 = \{v_{3i+2}\}$ and $V_3 = \{v_{3i+3}\}$, for $0 \leq i \leq k-1$. Now for each $1 \leq i \leq 3$, we partition V_i into two nonempty sets $V_{i,1}$ and $V_{i,2}$. Observe that $V_{i,1}$ and $V_{i,2}$ are ic-partners. Hence, by Lemma 3.8 and Observation 2.3, we have $IC(C_n) = 6$. □

Lemma 3.11. *For any cycle C_n with $n \geq 8$ and $n \equiv 5 \pmod 6$, it holds that $IC(C_n) = 6$.*

Proof. Assume $n = 6k - 1$, $(k \geq 2)$. Let $V(C_n) = \{v_1, v_2, \ldots, v_n\}$. Consider the sets

$$A = \bigcup_{i=0}^{k-1} \{v_{3i+1}\},\ A_1 = \bigcup_{i=k}^{2k-1} \{v_{3i+1}\},\ A_2 = \bigcup_{i=k}^{2k-1} \{v_{3i}\},$$

$$B = \bigcup_{i=k}^{2k} \{v_{3i-1}\},\ B_1 = \bigcup_{i=1}^{k-1} \{v_{3i-1}\},\ B_2 = \bigcup_{i=1}^{k-1} \{v_{3i}\}.$$

Let $\pi = \{A, A_1, A_2, B, B_1, B_2\}$. One can observe that π is an ic-partition of C_n, where A_1 and A_2 are ic-partners of A, and B_1 and B_2 are ic-partners of B. Now using Lemma 3.8 and Observation 2.3, we have $IC(C_n) = 6$. □

Lemma 3.12. *For any cycle C_n with $n \geq 8$ and $n \equiv 1 \pmod 6$, it holds that $IC(C_n) = 6$.*

Proof. Assume $n = 6k + 1$, $(k \geq 2)$. Let $V(C_n) = \{v_1, v_2, \ldots, v_n\}$. Consider the sets

$$A = \left(\bigcup_{i=0}^{k}\{v_{3i+1}\}\right) \cup \{v_{3k+3}\},\ A_1 = \bigcup_{i=k+2}^{2k}\{v_{3i}\},\ A_2 = \bigcup_{i=k+2}^{2k}\{v_{3i-1}\},$$

$$B = \left(\bigcup_{i=k+1}^{2k}\{v_{3i+1}\}\right) \cup \{v_{3k+2}\},\ B_1 = \bigcup_{i=1}^{k}\{v_{3i-1}\},\ B_2 = \bigcup_{i=1}^{k}\{v_{3i}\}.$$

Let $\pi = \{A, A_1, A_2, B, B_1, B_2\}$. One can observe that π is an ic-partition of C_n, where A_1 and A_2 are ic-partners of A, and B_1 and B_2 are ic-partners of B. Now using Lemma 3.8 and Observation 2.3, we have $IC(C_n) = 6$. □

Theorem 3.13. *For the cycle C_n,*

$$IC(C_n) = \begin{cases} n & \text{if } n \leq 6; \\ 5 & \text{if } n = 7; \\ 6 & \text{if } n \geq 8. \end{cases}$$

Proof. If $1 \leq n \leq 6$, then it is easy to check that $IC(C_n) = n$. Now assume $n = 7$. Consider the cycle C_7 with $V(C_7) = \{v_1, v_2, v_3, v_4, v_5, v_6, v_7\}$. First we show that $IC(C_7) \neq 6$. Suppose, to the contrary, that $IC(C_7) = 6$. Let π be an $IC(C_7)$-partition. We note that π consists of five singleton sets and a set of cardinality 2 (name A). By Theorem 2.5, A has at most two ic-partners. Hence, π contains two singleton sets that are ic-partners, which contradicts the fact that $\gamma_i(C_7) = 3$. The partition $\{\{v_1, v_3\}, \{v_5\}, \{v_6\}, \{v_4, v_7\}, \{v_2\}\}$ is an ic-partition of C_7, so $IC(C_7) = 5$. Furthermore, by Lemmas 3.9, 3.10, 3.11 and 3.12 we have $IC(C_n) = 6$, for $n \geq 8$. □

4 Graphs with small independent coalition number

In this section we investigate graphs G with $IC(G) \in \{1, 2, 3, 4\}$. We will make use of the following two lemmas.

