

Strength of Pressure Vessels with Ellipsoidal Heads

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A method for stress analysis in cylindrical pressure vessels with ellipsoidal heads, based on the axisymmetric shell theory, was proposed. The starting point were the approximate solutions of the differential equation system that were used to get mathematical expressions for determining internal forces, moments and displacements in the vessel walls.

Final expressions that can be applied were acquired by joining the membrane and moment theory and by setting and solving equations of boundary conditions. Diagrams that show distribution of internal forces and moments are in dimensionless form which enables their use regardless of dimensions and load. These expressions were used to develop a method for testing strength of pressure vessels with ellipsoidal heads in the design phase. Application of the method was shown on a selected numerical example, while a special computer programme was created for calculation purposes.

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0 INTRODUCTION

Because of the increasing use of pressure vessels in all types of industry (e.g. oil-refining industry, gas-processing industry, power plant industry, etc.), in transportation, and other processing facilities, and because of their many side effects on the environment, there is a need for accurate analysis of the shells in the design phase. The designer must know the real distribution of stresses, i.e. he must make calculations and analysis of stress in all critical parts of the vessel in order to make an adequate pressure vessel design. Stress is highest at the points of geometry parameter change, e.g. the transition from the shell of the vessel to the head, where higher wall bending usually occurs.

Simple empirical forms determined by norms are used for making calculations of stress in the walls of pressure vessels. This type of calculation does not give the real distribution of stress, the highest values or places where stress occurs. More accurate stress analysis can be done by using finite element method (FEM) or analytical method.

Analytical method for calculating strength of the thin-walled pressure vessels is based on the theory of axisymmetric shells. This theory has a system of the fourth order differential equation with internal forces, moments and displacements as unknown functions. The shell theory was established by famous researchers (Lourye 1947,

Goldenveiser 1953, Novozhilov 1962 and others) who developed analytical methods for determining general solutions of the system. Accurate solutions of the system can be acquired in general form only for pressure vessels of simple geometry (e.g. cylindrical, conical and spherical). In order to apply those solutions in the process of designing, we must calculate numerous complex functions and infinite orders with complex variables, and we must also set and solve an equation of boundary conditions. These solutions can be used to make calculations on a computer, but the calculation procedure and the algorithm structure are very complex.

Far simpler analytical procedure for calculating pressure vessel strength can be developed on the basis of approximate solutions of axisymmetric shell theory equation system [1]. It is of vital importance because the approximate solutions contain an error of the same order of magnitude as do the correct solutions. Relative error of those solutions is of the order h/R (ratio of the wall thickness and the curvature radius), which is less than $1/20$ at thin-walled pressure vessels, i.e. 5%, which can be neglected in engineering calculations.

Approximate solutions of the shell theory can be applied to pressure vessels with more complex shells.

This paper presents solutions for a cylindrical vessel with ellipsoidal heads. Final mathematical expressions for calculating internal

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forces, moments and displacements in the walls of head and cylindrical part were derived from approximate solutions. By using those expressions, and by connecting membrane and moment theory, a method for determining strength of pressure vessels with ellipsoidal heads, which is suitable for designing, was developed. A special computer programme was created for the application of this method. Computer calculation was done on a selected numerical example, and the analysis results were shown in a diagram.

1 MEMBRANE FORCES IN ELLIPSOIDAL HEAD

Stresses in the walls of pressure vessels occur due to different types of loads, depending on the purpose of the vessel and on different influences that a vessel is subjected to during exploitation. Internal pressure has the biggest influence on the amount of stress, so all other types of loads are considered to be less important.

Solutions of the shell theory equation show that internal forces which occur in the walls of the vessel can be, under certain conditions, determined by superimposing of two kinds of loads, membrane and moment. These conditions given in [2] are true for curvature radius derivations of vessel meridian R_1 and surface load q_n .

At membrane stress state, it is assumed that only normal forces occur in the wall, while shear forces and bending moments are not considered. In the design phase, pressure vessels should be formed so that the real stress is approximately the same as the membrane state in order to avoid bending of the walls and high stresses due to bending.