Lemma 4.1. *Let G be a graph of order n containing $r \geq 1$ full vertices, and let $F = \{v_1, v_2, \ldots, v_r\}$ be the set of full vertices of G. Then $IC(G) = k$, if and only if $IC(G[V \setminus F]) = k - r$, where $r < k \leq n$.*

Proof. Assume first that $IC(G[V \setminus F]) = k - r$. Let $\pi = \{V_1, V_2, \ldots, V_{k-r}\}$ be an $IC(G[V \setminus F])$-partition. Now the partition $\pi' = \{V_1, V_2, \ldots, V_{k-r}, \{v_1\}, \{v_2\}, \ldots, \{v_r\}\}$, is an ic-partition of G, so $IC(G) \geq k$. Now we prove that $IC(G) = k$. Suppose, to the contrary, that $IC(G) > k$. Let π be an $IC(G)$-partition. Now the partition $\pi' = \pi \setminus \{\{v_1\}, \{v_2\}, \ldots, \{v_r\}\}$ is an ic-partition of $G[V \setminus F]$ such that $|\pi'| > k - r$, a contradiction. Hence, $IC(G) = k$. Conversely, assume that $IC(G) = k$. Let π be an $IC(G)$-partition. Now the partition $\pi' = \pi \setminus \{\{v_1\}, \{v_2\}, \ldots, \{v_r\}\}$ is an ic-partition of $G[V \setminus F]$, so $IC(G[V \setminus F]) \geq k - r$. Now we prove that $IC(G[V \setminus F]) = k - r$. Suppose, to the contrary, that $IC(G[V \setminus F]) > k - r$. Let π be an $IC(G[V \setminus F])$-partition. Now the partition $\pi' = \pi \cup \{\{v_1\}, \{v_2\}, \ldots, \{v_r\}\}$ is an ic-partition of G such that $|\pi'| > k$, a contradiction. Hence, $IC(G[V \setminus F]) = k - r$. □

Lemma 4.2. *Let G be a graph containing a nonempty set of isolated vertices I. If $IC(G) \geq 3$, then for any $IC(G)$-partition π, there is a set $V_r \in \pi$ such that $V_r = I$.*

Proof. First we show that all vertices in I are in the same set of π. Suppose, to the contrary, that there are sets $V_i \in \pi$ and $V_j \in \pi$ such that both V_i and V_j contain isolated vertices. Let $V_k \in \pi$ be an arbitrary set in π such that $V_k \notin \{V_i, V_j\}$. (Since $IC(G) \geq 3$, such a set exists). Then V_k has no ic-partner, a contradiction. Now let V_r be the set in π containing isolated vertices. Further, let v be an arbitrary vertex in V_r, and let $u \in V(G)$ be an arbitrary vertex such that $u \neq v$. If $u \in V_r$, then u is not adjacent to v. Otherwise, the set in π containing u is an ic-partner of V_r, which again implies that u is not adjacent to v. Hence, we have $\deg(v) = 0$. Choosing v arbitrarily, we conclude that $V_r = I$. □

Proposition 4.3. *Let G be a graph of order n. Then*

(1) *$IC(G) = 1$ if and only if $G \simeq K_1$.*

(2) *$IC(G) = 2$ if and only if $G \simeq K_2$ or $G \simeq \overline{K}_n$, for some $n \geq 2$.*

Proof. (1) It is clear that $IC(G) = 1$ if and only if $G \simeq K_1$.
(2) If $G \simeq K_2$, then we clearly have $IC(G) = 2$. Now assume $G \simeq \overline{K}_n$, for some $n \geq 2$. Let π be an ic-partition of G. Note that no more than two sets in π contain isolated vertices, for otherwise, no two sets in π are ic-partners. Thus, $|\pi| \leq 2$. Partitioning vertices of G into two nonempty sets yields an ic-partition of G. Hence, $IC(G) = 2$. Conversely, suppose that $IC(G) = 2$. Let $\pi = \{V_1, V_2\}$ be an $IC(G)$-partition. If both V_1 and V_2 are singleton dominating sets, then $G \simeq K_2$. Hence, we may assume that at least one of them (say V_1) is not a singleton dominating set. It follows that V_2 is not a singleton dominating set either, for otherwise, G is a star, and so by Observation 3.2, we have $IC(G) = 3$. Hence, V_1 and V_2 are ic-partners, and so $V = V_1 \cup V_2$ is an independent set. Hence, $G \simeq \overline{K}_n$, for some $n \geq 2$. □

Definition 4.4. Let $\mathcal{B}_1$ represent the family of bipartite graphs H with partite sets H_1 and H_2 such that $|H_1| \geq 2$, $|H_2| \geq 2$, $\delta(H) \geq 1$ and $id(H) = 2$.

Definition 4.5. For $m \geq 1$, let $\mathcal{B}_2$ represent the family of graphs $H \cup \overline{K}_m$, where H is a bipartite graph with $\delta(H) \geq 1$ and $id(H) = 2$.

Definition 4.6. For $m \geq 1$, let $\mathcal{B}_3$ represent the family of graphs $H \cup \overline{K}_m$, where H is a 3-partite graph with $\delta(H) \geq 1$ and $id(H) = 3$.

Proposition 4.7. *Let G be a graph. Then $IC(G) = 3$, if and only if $G \in \{K_3, K_{1,n-1}\} \cup \mathcal{B}_2$.*

Proof. Observations 3.1 and 3.2 imply that $IC(K_3) = 3$ and that $IC(K_{1,n-1}) = 3$, respectively. Now let $G \in \mathcal{B}_2$. Let I be the set of isolated vertices of G, and let $\{H_1, H_2\}$ be a partition of $G - I$ into its partite sets. We observe that the partition $\{I, H_1, H_2\}$ is an ic-partition of G, so $IC(G) \geq 3$. Now we show that $IC(G) = 3$. Suppose, to the contrary, that $IC(G) \geq 4$. Let $\pi = \{V_1, V_2, \ldots, V_k\}$ be an $IC(G)$-partition. By Lemma 4.2, we have $I \in \{V_1, V_2, \ldots, V_k\}$. Assume, without loss of generality, that $I = V_1$. Now for each $2 \leq i \leq k$, V_i forms an independent coalition with V_1, and so V_i dominates H. Hence, the partition $\{V_2, \ldots, V_k\}$ is an idomatic partition of H, which contradicts the assumption. Hence, $IC(G) = 3$. Conversely, let G be a graph with $IC(G) = 3$, and let

$\pi = \{V_1, V_2, V_3\}$ be an $IC(G)$-partition. We consider four cases depending on the number of full vertices of G.

Case 1. G has three full vertices. In this case, the sets V_1, V_2 and V_3 are all singleton dominating sets, so $G \simeq K_3$.

Case 2. G has two full vertices. Note that this case never occurs.

Case 3. G has one full vertex. Let v_1 be the full vertex of G. Lemma 4.1 implies that $IC(G - v_1) = 2$. Thus, by Proposition 4.3, either $G - v_1 \simeq K_2$, implying that $G \simeq K_3$, or $G - v_1 \simeq \overline{K}_n$, for some $n \geq 2$, which implies that $G \simeq K_{1,n-1}$, for some $n \geq 3$.

Case 4. G has no full vertex. Let I be the set of isolated vertices of G. First we note that V_1, V_2 and V_3 are not pairwise ic-partners, for otherwise, we have $G \simeq \overline{K}_n$, and so by Proposition 4.3, we have $IC(G) = 2$, a contradiction. Hence, π contains a set (say V_1) that forms an independent coalition with V_2 and V_3, while V_2 and V_3 are not ic-partners. Therefore, each vertex in V_1 is an isolated vertex, so it follows from Lemma 4.2 that $I = V_1$. Further, the sets V_2 and V_3 are independent dominating sets of $G[V_2 \cup V_3]$, implying that $id(G[V_2 \cup V_3]) \geq 2$. It remains to show that $id(G[V_2 \cup V_3]) = 2$. Suppose, to the contrary, that $id(G[V_2 \cup V_3]) \geq 3$. Let $\pi' = \{U_1, U_2, \ldots, U_k\}$, ($k \geq 3$), be an idomatic partition of $G[V_2 \cup V_3]$. Then the partition $\pi'' = \{U_1, U_2, \ldots, U_k, V_1\}$ is clearly an ic-partition of G, implying that $IC(G) \geq 4$, a contradiction. Hence, $G \in \mathcal{B}_2$. □

Proposition 4.8. *Let G be a graph. If $IC(G) = 4$, then $G \in \{K_4, K_2 + \overline{K}_n, K_1 + B\} \cup \mathcal{B}_1 \cup \mathcal{B}_3$, where $n \geq 2$ and $B \in \mathcal{B}_2$.*

Proof. Let $\pi = \{V_1, V_2, V_3, V_4\}$ be an ic-partition of G. We consider two cases.

Case 1. G has a full vertex. Let v_1 be a full vertex of G. Lemma 4.1 implies that $IC(G - v_1) = 3$. Thus, by Proposition 4.7, we have $G - v_1 \simeq K_3$, implying that $G \simeq K_4$, or $G - v_1 \simeq K_{1,n}$, for some $n \geq 2$, implying that $G \simeq K_2 + \overline{K}_n$, for some $n \geq 2$, or $G - v_1 \in \mathcal{B}_2$, which implies that $G \simeq K_1 + B$, where $B \in \mathcal{B}_2$.