Cylindrical pressure vessels usually have semi-spherical, toroidal-spherical or ellipsoidal heads. Stress is best distributed in ellipsoidal heads. The middle plane of ellipsoidal head is shaped like a half of the ellipsoid of revolution which is the result of ellipse arc rotation around its minor axis. A major ellipsoid semi-axis is equal to cylindrical part radius ($a = R$), and a minor semi-axis is equal to the head height ($b = H$). A parameter determining ellipse form is $\gamma = a^2/b^2 - 1$, and $R_0 = a \cdot \sqrt{1+\gamma}$ is curvature radius of head vertex. Figure 1 shows the basic measures of cylindrical pressure vessels, closed by heads shaped like revolution ellipsoid half.

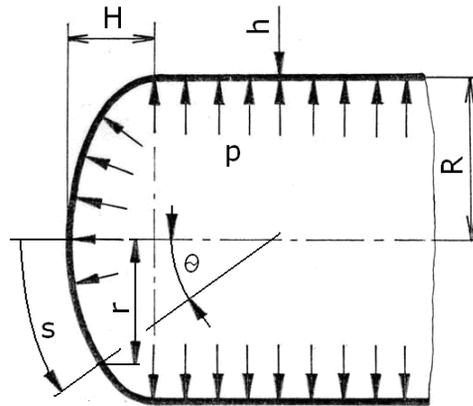


Fig.1. Pressure vessel with ellipsoidal heads

Internal forces in the head walls are determined according to the membrane theory [2] by using the following expressions:

$$N_s^{(0)} = \frac{p R}{2} \left(\frac{1+\gamma}{1+\gamma \sin^2 \theta} \right)^{1/2}, \quad (1)$$

$$N_t^{(0)} = (1-\gamma \sin^2 \theta) N_s^{(0)}, \quad (2)$$

Membrane component of circular force has (in places of shell connection) tensile character on the cylinder, and compressive character on the head, which leads to difference in radial displacements according to [3]:

$$u_r^{(0)''} - u_r^{(0)'}, = \frac{p R^2}{2 E h} (1+\gamma), \quad (3)$$

where:

$N_s^{(0)}$, - meridian normal force in the head wall,

$N_t^{(0)}$, - circular normal force in the head wall,

$u_r^{(0)'}$, - radial displacement of the wall (at the head),

$u_r^{(0)''}$, - radial displacement of the wall (at the cylindrical part),

E - modulus of elasticity of the pressure vessel wall,

h - thickness of the wall,

p - pressure in the vessel,

θ - meridian angle,

R - radius of cylindrical part of the vessel.

Superscript ⁽⁰⁾ denotes values pertaining to a membrane stress condition. Corresponding boundary condition values will be denoted by the superscript ⁽¹⁾, and those without the superscript refer to total solutions.

As can be seen from the expression (1), meridian normal force of ellipsoidal head changes without interruption in the points of transition from the ellipsoid to the cylinder, which is very important for achieving the membrane state of stress. Both normal forces (meridian and circular) reach their maximum value in the head vertex. Value $\gamma = 3$ should be chosen for ellipse parameter, because maximum values of internal forces in the head are the same as values of circular force in the cylindrical part of the vessel.

There is a discontinuity of the membrane component of circular force in the places where head and the cylinder are connected. Since total displacements of the vessel walls should stay intact, additional forces and moments occur in the vessel walls, which lead to bending of the walls. Expressions for determining their values can be acquired from general solutions of shell theory differential equation system.

2 FORCES AND MOMENTS OF BOUNDARY EFFECT

Thin-walled axisymmetric shells can be divided according to the moment theory into short and long. The shells are considered long if:

$$\lambda = \int_0^{s_1} \beta \cdot ds \geq 3, \quad \beta = \frac{\sqrt[4]{3(1-\nu^2)}}{\sqrt{R_2 \cdot h}} \quad (4)$$

With long shells we can disregard the influence of the load of one end of the shell on the internal forces and displacements on the other end. Each end of the shell can be observed independently (i.e. without considering the conditions at the opposite end). Bending of the shell occurs due to radial forces F_0 and bending moments M_0 at the boundary of the shell, which is called boundary load. Boundary load of the long axisymmetric shell is shown in Figure 2.