Case 2. G has no full vertex. First assume that G contains a nonempty set I of isolated vertices. Then by Lemma 4.2, we have $I \in \pi$. Without loss of generality, assume that $I = V_4$. Now for each $1 \leq i \leq 3$, V_i must form an independent coalition with I. Thus, $U = G[V_1 \cup V_2 \cup V_3]$ is a 3-partite graph with $id(U) \geq 3$. Since $IC(G) = 4$, the case $id(U) > 3$ is impossible. Hence, $id(U) = 3$, and so $G \in \mathcal{B}_3$. Now assume that G contains no isolated vertex. Since G has neither full vertices nor isolated vertices, each set of π has either one or two ic-partners. If there is a set of π, (say V_1) having one ic-partner, (say V_2), then it follows that V_3 and V_4 are ic-partners, and so G is a bipartitie graph with partite sets $V_1 \cup V_2$ and $V_3 \cup V_4$. Otherwise, assume, without loss of generality, that V_2 and V_3 are ic-partners of V_1. It follows that V_4 has an ic-partner in $\{V_2, V_3\}$. By symmetry, we may assume that V_4 and V_3 are ic-partners. Then G is again a bipartitie graph with partite sets $V_1 \cup V_2$ and $V_3 \cup V_4$. Now using Theorem 2.6, we have $id(G) = 2$, and so $G \in \mathcal{B}_1$. □

5 Graphs with large independent coalition number

Our main goal in this section is to investigate structure of graphs G of order n with $IC(G) = n$, under specified conditions. In addition, we will characterize all trees T of order n with $IC(T) = n - 1$. Let us begin with an observation that characterizes all disconnected graphs G of order n with $IC(G) = n$.

Observation 5.1. Let G be a disconnected graph of order n. Then $IC(G) = n$ if and only if $G \simeq K_s \cup K_r$, for some $s \geq 1$, and $r \geq 1$.

Now we introduce two sufficient conditions for a graph G of order n to have independent coalition number n.

Observation 5.2. If G is a graph of order n with $\alpha(G) = 2$, then $IC(G) = n$.

Proof. Let G be a graph of order n such that $\alpha(G) = 2$. Consider the singleton partition π_1 of G. Note that for any two non-adjacent vertices v and u in $V(G)$, the sets $\{v\}$ and $\{u\}$ in π_1, are ic-partners. Hence, π_1 is an ic-partition of G, and so $IC(G) = n$. □

Observation 5.3. Let G be a graph of order n. If G admits a partition of its vertices into two maximal cliques, then $IC(G) = n$.

5.1 Graphs G with $\delta(G) = 1$ and $IC(G) = n$

In this subsection, we characterize graphs G of order n with $\delta(G) = 1$ and $IC(G) = n$. We need the following definition.

Definition 5.4. Let G be a graph of order n, $(n \geq 3)$, and let $\delta(G) = 1$. Furthermore, let x be a pendant vertex of G, and let y be the support vertex of x. Then $G \in \mathcal{F}$ if and only if $V(G) \setminus \{x, y\}$ induces a clique.

Theorem 5.5. *Let G be a graph of order n with $\delta(G) = 1$. Then $IC(G) = n$ if and only if either $G \simeq K_2$, or $G \in \mathcal{F}$.*

Proof. Obviously, $IC(K_2) = 2$. Now assume that $G \in \mathcal{F}$. Let x be a pendant vertex of G, and let y be the support vertex of x. Further, let $U = V(G) \setminus \{x, y\}$. Note that U contains no full vertex. If y is a full vertex, then G is obtained from the complete graph K_{n-1}, where one of its vertices is adjacent to a leaf. In this case, we clearly have $IC(G) = n$. Thus, we may assume that y is not a full vertex, that is, there is a vertex $u \in U$ such that u is not adjacent to y. Then it is easy to verify that the sets $\{y\}$ and $\{u\}$ are ic-partners, and that each vertex in $U \setminus \{u\}$ forms an independent coalition with $\{x\}$. Therefore, $IC(G) = n$. Conversely, suppose that G is a graph with $\delta(G) = 1$ and $IC(G) = n$. Let x be a leaf of G, and let y be the support vertex of x. Consider the singleton partition π_1 of G. Note that each set in $\pi_1 \setminus \{\{x\}, \{y\}\}$ must be an ic-partner of $\{x\}$ or $\{y\}$, to dominate x. Let $A = N(y) \setminus \{x\}$, and $B = V(G) \setminus (\{x, y\} \cup A)$. We consider four cases.

Case 1. $A = \emptyset$ and $B = \emptyset$. In this case, we have $G \simeq K_2$.

Case 2. $A = \emptyset$ and $B \neq \emptyset$. By Observation 5.1, we have $G \simeq K_2 \cup K_r$, for some $r \geq 1$. Thus, $G \in \mathcal{F}$.

Case 3. $A \neq \emptyset$ and $B = \emptyset$. For each $v \in A$, the set $\{v\}$ cannot be an ic-partner of $\{y\}$, so it must be an ic-partner of $\{x\}$. This implies that A induces a clique. Hence, $G \in \mathcal{F}$.