The condition (4) is regularly achieved for the heads and cylindrical parts of thin-walled pressure vessels.

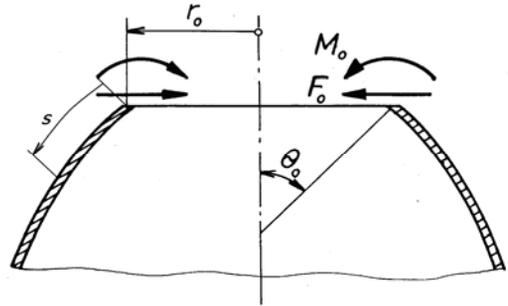


Fig.2. Boundary load of the long axisymmetric shell

With those conditions, general solutions of axisymmetric shell theory differential equation system, acquired by approximate solutions of the system by [4] can be reduced to:

$$u_r^{(1)} = -\frac{F_0}{2D\beta_0^3} \sin^2 \theta_0 \cdot e^{-x} \cos x - \frac{M_0}{2D\beta_0^2} \sin \theta_0 \cdot e^{-x} (\cos x - \sin x), \quad (5)$$

$$g^{(1)} = -\frac{F_0}{2D\beta_0^2} \sin \theta_0 \cdot e^{-x} (\cos x + \sin x) - \frac{M_0}{D\beta_0} \cdot e^{-x} \cos x, \quad (6)$$

$$F_r^{(1)} = F_0 \cdot e^{-x} (\cos x - \sin x) - \frac{2M_0 \beta_0}{\sin \theta_0} \cdot e^{-x} \sin x, \quad (7)$$

$$M_s = \frac{F_0}{\beta_0} \sin \theta_0 \cdot e^{-x} \sin x + M_0 \cdot e^{-x} (\cos x + \sin x), \quad (8)$$

$$M_t = \nu \cdot M_s, \quad N_t^{(1)} = \frac{Eh}{r} \cdot u_r^{(1)}, \quad N_s^{(1)} = F_r^{(1)} \cdot \cos \theta, \quad (9)$$

where:

$$x = \int_0^s \beta \cdot ds, \quad D = \frac{Eh^3}{12(1-\nu^2)}. \quad (10)$$

In expressions (4) to (10) the following was used:

- $F_r^{(1)}$ - radial force in the wall at boundary load,
- $N_s^{(1)}$ - meridian normal force in the wall,
- $N_t^{(1)}$ - circular normal force in the wall,
- $u_r^{(1)}$ - radial displacement of the wall,
- $\vartheta^{(1)}$ - wall twist angle,
- $M_s^{(1)}$ - meridian moment of bending,
- $M_t^{(1)}$ - circular moment of bending,
- ν - Poisson's ratio,
- s - meridian arc length from the shell boundary to a certain point,
- s_1 - total shell length,
- r - radius of the middle plane in a certain point of the wall,
- R - circular radius of wall curvature.

All expressions with a subscript $_0$ are values for the shell boundary. For example, θ_0 and β_0 are values of θ and β in points $x = 0$, at the shell boundary.

In order to apply expressions (5) to (8) it is necessary to determine the values of the force F_0 and moment M_0 of boundary load. They are, by their nature, internal forces of moment theory at the joint of two different shells that superimpose forces of membrane theory, in order to achieve inner mechanical balance.

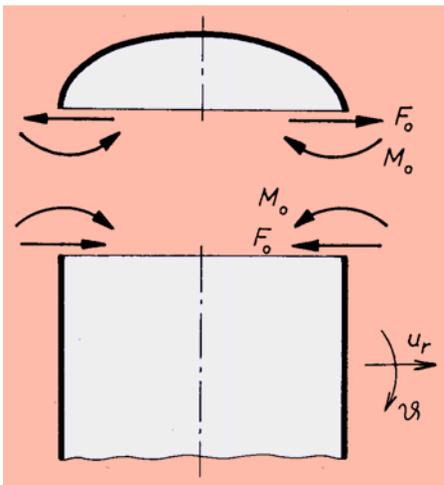


Fig.3. Forces and moments of boundary effect

Their values for the example under consideration (cylindrical part of the vessel with ellipsoidal heads) in Figure 3 will be determined by setting equations of boundary conditions.

Strain continuity at the joint (i.e. place where cylinder and ellipsoid are connected) depends on equal total displacements and twist angles of joined parts, which is expressed by the following equations:

$$u_r^{(0)'} + F_0 \cdot \delta_{11}' - M_0 \cdot \delta_{12}' = u_r^{(0)''} - F_0 \cdot \delta_{11}'' - M_0 \cdot \delta_{12}'' \tag{11}$$

$$-F_0 \cdot \delta_{21}' + M_0 \cdot \delta_{22}' = -F_0 \cdot \delta_{21}'' - M_0 \cdot \delta_{22}'' \tag{12}$$

Membrane components of the wall twist angle, both for ellipsoidal and cylindrical shell, equal null at the angle $\theta = \pi/2$, which can be shown by using membrane theory [5]. Coefficient values of generalized forces influence (F_0 and M_0) on generalized strain (u_r and ϑ), with long shells, according to [4] are:

$$\delta_{11}' = \delta_{11}'' = \frac{\sin^2 \theta_0}{2D\beta_0^3}, \quad \delta_{12}' = \delta_{12}'' = \frac{\sin \theta_0}{2D\beta_0^2},$$

$$\delta_{22}' = \delta_{22}'' = \frac{1}{D\beta_0} \tag{13}$$

where:

$$\beta_0 = \frac{\sqrt[4]{3(1-\nu^2)}}{\sqrt{R h}}$$

By solving equations system (11) and (12), and by considering values (3) we get:

$$M_0 = 0, \quad F_0 = \frac{P}{8\beta_0} (1 + \gamma) \tag{14}$$

By substituting solution (14) into expressions (5) to (10), we get formulas for internal forces and displacements that occur due to boundary effect in the walls of ellipsoidal (') and cylindrical (") part of the vessel:

$$u_r^{(1)'} = \frac{p R^2}{4 E h} (1+\gamma) e^{-x} \cos x, \quad (15)$$

$$\text{where: } x = \int_0^s \beta ds, \quad \beta = \frac{\sqrt[4]{3(1-\nu^2)}}{\sqrt{R_2 h}}. \quad (23)$$

$$u_r^{(1)''} = -\frac{p R^2}{4 E h} (1+\gamma) e^{-x} \cos x, \quad (16)$$

$$M_s' = -\frac{p}{8 \beta_0^2} (1+\gamma) e^{-x} \sin x, \quad (17)$$

$$M_s'' = \frac{p}{8 \beta_0^2} (1+\gamma) e^{-x} \sin x, \quad (18)$$

$$N_t^{(1)'} = \frac{pR}{4} (1+\gamma) e^{-x} \cos x, \quad (19)$$

$$N_t^{(1)''} = -\frac{pR}{4} (1+\gamma) e^{-x} \cos x, \quad (20)$$

$$F_r^{(1)'} = -\frac{p\sqrt{2}}{8 \beta_0} (1+\gamma) e^{-x} \cos(x+\pi/4), \quad (21)$$

$$F_r^{(1)''} = \frac{p\sqrt{2}}{8 \beta_0} (1+\gamma) e^{-x} \cos(x+\pi/4), \quad (22)$$

Distribution of internal forces and moments along the meridian length of the pressure vessel wall, according to expressions (15) to (22), is shown in the diagrams (Figures 4, 5, and 6).

According to the diagram, circular force $N_t^{(1)}$ (Fig.4) and meridian bending moment M_s (Fig. 5) have the biggest influence on stress. Circular moment M_t , according to (9), has smaller values, and components of radial force $F_r^{(1)}$ are of lower order of magnitude so that they can be disregarded in stress calculation.