Case 4. $A \neq \emptyset$ and $B \neq \emptyset$. For each $v \in A$, the set $\{v\}$ cannot be an ic-partner of $\{y\}$, so it must be an ic-partner of $\{x\}$. This implies that $[A, B]$ is full and that A induces a clique. Now for each vertex $u \in B$, in order for the set $\{u\}$ to be an ic-partner of $\{x\}$ or $\{y\}$, u must be adjacent to all other vertices in B. Hence, B induces a clique, and so $G \in \mathcal{F}$, which completes the proof. □

As an immediate result from Theorem 5.5 we have:

Corollary 5.6. *Let T be a tree of order n. Then $IC(T) = n$ if and only if $T \in \{P_1, P_2, P_3, P_4\}$.*

5.2 Triangle-free graphs G with $IC(G) = n$

In this subsection, we characterize graphs G of order n with $g(G) = 4$ and $IC(G) = n$. This will lead to characterization of all triangle-free graphs G of order n with $IC(G) = n$. We will make use the following lemmas.

Lemma 5.7. *Let G be a triangle-free graph of order n with $IC(G) = n$. Then $g(G) \leq 6$.*

Proof. Let G be a graph of order n with $IC(G) = n$, and suppose, to the contrary, that $g(G) \geq 7$. Let $C \subseteq G$ be a cycle of order $g(G)$. Consider an arbitrary vertex $v \in V(C)$. Note that $\gamma_i(C) \geq 3$, and so $\{v\}$ is not an ic-partner of any set $\{u\} \subset V(C)$. Therefore, it must be an ic-partner of a set $\{u\} \subseteq V(G) \setminus V(C)$. It follows that, $\{u\}$ dominates $V(C) \setminus N_c[v]$, which implies that G contains triangles, a contradiction. □

Lemma 5.8. *Let G be a graph of order n with $g(G) = 6$. Then $IC(G) = n$ if and only if $G \simeq C_6$.*

Proof. Let G be a graph of order n with $g(G) = 6$. If $G \simeq C_6$, then by Theorem 3.13, we have $IC(G) = 6$. Conversely, assume that $IC(G) = n$. Let $C \subseteq G$ be a cycle of order 6, and suppose, to the contrary, that $V(G) \setminus V(C) \neq \emptyset$. Consider an arbitrary vertex $v \in V(G) \setminus V(C)$. If $\{v\}$ is an ic-partner of a set $\{u\} \subset V(C)$, then $\{v\}$ must dominate $V(C) \setminus N_c[u]$, which implies that G contains triangles, a contradiction. Otherwise, $\{v\}$ must be an ic-partner of a set $\{u\} \subset V(G) \setminus V(C)$. Now since $\{u, v\}$ dominates C, it follows that G contains triangles, or induces cycles of order 4, a contradiction. □

Our next result can be established almost the same way as Lemma 5.8, so we state it without proof.

Lemma 5.9. *Let G be a graph of order n with $g(G) = 5$. Then $IC(G) = n$ if and only if $G \simeq C_5$.*

In order to characterize graphs G of order n with $IC(G) = n$ and $g(G) = 4$, we need the following definitions.

Definition 5.10. Let $\mathcal{K}_0$ represent a bipartite graph with partite sets $H_1 = \{v_1, v_2, v_3, v_4\}$ and $H_2 = \{u_1, u_2, u_3, u_4\}$ such that for each $1 \leq i \leq 4$, v_i is adjacent to all vertices in H_2, except u_i (see Figure 2).

Definition 5.11. Let $\mathcal{K}$ represent a family of 4-partite graphs with partite sets $H_1 = \{v_1, v_2, v_3, v_4\}$, $H_2 = \{u_1, u_2, u_3, u_4\}$, $H_3 = \{n_1, n_2, \ldots, n_k\}$ and $H_4 = \{m_1, m_2, \ldots, m_k\}$, for $k \geq 1$, with the following properties:

- $[H_1, H_3]$ is full and $[H_2, H_4]$ is full,
- $[H_1, H_4]$ is empty and $[H_2, H_3]$ is empty,
- For each $1 \leq i \leq 4$, v_i is adjacent to all vertices in H_2, except u_i,
- For each $1 \leq i \leq k$, n_i is adjacent to all vertices in H_4, except m_i.

Figure 3 illustrates such a graph for $k = 3$.

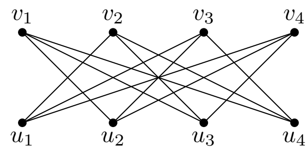

Figure 2: The graph $\mathcal{K}_0$.

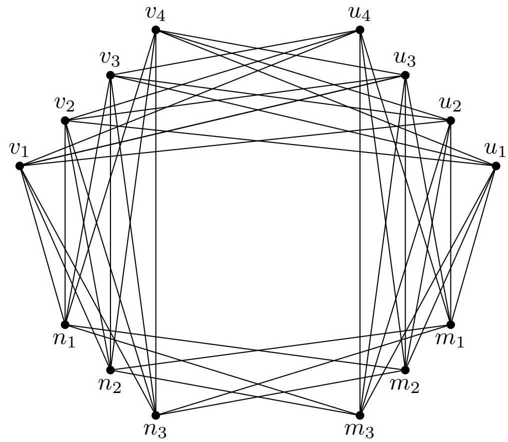

Figure 3: The graph in $\mathcal{K}$ for $k = 3$.

Theorem 5.12. *Let G be a graph of order n with $g(G) = 4$. Then $IC(G) = n$ if and only if $G \in \{C_4, \mathcal{K}_0\} \cup \mathcal{K}$.*

Proof. It is easy to check that $IC(C_4) = 4$ and that $IC(\mathcal{K}_0) = 8$. Now let $G \in \mathcal{K}$. We observe that for each $1 \leq i \leq 4$, $\{v_i\}$ and $\{u_i\}$ are ic-partners, and that for each $1 \leq i \leq k$, $\{n_i\}$ and $\{m_i\}$ are ic-partners. Thus, $IC(G) = n$. Conversely, let G be a graph of order n with $g(G) = 4$ and $IC(G) = n$, and let C be a cycle of G of order 4 with $V(C) = \{x, y, z, t\}$ and $E(C) = \{xy, yz, zt, tx\}$. If $G = C$, then the desired result follows. Hence, we assume that $G \neq C$. Since x is adjacent to y and t, neither $\{y\}$ nor $\{t\}$ is an ic-partner of $\{x\}$. Now consider two cases.

Case 1. $\{x\}$ and $\{z\}$ are ic-partners. In this case, G is dominated by $\{x, z\}$. Let $A = N(x) \setminus \{y, t\}$ and $B = N(z) \setminus \{y, t\}$. If $A \neq \emptyset$, (say $v \in A$), then it is not difficult to check that $\{v\}$ has no ic-partner. Thus, $A = \emptyset$, and so by symmetry, we have $B = \emptyset$. Hence, $G \simeq C_4$.

Case 2. $\{x\}$ and $\{z\}$ are not ic-partners. Let $\{e\}$ be an ic-partner of $\{x\}$. Since $\{x, e\}$ dominates G and z is not adjacent to x, it must be adjacent to e. Let $A = N(x) \setminus \{y, t\}$ and $B = N(e) \setminus \{z\}$. It is not difficult to verify that $A \cap B = \emptyset$. Now if $A = \emptyset$, then $\{z\}$ cannot form an independent coalition with any other set, so $A \neq \emptyset$. Let $\{f\} \subseteq A$ be an ic-partner of $\{z\}$. We note that if a set $\{g\}$ forms an independent coalition with $\{y\}$, then $g \in B$. Further, if a set $\{h\}$ forms an independent coalition with $\{t\}$ then $h \in B$. Let $\{g\}$ and $\{h\}$ be ic-partners of $\{y\}$ and $\{t\}$, respectively. Observe that $\{g\} \neq \{h\}$. Now let $A' = A \setminus \{f\}$ and $B' = B \setminus \{g, h\}$. There exist the following subcases.

Subcase 2.1. $A' = \emptyset$ and $B' = \emptyset$. In this case, we have $G \simeq \mathcal{K}_0$.

Subcase 2.2. $A' = \emptyset$ and $B' \neq \emptyset$. Let $v \in B'$. One can verify that $\{v\}$ cannot form an independent coalition with any other set. Thus, this case is impossible.

Subcase 2.3. $A' \neq \emptyset$ and $B' = \emptyset$. Let $v \in A'$. One can verify that $\{v\}$ cannot form an independent coalition with any other set. Thus, this case is impossible.

Subcase 2.4. $A' \neq \emptyset$ and $B' \neq \emptyset$. Let $v \in A'$. If a set $\{u\}$ forms an independent coalition with $\{v\}$, then $u \in B'$. Furthermore, for each vertex $u \in B'$, $\{u\}$ cannot form an independent coalition with more than one sets $\{v\} \subseteq A'$. Thus, $|A'| \leq |B'|$. Using a similar argument, we deduce that $|B'| \leq |A'|$, and so $|A'| = |B'|$. Consequently, the following statements hold in the graph G:

- $G[\{x, y, z, t, e, f, g, h\}]$ is a bipartite graph with partite sets $V_1 = \{x, z, g, h\}$ and $V_2 = \{y, t, e, f\}$, which is isomorphic to $\mathcal{K}_0$,
- $[V_1, A']$ is full and $[V_2, B']$ is full,
- $[V_1, B']$ is empty and $[V_2, A']$ is empty,
- $G[A' \cup B']$ is a bipartite graph with partite sets A' and B' such that $\deg_{G[A' \cup B']}(v) = |A'| - 1 = |B'| - 1$, for each $v \in A' \cup B'$.