4 METHOD FOR TESTING STRENGTH

Total values of internal forces in the pressure vessel walls can be acquired by adding components of membrane and moment theory:

$$N_t = N_t^{(0)} + N_t^{(1)}, \quad N_s = N_s^{(0)} + N_s^{(1)}. \quad (24)$$

If we analyse expressions (17) to (22), the following can be concluded. For designing purposes values of internal forces and moments can be observed as dimensionless magnitudes, e.g. $N_s/(p \cdot R)$ and $M_s/(p \cdot R \cdot h)$. Beside the position of the point on the vessel wall, they depend also on design parameters, i.e. on the head shape γ and ratio R/h .

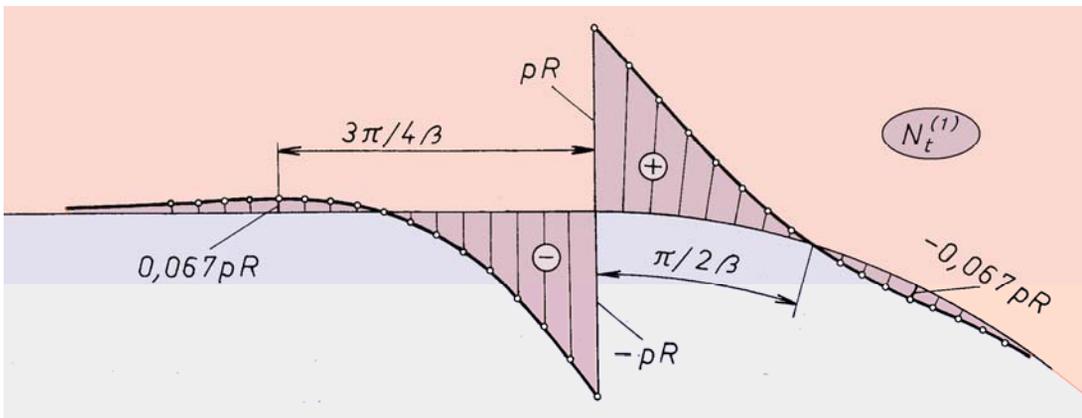


Fig.4. Distribution of circular forces along the vessel wall determined by moment theory

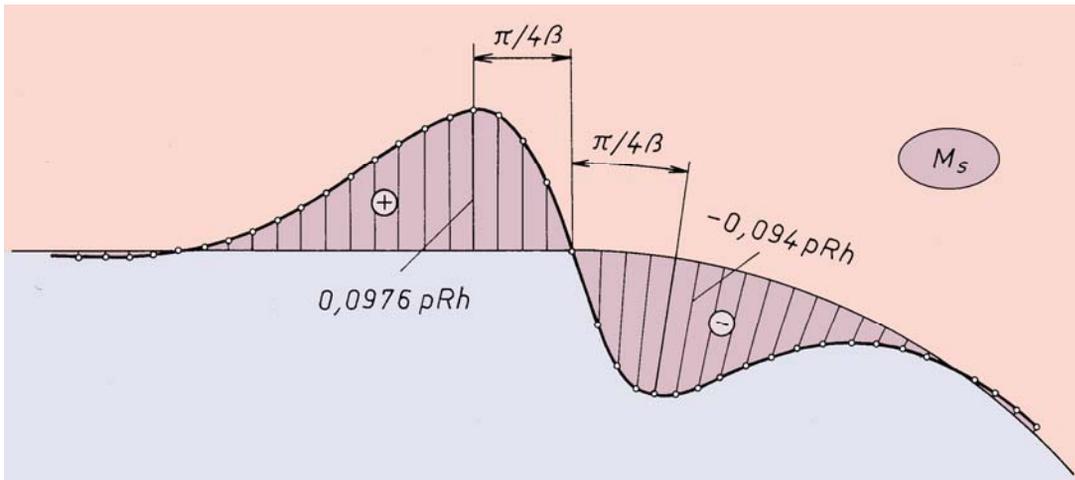


Fig.5. Distribution of meridian moments along the vessel wall determined by moment theory

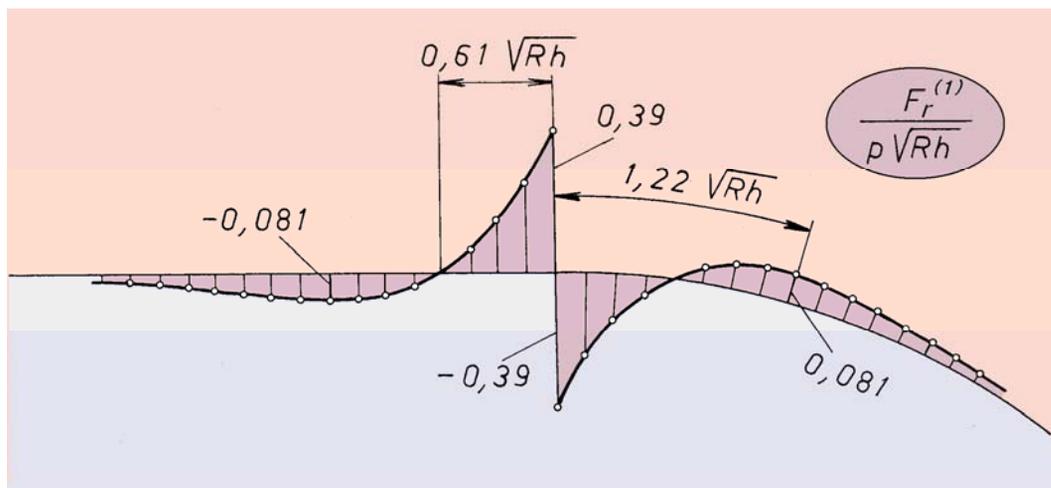


Fig.6. Distribution of radial forces along the vessel wall determined by moment theory

Therefore, by choosing values of these parameters we can test strength before determining final dimensions of the vessel.

In order to perform this procedure successfully, a special computer programme (in Fortran 77) [6], based on mathematical expressions, was created.

This programme can be used to calculate internal forces and moments by using expressions (1) and (2), (15) to (24) in a number of points distributed along the meridian of the vessel. It can

be also used to determine values of main stresses on the inner and outer surface of the wall and to calculate equivalent stresses.

The programme is made in such a way that calculation is performed for the chosen ratio R/h and factor of head shape γ . It was carried out on the example $R/h = 87$ and $\gamma = 3$, and calculated values of internal forces and moments are shown in Table 1 only for a limited number of points. Distribution of total circular force, according to calculated values, is shown in Fig. 7.

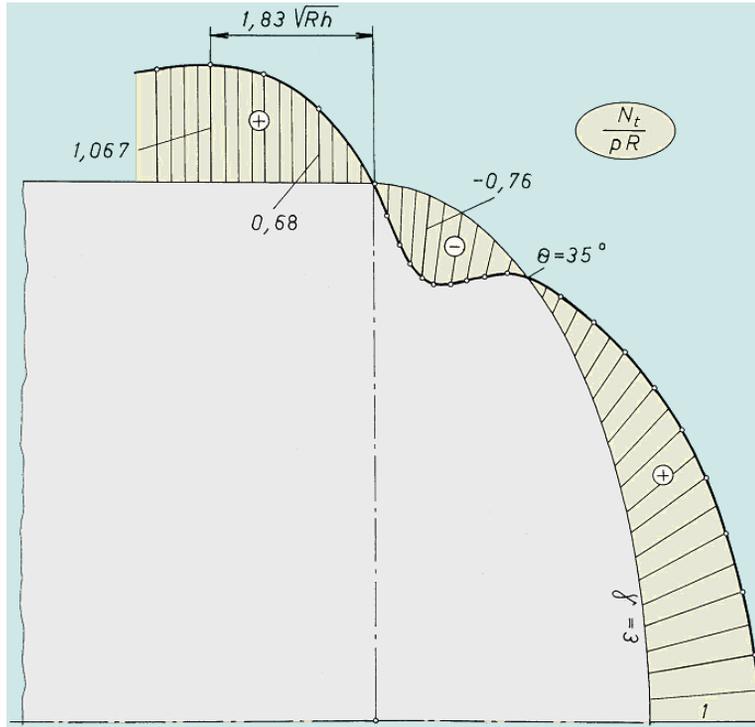


Fig.7. Distribution of total circular forces of the cylindrical vessel with ellipsoidal heads