Hence, $G \in \mathcal{K}$ and the proof is complete. □

Using Observation 5.1, Corollary 5.6, Lemmas 5.7, 5.8 and 5.9, and Theorem 5.12, we infer the following result.

Corollary 5.13. *Let G be a triangle-free graph of order n. Then $IC(G) = n$ if and only if $G \in \{C_4, C_5, C_6, P_1, P_2, P_3, P_4, \overline{K}_2, K_1 \cup K_2, K_2 \cup K_2, \mathcal{K}_0\} \cup \mathcal{K}$.*

5.3 Trees T with $IC(T) = n - 1$

The following theorem characterizes all trees T of order n with $IC(T) = n - 1$.

Theorem 5.14. *Let T be a tree of order n. Then $IC(T) = n - 1$ if and only if $T \in \{P_5, P_6, S_{1,2}, K_{1,3}\}$.*

Proof. By Theorem 3.6, we have $IC(P_5) = 4$ and $IC(P_6) = 5$. Further, by Observation 3.2, we have $IC(K_{1,3}) = 3$ and by Observation 3.3, we have $IC(S_{1,2}) = 4$. Conversely, let T be a tree of order n with $IC(T) = n - 1$, where x is a leaf, and y is the support vertex of x. Define $A = N(y) \setminus \{x\}$ and $B = V(G) \setminus (\{x, y\} \cup A)$. Further, let π be an $IC(T)$-partition. Note that π contains a set of cardinality 2 (say $V_1 = \{u, v\}$) and $n - 2$ singleton sets. Since x and y are adjacent, we have $V_1 \neq \{x, y\}$. Note as well that any set in π must be an ic-partner of the set containing x, or the set containing y, to dominate x. We consider two cases.

Case 1. $B = \emptyset$. If $A = \emptyset$, then we have $T \simeq K_2$, and so $IC(T) = 2 \neq n - 1$. Hence, $A \neq \emptyset$, and so $T \simeq K_{1,n-1}$, for some $n \geq 3$. Now by Lemma 3.7, we have $IC(T) = 3$. Hence, $T \simeq K_{1,3}$.

Case 2. $B \neq \emptyset$. Since T is connected, we have $A \neq \emptyset$. We divide this case into some subcases.

Subcase 2.1. $u \in A$ and $v \in B$. We first show that $|A| = 1$. Suppose, to the contrary, that $|A| \geq 2$. Then there is a vertex $z \in A$ such that $z \neq u$. Since z and y are adjacent,

$\{z\}$ cannot be an ic-partner of $\{y\}$, so it must be an ic-partner of $\{x\}$. Since $\{x\}$ does not dominate u, u must be adjacent to z, which is a contradiction, since y, z and u induce a triangle. Now $\{y\}$ cannot be an ic-partner of $\{x\}$ or $\{u, v\}$, so it must have an ic-partner in B. This implies that $|B| \geq 2$. Let $\{t\} \subset B$ be an ic-partner of $\{y\}$. we show that $B \setminus \{v, t\} = \emptyset$. Suppose that $B \setminus \{v, t\} \neq \emptyset$. Let $z \in B \setminus \{v, t\}$. Note that v and t are adjacent. Now $\{z\}$ must be an ic-partner of $\{x\}$ or $\{y\}$, so z must be adjacent to t and v, which is a contradiction, since z, t and v induce a triangle. Hence, $B = \{v, t\}$ and so $T \simeq P_5$.

Subcase 2.2. $\{u, v\} \subseteq B$. An argument similar to the one presented above implies that $|A| = 1$. Now we show that $B \setminus \{u, v\} = \emptyset$. Suppose that $B \setminus \{u, v\} \neq \emptyset$. Let $z \in B \setminus \{u, v\}$, and let $A = \{t\}$. Since t and y are adjacent, $\{t\}$ cannot be an ic-partner of $\{y\}$, so it must be an ic-partner of $\{x\}$. Thus, t must be adjacent to u, v and z. Now $\{z\}$ must be an ic-partner of $\{x\}$ or $\{y\}$, so z must be adjacent to u and v, which is a contradiction, since z, u and t induce a triangle. Hence, $T \simeq S_{1,2}$.