Equivalent stress according to HMM theory of strength [7] was chosen as a criterion for testing strength of pressure vessel. This programme can be used to calculate equivalent stresses at different points of the wall and to find critical points. Maximum value of equivalent stress was determined for the above mentioned example:

$$(\sigma_e)_{\max} = 1.42 \cdot \frac{p \cdot R}{h} \quad (25)$$

which occurs in the points at the inner wall surface at meridian angle $\theta = 69.61^\circ$ (which is equal to polar angle $\varphi = 5.31^\circ$). On the outer surface of the wall, maximum stress occurs at the head vertex, and it is much lower in the mentioned example.

Stress determined by the expression (25) is the basis for the choice of pressure vessel material (concerning the necessary strength), after the shape and dimensions of the pressure vessel have been determined. These two steps in the design phase can be done iteratively, i.e. by varying design R/h and γ , as input data, we can

get values of maximum equivalent stresses for different dimension ratios.

5 CONCLUSION

Approximate solutions of axisymmetric shell theory used for expressions (15) to (22) are valid for steep shells, i.e. those with big θ angle. This condition is found in the vicinity of the place where the head and the cylinder are joined (i.e. where angle θ slightly differs from 90°). It is better to use these expressions instead of exact solutions because they consist of simple mathematical functions, so that there is no need to consider boundary conditions, since they are already included in final expressions.

These expressions clearly show the law of distribution of internal forces and moments in the walls by using diagrams, which was done for ellipsoidal heads with $R/H=2$ ratio, since they are commonly used in praxis. These diagrams can be used for analysis of stress and for choosing dimensions during the design phase of such heads. Computer programme enables quick acquisition of data and representation of such diagrams for other R/H ratios.

Table 1. Values of internal forces and moments in the ellipsoidal head of cylindrical pressure vessel

Cylindrical vessel with ellipsoidal heads, $\gamma = 3$, $R/h = 87$					
$\hat{\theta}$	\hat{x}	$\frac{N_t}{p \cdot R}$	$\frac{N_s}{p \cdot R}$	$\frac{M_t}{p \cdot R \cdot h}$	$\frac{M_s}{p \cdot R \cdot h}$
0	11.59	1.00	1.00	0	0
$\pi/32$	9.950	0.9574	0.9859	0.000	0.000
$\pi/16$	8.417	0.8391	0.9474	0.000	0.000
$3\pi/32$	7.063	0.6682	0.8934	0.000	0.000
$\pi/8$	5.910	0.4699	0.8335	0.000	0.000
$5\pi/32$	4.947	0.2599	0.7746	0.000	0.0021
$3\pi/16$	4.145	0.0448	0.7206	0.0012	0.0040
$7\pi/32$	3.472	-0.1689	0.6731	0.0009	0.0031
$\pi/4$	2.901	-0.3696	0.6325	-0.0012	-0.0040
$9\pi/32$	2.408	-0.5412	0.5984	-0.0055	-0.0182
$5\pi/16$	1.975	-0.6672	0.5704	-0.0116	-0.0386
$11\pi/32$	1.588	-0.7338	0.5477	-0.0186	-0.0618
$3\pi/8$	1.234	-0.7309	0.530	-0.0249	-0.0831
$13\pi/32$	0.9056	-0.6531	0.5166	-0.0289	-0.0963
$7\pi/16$	0.5946	-0.4996	0.5073	-0.0281	-0.0935
$15\pi/32$	0.2946	-0.2764	0.5018	-0.0196	-0.0654
$\pi/2$	0	0	0.5	0	0

The described method for calculating strength, performed by a computer programme, gives data on critical stress in the form (25), in a short period of time, for arbitrarily chosen geometry parameters of ellipsoidal head, which makes this method very suitable for designing purposes.

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