Subcase 2.3. $\{u, v\} \subseteq A$. We first show that $A \setminus \{u, v\} = \emptyset$. Suppose that $A \setminus \{u, v\} \neq \emptyset$. Let $z \in A \setminus \{u, v\}$. Since z is adjacent to y, $\{z\}$ must be an ic-partner of $\{x\}$, so z must be adjacent to u and v, which is a contradiction, since z, u and y induce a triangle. Now we show that $|B| = 1$. Suppose that $|B| \neq 1$. First assume $|B| \geq 3$. Let $z, t, w \in B$. Now z, t and w induce a triangle, since the sets containing each of them, must be an ic-partner of $\{x\}$ or $\{y\}$, a contradiction. Now assume $|B| = 2$. Let $B = \{z, t\}$. Each of the sets $\{z\}$ and $\{t\}$ must be an ic-partner of $\{x\}$ or $\{y\}$. Thus, z must be adjacent to t. Now $\{u, v\}$ must be an ic-partner of $\{x\}$, so z and t must be dominated by $\{u, v\}$. Now the induced subgraph $T[\{u, v, z, t\}]$ contains at least one cycle, a contradiction. Hence, we have $T \simeq S_{1,2}$.

Subcase 2.4. $u = y$ and $v \in B$. we first show that $|A| = 1$. Suppose that $|A| \geq 2$. Let $z, t \in A$. Since z and t are adjacent to y, $\{z\}$ and $\{t\}$ cannot be an ic-partner of y, so each of them must be an ic-partner of $\{x\}$. Thus, z must be adjacent to t, which is a contradiction, since z, t and y induce a triangle. Now we show that $|B| = 2$. Suppose that $|B| \neq 2$. If $|B| = 1$, then $T \simeq P_4$, a contradiction. Otherwise, let $\{v, z, t\} \subseteq B$ and $A = \{w\}$. Now $\{w\}$ cannot be an ic-partner of $\{u, v\}$, so it must be an ic-partner of $\{x\}$. Thus, w must be adjacent to v, z and t. Now observe that $\{u, v\}$ must have an ic-partner in B. Assume, without loss of generality, that $\{u, v\}$ and $\{z\}$ are ic-partners. This implies that t is adjacent to z or v, which is impossible, since both cases lead to existence of an induced triangle. Hence, $T \simeq S_{1,2}$.

Subcase 2.5. $u = x$ and $v \in A$. We first show that $|B| \leq 2$. Suppose that $|B| \geq 3$. Let $\{z, t, w\} \subseteq B$. Note that $\{y\}$ must have an ic-partner in B. Assume, without loss of generality, that $\{y\}$ and $\{z\}$ are ic-partners. It follows that z is adjacent to t and w. Now if $\{y\}$ is an ic-partner of $\{t\}$ or $\{w\}$, then t must be adjacent to w, which is impossible, since z, t and w induce a triangle. Hence, both t and w must be ic-partners of $\{u, v\}$, which implies that t is adjacent to w. Now z, t and w induce a triangle, a contradiction. Now we show that $|A| \leq 2$. Suppose that $|A| \geq 3$. Let $\{z, t, v\} \subseteq A$. The sets $\{z\}$ and $\{t\}$ must be ic-partners of $\{u, v\}$. This implies that z is adjacent to t. Now z, t and y induce a triangle, a contradiction. Further, we observe that the case $|A| = |B| = 2$ is impossible. Hence, either $|A| = 2$ and $|B| = 1$, which implies that $T \simeq S_{1,2}$, or $|A| = 1$ and $|B| = 2$, which implies that $T \simeq P_5$.

Subcase 2.6. $u = x$ and $v \in B$. We first show that $|A| = 1$. Suppose that $|A| \geq 2$. Let $\{z, t\} \subseteq A$. Now each of the sets $\{z\}$ and $\{t\}$ must be an ic-partner of $\{u, v\}$. This implies that z is adjacent to t, which is a contradiction, since y, z and t induce a triangle. Now we show that $|B| \leq 3$. Suppose that $|B| \geq 4$. Let $\{v, t, w, h\} \subseteq B$ and $A = \{z\}$. Note that $\{y\}$ must have an ic-partner in B. Assume, without loss of generality, that $\{y\}$ and $\{t\}$ are ic-partners. It follows that t is adjacent to w, h and v. Now $\{w\}$ must be an ic-partner of $\{y\}$ or $\{u, v\}$. One can observe that both cases lead to contradiction. Hence, either $|B| = 2$, which implies that $T \simeq P_5$, or $|B| = 3$, which implies that $T \simeq P_6$. □

6 Discussion and conclusions

In Proposition 2.2, we introduced a family of graphs admitting no ic-partition. This result motivates the following problem:

Problem 6.1. Characterize graphs admitting an ic-partition.

In Observations 2.3 and 2.4, we presented the sharp inequalities $IC(G) \leq C(G)$ and $IC(G) \geq \chi(G)$. This raises the following problems:

Problem 6.2. Characterize graphs G in which the equality $IC(G) = C(G)$ holds.

Problem 6.3. Characterize graphs G in which the equality $IC(G) = \chi(G)$ holds.

In Theorem 5.14, trees T of order n, with $IC(T) = n - 1$ have been characterized. This raises the following problem:

Problem 6.4. Characterize graphs G of order n with $IC(G) = n - 1$.

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